Interfacial waves with surface elasticity

Lixin Hu
Department of Mechanical Aerospace Engineering
Rutgers University
Piscataway, NJ 08817
Email:lixin.phd@rutgers.edu

Liping Liu
Department of Mechanical Aerospace Engineering
Department of Mathematics
Rutgers University
Piscataway, NJ 08817
Email:liu.liping@rutgers.edu

In this paper we study the existence and uniqueness of interfacial waves in account of surface elasticity at the interface. A sufficient condition for the existence and uniqueness of a subsonic interfacial wave between two elastic half-spaces is obtained for general anisotropic materials. Further, we explicitly calculate the dispersion relations of interfacial waves for interfaces between two solids and solid & fluid and parametrically study the effects of surface elasticity on the dispersion relations. We observe that the dispersion relations of interfacial waves are nonlinear at the presence of surface elasticity and depend on surface elastic properties. This nonlinear feature can be used for probing the bulk and surface properties by acoustic measurements and designing waves guides or filters.

Article published at *ASME J. Appl. Mech. 81(8), 081007 (May 15, 2014)*

1 Introduction

Interfacial waves refer to localized wave modes that propagate along the interface of two materials and decay away from the interface. The dispersion relation of interfacial waves is important for probing material properties and designing wave guides for a number of applications. For two isotropic elastic materials, Stoneley (1924) first derived explicit solutions of interfacial waves that are subsequently named as *Stoneley waves*. Barnett *et al.* (1985) explored interfacial waves between general anisotropic solids and found sufficient conditions for the existence and uniqueness of *subsonic* interfacial waves in terms of surface impedance tensor.

At the advent of modern nanotechnology, it is widely speculated that elastic energy associated with a surface, or surface elasticity, will play an important role in determining the size-dependent behaviors at the length scale of submicron and below (Sharma et al., 2003; Miller and Shenoy, 2000). A widely used model of surface elasticity has been established by Gurtin and Murdoch (1975, 1976) where surface/interface is idealized as a two-dimensional body Γ with elastic stored energy postulated as

$$U_s[\mathbf{y}] = \int_\Gamma W_s(\nabla \mathbf{y}) ds, \tag{1}$$

where $W_s : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is the surface elastic energy density and $\mathbf{y} : \mathbb{R}^3 \to \mathbb{R}^3$ is the deformation. For a homogeneous continuum body the above surface elastic energy may be regarded as the next order of approximation of total internal energy beyond the bulk elastic energy. This is to some extent justified from the fundamental atomistic models in Blanc et al. (2002) and the elastic properties of surface have been calculated according to this viewpoint (Shenoy, 2005; Mi et al., 2008). From this standing point, it is anticipated that surface elasticity is particularly important for small bodies.

Surface elasticity may emerge from other considerations. First of all, as noticed in Mohammadi et al. (2013), an elastic surface may arise solely from the roughness of surfaces/interfaces and bulk elasticity even if the pristine flat surface is assumed to be free of surface elasticity. Also, for some heterogeneous structures, e.g., a sandwich structure with soft thick core and stiff thin face plates (Liu and Bhattacharya, 2009), the overall structure may be well modelled by a single elastic body with elastic surfaces. For these problems, it is worth noticing that the significance of "surface elasticity" prevails at all length scales instead of being limited to small bodies, which, consequently, broadens the applications of the model of surface elasticity and the results presented in this paper.

The ramifications of surface elasticity have been examined in several contexts, e.g., the *effective* bulk stress-strain

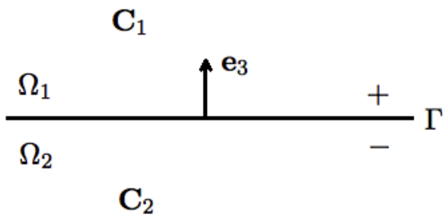

Fig. 1. An elastic interface between two half spaces.

relation due to nano-inclusions (Sharma et al., 2003; Duan et al., 2005), the sensing and vibration of nano-beams and plates (Miller and Shenoy, 2000; Bar On et al., 2010), wave in thin film attached on substrate (Steigmann and Ogden, 2007), and the free surface waves (Murdoch, 1976). The interested reader is also referred to Steigmann and Ogden (1997) for a generalization of surface elasticity incorporating curvature dependence of energy, Huang and Sun (2007) for further clarification of the formulation, and Altenbach et al. (2011) for a mathematical proof of existence and uniqueness theorem of boundary value problems with surface elasticity.

In this paper we study interfacial waves at the presence of surface elasticity. Since the energy of interfacial waves concentrates around the interface, we anticipate surface elasticity may have a significant effect on the dispersion relation of interfacial waves. In addition, it is of fundamental interest to prove whether an interfacial wave exists, and if so, is unique for a given frequency. These problems will be addressed by techniques developed in the study of classic free surface waves and interfacial waves in the absence of surface elasticity (Rayleigh 1885; Stoneley, 1924; Chadwick, 1977; Barnett et al., 1985). In particular, we obtain a similar existence and uniqueness theorem for *subsonic* interfacial waves between general anisotropic solids and interfaces. In addition, we explicitly calculate the dispersion relations of interfacial waves at the presence of surface elasticity for isotropic materials. A critical observation lies in that the interfacial wave is now dispersive and depends on the surface elastic properties. This distinguishing characteristics may be used to probe both the bulk and surface properties by acoustic measurements (McSkimin, 1964; Aussel and Monchalin, 1989; Every and Sachse, 1990; Chu and Rokhlin 1992). Further, upon specializing the bulk properties to various limits, the results of this paper can recover the classic interfacial waves in the absence of surface elasticity and be used to calculate the interfacial waves between fluid and solid.

The paper is organized as follows. We formulate the problem for interfacial waves with surface elasticity in section 2. In section 3 we present a sufficient condition for the existence of interfacial waves in terms of the *interface impedance matrix*. In section 3.2, by numerical calculations we analyze dependance of the existence and wave speed of interfacial waves on bulk and interface elastic constants. We conclude and provide an outlook of future work in section 4.

