Polynomial eigenstress inducing polynomial strain of the same degree in an ellipsoidal inclusion and its applications

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Abstract

We present a rigorous proof of polynomial eigenstress inducing polynomial strain of the same degree in an ellipsoidal inclusion. The coefficients of the induced polynomial strain are explicitly given in terms of elliptic integrals. The analogous Eshelby's tensor for polynomial eigenstress is also computed, and applied to solve the inhomogeneous problem as an example of applications.

1 Introduction

Since the seminal works of Eshelby (1957; 1961), the inclusion problem has played a critical role in the development of predictive material models (Mura, 1987; Nemat-Nasser and Hori, 1999). In a broader physical context, a number of problems of practical interest can be formulated in a similar form as the Eshelby inclusion problem in linear elasticity, including models in electrostatics, magnetostatics, piezoelectrics among many others. The remarkable *uniformity* property of ellipsoids, i.e., uniform eigenstress inducing uniform strain inside the inclusion, allows for explicit and closed-form predictions to important physical quantities such as stress concentration factor, force and torque on the inclusion, and effective material properties in the dilute limit which may be extended to finite volume fractions by the mean-field type theory of Mori-Tanaka (Mori and Tanaka 1973). Motivated by the uniformity propoerty of ellipsoids, the author and coworkers have recently constructed new shapes called *E-inclusions* with similar uniformity property but for different boundary conditions (Liu *et al.*, 2007; 2008). Much of the analysis based on the Eshelby's solutions can be applied to E-inclusions and account for interactions between inclusions (Liu 2009; 2010).

At the advent of modern nanotechnology, there is a renewed interest in the Eshelby inclusion problems, especially for nonuniform eigenstress. A particular application motivating this work is to find the force and torque on an ellipsoidal particle subjected to a nonuniform applied field in electrostatics or magnetostatics or elasticity. This problem arises from a number applications. For example, in the design of magnetic nanotweezers (Neuman and Nagy 2008; Gosse and Croquette 2002), it is critical to relate the force and torque on the particle with the applied nonuniform field so as to achieve precise control and manipulation of nano-particles. Also, to understand the

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mechanism of clustering or segregation of particles in a solution, an accurate account of interaction forces between neighboring particles requires replacing the assumption of uniform induced field by a nonuniform one in the particle (Sun et al., 2000; Pankhurst et al., 2003). In fracture mechanics, the micro-crack models in the process zone can no longer be analyzed by the Eshelby's solutions (Hori and Nemat-Nasser 1985; 1987).

Asaro and Barnett (1975) presented a generalization of the *uniformity* property of ellipsoids. They concluded that for general anisotropic solids a polynomial eigenstrain on an ellipsoidal inclusion induces a polynomial strain of the same degree inside the inclusion. Though widely used, some steps in their argument, e.g., switching the order of integrations for integrands which are not integrable, may require careful justification. Also, it is peculiar that the argument appears to work only in three (or odd) dimensional space. For example, in two dimensions the argument requires evaluating the critical integral $\int_{B_2}(\mathbf{x}')^\alpha \delta(\hat{\mathbf{k}}\cdot(\mathbf{x}'-\mathbf{x}))d\mathbf{x}'$ for a circle $B_2 := \{\mathbf{x}\in\mathbb{R}^2 : |\mathbf{x}|\le 1\}$ and unit vector $\hat{\mathbf{k}} = (\hat{k}_1, \hat{k}_2)$. Yet, by analogous argument we obtain ($\hat{\mathbf{k}}^\perp = (\hat{k}_2, -\hat{k}_1)$)

$$\int_{B_2}(\mathbf{x}')^\alpha \delta(\hat{\mathbf{k}}\cdot(\mathbf{x}'-\mathbf{x}))d\mathbf{x}' = \int_{-\sqrt{1-(\hat{\mathbf{k}}\cdot\mathbf{x})^2}}^{\sqrt{1-(\hat{\mathbf{k}}\cdot\mathbf{x})^2}}((\hat{\mathbf{k}}\cdot\mathbf{x})\hat{\mathbf{k}} + s\hat{\mathbf{k}}^\perp)^\alpha ds,$$

which does not yield the desired results that the above integral is a polynomial of $\mathbf{x}$. Similar issues exist in Mura and Kinoshita (1978).

In this paper we present a rigorous proof of polynomial eigenstress (or eigenstrain) inducing polynomial strain of the same degree for the *homogeneous* inclusion problem. Further, the coefficients of the induced polynomial strain are explicitly and systematically calculated in terms of elliptic integrals. In a similar manner as for uniform eigenstress, the equivalent inclusion method can then be applied to solve the *inhomogeneous* inclusion problem subjected to a nonuniform polynomial far field and to address the problem concerning the interaction between two spherical inhomogeneities (Moschovidis and Mura, 1975; Rodin and Hwang, 1991). Solutions to the inhomogeneous inclusion problems are the foundations of many material models concerning, for example, composite materials, solid-to-solid phase transformations, cracks and dislocations.

The paper is organized as follows. In § 2 we focus on simple p-harmonic problems. By utilizing spherical symmetry we explicitly solve p-harmonic problems for uniform sources in § 2.1 and nonuniform polynomial sources on a unit ball in § 2.2. In § 2.3 the solutions are extended to ellipsoids by observing that ellipsoids are linear transformations of the unit ball. In § 3 we solve the *homogeneous* Eshelby inclusion problem for general nonuniform polynomial eigenstress, which is then used to solve the *inhomogeneous* Eshelby inclusion problem by the analogous equivalent inclusion mehtod in § 4.

Finally, we remark that although terminologies from linear elasticity are employed for convenience, the solution techniques and results apply to problems in electrostatics, magnetostatics among many other settings where the corresponding fields are governed by second-order linear elliptic partial differential equations.

Notation. For an n-tuple nonnegative integer index $(\alpha_1, \cdots, \alpha_n)$ and a vector $\mathbf{x} = (x_1, \cdots, x_n)$, we denote by

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \mathbf{x}^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \qquad D_\mathbf{x}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots \partial x_n^{\alpha_n}}.$$

The Greek letters α, β, γ will be reserved for such multi-index; $\gamma \le \alpha$ if and only if $\gamma_i \le \alpha_i$ for all $i = 1, \cdots, n$. The number of different α for fixed $|\alpha| = p$ is given by $\frac{(p+n-1)!}{p!(n-1)!}$ (Hazewinkel, 2001).

For any function $f : \mathbb{R}^n \to \mathbb{C}$ we denote by $\hat{f}$ its Fourier transformation:

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^n} f(\mathbf{x}) \exp(-i\mathbf{k} \cdot \mathbf{x}) d\mathbf{x}. \tag{1}$$

For a nonzero vector $\mathbf{k} \in \mathbb{R}^n$, denote by $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ the associated unit vector with components $(\hat{k}_1, \cdots, \hat{k}_n)$. Also, we recall the Leibniz formula (Rudin 1991, p. 159, 177)

$$D^\alpha(fg) = \sum_{\gamma \le \alpha} c_{\alpha\gamma} D^{\alpha-\gamma} f D^\gamma g, \tag{2}$$

where $c_{\alpha\gamma} = \prod_{i=1}^n \frac{\alpha_i!}{\gamma_i!(\alpha_i-\gamma_i)!}$.

2 Solutions to the p-harmonic problems and their implications

2.1 Uniform sources on a ball

We first consider problems with uniform sources supported on a unit ball. Let $B_n \subset \mathbb{R}^n$ be the unit ball centered at the origin and χ_{B_n} ($= 1$ on B_n; $= 0$ otherwise) be the characteristic function of B_n. We shall modify the source such that all integrals arising from Fourier transformations can be interpreted in the sense of Riemann or Lebesgue (Rudin, 1987). For any $\eta > 0$, let $w_\eta : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with the following properties:

- Spherical symmetry: $w_\eta = w_\eta(r)$, $r = |\mathbf{x}|$,
- $w_\eta(r) = 1$ if $r \le 1$ and $w_\eta(r) = 0$ if $r > 1 + \eta$,
- As $\eta \to 0$, $\int_{\mathbb{R}^n} |w_\eta - \chi_{B_n}|^q \to 0$ for any $q \ge 1$.

