1.2 The divergence theorem

Tensor Fields Let $\Omega \subset I\!R^n$ be an open domain.

- $\varphi : \Omega \to I\!R$ is a scalar field;
- $\mathbf{v} : \Omega \to I\!R^m$ is a vector field;
- $\mathbf{T} : \Omega \to \text{Lin}(I\!R^m, I\!R^k)$ is a tensor field.

Notation convention: it is often convenient to denote vectors and tensors in index notation, e.g., v_i ($v_i = \mathbf{v} \cdot \mathbf{e}_i$) and T_{pi} ($T_{pi} = \hat{\mathbf{e}}_p \mathbf{T} \mathbf{e}_i$), where the bases $\{\hat{\mathbf{e}}_p, p = 1, \cdots, m\}$ and $\{\mathbf{e}_i, i = 1, \cdots, n\}$ are usually not specified but tacitly understood.

Differentiation Let φ be a scalar field on $\Omega \subset I\!R^n$. For any $\mathbf{a} \in I\!R^n$,

$$D\varphi(\mathbf{x}) : I\!R^n \to I\!R,$$
$$(D\varphi(\mathbf{x}))(\mathbf{a}) = \lim_{\varepsilon \to 0} \frac{\varphi(\mathbf{x} + \varepsilon \mathbf{a}) - \varphi(\mathbf{x})}{\varepsilon}.$$

Definition: φ is differentiable on Ω if $D\varphi(\mathbf{x}) \in \text{Lin}(I\!R^n, I\!R)$ for all $\mathbf{x} \in \Omega$

$$\begin{aligned} \varphi(\mathbf{x} + \varepsilon \mathbf{a}) &= \varphi(\mathbf{x}) + \varepsilon (D\varphi(\mathbf{x}))(\mathbf{a}) + o(\varepsilon) \\ &= \varphi(\mathbf{x}) + \varepsilon \nabla \varphi(\mathbf{x}) \cdot \mathbf{a} + o(\varepsilon) \qquad \forall \mathbf{x} \in \Omega, \mathbf{a} \in I\!R^n. \end{aligned}$$

and

$$\nabla \varphi(\mathbf{x}) = \sum_{i=1}^n \varphi_{,i} \mathbf{e}_i, \qquad \varphi_{,i} = \mathbf{e}_i \cdot \nabla \varphi = \lim_{\varepsilon \to 0} \frac{\varphi(\mathbf{x} + \varepsilon \mathbf{e}_i) - \varphi(\mathbf{x})}{\varepsilon} = \frac{\partial \varphi(x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n)}{\partial x_i}.$$

Definition: $\mathbf{v} : \Omega \to I\!R^m$ is differentiable on Ω if every component is differentiable

$$\mathbf{v}(\mathbf{x}) = \sum_p v_p(\mathbf{x}) \hat{\mathbf{e}}_p \, .$$

Definition:

$$\nabla \mathbf{v}(\mathbf{x}) = D\mathbf{v}(\mathbf{x}) : I\!R^n \to I\!R^m$$
$$(\nabla \mathbf{v}(\mathbf{x}))(\mathbf{a}) = \sum_{p=1}^m \hat{\mathbf{e}}_p \nabla v_p(x) \cdot \mathbf{a}$$

Thus,

$$\nabla \mathbf{v}(\mathbf{x}) = \sum_{p,i} v_{p,i} \hat{\mathbf{e}}_p \otimes \mathbf{e}_i \, .$$

Divergence: If $m = n$, $\text{div}(\mathbf{v}) = \text{Tr}(\nabla \mathbf{v})$, i.e.,

$$\text{div}(\mathbf{v}) = v_{i,i} \mathbf{e}_i \cdot \mathbf{e}_i = v_{i,i}.$$

Further, if $\mathbf{T} : \Omega \to \text{Lin}(I\!R^n, I\!R^m)$ and

$$\mathbf{T}(x) = T_{pi} \hat{\mathbf{e}}_p \otimes \mathbf{e}_i.$$

Then

$$\operatorname{div}(\mathbf{T}) : \mathbb{R}^m \to \mathbb{R}$$
$$\operatorname{div}(\mathbf{T})(\mathbf{a}) = T_{pi,i}\mathbf{a} \cdot \hat{\mathbf{e}}_p.$$

One may identify $\operatorname{div}(\mathbf{T})$ with a vector field $\Omega \to \mathbb{R}^m$ (instead of $\operatorname{Lin}(\mathbb{R}^m, \mathbb{R})$). With an abuse of notation, we write

$$\operatorname{div}(\mathbf{T}) = T_{pi,i}\hat{\mathbf{e}}_p.$$

Field of class $C^0, C^1, C^2, \cdots, C^\infty$

♦ CLAIM: Assume $\varphi,\ \mathbf{v},\ \mathbf{u},\ \mathbf{T} : \Omega \to \mathbb{R}, \mathbb{R}^n, \mathbb{R}^n, \operatorname{Lin}(\mathbb{R}^n, \mathbb{R}^m),\ \mathbf{w} : \Omega \to \mathbb{R}^m$ are smooth fields on Ω. The following identities hold:

1. $\nabla(\varphi \mathbf{v}) = \mathbf{v} \otimes (\nabla \varphi) + \varphi \nabla \mathbf{v}$;
2. $\operatorname{div}(\varphi \mathbf{v}) = (\nabla \varphi) \cdot \mathbf{v} + \varphi \operatorname{div}\mathbf{v}; \quad \nabla \cdot \mathbf{v} = \operatorname{div}\mathbf{v}$
3. $\nabla(\mathbf{v} \cdot \mathbf{u}) = (\nabla \mathbf{v})^T \mathbf{u} + (\nabla \mathbf{u})^T \mathbf{v}$
4. $\operatorname{div}(\mathbf{v} \otimes \mathbf{u}) = \mathbf{v}\operatorname{div}(\mathbf{u}) + (\nabla \mathbf{v})\mathbf{u}$
5. $\operatorname{div}(\mathbf{T}^T \mathbf{w}) = \mathbf{T} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \operatorname{div}\mathbf{T}$
6. $\operatorname{div}(\varphi \mathbf{T}) = \varphi \operatorname{div}\mathbf{T} + \mathbf{T}\nabla \varphi$

Proof: Tacitly, an orthonormal basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\} \subset \mathbb{R}^n$ and an orthonormal basis $\{\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_m\} \subset \mathbb{R}^m$ are chosen and fixed. Notation: Einstein summation, i.e., summation over double index is understood. For example, to show 5, we have

$$\operatorname{div}(\mathbf{T}^T \mathbf{w}) = (T_{pi}w_p)_{,i} = T_{pi,i}w_p + T_{pi}w_{p,i} = \mathbf{w} \cdot \operatorname{div}\mathbf{T} + \mathbf{T} \cdot \nabla \mathbf{w}$$

■

Curl operator: Let $\mathbf{v} : \Omega \to \mathbb{R}^3$.

$$\operatorname{curl}\mathbf{v} = \nabla \wedge \mathbf{v} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \sum_{i=1}^{3} \mathcal{E}_{ijk} v_{k,j} \mathbf{e}_i,$$

where *Levi-Civita* symbol is defined as

$$\mathcal{E}_{ijk} = \begin{cases} 1 & \text{if } (ijk) = (123),\ (231),\ (312), \\ -1 & \text{if } (ijk) = (132),\ (213),\ (321), \\ 0 & \text{otherwise.} \end{cases}$$

We notice that $\mathcal{E}_{ijk}$ is antisymmetric, i.e.,

$$\mathcal{E}_{ijk} = -\mathcal{E}_{ikj}, \quad \mathcal{E}_{ijk} = -\mathcal{E}_{jik}, \quad etc.$$

A useful identity between Kronecker symbol and Levi-Civita symbol is

$$\mathcal{E}_{pij}\mathcal{E}_{pkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Let $\Omega \subset I\!R^3$ be a domain in $I\!R^3$. Assume that $\mathbf{v} : \Omega \to I\!R^3$, $\varphi : \Omega \to I\!R$ are smooth fields.

♦ CLAIM: the following identities hold:

1. $\nabla \wedge \nabla\varphi = 0$.
2. $\mathrm{div}(\nabla \wedge \mathbf{v}) = 0$.
3. If $a, b, c \in I\!R^3$, $a \wedge (b \wedge c) = (a \cdot c)b - (a \cdot b)c$
4. $\nabla \wedge (\nabla \wedge \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \Delta\mathbf{v}$.

Proof:

■

Divergence Theorem

Let Ω be a smooth simply connected domain in $I\!R^n$, $\mathbf{v} : \Omega \to I\!R^m$ is a smooth vector field on Ω. Then we have

$$\int_{\partial\Omega} \mathbf{v} \otimes \mathbf{n}\, da = \int_{\Omega} \nabla\mathbf{v} dv, \tag{2}$$

where $\mathbf{n} : \partial\Omega \to I\!R^n$ is the outward unit normal on the boundary $\partial\Omega$. If $m = n$, take the trace of Eq. (2), we have

$$\int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, da = \int_{\Omega} \mathrm{div}\mathbf{v} dv. \tag{3}$$

For a smooth tensor field $\mathbf{T} : \Omega \to \mathrm{Lin}(I\!R^n; I\!R^m)$, we have

$$\int_{\partial\Omega} \mathbf{T}\mathbf{n}\, da = \int_{\Omega} \mathrm{div}\mathbf{T} dv. \tag{4}$$

♦ PROVIDE a heuristic proof for (2) with Ω being a rectangle in two dimensions.

Implications of divergence theorem in physics and mechanics

1. Gauss theorem
2. Stokes theorem
3. ......

1.3 Curvilinear coordinate systems

⧫ Introduce the cylindrical coordinate system and spherical coordinate system in 3D.