2 Problem formulation

Consider an infinite elastic medium with an interface $\Gamma = \{(x_1,x_2,x_3) : x_3 = 0\}$ between two half spaces: $\Omega_1 = \{(x_1,x_2,x_3)|x_3 > 0\}$ and $\Omega_2 = \{(x_1,x_2,x_3)|x_3 < 0\}$ (see Fig. 1). The bulk elastic properties of the two half spaces are described by the *bulk* stiffness tensors:

$$\mathbf{C}(\mathbf{x}) = \mathbf{C}_\alpha \qquad \text{if } \mathbf{x} \in \Omega_\alpha,\ \alpha = 1,2,$$

where the fourth-order tensor $\mathbf{C}_\alpha$ ($\alpha = 1,2$) satisfy the usual major and minor symmetries:

$$(\mathbf{C}_\alpha)_{piqj} = (\mathbf{C}_\alpha)_{pijq} = (\mathbf{C}_\alpha)_{qjpi}, \tag{2}$$

and the convexity condition:

$$\mathbf{A} \cdot \mathbf{C}_\alpha \mathbf{A} > 0, \quad \forall 0 \neq \mathbf{A} \in \mathbb{R}^{3\times3}_{\text{sym}}. \tag{3}$$

To account for the elastic effects of the interface Γ, we model the interface as an elastic massless membrane bonded with the two half spaces without slip. Starting from the postulation (1) and following the paradigm of classic nonlinear elasticity, upon linearization one can show that the above postulation implies the following *surface stress-strain relation*:

$$\boldsymbol{\sigma}_s = \mathbf{C}_s \nabla \mathbf{u} + \boldsymbol{\sigma}_s^0, \tag{4}$$

where $\mathbf{C}_s : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ is the fourth-order surface stiffness tensor satisfying the similar major and minor symmetries in (2) as a bulk stiffness tensor, $\mathbf{u}$ is the displacement, and $\boldsymbol{\sigma}_s^0$ is the residual surface stress. We remark that since interfaces are of two dimensions, the surface elastic energy shall depend only on the stretching within the interface. Therefore, surface strain, surface stress and surface stiffness tensor "live only on the surface" in the sense that

$$\boldsymbol{\sigma}_s,\ \boldsymbol{\sigma}_s^0 \in \mathbb{M} \quad \text{and} \quad \mathbf{C}_s \mathbf{M}^\perp = 0 \quad \forall \mathbf{M}^\perp \in \mathbb{M}^\perp,$$

where $\mathbb{M} = \{\mathbf{M} \in \mathbb{R}^{3\times3} : \mathbf{M}\mathbf{n} = 0,\ \mathbf{M}^T\mathbf{n} = 0\}$, $\mathbf{n}$ is the unit normal on the surface Γ, and $\mathbb{M}^\perp = \{\mathbf{M}^\perp \in \mathbb{R}^{3\times3} : \mathbf{M}^\perp \cdot \mathbf{M} = 0,\ \forall \mathbf{M} \in \mathbb{M}\}$.

The elastodynamic equation for small deformation in the two bulk half spaces is standard and given by

$$\begin{cases} \operatorname{div}[\mathbf{C}_1 \nabla \mathbf{u}(\mathbf{x},t)] = \rho_1 \frac{\partial^2}{\partial t^2}\mathbf{u}(\mathbf{x},t) & \text{for } x_3 > 0, \\ \operatorname{div}[\mathbf{C}_2 \nabla \mathbf{u}(\mathbf{x},t)] = \rho_2 \frac{\partial^2}{\partial t^2}\mathbf{u}(\mathbf{x},t) & \text{for } x_3 < 0, \end{cases} \tag{5}$$

where ρ_α ($\alpha = 1,2$) denote the mass densities. Further, the balance of linear momentum for any subsurface on Γ implies

$$\begin{aligned} &\operatorname{div}_s[\mathbf{C}_s \nabla \mathbf{u}(\mathbf{x},t) + \sigma_0] \\ &\quad + [\mathbf{C}_1 \nabla \mathbf{u}(\mathbf{x}^+,t) - \mathbf{C}_2 \nabla \mathbf{u}(\mathbf{x}^-,t)]\mathbf{e}_3 = \mathbf{0} \quad \forall\, \mathbf{x} \in \Gamma, \end{aligned} \tag{6}$$

where div_s denotes the surface divergence (Gurtin and Murdoch, 1975), and $\mathbf{x}^+$ ($\mathbf{x}^-$) denotes the boundary value approached from the top (bottom) of the interface. We remark that the above equation (6) can be regarded as the generalized Young-Laplace equation for the solid elastic surface Γ.

We define localized *interfacial waves* as solutions to (5) and (6) satisfying the boundary conditions:

$$\mathbf{u}(\mathbf{x},t) \to 0 \qquad \text{as } x_3 \to \pm\infty. \tag{7}$$

The presence of heterogeneity and the elastic interface Γ may give rise to interfacial waves that are important for interface characterization and the overall dynamic behaviors of the body. Below we explore the properties of interfacial waves propagating along interface between two half-spaces including the existence, uniqueness and dispersion relations of interfacial waves.