Using integration by parts and change of variables, we can easily show that the Fourier transformation $\hat{w}_\eta$ is smooth, real-valued, spherically symmetric, and decays faster than any polynomial (Rudin, 1991, p. 184). For an integer $q < n$, define a function $\Lambda_\eta^q : \mathbb{R} \to \mathbb{C}$ as

$$\Lambda_\eta^q(t) := \frac{1}{(2\pi)^n} \int_0^\infty k^{n-q-1} \exp(ikt) \hat{w}_\eta(k) dk = \Lambda_{\eta e}^q(t) + i\Lambda_{\eta o}^q(t), \tag{3}$$

where subscript "$_e$" ("$_o$") refers to "even" ("odd"),

$$(\Lambda_{\eta e}^q(t),\ \Lambda_{\eta o}^q(t)) = \frac{1}{(2\pi)^n} \int_0^\infty k^{n-q-1} [\cos(kt),\ \ \sin(kt)] \hat{w}_\eta(k) dk. \tag{4}$$

It is clear that

$$\Lambda_{\eta e}^q(t) = \Lambda_{\eta e}^q(-t), \qquad \Lambda_{\eta o}^q(t) = -\Lambda_{\eta o}^q(-t). \tag{5}$$

By differentiation,

$$\frac{d}{dt}\Lambda_\eta^q(t) = i\Lambda_\eta^{q-1}(t), \quad \text{i.e.,} \quad \frac{d}{dt}\Lambda_{\eta e}^q(t) = -\Lambda_{\eta o}^{q-1}(t), \quad \frac{d}{dt}\Lambda_{\eta o}^q(t) = \Lambda_{\eta e}^{q-1}(t). \tag{6}$$

We remark that the functions defined in (3)-(4) are smooth on $\mathbb{R}$ and will play a pivotal role in our subsequent analysis.

To explore the properties of Λ_η^q, for any positive integer p we consider the p-harmonic problem

$$\Delta^p \psi_p = -w_\eta \qquad \text{on } \mathbb{R}^n. \tag{7}$$

It can be shown that solutions to the above equation exist in

$$\mathbb{H}^{2p} := \{u : \int_{\mathbb{R}^n} \sum_{|\alpha|=2p} |D^\alpha u|^2 < +\infty\}, \tag{8}$$

and that the solution is unique within a polynomial of degree $2p-1$, see Gilbarg and Trudinger (1983).

By spherical symmetry we seek a special solution that can be written as $\psi_p = \psi_p(r)$ and rewrite (7) as the following ordinary differential equation:

$$[\frac{1}{r^{n-1}} \frac{d}{dr} r^{n-1} \frac{d}{dr}]^p \psi_p(r) = -w_\eta(r) \qquad \forall\, r > 0. \tag{9}$$

If $n > 2p$, we can enforce the boundary condition

$$\psi_p(r) \to 0 \qquad \text{as } r \to +\infty, \tag{10}$$

which eliminates the arbitrary polynomial of degree $2p-1$. In other words, if $n > 2p$, equations (9)-(10) admit a unique solution which belongs to $\mathbb{H}^{2p}$ and satisfies (7).

It will be useful to explicitly solve (9)-(10) for $n > 2p$. The results are as follows:

Lemma 1 *If $2p < n$, as $\eta \to 0$ the solution to (9)-(10) is given by*

$$\psi_p(r) = \begin{cases} \sum_{s=0}^{p} A_n^{p,s} r^{2s} & \text{if } r \le 1, \\ \sum_{s=0}^{p-1} B_n^{p,s} r^{2s+2-n} & \text{if } r > 1, \end{cases} \tag{11}$$

where the coefficients $A_n^{p,s}$ and $B_n^{p,s}$ are determined recursively by

$$\begin{aligned} &A_n^{1,1} = -\frac{1}{2n}, \quad A_n^{1,0} = \frac{1}{2(n-2)}, \quad B_n^{1,0} = \frac{1}{n(n-2)}, \\ &A_n^{p,s} = 2s(2s+n-2)A_n^{p-1,s-1}, \qquad B_n^{p,s} = 2s(2s+2-n)B_n^{p-1,s-1}, \\ &\sum_{s=0}^{p} A_n^{p,s} = \sum_{s=0}^{p-1} B_n^{p,s}, \qquad \sum_{s=1}^{p} 2sA_n^{p,s} = \sum_{s=0}^{p-1} (2s+2-n) B_n^{p,s}. \end{aligned} \tag{12}$$

Proof: As $\eta \to 0$, $w_\eta \to 1$ on B_n and vanishes otherwise. In this limit, by induction on p we conclude that the solution to (9)-(10) is necessarily given by a finite series of form (11). To determine the coefficients in (11), we first notice that if $p = 1$,

$$\psi_1(r) = \begin{cases} -\frac{1}{2n} r^2 + \frac{1}{2(n-2)} & \text{if } r \le 1, \\ \frac{1}{n(n-2) r^{n-2}} & \text{if } r > 1, \end{cases}$$

which confirms $(12)_1$. By (7) we have $\Delta \psi_p = \psi_{p-1}$, and upon differentiating (11) we obtain $(12)_2$. Finally, the continuities of ψ_p and $\frac{d}{dr}\psi_p$ at $r = 1$ imply $(12)_3$.

■

We remark that the above recursive formulae (12) are sufficient to determine all coefficients $A_n^{p,s}$ and $B_n^{p,s}$. In particular, we have

$$A_n^{p,p} = \frac{-1}{2^p p! \prod_{s=1}^p (2s-2+n)},$$

and hence

$$D_{\mathbf{x}}^\alpha \psi_p(r) = \frac{-1}{2^p p! \prod_{s=1}^p (2s-2+n)} D_{\mathbf{x}}^\alpha r^{2p} \qquad \forall\, r<1 \quad \text{if } |\alpha| = 2p. \tag{13}$$

Further, since the solution to (7) in $\mathbb{H}^{2p}$ is unique within a polynomial of degree $2p-1$, it can be shown that equation (13) applies to any $p \geq 1$ and $n \geq 2$.

On the other hand, by Fourier transformation we can formally represent the solution to (7) in $\mathbb{H}^{2p}$ as

$$\psi_p(\mathbf{x}) = \frac{-1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{i^{2p}|\mathbf{k}|^{2p}} \exp(i\mathbf{k}\cdot\mathbf{x}) \hat{w}_\eta(\mathbf{k}) d\mathbf{k}, \tag{14}$$

and hence for any multi-index α with $|\alpha| > 2p-n$,

$$\begin{aligned} D_{\mathbf{x}}^\alpha \psi_p(\mathbf{x}) &= \frac{-i^{|\alpha|-2p}}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\mathbf{k}}^\alpha k^{|\alpha|-2p} \exp(i\mathbf{k}\cdot\mathbf{x}) \hat{w}_\eta(\mathbf{k}) d\mathbf{k} \\ &= \frac{-i^{|\alpha|-2p}}{(2\pi)^n} \int_{S^{n-1}} \hat{\mathbf{k}}^\alpha \int_0^\infty k^{|\alpha|-2p+n-1} \exp(ik\hat{\mathbf{k}}\cdot\mathbf{x}) \hat{w}_\eta(k) dk d\mu(\hat{\mathbf{k}}) \\ &= -i^{|\alpha|-2p} \int_{S^{n-1}} \hat{\mathbf{k}}^\alpha \Lambda_\eta^{2p-|\alpha|}(\hat{\mathbf{k}}\cdot\mathbf{x}) d\mu(\hat{\mathbf{k}}). \end{aligned} \tag{15}$$

