Geometries of inhomogeneities with minimum field concentration

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Abstract

This paper is devoted to the study of geometries of inhomogeneities with minimum strain or stress concentration. The solutions are achieved by the indirect method of first deriving lower bounds and then constructing the geometries to attain the lower bounds. In particular, we show that a new class of geometries, namely, E-inclusions and periodic E-inclusions, are the optimal geometries with minimum field concentrations. We also obtain the explicit relation between the shape matrix of E-inclusion and remote applied strain which will be convenient for engineering applications of these new geometries.

Keywords: Stress concentration, E-inclusion, Optimal design

This work is dedicated to Lewis Wheeler with respect and admiration on the occasion of his 73rd birthday

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1. Introduction

The failure criteria of materials are often formulated in terms of "yield stress" or "ultimate stress", meaning that the maximum stress sustained by the material cannot exceed these critical values. As is well-known, inhomogeneities such as holes or inclusions inevitably increase the local stress and strain in an elastic body (Wheeler and Kunin, 1986; Mura, 1987; Cherkaev *et al.*, 1998; Nemat-Nasser and Hori, 1999; Vigdergauz, 2006). On the other hand, it is necessary to introduce inhomogeneities such as holes for adaptivity or desired geometry. For instance, it is common to use rivets or bolts to assemble small structural members into large, sometimes gigantic, structures such as airplanes, buildings and bridges. Also, second-phase precipitates often emerge for the coexistence of different phases of the same materials whose microstructure may be engineered to improve mechanical properties of the material (Schneider *et al.*, 1997; Jou *et al.,* 1997). In microelectronics, a similar dilemma occurs. To miniaturize microelectronic devices, it is desirable to use smaller conducting interconnects for realizing desired functionality. However, nuclei migrates under the bombardment of electric currents or flow of electrons and under certain critical currents or the driving force on the

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electrons (i.e., electric field), the migrations of nuclei become so severe that the material fails permanently (Christou, 1994).

From the above examples, it is clear that for practical engineering one needs to balance between lowering the magnitude of local fields such as stress, strain or electric field and maintaining the functionality or fulfilling the geometric constraints among others. Therefore, a precise analysis of *field concentration* is critical for the safety and reliability of the overall structure. In order to maintain the fields within safe limits, we are interested in the optimization problems of minimizing field concentration with respect to the geometries of inhomogeneities. A dimensionless quantity, namely, the *field concentration factor*, may be introduced to evaluate the severity of local field concentration in the body. Then a generic design problem is to find the optimal geometries of inhomogeneities such that the field concentration factor is minimized.

From a mathematical viewpoint, the dependence of field concentration on the geometries of inhomogeneities is rather complicated; one has to *a priori* solve the governing partial differential equation to evaluate the concentration factor for given geometries. In other words, the concentration factor depends *nonlocally* on the geometries of inhomogeneities. Therefore, the prevailing direct method of calculus of variation is not applicable. A conventional approach to such optimal design problems is based on an iterative process: trial geometries of inhomogeneities are chosen, the field concentration is evaluated upon a full solution of the underlying boundary value problem, and then a change of geometries is proposed to lower the field concentration via a sensitivity analysis (Haftka and Grandhi, 1986; Allaire and Jouve, 2008). This process is iterated until a local minimum of field concentration is achieved. This approach is computationally intensive and the final result, though could be satisfactory for a target application, cannot give a definitive answer to the global minimum. For the global minimum, one has to use the indirect method of first finding a lower bound on the concentration factor and then construct geometries to attain the lower bound.

In the context of linear elasticity, the problem of minimum stress or strain concentration has been discussed and reviewed by Sternberg (1958) and Wheeler (1992). Recently, there has been significant progress on a general theory concerning minimum field concentration for general measures of local fields that include the local Von Mises stress and strain (Alali and Lipton, 2009), hydrostatic stress and strain (Lipton, 2005; 2006), and local mixed modes of stress and strain (Alali and Lipton, 2012). The theory has also been established in much broader physical contexts including thermo-elastic composites (Chen and Lipton, 2010) and conductive composites (Lipton, 2003; 2004). The existing results concerning optimal geometries clearly suggest that the uniformity of field in the inhomogeneities is intimately related with the optimality of the geometries for minimum field concentration. Also, the optimal microstructures such as coated ellipsoids that achieve minimum field concentration, under suitable algebraic assumptions about the material properties, turn out to be optimal microstructures attaining the Hashin-Shtrikman's bounds of the effective properties of two-phase composites. As shown in recent works of Liu *et al.* (2007; accepted), a new class of geometries, namely, E-inclusions and periodic E-inclusions,[1] have similar uniformity property as ellipsoids and achieve the Hashin-Shtrikman bounds for composites. One may wonder if

[1]E-inclusions or periodic E-inclusions in two dimensions are first constructed by Vigdergauz (1976, 1986).

they are also the optimal geometries that minimize the field concentrations. Our main goal here is to report that the answer to the above question is affirmative. We also find explicitly the relationship between the average applied strain $\overline{\mathbf{E}}$ and the shape matrix $\mathbf{Q}$ of the E-inclusions with minimum strain or stress concentration (cf., (40)). Since the shape matrices of E-inclusions have to be positive semi-definite, E-inclusions being the solutions requires that the average applied strain has to satisfy some algebraic conditions. Beyond this region, the reader is referred to Cherkaev *et al.* (1998) and Vigdergauz (2006; 2008) for approximate solutions and important insight.

The paper is organized as follows. In Section 2 we formulate and state the mathematical optimization problem in the context of linear elasticity. The formulation allows for simultaneous consideration of finite many inhomogeneities and periodic array of inhomogeneities and in both two and three dimensions. The lower bounds for stress and strain concentration factors are derived in Section 3. In Section 4 we show that E-inclusions indeed achieve the lower bounds of minimum stress or strain concentration. We conclude and provide an outlook of potential engineering applications in Section 5.

