Theory of Elasticity

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Contents

1 Prelimilaries

1.1 A review of linear algebra

Vector Space Let $I\!R$ be the scalar field of real numbers. We consider only real vector spaces. Let V_n be a set. V_n is a vector space (also called a linear space) if it is equipped with two operations:

$$\text{scalar product} \quad I\!R \times V_n \to V_n,$$
$$\text{vector addition} \quad V_n \times V_n \to V_n,$$

and it is closed under these two operations. That is, V_n is a vector space if $\forall\, \alpha, \beta \in I\!R$ & $\forall\, \mathbf{a}, \mathbf{b} \in V_n$,

$$\alpha\mathbf{a} + \beta\mathbf{b} \in V_n.$$

The vector space V_n is n**-dimensional** if we can find a basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\} \subset V_n$ such that for any $\mathbf{a} \in V_n$, we have a unique decomposition

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i,$$

where $a_i \in I\!R$ $(i = 1, \cdots n)$ are the components (coordinates) of vector $\mathbf{a}$ under the basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$.
Tensor Space Let V_n (V_m) be n-dimensional (m-dimensional) vector space. A mapping $\mathbf{A}: V_n \to V_m$ is a tensor if $\mathbf{A}$ is linear. That is, $\forall\, \alpha, \beta \in I\!R$ & $\forall\, \mathbf{a}, \mathbf{b} \in V_n$,

$$\mathbf{A}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathbf{A}(\mathbf{a}) + \beta\mathbf{A}(\mathbf{b}). \tag{1}$$

Let $\mathrm{Lin}(V_n, V_m)$ be the collection of all linear mappings (i.e., tensors) with domain V_n and range V_m. For any $\alpha \in I\!R$ and any $\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{Lin}(V_n, V_m)$, define two operations

$$\text{scalar product} \quad (\alpha\mathbf{A}_1)(\mathbf{a}) = \alpha\mathbf{A}_1(\mathbf{a}) \qquad \forall\, \mathbf{a} \in V_n,$$
$$\text{vector addition} \quad (\mathbf{A}_1 + \mathbf{A}_2)(\mathbf{a}) = \mathbf{A}_1(\mathbf{a}) + \mathbf{A}_2(\mathbf{a}) \qquad \forall\, \mathbf{a} \in V_n.$$

♦ CLAIM: For any $\alpha, \beta \in I\!R$ and any $\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{Lin}(V_n, V_m)$, $\alpha\mathbf{A}_1 + \beta\mathbf{A}_2$ is a linear mapping (from V_n to V_m).

The above claim implies that the set $\mathrm{Lin}(V_n, V_m)$ is also a vector space.
Inner Product We equip a n-dimensional vector space V_n with a mapping $V_n \times V_n \to I\!R$, called inner product such that for any $\alpha, \beta \in I\!R$ and any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V_n$, the inner product is

1. Positive-definite: $\mathbf{a} \cdot \mathbf{a} \geq 0$; $\mathbf{a} \cdot \mathbf{a} = 0 \iff \mathbf{a} = 0$,
2. Linear: $\mathbf{a} \cdot (\alpha\mathbf{b} + \beta\mathbf{c}) = \alpha\mathbf{a} \cdot \mathbf{b} + \beta\mathbf{a} \cdot \mathbf{c}$,
3. Symmetric: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

Geometric interpretations:

- Length of a vector: $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$,
- Angle between two vectors: $\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

Subspace of $I\!R^n$, Orthogonal Subspace A subset $M \subset I\!R^n$ is a subspace if $\forall\, \alpha, \beta \in I\!R$ & $\forall\, \mathbf{a}, \mathbf{b} \in M$,

$$\alpha\mathbf{a} + \beta\mathbf{b} \in M.$$

Let $M^\perp = \{\mathbf{b} : \mathbf{b} \cdot \mathbf{a} = 0 \;\forall\; \mathbf{a} \in M\}$.

⧫ Claim: $M^\perp$ is a subspace of $I\!R^n$ if M is a subspace.

Projection Theorem Let M be a subspace of $I\!R^n$. For any $\mathbf{x} \in I\!R^n$, we have

$$\mathbf{x} = \mathbf{y} + \mathbf{z} \text{ where } \mathbf{y} \in M,\; \mathbf{z} \in M^\perp.$$

The vector $\mathbf{y}$, $\mathbf{z}$ are uniquely determined by $\mathbf{x}$.

⧫ Proof:

Euclidean Space $I\!R^n$ For a n-dimensional vector space V_n equipped with an inner product, we ⧫ Claim that there exists an orthonormal basis $\{\mathbf{e}_i : i = 1, \cdots, n\}$ such that for all $i, j = 1, \cdots, n$,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where δ_{ij} is called *Kronecker delta.*

With respect to this basis, for any vector $\mathbf{a} \in V_n$, we find its components $(a_1, \cdots, a_n)$ (or coordinates if a is a point in space)

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i, \qquad a_i = \mathbf{a} \cdot \mathbf{e}_i \in I\!R \quad \forall\, i = 1, \cdots, n.$$

We can further identify the space V_n with the familiar Euclidean space $I\!R^n$. However, one shall keep in mind, $I\!R^n$, as a vector space equipped with an inner product, is more than a collection of arrays of real numbers. One should not think of a vector in $I\!R^n$ as an array of real numbers unless we specify a basis or a frame.

Tensor Product For vectors $\mathbf{a} \in I\!R^n$ and $\mathbf{b} \in I\!R^m$, the tensor product $\mathbf{b} \otimes \mathbf{a}$ is a linear mapping:

$$\begin{aligned} &\mathbf{b} \otimes \mathbf{a} : V_n \to V_m \\ &(\mathbf{b} \otimes \mathbf{a})(\mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} \quad \forall\, \mathbf{c} \in I\!R^n. \end{aligned}$$

⧫ Claim: For any $\mathbf{a} \in I\!R^n$ and $\mathbf{b} \in I\!R^m$, the mapping $\mathbf{b} \otimes \mathbf{a}$ (from V_n to V_m) defined above is linear.

Einstein convention For many calculations, it is convenient to use index notation and Einstein convention of summation over repeated index. For example, upon fixing an orthonormal basis, a vector $\mathbf{a}$ can be identified with its components a_i whereas $\mathbf{a}\cdot\mathbf{a}$ can be identified with summation of$a_i a_i$ over $i = 1, \cdots, n$:

$$\begin{aligned} &\mathbf{a} \text{ is repsented by its components } a_i, \\ &\mathbf{a}\cdot\mathbf{a} = \sum_{i=1}^{n} a_i a_i = a_i a_i, \end{aligned}$$

where in the last equality we drop the summation symbol "$\sum_{i=1}^{n}$ since the index "i" is repeated. This is precisely the *Einstein convention*. When use Einstein convention, we notice the following:

1. Every index can appear only once (free index) or twice (dummy index).
2. The symbols used for free index or dummy index are irrelevant.
3. The free index on two sides of an equation must be identical.

♦ CLAIM: Let $\{\mathbf{e}_i : i = 1, \cdots n\}$ be an orthonormal basis of $I\!R^n$ and $\{\hat{\mathbf{e}}_p : p = 1, \cdots m\}$ be an orthonormal basis of $I\!R^m$. Show that

$$\{\hat{\mathbf{e}}_p \otimes \mathbf{e}_i : i = 1, \cdots, n,\ p = 1, \cdots m\} \subset \mathrm{Lin}(I\!R^n, I\!R^m)$$

forms a basis of the linear space $\mathrm{Lin}(I\!R^n, I\!R^m)$.

Transpose of a Tensor Let $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R^m)$, $\{\mathbf{e}_i : i = 1, \cdots n\}$ be an orthonormal basis of $I\!R^n$ and $\{\hat{e}_p : p = 1, \cdots m\}$ be an orthonormal basis of $I\!R^m$. Then $\mathbf{A}$ admits the following decomposition

$$\mathbf{A} = A_{pi}\hat{\mathbf{e}}_p \otimes \mathbf{e}_i \qquad \text{where } A_{pi} = \hat{\mathbf{e}}_p \cdot \mathbf{A}(\mathbf{e}_i)\ \ \forall\, i = 1, \cdots, n, p = 1, \cdots, m.$$

Define

$$\begin{aligned} &\mathbf{A}^T : I\!R_m \to I\!R^n, \\ &\mathbf{A}^T = A_{pi}\mathbf{e}_i \otimes \hat{\mathbf{e}}_p \in \mathrm{Lin}(I\!R^m, I\!R^n). \end{aligned}$$

♦ CLAIM: For any $\mathbf{a} \in I\!R^n$ and $\mathbf{b} \in I\!R^m$,

$$\mathbf{b}\cdot\mathbf{A}(\mathbf{a}) = \mathbf{a}\cdot\mathbf{A}^T(\mathbf{b}).$$

Symmetric and Skew-symmetric Tensor Let $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R^n)$. $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^T$; $\mathbf{A}$ is skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$.

Let $\{\mathbf{e}_i : i = 1, \cdots n\}$, $\{\hat{\mathbf{e}}_p : p = 1, \cdots n\}$ be two orthonormal bases of $I\!R^n$. We have shown

$$\mathbf{A} = A_{pi}\hat{\mathbf{e}}_p \otimes \mathbf{e}_i \qquad \text{where } A_{pi} = \hat{\mathbf{e}}_p \cdot \mathbf{A}(\mathbf{e}_i)\ \ \forall\, p, i = 1, \cdots, n.$$

♦ CLAIMS :

1. For any $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R^n)$, we have a unique decomposition $\mathbf{A} = \mathbf{E} + \mathbf{W}$, where $\mathbf{E} = \mathbf{E}^T$ and $\mathbf{W} = -\mathbf{W}^T$.

2. $\mathbf{A} = \mathbf{A}^T$ if and only if for any $\mathbf{a}, \mathbf{b} \in I\!R^n$,

$$\mathbf{b} \cdot \mathbf{A}(\mathbf{a}) = \mathbf{a} \cdot \mathbf{A}(\mathbf{b}).$$

3. If $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{a} \cdot \mathbf{A}(\mathbf{a}) = 0$ for any $\mathbf{a} \in I\!R^n$, then $\mathbf{A} = 0$.

4. There exists a nonzero tensor $\mathbf{A}$ such that

$$\mathbf{a} \cdot \mathbf{A}\mathbf{a} = 0 \qquad \forall\, \mathbf{a} \in I\!R^n,\ n \geq 2.$$

5. Assume that $(\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_n) = (\mathbf{e}_1, \cdots, \mathbf{e}_n)$. If $\mathbf{A} = \mathbf{A}^T$, then $A_{pi} = A_{ip}$ for all $p, i = 1, \cdots, n$; if $\mathbf{A} = -\mathbf{A}^T$, then $A_{pi} = -A_{ip}$.

Product of tensors Let $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R^m)$, $\mathbf{B} \in \mathrm{Lin}(I\!R^m, I\!R^k)$. Then

$$\mathbf{BA} : I\!R^n \to I\!R^k,$$
$$\mathbf{BA}(\mathbf{a}) = \mathbf{B}(\mathbf{A}(\mathbf{a})).$$

Orthogonal Tensor Let $\mathbf{Q} \in \mathrm{Lin}(I\!R^n, I\!R^n)$. The tensor $\mathbf{Q}$ is orthogonal if $\mathbf{Qa} \cdot \mathbf{Qb} = \mathbf{a} \cdot \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in I\!R^n$. From the definition we see that orthogonal tensor operating on vectors preserves the length of a vector and the angle between two vectors since

1. $|\mathbf{a}| = |\mathbf{Qa}|$, and

2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{Qa} \cdot \mathbf{Qb}$.

♦Claim: A tensor $\mathbf{Q} : I\!R^n \to I\!R^n$ is orthogonal if and only if $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$, where $\mathbf{I}$ is the identity mapping from $I\!R^n$ to $I\!R^n$.

Trace and determinant of a tensor Let $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R^n)$ and $\{\mathbf{e}_i : i = 1, \cdots, n\}$ be an orthonormal basis. Then we have $\mathbf{A} = A_{pi}\mathbf{e}_p \otimes \mathbf{e}_i$ and refer to $\mathrm{Tr}(\mathbf{A}) = A_{pp}$ as the trace of $\mathbf{A}$, $\det A = \det[A_{pi}]$ as the determinant of $\mathbf{A}$.

♦ Claim Tr, det $: \mathrm{Lin}(I\!R^n, I\!R^n) \to I\!R$ is independent of the choice of orthonormal basis.

Rigid Rotation Tensor An orthogonal tensor $\mathbf{R} \in \mathrm{Lin}(I\!R^n, I\!R^n)$ is a rigid rotation if $\det \mathbf{R} = +1$.
Representation theorem: For any $\mathbf{A} \in \mathrm{Lin}(I\!R^n, I\!R)$, there is an $\mathbf{a} \in I\!R^n$ such that $\mathbf{A}(\mathbf{b}) = \mathbf{b} \cdot \mathbf{a}$ $\forall\, \mathbf{b} \in I\!R^n$. Explicitly, if

$$\mathbf{A} = \sum_i A_{1i} \hat{\mathbf{e}}_1 \otimes \mathbf{e}_i, \quad \hat{\mathbf{e}}_1 = 1,$$

then

$$\mathbf{a} = \sum_{i=1}^{n} A_{1i} \mathbf{e}_i.$$

Cross product in $I\!R^3$: For $\mathbf{a},\ \mathbf{b} \in I\!R^3$,

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{W}(\mathbf{b}),$$

where $\mathbf{W} = \sum_{p,i} W_{pi} \mathbf{e}_p \otimes \mathbf{e}_i$,

$$[W_{p,i}] = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

⧫ CLAIM: The following properties of cross products holds:

1. $\mathbf{b} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{b}, \quad \mathbf{a} \cdot (\mathbf{a} \wedge \mathbf{b}) = 0$, $\mathbf{b} \cdot (\mathbf{a} \wedge \mathbf{b}) = 0$.
2. $(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \wedge \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \wedge \mathbf{a}) \cdot \mathbf{b}$.
3. Geometric interpretation: show that $|\mathbf{a} \wedge \mathbf{b}|$ =area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$; $|\mathbf{c} \cdot (\mathbf{a} \wedge \mathbf{b})|$= volume of the parallelepiped formed by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

1.2 The divergence theorem

Tensor Fields Let $\Omega \subset I\!R^n$ be an open domain.

- $\varphi : \Omega \to I\!R$ is a scalar field;
- $\mathbf{v} : \Omega \to I\!R^m$ is a vector field;
- $\mathbf{T} : \Omega \to \mathrm{Lin}(I\!R^m, I\!R^k)$ is a tensor field.

Notation convention: it is often convenient to denote vectors and tensors in index notation, e.g., v_i ($v_i = \mathbf{v} \cdot \mathbf{e}_i$) and T_{pi} ($T_{pi} = \hat{\mathbf{e}}_p \mathbf{T} \mathbf{e}_i$), where the bases $\{\hat{\mathbf{e}}_p, p = 1, \cdots, m\}$ and $\{\mathbf{e}_i, i = 1, \cdots, n\}$ are usually not specified but tacitly understood.
Differentiation Let φ be a scalar field on $\Omega \subset I\!R^n$. For any $\mathbf{a} \in I\!R^n$,

$$D\varphi(\mathbf{x}) : I\!R^n \to I\!R,$$
$$(D\varphi(\mathbf{x}))(\mathbf{a}) = \lim_{\varepsilon \to 0} \frac{\varphi(\mathbf{x} + \varepsilon \mathbf{a}) - \varphi(\mathbf{x})}{\varepsilon}.$$

Definition: φ is differentiable on Ω if $D\varphi(\mathbf{x}) \in \mathrm{Lin}(I\!R^n, I\!R)$ for all $\mathbf{x} \in \Omega$

$$\begin{aligned} \varphi(\mathbf{x} + \varepsilon \mathbf{a}) &= \varphi(\mathbf{x}) + \varepsilon (D\varphi(\mathbf{x}))(\mathbf{a}) + o(\varepsilon) \\ &= \varphi(\mathbf{x}) + \varepsilon \nabla \varphi(\mathbf{x}) \cdot \mathbf{a} + o(\varepsilon) \qquad \forall \mathbf{x} \in \Omega, \mathbf{a} \in I\!R^n. \end{aligned}$$

and

$$\nabla \varphi(\mathbf{x}) = \sum_{i=1}^{n} \varphi_{,i} \mathbf{e}_i, \qquad \varphi_{,i} = \mathbf{e}_i \cdot \nabla \varphi = \lim_{\varepsilon \to 0} \frac{\varphi(\mathbf{x} + \varepsilon \mathbf{e}_i) - \varphi(\mathbf{x})}{\varepsilon} = \frac{\partial \varphi(x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n)}{\partial x_i}.$$

Definition: $\mathbf{v} : \Omega \to I\!R^m$ is differentiable on Ω if every component is differentiable

$$\mathbf{v}(\mathbf{x}) = \sum_p v_p(\mathbf{x}) \hat{\mathbf{e}}_p \ .$$

Definition:

$$\nabla \mathbf{v}(\mathbf{x}) = D\mathbf{v}(\mathbf{x}) : I\!R^n \to I\!R^m$$
$$(\nabla \mathbf{v}(\mathbf{x}))(\mathbf{a}) = \sum_{p=1}^{m} \hat{\mathbf{e}}_p \nabla v_p(x) \cdot \mathbf{a}$$

Thus,

$$\nabla \mathbf{v}(\mathbf{x}) = \sum_{p,i} v_{p,i} \hat{\mathbf{e}}_p \otimes \mathbf{e}_i \ .$$

Divergence: If $m = n$, $\mathrm{div}(\mathbf{v}) = \mathrm{Tr}(\nabla \mathbf{v})$, i.e.,

$$\mathrm{div}(\mathbf{v}) = v_{i,i} \mathbf{e}_i \cdot \mathbf{e}_i = v_{i,i}.$$

Further, if $\mathbf{T} : \Omega \to \mathrm{Lin}(I\!R^n, I\!R^m)$ and

$$\mathbf{T}(x) = T_{pi} \hat{\mathbf{e}}_p \otimes \mathbf{e}_i.$$

Then

$$\mathrm{div}(\mathbf{T}) : I\!R^m \to I\!R$$
$$\mathrm{div}(\mathbf{T})(\mathbf{a}) = T_{pi,i}\mathbf{a} \cdot \hat{\mathbf{e}}_p.$$

One may identify $\mathrm{div}(\mathbf{T})$ with a vector field $\Omega \to I\!R^m$ (instead of $\mathrm{Lin}(I\!R^m, I\!R)$). With an abuse of notation, we write

$$\mathrm{div}(\mathbf{T}) = T_{pi,i}\hat{\mathbf{e}}_p.$$

Field of class $C^0, C^1, C^2, \cdots, C^\infty$

♦ CLAIM: Assume $\varphi,\ \mathbf{v},\ \mathbf{u},\ \mathbf{T} : \Omega \to I\!R, I\!R^n, I\!R^n, \mathrm{Lin}(I\!R^n, I\!R^m),\ \mathbf{w} : \Omega \to I\!R^m$ are smooth fields on Ω. The following identities hold:

1. $\nabla(\varphi\mathbf{v}) = \mathbf{v} \otimes (\nabla\varphi) + \varphi\nabla\mathbf{v}$;
2. $\mathrm{div}(\varphi\mathbf{v}) = (\nabla\varphi) \cdot \mathbf{v} + \varphi\mathrm{div}\mathbf{v}$; $\quad \nabla \cdot \mathbf{v} = \mathrm{div}\mathbf{v}$
3. $\nabla(\mathbf{v} \cdot \mathbf{u}) = (\nabla\mathbf{v})^T\mathbf{u} + (\nabla\mathbf{u})^T\mathbf{v}$
4. $\mathrm{div}(\mathbf{v} \otimes \mathbf{u}) = \mathbf{v}\mathrm{div}(\mathbf{u}) + (\nabla\mathbf{v})\mathbf{u}$
5. $\mathrm{div}(\mathbf{T}^T\mathbf{w}) = \mathbf{T} \cdot \nabla\mathbf{w} + \mathbf{w} \cdot \mathrm{div}\mathbf{T}$
6. $\mathrm{div}(\varphi\mathbf{T}) = \varphi\mathrm{div}\mathbf{T} + \mathbf{T}\nabla\varphi$

Proof: Tacitly, an orthonormal basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\} \subset I\!R^n$ and an orthonormal basis $\{\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_m\} \subset I\!R^m$ are chosen and fixed. Notation: Einstein summation, i.e., summation over double index is understood. For example, to show 5, we have

$$\mathrm{div}(\mathbf{T}^T\mathbf{w}) = (T_{pi}w_p)_{,i} = T_{pi,i}w_p + T_{pi}w_{p,i} = \mathbf{w} \cdot \mathrm{div}\mathbf{T} + \mathbf{T} \cdot \nabla\mathbf{w}$$

■

Curl operator: Let $\mathbf{v} : \Omega \to I\!R^3$.

$$\mathrm{curl}\mathbf{v} = \nabla \wedge \mathbf{v} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \sum_{i=1}^{3} \mathcal{E}_{ijk}v_{k,j}\mathbf{e}_i,$$

where *Levi-Civita* symbol is defined as

$$\mathcal{E}_{ijk} = \begin{cases} 1 & \text{if } (ijk) = (123),\ (231),\ (312), \\ -1 & \text{if } (ijk) = (132),\ (213),\ (321), \\ 0 & \text{otherwise.} \end{cases}$$

We notice that $\mathcal{E}_{ijk}$ is antisymmetric, i.e.,

$$\mathcal{E}_{ijk} = -\mathcal{E}_{ikj}, \quad \mathcal{E}_{ijk} = -\mathcal{E}_{jik}, \quad etc.$$

A useful identity between Kronecker symbol and Levi-Civita symbol is

$$\mathcal{E}_{pij}\mathcal{E}_{pkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Let $\Omega \subset \mathbb{R}^3$ be a domain in $\mathbb{R}^3$. Assume that $\mathbf{v} : \Omega \to \mathbb{R}^3$, $\varphi : \Omega \to \mathbb{R}$ are smooth fields.

