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Existence of Surface Waves and Band Gaps in Periodic Heterogeneous Half-spaces

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Abstract We find a sufficient condition for the existence of surface (Rayleigh) waves based on the Rayleigh-Ritz variational method. When specialized to a homogeneous half-space, the sufficient condition recovers the known criterion for the existence of subsonic surface waves. A simple existence criterion in terms of material properties is obtained for periodic half-spaces of general anisotropic materials. Further, we numerically compute the dispersion relation of the surface waves for a half-space of periodic laminates of two materials and demonstrate the existence of surface wave band gaps.

Keywords Surface waves · Periodic half-space · Band gaps

Mathematics Subject Classification (2000) 74J15 · 74J05 · 35C07

1 Introduction

Free surface waves are traveling waves that propagate along the surface of a half-space. They satisfy the traction-free boundary condition on the surface and decay exponentially away from the surface. Lord Rayleigh [25] first obtained a mathematical solution of surface waves in an isotropic half-space, which are subsequently named after him. Stoneley [28, 29] realized that not all homogeneous anisotropic half-spaces admit a Rayleigh-type solution. The existence and uniqueness of *subsonic* surface waves in a general anisotropic half-space have now been resolved, see, e.g., Barnett et al. [3], Chadwick and Smith [6] Barnett and Lothe [2], Fu and Mielke [14, 15]. The proof employed in these works relies on the known form of waves implied by the translational invariance of a homogeneous medium, and hence cannot be applied to a heterogeneous medium.

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In this paper we address the surface waves in a heterogeneous half-space. We first present a sufficient condition for the existence of surface waves. Although it is widely accepted and numerically verified that inhomogeneities in general promote localized waves and in particular surface waves which are localized near the surface [4, 20, 21], the literature lacks a proof even for some simple situations. Our approach is based on the Rayleigh-Ritz variational formulae. Roughly speaking, surface waves localized near the surface can be identified as eigenfunctions of a differential operator. If one could derive a lower bound for the essential spectrum of this differential operator, then the existence of an eigenfunction or a localized wave would follow from the existence of a test function whose Rayleigh quotient is strictly below the essential spectrum. This variational approach has been used by Kamotskii and Kiselev [17] for the existence of surface waves in a homogeneous half-space. Such a variational approach has also been widely used in the existence proof of localized modes of water waves, electromagnetic waves, and electronic waves [5, 19, 26]. In these problems, the differential operator concerns the negative Laplacian in semi-infinite domains with various boundary conditions. The localized waves originate either from a diffraction grating on the surface or from the non-planar geometry of the surface. Elastic surface waves in a half-space, however, can originate from the very vectorial nature of elasticity problems besides the presence of inhomogeneities and non-planar geometry of the surface.

The technical difficulty of this approach arises from estimating the lower bound of the essential spectrum of the associated differential operator. Physically the essential spectrum corresponds to bulk waves that can penetrate the half-space and radiate energy into the half-space. To obtain such an estimate, inevitably we have to make some structural assumption about the elastic half-space. A common and application relevant assumption is that the heterogeneous half-space is obtained by cutting an infinite periodic medium along a plane. It has been shown in Allaire and Conca [1] that the propagating bulk waves in a periodic half-space are exactly the Bloch waves in the infinite periodic medium, see also Eastham [11, 12]. We therefore derive a good estimate about the essential spectrum based on our knowledge on the Bloch waves in the infinite periodic medium. Note that the bulk waves are explicitly known for a homogeneous medium. Therefore, the sufficient condition, when specialized to homogeneous half-spaces, becomes necessary for the existence of *subsonic* surface waves.

The advantage of this variational method lies in its simplicity and flexibility. It is even possible to conclude the existence of surface waves in some heterogeneous half-spaces regardless of the microstructure of the periodic media, see Sect. 4. Also this method yields an upper bound for the phase speed of surface waves. However, this method cannot handle *supersonic* surface waves. This limitation is intrinsic and difficult to overcome in such variational methods.

As an example, we numerically compute the dispersion relation of surface waves in a half-space consisting of laminates of two materials. The results, unsurprisingly, show the existence of band gaps of surface waves. The existence of surface wave band gaps implies that surface waves in phononic crystals, in addition to bulk waves, can be used to control and manipulate elastic waves. Phononic devices based on surface waves are attractive since surface waves are easier to be excited and detected compared with bulk waves. Also, reducing dissipation and guided propagation are less challenging for the localized characteristics of surface waves.

The paper is organized as follows. In Sect. 2 we state the mathematical problems and some facts from the spectral theory concerning closed quadratic forms and the associated self-adjoint operators. The important result of this section is Theorem 1, whose proof follows from Allaire and Conca [1] and Eastham [11, 12]. In Sect. 2.1 we present the main existence theorems of this paper. In Sect. 3 we specify the obtained existence theorem to homogeneous half-spaces and show the obtained existence condition is necessary and sufficient for subsonic surface waves. In Sect. 4 we illustrate the application of the existence theorem by an example. We present a numerical example of surface waves in Sect. 5. In Sect. 6 we summarize our results and make a few remarks about guided waves on periodically grated surface and interfacial waves between two periodic half-spaces.

