642.581 Problem Set 5 (final)

1. A stronger version of Turán's Theorem: Show that if $G \not\supseteq K_{r+1}$, then there is an r-partite H on $V = V(G)$ with $d_H(v) \geq d_G(v)$ for every $v \in V$.

[Try induction, starting with a vertex of maximum degree. If you have the construction, you can go light on the justification.]

2. Let H be a graph with $v_H = h$ and $\chi(H) = r+1$; let u be the minimum possible size of a color class in a (proper) $(r+1)$-coloring of H; and set

$$\gamma = \gamma(H) = (h-u)/(rh).$$

For a graph G with $h|n = v_G$, an *H-factor* of G is n/h vertex-disjoint copies of H. Show that (for n divisible by h)

$$T(n,H) := \min\{t : v_G = n \text{ and } \delta(G) \geq t \text{ imply } G \text{ has an } H\text{-factor}\}$$

is at least $(1-\gamma)n$.

3. Show that (as mentioned in class) $\mathrm{ex}(n, C_4) = O(n^{3/2})$.

4. Improve Erdős' lower bound on $R(k,k)$ to $(1-o(1))(k/e)2^{k/2}$.

5. Define the *Hadwiger number* of G to be $\tau(G) = \max\{m : G \succ K_m\}$. Show that for $G = G_{n,1/2}$ and a suitable fixed C,

$$\tau(G) < Cn/\sqrt{\log n} \text{ w.h.p.}$$

[In fact $\tau(G) = \Theta(n/\sqrt{\log n})$ w.h.p. You could try this, but it's a little harder and not part of the problem.

Hint: Show that, for suitable C and $m = Cn/\sqrt{\log n}$, w.h.p. every partition $V_1 \cup \cdots \cup V_m$ of $V = V(G)$ has $\nabla_G(V_i, V_j) = \emptyset$ for some (distinct) i and j.]

[A neat, maybe (or maybe not) challenging last problem:]

6. Say G is *H-Ramsey* if every coloring $E(G) = R \cup B$ has a monochromatic copy of H. Let $r(H) = \min\{n : K_n$ is H-Ramsey and

$$r_2(H) = \min\{e_G : G \text{ is } H\text{-Ramsey}\}.$$

Show that (for any s) $r_2(K_s) \geq \binom{r(K_s)}{2}$.

[So this is actually an equality since $K_{r(K_s)}$ is K_s-Ramsey. Hint: What can you say about $\chi(G)$ if G is K_s-Ramsey?]