♦ CLAIM: the following identities hold:

1. $\nabla \wedge \nabla\varphi = 0$.
2. $\mathrm{div}(\nabla \wedge \mathbf{v}) = 0$.
3. If $a, b, c \in \mathbb{R}^3$, $a \wedge (b \wedge c) = (a \cdot c)b - (a \cdot b)c$
4. $\nabla \wedge (\nabla \wedge \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \Delta\mathbf{v}$.

Proof:

■

Divergence Theorem

Let Ω be a smooth simply connected domain in $\mathbb{R}^n$, $\mathbf{v} : \Omega \to \mathbb{R}^m$ is a smooth vector field on Ω. Then we have

$$\int_{\partial\Omega} \mathbf{v} \otimes \mathbf{n}\, da = \int_\Omega \nabla\mathbf{v} dv, \tag{2}$$

where $\mathbf{n} : \partial\Omega \to \mathbb{R}^n$ is the outward unit normal on the boundary $\partial\Omega$. If $m = n$, take the trace of Eq. (2), we have

$$\int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, da = \int_\Omega \mathrm{div}\mathbf{v} dv. \tag{3}$$

For a smooth tensor field $\mathbf{T} : \Omega \to \mathrm{Lin}(\mathbb{R}^n; \mathbb{R}^m)$, we have

$$\int_{\partial\Omega} \mathbf{T}\mathbf{n}\, da = \int_\Omega \mathrm{div}\mathbf{T} dv. \tag{4}$$

♦ PROVIDE a heuristic proof for (2) with Ω being a rectangle in two dimensions.

Implications of divergence theorem in physics and mechanics

1. Gauss theorem
2. Stokes theorem Let $\mathcal{C}$ be a closed curve given by $\{\hat{\mathbf{x}}(s) : s \in [0, 1]\}$.

$$\int_{\mathcal{C}} \varphi d\mathbf{x} := \int_0^1 \varphi(\hat{\mathbf{x}}(s))\frac{d\hat{\mathbf{x}}(s)}{ds} ds = \int_{\mathcal{S}} \mathbf{n} \wedge \nabla\varphi,$$
$$\int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{x} := \int_0^1 \mathbf{v} \cdot (\hat{\mathbf{x}}(s))\frac{d\hat{\mathbf{x}}(s)}{ds} ds = \int_{\mathcal{S}} \mathbf{n} \cdot \mathrm{curl}\mathbf{v}.$$

3. ......

1.3 Curvilinear coordinate systems

⧫ INTRODUCE the cylindrical coordinate system and spherical coordinate system in 3D.

2 Kinematics

Consider a continuum deformable body subject to the applications of external loads and a set of geometric constraints. To describe the change of shape or *deformation* of this body, we introduce the concepts of *reference configuration* and *current configuration*.

- Reference configuration Ω: a configuration of body. Generally speaking, the reference configuration may be selected arbitrarily. However, in elasticity the reference configuration is typically selected to be the state when the body is subject to no external loads at all, i.e., the *natural* or *stress-free* state.

- Current configuration $\mathbf{y}(\Omega)$: the configuration after the body is deformed by external loads.

We expect that a material point $\mathbf{x}$ in the reference configuration would move to a new poisition $\mathbf{y}$, which enable us to establish a mapping $\mathbf{y} : \Omega \to \mathbf{y}(\Omega)$. This mapping is referred to as *deformation*. Based on physical ground, we assume that $\mathbf{y} : \Omega \to \mathbf{y}(\Omega)$ is one-to-one, Lipschitz continuous and $\det(\nabla \mathbf{y}) > 0$ on Ω, as illustrated in Fig. 1.

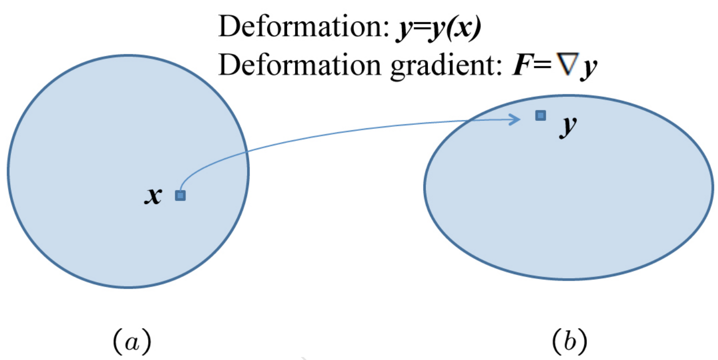

Figure 1: Deformation $\mathbf{y} : \Omega \to I\!R^3$ and deformation gradient $\mathbf{F} = \nabla \mathbf{y}$. (a) Reference configuration; (b) current configuration.

2.1 Geometric interpretation of the deformation gradient

We now explore the geometric interpretation of deformation gradient. For simplicity, we first consider a homogeneous *deformation* $\mathbf{y} : \Omega \to I\!R^3$ with *deformation gradient* $\nabla \mathbf{y} = \mathbf{F} \in I\!R^{3\times 3}$ and $\det(\mathbf{F}) > 0$. Note that the deformation gradient $\mathbf{F}$ is independent of the position $\mathbf{x}$.

1. The length of a material line. Let $\mathbf{p}_0, \mathbf{p}_1$ be two points in Ω, $L = |\mathbf{p}_1 - \mathbf{p}_0|$ the length of the line $\mathbf{p}_0\mathbf{p}_1$ in the reference configuration. After deformation, the deformed line is between $\mathbf{F}\mathbf{p}_0$ and $\mathbf{F}\mathbf{p}_1$, and the length of the deformed line is

$$L' = |\mathbf{F}(\mathbf{p}_1 - \mathbf{p}_0)| = [(\mathbf{p}_1 - \mathbf{p}_0) \cdot \mathbf{F}^T \mathbf{F}(\mathbf{p}_1 - \mathbf{p}_0)]^{1/2}, \tag{5}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is called the CAUCHY-GREEN strain tensor.

2. The area of a material surface. Let $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ be three points in Ω, $A = \frac{1}{2}|(\mathbf{p}_2 - \mathbf{p}_0) \wedge (\mathbf{p}_1 - \mathbf{p}_0)| \neq 0$ the area of the triangle $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$. After deformation, the deformed triangle have $\mathbf{F}\mathbf{p}_0, \mathbf{F}\mathbf{p}_1, \mathbf{F}\mathbf{p}_2$ as vertices and the area of the deformed triangle is

$$A' = |\mathbf{F}(\mathbf{p}_2 - \mathbf{p}_0) \wedge \mathbf{F}(\mathbf{p}_1 - \mathbf{p}_0)| = \det(\mathbf{F})[\mathbf{q} \cdot \mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{q}]^{1/2}. \tag{6}$$

where $\mathbf{q} = (\mathbf{p}_2 - \mathbf{p}_0) \wedge (\mathbf{p}_1 - \mathbf{p}_0)$.

To show the identity (6), we consider three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in I\!R^3$. An use of (8) yields

$$\mathbf{F}\mathbf{a} \cdot (\mathbf{F}\mathbf{b} \wedge \mathbf{F}\mathbf{c}) = \det(\mathbf{F})\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) \Rightarrow \mathbf{a} \cdot \mathbf{F}^T(\mathbf{F}\mathbf{b} \wedge \mathbf{F}\mathbf{c}) = \det(\mathbf{F})\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}).$$

Since $\mathbf{a}$ is arbitrary, we conclude

$$\mathbf{F}^T(\mathbf{F}\mathbf{b} \wedge \mathbf{F}\mathbf{c}) = \det(\mathbf{F})(\mathbf{b} \wedge \mathbf{c}) \Rightarrow \mathbf{F}\mathbf{b} \wedge \mathbf{F}\mathbf{c} = \det(\mathbf{F})\mathbf{F}^{-T}(\mathbf{b} \wedge \mathbf{c}). \tag{7}$$

3. The volume of a material volume. Let $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ be four points in Ω, $V = \frac{1}{2}|(\mathbf{p}_3 - \mathbf{p}_0) \cdot [(\mathbf{p}_2 - \mathbf{p}_0) \wedge (\mathbf{p}_1 - \mathbf{p}_0)]| \neq 0$ the volume of the tetrahedron with $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ as vertexes. After deformation, the deformed tetrahedron have $\mathbf{F}\mathbf{p}_0, \mathbf{F}\mathbf{p}_1, \mathbf{F}\mathbf{p}_2, \mathbf{F}\mathbf{p}_3$ as vertexes and the volume of the deformed tetrahedron is

$$V' = \frac{1}{2}|(\mathbf{F}\mathbf{p}_3 - \mathbf{F}\mathbf{p}_0) \cdot [(\mathbf{F}(\mathbf{p}_1 - \mathbf{p}_0) \wedge \mathbf{F}(\mathbf{p}_2 - \mathbf{p}_0)]| = \det(\mathbf{F})V. \tag{8}$$

2.2 Small strain and linearization

Assume the deformation gradient is small: $\mathbf{F} = \mathbf{I} + \varepsilon\mathbf{H}$ with $\varepsilon << 1$ and $\mathbf{I}$ being the identity matrix. Let

$$\mathbf{E} = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \mathbf{I} = \varepsilon\frac{1}{2}(\mathbf{H} + \mathbf{H}^T).$$

Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in I\!R^3$ be three points in Ω. Without loss of generality we choose $\mathbf{p}_0 = 0$ be the origin.

♦ Claim: Show the following identities (9), (10) and (11). (Hints: Taylor expansion with respect to the small number ε)

1. The change of length per unit lenght (in the reference configuration), to the leading order, is given by

$$\epsilon_{11} = \frac{L' - L}{L} = \hat{\mathbf{p}}_1 \cdot \mathbf{E}\hat{\mathbf{p}}_1 + o(\varepsilon),$$

where ϵ_{11} is called the normal strain along direction $\hat{\mathbf{p}}_1 = \mathbf{p}_1/|\mathbf{p}_1|$. That is,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[(\mathbf{p}_1 \cdot \mathbf{F}^T\mathbf{F}\mathbf{p}_1)^{1/2} - |\mathbf{p}_1|] = \frac{1}{2}\mathbf{p}_1 \cdot (\mathbf{H} + \mathbf{H}^T)\mathbf{p}_1/|\mathbf{p}_1|. \tag{9}$$

2. The change of volume per unit volume (in the reference configuration), to the leading order, is given by

$$\frac{V' - V}{V} = \mathrm{Tr}(\mathbf{E}) + o(\varepsilon).$$

That is,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\det(\mathbf{F}) - 1) = \mathrm{Tr}(\mathbf{H}). \tag{10}$$

3. The change of area per unit area (in the reference configuration), to the leading order, is given by

$$\frac{A'-A}{A} = \mathrm{Tr}(\mathbf{E}) - \hat{\mathbf{q}}\cdot\mathbf{E}\hat{\mathbf{q}} + o(\varepsilon) \quad \text{where} \quad \mathbf{q} = \mathbf{p}_2 \wedge \mathbf{p}_1 \neq 0, \quad \hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|.$$

That is,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[|\mathbf{F}\mathbf{p}_2 \wedge \mathbf{F}\mathbf{p}_1| - |\mathbf{p}_2\wedge\mathbf{p}_1|] = \mathrm{Tr}(\mathbf{H})|\mathbf{q}| - \frac{1}{2}\mathbf{q}\cdot(\mathbf{H}+\mathbf{H}^T)\mathbf{q}/|\mathbf{q}|, \tag{11}$$

where we have noticed that

$$\mathbf{F}^{-1} = (\mathbf{I}+\varepsilon\mathbf{H})^{-1} \approx \mathbf{I} - \varepsilon\mathbf{H}. \tag{12}$$

2.3 Transformations of a deformation gradient

Consider a homogeneous deformation with $\mathbf{y} = \mathbf{F}\mathbf{x}$ and $\mathbf{F} \in I\!R^{3\times 3}$. For the reference configuration we choose the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$; for the current configuration we choose the orthornormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. In these bases, we have

$$\mathbf{y} = y_p\hat{\mathbf{e}}_p, \qquad \mathbf{x} = x_i\mathbf{e}_i.$$

Let $F_{pi} = \frac{\partial y_p}{\partial x_i}$ be the numerical matrix of the deformation gradient. Then the tensor

$$\mathbf{F} = \nabla\mathbf{y} = F_{pi}\hat{\mathbf{e}}_p \otimes \mathbf{e}_i.$$

1. Passive viewpoint. We have a change of the bases:

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \to \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}, \qquad \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\} \to \{\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3\}.$$

We assume the new bases are orthonormal as well. In terms of the orginal bases, the new bases can be expressed as

$$\mathbf{e}'_i = H_{ij}\mathbf{e}_j, \qquad \hat{\mathbf{e}}'_p = Q_{pq}\hat{\mathbf{e}}_q.$$

♦ Claim:

$$\begin{aligned} H_{ij} &= \mathbf{e}'_i\cdot\mathbf{e}_j, \qquad H_{ik}H_{jk} = \delta_{ij} \text{ or } \mathbf{H}\mathbf{H}^T = \mathbf{I}; \\ Q_{pq} &= \hat{\mathbf{e}}'_p\cdot\hat{\mathbf{e}}_q, \qquad Q_{pr}Q_{qr} = \delta_{pq} \text{ or } \mathbf{Q}\mathbf{Q}^T = \mathbf{I}. \end{aligned}$$

In these new bases, the *same* tensor $\mathbf{F}$ admits the following decompositions:

$$\begin{aligned} \mathbf{F} &= F'_{pi}\hat{\mathbf{e}}'_p\otimes\mathbf{e}'_i = F'_{pi}Q_{pq}\hat{\mathbf{e}}_q\otimes H_{ij}\mathbf{e}_j \\ &= F'_{pi}Q_{pq}H_{ij}\hat{\mathbf{e}}_q\otimes\mathbf{e}_j = F_{pi}\hat{\mathbf{e}}_p\otimes\mathbf{e}_i. \end{aligned} \tag{13}$$

Since the representation of a tensor in a basis is unique, by (13) and $\mathbf{Q}\mathbf{Q}^T = \mathbf{H}\mathbf{H}^T = \mathbf{I}$ we obtain

$$F'_{pi}Q_{pq}H_{ij} = F_{qj} \quad \text{or} \quad F'_{pi} = Q_{pq}F_{qk}H_{ik}. \tag{14}$$

2. Active viewpoint. Assume $\mathbf{H}, \mathbf{Q} \in So(3)$ be two rigid rotation matrix. Here we consider the following composition of mappings:

$$\begin{aligned}
\mathbf{x}' = \mathbf{H}\mathbf{x} \Leftrightarrow \mathbf{x} = \mathbf{H}^T\mathbf{x}' \quad &\longleftrightarrow \quad \text{rotate the reference configuration by } \mathbf{H}, \\
\mathbf{y}' = \mathbf{Q}\mathbf{y} \Leftrightarrow \mathbf{y} = \mathbf{Q}^T\mathbf{y}' \quad &\longleftrightarrow \quad \text{rotate the current configuration by } \mathbf{Q}.
\end{aligned}$$

Then with respect to the old deformation $\Omega \ni \mathbf{x} \mapsto \mathbf{y}(\mathbf{x})$, we consider the new deformation given by

$$\Omega' \ni \mathbf{x}' \mapsto \mathbf{y}'(\mathbf{x}') := \mathbf{Q}\mathbf{y}(\mathbf{x}) = \mathbf{Q}\mathbf{y}(\mathbf{H}^T\mathbf{x}').$$

If since the old deformation is homogeneous, $\mathbf{y} = \mathbf{F}\mathbf{x}$, we have

$$\mathbf{F}' = \nabla_{\mathbf{x}'}\mathbf{y}'(\mathbf{x}') = \mathbf{Q}\mathbf{F}\mathbf{H}^T \quad \text{or} \quad \mathbf{F}' = F'_{pi}\hat{\mathbf{e}}_p \otimes \mathbf{e}_i = Q_{pq}F_{qk}H_{ik}\hat{\mathbf{e}}_p \otimes \mathbf{e}_i, \tag{15}$$

which implies

$$F'_{pi} = Q_{pq}F_{qk}H_{ik}. \tag{16}$$

Note that the relation between the new matrix F'_{pi} and the old matrix F_{pi} of the deformation gradients are exactly the same for the two viewpoints, see (14) and (16).

♦ Consider a vector $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$ in 2D and a rotation matrix

$$\mathbf{Q} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Draw two diagrams that show the differences between the passive viewpoint and the active viewpoint. (Hints: In one diagram, plot $\mathbf{v}$, its components in the bases $\{\mathbf{e}_1, \mathbf{e}_2\}$, $\{\mathbf{Q}\mathbf{e}_1, \mathbf{Q}\mathbf{e}_2\}$; in the other, plot $\mathbf{v}$ and its rotation $\mathbf{Q}\mathbf{v}$, and their components in the bases $\{\mathbf{e}_1, \mathbf{e}_2\}$.)

2.4 Compatibility conditions

Let Ω be a simply-connected open domain in $I\!R^3$, Ω_1, Ω_2 two disjoint open set, and Γ an interface such that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$.

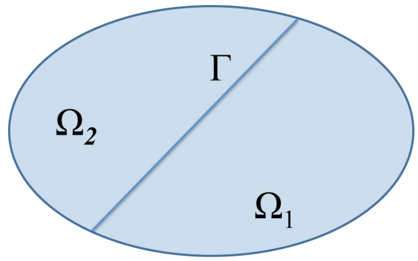

Figure 2: Compatibility conditions.

1. Compatibility conditions for a vector field $\mathbf{v} : \Omega \to I\!R^3$ to be a gradient of a scalar potential.
   (a) Assume $\mathbf{v} : \Omega \to I\!R^3$ is differentiable.
   - A necessary condition (curl-free condition). If $\mathbf{v} = \nabla\varphi$, then

     $$\nabla \wedge \mathbf{v} = \nabla \wedge \nabla\varphi = \mathcal{E}_{ijk}\varphi_{,jk} = 0,$$

     That is,

     $$\nabla \wedge \mathbf{v} = 0 \qquad \text{on } \ \Omega. \tag{17}$$

   - A sufficient condition. It turns out that equation (17) is also sufficient. To see this, let us define

     $$\varphi(\mathbf{x}, \gamma) = \int_{\gamma(\mathbf{x}_0, \mathbf{x}')} v_i(\mathbf{x})\mathbf{e}_i \cdot (dx'_j \mathbf{e}_j) = \int_{\gamma(\mathbf{x}_0, \mathbf{x})} v_i(\mathbf{x}')dx'_i.$$

     The above integral in general depends on the integration path γ. However, if equation (17) is satisfied, by the Stoke's theorem we have that for any closed contour γ_c,

     $$\oint_{\gamma_c} V_i(x')dx'_i = \int_{\Omega} \mathbf{n} \cdot (\nabla \wedge \mathbf{v}) = 0.$$

     Assume there is a second path γ' between x_0 and x_1. By the above equation we have

     $$\int_{\gamma' - \gamma} v_i(\mathbf{x}')dx'_i = \oint_{\gamma_c} v_i(\mathbf{x}')dx'_i = 0, \quad \text{i.e.,} \quad \varphi(\mathbf{x}, \gamma) = \varphi(\mathbf{x}, \gamma') =: \varphi(\mathbf{x}).$$

     Further, we verify that, indeed,

     $$\nabla\varphi = \mathbf{v} \qquad \text{on } \ \Omega.$$

     Henceforth, we conclude that the curl-free condition (17) is a necessary and sufficient condition for a differentiable vector field $\mathbf{v} : \Omega \to I\!R^3$ to be a gradient of a scalar potential $\varphi : \Omega \to I\!R$.

Remark: In higher dimension, the necessary and sufficient condition is that the antisymmetrization of $\nabla \mathbf{v}$ vanishes:

$$\frac{1}{2}(\nabla \mathbf{v} - (\nabla \mathbf{v})^T) = 0 \qquad \text{on } \Omega.$$

(b) Assume that $\mathbf{v} : \Omega \to I\!R^3$ is differentiable on $\Omega_1 \cup \Omega_2$ but discontinuous across Γ.

- A necessary and sufficient condition for $\mathbf{v}$ being a gradient field:

$$\nabla \wedge \mathbf{v} = 0 \qquad \text{on } \Omega_1 \cup \Omega_2, \qquad [\![\mathbf{v}]\!] \cdot \mathbf{t} = 0 \text{ or } [\![\mathbf{v}]\!] = c(\mathbf{x})\mathbf{n} \quad \text{on } \Gamma, \tag{18}$$

where $c : \Gamma \to I\!R$ is a scalar function on Γ, $\mathbf{n}$ is the unit normal on Γ and $\mathbf{t}$ is any unit vector parallel to the surface, i.e., $\mathbf{t} \cdot \mathbf{n} = 0$.

♦ Claim: Assume the vector field $\mathbf{v} = \nabla \varphi$, where $\varphi : \Omega \to I\!R$ is continuous. If $\mathbf{v}$ is discontinuous across Γ, then

$$[\![\mathbf{v}]\!] \cdot \mathbf{t} = 0 \text{ or } [\![\mathbf{v}]\!] = c(\mathbf{x})\mathbf{n} \quad \text{on } \Gamma.$$

(Hint: Use equation (17) and the Stoke's theorem.)

2. Compatibility conditions for a tensor field $\mathbf{F} : \Omega \to I\!R^3$ to be a gradient of a vector field, e.g., a deformation gradient.