If $|\alpha| = 2p$, by (13) we see that $D_{\mathbf{x}}^\alpha \psi_p(\mathbf{x})$ is uniform on B_n, and hence

$$-\int_{S^{n-1}} \hat{\mathbf{k}}^\alpha \Lambda_\eta^0(\hat{\mathbf{k}}\cdot\mathbf{x}) d\mu(\hat{\mathbf{k}}) = -\int_{S^{n-1}} \hat{\mathbf{k}}^\alpha \Lambda_{\eta e}^0(\hat{\mathbf{k}}\cdot\mathbf{x}) d\mu(\hat{\mathbf{k}}) = -\omega_n \int_{S^{n-1}} \hat{\mathbf{k}}^\alpha d\mu(\hat{\mathbf{k}}) \qquad \mathbf{x}\in B_n, \tag{16}$$

where

$$\omega_n = \Lambda_{\eta e}^0(0) = \frac{1}{(2\pi)^n} \int_0^\infty k^{n-1} \hat{w}_\eta(k) dk.$$

Upon evaluating (15) for Δ^p at $\mathbf{x} = 0$, we find that

$$-1 = \Delta^p \psi(\mathbf{x})\Big|_{\mathbf{x}=0} = -\omega_n \int_{S^{n-1}} d\mu(\hat{\mathbf{k}}) \quad \Longrightarrow \quad \omega_n = \frac{1}{\text{Area}(\ S^{n-1})}. \tag{17}$$

Therefore, by (13) and (15) we have that if $|\alpha| = 2p$,

$$⨍_{S^{n-1}} \hat{\mathbf{k}}^\alpha d\mu(\hat{\mathbf{k}}) = \frac{1}{2^p p! \prod_{s=1}^p (2s-2+n)} D_{\mathbf{x}}^\alpha r^{2p},$$

where $⨍$ denotes the average value of the integrand on the integration domain. In particular, for $\alpha = (2p, 0, \cdots, 0)$ we have

$$⨍_{S^{n-1}} \hat{k}_1^{2p} d\mu(\hat{\mathbf{k}}) = \frac{(2p)!}{2^p p! \prod_{s=1}^p (2s-2+n)}. \tag{18}$$

Since equation (16) holds for any α with $|\alpha| = 2p$ ($p = 1, 2, \cdots$), one may see that $\Lambda_{\eta e}^0(t)$ is, in fact, constant for $t \in (-1, 1)$. This can be used to show the remarkable uniformity properties of ellipsoids in the context of second-order constant-coefficient partial differential equations. For future convenience, we summarize below.

Lemma 2 *Let $\Lambda^q_\eta, \Lambda^q_{\eta e}, \Lambda^q_{\eta o} : \mathbb{R} \to \mathbb{C}$ be defined as in (3) and (4). Then for any $t \in (-1, 1)$,*

(i) if $q = 0$, $\Lambda^0_{\eta e}(t) = \omega_n$;

(ii) if $q < 0$ is even, $\Lambda^q_{\eta e}(t) = 0$; if $q < 0$ is odd, $\Lambda^q_{\eta o}(t) = 0$;

(iii) if $0 \le q = 2p < n$,

$$\Lambda^{2p}_{\eta e}(t) = \sum_{m=0}^{p} C^{p,m}_{\eta e} \frac{\omega_n}{(2m)!} t^{2m}, \qquad \Lambda^{2p-1}_{\eta o}(t) = \sum_{m=0}^{p-1} C^{p,m}_{\eta o} \frac{\omega_n}{(2m+1)!} t^{2m+1}, \tag{19}$$

where the coefficients $C^{p,m}_{\eta e}$ and $C^{p,m}_{\eta o}$ satisfy

$$C^{p,m}_{\eta e} = -C^{p,m-1}_{\eta o} = -C^{p-1,m-1}_{\eta e}. \tag{20}$$

Proof: First, we notice that (ii) and (iii) follow from (i), (5) and (6). The requirement that $q < n$ arises from the convergence of the integrals, i.e., (3) and (4). Property (iii) might prevail if the definition of the function Λ^q_η could be appropriately generalized for $q \ge n$, which, however, will not be pursued here.

To show (i), we simply notice that (16) holds for any α with $|\alpha| = 2p$ ($p = 1, 2, \cdots$). For any $\mathbf{x} \in B_n$, $\hat{\mathbf{k}} \mapsto \Lambda^0_{\eta e}(\hat{\mathbf{k}} \cdot \mathbf{x})$ is even and smooth on S^{n-1} while the vector space spanned by $\{\hat{\mathbf{k}}^\alpha : |\alpha| = p,\ p = 1, 2, \cdots\}$ is dense in $C(S^{n-1})$. Therefore, by the localization theorem we conclude that $\Lambda^0_{\eta e}(\hat{\mathbf{k}} \cdot \mathbf{x})$ is independent of $\hat{\mathbf{k}}$, and hence $\Lambda^0_{\eta e}(t)$ is constant for $t \in (-1, 1)$.

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It will be of interest to explicitly compute the coefficients $C^{p,m}_{\eta e}$ and $C^{p,m}_{\eta o}$ defined in (19). First, by (20), it is sufficient to determine all $C^{p,m}_{\eta e}$ and $C^{p,m}_{\eta o}$ if $C^{p,0}_{\eta e}$ are known for $p \ge 0$. By (i) of Lemma 2, $C^{0,0}_{\eta e} = 1$. Further, by (14) we find that if $2p < n$,

$$\psi_p(\mathbf{x}) = \frac{-1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{i^{2p} |\mathbf{k}|^{2p}} \exp(i\mathbf{k} \cdot \mathbf{x}) \hat{w}_\eta(\mathbf{k}) d\mathbf{k} = (-1)^{p+1} \int_{S^{n-1}} \Lambda^{2p}_{\eta e}(\hat{\mathbf{k}} \cdot \mathbf{x}) d\mu(\hat{\mathbf{k}}).$$

Therefore, by $(12)_1$ we have that as $\eta \to 0$,

$$C^{p,0}_{\eta e} \to (-1)^{p+1} \psi_p(\mathbf{x})\Big|_{\mathbf{x}=0} = (-1)^{p+1} A^{p,0}_n, \quad (\text{ if } p = 1) \ = A^{1,0}_n = \frac{1}{2(n-2)}. \tag{21}$$

2.2 Nonuniform polynomial sources on a ball

The problem defined by (7) essentially concerns a uniform source term supported on the unit ball B_n. We now consider a nonuniform polynomial source term. Since our interested problems are linear, it is sufficient to consider monomial $\mathbf{x}^\alpha$ sources on the unit ball. In parallel to (7) we consider the p-harmonic problem

$$\Delta^p \psi_p = -\mathbf{x}^\alpha w_\eta \qquad \text{on } \mathbb{R}^n. \tag{22}$$

It can be shown that the above equation admits a solution in $\mathbb{H}^{2p}$, and the solution is unique within a polynomial of degree $2p - 1$ (Gilbarg and Trudinger, 1983). In other words, all derivatives $D^\alpha_{\mathbf{x}} \psi_p$ are uniquely defined if $|\alpha| \ge 2p$.

