Gluing Methods and Entire Solutions of Nonlinear Field Equations

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Entire Solutions of Nonlinear Elliptic Equations

Classifying entire solutions of semilinear elliptic equations

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^N$$

is a problem that has always been at the center of nonlinear PDE research.
This is the context of various classical results in PDE literature like

Liouville Theorem

$$\Delta u = 0 \text{ in } \mathbb{R}^N, |u| \leq C \implies u \equiv C$$

Gidas-Ni-Nirenberg Theorem

$$\Delta u + f(u) = 0, u > 0 \text{ in } \mathbb{R}^N, \lim_{|x| \to +\infty} u(x) = 0 \implies u(x) = u(|x - x_0|)$$

Caffarelli-Gidas-Spruck Theorem

$$\Delta u + u^{\frac{N+2}{N-2}} = 0, u > 0 \quad \implies u = c_N \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2}\right)^{\frac{N-2}{2}}$$

In this series of lectures, I will introduce our recent research program of bringing geometry into the constructions and classifications of entire solutions of some classical semilinear elliptic equations.

Two objectives of this mini-course

•• uncover deep connection between geometry and nonlinear field equations
•• introduce the gluing methods: finite dimensional as well as infinite dimensional reduction method

Outline of Mini-course

Lecture 1: New Entire Solutions to Nonlinear Schrödinger Equation: An Overview

(NLS) $$\Delta u - u + u^3 = 0 \quad \text{in } \mathbb{R}^N$$

Lecture 2-3: Finite Dimensional Liapunov-Schmidt Reduction Method and Applications to Finite Energy Solutions of (NLS)

Lecture 4: New Entire Solutions to Allen-Cahn Equation: An Overview

(AC) $$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N$$

Lecture 5-6: Infinite Dimensional Reduction Method and Applications to High Dimensional Concentrations

Lecture One
Entire Solutions of Nonlinear Schrödinger Equations: An Overview

Entire Solutions of NLS

This first lecture deals with entire solutions of

(NLS) $$\Delta u - u + u^p = 0 \quad \text{in } \mathbb{R}^N$$

Nonlinear Schrödinger equation (NLS)

Problem (NLS) arises as

•• standing wave of nonlinear Schrödinger equation

$$-\sqrt{-1}\ \frac{\partial u}{\partial t} = \Delta u + |u|^{p-1} u$$

Let $\boxed{\ u = e^{i\lambda t} v(x)}$

Then v satisfies

$$\Delta\ v - \lambda\ v + |v|^{p-1}\ v = 0$$

λ can be scaled out to be one.

This is an important equation is soliton dynamics.

Or

•• Gierer-Meinhardt system of biological pattern formation

$$\begin{cases} a_t = \varepsilon^2 \Delta\ a - a + \dfrac{a^p}{h^q} \\ \tau h_t = D \Delta\ h - h + \dfrac{a^r}{h^s} \end{cases} \quad \text{Gierer-Meinhardt System}$$

$$\begin{cases} a_t = \varepsilon^2 \Delta\ a - a + \dfrac{a^p}{\xi^q} \\ \tau \xi_t = -\xi + \dfrac{\frac{1}{|\Omega|} \int_\Omega a^r}{\xi^s} \end{cases} \quad \text{shadow System}$$

Ward-Wei, Winter-Wei, Ni-Takagi-Yanagida, etc.

$$\varepsilon^2 \Delta u - u + u^p = 0, u > 0 \quad \text{in } \Omega, \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \qquad (\mathrm{P})_\varepsilon.$$

Ni-Takagi, Gui-Wei, Lin-Ni-Wei, Ao-Wei-Zeng etc.

I.1. Previous Known Entire Solutions

$$\boxed{\triangle u - u + u^p = 0, u > 0 \text{ in } \mathbb{R}^2 \quad (NLS)}$$

It is well-known that if we impose **decaying** assumption

(1) $u(x) \to 0$ as $|x| \to +\infty$ uniformly

then Gidas-Ni-Nirenberg theorem implies that

(2) u is radially symmetric around some point x_0.