(a) Assume $\mathbf{F} : \Omega \to I\!R^{3\times 3}$ is differentiable.

- A necessary condition. If $\mathbf{F} = \nabla \mathbf{y}$, we have

$$\mathcal{E}_{ijk} F_{pj,k} = \mathcal{E}_{ijk} y_{p,jk} = 0 \qquad \forall\, i = 1,2,3 \,\&\, p = 1,2,3 \qquad \text{on } \Omega. \tag{19}$$

- A sufficient condition. First, we may write $\mathbf{F} = F_{pi} \hat{\mathbf{e}}_p \otimes \mathbf{e}_i$ and define

$$\mathbf{v}_p = \mathbf{F}^T(\hat{\mathbf{e}}_p) = F_{pi}\mathbf{e}_i : \Omega \to I\!R^3.$$

From (19), we have

$$\nabla \wedge \mathbf{v}_p = \mathcal{E}_{ijk}(V_p \cdot \mathbf{e}_j)_{,k} = \mathcal{E}_{ijk} F_{pi,k} = 0.$$

Applying (17) to each of the vector field V_p for $p = 1,2,3$, we see that (19) is also sufficient and

$$y_p(\mathbf{x}) = \int_{\gamma(\mathbf{x}_0,\mathbf{x})} \mathbf{v}_p(\mathbf{x}') \cdot dx'_j \mathbf{e}_j = \int_{\gamma(\mathbf{x}_0,\mathbf{x})} F_{pi}(\mathbf{x}') dx'_i.$$

(b) Assume $\mathbf{F} : \Omega \to I\!R^{3\times 3}$ is differentiable on $\Omega_1 \cup \Omega_2$ but discontinuous across Γ.

- A necessary and sufficient condition

$$\nabla \wedge \mathbf{F}^T(\hat{\mathbf{e}}_p) = 0 \quad \text{or (19)} \;\; \text{on } \Omega_1 \cup \Omega_2, \qquad [\![\mathbf{F}]\!]\mathbf{t} = 0 \text{ or } [\![\mathbf{F}]\!] = \mathbf{a} \otimes \mathbf{n} \quad \text{on } \Gamma. \tag{20}$$

3. Compatibility conditions for a symmetric tensor field $\mathbf{E} : \Omega \to I\!R^{3\times 3}_{sym}$ to be the symmetrized gradient of a vector field

$$\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad \mathbf{u} : \Omega \to I\!R^3. \tag{21}$$

(a) Assume $\mathbf{E} : \Omega \to I\!R^{3\times 3}_{sym}$ is of C^2

$$\mathbf{E} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} u_{x,x} & \frac{1}{2}(u_{x,y} + u_{y,x}) & \frac{1}{2}(u_{x,z} + u_{z,x}) \\ \frac{1}{2}(u_{x,y} + u_{y,x}) & u_{y,y} & \frac{1}{2}(u_{y,z} + u_{z,y}) \\ \frac{1}{2}(u_{x,z} + u_{z,x}) & \frac{1}{2}(u_{y,z} + u_{z,y}) & u_{z,z} \end{bmatrix}.$$

Note that $\mathbf{E} = \mathbf{E}^T$.

- A necessary condition:

$$\begin{aligned} \frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x}\frac{\partial^2 u_x}{\partial y \partial z} \\ &= \frac{\partial}{\partial x}\left[\frac{\partial \epsilon_{zy}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial x}\right] \\ \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} &= \frac{1}{2}\left(\frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial^3 u_y}{\partial y \partial x^2}\right) \\ &= \frac{1}{2}\left[\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2}\right] \end{aligned} \tag{22}$$

  Similarly, we have (♦ (2pt) 17. Complete the right hand side of the following equations):

$$\begin{aligned} \frac{\partial^2 \epsilon_{yy}}{\partial x \partial z} &= \\ \frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} &= \\ \frac{\partial^2 \epsilon_{xz}}{\partial x \partial z} &= \\ \frac{\partial^2 \epsilon_{yz}}{\partial y \partial z} &= \end{aligned} \tag{23}$$

- A sufficient condition: it can be shown that the above conditions are also sufficient for the existence of a displacement such that equation (21) admits a solution $\mathbf{u}$ for a given strain field that satisfies the compatibility equations (22)-(23). This was first shown by VOLTERRA (1907).

(b) Assume $\mathbf{E} : \Omega \to I\!R^{3\times 3}$ is of C^2 on $\Omega_1 \cup \Omega_2$ but discontinuous across Γ.

- A necessary and sufficient condition for the existence of a continuous displacement $\mathbf{u} : \Omega \to I\!R^3$ such that (21) is satisfied is

$$\begin{aligned} &\mathbf{E} \quad \text{satisfies the compatibility equations (22)-(23)} \quad \text{on } \Omega_1 \cup \Omega_2, \quad \text{and} \\ &[\![\mathbf{E}]\!] = \frac{1}{2}(\mathbf{a} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{a}) \quad \text{on } \Gamma, \end{aligned} \tag{24}$$

  where $\mathbf{a} : \Gamma \to I\!R^3$ is a vector field on Γ, and the jump condition is called HADAMARD's jump conditions.

4. Assume $\mathbf{v} : \Omega \to I\!R^3$ is differentiable on $\Omega_1 \cup \Omega_2$ but discontinuous across Γ. Let $b : \Omega \to I\!R$ is continuous and bounded. By

$$\text{div}\mathbf{v} = b \qquad \text{on } \ \Omega, \tag{25}$$

we mean the following

$$\text{div}\mathbf{v} = b \qquad \text{on } \ \Omega_1 \cup \Omega_2 \tag{26}$$

and

$$[\![\mathbf{v}]\!] \cdot \mathbf{n} = 0 \qquad \text{on } \ \Gamma. \tag{27}$$

Note that $\mathbf{v}$ is discontinuous across Γ, $\mathbf{v}$ is not differentiable and so div$\mathbf{v}$ is not literally well-defined on Γ.

5. Assume $\sigma : \Omega \to I\!R^{3\times 3}$ is of C^1 on $\Omega_1 \cup \Omega_2$ but discontinuous across Γ. Let $\mathbf{b} : \Omega \to I\!R^3$ be continuous and bounded. By

$$\text{div}\sigma = \mathbf{b} \qquad \text{on } \ \Omega, \tag{28}$$

we mean the following

$$\text{div}\sigma = \mathbf{b} \qquad \text{on } \ \Omega_1 \cup \Omega_2 \tag{29}$$

and

$$[\![\sigma]\!]\mathbf{n} = 0 \qquad \text{on } \ \Gamma. \tag{30}$$

To see the motivation behind the definition, e.g., (25)-(27), we assume the divergence theorem is valid even though $\mathbf{v}$ and σ is discontinuous across Γ. That is, we have for any subdomain $D \subset \Omega$,

$$\int_D \text{div}\mathbf{v} = \int_D \mathbf{b} = \int_{\partial D} \mathbf{V} \cdot \mathbf{n}.$$

Choosing D to be a domain as shown in the figure, we obtain (27).

3 Concepts of Stress

3.1 Cauchy stress and balance laws

Cauchy hypothesis: On any interface in the material body, there exists contact force between the two parts separated by the interface. This contact force on an infinitesimal area da can be expressed as

$$\Sigma(\mathbf{x}, \mathbf{n})da,$$

where $\mathbf{x}$ is the position of the area element da and $\mathbf{n}$ is the unit normal on da, see the following figure.

Cauchy theorem:

Theorem 1 *The contact force can be expressed as*

$$\Sigma(\mathbf{x}, \mathbf{n})da = \sigma(\mathbf{x})\mathbf{n}da,$$

where $\sigma : \Omega \to \mathbb{R}^{3\times 3}$ *is a tensor field.*

The meaning of this theorem is that the dependence of $\Sigma(\mathbf{x}, \mathbf{n})$ on the unit normal $\mathbf{n}$ is linear.

Balance laws and their implications:

First, let us assume the following theorem.

Theorem 2 (Localization theorem) *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded domain. Assume* f *is a continuous function.*

1. *If*

$$\int_\Omega fg = 0 \qquad \forall \text{ continuous functions } g : \Omega \to \mathbb{R}, \text{ or} \tag{31}$$

2. *if*

$$\int_D f = 0 \qquad \forall \text{ subdomains } D \subset \Omega, \tag{32}$$

then

$$f = 0.$$

Proof: We prove it by contradiction. Assume $f \neq 0$ at $\mathbf{x}_0 \in \Omega$. By continuity of f, we know $f > 0$ on a neighborhood U of x_0. Then equation (31) is violated for the choice of g which is a continuous positive function and vanishes outside U or the subdomain $D = U$.

■

Let Ω be the material body. Assume the body is subject to a body force $\mathbf{b} : \Omega \to \mathbb{R}^3$ and traction $\mathbf{t} : \partial\Omega \to \mathbb{R}^3$ and is in equilibrium. Consider a part of the body $D \subset \Omega$ with $\partial D \cap \partial\Omega = \emptyset$. Then the balance of linear momentum implies

$$\int_{\partial D} \sigma \mathbf{n} da + \int_D \mathbf{b} = 0 \implies \int_D (\mathrm{div}\sigma + \mathbf{b}) = 0; \tag{33}$$

the balance of angular momentum implies

$$\int_{\partial D} \mathbf{x} \wedge (\sigma \mathbf{n}) da + \int_D \mathbf{x} \wedge \mathbf{b} = 0 \implies \int_D [\mathbf{x} \wedge (\mathrm{div}\sigma + \mathbf{b}) + \mathcal{E}_{ijk}\sigma_{kj}] = 0. \tag{34}$$

From (33), (34), and part 2) of Theorem 2, we arrive at the equilibrium equation

$$\mathrm{div}\sigma + \mathbf{b} = 0 \qquad \text{and} \qquad \sigma = \sigma^T \qquad \text{on } \Omega. \tag{35}$$

Thus, the CAUCHY stress is *symmetric*.

3.2 Implications of the first and second laws of thermodynamics

.

We begin from the following hypothesis

H1. Assume the homogeneous material body Ω has a deformation $\mathbf{y} : \Omega \to \mathbb{R}^3$. Let $\mathbf{F} = \nabla \mathbf{y}$ be the deformation gradient. At a constant temperature, the internal energy of the body can be expressed as

$$U(\mathbf{y}) = \int_\Omega W(\nabla \mathbf{y}). \tag{36}$$

Assume that the body is subject to a body force $\mathbf{b} : \Omega \to \mathbb{R}^3$, a traction $\mathbf{t} : \Gamma_N \to \mathbb{R}^3$, and $\mathbf{y} = \mathbf{x}$ on Γ_D, where $\Gamma_N \subset \partial\Omega$ and $\Gamma_D = \partial\Omega \subset \Gamma_N$. In equilibrium, the deformation of the body is $\mathbf{y} : \Omega \to \mathbb{R}^3$. Now let us imagine the deformation is perturbed infinitesimally with new deformation $\mathbf{y}_\varepsilon = \mathbf{y} + \varepsilon \mathbf{z}$ with $\mathbf{z} = 0$ on Γ_D. By the first law of thermodynamics, within the leading order we have

$$\begin{gathered} U(\mathbf{y}_\varepsilon) - U(\mathbf{y}) = \varepsilon \int_\Omega \mathbf{b} \cdot \mathbf{z} + \varepsilon \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{z} \;\Rightarrow \\ \int_\Omega [\nabla \mathbf{z} \cdot \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{y}) - \mathbf{b} \cdot \mathbf{z}] - \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{z} = 0. \end{gathered} \tag{37}$$

Note that an immediate implication of the above formula is that the body is not "moving", i.e., in equilibrium, since we have not included the kinetic energy in (37). Define $\mathbf{S} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ as the first order derivative of the scalar function $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} \quad \text{or} \quad S_{pi} = \frac{\partial W}{\partial F_{pi}}. \tag{38}$$

For the deformation gradient $\nabla \mathbf{y} : \Omega \to \mathbb{R}^{3\times 3}$, we evaluate the function $\mathbf{S}$ and obtain the Piola-Kirchhoff stress on point $\mathbf{x} \in \Omega$ as

$$\mathbf{S}(\nabla \mathbf{y}(\mathbf{x})).$$

Further, by an use of the divergence theorem, equation (37) can be rewritten as

$$\int_\Omega \mathbf{z} \cdot [-\mathrm{div}\mathbf{S}(\nabla\mathbf{y}) - \mathbf{b}] + \int_{\Gamma_N} [\mathbf{S}(\nabla\mathbf{y})\mathbf{n} - \mathbf{t}] \cdot \mathbf{z} = 0. \tag{39}$$

If we a priori assume that $[-\mathrm{div}\mathbf{S}(\nabla\mathbf{y}) - \mathbf{b}]$ and $[\mathbf{S}(\nabla\mathbf{y})\mathbf{n} - \mathbf{t}]$ are continuous functions on Ω and Γ_N, respectively, by the part 1) of the localization theorem 2, we conclude the equilibrium equation and boundary conditions

$$\begin{cases} \mathrm{div}(\mathbf{S}(\nabla\mathbf{y})) = -\mathbf{b} & \text{on } \Omega, \\ \mathbf{S}(\nabla\mathbf{y})\mathbf{n} = \mathbf{t} & \text{on } \Gamma_N, \\ \mathbf{y} = \mathbf{x} & \text{on } \Gamma_D. \end{cases} \tag{40}$$

♦ (6pt) 18. By index notation, carry out the derivation of (39) in details.

The second law of thermodynamics implies that when the body is in equilibrium, the GIBBS free energy

$$G(\mathbf{y}') := \int_\Omega [W(\nabla\mathbf{y}') - \mathbf{b}\cdot\mathbf{y}'] - \int_{\Gamma_N} \mathbf{y}'\cdot\mathbf{t}. \tag{41}$$

is minimized among all admissible deformation satisfying $\mathbf{y}' = \mathbf{x}$ on Γ_D.

♦ (6pt) 19. Assume that $\mathbf{y}:\Omega\to I\!R^3$ is a minimizer of the energy functional $G(\mathbf{y}')$:

$$G(\mathbf{y}) = \min_{\mathbf{y}'} G(\mathbf{y}'). \tag{42}$$

Consider a small perturbation of $\mathbf{y}$, i.e., $\mathbf{y}_\varepsilon = \mathbf{y}+\varepsilon\mathbf{z}$ with $\mathbf{z}=0$ on Γ_D. Show that the minimizer $\mathbf{y}$ necessarily satisfies (37), and hence (40) if the relevant fields are a priori assumed to be continuous. The PDE (40) is called the EULER-LAGRANGE equation of the variational principle (42).

Pialo-Kirchhoff stress vs Cauchy stress:

♦ (2pt) 20. Explain in words the difference between PIALO-KIRCHHOFF stress and CAUCHY stress. Draw a diagram if necessary.

4 Constitutive Laws

4.1 Hooke's law

To complete our theory of elasticity, we need an additional equation that describes the properties of the material. Such a equation or a "law" is called the constitutive law (of the material). The classic Hooke's law states that the stress depends on the strain linearly

$$\sigma = \mathbf{C}\mathbf{E}, \tag{43}$$

where σ is the (Cauchy) stress tensor, $\mathbf{E}$ is the symmetrized strain, and $\mathbf{C} : \mathbb{R}^{3\times 3}_{sym} \to \mathbb{R}^{3\times 3}_{sym}$ is called stiffness tensor.

A constitutive law describes the behaviors of a material. At the continuum level, a constitutive law cannot be "derived" from the laws of physics. Instead, we should think of the constitutive law is obtained by experimental measurements. Nevertheless, the symmetries of the underlying material and the physical space and the laws of physics place nontrivial restrictions on the forms of the stiffness tensor $\mathbf{C}$. Though these restrictions are realized long time ago, a systematic derivation of these restrictions mature after the development of the general framework of continuum mechanics. The modern viewpoint on the Hooke's law (43) is that it is the first-order approximation of the actual behavior of the material, and hence valid only for *small* strain. For materials, e.g., rubber that remain to be elastic for large strain, the Hooke's law fails completely.

4.2 Frame indifference and material symmetry

The modern viewpoint of the constitutive laws, instead of specifying the stress-strain relation, specifies the stored energy function $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$, see (36). Consider a material body Ω with a homogenous deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$. From (36), we see that the stored energy in the material is

$$U = W(\mathbf{F})|\Omega|,$$

where $|\Omega|$ is the volume of Ω. Now let us consider a new deformation $\mathbf{y}' = \mathbf{Q}\mathbf{y}(\mathbf{x})$ with $\mathbf{Q} \in So(3)$, i.e., the new configuration of the material body is an rigid rotation of $\mathbf{y}(\Omega)$. The principle of relativity (Galilean invariance) tells us the stored energy in the material shall be the same as before, i.e., (c.f. (15))

$$W(\mathbf{F}) = W(\mathbf{Q}\mathbf{F}) \qquad \forall\, \mathbf{F} \in \mathbb{R}^{3\times 3} \;\&\; \mathbf{Q} \in So(3). \tag{44}$$

The above equation is referred as the principle of material frame indifference, which is in fact valid in a much broader context.

Material symmetry. Without loss of generality we may assume Ω is a perfect sphere. Before applying the deformation $\mathbf{y} : \Omega \to \mathbf{y}(\Omega)$, we transform the reference configuration Ω to $\Omega' = \mathbf{R}\Omega$. If the new reference configuration Ω' is "exactly" the same as Ω, then we will have

$$W(\mathbf{F}) = W(\mathbf{F}\mathbf{R}^T) \qquad \forall\, \mathbf{F} \in \mathbb{R}^{3\times 3} \;\&\; \mathbf{R} \in \mathcal{G}, \tag{45}$$

where $\mathcal{G}$ is called the symmetry group of the material. If the material is a crystal with a Bravais lattice

$$\mathcal{L} := \{\sum_{i=1}^{3} \nu_i \mathbf{a}_i : \ \nu_i \text{ are integers and } \mathbf{a}_i \text{ are lattice vectors}\},$$

then $\mathcal{G}$ is the point group $\mathcal{L}$, which is the collection of orthogonal matrices $\mathbf{R}$ such that

$$\mathcal{L} = \mathbf{R}\mathcal{L}.$$

Multiplying the above equation by $\mathbf{R}^T$, we see that if $\mathbf{R} \in \mathcal{G}$, then $\mathbf{R}^T \in \mathcal{G}$.

Note that the frame-indifference (44) is material-independent, but equation (45) depends on the material through the group $\mathcal{G}$.

A material is *isotropic* if $\mathcal{G} \supset So(3)$.

♦ (1pt) 21. Read the article http://en.wikipedia.org/wiki/Crystal_system and write below the names of seven crystal systems.

4.3 Approximation of the stored energy functions

Clearly, the stored energy function $W : I\!R^{3\times 3} \to I\!R$ depends on the reference configuration. We now make a choice of a reference configuration. Assume the body is free of body force and traction. From the variational principle (42), we see that the deformation gradient shall take the matrix that minimizes the energy function

$$W(\mathbf{F}^*) = \min_{\mathbf{F}\in I\!R^{3\times 3}, \det(\mathbf{F})>0} W(\mathbf{F}).$$

We assume the minimizer exists. If a minimizer does not exist, our solid won't be stable by the second law of thermodynamics. If the minimizer $\mathbf{F}^* \neq \mathbf{I}$ ($\mathbf{I}$ is the identity matrix), we redefine our energy function as $\overline{W}(\mathbf{F}) = W(\mathbf{F}^*\mathbf{F})$. So, without loss of generality, we assume $\mathbf{F}^* = \mathbf{I}$, i.e., $\mathbf{y} = \mathbf{x}$ is the minimizing deformation when the material is free of body force and surface traction. Upon a Taylor expansion of the function W at a neighborhood of the point $\mathbf{I}$, we have

$$W(\mathbf{I} + \varepsilon\mathbf{H}) = W(\mathbf{I}) + \varepsilon\mathbf{H} \cdot \frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) + \varepsilon^2 \frac{1}{2}\mathbf{H} \cdot \mathbf{C}\mathbf{H} + o(\varepsilon^2), \tag{46}$$

where

$$\mathbf{C} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}}(\mathbf{I}) \quad \text{or} \quad C_{piqj} = \frac{\partial^2 W}{\partial F_{pi} \partial F_{qj}}(\mathbf{I}). \tag{47}$$

♦ 22. Answer the following questions:

(i) (2pt) Explain why

$$\frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) = 0 \qquad \text{and} \qquad \mathbf{C} \text{ is positive semi-definite, i.e., } \mathbf{H} \cdot \mathbf{C}\mathbf{H} \geq 0 \quad \forall\, \mathbf{H} \in I\!R^{3\times 3}.$$

(ii) (3pt) Read the article http://en.wikipedia.org/wiki/Matrix_exponential and show that for any skew-symmetric matrix $\mathbf{W}$, $\mathbf{Q} = \exp(\mathbf{W})$ is a rigid rotation, i.e.,

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I} \qquad \text{and} \qquad \det(\mathbf{Q}) = 1.$$

(iii) (2pt) Using the frame-indifference equation (44), show that the stiffness tensor $\mathbf{C}$ defined by (47) satisfies (Hint: consider rigid rotation $\mathbf{Q} = \exp(\varepsilon\mathbf{W})$ and (46))

$$\mathbf{CW} = 0 \quad \forall\, \mathbf{W}^T = -\mathbf{W} \quad \text{or} \quad C_{piqj} = C_{pijq}.$$

(iv) (2pt) Let $\mathcal{G}$ be the symmetry group of the material. By (45), (44) and (46), show that

$$\mathbf{H} \cdot \mathbf{CH} = (\mathbf{RHR}^T) \cdot \mathbf{C}(\mathbf{RHR}^T) \qquad \forall\, \mathbf{H} \in I\!R^{3\times 3}\ \&\ \mathbf{R} \in (\mathcal{G} \cap So(3)). \tag{48}$$

(Hint: first show that $W(\mathbf{I} + \varepsilon\mathbf{H}) = W(\mathbf{R}(\mathbf{I} + \varepsilon\mathbf{H})\mathbf{R}^T)$, and then use (46))

(v) (2pt) Assume the 180^0 rigid rotation around z-axis

$$\mathbf{R} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{G}.$$

By (48), show that

$$C_{1113} = C_{2213} = C_{1213} = C_{1123} = C_{2223} = C_{1223} = 0.$$

(vi) (2pt) If $\mathcal{G} = So(3)$, from similar calculations as in the last problem, one can show that the stiffness tensor must take the following form

$$C_{piqj} = \mu(\delta_{pq}\delta_{ij} + \delta_{pj}\delta_{iq}) + \lambda\delta_{pi}\delta_{qj}, \tag{49}$$

where μ and λ are called LAMÉ constants. Show that the stored energy function

$$W(\mathbf{F}) = W(\mathbf{I}) + \frac{1}{2}(\mathbf{F} - \mathbf{I}) \cdot \mathbf{C}(\mathbf{F} - \mathbf{I}), \tag{50}$$

with $\mathbf{C}$ given by (49) violates the frame-indifference requirement (44).

