581 PS5 Solutions

1. Choose x with $d_G(x)$ maximum, say with $N_x = N$ and $V \setminus (N \cup \{x\}) = Y$. Then for H, retain the edges at x, delete the edges of $G[Y]$, and replace:

- $G[N]$ by an $(r-1)$-partite $H[N]$ with $d_{H[N]}(v) \geq d_{G[N]}(v)\ \forall v \in N$ (this is where we use induction), and
- $G[N,Y]$ by $K_{N,Y}$.

2. With $m \in [n/(r+1), n/r)$ TBA, let G be complete $(r+1)$-partite with r parts of size m; so the remaining part, say X, has size $n - rm \leq m$. Then $\delta_G = n - m$, so it's ETS

if $m > \gamma n$, then G has no H-factor.

Proof. The n/h copies of H in an H-factor would use at least un/h vertices of X, but if $m > \gamma n$ then

$$|X| = n - rm < n(1 - (h-u)/h) = un/h.$$

3. Assume $G \not\supseteq C_4$ and write d_x for $d_G(x)$. Then

$$\sum \tbinom{d_x}{2} \leq \tbinom{n}{2}$$

(since *each pair of distinct vertices has at most one common neighbor*), and it follows (using Cauchy-Schwarz and $d_x - 1 \leq (n-1)d_x/n$) that

$$(2e_G)^2 = (\sum d_x)^2 \leq n \sum d_x^2 \leq n^3.$$

4. With n TBA, choose $E(K_n) = R \sqcup B$ uniformly at random and delete a vertex of each monochromatic K_k. With Z the number of monochromatic K_k's, this gives

$$R(k,k) > n - \mathbb{E}Z = n - \binom{n}{k} 2^{1-\binom{k}{2}} > n - \frac{n^k}{k!} 2^{1-\binom{k}{2}} =: y,$$

so we should choose n to make y large. Noting that

$$\frac{dy}{dn} = 1 - \frac{n^{k-1}}{(k-1)!} 2^{1-\binom{k}{2}},$$

take $n = [(k-1)!2^{\binom{k}{2}-1}]^{\frac{1}{k-1}}$ (or its integer part), and *check* that then

$$(n^k/k!)2^{1-\binom{k}{2}} \sim e^{-1}2^{k/2}$$

and $y \sim (k/e)2^{k/2}$.

5. We know that if $G \succ K_m$ then there is a partition

$$[n] = V(G) = V_1 \cup \cdots \cup V_m \tag{1}$$

with

$$E(V_i, V_j) \neq \emptyset \quad \forall\ 1 \leq i < j \leq m. \tag{2}$$

For a given partition as in (1), say with $|V_i| = t_i$ and $t = n/m$ the average of the t_i's, the probability of (2) is (with product and sum over $1 \leq i < j \leq m$)

$$\prod(1 - 2^{-t_i t_j}) < \exp[-\sum 2^{-t_i t_j}] \leq \exp[-\binom{m}{2}2^{-t^2}],$$

where (*exercise*) the second inequality follows from the AM-GM inequality.

But, the number of such partitions is less than m^n, so the probability that at least one of them satisfies (2) (*a fortiori* that $G \succ K_m$) is less than $\exp[n \log n - \binom{m}{2}2^{-(n/m)^2}]$, and (check) this is $o(1)$ if $m \geq cn/\log_2 n$ with c any constant greater than 1.

6. Main point: if G is K_s-Ramsey then $\chi(G) \geq r(K_s)$.

Proof: If not, let $V = V(G) = V_1 \cup \cdots \cup V_m$ be a partition into independent sets, with $m = \chi(G)$, and let $R^* \cup B^*$ be a coloring of $E(K_m)$ with no monochromatic K_s. Then $E(G) = R \cup B$ defined by taking $\nabla(V_i, V_j) \subseteq R$ if $ij \in R^*$, and similarly for B and B^*, shows that G is not K_s-Ramsey. $\square$

To finish just note that $\chi(G) \geq r$ implies $e_G \geq \binom{r}{2}$. (Why?)