2 An Existence Theorem for Surface Waves

Let $\{e_1, \dots, e_n\} \subset \mathbb{R}^n$ be the canonical rectangular bases and $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ be the upper half-space, see Fig. 1. The periodic medium, described by the elasticity tensor $L(x)$ and density $\rho(x)$, satisfies

$$L(x+y) = L(x) \qquad \text{and} \qquad \rho(x+y) = \rho(x) \tag{1}$$

for all $x \in \mathbb{R}^n$ and all $y \in \mathcal{L}$, where

$$\mathcal{L} := \left\{ \sum_{i=1}^{n} \nu_i a_i : (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n, \ \{a_1, \dots, a_n\} \subset \mathbb{R}^n \text{ are linearly independent} \right\}$$

is a Bravais lattice describing the periodicity of the medium. The elasticity tensor $L(x)$ satisfies the usual symmetries of the elastic moduli: $L_{piqj}(x) = L_{qjpi}(x) = L_{pijq}(x)$, and there exists a constant $c > 0$ such that for any $x \in \mathbb{R}^n$,

$$c^{-1}|X|^2 \le X \cdot L(x)X \le c|X|^2 \quad \forall X \in \mathbb{R}^{n\times n}_{sym} \qquad \text{and} \qquad c^{-1} \le \rho(x) \le c. \tag{2}$$

The elastodynamic equations for the half-space can be written as:

$$\begin{cases} -(L_{piqj}(x)u_{q,j}(x))_{,i} = \omega^2 \rho(x) u_p(x) & \text{in } \mathbb{R}^n_+ \\ L_{pnqj}(x)u_{q,j}(x) = 0 & \text{on } \Gamma_0 \end{cases} \quad (p = 1, \dots, n), \tag{3}$$

where $\Gamma_0 := \{x : x_n = 0\}$ is the free surface, $u : \mathbb{R}^n_+ \to \mathbb{C}^n$ is the complex displacement and $\omega > 0$ is the frequency. However, a solution of (3) cannot be a "surface wave" if it does not decay as $x_n \to +\infty$. To define surface waves, we introduce the tube problem and the quasi-periodic problem.

To proceed, we introduce a few notations. It is assumed that the lattice vectors $\{a_1, \dots, a_{n-1}\}$ are all contained in the plane Γ_0. Let $Y := \{\sum_{i=1}^{n} x_i a_i : (x_1, \dots, x_n) \in (-\frac{1}{2}, \frac{1}{2})^n\}$ be a unit cell, $Y' := Y \cap \Gamma_0$ be an in-plane unit cell, and $T = \{\sum_{i=1}^{n} x_i a_i : (x_1, \dots, x_{n-1}) \in (-\frac{1}{2}, \frac{1}{2})^{n-1}, \ x_n > 0\}$ be a semi-infinite tube. A function is Y-periodic (resp. Y'-periodic) if it is invariant under all translations in $\mathcal{L}$ (resp. $\mathcal{L}' = \mathcal{L} \cap \Gamma_0$). The reciprocal lattice of $\mathcal{L}$ is given by

$$\mathcal{K} = \left\{ \sum_{i=1}^{n} \nu_i b_i : (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n \right\}, \tag{4}$$

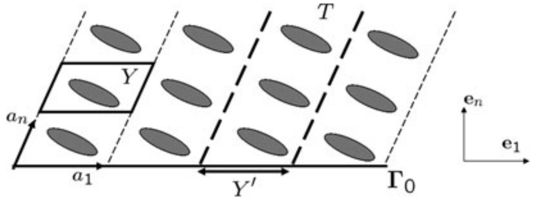


Fig. 1 A halfspace of periodic heterogeneous medium. Y (resp. Y') is a (resp. in-plane) unit cell, T is a semi-infinite tube, and Γ_0 is the free surface

where the *reciprocal vectors* $\{b_1, \ldots, b_n\}$ of $\{a_1, \ldots, a_n\}$ is determined by

$$b_i \cdot a_j = 2\pi\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 2\pi & \text{if } i = j \end{cases} \quad (i, j = 1, \ldots, n).$$

The first (resp. in-plane) *Brillouin zone* is

$$Z := \left\{ t \in \mathbb{R}^n : |t| \leq \inf_{k \in \mathcal{K}\setminus\{0\}} |t - k| \right\} \quad (\text{resp. } Z' := Z \cap \Gamma_0). \tag{5}$$

We now state the tube problem and the quasi-periodic problem, see the textbooks of Davies [8] and Schechter [26] for details.

(i) *The tube problem* concerns solving equation (3) subjected to quasi-periodic boundary conditions on the side faces of the tube T for an in-plane wave vector $t' \in Z'$. Associated with this tube problem, we define a self-adjoint differential operator

$$\begin{cases} \mathcal{A}_{t'} : \text{Dom}(\mathcal{A}_{t'}) & \to \exp(\mathring{\imath} t' \cdot x)[L^2_{per}(T)]^n \\ \qquad\qquad u & \mapsto f \end{cases}$$

by

$$\begin{cases} -(L_{piqj}(x)u_{q,j}(x))_{,i} = \rho(x) f_p(x) & \text{in } \mathbb{R}^n_+ \\ L_{pnqj}(x)u_{q,j}(x) = 0 & \text{on } \Gamma_0 \end{cases} \quad (p = 1, \ldots, n), \tag{6}$$

where $\text{Dom}(\mathcal{A}_{t'}) \subset \exp(\mathring{\imath} t' \cdot x)[W^{1,2}_{per}(T)]^n$, and the function space $L^2_{per}(T)$ (resp. $W^{1,2}_{per}(T)$) is the collection of all complex-valued functions φ that are Y'-periodic and have finite $\|\varphi\|_{L^2_{per}(T)} = [\int_T |\varphi|^2 dx]^{1/2}$ (resp. $\|\varphi\|_{W^{1,2}_{per}(T)} = [\int_T (|\nabla\varphi|^2 + |\varphi|^2) dx]^{1/2}$). We denote by $\sigma(\mathcal{A}_{t'})$ (resp. $\sigma_e(\mathcal{A}_{t'})$) the (resp. essential) spectrum of $\mathcal{A}_{t'}$.