Is there a contradiction? Why? In fact, there is no quadratic energy function as in (50) can fulfill the frame-indifference requirement (44). This is why you may see statements such as "there exists no linear material in nature" or terminology such as "linearized elasticity".

5 Formulations of Linear (Linearized) Elasticity Problems

5.1 PDE formulation, Variational formulation, Weak formulation

Assume $\mathbf{y} = \mathbf{x} + \mathbf{u}$ with $\nabla\mathbf{u}$ being small. Neglect the higher order terms in the energy function expansion (46), we can write the stored energy function as in (50), where the stiffness tensor is positive semi-definite and satisfies

$$C_{piqj} = C_{qjpi} = C_{pijq}.$$

Based on stability requirement, we shall further assume that for some $c > 0$,

$$\mathbf{E} \cdot \mathbf{CE} \geq c|\mathbf{E}|^2 \qquad \forall\, \mathbf{E}^T = \mathbf{E}.$$

Collecting results from previous chapters, in particular, by (38), (40), (50) we have the PDE formulation of the linear elasticity problem

$$\begin{cases} \operatorname{div}[\mathbf{C}\nabla\mathbf{u}] = -\mathbf{b} & \text{on } \Omega, \\ (\mathbf{C}\nabla\mathbf{u})\mathbf{n} = \mathbf{t} & \text{on } \Gamma_N, \\ \mathbf{u} = 0 & \text{on } \Gamma_D. \end{cases} \tag{51}$$

By (38), (39), (50) we have the weak or integral formulation of the linear elasticity problem that for any $\mathbf{z} = 0$ on Γ_D,

$$\int_\Omega \mathbf{z} \cdot [-\operatorname{div}(\mathbf{C}\nabla\mathbf{u}) - \mathbf{b}] + \int_{\Gamma_N} [(\mathbf{C}\nabla\mathbf{u})\mathbf{n} - \mathbf{t}] \cdot \mathbf{z} = 0. \tag{52}$$

By (42), (39), (50) we have the variational formulation of the linear elasticity problem that $\mathbf{u}$ is the minimizer among all admissible $\mathbf{v}$ satisfying $\mathbf{v} = 0$ on Γ_D

$$G_l(\mathbf{u}) = \min_{\mathbf{v}} \Big\{ G_l(\mathbf{v}) := \int_\Omega [\frac{1}{2}\nabla\mathbf{v} \cdot \mathbf{C}\nabla\mathbf{v} - \mathbf{b}\cdot\mathbf{v}] - \int_{\Gamma_N} \mathbf{t}\cdot\mathbf{u} \Big\}. \tag{53}$$

The equivalence between the PDE formulation (51) and the weak formulation (52) follows from part 2) of the localization Theorem 2 if all relevant quantities are a priori assumed to be continuous.

♦ (5pt) 23. Show that if $\mathbf{u}$ is a solution (minimizer) of the variation problem (53), then $\mathbf{u}$ necessarily satisfies that for any $\mathbf{z} \in C^1$ with $\mathbf{z} = 0$ on Γ_D,

$$\int_\Omega [\nabla\mathbf{z} \cdot \mathbf{C}\nabla\mathbf{u} - \mathbf{z}\cdot\mathbf{b}] - \int_{\Gamma_N} \mathbf{t}\cdot\mathbf{z} = 0,$$

which, by the divergence theorem, implies (52), and hence (51).

5.2 Uniqueness of the solution

♦ (6pt) 24. Show that if $\mathbf{u}$ and $\mathbf{u}'$ are both minimizers of the variation problem (53), then $\mathbf{u}$ and $\mathbf{u}'$ necessarily satisfies

$$\int_\Omega |\nabla(\mathbf{u}-\mathbf{u}') + [\nabla(\mathbf{u}-\mathbf{u}')]^T|^2 = 0,$$

which implies

$$\frac{1}{2}\{\nabla(\mathbf{u}-\mathbf{u}') + [\nabla(\mathbf{u}-\mathbf{u}')]^T\} = 0 \qquad \text{on } \Omega. \tag{54}$$

♦ (3pt) 25. By (54), show that there is a skew-symmetric matrix $\mathbf{W} \in I\!R^{3\times 3}$ and a constant vector $\mathbf{c} \in I\!R^3$ such that

$$\mathbf{u}-\mathbf{u}' = \mathbf{W}\mathbf{x} + \mathbf{c} \qquad \text{on } \Omega.$$

<u>Well-posedness of a problem</u> means the following three properties

- Existence. The existence theorem is usually the most important question in a theory. However, a proof of the existence theorem is usually difficult. We will not address the existence problem in this course.
- Uniqueness. Note that, though the displacement may not be unique in linear elasticity, the strain and stress fields are unique.
- Stability.

5.3 Plane strain problem

In a plane strain problem, we consider an infinite cylindrical body $\Omega = D \times (-\infty, +\infty)$, where $D \subset I\!R^2$ is a two dimensional region on xy-plane. Consider the elasticity problem

$$\begin{cases} -\text{div}[\mathbf{C}\nabla\mathbf{u}] = b_x(x,y)\mathbf{e}_x + b_y(x,y)\mathbf{e}_y & \text{on } \Omega, \\ (\mathbf{C}\nabla\mathbf{u})\mathbf{n} = t_x(x,y)\mathbf{e}_x + t_y(x,y)\mathbf{e}_y & \text{on } \Gamma'_N \times (-\infty,+\infty), \\ \mathbf{u} = u_x^0(x,y)\mathbf{e}_x + u_y^0(x,y)\mathbf{e}_y & \text{on } \Gamma'_D \times (-\infty,+\infty), \end{cases} \tag{55}$$

where $b_x, b_y, t_x, t_y, u_x^0, u_y^0$ are given data and independent of z, $\mathbf{C}$ is independent of x, y, z, i.e., the material is homogeneous, Γ'_N, Γ'_D are a mutual disjoint subdivision of ∂D, see the following figure.

♦ (2pt) 26 (i). Show that if $\mathbf{u}(x,y,z)$ satisfies (55), then for any $z_0 \in I\!R$,

$$\mathbf{u}'(x,y,z) = \mathbf{u}(x,y,z+z_0) \tag{56}$$

satisfies (55) as well.

From the uniqueness theorem, by (56) we conclude that $\mathbf{u}'(x,y,z) = \mathbf{u}(x,y,z+z_0) = \mathbf{u}(x,y,z)$ (the arbitrary translation is ignored). Therefore, any solution to (55) is in fact independent of z. Thus,

$$\frac{\partial}{\partial z}\mathbf{u} = 0. \tag{57}$$

In another word,

$$\nabla\mathbf{u} = \begin{bmatrix} u_{x,x} & u_{x,y} & 0 \\ u_{y,x} & u_{y,y} & 0 \\ u_{z,x} & u_{z,y} & 0 \end{bmatrix}.$$

Let $\alpha, \beta, \alpha', \beta' \in \{1,2\}$ or $\{x,y\}$, i.e., the in-plane indices. By (57), the stress can be written as (summation over double indices α, β)

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} C_{11\alpha\beta}u_{\alpha,\beta} + C_{113\beta}u_{3,\beta} & C_{12\alpha\beta}u_{\alpha,\beta} + C_{123\beta}u_{3,\beta} & C_{13\alpha\beta}u_{\alpha,\beta} + C_{133\beta}u_{3,\beta} \\ C_{21\alpha\beta}u_{\alpha,\beta} + C_{213\beta}u_{3,\beta} & C_{22\alpha\beta}u_{\alpha,\beta} + C_{223\beta}u_{3,\beta} & C_{23\alpha\beta}u_{\alpha,\beta} + C_{233\beta}u_{3,\beta} \\ C_{31\alpha\beta}u_{\alpha,\beta} + C_{313\beta}u_{3,\beta} & C_{32\alpha\beta}u_{\alpha,\beta} + C_{323\beta}u_{3,\beta} & C_{33\alpha\beta}u_{\alpha,\beta} + C_{333\beta}u_{3,\beta} \end{bmatrix},$$

and the equations in (55) are

$$\begin{cases} C_{\alpha\beta\alpha'\beta'}u_{\alpha',\beta'\beta} + C_{\alpha\beta3\beta'}u_{3,\beta'\beta} = -b_\alpha & \text{on } D, \\ C_{3\beta\alpha'\beta'}u_{\alpha',\beta'\beta} + C_{3\beta3\beta'}u_{3,\beta'\beta} = 0 & \text{on } D, \\ C_{\alpha\beta\alpha'\beta'}u_{\alpha',\beta'}n_\beta + C_{\alpha\beta3\beta'}u_{3,\beta'}n_\beta = t_\alpha & \text{on } \Gamma'_N, \\ C_{3\beta\alpha'\beta'}u_{\alpha',\beta'}n_\beta + C_{3\beta3\beta'}u_{3,\beta'}n_\beta = 0 & \text{on } \Gamma'_N, \\ u_\alpha = u^0_\alpha(x,y), \quad u_3 = 0 & \text{on } \Gamma'_D. \end{cases} \tag{58}$$

For plane strain, we seek a solution satisfying

$$u_3(x,y) = 0 \qquad \text{on } D. \tag{59}$$

Using (55), equation (58) can be written as (the in-plane components)

$$\begin{cases} C_{\alpha\beta\alpha'\beta'}u_{\alpha',\beta'\beta} = -b_\alpha & \text{on } D, \\ C_{\alpha\beta\alpha'\beta'}u_{\alpha',\beta'}n_\beta = t_\alpha & \text{on } \Gamma'_N, \\ u_\alpha = u^0_\alpha(x,y) & \text{on } \Gamma'_D, \end{cases} \tag{60}$$

plus the out-of-plane equations

$$\begin{cases} C_{3\beta\alpha'\beta'}u_{\alpha',\beta'\beta} = 0 & \text{on } D, \\ C_{3\beta\alpha'\beta'}u_{\alpha',\beta'}n_\beta = 0 & \text{on } \Gamma'_N, \\ u_3 = 0 & \text{on } \Gamma'_D. \end{cases} \tag{61}$$

Note that (61) is not automatically satisfied for all materials. However, if the stiffness tensor satisfies

$$C_{3\alpha'\alpha\beta} = 0 \qquad \forall\, \alpha', \alpha, \beta \in \{1,2\}, \tag{62}$$

then equation (61) is trivial.

The equation (60) is the plane strain problem. For given data $b_x, b_y, t_x, t_y, u^0_x, u^0_y$ and domain D, it can be shown it is a well-posed problem, i.e., there exists a solution, the strain is unique and the solution is stable with respect to given data.

5.4 Antiplane shear problem

The antiplane shear problem is similar to the plane strain problem. The difference is that the given body force $\mathbf{b}$, traction $\mathbf{t}$, and displacement $\mathbf{u}^0$ have only $\mathbf{e}_z$ component. More precisely, we pose the antiplane shear problem as

$$\begin{cases} -\mathrm{div}[\mathbf{C}\nabla\mathbf{u}] = b_z(x,y)\mathbf{e}_z & \text{on } \Omega, \\ (\mathbf{C}\nabla\mathbf{u})\mathbf{n} = t_z(x,y)\mathbf{e}_z & \text{on } \Gamma'_N \times (-\infty, +\infty), \\ \mathbf{u} = u^0_z(x,y)\mathbf{e}_z & \text{on } \Gamma'_D \times (-\infty, +\infty). \end{cases} \tag{63}$$

By similar arguments as before, we conclude that $\mathbf{u}$ is independent of z. Further, we seek a solution satisfying

$$u_x(x,y) = u_y(x,y) = 0 \qquad \forall\,(x,y) \in D. \tag{64}$$

By the above equations, the antiplane shear problem (63) can be written as (c.f. (58))

$$\begin{cases} C_{\alpha\beta 3\beta'} u_{3,\beta'\beta} = 0 & \text{on } \; D, \\ C_{3\beta 3\beta'} u_{3,\beta'\beta} = -b_z & \text{on } \; D, \\ C_{\alpha\beta 3\beta'} u_{3,\beta'} n_\beta = 0 & \text{on } \; \Gamma'_N, \\ C_{3\beta 3\beta'} u_{3,\beta'} n_\beta = t_z & \text{on } \; \Gamma'_N, \\ u_3 = u_z^0(x,y) & \text{on } \; \Gamma'_D. \end{cases} \tag{65}$$

That is,

$$\begin{cases} C_{3\beta 3\beta'} u_{3,\beta'\beta} = -b_z & \text{on } \; D, \\ C_{3\beta 3\beta'} u_{3,\beta'} n_\beta = t_z & \text{on } \; \Gamma'_N, \\ u_3 = u_z^0(x,y) & \text{on } \; \Gamma'_D. \end{cases} \tag{66}$$

plus the in-plane components

$$\begin{cases} C_{\alpha\beta 3\beta'} u_{3,\beta'\beta} = 0 & \text{on } \; D, \\ C_{\alpha\beta 3\beta'} u_{3,\beta'} n_\beta = 0 & \text{on } \; \Gamma'_N. \end{cases} \tag{67}$$

Note again that equation (67) is not always trivial. If

$$C_{\alpha\beta 3\beta'} = 0 \qquad \forall\,\alpha,\beta,\beta' \in \{1,2\}, \tag{68}$$

then indeed equation (67) is automatically satisfied regardless what values is $u_3(x,y)$. In this case, the equation (66) is the governing equation for antiplane shear.

⧫ (6pt) 26 (ii). Consider an isotropic stiffness tensor $\mathbf{C}$ with Lamé constants μ, λ.

(i) Is equation (62) equivalent to (68)? If yes, please explain why. Does the isotropic stiffness tensor $\mathbf{C}$ satisfies (62)?

(ii) Write down the equations in (60) in terms of μ, λ (instead of $C_{\alpha\beta,\alpha'\beta'}$).

(iii) Write down the equations in (66) in terms of μ, λ (instead of $C_{\alpha\beta,\alpha'\beta'}$).

5.5 Plane stress problem

In a plane stress problem, we consider a thin body $\Omega = D \times (-h/2, h/2)$, where $h << 1$ and $D \subset I\!R^2$ is a two dimensional region on xy-plane. Consider the elasticity problem

$$\begin{cases} -\mathrm{div}[\mathbf{C}\nabla\mathbf{u}] = b_x(x,y)\mathbf{e}_x + b_y(x,y)\mathbf{e}_y & \text{on } \Omega, \\ (\mathbf{C}\nabla\mathbf{u})\mathbf{n} = t_x(x,y)\mathbf{e}_x + t_y(x,y)\mathbf{e}_y & \text{on } \Gamma'_N \times (-h/2, h/2), \\ (\mathbf{C}\nabla\mathbf{u})\mathbf{e}_z = 0 & \text{on } \{\mathbf{x} : z = -h/2\} \cup \{\mathbf{x} : z = h/2\}, \\ \mathbf{u} = u_x^0(x,y)\mathbf{e}_x + u_y^0(x,y)\mathbf{e}_y & \text{on } \Gamma'_D \times (-h/2, h/2), \end{cases} \tag{69}$$

where $b_x, b_y, t_x, t_y, u_x^0, u_y^0$ are given data and independent of z, $\mathbf{C}$ is independent of x, y, z, i.e., the material is homogeneous, Γ'_N, Γ'_D are a mutual disjoint subdivision of ∂D.

Based on the boundary condition $(\mathbf{C}\nabla\mathbf{u})\mathbf{e}_z = 0$ on $\{\mathbf{x} : z = -h/2\} \cup \{\mathbf{x} : z = h/2\}$ and the fact that the body is a thin ($h << 1$), we seek a solution with stress field of the following form

$$\sigma = \begin{bmatrix} \sigma_{xx}(x,y) & \sigma_{xy}(x,y) & 0 \\ \sigma_{yx}(x,y) & \sigma_{yy}(x,y) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{70}$$

Note that the stress field is assumed to be independent of z. Let $\mathbf{S} = \mathbf{C}^{-1}$ is the compliance tensor. Then in terms of the compliance tensor, the strain can be written as

$$\epsilon_{\alpha\beta} = S_{\alpha\beta\alpha'\beta'}\sigma_{\alpha'\beta'}, \quad \epsilon_{3\beta} = S_{3\beta\alpha'\beta'}\sigma_{\alpha'\beta'}, \quad \epsilon_{33} = S_{33\alpha'\beta'}\sigma_{\alpha'\beta'}, \tag{71}$$

which is independent of z. Therefore, the displacement $\mathbf{u}$ is also independent of z. The first of (71) defines a linear mapping $\mathbf{S}' : I\!R^{2\times 2}_{sym} \to I\!R^{2\times 2}_{sym}$ with components given by $S_{\alpha\beta\alpha'\beta'}$. It can be shown $\mathbf{S}'$ is invertible and we denote by $\mathbf{C}' = \mathbf{S}'^{-1}$. Therefore, if $\mathbf{u}' = (u_x, u_y)$,

$$\nabla'\mathbf{u}' = \begin{bmatrix} u_{x,x} & u_{x,y} \\ u_{y,x} & u_{y,y} \end{bmatrix}, \tag{72}$$

then the in-plane components of (69) can be written as

$$\begin{cases} C'_{\alpha\beta\alpha'\beta'} u_{\alpha',\beta'\beta} = -b_\alpha & \text{on } D, \\ C'_{\alpha\beta\alpha'\beta'} u_{\alpha',\beta'} n_\beta = t_\alpha & \text{on } \Gamma'_N, \\ u_\alpha = u_\alpha^0(x,y) & \text{on } \Gamma'_D \times (-h/2, h/2). \end{cases} \tag{73}$$

♦ (6pt) 27. Consider an isotropic stiffness tensor $\mathbf{C}$ with Lamé constants μ, λ.

(i) Calculate $\mathbf{C}' = (\mathbf{S}')^{-1}$. That is, for a given 2×2 symmetric matrix $\epsilon_{\alpha\beta}$, what are $C'_{\alpha\beta\alpha'\beta'}\epsilon_{\alpha'\beta'}$? (Hint: see the handout I gave in class)

(ii) Write down the equations in (73) in terms of μ, λ (instead of $C'_{\alpha\beta,\alpha'\beta'}$).

(iii) Compared with plane strain problem, list at least four differences between plane strain problem and plane stress problem (Hint: geometric feature of the bodies, out-of-plane component of strain and stress, boundary conditions, the modulus, etc)

6 A Brief Introduction to Finite Element Method

6.1 Basis

6.2 Shape fucntions

7 St Venant's Problems and Semi-inverse Method

St Venant's Principle: The elastic field produced by a self-balanced force system on a *local* region on the body is also *local*. More precisely, consider the Dirichlet problem or the Neumann problem

$$\begin{cases} \operatorname{div}[\mathbf{C}\nabla\mathbf{u}] = -\mathbf{b} & \text{on } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{or} \quad \begin{cases} \operatorname{div}[\mathbf{C}\nabla\mathbf{u}] = 0 & \text{on } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma_D, \\ [\mathbf{C}\nabla\mathbf{u}]\mathbf{n} = \mathbf{t} & \text{on } \Gamma_N. \end{cases} \tag{74}$$

Let us denote the support of $\mathbf{b}$ or $\mathbf{t}$ by D

$$D = \operatorname{supp}\mathbf{b} := \{\mathbf{x} \in \Omega : |\mathbf{b}(\mathbf{x})| \neq 0\} \quad \text{or} \quad = \{\mathbf{x} : |\mathbf{t}(\mathbf{x})| \neq 0\}.$$

If D is small and

$$\int_D \mathbf{b} = 0 \quad \text{and} \quad \int_D \mathbf{x} \wedge \mathbf{b} = 0 \quad \text{or} \quad \int_{\Gamma_N} \mathbf{t} = 0 \quad \text{and} \quad \int_{\Gamma_N} \mathbf{x} \wedge \mathbf{t} = 0$$

then $\mathbf{u}(\mathbf{x})$ is small for points $\mathbf{x}$ that is far away from D.

St Venant's Problem: Consider a long prismatic bar $D \times (0, L)$, where D is the two dimensional cross-section. Denote by the two end surfaces by S_0 and S_L, and the side surface by S_s:

$$S_0 = \{\mathbf{x} \in \partial\Omega : z = 0\}, \quad S_L = \{\mathbf{x} \in \partial\Omega : z = L\}, \quad S_s = \partial D \times (0, L),$$

see the following figure.