Notation. Since stress and strain are symmetric tensor fields, we introduce the following l^p-norm of a symmetric matrix $\mathbf{M} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ for $p \in [1, \infty]$:

$$\|\mathbf{M}\|_{l^p} := (\sum_{i=1}^{n} |\lambda_i(\mathbf{M})|^p)^{1/p}, \tag{1}$$

where $\lambda_1(\mathbf{M}) \le \cdots \le \lambda_n(\mathbf{M})$ are the ordered eigenvalues of the symmetric $\mathbf{M}$. We remark that $\|\mathbf{M}\|_{l^p} = (\mathbf{M}\cdot\mathbf{M})^{1/2}$ is the usual Euclidean norm for $p = 2$, $\|\mathbf{M}\|_{l^p} = \sum_{i=1}^{n} |\lambda_i(\mathbf{M})|$ if $p = 1$, and $\|\mathbf{M}\|_{l^p} = \max\{|\lambda_i(\mathbf{M})| : i = 1, \cdots, N\}$ if $p = \infty$.

2. Problem statement

Consider an infinite homogeneous elastic body occupying the entire Euclidean space $\mathbb{R}^n$ ($n = 2$ or 3). Let $\mathbf{C}_0 : \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ be the fourth-order stiffness tensor of the body, $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$ be the displacement, and $\boldsymbol{\sigma} : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ be the stress. Assume that the body is under the application of an average strain for some $\overline{\mathbf{E}} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$:

$$\mathbf{u}(\mathbf{x}) = \overline{\mathbf{E}}\mathbf{x} + O(1) \qquad \text{as } |\mathbf{x}| \to +\infty. \tag{2}$$

In the absence of body force, the equilibrium state of the body requires that

$$\mathrm{div}\boldsymbol{\sigma} = 0 \qquad \text{in } \mathbb{R}^n. \tag{3}$$

Also, it is clear that a solution to the above equation with the boundary condition (2) is given by

$$\mathbf{u} = \mathbf{u}_0 := \overline{\mathbf{E}}\mathbf{x} \qquad \text{in } \mathbb{R}^n.$$

Let $Y \subset \mathbb{R}^n$ be a "representative volume element" of the body. We now introduce N mutually disjoint inhomogeneities $\Omega_\alpha \subset Y$ ($\alpha = 1, \cdots, N$) of materials with stiffness tensor $\mathbf{C}_\alpha$. Two scenarios will be considered: (i) the inhomogeneities are distributed in a bounded region in $\mathbb{R}^n$, and (ii) the inhomogeneities are distributed periodically in the whole space $\mathbb{R}^n$. The representative volume element Y is taken as the entire space $\mathbb{R}^n$ for the former case whereas, without loss of generality, can be assumed to be $Y = (0,1)^n$ for the latter case. We remark that the latter case corresponds to a periodic composite with infinitely many inhomogeneities occupying $\{\Omega_\alpha + \sum_{i=1}^n k_i \mathbf{f}_i : \alpha = 1, \cdots, N; k_1, \cdots, k_n \text{ are integers}\}$. ($\mathbf{f}_1, \cdots, \mathbf{f}_n$ is the basis of our rectangular coordinate system for $\mathbb{R}^n$.) Further, we denote by $\Omega = (\Omega_1, \cdots, \Omega_N)$ and $\Omega_0 = Y \setminus (\cup_{\alpha=1}^N \Omega_\alpha)$, and henceforth the stress-strain relation of the inhomogeneous body is now given by

$$\boldsymbol{\sigma} = \mathbf{C}_\alpha \mathbf{E}, \qquad \mathbf{E} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \qquad \text{in } \Omega_\alpha, \ \alpha = 0, \cdots, N. \tag{4}$$

Upon inspecting (2), (3) and (4) , it is clear that if $\overline{\mathbf{E}} \to \overline{\mathbf{E}}' = a\overline{\mathbf{E}}$ for any constant $a \in \mathbb{R}$, then $\mathbf{u} \to \mathbf{u}' = a\mathbf{u}$. Therefore, without loss of generality we assume $\mathrm{Tr}\overline{\mathbf{E}} = 1$ subsequently. Further, we define the average dilatational strain and stress on the matrix region Ω_0 as

$$\bar{\vartheta}_0 = \fint_{\Omega_0} \mathrm{Tr}\mathbf{E}, \qquad \bar{\theta}_0 = \fint_{\Omega_0} \mathrm{Tr}\boldsymbol{\sigma}.$$

Here and subsequently, $\fint_V$ denote the average of the integrand over domain V. It is clear that $\bar{\vartheta}_0 = 1$ if $Y = \mathbb{R}^n$.

The stress and strain concentrations are measured by the following *concentration factors*:

$$\sigma^*\{\Omega\} := \frac{1}{\bar{\theta}_0} \sup\{\|\boldsymbol{\sigma}(\mathbf{x})\|_{l^\infty} : \mathbf{x} \in \mathbb{R}^n\}, \qquad e^*\{\Omega\} := \frac{1}{\bar{\vartheta}_0} \sup\{\|\mathbf{E}(\mathbf{x})\|_{l^\infty} : \mathbf{x} \in \mathbb{R}^n\}. \tag{5}$$

We remark that one may choose other suitable measures of the state of stress and strain concentration, e.g., the norm that is consistent with the von Mises yield criterion (Eldiwany and Wheeler, 1986). For fixed material properties (i.e., the stiffness tensor $\mathbf{C}_\alpha$) of each inhomogeneity, the formal mathematical problems can be stated as

$$\text{minimum stress concentration:} \qquad \inf\{\sigma^*\{\Omega\} : \emptyset \neq \Omega_\alpha \subset \mathbb{R}^n, \ \alpha = 1, \cdots, N\}; \tag{6}$$

$$\text{minimum strain concentration:} \qquad \inf\{e^*\{\Omega\} : \emptyset \neq \Omega_\alpha \subset \mathbb{R}^n, \ \alpha = 1, \cdots, N\}. \tag{7}$$