(ii) *The quasi-periodic problem* concerns solving equation (3) subjected to quasi-periodic boundary conditions on all faces of the unit cell Y for a wave vector $t \in Z$. Associated with this quasi-periodic problem, we define a self-adjoint differential operator

$$\begin{cases} \mathcal{B}_t : \text{Dom}(\mathcal{B}_t) & \to \exp(\mathring{\imath} t \cdot x)[L^2_{per}(Y)]^n \\ \qquad\qquad u & \mapsto f \end{cases}$$

by

$$-(L_{piqj}(x)u_{q,j}(x))_{,i} = \rho(x) f_p(x) \quad \text{in } \mathbb{R}^n \quad (p = 1, \ldots, n), \tag{7}$$

where $\text{Dom}(\mathcal{B}_t) \subset \exp(\mathring{\imath} t \cdot x)[W^{1,2}_{per}(Y)]^n$, and the function space $L^2_{per}(Y)$ (resp. $W^{1,2}_{per}(Y)$) is the collection of all complex-valued functions φ that are Y-periodic and have finite $\|\varphi\|_{L^2_{per}(Y)} = [\int_Y |\varphi|^2 dx]^{1/2}$ (resp. $\|\varphi\|_{W^{1,2}_{per}(Y)} = [\int_Y (|\nabla\varphi|^2 + |\varphi|^2) dx]^{1/2}$). We denote by $\sigma(\mathcal{B}_t)$ the spectrum of this differential operator. We remark that the operator $\mathcal{B}_t$ has compact resolvent, and hence its spectrum is discrete. The corresponding eigenfunctions form a complete basis of the Hilbert space $\exp(\mathring{\imath} t \cdot x) \times [L^2_{per}(Y)]^n$ and are referred to as Bloch waves with a reduced wave vector $t \in Z$.

We have the following theorem, see Allaire and Conca [1] and Eastham [11, 12] for proofs. This theorem enables us to bound from below the essential spectrum of the tube problem by those of the quasi-periodic problems and separate subsonic surface waves from bulk waves.

Theorem 1 *Consider the self-adjoint differential operators $\mathcal{A}_{t'}$ and $\mathcal{B}_t$ defined by* (6) *and* (7). *Let $\sigma(\mathcal{A}_{t'})$ and $\sigma(\mathcal{B}_t)$ be their spectrums.*

A. *If $u \in \mathrm{Dom}(\mathcal{A}_{t'})$ is an eigenfunction of the operator $\mathcal{A}_{t'}$, i.e., $\mathcal{A}_{t'}u = \lambda u$ for some $\lambda \in \mathbb{R}$, then there exist constants $C(j)$ such that*

$$\int_T |x_n^j u|^2 dx \le C(j) \tag{8}$$

for all positive integers j.

B. *For any given in-plane wave vector $t' \in Z'$, let $S_{t'} = \{t \in Z : t - (t \cdot e_n)e_n = t'\}$ and $\sigma_e(\mathcal{A}_{t'})$ be the essential spectrum of $\mathcal{A}_{t'}$. Then*

$$\sigma_e(\mathcal{A}_{t'}) = \text{closure}\left\{\bigcup_{\mathrm{t}\in \mathrm{S}_{\mathrm{t}'}} \sigma(\mathcal{B}_{\mathrm{t}})\right\}. \tag{9}$$

2.1 Existence of Surface Waves

We identity a surface wave as an eigenfunction $u \in \mathrm{Dom}(\mathcal{A}_{t'})$ of the operator $\mathcal{A}_{t'}$ for some $t' \in Z'$. Since $u\exp(-it' \cdot x) \in [W^{1,2}_{per}(T)]^n$, the energy propagating away from the surface Γ_0 per unit area satisfies

$$\fint_{T\cap\{x:\, x_n=h\}} L_{pnqj}(x) u_p \overline{u_{q,j}} dx \to 0 \quad \text{as } h \to +\infty.$$

Further, from equation (8) we see that $u(x)$ decays away from Γ_0 faster than x_n^{-j} for any positive integer j. These properties of an eigenfunction justify our definition of "surface waves".

We will need the following *Rayleigh-Ritz variational formulae*, see Davies [8, p. 91] for details.

Theorem 2 *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \to \mathcal{H}$ be a self-adjoint operator, $Q : \mathrm{Dom}(Q) \times \mathrm{Dom}(Q) \to \mathbb{R}$ be the associated closed quadratic form, and $\sigma(\mathcal{A})$ (resp. $\sigma_e(\mathcal{A})$) be the (resp. essential) spectrum of $\mathcal{A}$. Further, for a subspace M of $\mathrm{Dom}(Q)$, let $\dim(M)$ be the dimensions of M,*

$$\lambda(M) := \sup\{Q(u,u) :\ u \in M \text{ and } \|u\|_{\mathcal{H}} = 1\},$$

and

$$\lambda_m := \inf\{\lambda(M) :\ M \subset \mathrm{Dom}(Q) \text{ and } \dim(M) = m\}.$$

Then there exist at least m linearly independent eigenfunctions in $\mathrm{Dom}(\mathcal{A})$ if

$$\lambda_m < \inf\{\lambda :\ \lambda \in \sigma_e(\mathcal{A})\}. \tag{10}$$

To apply the Rayleigh-Ritz variational formulae to the tube problem, we identity the Hilbert space $\mathcal{H}$ as $\exp(\mathring{\imath}t' \cdot x)[L^2_{per}(T)]^n$ equipped with the norm $\|\cdot\|_{\mathcal{H}} = [\int_T \rho(x)|\cdot$

$|^2 dx]^{1/2}$. Note that this norm $\|\cdot\|_{\mathcal{H}}$ is an equivalent norm as the usual norm $\|\cdot\|_{L^2_{per}(T)}$. The associated closed quadratic form is given by

$$Q_{t'}(u,u) = \int_T L_{piqj}(x) u_{p,i} \overline{u_{q,j}} dx \tag{11}$$

with $\mathrm{Dom}(Q_{t'}) = \{u : u \in \exp(\mathring{\imath} t' \cdot x)[W^{1,2}_{per}(T)]^n\}$. Similarly, for the quasi-periodic problem the Hilbert space $\mathcal{H}$ is identified as $\exp(\mathring{\imath} t \cdot x)[L^2_{per}(Y)]^n$ equipped with the norm $\|\cdot\|_{\mathcal{H}} = [\int_Y \rho(x)|\cdot|^2 dx]^{1/2}$. The associated closed quadratic form is given by

$$Q_t(u,u) = \int_Y L_{piqj}(x) u_{p,i} \overline{u_{q,j}} dx \tag{12}$$

with $\mathrm{Dom}(Q_t) = \{u : u \in \exp(\mathring{\imath} t \cdot x)[W^{1,2}_{per}(Y)]^n\}$. Then from Theorem 1 and Theorem 2, we have the following.