3 Interfacial waves

3.1 Existence and uniqueness

Without loss of generality we assume the wave propagates in $\mathbf{e}_1$-direction. By translational invariance we seek a solution to (5)-(6) that can be written as

$$\mathbf{u}(\mathbf{x},t) = \hat{\mathbf{u}}(kx_3)e^{i(kx_1-\omega t)}, \tag{8}$$

where $\hat{\mathbf{u}} : \mathbb{R} \to \mathbb{C}^3$ describes the mode shape along $\mathbf{e}_3$-axis, $k > 0$ is the wave number along $\mathbf{e}_1$-axis, and $\omega > 0$ is the frequency. Let $y = kx_3$. Inserting the above equation into (5) and (6) we obtain

$$\begin{cases} (\rho_1 v^2\mathbf{I} - \mathbf{Q}_1)\hat{\mathbf{u}}(y) + i(\mathbf{R}_1 + (\mathbf{R}_1)^T)\hat{\mathbf{u}}'(y) \\ \qquad\qquad +\mathbf{T}_1\hat{\mathbf{u}}''(y) = 0 \quad \text{for} \quad x_3 > 0, \\ (\rho_2 v^2\mathbf{I} - \mathbf{Q}_2)\hat{\mathbf{u}}(y) + i(\mathbf{R}_2 + (\mathbf{R}_2)^T)\hat{\mathbf{u}}'(y) \\ \qquad\qquad +\mathbf{T}_2\hat{\mathbf{u}}''(y) = 0 \quad \text{for} \quad x_3 < 0, \\ -k\mathbf{Q}_s\hat{\mathbf{u}}(0) + i(\mathbf{R}_1^T - \mathbf{R}_2^T)\hat{\mathbf{u}}(0) \\ \qquad\qquad + \mathbf{T}_1\hat{\mathbf{u}}'(0^+) - \mathbf{T}_2\hat{\mathbf{u}}'(0^-) = 0, \end{cases} \tag{9}$$

where $v = \omega/k$ is the wave speed, $(\,)' = \frac{d}{dy}$, and $(\alpha = 1,2)$

$$(\mathbf{R}_\alpha)_{pq} = (\mathbf{C}_\alpha)_{p1q3}, \quad (\mathbf{T}_\alpha)_{pq} = (\mathbf{C}_\alpha)_{p3q3},$$
$$(\mathbf{Q}_\alpha)_{pq} = (\mathbf{C}_\alpha)_{p1q1}, \quad (\mathbf{Q}_s)_{pq} = (\mathbf{C}_s)_{p1q1}.$$

From symmetry condition (2) and convexity condition (3), it is clear that $\mathbf{Q}_\alpha$, $\mathbf{T}_\alpha$, and $\mathbf{Q}_s$ are all 3×3 symmetric matrices and that $\mathbf{Q}_\alpha$ and $\mathbf{T}_\alpha$ are all positive definite and invertible for $\alpha = 1,2$. By the theory of ordinary differential equations (Coddington & Levinson; 1984), a general solution to $(9)_{1,2}$ is given by

$$\hat{\mathbf{u}}(kx_3) = \begin{cases} e^{-x_3k\mathbf{E}_1}\hat{\mathbf{u}}_1 & \text{for } x_3 > 0, \\ e^{x_3k\mathbf{E}_2}\hat{\mathbf{u}}_2 & \text{for } x_3 < 0 \end{cases} \tag{10}$$

for some $\mathbf{E}_1, \mathbf{E}_2 \in \mathbb{C}^{3\times 3}$ and vector $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2 \in \mathbb{C}^3$. From the displacement continuity at $x_3 = 0$, we clearly have $\hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_2$. To satisfy $(9)_{1,2}$, it is sufficient to have

$$\begin{cases} \mathbf{T}_1\mathbf{E}_1^2 - i[\mathbf{R}_1 + (\mathbf{R}_1)^T)]\mathbf{E}_1 + \rho_1 v^2\mathbf{I} - \mathbf{Q}_1 = 0, \\ \mathbf{T}_2\mathbf{E}_2^2 + i[\mathbf{R}_2 + (\mathbf{R}_2)^T)]\mathbf{E}_2 + \rho_2 v^2\mathbf{I} - \mathbf{Q}_2 = 0. \end{cases} \tag{11}$$

Moreover, by (10) equation $(9)_3$ can be rewritten as

$$[-k\mathbf{Q}_s + i(\mathbf{R}_1^T - \mathbf{R}_2^T) - (\mathbf{T}_1\mathbf{E}_1 + \mathbf{T}_2\mathbf{E}_2)]\hat{\mathbf{u}}_1 = 0. \tag{12}$$

Further, in account of (7) we shall require that

$$\mathrm{eig}(\mathbf{E}_1), \mathrm{eig}(\mathbf{E}_2) \subset \mathbb{C}_+, \tag{13}$$

where $\mathrm{eig}(\cdot)$ denotes the set of eigenvalues, and $\mathbb{C}_+$ is the set of all complex numbers with positive real parts.

We remark that equations in (11) can be identified as *algebraic Riccati equations*. To solve for $\mathbf{E}_\alpha$, we assume that $\lambda_\alpha \in \mathbb{C}$ and $\mathbf{a}_\alpha \in \mathbb{C}^3$ are a pair of eigenvalue and eigenvector of $\mathbf{E}_\alpha$:

$$\mathbf{E}_\alpha\mathbf{a}_\alpha = \lambda_\alpha\mathbf{a}_\alpha, \qquad \mathbf{a}_\alpha \neq 0.$$

Operating the left hand sides of (11) on the eigenvector $\mathbf{a}$ we find that

$$[\mathbf{T}_\alpha\lambda_\alpha^2 + (-1)^\alpha i(\mathbf{R}_\alpha + \mathbf{R}_\alpha^T)\lambda + \rho v^2\mathbf{I} - \mathbf{Q}_\alpha]\mathbf{a}_\alpha = 0. \tag{14}$$

Taking complex conjugate of (14), we observe that if $(\lambda_\alpha, \mathbf{a}_\alpha)$ satisfies (14), so does $(-\bar{\lambda}_\alpha, \bar{\mathbf{a}}_\alpha)$.