3. Lower bounds of field concentration

Though the following argument may be generalized for anisotropic inhomogeneities of different material properties and to include the conductivity problems (Liu, 2010), for simplicity we assume all inhomogeneities are isotropic and of the same material property. In this case, we simply regard Ω as the union of Ω_α, i.e., $\Omega = \cup_{i=1}^N \Omega_\alpha$. For future convenience, let

$$\theta = \mathrm{Tr}(\boldsymbol{\sigma}), \qquad \vartheta = \mathrm{Tr}(\mathbf{E})$$

be the dilatational stress and strain, respectively. The Hooke's law of the body can be written as

$$\boldsymbol{\sigma} = 2\mu \mathbf{E} + \lambda \vartheta \mathbf{I}, \qquad (\mu, \lambda) = \begin{cases} (\mu_1, \lambda_1) & \text{in } \Omega, \\ (\mu_0, \lambda_0) & \text{in } \Omega_0, \end{cases} \tag{8}$$

where $(\mu_\alpha, \lambda_\alpha)$ $(\alpha = 1, 2)$ are the Lamé constants if $n = 3$ and $\mathbf{I}$ is the identity matrix in $\mathbb{R}^{n\times n}$. Then by (3) the equilibrium equation in the matrix phase can be written as

$$\mu_0 \Delta \mathbf{u} + (\mu_0 + \lambda_0) \nabla(\nabla \cdot \mathbf{u}) = 0 \qquad \text{in } \Omega_0. \tag{9}$$

We remark that in two dimensions $(n = 2)$, for plane stress the constants (μ, λ) are different from the Lamé constants of the material. Instead, they are given by

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{\nu E}{1-\nu^2},$$

where E, ν are the Young's modulus and Poisson's ratio, respectively. Also, we have

$$\theta = n\kappa\vartheta, \qquad \kappa = \begin{cases} \kappa_1 = 2\mu_1/n + \lambda_1 & \text{in } \Omega, \\ \kappa_0 = 2\mu_0/n + \lambda_0 & \text{in } \Omega_0, \end{cases}$$

where κ is referred to as the "bulk modulus" (not the actual bulk modulus for plane stress or plane strain).

We now derive the lower bounds of stress and strain concentrations following the argument of Wheeler (2004). There will be two cases that require separate considerations.

3.1. Finite number of bounded inhomogeneities

If $Y = \mathbb{R}^n$ and all inhomogeneities Ω_α are bounded, as shown in the next section the argument of Wheeler (2004) applies to two dimensions as well as three dimensions. From Wheeler (2004), we assert that for any geometries of a nonempty region Ω, the stress and strain concentration factors are bounded from below as $(\bar{\vartheta}_0 = 1, \bar{\theta}_0 = n\kappa_0)$:

$$\begin{cases} e^*\{\Omega\} \geq \hat{e}^\star & \text{if } \mu_0 > \mu_1, \\ \sigma^*\{\Omega\} \geq \hat{\sigma}^\star & \text{if } \mu_1 > \mu_0, \end{cases} \tag{10}$$

where

$$\hat{e}^\star = \frac{2\mu_0 + \lambda_0}{n(\kappa_1 - \kappa_0 + 2\mu_0 + \lambda_0)} = \frac{2\mu_0 + \lambda_0}{2(n-1)\mu_0 + 2\mu_1 + n\lambda_1}, \qquad \hat{\sigma}^\star = \frac{\kappa_1}{\kappa_0}\hat{e}^\star. \tag{11}$$

3.2. Periodic array of inhomogeneities

A second interesting case concerns composites of two materials that may be modeled as a periodic array of inhomogeneities embedded in the infinite matrix material. In this case, we shall interpret the boundary condition (2) as

$$\fint_Y \mathbf{E}(\mathbf{x})dv = \overline{\mathbf{E}}.$$

Further, from the homogenization theory we have

$$\fint_Y \theta(\mathbf{x})dv = n\kappa^e \fint_Y \mathbf{E}(\mathbf{x})dv = n\kappa^e \mathrm{Tr}(\overline{\mathbf{E}}) = n\kappa^e, \tag{12}$$

where κ^e is the effective bulk modulus and satisfies the Hashin-Shtrikman's bounds (Hashin and Shtrikman, 1962; Milton, 2002):

$$\kappa_L^{\mathrm{HS}} \le \kappa^e \le \kappa_U^{\mathrm{HS}}.$$

Note that the expressions of the lower (κ_L^{HS}) and upper (κ_U^{HS}) Hashin-Shtrikman bounds depend on the orderness of μ_0, μ_1 and κ_0, κ_1. In addition, we have

$$\fint_Y \vartheta(\mathbf{x})dv = \frac{\eta}{n\kappa_1}\fint_\Omega \theta(\mathbf{x})dv + \frac{(1-\eta)}{n\kappa_0}\fint_{\Omega_0} \theta(\mathbf{x})dv = \mathrm{Tr}(\overline{\mathbf{E}}) = 1, \tag{13}$$

where $\eta = \mathrm{vol}(\Omega)/\mathrm{vol}(Y)$ is the volume fraction of the inhomogeneities. Combining (12) with (13), we can solve for the average $\theta(\mathbf{x})$ restricted to the inhomogeneities and the matrix as follows:

$$\begin{aligned} \bar\theta_1 &:= \fint_\Omega \theta(\mathbf{x})dv = \frac{n\kappa_1(\kappa_0-\kappa^e)}{\eta(\kappa_0-\kappa_1)}, \qquad \bar\vartheta_1 = \frac{\kappa_0-\kappa^e}{\eta(\kappa_0-\kappa_1)}, \\ \bar\theta_0 &:= \fint_{\Omega_0} \theta(\mathbf{x})dv = \frac{n\kappa_0(\kappa_1-\kappa^e)}{(1-\eta)(\kappa_1-\kappa_0)}, \qquad \bar\vartheta_0 = \frac{\kappa_1-\kappa^e}{(1-\eta)(\kappa_1-\kappa_0)}. \end{aligned} \tag{14}$$

Note that $\bar\theta_0, \bar\vartheta_0 > 0$. Further, by (9) it is straightforward to verify that

$$\nabla^2\theta = 0 \qquad \text{in } \Omega_0, \qquad \fint_{\Omega_0} \theta = \bar\theta_0.$$

By the maximum principle we conclude that

$$\sup_{\partial\Omega} \theta^+ \ge \bar\theta_0, \tag{15}$$

where θ^+ denote the boundary value of θ on $\partial\Omega$ approached from the matrix phase Ω_0.