Consider the problem

$$\begin{cases} \operatorname{div}[\mathbf{C}\nabla\mathbf{u}] = 0 & \text{on } \Omega, \\ [\mathbf{C}\nabla\mathbf{u}]\mathbf{n} = 0 & \text{on } S_s, \\ [\mathbf{C}\nabla\mathbf{u}]\mathbf{n} = \mathbf{t} & \text{on } S_L, \\ \mathbf{u} = 0 & \text{on } S_0. \end{cases} \tag{75}$$

From the general theory of linear elasticity, we know the above problem admits a unique solution. By St Venant's principle, we can instead consider a relaxed problem

$$\begin{cases} \operatorname{div}[\mathbf{C}\nabla\mathbf{u}] = 0 & \text{on } \Omega, \\ [\mathbf{C}\nabla\mathbf{u}]\mathbf{n} = 0 & \text{on } S_s, \\ \int_{S_L} [\mathbf{C}\nabla\mathbf{u}]\mathbf{n} = \int_{S_L} \mathbf{t} = \mathbf{R}, \quad \int_{S_L} \mathbf{x} \wedge [\mathbf{C}\nabla\mathbf{u}]\mathbf{n} = \int_{S_L} \mathbf{x} \wedge \mathbf{t} = \mathbf{M}. \end{cases} \tag{76}$$

Clearly, the solution of (75) satisfies (76). In general equation (76) admits infinitely many solutions. However, since the scale of the cross-section D is small compared with L, by the St Venant's principle, we conclude that any solution of (76) is a good representation of the true solution of the original problem (75). In practice, we of course shall focus on the solution of (76) that can be made as simple as possible.

Classification of St Venant's Problems: For St Venant's problems, what is given is the domain $\Omega = D \times (0, L)$, and the total force $\mathbf{R} \in I\!R^3$ and torque $\mathbf{M} \in I\!R^3$ applied on one end of the bar, say, S_L. According to the directions of vector $\mathbf{R}$ and $\mathbf{M}$, we classify the St Venant's problems as follows.

(i) Simple extension: $\mathbf{R} \parallel \mathbf{e}_z$, $\mathbf{M} = 0$.

(ii) Pure bending: $\mathbf{R} = 0$, $\mathbf{M} \perp \mathbf{e}_z$.

(iii) Torsion: $\mathbf{R} = 0$, $\mathbf{M} \parallel \mathbf{e}_z$.

(iv) Bending: $\mathbf{R} \perp \mathbf{e}_z$, $\mathbf{M} = 0$.

To solve the St Venant's problems (76), we often focus on the stress field $\sigma : \Omega \to I\!R^{3\times3}_{sym}$. In terms of stress field (instead of the displacement), the St Venant's problems (76) can be rewritten as

$$\begin{cases} \text{div}\sigma = 0 & \text{on } \Omega, \\ \sigma\mathbf{n} = 0 & \text{on } S_s, \\ \int_{S_L} \sigma\mathbf{n} = \int_{S_L} \mathbf{t} = \mathbf{R}, & \int_{S_L} \mathbf{x} \wedge \sigma\mathbf{n} = \int_{S_L} \mathbf{x} \wedge \mathbf{t} = \mathbf{M}. \end{cases} \tag{77}$$

plus the compatibility equation: the strain field

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \text{ satisfies (22)-(23), where } \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \mathbf{C}^{-1}\sigma. \tag{78}$$

If $\mathbf{C}$ is homogeneous and isotropic with Young's modulus E and Poisson's ration ν, then

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} \sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) & (1+\nu)\sigma_{xy} & (1+\nu)\sigma_{xz} \\ (1+\nu)\sigma_{yx} & (1+\nu)\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}) & (1+\nu)\sigma_{yz} \\ (1+\nu)\sigma_{zx} & (1+\nu)\sigma_{zy} & \sigma_{zz} - \nu(\sigma_{yy} + \sigma_{xx}) \end{bmatrix}. \tag{79}$$

Thus, in terms of the stress field, the compatibility equations (22)-(23) can be written as

$$\begin{aligned} \frac{\partial^2}{\partial y \partial z}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) &= (1+\nu)\frac{\partial}{\partial x}[\frac{\partial \sigma_{zy}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial y} - \frac{\partial \sigma_{xy}}{\partial x}] \\ (1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} &= \frac{1}{2}[\frac{\partial^2}{\partial y^2}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) + \frac{\partial^2}{\partial x^2}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}))] \\ \vdots \quad &\vdots \quad \vdots \end{aligned} \tag{80}$$

In particular, if there is no body force, i.e., $\text{div}\sigma = 0$ on Ω, then the above equation can be rewritten as

$$\nabla^2\sigma + \frac{1}{1+\nu}\nabla\nabla\text{Tr}(\sigma) = 0 \qquad \text{on } \Omega. \tag{81}$$

This equation is called MICHELL stress compatibility equation.

7.1 Simple extension

$\mathbf{R} = R_z\mathbf{e}_z$ and $\mathbf{M} = 0$. By the last of (77), we make a guess on the stress field $\sigma : \Omega \to I\!R^{3\times 3}_{sym}$

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{R_z}{|D|} \end{bmatrix}, \tag{82}$$

where $|D|$ denotes the area of the cross-section D. Note that σ is constant throughout the domain D. Further, the equilibrium equation (the first of (77)) and the compatibility equation (78) is automatically satisfied. Thus, we conclude that (82) is indeed a solution of the St Venant's problem (77) and (78) with $\mathbf{R} = R_z\mathbf{e}_z$ and $\mathbf{M} = 0$ for general **anisotropic** medium.

To find strain and displacement when the medium is isotropic, by (82) and (79) we have

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} -\nu\frac{R_z}{|D|} & 0 & 0 \\ 0 & -\nu\frac{R_z}{|D|} & 0 \\ 0 & 0 & \frac{R_z}{|D|} \end{bmatrix}. \tag{83}$$

Thus, a solution of displacement (u_x, u_y, u_z) to $\frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) = \mathbf{E}$ is

$$\begin{cases} u_x = -\nu\frac{R_z}{E|D|}x \\ u_y = -\nu\frac{R_z}{E|D|}y \\ u_z = \frac{R_z}{E|D|}z \end{cases}. \tag{84}$$

7.2 Pure bending

$\mathbf{R} = 0$ and $\mathbf{M} = M_x\mathbf{e}_x + M_y\mathbf{e}_y$. By the last of (77), we make a guess on the stress field $\sigma : \Omega \to I\!R^{3\times 3}_{sym}$

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_x x + A_y y \end{bmatrix}, \tag{85}$$

where A_x, A_y are constants to be determined by the third of (77). Note that σ is linear throughout the domain D. Thus, the equilibrium equation (the first of (77)) and the compatibility equation (78) is automatically satisfied. Thus, we conclude that (82) is indeed a solution of the St Venant's problem (77) and (78) with $\mathbf{R} = 0$ and $\mathbf{M} = M_x\mathbf{e}_x + M_y\mathbf{e}_y$ for general **anisotropic** medium if

$$\begin{cases} \mathbf{R} = \int_{S_L}(A_x x + A_y y)\mathbf{e}_z = 0, \\ \mathbf{M} = \int_{S_L}\mathbf{x}\wedge(A_x x + A_y y)\mathbf{e}_z = M_x\mathbf{e}_x + M_y\mathbf{e}_y. \end{cases} \tag{86}$$

Choose the centroid of S_L as our origin, then the first of (86) is automatically satisfied. Choose $\mathbf{e}_x, \mathbf{e}_y$ such that the momentum of inertia is diagonized

$$\int_{S_L}\begin{bmatrix} x^2 & xy \\ yx & y^2 \end{bmatrix} = \mathrm{diag}[I_{xx}, I_{yy}].$$

By the second of (86), we obtain

$$A_x = -\frac{M_y}{I_{xx}}, \qquad A_y = \frac{M_x}{I_{yy}}. \tag{87}$$

To find strain and displacement when the medium is isotropic, by (85) and (79) we have

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} -\nu(-x\frac{M_y}{EI_{xx}} + y\frac{M_x}{EI_{yy}}) & 0 & 0 \\ 0 & -\nu(-x\frac{M_y}{EI_{xx}} + y\frac{M_x}{EI_{yy}}) & 0 \\ 0 & 0 & -x\frac{M_y}{EI_{xx}} + y\frac{M_x}{EI_{yy}} \end{bmatrix}. \tag{88}$$

Thus, a solution of displacement (u_x, u_y, u_z) to $\frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) = \mathbf{E}$ is

$$\begin{cases} u_x = -\frac{M_y}{2EI_{xx}}(-z^2 - \nu(x^2 - y^2)) - \nu xy\frac{M_x}{EI_{yy}}, \\ u_y = \frac{M_x}{2EI_{yy}}(-z^2 + \nu(x^2 - y^2)) + \nu xy\frac{M_y}{EI_{xx}}, \\ u_z = -xz\frac{M_y}{EI_{xx}} + yz\frac{M_x}{EI_{yy}}. \end{cases} \tag{89}$$

7.3 Torsion

$\mathbf{R} = 0$ and $\mathbf{M} = M_z\mathbf{e}_z$. We assume the material is isotropic. By physical intuition, we make a guess on the stress field $\sigma : \Omega \to I\!R^{3\times3}_{sym}$

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & 0 \end{bmatrix}, \tag{90}$$

where $\sigma_{xz}, \sigma_{yz} : D \to I\!R$, depending on (x, y) but independent of z, are shear stress to be determined. By (77) we have

$$\begin{cases} \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 & \text{on } D, \\ \sigma_{xz}n_x + \sigma_{yz}n_y = 0 & \text{on } \partial D, \\ \int_{S_L}(\sigma_{xz}\mathbf{e}_x + \sigma_{yz}\mathbf{e}_y) = 0, & \int_{S_L} \mathbf{x} \wedge (\sigma_{xz}\mathbf{e}_x + \sigma_{yz}\mathbf{e}_y) = M_z\mathbf{e}_z. \end{cases} \tag{91}$$

Note that

$$\int_{S_L} \mathbf{x} \wedge (\sigma_{xz}\mathbf{e}_x + \sigma_{yz}\mathbf{e}_y) = \int_{S_L} -z\sigma_{yz}\mathbf{e}_x + z\sigma_{xz}\mathbf{e}_y + (x\sigma_{yz} - y\sigma_{xz})\mathbf{e}_z.$$

Further, by the compatibility equation (81), we have

$$\nabla^2\sigma_{yz} = \nabla^2\sigma_{xz} = 0 \qquad \text{on } \Omega. \tag{92}$$

By the first of (91), we have $\nabla \wedge [-\sigma_{yz}, \sigma_{xz}, 0] = 0$, i.e., the vector field $[-\sigma_{yz}, \sigma_{xz}]$ is curl-free. Therefore, the following scalar potential

$$F(x, y) = \int_{(x_0,y_0)}^{(x,y)} [-\sigma_{yz}, \sigma_{xz}] \cdot d\mathbf{l}.$$

is well-defined and

$$\sigma_{xz} = \frac{\partial}{\partial y}F(x, y), \qquad \sigma_{yz} = -\frac{\partial}{\partial x}F(x, y) \qquad \forall\, (x, y) \in D. \tag{93}$$

Plugging the above equation to the stress compatibility equation (92) we obtain

$$\frac{\partial}{\partial y}\nabla^2 F = 0, \qquad -\frac{\partial}{\partial x}\nabla^2 F = 0 \qquad \text{on } D.$$

We thus conclude

$$\nabla^2\Psi = -2, \quad F = \mu\alpha\Psi \qquad \text{on } D, \tag{94}$$

where μ is the shear modulus, α is a constant to be determined, and Ψ is called PRANDL stress function. Let (n_x, n_y) be the unit normal on ∂D at a point $p \in \partial D$ and (t_x, t_y) be the unit tangential unit vector along ∂D at the same point p. Clearly,

$$[t_x, t_y] \cdot [n_x, n_y] = 0 \Longrightarrow (t_x, t_y) = \pm(-n_y, n_x). \tag{95}$$

Further, from the second of (91), (93) and (95) we have

$$[t_x, t_y] \cdot \nabla F = t_x \frac{\partial F}{\partial x} + t_y \frac{\partial F}{\partial x} = \pm(n_y\sigma_{yz} + n_x\sigma_{xz}) = 0 \qquad \text{on } \partial D.$$

Therefore, we conclude that for some $c \in I\!R$,

$$F = c \qquad \text{on } \partial D. \tag{96}$$

To see the consequence of the last of (91), we first verify the total force

$$\int_{S_L} \sigma_{xz} = \int_{S_L} \frac{\partial F}{\partial y} = \int_{\partial D} [n_x, n_y] \cdot [0, 1] F = c \int_{\partial D} n_y = c \int_{S_L} \mathrm{div}[0, 1] = 0.$$

Similarly, we have $\int_{S_L} \sigma_{yz} = 0$. Second, we calculate the total torque in $\mathbf{e}_z$ direction

$$\begin{aligned} \int_{S_L} (x\sigma_{yz} - y\sigma_{xz}) &= -\mu\alpha \int_{S_L} (x\frac{\partial \Psi}{\partial x} + y\frac{\partial \Psi}{\partial x}) = -\mu\alpha \int_{S_L} (\frac{\partial x\Psi}{\partial x} + \frac{\partial y\Psi}{\partial x} - 2\Psi) \\ &= -\mu\alpha[\int_{\partial S_L} (n_x x\Psi + n_y y\Psi) - \int_{S_L} 2\Psi] = 2\mu\alpha\chi = M_z, \end{aligned} \tag{97}$$

where

$$\chi = \int_{S_L} \Psi - \frac{1}{2} \int_{\partial S_L} (n_x x\Psi + n_y y\Psi) \tag{98}$$

is called torsion stiffness.

Note that torsion stiffness is an geometric property of the cross-section D, i.e., $S_L = D$. To calculate the torsion stiffness, from (94) and (96) we notice that the stress function Ψ satisfies that for some constant $c' \in I\!R$,

$$\begin{cases} \Delta\Psi = -2 & \text{on } D, \\ \Psi = c' & \text{on } \partial D. \end{cases} \tag{99}$$

♦ 28. Consider the torsion of a long prismatic bar $\Omega = D \times (0, L)$, i.e., the problem (75) with $\mathbf{R} = 0$ and $\mathbf{M} = M_z\mathbf{e}_z$. Assume the material is isotropic with LAMÉ constants μ, λ. As discussed above, the torsion stiffness is determined by (99) and (98).

(i) (2pt) Show that the torsion stiffness is independent of the constant c'.

(ii) (2pt) Assume $D = \{(x, y) : x^2 + y^2 < R^2\}$ is the circle centered at the origin and with radius R. Solve (99) for $c' = 0$ and calculate the torsion stiffness χ.

(iii) (4pt) By (93) and (94), calculate the stress field σ_{xz}, σ_{yz}, the strain field, and the displacement (Assume $\mathbf{u} = 0$ if $z = 0$). (Note that the "fields" are functions defined on Ω).

(iv) (2pt) If we express the displacement u_x, u_y in the following form

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \mathbf{W} \begin{bmatrix} zx \\ zy \end{bmatrix},$$

where $\mathbf{W} \in I\!R^{2\times 2}$. Is it true $\mathbf{W}^T = -\mathbf{W}$? Sketch a figure that illustrates the geometric meaning of the constant α in (94). What is the relation between α and M_z, χ?

7.4 Bending

8 Important Problems and their Solutions

8.1 The fundamental solution —— Kelvin's solution

Formulation of the problem. Consider an infinite homogeneous body with an elastic stiffness tensor $\mathbf{C}$ in $\mathbb{R}^n$. At the origin, there is a concentrated body force $\mathbf{b}(\mathbf{x}) = \mathbf{b}^0\delta(\mathbf{x})$, where $\delta(\mathbf{x})$ is the Dirac function that satisfies

$$\int_{\mathbb{R}^n} f(\mathbf{x})\delta(\mathbf{x}) = f(0) \qquad \forall\, f \in C_0^\infty(\mathbb{R}^n).$$

We are interested in finding the elastic field, i.e., solving the following problem for $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$,

$$\begin{cases} \operatorname{div}[\mathbf{C}\nabla\mathbf{u}] = -\mathbf{b}^0\delta(\mathbf{x}) & \text{on } \mathbb{R}^n, \\ |\nabla\mathbf{u}| \to 0 & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{100}$$

Solution. We will solve this problem by the Fourier method. First let us recall the definition of Fourier transformation and the inversion theorem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a "generalized" function (Precisely, f needs to be a "tempered distribution". We notice that the Fourier transformation and the inversion theorem are valid for almost all functions we encountered in elasticity, including the Dirac function and its derivatives). Then the **Fourier transformation** of f, denoted by $\hat{f} : \mathbb{R}^n \to \mathbb{R}$, is given by

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^n} f(\mathbf{x}) \exp(-i\mathbf{k}\cdot\mathbf{x}) d\mathbf{x}. \tag{101}$$

Further, the **Fourier inversion theorem** holds

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}) d\mathbf{k}. \tag{102}$$

♦ 29 (i). (2pt) Let $\hat{\mathbf{u}}(\mathbf{k})$ be the Fourier transformation of the solution of (100). Show that $\hat{\mathbf{u}}(\mathbf{k})$ satisfies

$$\hat{\mathbf{u}}(\mathbf{k}) = \mathbf{N}(\mathbf{k})\mathbf{b}^0, \tag{103}$$

where $\mathbf{N}(\mathbf{k}) \in \mathbb{R}^{n\times n}_{sym}$ is the inverse of the matrix $C_{piqj}k_ik_j$.

Equation (103) is the $\mathbf{k}$-space equation corresponding to (100). In effect, the Fourier transformation converts a partial differential equation, e.g., (100) into an algebraic equation, e.g. (103). This is possible because our problem is *homogeneous* in the sense that the material properties $\mathbf{C}$ is independent of the position $\mathbf{x}$ and the domain is special, i.e., $\mathbb{R}^n$.

An application of the inversion theorem to (103) yields

$$\mathbf{u}(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathbf{N}(\mathbf{k})\mathbf{b}^0 \exp(i\mathbf{k}\cdot\mathbf{x}) d\mathbf{k}. \tag{104}$$

Further, if $\mathbf{C}$ is an isotropic stiffness tensor with Lamé constants μ, λ (see (49)), we find

$$\mathbf{N}(\mathbf{k}) = \frac{\alpha}{|\mathbf{k}|^2}\mathbf{I} - \frac{\beta}{|\mathbf{k}|^4}\mathbf{k}\otimes\mathbf{k}. \tag{105}$$

♦ 29 (ii). (2pt) Calculate the constants α, β in terms of μ, λ.

Plugging (105) into (104), we arrive at

$$\mathbf{u}(\mathbf{x}) = \alpha\phi(\mathbf{x})\mathbf{b}^0 - \beta[\nabla\nabla\psi(\mathbf{x})]\mathbf{b}^0, \tag{106}$$

where

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{1}{|\mathbf{k}|^2}\exp(i\mathbf{k}\cdot\mathbf{x})d\mathbf{k},$$
$$\psi(\mathbf{x}) = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{1}{|\mathbf{k}|^4}\exp(i\mathbf{k}\cdot\mathbf{x})d\mathbf{k}.$$

♦ 29 (iii). (2pt) Show that the above functions ϕ and ψ satisfy

$$\Delta\phi(\mathbf{x}) = -\delta(0), \qquad \Delta\Delta\psi(\mathbf{x}) = -\delta(0). \tag{107}$$

By symmetry, we seek solutions to (107) that can be written as

$$\phi = \phi(r), \qquad \psi = \psi(r),$$

where $r = (x_1^2 + \cdots + x_n^2)^{1/2}$. In this case, we have

$$\Delta\phi(r) = \phi(r)_{,x_ix_i} = (\phi'(r)r_{,x_i})_{,x_i} = \phi''(r) + \frac{n-1}{r}\phi'(r), \tag{108}$$

where $(\ )' = \frac{d}{dr}(\ )$.

♦ 29 (iv). (2pt) Show in detail the identity $\Delta\phi(r) = (\phi'(r)r_{,x_i})_{,x_i} = \phi''(r) + \frac{n-1}{r}\phi'(r)$.

