581 PS3 Solutions

1. Apply Menger's Theorem (arc version) to the digraph $D = (V, A)$, where $A = \{uv : \{u, v\} \in E;\ d_G(x, v) = d_G(x, u) + 1\}$.

2. This uses only the fact, which follows from the degree assumption, that

$$d(x, y) \leq 2 \ \ \forall x, y \in V(G).$$

Let $F = \nabla(X)$ be a smallest edge separator in G. Then $V(F)$ contains one of X, $\overline{X}$, since if $x \in X \setminus V(F)$ and $y \in \overline{X} \setminus V(F)$, then $d(x, y) \geq 3$. So suppose w.l.o.g. that $X \subseteq V(F)$, and choose $x \in X$ with $d_F(x)$ minimum. Then with $|X| = m$ we have (as in an argument from class)

$$\delta(G) \leq d_G(x) \leq m - 1 + d_F(x) \leq m - 1 + |F|/m \leq |F|.$$

3. Let C be a cycle with $V(C) \cap X$ as large as possible; w.l.o.g. $V(C) \cap X = \{x_1, \ldots, x_l\}$ with $l < k$. Let the paths $P_1, \ldots, P_m$ form an $(x_k, V(C))$-fan, with y_i the end of P_i in C and $m = \min\{k, |C|\}$; so either $m > l$ or $V(C) = \{x_1, \ldots, x_l\}$. In either case there are $i \neq j$ such that one of the two arcs (say Q) of C joining y_i, y_j has no internal vertices in X. But then $(C \setminus Q) \cup P_i \cup P_j$ is a cycle containing $\{x_1, \ldots, x_l, x_k\}$, contradicting our choice of C.

4. Answer: $\lceil \log_r k \rceil =: t$. To see that t is an upper bound, let $(V_\alpha : \alpha \in [r]^t)$ be a partition of V into independent sets (exists since $r^t \geq k$), and for $x \in V$ let $\alpha(x) = \alpha$ if $x \in V_\alpha$. Then for $i \in [t]$ let $V(G_i) = V(G)$ and

$$E(G_i) = \{xy \in E(G) : \alpha_i(x) \neq \alpha_i(y)\}.$$

To see that t is a lower bound: Given $G = \cup_{j=1}^m G_j$ with $\chi(G_j) \leq r$ (WMA $V(G_j) = V$), let σ_j be an r-coloring of G_j and set $\sigma(v) = (\sigma_1(v), \ldots, \sigma_m(v))$. Then σ is a coloring of G using at most r^m colors, so $m \geq \log_r \chi$. (The real statement here is: if $G = \cup G_i$ then $\chi(G) \leq \prod \chi(G_i)$.)

5.(a) The statement is trivial for $n = 2$, so let $n \geq 3$ and proceed by induction. WMA G is connected. Fix $x \in V$ and let $V_k = \{y : d(x, y) = k\}$. The main observation is that $G[V_k] \not\succ K_{n-1}$: otherwise we get a K_n-minor of G by contracting $V_0 \cup \cdots \cup V_{k-1}$ to a single vertex (and deleting superfluous edges and vertices). So (by induction) we can use 2^{n-3} colors to color $\cup_{i \geq 0} V_{2i}$ and 2^{n-3} *other* colors to color $\cup_{i \geq 0} V_{2i+1}$.

Alternate: We know (see *coloring number*) that if $\chi(G) > 2^{n-2}$ then $G \supseteq H$ with $\delta_H \geq 2^{n-2}$; so it is ETS (for $n \geq 2$, with $\bar{d}$ denoting average degree)

$$\overline{d}_H \geq 2^{n-2} \Rightarrow H \succeq K_n.$$

We show this by induction on n (with $n = 2$ trivial). If there is some $uv \in E(H)$ with $|N_u \cap N_v| \leq 2^{n-3}-1$, then H/uv (with multiple edges deleted) has $\bar{d}$ at least $(|V(H)|-1)^{-1}[|V(H)|2^{n-2} - 2^{n-2}] = 2^{n-2}$ (right?) and induction gives $H \succeq H/uv \succeq K_n$. But if there is any u with $N_u \cap N_v \geq 2^{n-3}$ $\forall v \sim u$, then induction gives $H[N_u] \succeq K_{n-1}$ and adding u gives $H \succeq K_n$.

(b) Let G be a minimal counterexample. WMA (see the proof of Brooks' Theorem) that G is 3-connected and not complete. Then for any $x \in V$ there are a cycle C in $G - x$ and a 3-linking of x to C; and their union is a topological K_4 (which can be contracted to produce a K_4-minor).

6. Define $\varphi : \mathcal{A} \to \mathcal{B}$ sending $\sigma \in \mathcal{A}$ to σ', where we obtain σ' from σ by switching R and B in the (R, B)-component $(C(\sigma))$ of y in σ. Then:

(a) $x \notin C(\sigma)$ implies $\sigma' \in \mathcal{B}$;

(b) we obtain σ from σ' by switching R and B in the (R, B)-component of y in σ', so φ is an injection; and

(c) if $\tau \in \mathcal{B}$ and x, y are in the same (R, B)-component in τ, then τ does not belong to the image of φ (and there do exist such τ).