The above equation (14) can be identified as a generalized eigenvalue-eigenvector problem. Clearly, the eigenvalues λ_α can be determined as the roots of the polynomial:

$$\begin{aligned} P_\alpha(\lambda, v) := \det[&\mathbf{T}_\alpha\lambda^2 + (-1)^\alpha i(\mathbf{R}_\alpha + \mathbf{R}_\alpha^T)\lambda \\ &+ \rho v^2\mathbf{I} - \mathbf{Q}_\alpha] \end{aligned} \tag{15}$$

whereas the associated eigenvectors $\mathbf{a}_\alpha$ can be obtained as nonzero solutions to (14). In a generic case, we shall be able to find six eigenvalue-eigenvector pairs $(\lambda_\alpha^i, \mathbf{a}_\alpha^i)$ ($i = 1, \cdots, 6$) for a given $v > 0$. Let $\mathbf{A}_\alpha = [\mathbf{a}_\alpha^{i_1}; \mathbf{a}_\alpha^{i_2}; \mathbf{a}_\alpha^{i_3}] \in \mathbb{C}^{3\times 3}$ be the matrix formed by three of the (column) eigenvectors and $\mathbf{D}_\alpha = \mathrm{diag}[\lambda_\alpha^{i_1}, \lambda_\alpha^{i_2}, \lambda_\alpha^{i_3}]$ be the diagonal matrix formed by the corresponding eigenvalues. If $\det\mathbf{A}_\alpha \neq 0$, then a solution to $(11)_\alpha$ is given by

$$\mathbf{E}_\alpha = \mathbf{A}_\alpha\mathbf{D}_\alpha\mathbf{A}_\alpha^{-1}.$$

For interfacial waves, we shall focus on solutions to (11) that satisfy (13). Since the eigenvalues of (14) are symmetric about the imaginary axis, a solution $\mathbf{E}_\alpha \in \mathbb{C}^{3\times 3}$ to (11) satisfying (13) cannot be constructed by the above procedure if $P_\alpha(\lambda, v)$ has a pure imaginary solution. This motivates us to introduce the *limiting speed*:

$$\hat{v}_\alpha := \inf\{v > 0 : P_\alpha(\lambda, v) \text{ has a pure imaginary root}\}.$$

The reader is referred to Chadwick and Smith (1977, p. 339) for a neat geometrical interpretation of the limiting speed $\hat{v}_\alpha$ on the slowness section on the plane spanned by $\{\mathbf{e}_1, \mathbf{e}_3\}$. Let

$$\hat{v} = \min\{\hat{v}_1, \hat{v}_2\}.$$

In analogy with free surface waves, we refer to interfacial waves as *subsonic* if the phase speed $v < \hat{v}$, and *supersonic* if otherwise. Following Barnett & Lothe(1985) and Fu & Mielke (2002), it can be shown that if $0 \le v < \hat{v}$, both equations in (11) admit unique solutions $\mathbf{E}_\alpha(v)$ ($\alpha = 1,2$) satisfying (13). We can therefore define two new quantities

$$\begin{cases} \mathbf{M}_1(v) = \mathbf{T}_1\mathbf{E}_1(v) - i\mathbf{R}_1^T, \\ \mathbf{M}_2(v) = \mathbf{T}_2\mathbf{E}_2(v) + i\mathbf{R}_2^T, \end{cases} \tag{16}$$

which are known as *surface impedance matrices*. Replacing $\mathbf{E}_\alpha$ by $\mathbf{M}_\alpha$ in (11), we find that $\mathbf{M}_\alpha$ ($\alpha = 1,2$) satisfy the standard algebraic Riccati equations:

$$\begin{cases} (\mathbf{M}_1 - i\mathbf{R}_1^T)\mathbf{T}_1^{-1}(\mathbf{M}_1 + i\mathbf{R}_1^T) - \mathbf{Q}_1 + \rho_1 v^2 \mathbf{I} = 0, \\ (\mathbf{M}_2 + i\mathbf{R}_2^T)\mathbf{T}_2^{-1}(\mathbf{M}_2 - i\mathbf{R}_2^T) - \mathbf{Q}_2 + \rho_2 v^2 \mathbf{I} = 0. \end{cases} \tag{17}$$

We define the *interface impedance matrix* as

$$\mathbf{Z}(v) = \mathbf{M}_1(v) + \mathbf{M}_2(v) + k\mathbf{Q}_s. \tag{18}$$

Then equation (12) admits a nonzero solution $\hat{\mathbf{u}}_1$ if and only if

$$\det \mathbf{Z}(v) = 0. \tag{19}$$

We now introduce a few useful properties of the interface impedance matrix that follow from Barnett & Lothe (1985) and Fu & Mielke (2002).

Lemma 1. *Assume that $\mathbf{Q}_s$ is positive semi-definite. Then the interface impedance matrix $\mathbf{Z}(v)$ defined by* (18) *satisfies that*

(i) $\mathbf{Z}(v)$ *is Hermitian for* $v \in (0, \hat{v})$*;*
(ii) $\mathbf{Z}(0)$ *is positive definite;*
(iii) $\frac{d}{dv}\mathbf{Z}(v)$ *is negative definite for* $v \in (0, \hat{v})$*, i.e., every eigenvalue of* $\mathbf{Z}(v)$ *is monotonically decreasing as a function of* v*;*
(iv) $\mathbf{a} \cdot \mathbf{Z}(v)\mathbf{a} \ge 0, \quad \forall\, \mathbf{a} \in \mathbb{R}^3$ *and* $v \in (0, \hat{v})$.