By Fourier transformation we can formally represent the solution as

$$\psi_p(\mathbf{x}) = \frac{-1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{(-i)^{|\alpha|} i^{2p} |\mathbf{k}|^{2p}} \exp(i\mathbf{k} \cdot \mathbf{x}) D^\alpha_{\mathbf{k}} \hat{w}_\eta(\mathbf{k}) d\mathbf{k},$$

and hence any β-derivative of ψ_p with $|\beta| > 2p + |\alpha| - n$ can be represented as

$$\begin{aligned} D^{\beta}_{\mathbf{x}}\psi_p(\mathbf{x}) &= \frac{(-1)^{|\alpha|+1} i^{|\beta|-|\alpha|-2p}}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\mathbf{k}^{\beta}}{|\mathbf{k}|^{2p}} \exp(i\mathbf{k}\cdot\mathbf{x}) D^{\alpha}_{\mathbf{k}}\hat{w}_{\eta}(\mathbf{k})d\mathbf{k} \\ &= \frac{-i^{|\beta|-|\alpha|-2p}}{(2\pi)^n} \int_{\mathbb{R}^n} D^{\alpha}_{\mathbf{k}}\Big[\frac{\mathbf{k}^{\beta}}{|\mathbf{k}|^{2p}} \exp(i\mathbf{k}\cdot\mathbf{x})\Big] \hat{w}_{\eta}(\mathbf{k})d\mathbf{k} \\ &= \frac{-i^{|\beta|+|\gamma|-|\alpha|-2p}}{(2\pi)^n} \sum_{\gamma\le\alpha} c_{\alpha\gamma}\mathbf{x}^{\gamma} \int_{\mathbb{R}^n} D^{\alpha-\gamma}_{\mathbf{k}}\Big[\frac{\mathbf{k}^{\beta}}{|\mathbf{k}|^{2p}}\Big] \exp(i\mathbf{k}\cdot\mathbf{x})\hat{w}_{\eta}(\mathbf{k})d\mathbf{k} \\ &= -i^{|\beta|+|\gamma|-|\alpha|-2p} \sum_{\gamma\le\alpha} c_{\alpha\gamma}\mathbf{x}^{\gamma} \int_{S^{n-1}} g^{\beta}_{\alpha-\gamma}(\hat{\mathbf{k}}) \Lambda^{2p+|\alpha|-|\gamma|-|\beta|}_{\eta}(\hat{\mathbf{k}}\cdot\mathbf{x}) d\mu(\hat{\mathbf{k}}), \end{aligned} \tag{23}$$

where the second equality is obtained by integrating by parts, the third follows from the Leibniz formula (2), and

$$g^{\beta}_{\alpha-\gamma}(\mathbf{k}) = |\mathbf{k}|^{2p+|\alpha|-|\gamma|-|\beta|} D^{\alpha-\gamma}_{\mathbf{k}}\Big[\frac{\mathbf{k}^{\beta}}{|\mathbf{k}|^{2p}}\Big] = g^{\beta}_{\alpha-\gamma}(\hat{\mathbf{k}}). \tag{24}$$

If $|\beta| = 2p + |\alpha|$, $g^{\beta}_{\alpha-\gamma}(\hat{\mathbf{k}})$ is even (odd) if $|\gamma|$ is even (odd), and hence by Lemma 2 and (23) we have

$$D^{\beta}_{\mathbf{x}}\psi_p(\mathbf{x}) = -\int_{S^{n-1}} D^{\alpha}_{\mathbf{k}}\Big[\frac{\mathbf{k}^{\beta}}{|\mathbf{k}|^{2p}}\Big] \Lambda^{0}_{\eta e}(\hat{\mathbf{k}}\cdot\mathbf{x})d\hat{\mathbf{k}} = -⨍_{S^{n-1}} D^{\alpha}_{\mathbf{k}}\Big[\frac{\mathbf{k}^{\beta}}{|\mathbf{k}|^{2p}}\Big] d\mu(\hat{\mathbf{k}}) \qquad \forall\, \mathbf{x}\in B_n. \tag{25}$$

The above equation implies that $\psi_p(\mathbf{x})$ is necessarily a polynomial of degree $2p + |\alpha|$ in the unit ball B_n.

2.3 Extensions to ellipsoidal sources

Let $\Omega := \{\mathbf{x} :\ \sum_{i=1}^{n} x_i^2/a_i^2 < 1\}$ with $0 < a_1 \le a_2 \cdots \le a_n$ be an ellipsoid. We introduce the transformation:

$$w'_{\eta}(\mathbf{x}) = w_{\eta}(\mathbf{y}), \qquad \mathbf{y} = \mathbf{A}^{-1}\mathbf{x}, \tag{26}$$

where $\mathbf{A} = \mathrm{diag}[a_1, \cdots, a_n]$. It is clear that the source function $w'_{\eta} : \mathbb{R}^n \to \mathbb{R}$ has the following properties:

- $w'_{\eta}(\mathbf{x}) = 1$ if $\mathbf{x} \in \Omega$ and $w_{\eta}(\mathbf{x}) = 0$ if $\mathbf{x} \notin \Omega$ and $\mathrm{dist}(\mathbf{x}, \partial\Omega) := \min_{\mathbf{y}\in\partial\Omega} |\mathbf{x} - \mathbf{y}| > a_n\eta$.
- The Fourier transformation of w'_{η} satisfies

$$\hat{w}'_{\eta}(\mathbf{k}) = \int_{\mathbb{R}^n} w'_{\eta}(\mathbf{x}) \exp(-i\mathbf{x}\cdot\mathbf{k})d\mathbf{x} = \det(\mathbf{A})\hat{w}_{\eta}(\mathbf{A}\mathbf{k}) = \det(\mathbf{A})\hat{w}_{\eta}(|\mathbf{A}\mathbf{k}|). \tag{27}$$

In parallel to (22) we consider the p-harmonic problem

$$\Delta^p \psi_p = -\mathbf{x}^{\alpha} w'_{\eta} \qquad \text{on } \mathbb{R}^n. \tag{28}$$

The above equation similarly admits solutions in $\mathbb{H}^{2p}$ which are unique within a polynomial of degree $2p - 1$ (Gilbarg and Trudinger, 1983). If $|\beta| > 2p + |\alpha| - n$, in analogy with (23) we can represent the solution as

$$\begin{aligned} D^\beta_{\mathbf{x}}\psi_p(\mathbf{x}) &= \frac{(-1)^{|\alpha|+1} i^{|\beta|-|\alpha|-2p}}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{\mathbf{k}^\beta}{|\mathbf{k}|^{2p}}\exp(i\mathbf{k}\cdot\mathbf{x})D^\alpha_{\mathbf{k}}\hat{w}'_\eta(\mathbf{k})d\mathbf{k} \\ &= \frac{-i^{|\beta|+|\gamma|-|\alpha|-2p}}{(2\pi)^n}\sum_{\gamma\le\alpha}c_{\alpha\gamma}\mathbf{x}^\gamma\int_{\mathbb{R}^n}D^{\alpha-\gamma}_{\mathbf{k}}\Big[\frac{\mathbf{k}^\beta}{|\mathbf{k}|^{2p}}\Big]\exp(i\mathbf{k}\cdot\mathbf{x})\hat{w}'_\eta(\mathbf{k})d\mathbf{k}. \end{aligned} \tag{29}$$

Inserting (27) into the integral on the right-hand side of (29) and changing integration variables, we obtain (cf., (24))

$$\begin{aligned} &\int_{S^{n-1}} g^\beta_{\alpha-\gamma}(\hat{\mathbf{k}})\int_0^\infty k^{n-1-2p-|\alpha|+|\gamma|+|\beta|}\exp(ik\hat{\mathbf{k}}\cdot\mathbf{x})\hat{w}_\eta(k|\mathbf{A}\hat{\mathbf{k}}|)dk\,d\mu(\hat{\mathbf{k}}) \\ &\quad= \int_{S^{n-1}}\frac{\det(\mathbf{A})g^\beta_{\alpha-\gamma}(\hat{\mathbf{k}})}{|\mathbf{A}\hat{\mathbf{k}}|^{n-2p-|\alpha|+|\gamma|+|\beta|}}\int_0^\infty k'^{n-1-2p-|\alpha|+|\gamma|+|\beta|}\exp(ik'\frac{\hat{\mathbf{k}}\cdot\mathbf{x}}{|\mathbf{A}\hat{\mathbf{k}}|})\hat{w}_\eta(k')dk'd\mu(\hat{\mathbf{k}}) \\ &\quad= (2\pi)^n\int_{S^{n-1}}\frac{\det(\mathbf{A})g^\beta_{\alpha-\gamma}(\hat{\mathbf{k}})}{|\mathbf{A}\hat{\mathbf{k}}|^{n-2p-|\alpha|+|\gamma|+|\beta|}}\Lambda_\eta^{2p+|\alpha|-|\gamma|-|\beta|}(\frac{\hat{\mathbf{k}}\cdot\mathbf{x}}{|\mathbf{A}\hat{\mathbf{k}}|})d\mu(\hat{\mathbf{k}}). \end{aligned} \tag{30}$$

If $|\beta| = 2p+|\alpha|$, $g^\beta_{\alpha-\gamma}(\hat{\mathbf{k}})$ is even (odd) if $|\gamma|$ is even (odd), and hence by Lemma 2 and (29) we have

$$D^\beta_{\mathbf{x}}\psi_p(\mathbf{x}) = -\fint_{S^{n-1}}\frac{\det(\mathbf{A})}{|\mathbf{A}\hat{\mathbf{k}}|^n}D^\alpha_{\mathbf{k}}\Big[\frac{\mathbf{k}^\beta}{|\mathbf{k}|^{2p}}\Big]d\mu(\hat{\mathbf{k}}) \qquad \forall\,\mathbf{x}\in\Omega. \tag{31}$$

The above equation implies that $\psi_p(\mathbf{x})$ is again necessarily a polynomial of degree $2p+|\alpha|$ in the ellipsoid Ω.