Therefore, by (107) we have

$$\phi''(r) + \frac{n-1}{r}\phi'(r) = 0 \qquad \forall\, r > 0. \tag{109}$$

From the theory of *ordinary differential equation*, we know that a solution to (109) can be written as

$$\phi(r) = \begin{cases} C_2\log(r) + C_0 & \text{if } n = 2, \\ C_n\frac{1}{r^{n-2}} + C_0 & \text{if } n \geq 3, \end{cases} \tag{110}$$

where C_i are constants determined by the boundary conditions. In particular, the constant C_0 is immaterial since it does not affect the strain and stress field, see (106). It is the usual convention that we choose it to be zero. Further, to find the constants C_i $(i = 2, \cdots)$, we integrate the first of (107) over a unit ball centered at the origin B_1 and obtain

$$1 = \int_{B_1} \delta(0) = \int_{B_1} \nabla \cdot \nabla \phi(r) = \int_{\partial B_1} \mathbf{e}_r \cdot \nabla \phi(r) = \int_{\partial B_1} \phi'(r)$$

♦ 29 (v). (2pt) Plug (110) into the above identity and find out what is the constants C_2 and C_3. (Bonus problem: 1pt, find C_n for $n \geq 4$)

We now calculate the other potential function $\psi(r)$ in (107). From (107), (108), and (110), we see that

$$\Delta \psi(r) = \psi''(r) + \frac{n-1}{r} \psi'(r) = \phi(r) = \begin{cases} C_2 \log(r) & \text{if } n = 2, \\ C_n \frac{1}{r^{n-2}} & \text{if } n \geq 3. \end{cases} \tag{111}$$

Again, from the theory of *ordinary differential equation*, we find

$$\psi(r) = \begin{cases} D_2 r^2 (\log(r) - 1) & \text{if } n = 2, \\ D_n r^{4-n} & \text{if } n \geq 3, n \neq 4, \\ D_4 \log(r) & \text{if } n = 4. \end{cases} \tag{112}$$

♦ 29 (vi). (2pt) Verify that the above defined $\psi(r)$ indeed satisfies (111) for appropriately chosen constants $D_i \in I\!R$ and find the relation between the constants D_i and C_i for $i = 2, 3$. (Bonus problem: 1pt, find D_i for $i \geq 4$)

In summary, from (106), (110) and (112), we conclude that the solution to (100) is

$$\mathbf{u}(\mathbf{x}) = \mathbf{G}^{(n)}(\mathbf{x}) \mathbf{b}^0,$$

$$\mathbf{G}^{(n)}(\mathbf{x}) = \begin{cases} \{\alpha C_2 \log(r) \mathbf{I} - \beta D_2 [(1 - 2\log(r)) \mathbf{I} - 2\mathbf{e}_r \otimes \mathbf{e}_r]\} & \text{if } n = 2, \\ r^{2-n} \{\alpha C_n \mathbf{I} - \beta D_n [(4-n)(2-n) \mathbf{e}_r \otimes \mathbf{e}_r + (4-n) \mathbf{I}]\} & \text{if } n \geq 3, n \neq 4, \\ r^{2-n} \{\alpha C_n \mathbf{I} - \beta D_n [-2\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{I}]\} & \text{if } n = 4, \end{cases} \tag{113}$$

where $\mathbf{e}_r = \mathbf{x}/|\mathbf{x}|$, and we have used the identity

$$\nabla \nabla \psi(r) = (\psi''(r) - \frac{\psi'(r)}{r}) \mathbf{e}_r \otimes \mathbf{e}_r + \frac{\psi'(r)}{r} \mathbf{I}.$$

The above solution (113) is called the KELVIN's solution in three dimensions $(n = 3)$. It is also referred to as the GREEN's function or the fundamental solution.

Direct calculations verify that for any $\mathbf{b} \in [C_0(I\!R^n)]^n$, the solution to

$$\begin{cases} \operatorname{div}[\mathbf{C} \nabla \mathbf{u}] = -\mathbf{b}(\mathbf{x}) & \text{on } I\!R^n, \\ |\nabla \mathbf{u}| \to 0 & \text{as } |\mathbf{x}| \to \infty \end{cases} \tag{114}$$

is given by

$$\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^n} \mathbf{G}^{(n)}(\mathbf{x} - \mathbf{x}')\mathbf{b}(\mathbf{x}')d\mathbf{x}',$$

which is called the GREEN's formula.

8.2 Pressured vessel

Formulation of the problem. Consider a homogeneous body with an elastic stiffness tensor $\mathbf{C}$ on an annulus region $\Omega = \{\mathbf{x} : r_a < |\mathbf{x}| < r_b\}$ subject to a hydrostatic pressure p_a from inside and pressure p_b from outside. We are interested in finding the elastic field, i.e., solving the following problem for $\mathbf{u} : \Omega \to \mathbb{R}^n$,

$$\begin{cases} \text{div}[\mathbf{C}\nabla\mathbf{u}] = 0 & \text{on } \Omega, \\ -(\mathbf{C}\nabla\mathbf{u})\mathbf{e}_r = p_a\mathbf{e}_r & \text{if } r = r_a, \\ (\mathbf{C}\nabla\mathbf{u})\mathbf{e}_r = -p_b\mathbf{e}_r & \text{if } r = r_b. \end{cases} \tag{115}$$

Solution. For simplicity, we assume the stiffness $\mathbf{C}$ is isotropic. First, we show that if $\mathbf{u}$ is a solution to (115), then

$$\mathbf{u}'(\mathbf{x}') = \mathbf{Q}\mathbf{u}(\mathbf{x}), \qquad \mathbf{x}' = \mathbf{Q}\mathbf{x} \tag{116}$$

is also a solution to (115), where $\mathbf{Q} \in So(n)$ is any rigid rotation. To see this, by the chain rule we have

$$\begin{aligned} u_{q,x_j} &= Q_{sq}u'_{s,x_j} = Q_{sq}u'_{s,x'_k}x'_{k,x_j} = Q_{sq}Q_{kj}u'_{s,x'_k} \\ C_{piqj}u_{q,x_jx_i} &= (C_{piqj}Q_{sq}Q_{kj}u'_{s,x'_k})_{,x'_l}x'_{l,x_i} = C_{piqj}Q_{sq}Q_{kj}Q_{li}u'_{s,x'_kx'_l}. \end{aligned}$$

Since $\mathbf{u}$ satisfies the first of (115), by the second of the above equation we obtain

$$C_{piqj}Q_{sq}Q_{kj}Q_{li}Q_{rp}u'_{s,x'_kx'_l} = 0 \quad \text{i.e.,} \quad \text{div}_{\mathbf{x}'}[\mathbf{C}'\nabla_{\mathbf{x}'}\mathbf{u}'(\mathbf{x}')] = 0 \qquad \forall\, \mathbf{x}' \in Q\Omega = \Omega, \tag{117}$$

where

$$(\mathbf{C}')_{piqj} = (\mathbf{C})_{p'i'q'j'}Q_{pp'}Q_{qq'}Q_{ii'}Q_{jj'}.$$

Since $\mathbf{C}$ is an isotropic tensor, by (48) we see that $\mathbf{C}' = \mathbf{C}$, and hence equation (117) is identical to the first of (115). Similarly, we can verify $\mathbf{u}'$ satisfy the second and third of (115) as well. We thus conclude that $\mathbf{u}'$ is also a solution of (115) if $\mathbf{u}$ is so. From the uniqueness theorem, we infer that

$$\mathbf{u}(\mathbf{x}) = \mathbf{Q}\mathbf{u}(\mathbf{Q}\mathbf{x}) \quad \forall\, \mathbf{x} \in \Omega \;\&\; \mathbf{Q} \in So(n) \quad \Longrightarrow \quad \mathbf{u}(\mathbf{x}) = u_r(r)\mathbf{e}_r.$$

Therefore, the strain

$$u_{p,i} = (\frac{u_r}{r}x_p)_{,x_i} = (\frac{u_r}{r})'\frac{x_ix_p}{r} + \frac{u_r}{r}\delta_{pi}, \tag{118}$$

the stress

$$\sigma_{pi} = C_{piqj} u_{q,j} = 2\mu[(\frac{u_r}{r})' \frac{x_i x_p}{r} + \frac{u_r}{r}\delta_{pi}] + \lambda[(\frac{u_r}{r})' r + n\frac{u_r}{r}]\delta_{pi}, \tag{119}$$

and

$$\begin{aligned}\mathrm{div}[\mathbf{C}\nabla\mathbf{u}] = C_{piqj} u_{q,ji} &= 2\mu[(\frac{u_r}{r})' \frac{x_i x_p}{r}]_{,i} + 2\mu(\frac{u_r}{r})_{,p} + \lambda[(\frac{u_r}{r})' r + n\frac{u_r}{r}]_{,p} \\ &= \frac{x_p}{r}\Big\{2\mu[(\frac{u_r}{r})'' r + (n+1)(\frac{u_r}{r})'] + \lambda[(\frac{u_r}{r})'' r + (n+1)(\frac{u_r}{r})']\Big\}\end{aligned} \tag{120}$$

♦ 30 (i). (2pt) Start from (119). Show (120) in details.

Thus, the first of (115) implies

$$(\frac{u_r}{r})'' r + (n+1)(\frac{u_r}{r})' = 0.$$

From the theory of ordinary differential equation, we conclude that

$$\frac{u_r}{r} = -\frac{r^{-n}}{n} + C_1 r + C_0,$$

where the constants C_0, C_1 are determined by the boundary conditions in (115).

♦ 30 (ii). (3pt) Find the constants C_1, C_0 in terms of p_a, p_b, r_a, r_b.

♦ 30 (iii). (3pt) Let $n = 2$ and $\mathbf{e}_\theta = [-x_2, x_1]/r$. Calculate the stress tensor in the frame $\{\mathbf{e}_r, \mathbf{e}_\theta\}$, i.e., calculate the following matrix

$$\sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{\theta r} & \sigma_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_r \cdot \sigma\mathbf{e}_r & \mathbf{e}_\theta \cdot \sigma\mathbf{e}_r \\ \mathbf{e}_r \cdot \sigma\mathbf{e}_\theta & \mathbf{e}_\theta \cdot \sigma\mathbf{e}_\theta \end{bmatrix},$$

where the stress is given by (119) in the rectangular frame $\{\mathbf{e}_1, \mathbf{e}_2\}$.

♦ 30 (iv). (2pt) Assume $n = 2$, $r_b = c r_a$ $(c > 1)$, $p_b = 0$ but $p_a \neq 0$. What is $\hat{\sigma}_{\theta\theta} = \sigma_{\theta\theta}/p_a$ at $r = a$? Sketch the curve $\hat{\sigma}_{\theta\theta} = \hat{\sigma}_{\theta\theta}(c)$ below.

8.3 Half-space problem —— Boussinesq's solution

Formulation of the problem. Consider a homogeneous body with elastic stiffness tensor $\mathbf{C}$ in the half space $H = \{\mathbf{x} \in I\!R^n : z < 0\}$. Our problem is to solve for $\mathbf{u} : H \to I\!R^n$ $(n = 3)$ such that

$$\begin{cases} \mathrm{div}(\mathbf{C}\nabla\mathbf{u}) = 0 & \text{on } H, \\ (\mathbf{C}\nabla\mathbf{u})\mathbf{e}_z = \mathbf{t}(x,y) & \text{on } \partial H, \\ |\nabla\mathbf{u}| \to 0 & \text{as } z \to -\infty. \end{cases} \tag{121}$$

Solution. Let $\mathbf{k}' = (k_x, k_y)$, $\mathbf{x}' = (x, y) \in I\!R^{n-1}$ be the in-plane wave vector and position vector, respectively. Upon Fourier transformation on variable (x, y), the first of (121) can be written as

$$C_{p\alpha q\beta}(-\hat{u}_q k_\alpha k_\beta) + iC_{p\alpha q3}k_\alpha \frac{d\hat{u}_q}{dz} + iC_{p3q\beta}k_\beta \frac{d\hat{u}_q}{dz} + C_{p3q3}\frac{d^2\hat{u}_q}{dz^2} = 0, \tag{122}$$

where $\alpha, \beta \in \{1, 2\}$ are in-plane indices,

$$\hat{\mathbf{u}}(k_x, k_y, z) = \int_{I\!R^{n-1}} \mathbf{u}(x, y, z)\exp(-i\mathbf{k}' \cdot \mathbf{x}')dxdy. \tag{123}$$

Similarly, the second of (121) can be written as

$$iC_{p3q\alpha}\hat{u}_q k_\alpha + C_{p3q3}\frac{\hat{u}_q}{dz} = \hat{t}_p(k_x, k_y) = \int_{I\!R^{n-1}} t_p(x, y)\exp(-i\mathbf{k}' \cdot \mathbf{x}')dxdy. \tag{124}$$

Denote by

$$T_{pq} = C_{p3q3}, \quad R_{pq} = iC_{p\alpha q3}k_\alpha \ (i.e.,\ R_{qp} = iC_{p3q\beta}k_\beta), \quad Q_{pq} = -C_{p\alpha q\beta}k_\alpha k_\beta. \tag{125}$$

The equation (122) can be written as

$$T_{pq}\frac{d^2\hat{u}_q}{dz^2} + (R_{pq} + R_{qp})\frac{d\hat{u}_q}{dz} + Q_{pq}\hat{u}_q = 0. \tag{126}$$

Let

$$\hat{\boldsymbol{\tau}} = \mathbf{T}\frac{d\hat{\mathbf{u}}}{dz} + \mathbf{R}^T\hat{\mathbf{u}}, \qquad \phi = \begin{bmatrix} \hat{\mathbf{u}} \\ \hat{\boldsymbol{\tau}} \end{bmatrix}. \tag{127}$$

Note that $\hat{\tau}_p = R_{qp}\hat{u}_q + T_{pq}\frac{d\hat{u}_q}{dz} = iC_{p3q\alpha}\hat{u}_q k_\alpha + C_{p3q3}\frac{\hat{u}_q}{dz}$ is the Fourier transformation of the traction $(\mathbf{C}\nabla\mathbf{u})\mathbf{e}_z$. By (126) and (127) we arrive at

$$\frac{d\phi}{dz} = \mathbf{H}\phi, \qquad \mathbf{H} = \begin{bmatrix} -\mathbf{T}^{-1}\mathbf{R}^T & \mathbf{T}^{-1} \\ \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q} & -\mathbf{R}\mathbf{T}^{-1} \end{bmatrix}. \tag{128}$$

Thus, in general a solution to (122) can be expressed as

$$\hat{\mathbf{u}} = [\mathbf{I}, 0] \exp(z\mathbf{H})\phi_0, \tag{129}$$

where the constant vector ϕ_0 is determined by the boundary conditions at $z \to -\infty$ and $z = 0$.

Example 3 *For an isotropic material with the* Lamé *constants μ, λ (c.f. (49)), the matrices* $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ *in Eq. (125) can be identified as*

$$\mathbf{T} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 2\mu+\lambda \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0 & 0 & i\lambda k_x \\ 0 & 0 & i\lambda k_y \\ i\mu k_x & i\mu k_y & 0 \end{bmatrix},$$

$$\mathbf{Q} = -\begin{bmatrix} \mu|\mathbf{k}'|^2 + (\mu+\lambda)k_x^2 & (\mu+\lambda)k_xk_y & 0 \\ (\mu+\lambda)k_xk_y & \mu|\mathbf{k}'|^2 + (\mu+\lambda)k_y^2 & 0 \\ 0 & 0 & \mu|\mathbf{k}'|^2 \end{bmatrix}. \tag{130}$$

From Eqs. (128), direct calculations reveal that

$$\mathbf{H}(k_x, k_y) = \begin{bmatrix} 0 & 0 & -ik_x & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & -ik_y & 0 & \frac{1}{\mu} & 0 \\ \frac{k_x\lambda}{i(\lambda+2\mu)} & \frac{k_y\mu}{i(\lambda+2\mu)} & 0 & 0 & 0 & \frac{1}{2\mu+\lambda} \\ \frac{4\mu(\mu+\lambda)k_x^2}{2\mu+\lambda} + k_y^2 & \frac{\mu(2\mu+3\lambda)k_yk_x}{2\mu+\lambda} & 0 & 0 & 0 & \frac{k_x\lambda}{i(\lambda+2\mu)} \\ \frac{\mu(2\mu+3\lambda)k_yk_x}{2\mu+\lambda} & \frac{4\mu(\mu+\lambda)k_y^2}{2\mu+\lambda} + k_x^2 & 0 & 0 & 0 & \frac{k_y\lambda}{i(\lambda+2\mu)} \\ 0 & 0 & 0 & -ik_x & -ik_y & 0 \end{bmatrix}. \tag{131}$$

The Jordan normal form of $\mathbf{H}(k_x, k_y)$ *is* ($k_z = |\mathbf{k}'|$)

$$\mathbf{H}(k_x, k_y) = \mathbf{P} \begin{bmatrix} k_z & 0 & 0 & 0 & 0 & 0 \\ 0 & k_z & 1 & 0 & 0 & 0 \\ 0 & 0 & k_z & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_z & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_z & 1 \\ 0 & 0 & 0 & 0 & 0 & -k_z \end{bmatrix} \mathbf{P}^{-1},$$

where the column vectors in $\mathbf{P} \in I\!R^{6\times 6}$ *are the generalized eigenvectors. Note that the eigenvalues of* $\mathbf{H}(k_x, k_y)$ *are* $\pm|\mathbf{k}'|$ *and the above decomposition is unique. By (129) we conclude that*

$$\hat{\mathbf{u}} = (\mathbf{a} + z\mathbf{b}) \exp(k_z z), \tag{132}$$

and by (127)

$$\hat{\boldsymbol{\tau}} = [\mathbf{T}\mathbf{b} + (\mathbf{R}^T + k_z\mathbf{T})\mathbf{a} + z(\mathbf{R}^T + k_z\mathbf{T})\mathbf{b}] \exp(k_z z) \tag{133}$$

Plugging (132) and (133) back to (128) or (126) we find that for all $z < 0$,

$$z[k_z^2\mathbf{T} + k_z(\mathbf{R} + \mathbf{R}^T) + \mathbf{Q}]\mathbf{b} + [k_z^2\mathbf{T} + k_z(\mathbf{R} + \mathbf{R}^T) + \mathbf{Q}]\mathbf{a} + (2k_z\mathbf{T} + \mathbf{R} + \mathbf{R}^T)\mathbf{b} = 0,$$

which implies

$$\begin{cases} [k_z^2\mathbf{T} + k_z(\mathbf{R} + \mathbf{R}^T) + \mathbf{Q}]\mathbf{b} = 0 \\ [k_z^2\mathbf{T}k_z(\mathbf{R} + \mathbf{R}^T) + \mathbf{Q}]\mathbf{a} + (2k_z\mathbf{T} + \mathbf{R} + \mathbf{R}^T)\mathbf{b} = 0 \end{cases} \tag{134}$$

Direct calculations show that the rank of $k_z^2\mathbf{T}+k_z(\mathbf{R}+\mathbf{R}^T)+\mathbf{Q}$ *is two (because* k_z *is an eigenvalue of* $\mathbf{H}(k_x,k_y)$*,* $k_z^2\mathbf{T}+k_z(\mathbf{R}+\mathbf{R}^T)+\mathbf{Q}$ *is necessarily singular) and equation (134) is equivalent to*

$$\begin{cases} -b_1k_x - b_2k_y + ib_3k_z = 0 \\ b_1k_y - b_2k_x = 0 \\ k_xa_1 + k_ya_2 - ik_za_3 - i(3-4\nu)b_3 = 0 \end{cases} \tag{135}$$

Further, the boundary condition (124) implies

$$\mathbf{T}\mathbf{b} + (\mathbf{R}^T + k_z\mathbf{T})\mathbf{a} = \hat{\mathbf{t}}. \tag{136}$$

Equations (135)-(136) have six unknowns and six equations. The coefficient matrix is non-singular and hence they admit a unique solution for any given $\hat{\mathbf{t}}$*. In particular, if* $\hat{\mathbf{t}} = [0,0,\hat{t}_z]$*, we find that*

$$\begin{cases} a_1 = \frac{k_x(1-2\nu)\hat{t}_z}{2i\mu k_z^2}, \\ a_2 = \frac{k_y(1-2\nu)\hat{t}_z}{2i\mu k_z^2}, \\ a_3 = \frac{(1-\nu)\hat{t}_z}{\mu k_z}, \end{cases} \qquad \begin{cases} b_1 = \frac{k_x\hat{t}_z}{2i\mu k_z}, \\ b_2 = \frac{k_y\hat{t}_z}{2i\mu k_z}, \\ b_3 = -\frac{\hat{t}_z}{2\mu}. \end{cases} \tag{137}$$

Further, if $\mathbf{t}(\mathbf{x}) = -P\delta_2(0)\mathbf{e}_z$*, then*

$$\hat{t}(k_x,k_y) = -P\mathbf{e}_z \qquad \forall\,(k_x,k_y)\in I\!R^2.$$

Thus, by (132) and (137) we obtain

$$\hat{\mathbf{u}}(k_x,k_y,z) = -P\begin{bmatrix} \frac{k_x(1-2\nu)}{2i\mu k_z^2} \\ \frac{k_y(1-2\nu)}{2i\mu k_z^2} \\ \frac{(1-\nu)}{\mu k_z} \end{bmatrix}\exp(k_zz) - P\begin{bmatrix} \frac{k_x}{2i\mu k_z} \\ \frac{k_y}{2i\mu k_z} \\ -\frac{1}{2\mu} \end{bmatrix} z\exp(k_zz). \tag{138}$$

Let

$$\varphi(x,y,z) = \frac{1}{(2\pi)^2}\int_{I\!R^2}\frac{1}{k_x^2+k_y^2}\exp(ik_xx+ik_yy+(k_x^2+k_y^2)^{1/2}z)dk_xdk_y. \tag{139}$$

Then, by (138) we arrive at

$$\mathbf{u}(x,y,z) = \frac{1}{(2\pi)^2}\int_{I\!R^2}\hat{\mathbf{u}}(k_x,k_y,z)\exp(ik_xx+ik_yy)dk_xdk_y,$$

which, by (156), can be identified as

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = -P\begin{bmatrix} \frac{-(1-2\nu)}{2\mu}\varphi_{,x} \\ \frac{-(1-2\nu)}{2\mu}\varphi_{,y} \\ \frac{(1-\nu)}{\mu}\varphi_{,z} \end{bmatrix} - P\begin{bmatrix} \frac{1}{2\mu}z\varphi_{,zx} \\ \frac{1}{2\mu}z\varphi_{,zy} \\ -\frac{1}{2\mu}z\varphi_{,zz} \end{bmatrix}. \tag{140}$$

To evaluate φ*, let* $r = (x^2+y^2)^2$*,* $\cos\theta = (k_xx+k_yy)/rk_z$*, and the integral formula that for* $\xi\in\mathbb{C}$*,*

$$\int_0^\infty \exp(t\xi)dt = -\frac{1}{\xi}, \qquad \int_0^\infty \frac{1}{t}\exp(t\xi)dt = -\log\xi.$$

Thus,

$$\begin{aligned}\varphi(x,y,z) &= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\frac{1}{k_z}\exp(ik_z r\cos\theta + k_z z)d\theta dk_z\\ &= -\frac{1}{2(2\pi)^2}\int_0^{2\pi}\log(z+ir\cos\theta)d\theta\\ &= -\frac{1}{2(2\pi)^2}\int_0^{2\pi}\log(z^2+r^2\cos^2\theta)d\theta\end{aligned}$$

and hence

$$\begin{bmatrix}\varphi_{,x}\\ \varphi_{,y}\\ \varphi_{,z}\end{bmatrix} = \frac{1}{2\pi}\begin{bmatrix}-\frac{x(R+z)}{rR}\\ -\frac{y(R+z)}{rR}\\ \frac{1}{R}\end{bmatrix},$$

where $R=(z^2+r^2)^{1/2}$*. Finally, by (140) we conclude that for any* $(x,y,z)\in H$*,*

$$\begin{bmatrix}u_x\\ u_y\\ u_z\end{bmatrix} = \frac{P}{4\pi\mu}\begin{bmatrix}\frac{xz}{R^3}+(1-2\nu)\frac{x}{R(R-z)}\\ \frac{yz}{R^3}+(1-2\nu)\frac{y}{R(R-z)}\\ -\frac{z^2}{R^3}-(1-2\nu)\frac{1}{R}\end{bmatrix}. \tag{141}$$

From the above equation, if there is a distributed surface load $p(x,y)$ *in* $-\mathbf{e}_z$ *direction on* $\{\mathbf{x}: z=0\}$*, then the displacement is given by the Green's formula*

$$\begin{bmatrix}u_x\\ u_y\\ u_z\end{bmatrix} = \frac{1}{4\pi\mu}\begin{bmatrix}z\int\frac{p(x',y')(x-x')}{R'^3}+(1-2\nu)\int\frac{p(x',y')(x-x')}{R'(R'-z)}\\ z\int\frac{p(x',y')(y-y')}{R'^3}+(1-2\nu)\int\frac{p(x',y')(y-y')}{R'(R'-z)}\\ -z^2\int\frac{p(x',y')}{R'^3}-(1-2\nu)\int\frac{p(x',y')}{R'}\end{bmatrix}, \tag{142}$$

where

$$R'=[(x-x')^2+(y-y')^2+z^2]^{1/2}.$$

♦ 31. Consider a homogeneous half space $H = \{\mathbf{x} \in I\!R^3 : z < 0\}$. For given $k > 0$, let $\mathbf{u} : H \to I\!R^3$ be given by

$$\mathbf{u}(\mathbf{x}) = (\mathbf{a} + z\mathbf{b}) \exp(kz) \exp(ikx), \tag{143}$$

where $\mathbf{a}, \mathbf{b} \in I\!R^3$ are independent of position $\mathbf{x}$. Consider the equilibrium equation

$$\mathrm{div}(\mathbf{C}\nabla\mathbf{u}) = 0 \quad \text{on } \ H, \tag{144}$$

where $\mathbf{C}$ is an isotropic stiffness tensor, i.e., the above equation is equivalent to

$$\mu\Delta\mathbf{u} + (\mu + \lambda)\nabla(\nabla \cdot \mathbf{u}) = 0 \quad \text{on } \ H. \tag{145}$$

(i) (3pt) Plug (143) into (145). Find the conditions on the vectors $\mathbf{a}$ and $\mathbf{b}$ such that (145) is satisfied.