Theorem 3 *Consider the self-adjoint operators $\mathcal{A}_{t'}$, $\mathcal{B}_t$ (cf., (6), (7)) and the associated quadratic forms $Q_{t'}$, Q_t (cf., (11), (12)). Let*

$$R_T(u) = \frac{\int_T L_{piqj}(x) u_{p,i} \overline{u_{q,j}} dx}{\int_T \rho(x) u_p \overline{u_p} dx} \quad \textit{and} \quad R_Y(u) = \frac{\int_Y L_{piqj}(x) u_{p,i} \overline{u_{q,j}} dx}{\int_Y \rho(x) u_p \overline{u_p} dx} \tag{13}$$

be their Rayleigh quotients. *Also,*

$$\tilde{v}_s(L,\rho,t') = \inf\left\{ \frac{\sqrt{R_T(u)}}{|t'|} : u \in \mathrm{Dom}(Q_{t'}) \setminus \{0\} \right\} \tag{14}$$

and

$$\tilde{v}_*(L,\rho,t') = \inf\left\{ \frac{\sqrt{R_Y(u)}}{|t'|} : u \in \mathrm{Dom}(Q_t) \setminus \{0\},\ t = t' + t_n e_n,\ t_n \in \mathbb{R} \right\}. \tag{15}$$

Then there exists a surface wave with in-plane wave vector $t' \in Z'$ if

$$\tilde{v}_s(L,\rho,t') < \tilde{v}_*(L,\rho,t'). \tag{16}$$

Theorem 3 is not very enlightening in the sense that neither of the quantities in equation (16) is easy to evaluate. Our strategy to show the existence of surface waves is therefore to derive upper bounds for the left-hand side of (16) and lower bounds for the right-hand side of (16). Below we first derive a lower bound for $\tilde{v}_*(L,\rho,t')$, and then evaluate the Rayleigh quotients $R_T(u)$ for a simple class of test functions, which gives rise to an upper bound for $\tilde{v}_s(L,\rho,t')$.

2.2 A Lower Bound of the Essential Spectrum

By equation (9), we can use the quasi-periodic problem (7) to find a lower bound of $\sigma_e(\mathcal{A}_{t'})$. The nonzero in-plane wave vector $t' \in Z'$ is fixed in this section. Let $u \in \mathrm{Dom}(Q_t)$ and $t = t' + t_n e_n$, then $u \exp(-\mathring{\imath} x \cdot t)$ is Y-periodic and we have the following Fourier expansion

$$u_{p,i}(x) = \sum_{k \in \mathcal{K}} \mathring{\imath}(k_i + t_i) \hat{u}_p(k) \exp(\mathring{\imath} x \cdot k + \mathring{\imath} x \cdot t), \tag{17}$$

where $\hat{u}_p(k) = \fint_Y u_p(x)\exp(-\mathbb{i}x\cdot k - \mathbb{i}x\cdot t)dx$ and $\mathcal{K}$ is the reciprocal lattice defined in (4). Here and subsequently, $\fint_Y \cdot\, dx = \int_Y \cdot\, dx/volume(Y)$ denotes the average value of the integrand over the domain Y. Let $\hat{L}$ be a fourth-order symmetric tensor. If $L(x) - \hat{L}$ is positive semi-definite (resp. negative semi-definite) for all x, we write $L(x) \geq$ (resp. $\leq$)$\hat{L}$. Clearly, for any $L^l \leq L(x)$ we have

$$\fint_Y L_{piqj}(x)u_{p,i}\overline{u_{q,j}}dx \geq \fint_Y L^l_{piqj}u_{p,i}\overline{u_{p,i}}\,dx = \sum_{k\in\mathcal{K}} L^l_{piqj}\hat{u}_p(k)\overline{\hat{u}_q(k)}(k+t)_i(k+t)_j.$$

Let

$$b(L^l,\eta,\xi) := L^l_{piqj}\eta_p\overline{\eta}_q\xi_i\xi_j \tag{18}$$

and (cf., (4))

$$\mu(L^l,t') := \min_{k\in\mathcal{K}}\ \min_{|\eta|=1,\, t_n\in\mathbb{R}} \frac{b(L^l,\eta,k+t'+t_ne_n)}{|t'|^2}. \tag{19}$$

Then from the Parseval theorem we have

$$\fint_Y L_{piqj}(x)u_{p,i}\overline{u_{q,j}}dx \geq |t'|^2\mu(L^l,t')\sum_{k\in\mathcal{K}}|\hat{u}(k)|^2 = |t'|^2\mu(L^l,t')\fint_Y |u(x)|^2\,dx.$$

Thus,

$$\frac{\sqrt{R_Y(u)}}{|t'|} = \sqrt{\frac{\fint_Y L_{piqj}(x)u_{p,i}\overline{u_{q,j}}dx}{|t'|^2\fint_Y \rho(x)u_p\overline{u_p}dx}} \geq \sqrt{\frac{\mu(L^l,t')}{\rho^u}} =: v_*(L^l,\rho^u,t'), \tag{20}$$

where $\rho^u = \sup_{x\in Y}\rho(x)$. Therefore, from (15) we have

$$\tilde{v}_*(L,\rho,t') \geq \sup_{L^l\leq L(x)} v_*(L^l,\rho^u,t'). \tag{21}$$

Note that both $\mu(L^l,t')$ and $v_*(L^l,\rho^u,t')$ in general depend on the reciprocal lattice $\mathcal{K}$, which is suppressed in notation for simplicity.