Further, if $|\alpha|<n$ and $|\beta| = 2p$, by (30), equation (29) can be rewritten as

$$D^\beta_{\mathbf{x}}\psi_p(\mathbf{x}) = -i^{|\gamma|-|\alpha|}\sum_{\gamma\le\alpha}c_{\alpha\gamma}\mathbf{x}^\gamma\int_{S^{n-1}}\frac{\det(\mathbf{A})g^\beta_{\alpha-\gamma}(\hat{\mathbf{k}})}{|\mathbf{A}\hat{\mathbf{k}}|^{n-|\alpha|+|\gamma|}}\Lambda_\eta^{|\alpha|-|\gamma|}(\frac{\hat{\mathbf{k}}\cdot\mathbf{x}}{|\mathbf{A}\hat{\mathbf{k}}|})d\mu(\hat{\mathbf{k}}). \tag{32}$$

More explicitly, if $|\alpha| = 1$, $|\beta| = 2p$ and $n\ge 2$, by Lemma 2 and (32) we have that for $\mathbf{x}\in\Omega$,

$$D^\beta_{\mathbf{x}}\psi_p(\mathbf{x}) = -\fint_{S^{n-1}}\frac{\det(\mathbf{A})}{|\mathbf{A}\hat{\mathbf{k}}|^n}\Big[(\mathbf{x}\cdot\mathbf{k})D^\alpha_{\mathbf{k}}\hat{\mathbf{k}}^\beta + \mathbf{x}^\alpha\hat{\mathbf{k}}^\beta\Big]d\mu(\hat{\mathbf{k}}). \tag{33}$$

If $|\alpha| = 2$, $|\beta| = 2p$ and $n\ge 3$, by Lemma 2 and (32), similar calculations yield that for $\mathbf{x}\in\Omega$,

$$\begin{aligned} D^\beta_{\mathbf{x}}\psi_p(\mathbf{x}) = -\fint_{S^{n-1}}\frac{\det(\mathbf{A})}{|\mathbf{A}\hat{\mathbf{k}}|^n}\Big[&[\frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{x}})^2 - \frac{|\mathbf{A}\hat{\mathbf{k}}|^2}{2(n-2)}]|\mathbf{k}|^2 D^\alpha_{\mathbf{k}}\hat{\mathbf{k}}^\beta \\ &+ \sum_{|\gamma|=1,\gamma\le\alpha}c_{\alpha\gamma}\mathbf{x}^\gamma(\mathbf{k}\cdot\mathbf{x})D^{\alpha-\gamma}_{\mathbf{k}}\hat{\mathbf{k}}^\beta + \mathbf{x}^\alpha\hat{\mathbf{k}}^\beta\Big]d\mu(\hat{\mathbf{k}}). \end{aligned} \tag{34}$$

We remark that all coefficients associated with polynomials in (33) and (34) are elliptic integrals and can be conveniently evaluated (see Appendix). In addition, we can apply the above formulae (32)-(33) to spaces of dimension $d\le|\alpha|$ by considering the limit of ellipsoids in a higher-dimensional space of dimension $n = 1+|\alpha|$ with some of the semi-axes approaching infinity. Therefore, the solutions (32)-(33) are actually not restricted to $n>|\alpha|$.

3 Solutions to the homogeneous inclusion problem with nonuniform polynomial eigenstress

We now consider a second-order linear elliptic system which determines the relevant fields in a number of physical settings including elasticity, electrostatics and magnetostatics. Let $\mathbf{C}_0 : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ be a positive semi-definite symmetric tensor satisfying that for some $c > 0$,

$$(\mathbf{C}_0)_{piqj}(\mathbf{a})_i(\mathbf{a})_j(\mathbf{b})_p(\mathbf{b})_q \geq c|\mathbf{a}|^2|\mathbf{b}|^2 \qquad \forall\, \mathbf{a} \in \mathbb{R}^n,\ \mathbf{b} \in \mathbb{R}^m. \tag{35}$$

The collection of such tensors is denoted by $\mathbb{L}^+$. Consider the following problem for $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^m$,

$$\begin{cases} \mathrm{div}(\mathbf{C}_0\nabla\mathbf{u} + \mathbf{P}^* w'_\eta) = 0 & \text{on } \mathbb{R}^n, \\ |\nabla\mathbf{u}| \to 0 & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{36}$$

where

$$\mathbf{P}^* \in \mathcal{P}_q = \{\mathbf{P} : \mathbb{R}^n \to \mathbb{R}^{m\times n} : \text{every component of } \mathbf{P} \text{ is a polynomial of degree } \leq q\}.$$

We remark that $\mathcal{P}_q$ is a linear space of dimension $nm\sum_{s=0}^{q}\frac{(n+s-1)!}{(n-1)!s!}$. In the context of elasticity, this problem may be recognized as the *homogeneous* ESHELBY's inclusion problem; $\mathbf{C}_0$ ($\mathbf{P}^*$) is the elastic stiffness tensor (eigenstress).

The following theorem is presented in Asaro and Barnett (1975), Mura and Kinoshita (1978) and Mura (1987, p. 158) in three dimensions.

Theorem 3 *The solution to* (36) *satisfies that*

$$\nabla\mathbf{u}\Big|_{\mathbf{x}\in\Omega} \in \mathcal{P}_q. \tag{37}$$

Proof: Since (36) is linear, it is sufficient to show (37) for eigenstresses

$$\mathbf{P}^* = \mathbf{x}^\alpha\mathbf{P}^0 = x_1^{\alpha_1}\cdots x_n^{\alpha_n}\mathbf{P}^0 \qquad \forall\, \mathbf{P}^0 \in \mathbb{R}^{n\times n}. \tag{38}$$

For the homogeneous problem (36), the method of Fourier analysis can be conveniently used. Let

$$[\mathbf{D}(\mathbf{k})]_{pq} = (\mathbf{C}_0)_{piqj}(\mathbf{k})_i(\mathbf{k})_j, \quad [\mathbf{D}(\mathbf{k})]^{-1} = [\mathrm{cof}\,\mathbf{D}(\mathbf{k})]^T / \det(\mathbf{D}(\mathbf{k})). \tag{39}$$

By the first of (36) we find the Fourier transformation of $\mathbf{u}$ is given by

$$\hat{\mathbf{u}}(\mathbf{k}) = i^{|\alpha|+1}\frac{[\mathrm{cof}\,\mathbf{D}(\hat{\mathbf{k}})]^T\mathbf{P}^0\hat{\mathbf{k}}}{|\mathbf{k}|\det(\mathbf{D}(\hat{\mathbf{k}}))}D^\alpha_{\mathbf{k}}\hat{w}'_\eta(\mathbf{k}) \qquad \forall\, \mathbf{k} \in \mathbb{R}^n. \tag{40}$$