Ground State

That is

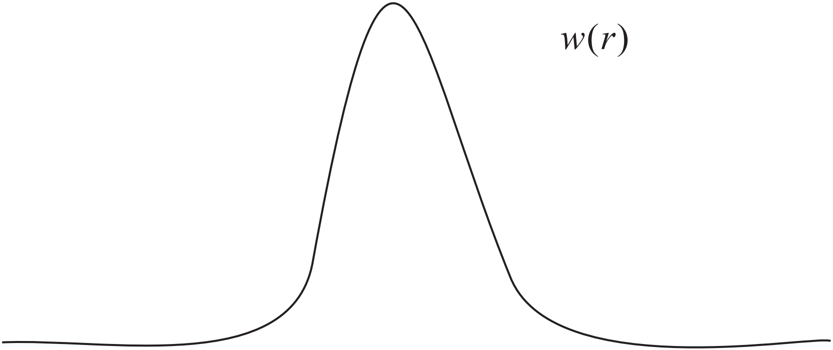
$$u(x) = w(|x - x_0|), \qquad w = w(r)$$

where w satisfies

$$w'' + \frac{1}{r}w' - w + w^p = 0$$

$w = w(r)$ – "ground state"
This solution is called spike. It has been shown to be unique (Kwong 1991)

Ground State



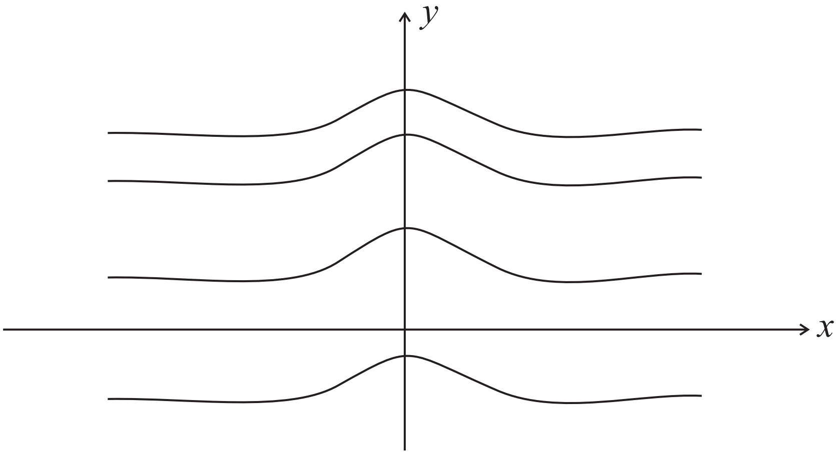
Second Solution

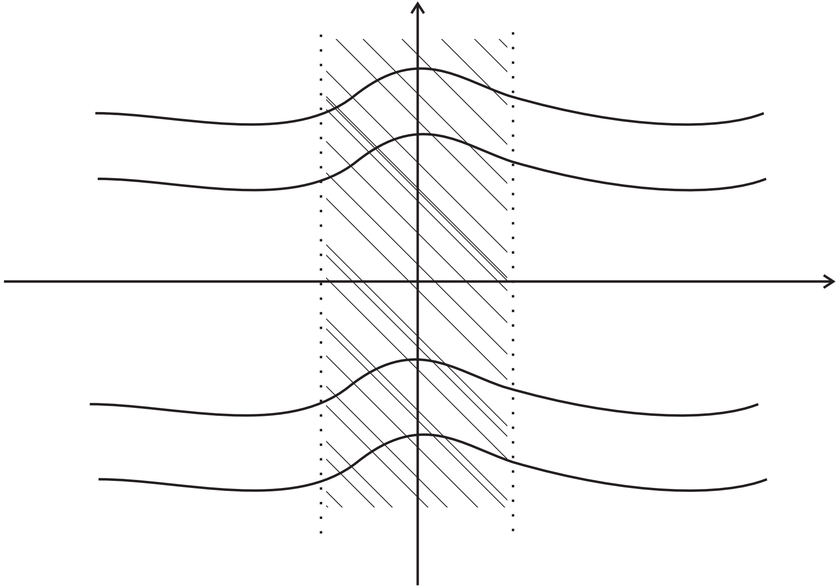
Another solution, which is obvious, canonical one, is the one-dimensional profile.