2.3 An Existence Theorem

We choose test functions

$$u(x) = \exp(|t'|x_nE)\tilde{u}\exp(\mathbb{i}t'\cdot x), \tag{22}$$

where $\tilde{u}\in\mathbb{C}^n$ and $E\in\mathbb{C}^{n\times n}$ with $\sigma(E)\subset\mathbb{C}_-$. Here $\sigma(E)$ denotes the set of eigenvalues (spectrum) of E, $\mathbb{C}_- = \{z\in\mathbb{C}: \operatorname{Re}(z)<0\}$, and the matrix exponential function is defined by $\exp(A) = \sum_{j=0}^{+\infty}\frac{A^j}{j!}$ for any square matrix A, see Coddington and Levinson [7, p. 62]. Note that, unless the matrix E is diagonalizable, $u(x)$ of form (22) in general cannot be written as

$$u(x) = \sum_{\alpha=1}^{n}\tilde{u}_\alpha\exp(\kappa_\alpha x_n)\exp(\mathbb{i}t'\cdot x') \tag{23}$$

with $\tilde{u}_\alpha\in\mathbb{C}^n$ and $\kappa_\alpha\in\mathbb{C}_-$. Indeed, there exist "non-Rayleigh-type" surface waves that cannot be put into the form of (23), as reviewed by Ting and Barnett [33]. For a test function of the form of (22), uniform density $\rho^l = \inf_{x\in Y}\rho(x)$, and a tensor $L^u \ge L(x)$, we have the following upper bound for the Rayleigh quotient:

$$\begin{aligned}
\frac{R_T(u)}{|t'|^2} &= \frac{\int_T L_{piqj}(x) u_{p,i}\overline{u_{q,j}}dx}{|t'|^2 \int_T \rho(x) u_p \overline{u_p} dx} \\
&\le \frac{\int_T L^u_{piqj} u_{p,i}\overline{u_{q,j}}dx}{|t'|^2 \int_T \rho^l u_p \overline{u_p} dx} = \frac{-\int_{\partial T\cap\Gamma_0} L^u_{pnqj} u_{q,j}\overline{u}_p dS - \int_T L^u_{piqj} u_{q,ij}\overline{u}_p dx}{|t'|^2 \int_T \rho^l u_p \overline{u_p} dx} \\
&= \frac{-\overline{\tilde{u}}\cdot\{(T^u E + \mathrm{i} R^{u\,T})/|t'| + \int_0^\infty \exp(yE^*)[T^u E^2 + \mathrm{i}(R^u + R^{u\,T})E - Q^u]\exp(yE)\,dy\}\tilde{u}}{\int_0^\infty \rho^l \overline{\tilde{u}}\cdot\exp(yE^*)\exp(yE)\tilde{u}\,dy} \\
&=: \Upsilon(L^u, \rho^l, t'; \tilde{u}, E),
\end{aligned} \tag{24}$$

where $(^*)$ denotes the conjugate transpose, and matrices T^u, R^u and Q^u are given by

$$T^u{}_{pq} = L^u_{pnqn}, \quad R^u{}_{pq} = L^u_{pjqn} t'_j/|t'|, \quad \text{and} \quad Q^u{}_{pq} = L^u_{piqj} t'_i t'_j/|t'|^2, \tag{25}$$

respectively. Upon minimizing $\Upsilon(L^u, \rho^l; E, \tilde{u})$ among all admissible E and $\tilde{u}$, we denote by

$$v_s(L^u, \rho^l, t') = \inf\{\sqrt{\Upsilon(L^u, \rho^l, t'; \tilde{u}, E)} : \sigma(E) \subset \mathbb{C}_-, \ \tilde{u} \in \mathbb{C}^n, \ |\tilde{u}| = 1\}, \tag{26}$$

which is clearly greater or equal to $\tilde{v}_s(L, \rho, t')$ in (16).

From (21), (26) and Theorem 3, we obtain the following sufficient condition for the existence of surface waves in a heterogeneous half-space.

Theorem 4 *Consider a periodic heterogeneous half-space of elasticity tensor $L(x)$ and density $\rho(x)$. For given tensors L^l, L^u, nonzero in-plane vector $t' \in Z'$, and densities $\rho^l = \inf_{x\in Y}\rho(x)$, $\rho^u = \sup_{x\in Y}\rho(x)$, let $v_*(L^l, \rho^u, t')$ and $v_s(L^u, \rho^l, t')$ be given by (20) and (26), respectively. Then there exists a surface wave with in-plane wave vector $t' \in Z'$ if*

$$\inf_{L^u \ge L(x)} v_s(L^u, \rho^l, t') < \sup_{L^l \le L(x)} v_*(L^l, \rho^u, t'). \tag{27}$$

3 Specification to Homogeneous Half-spaces

A homogeneous half-space can be regarded as a half-space of periodic medium with any lattice vectors $\{a_1, \dots, a_n\} \subset \mathbb{R}^n$. We denote by L^0 the elasticity tensor and by ρ^0 the density. Choosing $L^l = L^u = L^0$ in Theorem 4, we have the following result.

Corollary 5 *For a homogeneous half-space of elasticity tensor L^0 and density ρ^0, there exists a surface wave with nonzero in-plane wave vector $t' \in \Gamma_0$ if (cf., (20) and (26))*

$$v_s(L^0, \rho^0, t') < v_*(L^0, \rho^0, t'). \tag{28}$$

For a homogeneous medium, recall that a *transonic* wave corresponding to an in-plane wave vector $t' \in \Gamma_0$ are the bulk wave with smallest $v|t|/|t'|$ among all bulk waves with wave vectors in the plane spanned by t' and e_n, where v and t denote the phase speed and wave vector of the bulk wave, respectively. The minimum of $v|t|/|t'|$ is referred to as the *limiting* *wave speed*, see Chadwick and Smith [6, p. 339] for a geometrical interpretation of the limiting wave speed on the slowness section. A surface wave is *subsonic* (resp. *supersonic*) if its phase speed is less than (resp. greater or equal to) the limiting wave speed.