We now adapt the argument of Wheeler (2004) and derive lower bounds for stress and strain concentration factors. Consider first the stress concentration. By the kinematic compatibility and balance law across the interfaces $\partial\Omega$, we have that

$$[\![\boldsymbol{\sigma}]\!]\mathbf{n} = 0, \qquad \mathbf{t}\cdot[\![\mathbf{E}]\!]\mathbf{t} = 0 \qquad \text{on } \partial\Omega, \tag{16}$$

where $[\![\ ]\!]$ denote the jump across the interface, and $\mathbf{n}$ and $\mathbf{t}$ is the unit normal vector and any unit tangential vector on $\partial\Omega$, respectively. Set $\sigma_{\mathbf{nn}} = \mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n}$ and $e_{\mathbf{nn}} = \mathbf{n} \cdot \mathbf{E}\mathbf{n}$. The above equation implies that

$$[\![\sigma_{\mathbf{nn}}]\!] = 0, \qquad [\![\vartheta - e_{\mathbf{nn}}]\!] = 0 \qquad \text{on } \partial\Omega. \tag{17}$$

By the Hooke's law (8) we have $\sigma_{\mathbf{nn}} = 2\mu e_{\mathbf{nn}} + \lambda\vartheta$ and the second of (17) can be written as

$$[\![(1 + \frac{\lambda}{2\mu})\vartheta - \frac{1}{2\mu}\sigma_{\mathbf{nn}}]\!] = 0 \qquad \text{on } \partial\Omega. \tag{18}$$

Solving the above equation for $\sigma_{\mathbf{nn}}$, by the first of (17) we obtain

$$(\frac{1}{2\mu_0} - \frac{1}{2\mu_1})\sigma_{\mathbf{nn}} = (1 + \frac{\lambda_0}{2\mu_0})\frac{\theta^+}{n\kappa_0} - (1 + \frac{\lambda_1}{2\mu_1})\frac{\theta^-}{n\kappa_1} \qquad \text{on } \partial\Omega,$$

where θ^+ (θ^-) denotes the boundary values of θ outside (inside) Ω. Moreover, from the definition (5) we have

$$\frac{1}{n}\theta^-, \frac{1}{n}\theta^+, \sigma_{\mathbf{nn}} \leq \bar{\theta}_0\sigma^*\{\Omega\} \qquad \text{on } \partial\Omega.$$

Therefore, if $\mu_1 > \mu_0$ we have

$$\begin{aligned} \bar{\theta}_0\sigma^*\{\Omega\} &\geq (1 + \frac{\lambda_0}{2\mu_0})\frac{1}{n\kappa_0}\Big[\frac{1}{2\mu_0} - \frac{1}{2\mu_1} + (1 + \frac{\lambda_1}{2\mu_1})\frac{1}{\kappa_1}\Big]^{-1}\theta^+ \\ &\geq (1 + \frac{\lambda_0}{2\mu_0})\frac{1}{n\kappa_0}\Big[\frac{1}{2\mu_0} - \frac{1}{2\mu_1} + (1 + \frac{\lambda_1}{2\mu_1})\frac{1}{\kappa_1}\Big]^{-1}\bar{\theta}_0, \end{aligned} \tag{19}$$

where the second inequality follows from (15).

Next we consider the problem of minimum strain concentration. Since $\sigma_{\mathbf{nn}} = 2\mu e_{\mathbf{nn}} + \lambda\vartheta$, the first of (17) can be written as

$$[\![2\mu e_{\mathbf{nn}} + \lambda\vartheta]\!] = [\![2\mu(e_{\mathbf{nn}} - \vartheta) + (2\mu + \lambda)\vartheta]\!] = 0 \qquad \text{on } \partial\Omega.$$

Solving the above equation for $\vartheta - e_{\mathbf{nn}}$, by the second of (17) we obtain

$$(2\mu_0 - 2\mu_1)(\vartheta - e_{\mathbf{nn}}) = (2\mu_0 + \lambda_0)\vartheta^+ - (2\mu_1 + \lambda_1)\vartheta^-. \tag{20}$$

From the definition (5) we have

$$\frac{1}{n-1}(\vartheta - e_{\mathbf{nn}}),\ \frac{1}{n}\vartheta^- \leq \bar{\vartheta}_0 e^*\{\Omega\} \qquad \text{on } \partial\Omega.$$

Therefore, if $\mu_0 > \mu_1$, equation (20) implies

$$\begin{aligned} \bar{\vartheta}_0 e^*\{\Omega\} &\geq (2\mu_0 + \lambda_0)[2(n-1)(\mu_0 - \mu_1) + n(2\mu_1 + \lambda_1)]^{-1}\vartheta^+ \\ &\geq (2\mu_0 + \lambda_0)[2(n-1)(\mu_0 - \mu_1) + n(2\mu_1 + \lambda_1)]^{-1}\bar{\vartheta}_0. \end{aligned} \tag{21}$$

To summarize, for an infinite body with periodic array of inhomogeneities we have the following lower bounds on stress and strain concentration factors for any domain Ω:

$$\begin{cases} \sigma^*\{\Omega\} \geq \sigma^\star & \text{if } \mu_1 > \mu_0, \\ e^*\{\Omega\} \geq e^\star & \text{if } \mu_0 > \mu_1, \end{cases} \tag{22}$$

where we have noticed that (c.f. (11))

$$\begin{aligned} &(1+\frac{\lambda_0}{2\mu_0})\frac{1}{n\kappa_0}\Big[\frac{1}{2\mu_0}-\frac{1}{2\mu_1}+(1+\frac{\lambda_1}{2\mu_1})\frac{1}{\kappa_1}\Big]^{-1} = \hat{\sigma}^\star = \frac{\kappa_1}{\kappa_0}\hat{e}^\star, \\ &\frac{2\mu_0+\lambda_0}{2(n-1)(\mu_0-\mu_1)+n(2\mu_1+\lambda_1)} = \hat{e}^\star. \end{aligned} \tag{23}$$

We remark that the above lower bounds (22) recover the known results of Wheeler (2004) in the dilute limit $\eta \to 0$ (and hence $\kappa^e \to \kappa_0$).