Due to property (ii) and (iii), existence of an interfacial wave with phase speed $v_0 < \hat{v}$ satisfying (19) requires that $\mathbf{Z}(\hat{v}-)$ has at least one negative eigenvalue. In addition, the matrix $\mathbf{Z}(\hat{v}-)$ can have at most one negative eigenvalue since one could always find a vector $\mathbf{a} \in \mathbb{R}^3$ violating property (iv) if otherwise (Barnett & Lothe,1985). So in order for the interfacial wave to exist, eigenvalues of $\mathbf{Z}(\hat{v}-)$ should meet either of these two situations: (1) two positive and a negative eigenvalues; (2) one positive, one negative and one zero eigenvalues. In conclusion, we have the following existence theorem for subsonic interfacial waves:

Theorem 2. *Assume that the matrix $\mathbf{Q}_s$ is positive semi-definite. If $\det \mathbf{Z}(\hat{v}-) < 0$ or $(tr\, \mathbf{Z}(\hat{v}-))^2 - tr\, \mathbf{Z}^2(\hat{v}-) < 0$, there exist a unique subsonic interfacial wave. The phase speed $v = \omega/k \in (0, \hat{v})$ is determined by*

$$\det \mathbf{Z}(v) = 0.$$

We remark that the positive semi-definiteness of $\mathbf{Q}_s$ in the above theorem is a strong assumption. In fact, the above theorem applies as long as $\mathbf{Q}_s$ is such that $\mathbf{Z}(0)$ is positive definite. Therefore, the subsonic interfacial wave is unique for small k since $\mathbf{M}_1(0)$ and $\mathbf{M}_2(0)$ are both positive definite (Fu & Mielke, 2002).

3.2 Explicit solutions

An explicit solution (if exist) can be found when the interfacial wave is polarized in a symmetry plane (x_1-x_3 plane say) of both solids. A trial solution for this problem can be written as

$$\mathbf{u}(x_1, x_3, t) = \begin{cases} \hat{\mathbf{u}}_1 \exp(-ikp_1x_3)\exp(ikx_1 - \omega t) & \text{for } x_3 > 0, \\ \hat{\mathbf{u}}_2 \exp(ikp_2x_3)\exp(ikx_1 - \omega t) & \text{for } x_3 < 0. \end{cases}$$

Inserting the above trial solution into (5) one can find a quartic equation for p_α ($\alpha = 1$ or 2). For each half-space two pairs of complex conjugate solutions can be found from the quartic equation. Destrade and Fu (2006) have obtained analytic solutions of the quartic equations in terms of v and implemented a numerical method for calculating the interfacial wave speed from the condition at the interface without surface elasticity.

In particular, if both half-spaces are isotropic, by symmetry we observe that $\hat{\mathbf{u}}_1 \cdot \mathbf{e}_2 = \hat{\mathbf{u}}_2 \cdot \mathbf{e}_2 = 0$, and subsequently, omit components associated with $\mathbf{e}_2$-direction in matrices of (11) and (12). Removing trivial components associated $\mathbf{e}_2$-direction and with an abuse of notation, we find the material tensors defined by (10) as

$$\begin{aligned} \mathbf{Q}_\alpha &= \begin{bmatrix} 2\mu_\alpha + \lambda_\alpha & 0 \\ 0 & \mu_\alpha \end{bmatrix}, \quad \mathbf{R}_\alpha = \begin{bmatrix} 0 & \lambda_\alpha \\ \mu_\alpha & 0 \end{bmatrix}, \\ \mathbf{T}_\alpha &= \begin{bmatrix} \mu_\alpha & 0 \\ 0 & 2\mu_\alpha + \lambda_\alpha \end{bmatrix}, \quad \mathbf{Q}_s = \begin{bmatrix} Q_s & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{20}$$

where $Q_s = (\mathbf{Q}_s)_{11} = (\mathbf{C}_s)_{1111}$ is the surface elastic modulus. Since the surface impedance matrix is Hermitian, we can write it as

$$\mathbf{M}_1 = \begin{bmatrix} {}^1m_1 & {}^1m_3 + i\,{}^1m_4 \\ {}^1m_3 - i\,{}^1m_4 & {}^1m_2 \end{bmatrix}, \qquad \mathbf{M}_2 = \begin{bmatrix} {}^2m_1 & {}^2m_3 - i\,{}^2m_4 \\ {}^2m_3 + i\,{}^2m_4 & {}^2m_2 \end{bmatrix}, \tag{21}$$

where ${}^\alpha m_j (\alpha = 1,2; j = 1,\cdots,4) \in \mathbb{R}$. Solving (17) for $\mathbf{M}_\alpha$, we find that (Fu and Mielke, 2002)

$${}^\alpha m_1 = \sqrt{\mu_\alpha(2\mu_\alpha + \lambda_\alpha - \rho_\alpha v^2) - \frac{\mu_\alpha}{2\mu_\alpha + \lambda_\alpha}(\frac{\lambda_\alpha + \mu_\alpha}{1+\gamma_\alpha})^2},$$
$${}^\alpha m_2 = \gamma_\alpha \frac{2\mu_\alpha + \lambda_\alpha}{\mu_\alpha}{}^\alpha m_1, \quad {}^\alpha m_3 = 0, \quad {}^\alpha m_4 = \frac{\gamma_\alpha \lambda_\alpha - \mu_\alpha}{1+\gamma_\alpha},$$

where

$$\gamma_\alpha = \sqrt{\frac{\mu_\alpha(\mu_\alpha - \rho_\alpha v^2)}{(2\mu_\alpha + \lambda_\alpha)(2\mu_\alpha + \lambda_\alpha - \rho_\alpha v^2)}}, \qquad \alpha = 1,2.$$

Then equation (19) implies that

$$({}^1m_1 + {}^2m_1 + kQ_s)({}^1m_2 + {}^2m_2) - ({}^1m_3 + {}^2m_3)^2 - ({}^1m_4 + {}^2m_4)^2 = 0. \tag{22}$$

Now let

$$v_{\alpha l} = \left(\frac{\lambda_\alpha + 2\mu_\alpha}{\rho_\alpha}\right)^{\frac{1}{2}}, \qquad v_{\alpha t} = \left(\frac{\mu_\alpha}{\rho_\alpha}\right)^{\frac{1}{2}} \tag{23}$$

be the longitudinal bulk wave speed and transverse bulk wave (shear wave) speed ($\alpha = 1,2$), respectively, and

$$q_{\alpha l} = \sqrt{1 - \left(\frac{v}{v_{\alpha l}}\right)^2}, \qquad q_{\alpha t} = \sqrt{1 - \left(\frac{v}{v_{\alpha t}}\right)^2}. \tag{24}$$

