581 PS1 Solutions

[These will be pretty quick, and it may take a little thought to see why they're correct; but *all are solutions I'd accept without further justification.* (Later posted solutions may be sketchier.)]

1. We have $\sum_{v\in V} d(v) = 2|E(T)| = 2(n-1)$. On the other hand, if t is the number of vertices of degree 1, then

$$\sum_{v\in V} d(v) \geq \Delta + t + (n-t-1)2 = \Delta - t + 2(n-1).$$

So $t \geq \Delta$.

2. The necessary and sufficient condition is that G has no odd components. Necessity is obvious. For sufficiency:

WMA G is a tree (why?). We use induction (say on $n = |V(G)|$): If G has no even (degree) vertex then we are done. But if x is an even vertex then the two components of $G - e$ are even for at least one $e \ni x$. (Because: if the edges at x are $e_i = xx_i$, $i \in [2k]$, and the component of x_i in $G - e_i$ has t_i vertices, then $\sum_{i=1}^{2k} t_i = n - 1$ is odd, so at least one t_i is even.)

3.(a) We first specify a graph H and $u \in V(H)$: If all degrees in G are even, $H = G$ and u is arbitrary. Otherwise H is G plus a new vertex u with $N_H(u)$ consisting of the odd vertices of G; so H has all degrees even (why?).

By Euler's Theorem H has an Euler tour C. Start at u and follow C, coloring edges alternately R and B. Then each $x \in V(G)$ has $d_R(x) = d_B(x)$ (though $d_R(u) = d_B(u) \pm 2$ is possible if $H \neq G$; here we've used the fact that that $|E|$ is even if $H = G$). So the restriction of $R \cup B$ to $E(G)$ does what we want.

(b) If all degrees are even the desired partition requires that $d_R(v) = d_B(v)$ $\forall v$; but then $|R| = \sum d_R(v)/2 = \sum d_B(v)/2 = |B|$, so $|E|$ is even.