581 PS4 Solutions

1. We use induction. Since the condition is inherited by induced subgraphs, WMA $\chi(H) = \omega(H)$ for all *proper* induced subgraphs H.

Let $X \subseteq V$ be a minimal cutset.

Claim. $G[X]$ is a clique.

This gives $\chi(G) = \omega(G)$ via induction and the old observation that if $H \cap K$ is a clique, then $\chi(H \cup K) = \max\{\chi(H), \chi(K)\}$.

Proof of Claim. Suppose $x, y \in X$ and $x \not\sim y$. Minimality of X implies that for any component C of $G - X$, $G[V(C) \cup \{x, y\}]$ contains an (x, y)-path (since $C \cap N(x),\ C \cap N(y) \neq \emptyset$ and C is connected).

Let C, D be distinct components of $G - X$, and let P, Q be *shortest* (x, y)-paths in $G[V(C) \cup \{x, y\}]$, $G[V(D) \cup \{x, y\}]$. Then $P \cup Q$ is a chordless cycle of G of length at least 4, a contradiction.

2. (The conjecture is due to Albertson, Grossman and Haas, 1998)

Fix a proper coloring $\sigma : V \to [\chi]$ (where $V = V(G)$ and $\chi = \chi(G)$). Given lists $S(v)$ of size t, let $\gamma(s)$, $s \in S := \cup S(v)$, be chosen uniformly and independently from $[\chi]$, and let

$$T = \{v \in V : \gamma^{-1}(\sigma(v)) \cap S(v) \neq \emptyset\}.$$

Then (for any choice of γ) $G[T]$ admits an S-coloring. (Any coloring that assigns each $v \in T$ some color from $S(v) \cap \gamma^{-1}(\sigma(v))$ is proper.) On the other hand, $\mathbb{E}|T| = (1 - (1 - 1/\chi)^t)n$, since $\mathbb{P}(v \in T) = 1 - (1 - 1/\chi)^t$ for each $v \in V$ (etc.).

3.(a) Let $1 \ll l \ll d$ and let $\mathcal{C}$ be the family produced by the graph container theorem proved in class; thus:

- $|\mathcal{C}| < \binom{n}{\leq n/l} = 2^{o(n)}$;
- $C \in \mathcal{C} \Rightarrow |C| < (1 + l/d)n/2 + n/l = (1 + o(1))n/2$; and
- each $I \in \mathcal{I}(G)$ is contained in some $C \in \mathcal{C}$.

Since $I \in \mathrm{MIS}(G[C])$ if $I \in \mathrm{MIS}(G)$ and $C \supseteq I$, item (1) from the problem (with the above properties of $\mathcal{C}$) gives

$$\mathrm{mis}(G) \leq \sum_{C \in \mathcal{C}} \mathrm{mis}(G[C]) \leq 2^{o(n) + (1+o(1))n/4}.$$

(b) Let W, X, Y, Z be disjoint $(d-1)$-sets and let G be the graph with vertex set $W \cup X \cup Y \cup Z$ and edge set consisting of two perfect matchings, one between W and X, the other between Y and Z, plus the edges of the complete bipartite graphs $K_{W,Y}$ and $K_{X,Z}$. And *check:*

$$\mathrm{mis}(G) = 2 \cdot 2^{(d-1)/4} - 2 \ (\ = 2 \cdot 2^{n/4} - 2).$$

4. If for some i, $S_{x_i} \cap S_{y_i} \neq \emptyset$, then color x_i and y_i with the same color, delete this color from the remaining lists, and say induction. Otherwise use Hall's Theorem (how?).

5. Let $E_1 = E(W)$ and $E_2 = \nabla(W, V \setminus W)$; so $\sum_{x \in W} d(x) = 2|E_1| + |E_2|$. We know $|E_1| \leq 3k - 6$ and, since (V, E_2) is bipartite, $|E_2| \leq 2n - 4$. The result follows.

6. WMA G is a triangulation (since adding a (colored) edge can't decrease the number of color changes at a vertex). Note also that each color change at a vertex is also a color change between consecutive edges on the boundary of some face. So, since all faces are triangles, the total number of color changes is at most $2l = 4n - 8$ (with $l = 2n - 4$ the number of faces), whereas failure of the assertion in the problem would imply at least $4n$ changes.