By some tedious algebraic manipulation, equation (22) can be rewritten as

$$\begin{aligned} &((1 - q_{2l}q_{2t})\rho_1^2 - (q_{1t}q_{2l} + q_{1l}q_{2t} + 2)\rho_2\rho_1 \\ &+ (1 - q_{1l}q_{1t})\rho_2^2)v^4 + 4((q_{2l}q_{2t} - 1)\rho_1 \\ &+ (1 - q_{1l}q_{1t})\rho_2)(\rho_1 v_{1t}^2 - \rho_2 v_{2t}^2)v^2 \\ &+ 4(q_{1l}q_{1t} - 1)(q_{2l}q_{2t} - 1)(\rho_1 v_{1t}^2 - \rho_2 v_{2t}^2)^2 \\ &+ kQ_s((q_{2l}q_{2t} - 1)\rho_1 + (q_{1l}q_{1t} - 1)\rho_2)v^2 = 0, \end{aligned} \tag{25}$$

which determines the interfacial wave speed. Upon inspection it is clear that if the surface elasticity is ignored ($Q_s = 0$), the solution of v to the above equation is the wave speed of the classic Stoneley wave (Stoneley, 1924) and independent of the wave number k. At the presence of surface elasticity ($Q_s > 0$), a generic solution to the above equation clearly depends on k, meaning that the interfacial wave is dispersive. We also notice that a solution to equation (25) may not exist.

We now solve (25) numerically and results of interfacial wave speed versus frequency are shown in Fig. 2-4. In Fig. 2 the impact of surface elastic modulus is studied for two bulk materials with $\rho_1 = 500Kg/m^3$, $\rho_2 = 10000Kg/m^3$, $v_{1t} = v_{2t} = 1000m/s$, $v_{1l} = v_{2l} = 1450m/s$ (cf., (23)). Figure 2 shows that the wave speed v monotonically increases (resp. decreases) with respect to frequency ω for positive (resp. negative) Q_s. However, interfacial wave speed v becomes independent of Q_s at long wavelength limit ($\omega \to 0$). In Fig. 3 we show the dependence of interfacial waves on bulk densities for given surface elastic modulus of $Q_s = 10000J/m^2$ whereas the bulk wave speeds are specified as $v_{1t} = v_{2t} = 1000m/s$, $v_{1l} = v_{2l} = 1450m/s$. We remark that the surface elastic modulus $Q_s = 10000J/m^2$, though orders of magnitude larger than pristine surface of typical solid crystals, is realistic and physical for composite structures, e.g., a sandwich plate with thick soft core and stiff thin face plates. Curves with the same density ratio $\rho_1 : \rho_2 = 1 : 20$ intersects at $\omega = 0$, indicating that the wave speed at long wave length limit depends only on the ratio rather than the values of densities. This is in fact a property of the classic Stoneley waves. On the other hand, at any nonzero frequency larger densities correspond to greater interfacial wave speed. We also observe that interfacial waves are less likely to exist as the ratio gets closer to 1. Figure 4 shows the dependence of interfacial wave speed on bulk wave speeds for $Q_s = 10000J/m^2$, $\rho_1 = 500Kg/m^3$, $\rho_2 = 10000Kg/m^3$, $v_{1t} = v_{2t} = 1000m/s$, $v_{1l} = v_{2l}$. We observe that smaller difference between bulk longitudinal speeds and bulk shear speeds results in lower interfacial wave speed at the long wavelength limit, and also makes the interfacial wave speed depend more sensitively on frequency.

Further, we can study interfacial waves propagating along solid/fluid and two fluids interfaces in present framework. Assume that medium 2 is an inviscid fluid. Since the fluid cannot sustain shear force, we set the shear modulus to zero ($\mu_2 = 0$) for fluid phase. Then the condition at the interface shall be written as

$$\begin{cases} \hat{\mathbf{u}}_2 \cdot \mathbf{n} - \hat{\mathbf{u}}_1 \cdot \mathbf{n} = 0, \\ [\mathbf{M}_1(v) + k\mathbf{Q}_s]\hat{\mathbf{u}}_1 = -\mathbf{M}_2(v)\hat{\mathbf{u}}_2 = p\mathbf{n}, \end{cases} \tag{26}$$

where p is the pressure. By (21) components of $\mathbf{M}_2(v)$ are given by

$${}^2m_1 = {}^2m_3 = {}^2m_4 = 0, \quad {}^2m_2 = \rho_2 v^2 \sqrt{\frac{\lambda_2}{\lambda_2 - \rho_2 v^2}}. \tag{27}$$

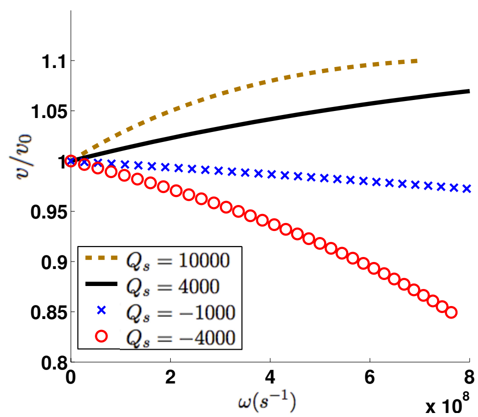

Fig. 2. Dependence of interfacial wave speed on surface elastic modulus $Q_s(J/m^2)$. v is normalized by the corresponding wave speed v_0 for $Q_s = 0$. ($\rho_1 = 500kg/m^3$, $\rho_2 = 10000kg/m^3$, $v_{1t} = v_{2t} = 1000m/s$, and $v_{1l} = v_{2l} = 1450m/s$)