Let

$$\mathbf{h}^\beta(\mathbf{k}) = \frac{(\mathbf{k})^\beta[\mathrm{cof}\,\mathbf{D}(\mathbf{k})]^T\mathbf{P}^0\mathbf{k}}{\det\mathbf{D}(\mathbf{k})}. \tag{41}$$

Note that $\mathbf{h}^\beta(t\mathbf{k}) = t^{|\beta|-1}\mathbf{h}^\beta(\mathbf{k})$. If $|\alpha| < n+|\beta|-1$, it is legitimate to integrate by parts and obtain

$$\begin{aligned} D^\beta_{\mathbf{x}}\mathbf{u}(\mathbf{x}) &= \frac{i^{|\beta|+|\alpha|+1}}{(2\pi)^n}\int_{\mathbb{R}^n}\mathbf{h}^\beta(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})D^\alpha_{\mathbf{k}}\hat{w}'_\eta(\mathbf{k})d\mathbf{k} \\ &= (-1)^{|\alpha|}\frac{i^{|\beta|+|\alpha|+1}}{(2\pi)^n}\int_{\mathbb{R}^n}D^\alpha_{\mathbf{k}}[\mathbf{h}^\beta(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})]\hat{w}'_\eta(\mathbf{k})d\mathbf{k} \\ &= (-1)^{|\alpha|}\frac{i^{|\beta|+|\alpha|+|\gamma|+1}}{(2\pi)^n}\sum_{\gamma\leq\alpha}c_{\alpha\gamma}\mathbf{x}^\gamma\int_{\mathbb{R}^n}D^{\alpha-\gamma}_{\mathbf{k}}\mathbf{h}^\beta(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})\hat{w}'_\eta(\mathbf{k})d\mathbf{k}, \end{aligned} \tag{42}$$

where the last equality follows from the Leibniz formula (2). Inserting (27) into the above equation we obtain

$$\begin{aligned} D_{\mathbf{x}}^{\beta}\mathbf{u}(\mathbf{x}) &= \frac{i^{|\beta|+|\alpha|+|\gamma|+1}}{(-1)^{|\alpha|}(2\pi)^n}\sum_{\gamma\le\alpha} c_{\alpha\gamma}\mathbf{x}^{\gamma}\int_{\mathbb{R}^n} D_{\mathbf{k}}^{\alpha-\gamma}\mathbf{h}^{\beta}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})\det(\mathbf{A})\hat{w}_{\eta}(|\mathbf{A}\hat{\mathbf{k}}|k)d\mathbf{k} \\ &= \frac{i^{|\beta|+|\alpha|+|\gamma|+1}}{(-1)^{|\alpha|}}\sum_{\gamma\le\alpha} c_{\alpha\gamma}\mathbf{x}^{\gamma}\int_{S^{n-1}} \frac{\det(\mathbf{A})\mathbf{f}_{\alpha-\gamma}^{\beta}(\hat{\mathbf{k}})}{|\mathbf{A}\hat{\mathbf{k}}|^{n-1-|\alpha|+|\gamma|+|\beta|}}\Lambda_{\eta}^{|\alpha|+1-|\beta|-|\gamma|}(\frac{\hat{\mathbf{k}}\cdot\mathbf{x}}{|\mathbf{A}\hat{\mathbf{k}}|})d\mu(\hat{\mathbf{k}}), \end{aligned} \tag{43}$$

where

$$\mathbf{f}_{\alpha-\gamma}^{\beta}(\mathbf{k}) := |\mathbf{k}|^{|\alpha|+1-|\beta|-|\gamma|}D_{\mathbf{k}}^{\alpha-\gamma}\mathbf{h}^{\beta}(\mathbf{k}) = \mathbf{f}_{\alpha-\gamma}^{\beta}(\hat{\mathbf{k}}).$$

If $|\beta| = |\alpha| + 1$, by Lemma 2 we obtain

$$D_{\mathbf{x}}^{\beta}\mathbf{u}(\mathbf{x}) = -\det(\mathbf{A})\fint_{S^{n-1}}\frac{\mathbf{f}_{\alpha}^{\beta}(\hat{\mathbf{k}})}{|\mathbf{A}\hat{\mathbf{k}}|^n}d\mu(\hat{\mathbf{k}}) \qquad \forall\,\mathbf{x}\in\Omega,$$

which completes the proof of (37).

■

By the above theorem we define a linear mapping $\mathbb{T} : \mathcal{P}_q \to \mathcal{P}_q$ by

$$\mathbb{T}(\mathbf{P}^*) = \nabla\mathbf{u}\Big|_{\mathbf{x}\in\Omega}. \tag{44}$$

As an example, below we explicitly calculate the linear mapping $\mathbb{T}$ defined by (44) for tensor $\mathbf{C}_0$ of the form:

$$m = n \qquad \text{and} \qquad (\mathbf{C}_0)_{piqj} = \mu_1\delta_{ij}\delta_{pq} + \mu_2\delta_{pj}\delta_{iq} + \lambda\delta_{ip}\delta_{jq}, \tag{45}$$

where δ_{ij} $(i,j = 1,\cdots,n)$ are the components of the identity matrix $\mathbf{I}$. By (35), the constants μ_1, μ_2, λ necessarily satisfy

$$\mu_1 \ge \mu_2, \qquad \mu_1 + \mu_2 > 0 \quad \text{and} \quad \lambda > -\frac{\mu_1+\mu_2}{n}. \tag{46}$$

The physical interpretations of the above tensor are versatile. For instance, (i) if $\mu_1 = \mu_2 = \mu > 0$, $\mathbf{C}_0$ can be recognized as an isotropic elasticity tensor, and (ii) if $\mu_2 = \lambda = 0$, $\mathbf{C}_0$ can be identified as an isotropic permittivity/permeability tensor in electrostatics/magnetostatics. Direct calculations show that $\mathbf{D}(\mathbf{k})$ defined by (39) is given by

$$\mathbf{D}(\mathbf{k}) = \mu_1|\mathbf{k}|^2\mathbf{I} + (\mu_2+\lambda)\mathbf{k}\otimes\mathbf{k}$$

and hence equation (40) can be rewritten as

$$\hat{\mathbf{u}}(\mathbf{k})\otimes(i\mathbf{k}) = i^{|\alpha|+2}\left[\frac{1}{\mu_1}\mathbf{P}^0\hat{\mathbf{k}}\otimes\hat{\mathbf{k}} - \frac{\mu_2+\lambda}{\mu_1(\lambda+\mu_1+\mu_2)}\hat{\mathbf{k}}\otimes\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{P}^0\hat{\mathbf{k}})\right]D_{\mathbf{k}}^{\alpha}\hat{w}_{\eta}'(\mathbf{k}) \qquad \forall\,\mathbf{k}\in\mathbb{R}^n. \tag{47}$$

Comparing the above equation with (34) for $|\beta| = 2p$, we conclude that

$$\nabla\mathbf{u} = \frac{1}{\mu_1}\mathbf{P}^0\nabla\nabla\psi_1 - \frac{\mu_2+\lambda}{\mu_1(\lambda+\mu_1+\mu_2)}(\nabla\nabla\nabla\nabla\psi_2)\mathbf{P}^0 \qquad \forall\,\mathbf{x}\in\mathbb{R}^n, \tag{48}$$

where ψ_p $(p = 1, 2)$ are the solutions to (22). Inserting (33) or (34) into the above equation, we immediately obtain the strain field explicitly. Moreover, by (48) we can write the strain field inside B_n in the usual form as

$$\nabla \mathbf{u}(\mathbf{x}) = -\mathbf{R}^{\alpha}(\mathbf{x})\mathbf{P}^0 \qquad \forall\, \mathbf{x} \in \Omega, \tag{49}$$

where

$$\mathbf{R}^{\alpha}(\mathbf{x}) = \frac{1}{\mu_1}[\mathbf{S}_1^{\alpha}(\mathbf{x}) - \frac{\mu_2 + \lambda}{\mu_1(\lambda + \mu_1 + \mu_2)}\mathbf{S}_2^{\alpha}(\mathbf{x})]. \tag{50}$$