4. E-inclusions as optimal shapes attaining the lower bounds

As observed by Wheeler (2004), inhomogeneities having uniform interface stress will be the optimal geometries for minimum stress or strain concentration. A well-known class of geometries that have uniform interface stress are ellipsoids (Eshelby, 1957; 1961). In the context of composites, microstructures with uniform stress in one of the phases include the construction of coated sphere (Hashin, 1962) and ellipsoids (Milton, 1981), and as shown by Lipton (2004), these constructions also have minimum field concentrations.

Based on these constructions, we observe that the uniformity property of ellipsoids in a variety of physical settings can be equivalently termed as that the solution of the Newtonian potential problem

$$\begin{cases} \nabla^2\xi = -\chi_\Omega & \text{in } \mathbb{R}^n, \\ |\nabla\xi| \to 0 & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{24}$$

satisfies that

$$\nabla\nabla\xi = -\mathbf{Q} \qquad \text{in } \Omega \tag{25}$$

for some $0 \leq \mathbf{Q} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ with $\mathrm{Tr}\mathbf{Q} = 1$. Here χ_Ω, equal to one in Ω and zero otherwise, denotes the characteristic function of domain Ω. Based on the above potential problem, we introduce a class of new geometries, namely, E-inclusions or periodic E-inclusions, that enjoy similar properties as an ellipsoid. In other words, an open bounded domain $\Omega \subset \mathbb{R}^n$ is an E-inclusion if a solution to (24) satisfies (25); an open bounded domain $\Omega \subset Y$ is a periodic E-inclusion if a solution to the periodic counterpart of (24), i.e.,

$$\begin{cases} \nabla^2\xi = \eta - \chi_\Omega & \text{in } Y, \\ \text{periodic boundary conditions} & \text{on } \partial Y, \end{cases} \tag{26}$$

satisfies that

$$\nabla\nabla\xi = -(1-\eta)\mathbf{Q} \qquad \text{in } \Omega. \tag{27}$$

For ellipsoids, the matrix $\mathbf{Q}$ depends only on the aspect ratios, i.e., characterizes the "shape" (but not the size) of domain Ω. Subsequently, we refer to the symmetric matrix $\mathbf{Q}$ as the shape matrix of E-inclusions or periodic E-inclusions

The existence of (periodic) E-inclusions has been reported in Liu *et al.* (2007; accepted). We now show that (periodic) E-inclusions are the optimal geometries with minimum stress or strain concentrations. We remark that Liu (2010) has shown that periodic E-inclusions are also the optimal geometries attaining the Hashin-Shtrikman's bounds for multiphase composites. Again we shall have separate discussions about E-inclusions and periodic E-inclusions.

4.1. Optimality of E-inclusions

Recall that $\mathrm{Tr}\overline{\mathbf{E}} = 1$. If $\Omega = \cup_{\alpha=1}^{N}\Omega_\alpha$ is an E-inclusion such that the overdetermined potential problem (24)-(25) admits a solution, and for some $a \in \mathbb{R}$,

$$[2(\mu_0-\mu_1)\overline{\mathbf{E}} + (\lambda_0-\lambda_1)\mathbf{I}] = a[2(\mu_0-\mu_1)\mathbf{Q} + (\lambda_0-\lambda_1)\mathbf{I} - (2\mu_0+\lambda_0)\mathbf{I}], \tag{28}$$

we claim that a solution to the elasticity boundary value problem (2), (3) and (8) is given by

$$\mathbf{u} = \overline{\mathbf{E}}\mathbf{x} + a\nabla\xi. \tag{29}$$

We remark that the above claim may be directly verified by inserting (29) into (2), (3) and (8). The idea behind this solution method tracks back to the celebrated work of Eshelby (1957) and the so-called *equivalent inclusion method* (Eshelby 1957; Mura, 1987).

To find the constant a in (28) and (29), we take the trace of (28) and obtain

$$a = \frac{\kappa_1-\kappa_0}{\kappa_1-\kappa_0+2\mu_0+\lambda_0}. \tag{30}$$

Inserting it back into (28) we find

$$\overline{\mathbf{E}} - a\mathbf{Q} = \frac{(2\mu_0+\lambda_1)a + \lambda_0 - \lambda_1}{2(\mu_1-\mu_0)}\mathbf{I} = \frac{2\mu_0+\lambda_0}{n(\kappa_1-\kappa_0+2\mu_0+\lambda_0)}\mathbf{I} = \hat{e}^\star\mathbf{I}. \tag{31}$$

By (28), (29) and (8), we find that the strain and stress in the body are given by

$$\mathbf{E} = \overline{\mathbf{E}} + a\nabla\nabla\xi = \begin{cases} \overline{\mathbf{E}} - a\mathbf{Q} = \hat{e}^\star\mathbf{I} & \text{in } \Omega, \\ \overline{\mathbf{E}} + a\nabla\nabla\xi & \text{in } \Omega_0, \end{cases}$$

and

$$\boldsymbol{\sigma} = \begin{cases} n\kappa_1\hat{e}^\star\mathbf{I} & \text{in } \Omega, \\ 2\mu_0\overline{\mathbf{E}} + \lambda_0(\mathrm{Tr}\overline{\mathbf{E}})\mathbf{I} + a(2\mu_0\nabla\nabla\xi) & \text{in } \Omega_0, \end{cases}$$

respectively. For any given unit vector $\mathbf{m} \in \mathbb{R}^n$, set

$$e_{\mathbf{m}} = \mathbf{m} \cdot \mathbf{E}\mathbf{m}, \qquad \sigma_{\mathbf{m}} = \mathbf{m} \cdot \boldsymbol{\sigma}\mathbf{m}.$$