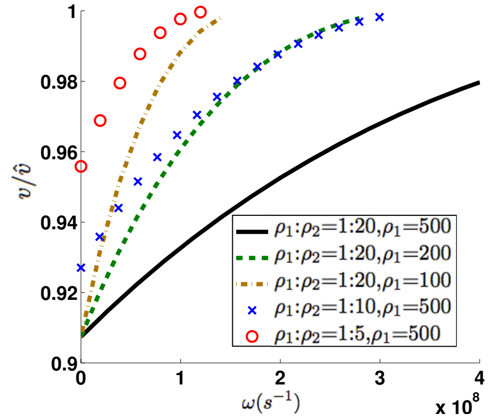

Fig. 3. Dependence of interfacial wave speed on densities ρ_1 and $\rho_2(kg/m^3)$. v is normalized by limiting wave speed $\hat{v}$. ($Q_s = 10000J/m^2$, $v_{1t} = v_{2t} = 1000m/s$, and $v_{1l} = v_{2l} = 1450m/s$)

Inserting (21) and (27) into (26) we have

$$({}^1m_1 + kQ_s)({}^1m_2 + {}^2m_2) - {}^1m_4^2 = 0. \qquad (28)$$

Substituting (23) and (24) into above equation (28) we have the secular equation interns of bulk wave speeds:

$$\begin{aligned}
&-2\rho_1 v_{1t}^2 v_{1l}^2[\rho_2(q_{1l}+q_{1t})+\rho_1 q_{2l}]v^4 \\
&+\rho_1 v_{1t}^2 v_{1l}^2(v_{1t}^2+v_{1l}^2)[\rho_1 q_{2l} - \rho_1 q_{1t} q_{1l} q_{2l} + \rho_2(q_{1l}+q_{1t})]v^2 \\
&+4\rho_1^2 v_{1t}^4 v_{1l}^2(v_{1l}^2 - v_{1t}^2 + v_{1t}^2 q_{1t} q_{1l})q_{2l} \\
&+kQ_s(v_{1t}^2 q_{qt} + v_{1l}^2 q_{1l})(\rho_1 v_{1t}^2 v_{1l}^2(q_{1l}+q_{1t})q_{1t}q_{2l} \\
&+\rho_2 v^2(v_{1t}^2 q_{1t} + v_{1l}^2 q_{1l})) = 0.
\end{aligned} \qquad (29)$$

By equation (29) we calculate the interfacial wave speed versus frequency for Aluminum/Water interface with surface elasticity ($\rho_1 = 2700kg/m^3$, $v_{1t} = 3040m/s$, $v_{1l} = 6420m/s$, $\rho_2 = 1000kg/m^3$, $v_{\text{water}} = 1484m/s$ and $Q_s = 100000J/m^2$). From Fig. 5, we observe that at the presence of surface elasticity the interfacial wave speed decreases as frequency increases. Also, we remark that interfacial wave speed is lower than the acoustic wave speed in water v_{water} and that the interfacial waves implied by (28) decay only in the solid but not the fluid phase (since we have ignored the viscosity).

4 Conclusion

In this paper we study interfacial waves that propagate at the interface between two half spaces and decay away from the interface. A sufficient condition for the existence and uniqueness of subsonic interfacial waves is obtained for general anisotropic half spaces. As examples, we present the explicit secular equation for determining the dispersion relation of subsonic interfacial waves for two isotropic half spaces with an isotropic interface. The secular equation can also be used to determine interfacial waves at the interfaces between solid and fluid. The important effects of surface elasticity on the dispersion relation of interfacial waves are then parametrically studied by explicitly solving the secular equation. In particular, we notice that the interfacial waves are now dispersive, strongly frequency-dependent, and surface-property dependent. We anticipate these fundamental results may have important applications in modeling dynamic behaviors of sandwich structures, designing acoustic wave guides and filters, and probing surface and bulk properties of materials among others.

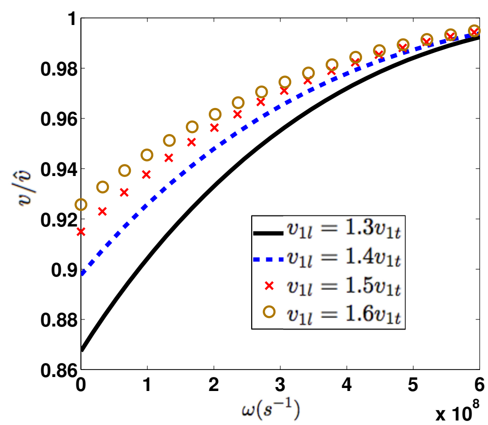

Fig. 4. Dependence of interfacial wave speed on bulk wave speeds for given limiting speed $\hat{v} = 2000m/s$. v is normalized by $\hat{v}$. ($Q_s = 10000J/m^2$, $\rho_1 = 500kg/m^3$, $\rho_2 = 10000kg/m^3$, $v_{1t} = v_{2t} = 1000m/s$, and $v_{1l} = v_{2l}$)

Acknowledgement: The authors acknowledge the support of NSF under Grant No. CMMI-1238835 and AFOSR (YIP-12).

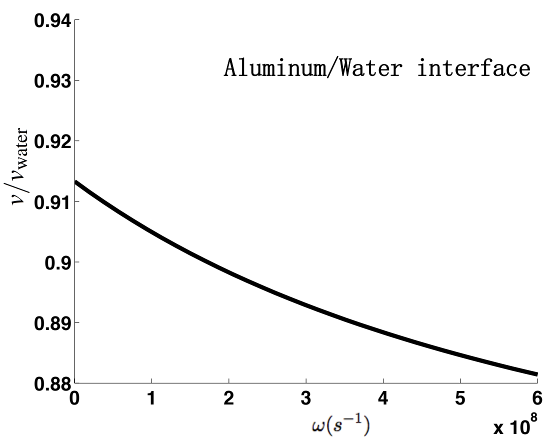

Fig. 5. Dispersion relation of interfacial wave at the interface of Aluminum ($\rho_1 = 2700kg/m^3$, $v_{1t} = 3040m/s$, and $v_{1l} = 6420m/s$) and water ($\rho_2 = 1000kg/m^3$ and $v_{\rm water} = 1484m/s$). Here surface elastic parameter is $Q_s = 100000J/m^2$ and interfacial wave speed v is normalized by speed of sound in water $v_{\rm water}$.