If $|\alpha| = 0$, the tensors $\mathbf{S}_1^0$ and $\mathbf{S}_2^0$ are uniform on Ω and given by $(\mathbf{S}_1^0)_{piqj} = \delta_{pq}(\mathbf{Q})_{ij}$,

$$(\mathbf{Q})_{ij} = \det(\mathbf{A}) \fint_{S^{n-1}} \frac{\hat{k}_i\hat{k}_j}{|\mathbf{A}\hat{\mathbf{k}}|^n} d\mu(\hat{\mathbf{k}}), \qquad (\mathbf{S}_2^0)_{piqj} = \det(\mathbf{A}) \fint_{S^{n-1}} \frac{\hat{k}_p\hat{k}_q\hat{k}_i\hat{k}_j}{|\mathbf{A}\hat{\mathbf{k}}|^n} d\mu(\hat{\mathbf{k}}). \tag{51}$$

If $|\alpha| = 1$ and $n \geq 2$, by (33) the tensors $\mathbf{S}_1^{\alpha}$ and $\mathbf{S}_2^{\alpha}$ are linear on Ω and given by

$$\begin{aligned}
(\mathbf{S}_1^{\alpha})_{piqj} &= \det(\mathbf{A})\delta_{pq} \fint_{S^{n-1}} \frac{(\mathbf{x}\cdot\mathbf{k})D_{\mathbf{k}}^{\alpha}(\hat{k}_i\hat{k}_j) + \mathbf{x}^{\alpha}\hat{k}_i\hat{k}_j}{|\mathbf{A}\hat{\mathbf{k}}|^n} d\mu(\hat{\mathbf{k}}), \\
(\mathbf{S}_2^{\alpha})_{piqj} &= \det(\mathbf{A}) \fint_{S^{n-1}} \frac{(\mathbf{x}\cdot\mathbf{k})D_{\mathbf{k}}^{\alpha}(\hat{k}_p\hat{k}_q\hat{k}_i\hat{k}_j) + \mathbf{x}^{\alpha}\hat{k}_p\hat{k}_q\hat{k}_i\hat{k}_j}{|\mathbf{A}\hat{\mathbf{k}}|^n} d\mu(\hat{\mathbf{k}}).
\end{aligned} \tag{52}$$

If $|\alpha| = 2$ and $n \geq 3$, by (34) the tensor $\mathbf{S}_1^{\alpha}$ and $\mathbf{S}_2^{\alpha}$ are quadratic on Ω and given by

$$\begin{aligned}
(\mathbf{S}_1^{\alpha})_{piqj} = \delta_{pq} \fint_{S^{n-1}} \frac{\det(\mathbf{A})}{|\mathbf{A}\hat{\mathbf{k}}|^n} \Big[&[\frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{x}})^2 - \frac{|\mathbf{A}\hat{\mathbf{k}}|^2}{2(n-2)}]|\mathbf{k}|^2 D_{\mathbf{k}}^{\alpha}(\hat{k}_i\hat{k}_j) \\
&+ \sum_{|\gamma|=1, \gamma\leq\alpha} c_{\alpha\gamma}\mathbf{x}^{\gamma}(\mathbf{k}\cdot\mathbf{x})D_{\mathbf{k}}^{\alpha-\gamma}(\hat{k}_i\hat{k}_j) + \mathbf{x}^{\alpha}(\hat{k}_i\hat{k}_j)\Big] d\mu(\hat{\mathbf{k}}), \\
(\mathbf{S}_2^{\alpha})_{piqj} = \fint_{S^{n-1}} \frac{\det(\mathbf{A})}{|\mathbf{A}\hat{\mathbf{k}}|^n} \Big[&[\frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{x}})^2 - \frac{|\mathbf{A}\hat{\mathbf{k}}|^2}{2(n-2)}]|\mathbf{k}|^2 D_{\mathbf{k}}^{\alpha}(\hat{k}_p\hat{k}_q\hat{k}_i\hat{k}_j) \\
&+ \sum_{|\gamma|=1, \gamma\leq\alpha} c_{\alpha\gamma}\mathbf{x}^{\gamma}(\mathbf{k}\cdot\mathbf{x})D_{\mathbf{k}}^{\alpha-\gamma}(\hat{k}_p\hat{k}_q\hat{k}_i\hat{k}_j) + \mathbf{x}^{\alpha}(\hat{k}_p\hat{k}_q\hat{k}_i\hat{k}_j)\Big] d\mu(\hat{\mathbf{k}}).
\end{aligned} \tag{53}$$

4 Solutions to the inhomogeneous inclusion problem

We now consider an inhomogeneous problem for $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$,

$$\begin{cases} \operatorname{div}\Big[\mathbf{C}(\mathbf{x})[\nabla\mathbf{u} + \mathbf{F}(\mathbf{x})]\Big] = 0 & \text{on } \mathbb{R}^n, \\ |\nabla\mathbf{u}| \to 0 & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{54}$$

where

$$\mathbf{C}(\mathbf{x}) = \begin{cases} \mathbf{C}_0 & \text{if } \mathbf{x} \in \Omega, \\ \mathbf{C}_1 & \text{if } \mathbf{x} \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{55}$$

and $\mathbf{F} \in \mathcal{P}_q$ is referred to the far applied field and satisfies

$$\operatorname{div}[\mathbf{C}_0\mathbf{F}(\mathbf{x})] = 0 \qquad \text{in } \mathbb{R}^n. \tag{56}$$

Let $\triangle \mathbf{C} = \mathbf{C}_1 - \mathbf{C}_0$ and assume $\triangle \mathbf{C}$ is invertible. By (56), equation (54) can be rewritten as

$$\begin{cases} \operatorname{div}\Big[\mathbf{C}_0[\nabla \mathbf{u} + \mathbf{F}(\mathbf{x})] + \triangle \mathbf{C}[\nabla \mathbf{u} + \mathbf{F}(\mathbf{x})]\chi_\Omega\Big] = 0 & \text{on } \mathbb{R}^n, \\ |\nabla \mathbf{u}| \to 0 & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{57}$$

Assume that Ω is an ellipsoid. Comparing the above equation with (36) and by (44) we see that if

$$\triangle \mathbf{C}[\mathbb{T}(\mathbf{P}^*) + \mathbf{F}] = \mathbf{P}^*, \qquad \text{i.e.,} \qquad \mathbb{T}(\mathbf{P}^*) = \triangle \mathbf{C}^{-1}[\mathbf{P}^* - \triangle \mathbf{C}\mathbf{F}], \tag{58}$$

then the solution to (36) (in the limit $\eta \to 0$) is also the solution to (57). If the linear mapping $\mathbb{T} - \triangle \mathbf{C}^{-1} : \mathcal{P}_q \to \mathcal{P}_q$ is invertible, we can formally write the equivalent eigenstress as

$$\mathbf{P}^* = -(\mathbb{T} - \triangle \mathbf{C}^{-1})^{-1} \triangle \mathbf{C}\mathbf{F}. \tag{59}$$

Further, an important physical quantity, i.e., energy arising from the presence of the inhomogeneity, is defined as

$$\begin{aligned} \mathcal{E}(\mathbf{x}_0, \boldsymbol{\theta}) &= \int_{\mathbb{R}^n} \Big\{ [\nabla \mathbf{u} + \mathbf{F}(\mathbf{x})] \cdot \mathbf{C}(\mathbf{x})[\nabla \mathbf{u} + \mathbf{F}(\mathbf{x})] - [\nabla \mathbf{u} + \mathbf{F}(\mathbf{x})] \cdot \mathbf{C}_0[\nabla \mathbf{u} + \mathbf{F}(\mathbf{x})] \Big\} d\mathbf{x} \\ &= \int_\Omega \Big\{ [\nabla \mathbf{u} + \mathbf{F}(\mathbf{x})] \cdot \triangle \mathbf{C}(\mathbf{x})[\nabla \mathbf{u} + \mathbf{F}(\mathbf{x})] \Big\} d\mathbf{x}, \end{aligned} \tag{60}$$

where $\mathbf{x}_0$ denotes the position of the center of the ellipsoid Ω and $\boldsymbol{\theta}$ describes the orientation of the ellipsoid. Again we notice that the energy arising from the presence of the inhomogeneity can be determined solely by the interior field inside the inclusion, and hence by the equivalent eigenstress (59) for given applied nonuniform field $\mathbf{F}(\mathbf{x})$.