It is easy to check that

$$\begin{cases} \Delta e_{\mathbf{m}} = 0 & \text{in } \Omega_0, \\ e_{\mathbf{m}} = \hat{e}^\star + a(\mathbf{m} \cdot \mathbf{n})^2 & \text{on } \partial\Omega^+, \end{cases} \qquad \begin{cases} \Delta \sigma_{\mathbf{m}} = 0 & \text{in } \Omega_0, \\ \sigma_{\mathbf{m}} = 2\mu_0[\hat{e}^\star + a(\mathbf{n} \cdot \mathbf{m})^2] + \lambda_0 & \text{on } \partial\Omega^+, \end{cases}$$

respectively. By the maximum principle, we conclude that for E-inclusions with the shape matrix $\mathbf{Q}$ satisfying (28), the maximum l^∞-norms of the strain and stress in the entire inhomogeneous body are given by

$$e^*\{\Omega\} = \max\{|\hat{e}^\star|, |\hat{e}^\star + a|\} = \begin{cases} \hat{e}^\star + a & \text{if } \kappa_1 > \kappa_0, \\ \hat{e}^\star & \text{if } \kappa_1 < \kappa_0, \end{cases} \tag{32}$$

and

$$n\kappa_0\sigma^*\{\Omega\} = \begin{cases} |2\mu_0(\hat{e}^\star + a) + \lambda_0| = n\kappa_1\hat{e}^\star = \hat{\sigma}^\star n\kappa_0 & \text{if } \kappa_1 > \kappa_0, \\ 2\mu_0\hat{e}^\star + \lambda_0 & \text{if } \kappa_1 < \kappa_0, \end{cases} \tag{33}$$

respectively. Comparing (32) and (33) with (10), for average strain $\overline{\mathbf{E}}$ with $\mathrm{Tr}\overline{\mathbf{E}} = 1$ we conclude that (i) E-inclusions with shape matrix $\mathbf{Q} = \frac{1}{a}(\overline{\mathbf{E}} - \hat{e}^\star\mathbf{I})$ are solutions to the optimization problem (7) with minimum strain concentration if $\kappa_1 < \kappa_0$ and $\mu_1 < \mu_0$, and are solutions to the optimization problem (6) with minimum stress concentration if $\kappa_1 > \kappa_0$ and $\mu_1 > \mu_0$.

4.2. Optimality of periodic E-inclusions

If $\Omega = \cup_{\alpha=1}^N \Omega_\alpha$ is a periodic E-inclusion with volume fraction η and shape matrix $\mathbf{Q}$, i.e., the overdetermined potential problem (26)-(27) admits a solution, and for some constant $a_\eta \in \mathbb{R}$,

$$[2(\mu_0 - \mu_1)\overline{\mathbf{E}} + (\lambda_0 - \lambda_1)\mathbf{I}] = a_\eta\{(1-\eta)[2(\mu_0 - \mu_1)\mathbf{Q} + (\lambda_0 - \lambda_1)\mathbf{I}] - (2\mu_0 + \lambda_0)\mathbf{I}\}, \tag{34}$$

we claim that a solution to the elasticity boundary value problem (2), (3) and (8) is given by

$$\mathbf{u} = \overline{\mathbf{E}}\mathbf{x} + a_\eta \nabla\xi. \tag{35}$$

Taking the trace of (34) we obtain

$$a_\eta = \frac{\kappa_1 - \kappa_0}{(\kappa_1 - \kappa_0)(1-\eta) + 2\mu_0 + \lambda_0}. \tag{36}$$

Inserting the above equation back into (34) we find that (cf., (14))

$$
\begin{aligned}
\overline{\mathbf{E}} - a_\eta(1-\eta)\mathbf{Q} &= \frac{(2\mu_0+\lambda_0)a_\eta + (\lambda_0-\lambda_1)(1-a_\eta(1-\eta))}{2(\mu_1-\mu_0)}\mathbf{I} \\
&= \frac{2\mu_0+\lambda_0}{n[(\kappa_1-\kappa_0)(1-\eta)+2\mu_0+\lambda_0]}\mathbf{I} \\
&= \frac{(\kappa_1-\kappa^e)\hat{e}^\star}{(1-\eta)(\kappa_1-\kappa_0)}\mathbf{I} = \hat{e}^\star\bar{\vartheta}_0\mathbf{I},
\end{aligned}
\tag{37}
$$

where

$$
\kappa^e = \kappa_0 + \frac{\eta(2\mu_0+\lambda_0)(\kappa_1-\kappa_0)}{(1-\eta)(\kappa_1-\kappa_0)+2\mu_0+\lambda_0}
$$

is the effective bulk modulus of the composite.

By (34), (35) and (8), we find that the strain and stress in the body are given by

$$
\mathbf{E} = \overline{\mathbf{E}} + a_\eta\nabla\nabla\xi = \begin{cases} \overline{\mathbf{E}} - a_\eta(1-\eta)\mathbf{Q} = \hat{e}^\star\bar{\vartheta}_0\mathbf{I} & \text{in } \Omega, \\ \overline{\mathbf{E}} + a_\eta\nabla\nabla\xi & \text{in } \Omega_0, \end{cases}
$$

and

$$
\boldsymbol{\sigma} = \begin{cases} \hat{\sigma}^\star\bar{\theta}_0\mathbf{I} & \text{in } \Omega, \\ 2\mu_0\overline{\mathbf{E}} + \lambda_0(\mathrm{Tr}\overline{\mathbf{E}})\mathbf{I} + a_\eta(2\mu_0\nabla\nabla\xi) & \text{in } \Omega_0, \end{cases}
$$

respectively. For any given unit vector $\mathbf{m} \in \mathbb{R}^n$, set

$$
e_{\mathbf{m}} = \mathbf{m}\cdot\mathbf{E}\mathbf{m}, \qquad \sigma_{\mathbf{m}} = \mathbf{m}\cdot\boldsymbol{\sigma}\mathbf{m}.
$$