(ii) (3pt) Calculate the traction on the plane $\{\mathbf{x} : z = 0\}$ in terms of $\mathbf{a}, \mathbf{b}$, i.e., the quantity

$$\mathbf{t}(x, y, z = 0) = (\mathbf{C}\nabla\mathbf{u})\mathbf{e}_z\big|_{z=0}$$

(iii) (4pt) Assume the traction

$$(\mathbf{C}\nabla\mathbf{u})\mathbf{e}_z\big|_{z=0} = \hat{\mathbf{t}}^0 \exp(ikx),$$

where $\hat{\mathbf{t}}^0 = [t_1^0, 0, t_3^0] \in I\!R^3$ is given. Solve for vectors $\mathbf{a}, \mathbf{b}$ in terms of $\hat{\mathbf{t}}^0$, k, μ, λ.

8.4 Contact problem —— Hertz's solution

Formulation of the problem. Consider a homogeneous body with an elastic stiffness tensor $\mathbf{C}$ on a half-sphere $\Omega = \{\mathbf{x}: \ x^2 + y^2 + (z + R_0)^2 < R_0^2, z > -R_0\}$. Assume the base of the body is hold still, i.e., $\mathbf{u} = 0$ on $\Gamma_D = \partial\Omega \cap \{\mathbf{x} : z = -R_0\}$ and Ω is initially in contact with a rigid plate Γ at the point $\mathbf{x} = 0$. Assume that there is no friction between the rigid plate and Ω. Now we push down the rigid plate Γ by a small amount h in z-direction. We are interested in the elastic field, i.e., solving for $\mathbf{u} : \Omega \to I\!R^3$,

$$\begin{cases} \mathrm{div}(\mathbf{C}\nabla\mathbf{u}) = 0 & \text{on } \ \Omega, \\ (\mathbf{C}\nabla\mathbf{u})\mathbf{n} = 0 & \text{on } \ \partial\Omega \setminus \Gamma_C, \\ \mathbf{u} = 0 & \text{on } \ \Gamma_D, \\ (\mathbf{x} + \mathbf{u}(\mathbf{x})) \cdot \mathbf{e}_z \le -h & \forall\, \mathbf{x} \in \Omega, \\ (\mathbf{C}\nabla\mathbf{u})\mathbf{n} = -p(x,y)\mathbf{e}_z & \text{on } \ \Gamma_C, \end{cases} \tag{146}$$

where the contact area

$$\Gamma_C = \{(x, y, z) : z + u_z(x, y, z) = -h\}. \tag{147}$$

Solution. Let $\Gamma'_C = \{(x, y) : (x, y, z) \in \Gamma_C\}$ be the contact area in the current configuration (projected to the xy-plane). Then for a point $(x, y, z) \in \Gamma_D$, we have

$$z \approx -\frac{1}{2}\kappa_{\alpha\beta}x_\alpha x_\beta = -\frac{1}{R_0}(x^2 + y^2). \tag{148}$$

Since $h << 1$, the contact area is small compared with the curvature of $\partial\Omega$ at the contact point. Therefore, near the contact point, the solution to (146) is given by the BOUSSINESQ's solution (142). In particular,

$$u_z(x, y, z = 0) = -\frac{1 - 2\nu}{4\pi\mu}\int_{\Gamma'_C}\frac{p(x', y')}{r'}dx'dy', \tag{149}$$

where

$$r' = [(x - x')^2 + (y - y')^2]^{1/2}.$$

From (147), (148) and (149), we have

$$-\frac{1}{2R_0}(x^2 + y^2) - \frac{1 - 2\nu}{4\pi\mu}\int_{\Gamma'_C}\frac{p(x', y')}{r'}dx'dy' = -h \qquad \forall\,(x, y) \in \Gamma'_C. \tag{150}$$

The above equation is peculiar in the sense that we need to determined the unknown function $p(x', y')$ and the unknown domain Γ'_C simultaneously. There is a general theory called *the variational inequalities* which addresses such kind of *free-boundary problems*, see e.g. FREEDMAN (1982). Below, by our knowledge of *potential theory and elliptic integrals*, we solve such an equation which includes (150) as a special case

$$\kappa_1 x^2 + \kappa_2 y^2 + \frac{1}{2\pi}\int_\Gamma \frac{f(x', y')}{r'} dx'dy' = h \qquad \forall\, (x, y) \in \Gamma, \tag{151}$$

where $\kappa_1, \kappa_2 > 0$ are two constants, $f : \Gamma \to I\!R$ and $\Gamma \subset I\!R^2$ are to be determined.

First, let us recall a theorem.

Theorem 4 *Let $\Omega = \{\mathbf{x} : \sum_{i=1}^n \frac{x_i^2}{a_i^2} = 1\}$ be an ellipsoid in $I\!R^n$. Consider a* POISSON*'s equation*

$$\Delta\varphi(\mathbf{x}) = -\rho\chi_\Omega(\mathbf{x}) \qquad \text{on } I\!R^n,$$

where $\rho \in I\!R$ is a constant and $\chi_\Omega = 1$ on Ω but vanishes otherwise. Then the solution

$$\varphi(\mathbf{x}) = \rho \int_\Omega \phi(\mathbf{x} - \mathbf{x}')d\mathbf{x}' \tag{152}$$

satisfies

$$\varphi(\mathbf{x}) = D_\Omega - \frac{1}{2}\mathbf{x} \cdot \mathbf{Q}_\Omega \mathbf{x} \qquad \forall\, \mathbf{x} \in \Omega, \tag{153}$$

where ϕ is the Green's function given by (110), the constant $D_\Omega > 0$ and the symmetric matrix $\mathbf{Q}_\Omega$ are given by (158) and (161), respectively.

Proof: By the Fourier method, we have that for any $\mathbf{x} \in I\!R^n$,

$$\begin{aligned}\varphi(\mathbf{x}) &= \frac{\rho}{(2\pi)^n}\int_{I\!R^n} \frac{1}{|\mathbf{k}|^2}\hat{\chi}_\Omega(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})d\mathbf{k},\\ \nabla\nabla\varphi(\mathbf{x}) &= \frac{-\rho}{(2\pi)^n}\int_{I\!R^n} \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\hat{\chi}_\Omega(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})d\mathbf{k},\end{aligned} \tag{154}$$

where

$$\hat{\chi}_\Omega(\mathbf{k}) = \int \chi_\Omega \exp(-i\mathbf{k}\cdot\mathbf{x})d\mathbf{x}.$$

Let

$$\begin{aligned}&k = |\mathbf{k}|, \quad \hat{\mathbf{k}} = \mathbf{k}/k, \quad \mathbf{A} = \mathrm{diag}[a_1, \cdots, a_n],\\ &B_n := \mathbf{A}^{-1}\Omega = \{\mathbf{A}^{-1}\mathbf{x} : \mathbf{x} \in \Omega\}, \quad S^{n-1} = \partial B_n.\end{aligned} \tag{155}$$

Note that B_n is the n-dimensional unit ball centered at the origin, and S^{n-1} the spherical surface of the unit ball B_n, which is (locally) *homeomorphic* with $(n-1)$-dimensional space.

Direct integration yields

$$\begin{aligned}\varphi(\mathbf{x}) &= \frac{\rho}{(2\pi)^n}\int_{S^{n-1}}\int_0^\infty \frac{k^{n-1}}{k^2}\int_\Omega \exp(ik\hat{\mathbf{k}}\cdot(\mathbf{x}-\mathbf{x}_1))d\mathbf{x}_1 dk d\hat{\mathbf{k}}\\ &= \frac{\rho}{(2\pi)^n}\int_{S^{n-1}}\int_0^\infty k^{n-3}\int_\Omega \exp(ik(\mathbf{A}\hat{\mathbf{k}})\cdot(\mathbf{A}^{-1}\mathbf{x}-\mathbf{A}^{-1}\mathbf{x}_1))d\mathbf{x}_1 dk d\hat{\mathbf{k}}\\ &= \frac{\rho}{(2\pi)^n}\int_{S^{n-1}}\int_0^\infty \frac{1}{|\mathbf{A}\hat{\mathbf{k}}|^{n-3}}(|\mathbf{A}\hat{\mathbf{k}}|k)^{n-3}\int_\Omega \exp(ik|\mathbf{A}\hat{\mathbf{k}}|\frac{(\mathbf{A}\hat{\mathbf{k}})}{|\mathbf{A}\hat{\mathbf{k}}|}\cdot(\mathbf{A}^{-1}\mathbf{x}-\mathbf{A}^{-1}\mathbf{x}_1))d\mathbf{x}_1 dk d\hat{\mathbf{k}}\\ &= \rho\int_{S^{n-1}} \frac{\det(\mathbf{A})}{|\mathbf{A}\hat{\mathbf{k}}|^{n-2}} g_{B_n}(\frac{\mathbf{A}\hat{\mathbf{k}}}{|\mathbf{A}\hat{\mathbf{k}}|}, \mathbf{A}^{-1}\mathbf{x})d\hat{\mathbf{k}}\end{aligned} \tag{156}$$

where

$$g_{B_n}(\hat{\mathbf{k}}, \mathbf{x}) = \frac{1}{(2\pi)^n} \int_0^\infty k^{n-3} \int_{B_n} \exp(ik\hat{\mathbf{k}} \cdot (\mathbf{x} - \mathbf{x}_1)) d\mathbf{x}_1 dk.$$

In particular, if $\mathbf{x} = 0$, we have that for any $\mathbf{R} \in So(n)$,

$$\begin{aligned} g_{B_n}(\hat{\mathbf{k}}, 0) &= \frac{1}{(2\pi)^n} \int_0^\infty k^{n-3} \int_{B_n} \exp(-ik\hat{\mathbf{k}} \cdot \mathbf{x}_1) d\mathbf{x}_1 dk \\ &= \frac{1}{(2\pi)^n} \int_0^\infty k^{n-3} \int_{B_n} \exp(-ik(\mathbf{R}\hat{\mathbf{k}})^T \cdot \mathbf{R}\mathbf{x}_1) d\mathbf{x}_1 dk = g_{B_n}(\mathbf{R}\hat{\mathbf{k}}, 0) =: \omega_1. \end{aligned} \tag{157}$$

Thus, $g_{B_n}(\hat{\mathbf{k}}, 0)$ is in fact independent of $\hat{\mathbf{k}}$ and

$$D_\Omega = \rho \det(\mathbf{A}) \omega_1 \int_{S^{n-1}} \frac{1}{|\mathbf{A}\hat{\mathbf{k}}|^{n-2}} d\hat{\mathbf{k}}. \tag{158}$$

Similarly, we have

$$\nabla\nabla\varphi(\mathbf{x}) = -\rho \det(\mathbf{A}) \int_{S^{n-1}} \frac{\hat{\mathbf{k}} \otimes \hat{\mathbf{k}}}{|\mathbf{A}\hat{\mathbf{k}}|^n} g'_{B_n}(\frac{\mathbf{A}\hat{\mathbf{k}}}{|\mathbf{A}\hat{\mathbf{k}}|}, \mathbf{A}^{-1}\mathbf{x}) d\hat{\mathbf{k}}, \tag{159}$$

where

$$g'_{B_n}(\hat{\mathbf{k}}, \mathbf{x}) = \frac{1}{(2\pi)^n} \int_0^\infty k^{n-1} \int_{B_n} \exp(ik\hat{\mathbf{k}} \cdot (\mathbf{x} - \mathbf{x}_1)) d\mathbf{x}_1 dk.$$

♦ 32 (i). (4pt) Show in details the above equation (159) (as in (156)).

It is a fundamental property of spherical domain that

$$g'_{B_n}(\hat{\mathbf{k}}, \mathbf{x}_1) = g'_{B_n}(\hat{\mathbf{k}}, \mathbf{x}_2) =: \omega_2 \qquad \forall \mathbf{x}_1, \mathbf{x}_2 \in B_n. \tag{160}$$

That is, the function $\mathbf{x} \mapsto g'_{B_n}(\hat{\mathbf{k}}, \mathbf{x})$ is a constant on B_n. Further, choosing $\mathbf{x} = 0$, by a similar argument as in (157), we can show the constant ω_2 is independent of $\hat{\mathbf{k}}$ as well. Thus, we identify

$$\mathbf{Q}_\Omega = \rho \det(\mathbf{A}) \omega_2 \int_{S^{n-1}} \frac{\hat{\mathbf{k}} \otimes \hat{\mathbf{k}}}{|\mathbf{A}\hat{\mathbf{k}}|^n} d\hat{\mathbf{k}}, \tag{161}$$

To find ω_1, ω_2, we consider the case Ω is the unit ball B_n and so $\mathbf{A} = \mathbf{I}$. Immediately, we have

$$\begin{aligned} D_{B_n} &= \omega_1 \int_{S^{n-1}} \frac{1}{|\hat{\mathbf{k}}|^{n-2}} d\hat{\mathbf{k}} \Longrightarrow \omega_1 = D_0 / |S^{n-1}|, \\ \mathbf{Q}_{B_n} &= \frac{1}{n} \mathbf{I}, \qquad \frac{1}{n\omega_2} = \int_{S^{n-1}} \hat{k}_1^2 d\hat{\mathbf{k}} = \cdots = \int_{S^{n-1}} \hat{k}_n^2 d\hat{\mathbf{k}} \Longrightarrow \omega_2 = 1/|S^{n-1}|, \end{aligned} \tag{162}$$

where $|S^{n-1}|$ denotes the area of the spherical surface S^{n-1}, and D_{B_n}, $\mathbf{Q}_{B_n}$ are determined by

$$\Delta\varphi_{B_n}(\mathbf{x}) = -\chi_{B_n}, \qquad \varphi_{B_n}(\mathbf{x}) = D_{B_n} - \frac{1}{2}\mathbf{x}\cdot\mathbf{Q}_{B_n}\mathbf{x} \quad \text{on } B_n. \tag{163}$$

■

♦ 32 (ii). (3pt) Assume $n = 3$. Calculate the value of D_{B_n} defined in (163).

Note that for any continuous function $f : S^{n-1} \to I\!R$,

$$\int_{S^{n-1}} f(\hat{k}_1,\cdots,\hat{k}_n)d\hat{\mathbf{k}} = \int_{B_{n-1}} \frac{f(\hat{k}_1,\cdots,\hat{k}_{n-1},(1-\hat{k}_1^2-\cdots-\hat{k}_{n-1}^2)^{1/2})}{(1-\hat{k}_1^2-\cdots-\hat{k}_{n-1}^2)^{1/2}}d\hat{k}_1\cdots d\hat{k}_{n-1}$$
$$+\int_{B_{n-1}} \frac{f(\hat{k}_1,\cdots,\hat{k}_{n-1},-(1-\hat{k}_1^2-\cdots-\hat{k}_{n-1}^2)^{1/2})}{(1-\hat{k}_1^2-\cdots-\hat{k}_{n-1}^2)^{1/2}}d\hat{k}_1\cdots d\hat{k}_{n-1}. \tag{164}$$

We now continue our solution to (151). From (152)-(153), in three dimensional space ($n = 3$) we have the identity

$$\frac{\rho}{4\pi}\int_\Omega \frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}dx'dy'dz' = D_\Omega - \frac{1}{2}\mathbf{x}\cdot\mathbf{Q}_\Omega\mathbf{x}. \tag{165}$$

Now send $a_3 \to 0$ while keep $\rho a_3 = 1$. Then the domain Ω approaches to a flat elliptic area $\Gamma = \{(x,y) : x_1^2/a_1^2 + y^2/a_2^2 = 1\}$ on the xy-plane, and we denote by

$$D_\Gamma = \lim_{a_3=1/\rho\to 0} D_\Omega = a_1a_2\omega_1 \lim_{a_3\to 0}\int_{S^{n-1}} \frac{1}{(a_1^2\hat{k}_1^2+a_2^2\hat{k}_2^2+a_3^2\hat{k}_3^2)^{(n-2)/2}}d\hat{\mathbf{k}}$$
$$= 2a_1a_2\omega_1\int_{B_{n-1}} \frac{1}{(a_1^2\hat{k}_1^2+a_2^2\hat{k}_2^2)^{(n-2)/2}(1-\hat{k}_1^2-\hat{k}_2^2)^{1/2}}d\hat{k}_1d\hat{k}_2, \tag{166}$$

where the last equality follows from (164). Similarly, we have

$$\mathbf{Q}_\Gamma = \lim_{a_3=1/\rho\to 0}\mathbf{Q}_\Omega = \mathrm{diag}[Q_\Gamma^1, Q_\Gamma^2, Q_\Gamma^3], \tag{167}$$

where

$$Q_\Gamma^i = a_1a_2\omega_2\lim_{a_3\to 0}\int_{S^{n-1}} \frac{\hat{k}_i^2}{(a_1^2\hat{k}_1^2+a_2^2\hat{k}_2^2+a_3^2\hat{k}_3^2)^{n/2}}d\hat{\mathbf{k}}$$
$$= 2a_1a_2\omega_2\int_{B_{n-1}} \frac{\hat{k}_i^2}{(a_1^2\hat{k}_1^2+a_2^2\hat{k}_2^2)^{n/2}(1-\hat{k}_1^2-\hat{k}_2^2)^{1/2}}d\hat{k}_1d\hat{k}_2, \tag{168}$$

where $\hat{k}_3^2 = 1 - \hat{k}_1^2 - \hat{k}_2^2$. Further, in this limit the l.h.s. of (165) can be written as

$$\begin{aligned}
&\lim_{a_3=1/\rho\to 0} \frac{\rho}{4\pi}\int_\Omega \frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}dx'dy'dz' \\
&= \lim_{a_3=1/\rho\to 0} \frac{\rho}{4\pi}\int_\Gamma\int_{-a_3 z(x',y')}^{a_3 z(x',y')} \frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}dz'dx'dy' \\
&= \frac{1}{2\pi}\int_\Gamma \frac{z(x',y')}{[(x-x')^2+(y-y')^2+z^2]^{1/2}}dx'dy' \qquad (169)
\end{aligned}$$

where

$$z(x,y) = (1 - x^2/a_1^2 - y^2/a_2^2)^{1/2}.$$

♦ 32 (iii). (1pt) Explain in words why the last equality in (169) holds.

Therefore, taking into account (165)-(169), choosing $z = 0$ we arrive at

$$\frac{1}{2\pi}\int_\Gamma \frac{z(x',y')}{[(x-x')^2+(y-y')^2]^{1/2}}dx'dy' = D_\Gamma - \frac{1}{2}(Q_\Gamma^1 x^2 + Q_\Gamma^2 y^2) \qquad \forall\,(x,y)\in\Gamma. \qquad (170)$$

Comparing (170) with (151), we see that if for some $l \in I\!R$,

$$h = lD_\Gamma, \quad \kappa_1 = lQ_\Gamma^1, \quad \kappa_2 = lQ_\Gamma^2, \qquad (171)$$

then equation (151) is satisfied with

$$\Gamma_C = \Gamma, \quad f(x',y') = lz(x',y').$$

We remark that equation (171) determines a_1, a_2 and the constant l uniquely.

