581 PS2 Solutions

1. The answer (call it $f(n)$) is n^2. For $f(n) \geq n^2$ take

$$V = \{x_1, \ldots, x_n, y_1 \ldots, y_n\}, \quad E = \{y_i y_j : 1 \leq i < j \leq n\} \cup \{x_i y_j : i \leq j\}.$$

For $f(n) \leq n^2$, let $\{e_1, \ldots, e_n\}$ be the perfect matching and *observe:* there are at most two edges between the ends of e_i and the ends of e_j. Thus

$$|E| \leq n + \tbinom{n}{2} 2 = n^2.$$

2.(a) Let the values be $a_1, \ldots, a_m$ and the columns $C_1, \ldots, C_m$, with a_i appearing t_{ij} times in C_j. Form a bipartite multigraph G with bipartition $\{a_1, \ldots, a_m\} \cup \{C_1, \ldots, C_m\}$ and t_{ij} edges joining a_i and C_j for each i, j. Then G is n-regular, so (by the corollary to Hall's Theorem given in class) has a perfect matching, which is equivalent to what we want.

(b) Fix a suit S and switch each of the cards from (a) with the card of the same value from S. (If the card is already in S then the "switch" does nothing.) This gives a card from S in each column; we can then remove these and say induction.

3. We use induction (on whatever). Suppose first that there is some edge $e = ab$ not in any p.m.s, and let $H = G - \{a, b\}$. This has no p.m., so by Hall's Theorem there is an $X \subseteq A \setminus \{a\}$ with $|N_H(X)| < |X|$. OTOH (since $\Phi(G) > 0$) $Y := N_G(X)$ has $|Y| \geq |X|$. It follows that: $Y \setminus N_H(X) = \{b\}$; $|Y| = |X|$; and each p.m. of G matches X with Y and $A \setminus X$ with $B \setminus Y$, so both $G_1 := G[X \cup Y]$ and $G_2 := G[(A \setminus X) \cup (B \setminus Y)]$ have p.m.s. But then G_1 satisfies our hypotheses, so by induction, $\Phi(G) = \Phi(G_1)\Phi(G_2) \geq d! \cdot 1 = d!$

Now suppose every edge is in a p.m. and let $a \in A$. For each $b \sim a$, $G_b := G - \{a, b\}$ has a p.m. and $d_{G_b}(a') \geq d - 1 \ \forall a' \in A \setminus \{a\}$; so induction gives $\Phi(G_b) \geq (d-1)!$. But $\Phi(G_b)$ is the number of p.m.s of G containing ab, so $\Phi(G) = \sum_{b \sim a} \Phi(G_b) \geq d(d-1)! = d!$

[Alternate: Skip "first" above and consider a minimal $S \subseteq A$ with $|N(S)| = |S|$ (noting that this holds when $S = A$).]

4. The conditions I had in mind are

$$|A| = |B| \quad \text{and} \quad |N(X)| > |X| \ \ \forall\, \emptyset \neq X \subset A. \tag{1}$$

Necessity. For w as in the problem, $|A| = w(E) = |B|$ and, for X as in (1),

$$|X| = w(\nabla(X)) < w(\nabla(N(X)) = |N(X)|,$$

where the inequality follows from $\nabla(X) \subset \nabla(N(X))$ (the containment is strict since otherwise $N(N(X)) = X$ and G is not connected).

Sufficiency. First notice that each $e = uv \in E$ is in a perfect matching (p.m.): equivalently $G' := G - \{u, v\}$ has a p.m., which is true since ((1) gives Hall's condition for G'. Let m_e (> 0) be the number of p.m.s containing e, m the number of p.m.s in G and $w(e) = m_e/m$ (and check this works).

5. (One of several ways to do this:) Let G be a balanced bigraph on (A, B) satisfying Hall's condition. Note that balance implies that we also have Hall's condition $T \subseteq B$, since $N(A \setminus N(T)) \subseteq B \setminus T$ and, by Hall's condition, $|B \setminus T| \geq |A \setminus N(T)|$, i.e. $|N(T)| \geq |T|$.

Now suppose Tutte's condition fails, and let $C \subseteq A \cup B$ be maximal with $q(G - C) > |C|$. Set $C \cap A = X, C \cap B = Y$. Since C is maximal, each component of $G - C$ is either balanced or a singleton (why?); so the *odd* components are the isolated vertices. Let S and T be the sets of isolated vertices in A and B (in $G - C$). Hall's condition gives $|S| \leq |Y|$ and $|T| \leq |X|$, so we have the contradiction

$$q(G - C) = |S| + |T| \leq |X| + |Y| = |C|.$$

6. *Claim:* Any orientation σ minimizing $\beta_\sigma := \sum(d^+_\sigma(v) - k(v))^+$ works (where, as usual, $x^+ = \max\{x, 0\}$).

Proof: Suppose instead that $d^+_\sigma(v) > k(v)$ and let W be the set of vertices reachable from v (in σ). Then $\nabla^+(W) = \emptyset$, implying $\sum_{w \in W} d^+_\sigma(w) = |E(W)| \leq \sum_{w \in W} k(w)$. So there must be some $w \in W$ with $d^+_\sigma(w) < k(w)$, and we can reverse edges on some (v, w)-path to improve β.

[Alternate: Define a bigraph H on $E \cup Y$ with Y consisting of $k(v)$ copies of v $\forall v \in V$ and e adjacent to all copies of its ends (in G). Then (*check*) H satisfies Hall's condition, so contains an E-perfect matching M, yielding orientation: v is the tail of e if M matches e with some copy of v.]

7. (WMA n is even, say $n = 2k$: then for n odd add $n+1$ and $a_{n+1} = n+1$.)

Let $M = \{\{2i - 1, 2i\} : i \in [k]\}$, $M' = \{\{a_{2i-1}, a_{2i}\} : i \in [k]\}$. Then $M \cup M'$ is (the edge set of) a bipartite graph, say with bipartition $X \cup Y$. Let $f(i)$ be 1 if $i \in X$ and -1 if $i \in Y$. Then $f(2i - 1) + f(2i) = f(a_{2i-1}) + f(a_{2i}) = 0$ $\forall i \in [k]$ and the result follows.

8. Define network $N = ((V, A), c, s, t)$ with $V = \{s, t, r_1, \ldots, r_m, c_1, \ldots, c_n\}$,

$$A = \{sr_i : i \in [m]\} \cup \{r_i c_j : i \in [m], j \in [n]\} \cup \{c_j t : j \in [n]\},$$

$c(sr_i) = a'$, $c(c_j t) = b'$ and $c(r_i c_j) = m_{ij}$.

Then f given by $f(sr_i) = a'$, $f(c_j t) = b'$ and $f(r_i c_j) = \delta m_{ij}$ is a max flow in N (e.g. since $\mathrm{val}(f) = \mathrm{cap}(\nabla(s))$); so there is an integer flow g of the same value, and we may take $m'_{ij} = g(r_i c_j)\ \forall i, j$.

9. Let K be the vertex set of a largest clique in G; say $|K| = r$. Maximality of K implies that for each $v \in V(G) \setminus K$, $\nabla(v, K)$ can be covered by a clique of size at most r. So, using induction for the second inequality, we have

$$\begin{aligned} \pi(G) &\leq \pi(G - K) + (n - r)r + |K| \\ &\leq 2\lfloor \tfrac{n-r}{2} \rfloor \lceil \tfrac{n-r}{2} \rceil + (n - r)r + r. \end{aligned}$$

And *check* this is at most $2\lfloor n/2 \rfloor \lceil n/2 \rceil$.