Again, we find that

$$
\begin{cases} \Delta e_{\mathbf{m}} = 0 & \text{in } \Omega_0, \\ e_{\mathbf{m}} = \hat{e}^\star\bar{\vartheta}_0 + a_\eta(\mathbf{m}\cdot\mathbf{n})^2 & \text{on } \partial\Omega^+, \end{cases} \qquad \begin{cases} \Delta\sigma_{\mathbf{m}} = 0 & \text{in } \Omega_0, \\ \sigma_{\mathbf{m}} = 2\mu_0[\hat{e}^\star\bar{\vartheta}_0 + a_\eta(\mathbf{n}\cdot\mathbf{m})^2] + \lambda_0 & \text{on } \partial\Omega^+, \end{cases}
$$

respectively. By the maximum principle, we conclude that for periodic E-inclusions with the shape matrix $\mathbf{Q}$ satisfying (28), the maximum l^∞-norms of the strain and stress in the entire inhomogeneous body are given by

$$
\bar{\vartheta}_0 e^*\{\Omega\} = \bar{\vartheta}_0\max\{|\hat{e}^\star|, |\hat{e}^\star + a_\eta|\} = \begin{cases} \hat{e}^\star\bar{\vartheta}_0 + a_\eta & \text{if } \kappa_1 > \kappa_0, \\ \hat{e}^\star\bar{\vartheta}_0 & \text{if } \kappa_1 < \kappa_0, \end{cases}
\tag{38}
$$

and

$$
\bar{\theta}_0\sigma^*\{\Omega\} = \begin{cases} |2\mu_0(\hat{e}^\star\frac{\bar{\theta}_0}{n\kappa_0} + a_\eta) + \lambda_0| = n\kappa_1\hat{e}^\star\frac{\bar{\theta}_0}{n\kappa_0} = \hat{\sigma}^\star\bar{\theta}_0 & \text{if } \kappa_1 > \kappa_0, \\ 2\mu_0\hat{e}^\star\frac{\bar{\theta}_0}{n\kappa_0} + \lambda_0 & \text{if } \kappa_1 < \kappa_0, \end{cases}
\tag{39}
$$

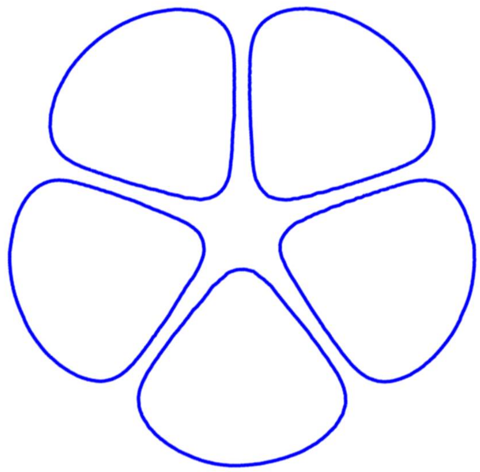

Figure 1: A five component E-inclusion in two dimensions with shape matrix $\mathbf{Q} = \frac{1}{2}\mathbf{I}$.

respectively. Comparing (38) and (39) with (22), for average strain $\overline{\mathbf{E}}$ with $\mathrm{Tr}\overline{\mathbf{E}} = 1$ we conclude that (i) periodic E-inclusions with shape matrix $\mathbf{Q} = \frac{1}{a_\eta(1-\eta)}(\overline{\mathbf{E}} - \bar{\vartheta}_0 \hat{e}^\star \mathbf{I})$ are solutions to the optimization problem (7) with minimum strain concentration if $\kappa_1 < \kappa_0$ and $\mu_1 < \mu_0$, and are solutions to the optimization problem (6) with minimum stress concentration if $\kappa_1 > \kappa_0$ and $\mu_1 > \mu_0$.

5. Examples of E-inclusions and proposed applications

E-inclusions (resp. periodic E-inclusions) are defined by the overdetermined problems (24)-(25) (resp. (26)-(27)). Upon specifying the shape matrix $\mathbf{Q}$, volume, and mutual distances and orientations, we may construct the E-inclusion by solving a variational inequality problem (Liu, 2008). For example, Figure 1 shows a five component E-inclusion corresponding to shape matrix $\mathbf{Q} = \frac{1}{2}\mathbf{I} \in \mathbb{R}^{2\times 2}$ whereas Fig. 2 (b)-(e) show a single periodic array of E-inclusions with period d, shape matrix $\mathbf{Q} = \mathrm{diag}[2/3; 1/3]$ and of different area A and different length a_x. The reader is referred to Vigdergauz (1976), Grabovsky and Kohn (1995b), Liu *et al* (2007; accepted) and Liu (2008) for more examples of (periodic) E-inclusions in two and three dimensions. Compared to ellipsoids, a critical advantage of E-inclusions lies in that E-inclusions can be disconnected, consists of a number of inclusions and retain the uniformity and optimality properties of ellipsoids. In other words, the interactions between inhomogeneities have been rigorously taken into account (Vigdergauz, 2008).

Being the optimal geometries with minimum field concentrations, E-inclusions and periodic E-inclusions have many potential applications in engineering when the field concentration is a major concern. For example, in using rivets to bond structural members, the bridges between neighboring holes sustain larger local stress and prone to fail (cf., Fig. 2(a)). To remedy this issue, holes of the shapes of E-inclusions can be used so that the stress concentration can be minimized. Depending the applied remote or average strain, the shape