Below we show that the criterion (28) is a necessary and sufficient condition for the existence of a *subsonic* surface wave. Since it is difficult to compute the exact values of v_s and v_* in (28), in practice we focus on the upper bounds for v_s and lower bounds for v_* which are much easier to compute and, in many situations, yield nontrivial and simple sufficient conditions for the existence of surface waves, see Corollary 7. Further, we remark that by the Stroh formalism [30–32], Barnett and Lothe [2, 3] (also see Chadwick and Smith [6]; Fu and Mielke [14]) have obtained a necessary and sufficient condition in terms of impedance matrix for the existence of subsonic surface waves. We also remark that Kamotskii and Kiselev [17] have used the variational approach to recover the necessary and sufficient conditions of Barnett and Lothe [2].

In a homogeneous medium, a bulk wave with its wave vector in the plane spanned by $t' \in \Gamma_0$ and e_n can be without loss of generality written as

$$u(x) = \tilde{u}^\star \exp(\mathring{\imath}\xi(t' \cdot x + t_n x_n)),$$

where $\xi \neq 0$ is a real number, $\tilde{u}^\star$ is a unit eigenvector of the $n \times n$ matrix $(L^0)_{piqj} t_i t_j$, and $t = t' + t_n e_n$. The phase speed of this wave is given by $v = \sqrt{b(L^0, \tilde{u}^\star, t)/\rho^0 |t|^2}$. Therefore, the limiting wave speed corresponding to the in-plane direction t' is

$$\min_{t_n \in \mathbb{R}, |\tilde{u}|=1} \sqrt{\frac{b(L^0, \tilde{u}, t' + t_n e_n)}{\rho^0 |t'|^2}}. \tag{29}$$

We notice that the limiting wave speed (29) can be identified as $v_*(L^0, \rho^0, t')$ in (20) if the outer minimization problem in equations (19) is minimized at $k = 0$. This is indeed the case if L^0 is isotropic or if the lengths of the lattice vectors a_i $(i = 1, \ldots, n)$ are small enough (i.e., the nonzero reciprocal lattice points in $\mathcal{K}$ are far away from the origin). For a homogeneous half-space, we can always choose the lattice vectors a_i $(i = 1, \ldots, n)$ such that the outer minimization problem in (19) is minimized at $k = 0$, and therefore identify $v_*(L^0, \rho^0, t')$ in (20) as the limiting wave speed (29). From the viewpoint of the tube problem (6), the quantity $|t'|^2 v_*^2(L^0, \rho^0, t')$ is the infimum of the essential spectrum of the operator $\mathcal{A}_{t'}$. In another word, the lower bound (21) is sharp for a homogeneous half-space.

To explore the physical meaning of $v_s(L^0, \rho^0, t')$, we evaluate the infimum of (26) as follows. Let Id_n be the $n \times n$ identity matrix, T^0, R^0 and Q^0 be the matrices defined in (25) with L^u replaced by L^0. If for some $0 < v < v_*(L^0, \rho^0, t')$, there exists a matrix $E \in \mathbb{C}^{n \times n}$ satisfying

$$\begin{cases} T^0 E^2 + \mathring{\imath}(R^0 + R^{0T})E - (Q^0 - \rho^0 v^2 \mathrm{Id}_n) = 0, \\ \sigma(E) \subset \mathbb{C}_- \quad \text{and} \quad \mathrm{rank}(T^0 E + \mathring{\imath} R^{0T}) < n, \end{cases} \tag{30}$$

then (24) yields

$$\sqrt{\Upsilon(L^0, \rho^0, t'; \tilde{u}^\star, E)} = v,$$

where $\tilde{u}^\star \in \mathbb{C}^n$ is a unit vector in the null space of $T^0 E + \mathring{\imath} R^{0T}$. From (26) we have $v_s(L^0, \rho^0, t') \le v < v_*(L^0, \rho^0, t')$, and hence the existence of the surface wave. Meanwhile, Fu and Mielke [14, 15] have shown that if (30) has a solution E for some $0 < v < v_*(L^0, \rho^0, t')$, then a subsonic surface wave exists and is unique with phase speed v, which then implies $v \le v_s(L^0, \rho^0, t')$. (Otherwise, by the Rayleigh-Ritz variational formulas, there would be a second subsonic surface wave with phase speed less or equal to $v_s(L^0, \rho^0, t') < v$.) Therefore, $v_s(L^0, \rho^0, t') = v$ if a subsonic surface wave exists. Therefore, we conclude that the sufficient condition (28) is necessary for the existence of a subsonic surface wave. Further, it explains why substantial difficulty is encountered if one tries to generalize previous and current approaches to *supersonic* surface waves, since proving the existence of an eigenvalue embedded in the essential spectrum is mathematically a much more difficult problem than proving the existence of an eigenvalue below the essential spectrum.

Two remarks are in order here. First, a subsonic surface wave does not always exist. Clearly $v_s(L^0, \rho^0, t') = v_*(L^0, \rho^0, t')$ if this is the case. Second, supersonic surface waves do exist for some anisotropic half-spaces, see Farnell [13] for examples.

4 Applications to Heterogeneous Half-spaces of Multiple Anisotropic Materials

We now consider a periodic half-space of N anisotropic materials with elasticity tensors L_α and densities ρ_α $(\alpha = 1, \dots, N)$. The dimension of space is fixed at $n = 3$ in this section. We will compare the generic elasticity tensors L_α $(\alpha = 1, \dots, N)$ with isotropic elasticity tensors. For brevity we denote an isotropic elasticity tensor with shear modulus μ and bulk modulus κ by $L^{iso}(\mu, \kappa)$. Further, we define the smallest and greatest densities ρ^l and ρ^u, the smallest and greatest shear moduli μ^l and μ^u, and the smallest and greatest bulk moduli κ^l and κ^u as