References

[1] Stoneley, R. 1924 Elastic waves at the surface of separation of two solids. *Proc. R. Soc. Lond. A* **106**, 416–428.

[2] Barnett, D. M., Lothe, J., Gavazza, S. D. & Musgrave, M. J. P. 1985 Considerations of the existence of interfacial (stoneley) waves in bonded anisotropic elastic half-spaces. *Proc. R. Soc. Lond. A* **402**, 153–166.

[3] Sharma, P., Ganti, S. & Bhate, N. 2003 Effect of surfaces on the size-dependent elastic state of nano-inhomogeneities. *Appl. Phys. Lett.* **82**, 535–537.

[4] Miller, R. E. & Shenoy, V. B. 2000 Size dependent elastic properties of structural elements. *Nanotechnology* **11**, 139–147.

[5] Gurtin, M. E. & Murdoch, A. I. 1975 A continuum theory of elastic material surfaces. *Arch. Rat. Mech. Anal.* **54**, 291–323.

[6] Gurtin, M. E. & Murdoch, A. I. 1976 Effect of surface stress on wave propagation in solids. *J. Appl. Phys.* **47**, 4414–4421.

[7] Blanc, X., Le Bris, C. & Lions, P. L. 2002 From molecular models to continuum mechanics. *Arch. Ration. Mech. Anal.* **164**, 341–381.

[8] Shenoy, V. B. 2005 Atomistic calculations of elastic properties of metallic fcc crystal surfaces. *Phys. Rev. B* **71**, 094104.

[9] Mi, C., Jun, S. & Kouris, D. A. 2008 Atomistic calculations of interface elastic properties in noncoherent metallic bilayers. *Phys. Rev. B* **77**, 075425.

[10] Mohammadi, P., Liu, L. P., Sharma, P. & Kukta, R. V. 2013 Surface energy, elasticity and the homogenization of rough surfaces. *J. Mech. Phys. Solids* **61**, 325–340.

[11] Liu, L. P. & Bhattacharya, K. 2009 Wave propagation in a sandwich structure. *Int. J. Solids Struct.* **46**, 3290–3300.

[12] Duan, H. L., Wang, J., Huang, Z. P. & Karihaloo, B. L. 2005 Size-dependant effective elastic constants of solids containing nano-inhomogeneities with interface stress. *J. Mech. Phys. Solids* **53**, 1574–1596.

[13] Bar On, B., Altus, E. & Tadmor, E. B. 2010 Surface effects in non-uniform nanobeams: continuum vs. atomistic modeling. *Int. J. Solids Struct.* **47**, 1243–1252.

[14] Murdoch, A. I. 1976 The propagation of surface waves in bodies with material boundaries. *J.Mech.Phys.Solids* **24**, 137–146.

[15] Steigmann, D. J. & Ogden, R. W. 2007 Surface waves supported by thin-film/substrate interactions. *IMA J. Appl. Math.* **72**, 730–747.

[16] Steigmann, D. J. & Ogden, R. W. 1997 Elastic surface-substrate interactions. *Proc. R. Soc. A* **455**, 437–474.

[17] Huang, Z. P. & Sun, L. 2007 Size-dependent effective properties of a heterogeneous material with interface energy effect: from finite deformation theory to infinitesimal strain analysis. *Acta Mech.* **190**, 151–163.

[18] Altenbach, H., Eremeyev, V. A. & Lebedev, L. P. 2011 On the spectrum and stiffness of an elastic body with surface stresses. *ZAMM* **91**, 699–710.

[19] Rayleigh, L. 1885 On waves propagated along the plane surface of an elastic solid. *Proc. London Math. Soc.* **s1-17**, 4–11.

[20] Chadwick, P. & Smith, G. D. 1977 Foundations of the theory of surface waves in anisotropic elastic materials. *Adv. Appl. Mech.* **17**, 303–376.

[21] Barnett, D. M. & Lothe, J. 1985 Free surface (rayleigh) waves in anisotropic elastic half-spaces: the surface impedance method. *Proc. R. Soc. Lond. A* **402**, 135–152.

[22] McSkimin, H. J. 1964 Ultrasonic methods for measuring the mechanical properties of liquids and solids. In *Physical Acoustics*(ed. W. P. Mason), vol. 1. New York: Academic.

[23] Aussel, J. D. & Monchalin, J. P. 1989 Precision laser-ultrasonic velocity measurement and elastic constant determination. *Ultrasonics* **27**, 165–177.

[24] Every, A. G. & Sachse, W. 1990 Determination of the elastic constants of anisotropic solids from acoustic-wave group-velocity measurements. *Phys. Rev. B* **42**, 8196–8205.

[25] Chu, Y. C. & Rokhlin, S. I. 1992 Determination of macro- and micromechanical and interfacial elastic properties of composites from ultrasonic data. *J. Acoust. Soc. Am.* **92**, 920–931.

[26] Coddington, E. A. & Levinson, N. 1984 *Theory of Ordinary Differential Equations*. Krieger Pub Co.

[27] Hu, L. X., Liu, L. P. & Bhattacharya, K. 2011 Existence of surface waves and band gaps in periodic heterogeneous half-spaces. *J. Elast.* **107**, 65–79.

[28] Fu, Y. B. & Mielke, A. 2002 A new identity for the surface-impedance matrix and its application to the determination of surface-wave speeds. *Proc. R. Soc. Lond. A* **458**, 2523–2543.

[29] Destrade, M. & Fu, Y.B. 2006 The speed of interfacial waves polarized in a symmetry plane. *Int. J. Eng. Sci.* **44**, 26–36