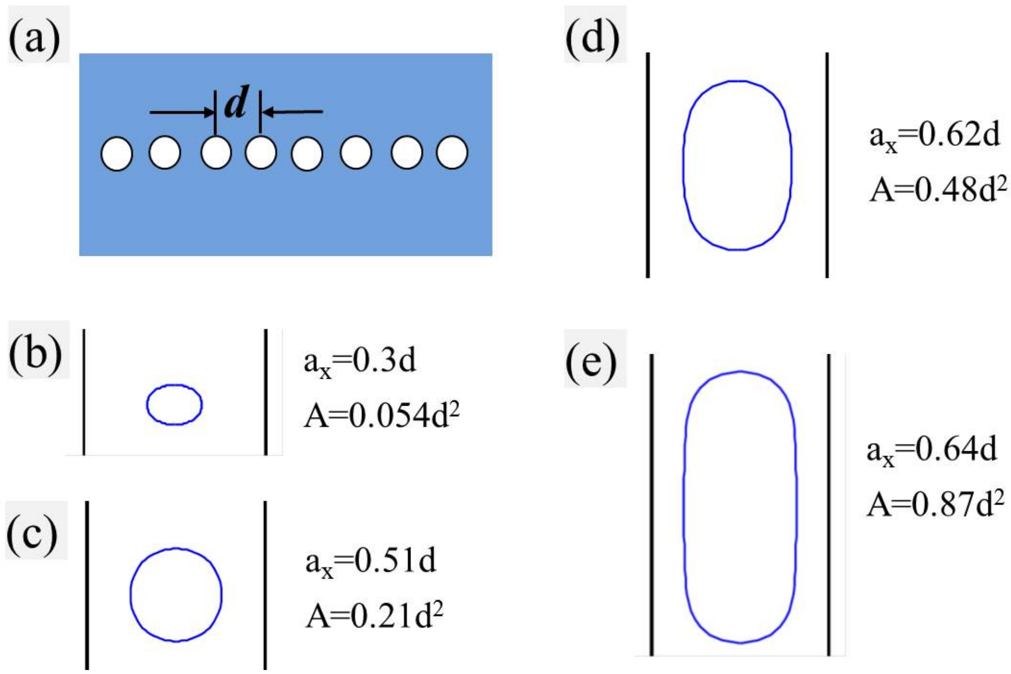

Figure 2: (a) A single period of holes of spacing d under the application of some remote strain. The optimal shapes such that strain concentration is minimized for different areas (A) and lengths (a_x): (b) $a_x = 0.3d$ and $A = 0.054d^2$; (c) $a_x = 0.51d$ and $A = 0.21d^2$; (d) $a_x = 0.62d$ and $A = 0.48d^2$; (e) $a_x = 0.64d$ and $A = 0.87d^2$. The shape matrix of (b)-(e) is $\mathbf{Q} =$ diag$[2/3, 1/3]$.

matrix $\mathbf{Q}$ of the E-inclusions satisfies ($\eta = 0$ if $Y = \mathbb{R}^n$)

$$\mathbf{Q} \propto \overline{\mathbf{E}} - \bar{\vartheta}_0 \hat{e}^\star \mathbf{I} = \overline{\mathbf{E}} - \frac{2\mu_0 + \lambda_0}{n[(1-\eta)(\kappa_1 - \kappa_0) + 2\mu_0 + \lambda_0]}\mathbf{I}. \tag{40}$$

Since E-inclusions exist only for positive semi-definite shape matrix $\mathbf{Q}$, the above equation admits a solution for average applied average strain $\overline{\mathbf{E}}$ with Tr$\overline{\mathbf{E}} = 1$ if

$$\mathrm{eig}_{\min}(\overline{\mathbf{E}}) \geq \frac{2\mu_0 + \lambda_0}{n[(1-\eta)(\kappa_1 - \kappa_0) + 2\mu_0 + \lambda_0]} \quad \text{or}$$
$$\mathrm{eig}_{\max}(\overline{\mathbf{E}}) \leq \frac{2\mu_0 + \lambda_0}{n[(1-\eta)(\kappa_1 - \kappa_0) + 2\mu_0 + \lambda_0]},$$

where $\mathrm{eig}_{\max}$ ($\mathrm{eig}_{\min}$) denotes the maximum (minimum) eigenvalues. If the average applied strain $\overline{\mathbf{E}}$ with Tr$\overline{\mathbf{E}} = 1$ does not satisfy the above inequality, the optimal design problems (6)-(7) concerning minimum strain or stress concentration remain open; the works of Cherkaev *et al.* (1998) and Vigdergauz (2006; 2008) provide approximate solutions and significant insight.

6. Summary and discussion

In this paper we have shown that (periodic) E-inclusions are optimal geometries for minimum strain or stress concentration for heterogeneous media or structures. It is worthwhile noticing that E-inclusions are not the only optimal geometries. In general, it can be shown that micro-geometries that attain the Hashin-Shtrikman bulk modulus bounds are also optimal for minimum stress or strain concentration, including the coated ellipsoids or spheres (Hashin 1962; Milton, 1984; Grabovsky and Kohn, 1995a), Vigdergauz microstructures or E-inclusions in two dimensions (Vigdergauz 1987; Grabovsky and Kohn, 1995b), and multi-rank laminations. The underlying reason can be understood from the optimal conditions. To be precise, we consider periodic array of inhomogeneities discussed in § 3.2 and assume that (i) the domain Ω is regular with smooth boundary $\partial\Omega$, and (ii) two stiffness tensors are strickly well-ordered. As one tracks back our argument for the lower bound in § 3.2, the first inequality in (22) hold as an equality if (i) $\theta = \bar{\theta}_0$ in $Y \setminus \Omega$ (cf., (15)) and, (ii) $\boldsymbol{\sigma} = \bar{\theta}_0 \hat{\sigma}^\star \mathbf{I}$ on Ω (cf., (19)). In other words, the actual displacement $\mathbf{u}$ shall satisfies the following equation:

$$
\begin{cases}
\mathrm{div}[\mathbf{C}(\mathbf{x})\nabla\mathbf{u}] = 0 & \text{in } Y \setminus \partial\Omega, \\
\frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] = \hat{e}^\star \frac{\bar{\theta}_0}{n\kappa_0}\mathbf{I} & \text{in } \Omega, \\
\nabla \cdot \mathbf{u} = \frac{\bar{\theta}_0}{n\kappa_0} & \text{in } Y \setminus \Omega, \\
⨍_Y \nabla\mathbf{u} = \overline{\mathbf{E}},
\end{cases}
\tag{41}
$$

where $\mathbf{C}(\mathbf{x})$ denotes the stiffness tensor of the periodic composite. For periodic microstructures with a scalar potential ξ satisfying the overdetermined problem (26)-(27), it is easy to verify that the above equations are satisfied by $\mathbf{u} = a_\eta \nabla\xi + \overline{\mathbf{E}}\mathbf{x}$. Moreover, it has been shown (Grabovsky, 1996; Liu, 2010) that any periodic microstructures attaining the Hashin-Shtrikman's bounds have to be such that the overdetermined problem (26)-(27) admits a solution. Similar argument can be applied to the second inequality in (22). Therefore, the classical constructions of coated ellipsoids, Vigdergauz microstructures and multi-rank laminates that attain the Hashin-Shtrikman's bounds are also optimal microstructures achieving bounds (22) with minimum stress or strain concentration. This motivates an interesting question: does equation (41) admitting a solution imply the overdetermined problem (26)-(27) admits a solution? A positive answer to this question will imply optimal microstructures for minimum stress or strain concentration have to be identical to optimal microstructures attaining the Hashin-Shtrikman's bounds. A proof, however, appears to be elusive and will not be discussed here.

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