$$
\begin{aligned}
&\rho^l = \min_{\alpha \in \{1,\dots,N\}} \rho_\alpha, \qquad \rho^u = \max_{\alpha \in \{1,\dots,N\}} \rho_\alpha, \\
&\mu^l = \min_{\substack{0 \ne X \in \mathbb{R}^{3\times 3}_{sym} \\ \mathrm{Tr}(X)=0 \\ \alpha \in \{1,\dots,N\}}} \frac{X \cdot L_\alpha X}{|X|^2}, \qquad \mu^u = \max_{\substack{0 \ne X \in \mathbb{R}^{3\times 3}_{sym} \\ \mathrm{Tr}(X)=0 \\ \alpha \in \{1,\dots,N\}}} \frac{X \cdot L_\alpha X}{|X|^2}, \\
&\kappa^l = \max_\kappa \{\kappa : L^{iso}(\mu^l, \kappa) \le L_\alpha \ \forall \alpha = 1, \dots, N\}, \\
&\kappa^u = \min_\kappa \{\kappa : L^{iso}(\mu^u, \kappa) \ge L_\alpha \ \forall \alpha = 1, \dots, N\},
\end{aligned}
\tag{31}
$$

respectively. Since $L_\alpha \ge L^l := L^{iso}(\mu^l, \kappa^l)$ for all $\alpha = 1, \dots, N$, from (19) and (20) we have

$$
v_*(L^l, \rho^u, t') = \sqrt{\frac{\mu^l}{\rho^u}}. \tag{32}
$$

Since $L_\alpha \le L^u := L^{iso}(\mu^u, \kappa^u)$ for all $\alpha = 1, \dots, N$, from Landau and Lifshitz [18, p. 96] and discussions in Sect. 3 we have

$$
v_s(L^u, \rho^l, t') = \xi(\nu^u)\sqrt{\frac{\mu^u}{\rho^l}}, \tag{33}
$$

where

$$
\nu^u = (3\kappa^u - 2\mu^u)/[2(3\kappa^u + \mu^u)] \tag{34}
$$

is the Poisson's ratio associated with L^u, and the constant $\xi(\nu^u) > 0$ satisfies [18, p. 96]),

$$\xi^6 - 8\xi^4 + \frac{8(2-\nu^u)}{1-\nu^u}\xi^2 - \frac{8}{1-\nu^u} = 0. \tag{35}$$

A quick numerical calculation shows that $\xi(\nu^u)^2$ monotonically increases from 0.4746 to 0.9126 as the Poisson's ratio ν^u varies from -1 to 0.5. Further, for the elasticity tensors L^l and L^u chosen above, by (32) and (33) the sufficient condition (27) for the existence of surface waves can be written as

$$\frac{\mu^l \rho^l}{\mu^u \rho^u} > \xi(\nu^u)^2. \tag{36}$$

Note that (36) is independent of the in-plane wave vector t' and intermediate shear moduli and densities. From Theorem 4 we conclude the following.

Theorem 6 *Consider a periodic half-space of materials with elasticity tensors L_α and densities ρ_α $(\alpha = 1, \ldots, N)$. Let ρ^l, ρ^u, μ^l, κ^l, μ^u, κ^u, ν^u be defined as in* (31) *and* (34), *and $\xi(\nu^u) > 0$ be a solution to* (35). *Then* (36) *implies the existence of surface waves for any nonzero in-plane wave vectors $t' \in Z'$.*

Specified to $N = 1$, i.e., a homogeneous half-space, the above theorem implies the following simple sufficient condition for the existence of free surface waves.

Corollary 7 *Let L_1 be an elasticity tensor, $\mu^l, \mu^u, \kappa^l, \kappa^u$, ν^u be defined as in* (31) *and* (34) *with $N = 1$, and $\xi(\nu^u) > 0$ be a solution to* (35). *If*

$$\frac{\mu^l}{\mu^u} > \xi(\nu^u)^2, \tag{37}$$

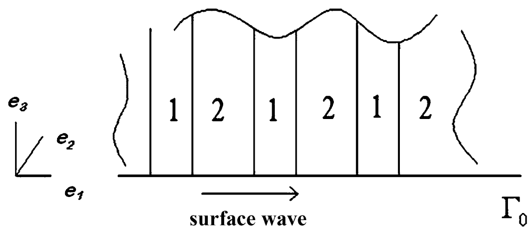
then there exist surface waves on a homogeneous half-space of L_1 for any nonzero in-plane wave vectors $t' \in Z'$.

We remark that the sufficient condition (37) for the existence of free surface waves on a homogeneous half-space is independent of the orientation of the free surface and the propagating directions and is easy to check, which is an advantage in applications.

5 Existence of Surface Wave Band Gaps

As an example, we numerically calculate the dispersion relation of the surface waves in an elastic half-space and demonstrate the existence of band gaps, which has been pointed out by Djafari-Rouhani et al. [10] based on the Fourier analysis. The half-space consists of alternating slabs of two materials as shown in Fig. 2. The slabs are perpendicular to the surface Γ_0 and the period in e_1-direction is one. We further assume that the materials are either copper (isotropic, Young's modulus $E_{\mathrm{Cu}} = 115$ Gpa, Poisson's ratio $\nu_{\mathrm{Cu}} = 0.355$, density $\rho_{\mathrm{Cu}} = 8.92$ g/cm^3) or aluminum (isotropic, Young's modulus $E_{\mathrm{Al}} = 69$ Gpa, Poisson's ratio $\nu_{\mathrm{Al}} = 0.334$, density $\rho_{\mathrm{Al}} = 2.7$ g/cm^3). We are interested in the dispersion relations of the surface waves propagating in e_1-direction. To this end, we numerically solve the eigenvalue problem (3). Since (3) is invariant under arbitrary translations in e_2-direction,

Fig. 2 A two-material periodic half-space in the numerical example



we restrict our attention to solutions of the form $u=(u_1(x_1,x_3),0,u_3(x_1,x_3))$. Inserting $u=(u_1(x_1,x_3),0,u_3(x_1,x_3))$ into (3), we obtain a two dimensional plane strain problem.