In particular, if $a_1 = a_2 = a_0$, by (168) we see that (By (162) $\omega_2 = 1/4\pi$)

$$Q_\Gamma^1 = Q_\Gamma^2 = \frac{1}{2\pi a_0}\int_{B_2} \frac{\hat{k}_1^2}{(\hat{k}_1^2+\hat{k}_2^2)^{3/2}(1-\hat{k}_1^2-\hat{k}_2^2)^{1/2}}d\hat{k}_1 d\hat{k}_2,$$

and

$$Q_\Gamma^1 + Q_\Gamma^2 = \frac{1}{2\pi a_0}\int_{B_2} \frac{1}{(\hat{k}_1^2+\hat{k}_2^2)^{1/2}(1-\hat{k}_1^2-\hat{k}_2^2)^{1/2}}d\hat{k}_1 d\hat{k}_2 = \frac{1}{a_0}\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2a_0},$$

where the last integral is found by using Mathematica (which you should learn how to use). Similarly, by (166), we find (By (162) $\omega_1 = 1/8\pi$)

$$D_\Gamma = \frac{a_0}{4\pi}\int_{B_{n-1}} \frac{1}{(\hat{k}_1^2+\hat{k}_2^2)^{1/2}(1-\hat{k}_1^2-\hat{k}_2^2)^{1/2}}d\hat{k}_1 d\hat{k}_2 = \frac{\pi a_0}{4}.$$

By (170), we obtain the formula that for $\Gamma = \{(x,y) : x^2+y^2 < a_0^2\}$ and $z(x,y) = (1 - x^2/a_0^2 - y^2/a_0^2)^{1/2}$,

$$\frac{1}{2\pi}\int_\Gamma \frac{z(x',y')}{[(x-x')^2+(y-y')^2]^{1/2}}dx'dy' = \frac{\pi a_0}{4} - \frac{\pi}{8a_0}(x^2+y^2) \qquad \forall\,(x,y)\in\Gamma. \qquad (172)$$

♦ 32 (iv). (6pt) Using the above formula (172) and the procedure outlined in (171), solve (150), i.e., find the unknown Γ'_C and $p(x', y')$. Further, let $F = \int_{\Gamma'_C} p(x', y') dx' dy'$. What is the functional relation between F and h? This relation, i.e., $F = F(h)$, is called the HERTZ contact force law.

8.5 Inclusion problem —— Eshelby's solution

Formulation of the problem. Let $\Omega := \{\mathbf{x} : \sum_{i=1}^n x_i^2/a_i^2 = 1\}$ be an ellipsoidal inclusion. Consider a homogeneous body with elastic stiffness tensor $\mathbf{C}_0$ in $I\!R^n$. We are interested in solving for $\mathbf{u} : I\!R^n \to I\!R^n$,

$$\begin{cases} \operatorname{div}(\mathbf{C}_0\nabla\mathbf{u} + \mathbf{P}^0\chi_\Omega) = 0 & \text{on } I\!R^n, \\ |\nabla\mathbf{u}| \to 0 & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{173}$$

where $\mathbf{P}^0 \in I\!R^{n\times n}$ is called eigenstress. This problem is called the *homogeneous* Eshelby's inclusion problem.

Solution. By Fourier transformation, we have

$$\begin{aligned} \nabla\mathbf{u}(\mathbf{x}) &= \frac{-1}{(2\pi)^n}\int_{I\!R^n} \frac{\hat{\mathbf{k}} \otimes \operatorname{cof}\mathbf{D}(\hat{\mathbf{k}})^T\mathbf{P}^0\hat{\mathbf{k}}}{\det(\mathbf{D}(\hat{\mathbf{k}}))}\int_\Omega \exp(i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_1))d\mathbf{x}_1 d\mathbf{k} \\ &= -\det(\mathbf{A})\int_{S^{n-1}} \frac{\hat{\mathbf{k}} \otimes \operatorname{cof}\mathbf{D}(\hat{\mathbf{k}})^T\mathbf{P}^0\hat{\mathbf{k}}}{\det(\mathbf{D}(\hat{\mathbf{k}}))|\mathbf{A}\hat{\mathbf{k}}|^n} g'_{B_n}(\frac{\mathbf{A}\hat{\mathbf{k}}}{|\mathbf{A}\hat{\mathbf{k}}|}, \mathbf{A}^{-1}\mathbf{x})d\hat{\mathbf{k}} \end{aligned}$$

where

$$[\mathbf{D}(\mathbf{k})]_{pq} = [\mathbf{C}_0]_{piqj}k_ik_j, \quad [\mathbf{D}(\mathbf{k})]^{-1} = \operatorname{cof}\mathbf{D}(\mathbf{k})^T/\det(\mathbf{D}(\mathbf{k})).$$

By (160), we find

$$\nabla\mathbf{u}(\mathbf{x}) = -\det(\mathbf{A})\omega_2\int_{S^{n-1}} \frac{\hat{\mathbf{k}} \otimes \operatorname{cof}\mathbf{D}(\hat{\mathbf{k}})^T\mathbf{P}^0\hat{\mathbf{k}}}{\det(\mathbf{D}(\hat{\mathbf{k}}))|\mathbf{A}\hat{\mathbf{k}}|^n}d\hat{\mathbf{k}} \qquad \forall\, \mathbf{x} \in \Omega.$$

That is, inside the ellipsoidal inclusion Ω, the induced strain is uniform and can be written as

$$\nabla\mathbf{u} = -\mathbf{R}\mathbf{P}^0 \qquad \text{on } \Omega, \tag{174}$$

where

$$[\mathbf{R}]_{piqj} = \det(\mathbf{A})\omega_2\int_{S^{n-1}} \frac{[\operatorname{cof}\mathbf{D}(\hat{\mathbf{k}})^T]_{pq}(\hat{\mathbf{k}})_i(\hat{\mathbf{k}})_j}{\det(\mathbf{D}(\hat{\mathbf{k}}))|\mathbf{A}\hat{\mathbf{k}}|^n}d\hat{\mathbf{k}} \tag{175}$$

is called the Eshelby's tensor. The uniformity of the induced strain is now called the Eshelby's uniformity property, which is remarkable since we solve a PDE but obtain a simple quadratic solution.

In particular, if $\mathbf{C}_0$ is isotropic with LAMÉ constants μ, λ, by direct calculations we find the above ESHELBY's tensor can be written as

$$\mathbf{R} = \frac{\omega_2}{\mu}[\mathbf{S}_1 - \frac{\mu+\lambda}{2\mu+\lambda}\mathbf{S}_2],$$

where

$$[\mathbf{S}_1]_{piqj} = \det(\mathbf{A}) \int_{S^{n-1}} \frac{\delta_{pq}\hat{k}_i\hat{k}_j}{|\mathbf{A}\hat{\mathbf{k}}|^n}, \quad [\mathbf{S}_2]_{piqj} = \det(\mathbf{A}) \int_{S^{n-1}} \frac{\hat{k}_p\hat{k}_q\hat{k}_i\hat{k}_j}{|\mathbf{A}\hat{\mathbf{k}}|^n}.$$

Note that the tensors $\mathbf{S}_1$ and $\mathbf{S}_2$ depends only on the geometric properties of ellipsoid.

A critical observation made by ESHELBY in his famous 1957 paper is that the solution the *inhomogeneous* inclusion problem

$$\begin{cases} \operatorname{div}(\mathbf{C}(\mathbf{x})\nabla\mathbf{u} + \mathbf{P}^*\chi_\Omega) = 0 & \text{on } I\!R^n, \\ |\nabla\mathbf{u}| \to 0 & \text{as } |\mathbf{x}| \to \infty, \end{cases} \qquad \mathbf{C}(\mathbf{x}) = \begin{cases} \mathbf{C}_1 & \text{if } \mathbf{x} \in \Omega, \\ \mathbf{C}_0 & \text{if } \mathbf{x} \in \bar{\Omega}^c \end{cases} \tag{176}$$

if the eigenstress for the homogeneous problem $\mathbf{P}^0$ is appropriately chosen. To see this, let us formally rewrite equations (173)and (176) in a less concise form as

$$\begin{cases} \operatorname{div}[\mathbf{C}_0\nabla\mathbf{u}] = 0 & \text{in } \bar{\Omega}^c, \\ \operatorname{div}[\mathbf{C}_0\nabla\mathbf{u}] = 0 & \text{in } \Omega, \\ [\![\mathbf{C}_0\nabla\mathbf{u} + \mathbf{P}^0\chi_\Omega]\!]\mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{177}$$

and

$$\begin{cases} \operatorname{div}[\mathbf{C}_0\nabla\mathbf{u}] = 0 & \text{in } \bar{\Omega}^c, \\ \operatorname{div}[\mathbf{C}_1\nabla\mathbf{u}] = 0 & \text{in } \Omega, \\ [\![\mathbf{C}(\mathbf{x})\nabla\mathbf{u} + \mathbf{P}^*\chi_\Omega)]\!]\mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{178}$$

respectively, where $[\![\cdot]\!]$ denotes the jump across the $\partial\Omega$. Clearly, a solution to (177) satisfies the first two of (178) automatically since on Ω, the first of (177) coincides with the first of (178), and the uniformity of (174) guarantees the second of (178). Finally, to verify the last of (178), we rewrite the last of (177) as

$$[\mathbf{C}_0\nabla\mathbf{u}^+ - \mathbf{C}_0\nabla\mathbf{u}^- - \mathbf{P}^0]\mathbf{n} = 0 \qquad \text{on } \partial\Omega, \tag{179}$$

and the last of (178) as

$$[\mathbf{C}_0\nabla\mathbf{u}^+ - \mathbf{C}_1\nabla\mathbf{u}^- - \mathbf{P}^*]\mathbf{n} = 0 \qquad \text{on } \partial\Omega, \tag{180}$$

where $\mathbf{u}^+$ ($\mathbf{u}^-$) is the boundary value approached from the outside (inside) of Ω. From (179) and (174), we have

$$[\mathbf{C}_0\nabla\mathbf{u}^+]\mathbf{n} = [\mathbf{C}_0\nabla\mathbf{u}^- + \mathbf{P}^0]\mathbf{n} = [-\mathbf{C}_0\mathbf{R}\mathbf{P}^0 + \mathbf{P}^0]\mathbf{n} \qquad \text{on } \partial\Omega.$$

Plugging the above equation into (180), we verify that equation (180) is satisfied as well if

$$[(\mathbf{C}_1 - \mathbf{C}_0)\mathbf{R} + I\!I]\mathbf{P}^0 = \mathbf{P}^*, \tag{181}$$

where $I\!I : I\!R^{n\times n} \to I\!R^{n\times n}$ is the identity mapping, i.e., for any $\mathbf{P}^0 \in I\!R^{n\times n}$, $I\!I\mathbf{P}^0 = \mathbf{P}^0$. From the definition of $\mathbf{R}$ (see (175)), we can show that the the above linear algebraic equation admits a unique solution $\mathbf{P}^0 \in I\!R^{n\times n}_{sym}$ for any given $\mathbf{P}^*_{sym}$. The above solutions are referred to as the ESHELBY's solutions.

♦ 33. Important physical quantities can be explicitly computed for the inhomogeneous inclusion problem (176).

(i) (2pt) Calculate the elastic energy in terms of the Eshelby's tensor $\mathbf{R}$, which is given by

$$\mathcal{E}(\mathbf{P}^*) = \frac{1}{2}\int_{I\!R^n} \nabla\mathbf{u}\cdot\mathbf{C}(\mathbf{x})\nabla\mathbf{u}.$$

(Hints: use the divergence theorem

$$\int_{I\!R^n} \nabla\mathbf{v}\cdot\sigma = -\int_{I\!R^n} \mathbf{v}\cdot \mathrm{div}\sigma.$$

In particular, you may choose $\sigma = \mathbf{C}(\mathbf{x})\nabla\mathbf{u} + \chi_\Omega\mathbf{P}^*$.)

Now let us consider a physical situation: let V be a finite but large elastic body containing the ellipsoidal inhomogeneity Ω. Assume the body is subjected to a uniform applied stress $\mathbf{P}^a \in I\!R^{n\times n}_{sym}$, i.e., the stress satisfies

$$\sigma\mathbf{n} = \mathbf{P}^a\mathbf{n} = \mathbf{t} \qquad \text{on } \partial V,$$

and the material properties are given by

$$\mathbf{C}(\mathbf{x}) = \begin{cases} \mathbf{C}_1 & \text{if } \mathbf{x}\in\Omega, \\ \mathbf{C}_0 & \text{if } \mathbf{x}\in\bar{\Omega}^c = V\setminus\bar{\Omega} \end{cases}$$

see the following figure.

(ii) (2pt) What are the governing equation and boundary conditions for the displacement?

(iii) (2pt) The displacement $\mathbf{u}$ may be written as

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1, \qquad \mathbf{u}_0 = [\mathbf{C}_0^{-1}\mathbf{P}^a]\mathbf{x}.$$

What is the governing equation and boundary conditions for $\mathbf{u}_1$?

(iv) (4pt) In particular, we notice that $\mathbf{u}_1$ satisfies

$$(\mathbf{C}_0\nabla\mathbf{u}_1)\mathbf{n} = 0 \qquad \text{on } \partial V.$$

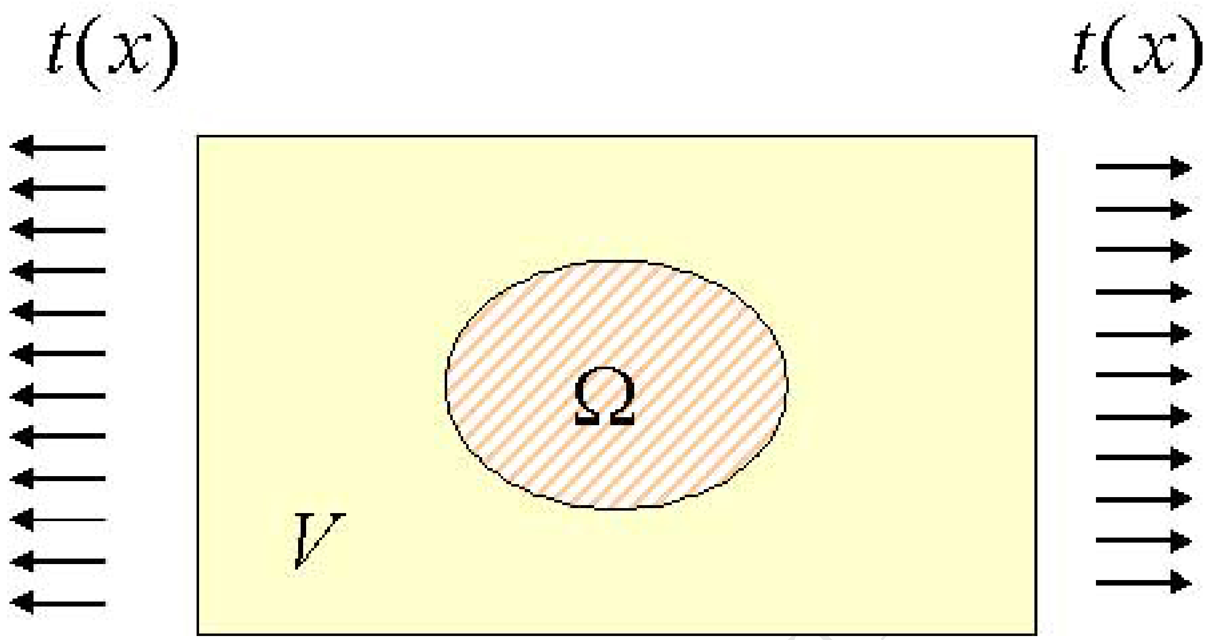

Figure 3: An elastic body with an ellipsoidal inhomogeneity

Since V is much larger than Ω, to solve the governing equation for $\mathbf{u}_1$, we may replace V by $I\!R^n$ and require the strain and stress approach to zero at infinity. Calculate the strain, i.e., $\nabla \mathbf{u}_1$, on Ω in terms of the applied stress $\mathbf{P}^a$ and the Eshelby's tensor. (Hint: identify the equivalent eigenstress $\mathbf{P}^0$ for the homogeneous problem such that

$$\text{div}[\mathbf{C}_0 \nabla \mathbf{u}_1 + \mathbf{P}^0 \chi_\Omega] = 0.$$

That is, find the relation between $\mathbf{P}^a$ and $\mathbf{P}^0$.)

(v) (3pt) Now let us calculate the total free energy of the system which can be defined as

$$
\begin{aligned}
\mathcal{F}(\mathbf{C}_1, \mathbf{R}; \mathbf{P}^a) &= \frac{1}{2}\int_V \nabla\mathbf{u}\cdot\mathbf{C}(\mathbf{x})\nabla\mathbf{u} - \int_{\partial V}\mathbf{u}\cdot\mathbf{P}^a\mathbf{n} \\
&= \frac{1}{2}\int_V \left\{\nabla\mathbf{u}\cdot[\mathbf{C}_0 + (\mathbf{C}_1 - \mathbf{C}_0)\chi_\Omega]\nabla\mathbf{u} - \nabla\mathbf{u}\cdot\mathbf{P}^a\right\} \\
&= \frac{1}{2}\int_V \left\{\nabla\mathbf{u}\cdot\mathbf{C}_0\nabla\mathbf{u} + \nabla\mathbf{u}\cdot(\mathbf{C}_1 - \mathbf{C}_0)\chi_\Omega]\nabla\mathbf{u} - \nabla\mathbf{u}\cdot\mathbf{P}^a\right\} \\
&= \frac{1}{2}\int_V \left\{\mathbf{P}^a\cdot\mathbf{C}_0^{-1}\mathbf{P}^a + \chi_\Omega\nabla\mathbf{u}\cdot(\mathbf{C}_1 - \mathbf{C}_0)\nabla\mathbf{u} - \nabla\mathbf{u}\cdot\mathbf{P}^a\right\} \\
&= \cdots
\end{aligned}
\tag{182}
$$

Complete the rest of the calculations, i.e., obtain the functional relation $\mathcal{F} = \mathcal{F}(\mathbf{C}_1, \mathbf{R}; \mathbf{P}^a)$. (Hint: $\int_V \nabla\mathbf{u}\cdot\mathbf{P}^a = \int_V(\mathbf{C}_0^{-1}\mathbf{P}^a + \nabla\mathbf{u}_1)\cdot\mathbf{P}^a = \mathbf{P}^a\cdot\mathbf{C}_0^{-1}\mathbf{P}^a + \mathbf{P}^a\cdot\int_V\nabla\mathbf{u}_1 = \mathbf{P}^a\cdot\mathbf{C}_0^{-1}\mathbf{P}^a$. Think for a while about the last equality.)

(vi) (20pt, bonus problem) In 2D, assume $\Omega = \{(x, y), x^2/a^2 + y^2/b^2 = 1\}$, $\mathbf{C}_0$ ($\mathbf{C}_1$) is isotropic with Young's modulus and Poisson's ratio $E_0, \nu_0 = 0.3$ ($E_0, \nu_1 = 0.3$), and $\mathbf{P}^a = \mathrm{diag}[0, t]$. Rewrite the above energy function as

$$\mathcal{F} = \mathcal{F}(E_0, E_1, a, b, t)$$

and define

$$\triangle\mathcal{F}(E_0, E_1, a, b, t) = \mathcal{F}(E_0, E_1, a, b, t) + \frac{1}{2}\mathbf{P}^a\cdot\mathbf{C}_0^{-1}\mathbf{P}^a = \frac{1}{2}\int_\Omega \nabla\mathbf{u}\cdot(\mathbf{C}_1 - \mathbf{C}_0)\nabla\mathbf{u}.$$

Plot the following curves using Mathematica or Matlab (submit your Matlab or Mathematica codes if you choose to work on this problem):

$$
\begin{aligned}
Curve1: &\quad y_1 = y_1(b) = \triangle\mathcal{F}(E_0 = 1, E_1 = 10, a = 1, b, t = 1), \\
Curve2: &\quad y_2 = y_2(b) = \triangle\mathcal{F}(E_0 = 1, E_1 = 10, a = 1, b, t = -1), \\
Curve3: &\quad y_3 = y_3(b) = \triangle\mathcal{F}(E_0 = 1, E_1 = 0.1, a = 1, b, t = 1), \\
Curve4: &\quad y_4 = y_4(b) = \triangle\mathcal{F}(E_0 = 1, E_1 = 0.1, a = 1, b, t = -1), \\
Curve5: &\quad y_5 = y_5(E_1) = \triangle\mathcal{F}(E_0 = 1, E_1, a = 1, b = 0.1, t = 1), \\
Curve6: &\quad y_6 = y_6(E_1) = \triangle\mathcal{F}(E_0 = 1, E_1, a = 1, b = 0.1, t = -1), \\
Curve7: &\quad y_7 = y_7(E_1) = \triangle\mathcal{F}(E_0 = 1, E_1, a = 1, b = 10, t = 1), \\
Curve8: &\quad y_8 = y_8(E_1) = \triangle\mathcal{F}(E_0 = 1, E_1, a = 1, b = 10, t = -1).
\end{aligned}
$$

9 Introduction to Micromechanics

9.1 Cracks

9.2 Composite materials

9.3 Dislocations

9.4 Plasticity

10 A Hierarchy of Plate Theories

10.1 Kirchhoff-Love plate theory

10.2 von Karman plate theory

10.3 Mindlin plate theory

11 Waves in Solid Structures

11.1 Bulk waves

11.2 Surface waves

11.3 Interfacial waves