Explicitly finding the equivalent eigenstress for a given nonuniform applied field $\mathbf{F}$ is algebraically formidable if $q > 0$ and will be postponed to a future report. To present an example of how the inhomogeneous problem (54) can be solved by the equivalent inclusion method, we solve the problem "backward" in the sense that we specify the eigenstress $\mathbf{P}^*$, and then by (58), find the correct nonuniform applied field $\mathbf{F}(\mathbf{x})$.

For simplicity, assume that $|\alpha| = 1$, $n = 3$ (three dimensions), and for some $\mathbf{P}^0 \in \mathbb{R}^{3\times 3}$,

$$\mathbf{P}^* = x_1 \mathbf{P}^0.$$

Then by (52) we have ($\alpha = (1, 0, 0)$),

$$\begin{aligned} &(\mathbf{S}_1^\alpha)_{piqj} = x_1 \mathbf{M}_1^\alpha + x_2 \mathbf{M}_2^\alpha + x_3 \mathbf{M}_3^\alpha, \qquad \mathbf{S}_2^\alpha = x_1 \mathbf{N}_1^\alpha + x_2 \mathbf{N}_2^\alpha + x_3 \mathbf{N}_3^\alpha, \\ &(\mathbf{M}_1^\alpha)_{piqj} = \delta_{pq}[(\mathbf{Q})_{ij} - 2(\mathbf{S}_2^0)_{11ij} + 2(\mathbf{Q})_{11}\delta_{1i}\delta_{1j}], \\ &(\mathbf{M}_2^\alpha)_{piqj} = \delta_{pq}[-2(\mathbf{S}_2^0)_{12ij} + (\mathbf{Q})_{22}(\delta_{1i}\delta_{2j} + \delta_{2i}\delta_{1j})], \\ &(\mathbf{M}_3^\alpha)_{piqj} = \delta_{pq}[-2(\mathbf{S}_2^0)_{13ij} + (\mathbf{Q})_{33}(\delta_{1i}\delta_{3j} + \delta_{3i}\delta_{1j})], \\ &(\mathbf{N}_1^\alpha)_{piqj} = [(\mathbf{S}_2^0)_{piqj} - 4(\mathbf{S}_3^0)_{11piqj} + (\mathbf{S}_2^0)_{1iqj}\delta_{1p} + (\mathbf{S}_2^0)_{1pij}\delta_{1q} + (\mathbf{S}_2^0)_{1pqj}\delta_{1i} + (\mathbf{S}_2^0)_{1pqi}\delta_{1j}], \\ &(\mathbf{N}_2^\alpha)_{piqj} = [-4(\mathbf{S}_3^0)_{12piqj} + (\mathbf{S}_2^0)_{2iqj}\delta_{1p} + (\mathbf{S}_2^0)_{2pij}\delta_{1q} + (\mathbf{S}_2^0)_{2pqj}\delta_{1i} + (\mathbf{S}_2^0)_{2pqi}\delta_{1j}], \\ &(\mathbf{N}_3^\alpha)_{piqj} = [-4(\mathbf{S}_3^0)_{13piqj} + (\mathbf{S}_2^0)_{3iqj}\delta_{1p} + (\mathbf{S}_2^0)_{3pij}\delta_{1q} + (\mathbf{S}_2^0)_{3pqj}\delta_{1i} + (\mathbf{S}_2^0)_{3pqi}\delta_{1j}], \end{aligned} \tag{61}$$

where $(\mathbf{Q})_{ij}, (\mathbf{S}_2^0)_{piqj}$ are defined in (51), and

$$(\mathbf{S}_3^0)_{piqjkl} = \det(\mathbf{A}) ⨍_{S^{n-1}} \frac{\hat{k}_p \hat{k}_q \hat{k}_i \hat{k}_j \hat{k}_k \hat{k}_l}{|\mathbf{A}\hat{\mathbf{k}}|^n} d\mu(\hat{\mathbf{k}}).$$

Therefore, by (44) and (49) we obtain

$$\mathbb{T}(x_1\mathbf{P}^0) = \Big[\frac{1}{\mu_1}(x_1\mathbf{M}_1^\alpha + x_2\mathbf{M}_2^\alpha + x_3\mathbf{M}_3^\alpha) - \frac{\mu_2+\lambda}{\mu_1(\lambda+\mu_1+\mu_2)}(x_1\mathbf{N}_1^\alpha + x_2\mathbf{N}_2^\alpha + x_3\mathbf{N}_3^\alpha)\Big]\mathbf{P}^0.$$

Consequently, by (58) we have

$$\mathbf{F}(\mathbf{x}) = \Big[x_1\triangle\mathbf{C}^{-1} - \frac{1}{\mu_1}(x_1\mathbf{M}_1^\alpha + x_2\mathbf{M}_2^\alpha + x_3\mathbf{M}_3^\alpha) + \frac{\mu_2+\lambda}{\mu_1(\lambda+\mu_1+\mu_2)}(x_1\mathbf{N}_1^\alpha + x_2\mathbf{N}_2^\alpha + x_3\mathbf{N}_3^\alpha)]\mathbf{P}^0.$$

Inserting the above equation into (60), we can compute the energy $\mathcal{E}$, how it depends on $\mathbf{x}_0$, $\boldsymbol{\theta}$, and the force and torque on the ellipsoid Ω upon differentiating $\mathcal{E}(\mathbf{x}_0,\boldsymbol{\theta})$ against positions $\mathbf{x}_0$ and angles $\boldsymbol{\theta}$.

Appendix: Evaluation of elliptic integrals

In the explicit solutions, the coefficients are given in terms of elliptic integrals of the form:

$$I^\beta = \fint_{S^{n-1}} \frac{\det(\mathbf{A})\hat{\mathbf{k}}^\beta}{|\mathbf{A}\hat{\mathbf{k}}|^n} d\mu(\hat{\mathbf{k}}). \tag{62}$$

For $|\alpha| = 0$ and $\beta = 2p$, equation (31) can be rewritten as

$$D_{\mathbf{x}}^\beta \psi_p(\mathbf{x}) = -\fint_{S^{n-1}} \frac{\det(\mathbf{A})\hat{\mathbf{k}}^\beta}{|\mathbf{A}\hat{\mathbf{k}}|^n} d\mu(\hat{\mathbf{k}}) = -I^\beta \qquad \forall\, \mathbf{x}\in\Omega. \tag{63}$$

Further, assume $\mathbf{A} = \operatorname{diag}[a_1, a_2, \cdots, a_n]$ is diagonal, meaning that the principle axes of the ellipsoid aligns with the coordinate frame. From the real space formulation, it is easy to check the following properties of ψ_p:

$$\psi_p(-x_1,\cdots) = \psi_p(x_1,\cdots),$$

where x_1 can be replaced by any other coordinates x_i ($i = 1,\cdots,n$). From (63) and the above symmetry, we infer the following properties of the above elliptic integrals I^α.

(i) If $|\alpha| = 0$, $I^0 = 1$; if any entry in the multi-index α is odd, $I^\alpha = 0$, and in particular, if $|\alpha|$ is odd, $I^\alpha = 0$.

(ii) If $|\alpha|$ is even , $\sum_{j=1}^n I^{\alpha+2\varsigma_j} = I^\alpha$, where ς_j is the multi-index with $|\varsigma_j| = 1$ and the only nonzero occurs at the jth entry.

A simple MATLAB program for computing the integral (62) is available at the author's homepage `http://math.rutgers.edu/~ll502/EllipticIntegrals/`.

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