In simulations we consider a truncated finite two dimensional tube $T_{truc}=(0,1)\times(0,10)$. For a given wave number $t_1\in(0,2\pi)$, by the standard finite element method we find the eigenfrequencies ω such that

$$\begin{cases} -(L_{piqj}(x_1,x_3)u_{q,j}(x_1,x_3))_{,i}=\omega^2\rho(x_1,x_3)u_p(x_1,x_3) & \text{on } T_{truc}, \\ L_{p3qj}(x_1,x_3)u_{q,j}(x_1,x_3)=0 & \text{if } x_3=0, \\ u_p=0 & \text{if } x_3=10, \\ u_p(1,x_3)=u_p(0,x_3)exp(-\mathring{\imath}t_1) & \forall\, x_3\in(0,10), \end{cases} \tag{38}$$

admits a nontrivial solution. Here, $L(x_1,x_3)$ ($\rho(x_1,x_3)$) takes the value of the elasticity tensor (density) of copper if $x_1\in(0,1/2)$ and that of aluminum if $x_1\in(1/2,1)$. From the spectrum theory, we see that the above eigenvalue problem in general has infinitely many eigenfrequencies, including those that do not correspond to surface waves. To eliminate these eigenfrequencies, we require that

$$U_b\geq 5U_t, \tag{39}$$

where U_b (U_t) is the strain energy in bottom (top) half of the truncated tube T_{truc}. Upon eliminating the eigenfrequencies violating (39), we are left with eigenfrequencies $\omega_1(t_1)<\omega_2(t_1)<\omega_3(t_1)<\cdots$, which are presumably eigenfrequencies of surface waves in different bands. We also compare the dispersion relation of the surface waves with that of the bulk waves propagating in the e_1-direction. The results are shown in Fig. 3, where the dashed curve "– –" shows the dispersion curve of the bulk waves, the solid line "—" shows that of the surface wave, and the cross signs "×" are data points from the simulations.

From Fig. 3, we observe a few interesting features of the dispersion curves. First, band gaps are present for both curves. Also, the bulk waves has a higher frequency than the surface waves for the same wave number. These features of the dispersion curves provide a potential method to manipulate elastic waves. For example, for excitations at frequencies at the band gaps of the bulk waves, surface waves are preferably excited and propagate along the free surface, instead of radiating into the half-space, whereas excitations at frequencies at the band gaps of surface waves tend to propagate into the half-space instead of along the surface. Also, at the long wavelength limit, i.e., $t_1\to 0$, it is anticipated the dispersion relation should be predicted by the homogenization theory. Indeed we numerically verify that the phase speed of the surface waves coincides with the surface waves of a homogeneous half-space with the effective elasticity tensor and the effective density. However, we are not aware of a rigorous proof of this fact for surface waves, though the homogenization theory has been well established for bulk waves in the long wave length limit (see, e.g., [16, 22]).

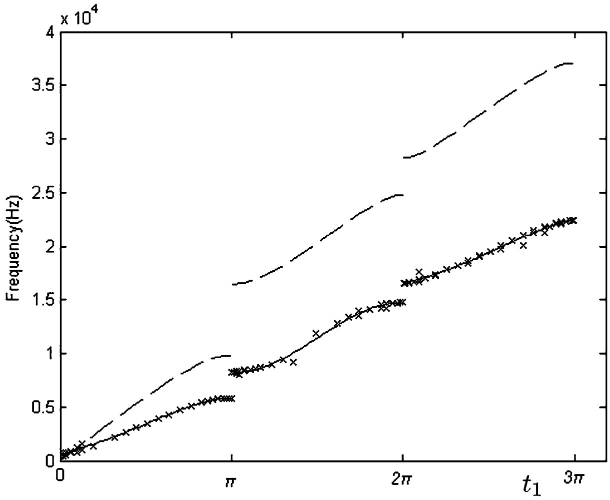


Fig. 3 Dispersion relations of the bulk waves and surface waves propagating in e_1-direction. The dashed curve "– –" shows the dispersion curve of the bulk waves; the solid line "—" shows that of the surface wave. The cross signs "×" denote the data points from the simulation

6 Summary and Discussion

We have addressed the existence of surface waves in periodic heterogeneous half-spaces using the variational method. When specialized to homogeneous media, we have shown that our existence condition (28) is necessary and sufficient for the existence of subsonic surface waves. For general periodic heterogeneous media of anisotropic materials, a simple criterion for the existence of surface waves is given in Theorem 6.

In general, the more is known about the medium, the tighter bounds we would be able to obtain for $\tilde{v}_s(L, \rho, t')$ and $\tilde{v}_*(L, \rho, t')$ in (16), and so an improved sufficient condition for the existence of surface waves would follow from Theorem 3. For instance, if $L(x)$ and $\rho(x)$ are known, the Rayleigh quotient of the test function (22)

$$\frac{R_T(u)}{|t'|^2} = \frac{\int_T L_{piqj}(x) u_{p,i} \overline{u_{q,j}} dx}{|t'|^2 \int_T \rho(x) u_p \overline{u_p} dx} =: \Upsilon(L, \rho, t'; \tilde{u}, E)$$

is fairly easy to evaluate, at least numerically. Then from Theorem 3 we obtain an improved sufficient condition for the existence of surface waves as

$$\inf\{\sqrt{\Upsilon(L, \rho, t'; \tilde{u}, E)} : \sigma(E) \subset \mathbb{C}_-, \ |\tilde{u}|^2 = 1\} < \sup_{L^l \le L(x)} v_*(L^l, \rho^u, t').$$

The quality of the sufficient condition (for the existence of surface waves) is mainly dictated by the quality of the test functions. The best test function is clearly the eigenfunction associated with the smallest eigenvalue of $\mathcal{A}_{t'}$ if such an eigenvalue indeed exists. The sufficient condition could be trivial if the quality of the test functions deteriorates on one hand, and on the other hand the condition could be too complicated to be informative if the test functions are too general to have a simple and explicit parametrization. Therefore, the detailed implementation of this approach for a particular problem can be delicate.

The variational method can also be adapted to address the existence of surface waves and interfacial waves [27] in other heterogeneous systems when the domain is unbounded and the coefficients or potentials are specified at the infinity. For instance, we may consider the Maxwell equation with a periodic permittivity and/or permeability [23, 24]; or the

Shrödinger equation with a periodic potential [9, 12, 26]. Following the same arguments as in this paper, we will be able to obtain sufficient conditions for the existence of surface/interfacial waves for these problems.

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