$$(3)\quad \begin{cases} w'' - w + w^p = 0, & w > 0 \text{ in } \mathbb{R} \\ w \to 0 \text{ at } +\infty \end{cases}$$

Let $u(x, y) = w(x)$ – solution to (NLS) with two "ends"

This solution is called front


y
x



Third Solution:Dancer's Solutions

The third type of solution is the so-called Dancer's solution. Such solutions satisfy

$$\begin{cases} u(x, y+T) = u(x,y) \\ u(x,y) \to 0 \text{ as } |x| \to +\infty \end{cases}$$

for some $T > 0$
How to find such solutions?

Bifurcations:

Linearize problem (I) around $w(x)$

$$L[\varphi] = \varphi_{xx} + \varphi_{yy} - \varphi + pw^{p-1}(x)\varphi$$

What are possible kernels?

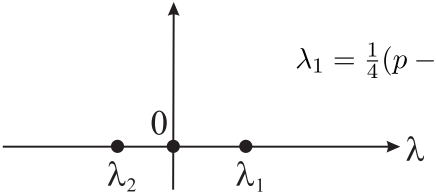
$$L_0(\varphi) = \varphi_{xx} - \varphi + pw^{p-1}(x)\varphi$$

$$\varphi_{xx} - \varphi + pw^{p-1}(x)\varphi = \lambda\varphi, \qquad \|\varphi\|_{L^\infty} < +\infty$$

Spectral Information in one-dimension

$$\varphi_{xx} - \varphi + p w^{p-1}(x)\varphi = \lambda\varphi, \qquad \|\varphi\|_{L^\infty} < +\infty$$

- $\lambda_1 > 0, \varphi_1 = (w(x))^{\frac{p+1}{2}} = Z(x)$
- $\lambda_2 = 0, \varphi_2 = w^{'}(x)$
- $\lambda_3 > 0$



$$\lambda_1 = \frac{1}{4}(p-1)(p+3),$$

Now we consider the full linearized operator

$$L(\varphi) := L_0(\varphi) + \varphi_{yy} \tag{1}$$

A simple computation shows that

$$w_x(x), Z(x)\cos\sqrt{\lambda_1}y, Z(x)\sin\sqrt{\lambda_1}y \tag{2}$$

satisfy $L(\varphi) = 0$. The following lemma shows these are the only kernels.

Lemma (del Pino, Kowalczyk, Pacard, Wei) Let φ be a bounded solution of the problem

$$\varphi_{xx} + \varphi_{yy} - \varphi + pw^{p-1}(x)\varphi = 0 \text{ in } \mathbb{R}^2 \tag{3}$$

Then

$$\varphi(x, y) = c_1 w_x(x) + c_2 Z(x)\cos(\sqrt{\lambda_1}y) + c_3 Z(x)\sin(\sqrt{\lambda_1}y) \tag{4}$$

Proof: (sketch)
Consider the Fourier transform in the y-variable

$$\hat{\varphi}(x,\xi) = \int \varphi(x,y)e^{i\xi y}dy$$

$\Rightarrow \hat{\varphi}_{xx} - \xi^2\hat{\varphi} - \hat{\varphi} + pw^{p-1}(x)\hat{\varphi} = 0$
by the spectral information

$$\text{Support } (\hat{\varphi}(x,\cdot)) \subset \{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0\}$$

By distribution theory

$$\varphi(x,y) = p_0(x,y) + p_1(x,y)\cos(\sqrt{\lambda_1}y) + p_2(x,y)\sin(\sqrt{\lambda_1}y)$$

where p_j are polynomials in y with coefficients depending on x

$$\varphi \text{ bounded} \quad \Rightarrow \quad \begin{aligned} p_0 &= c_1 w_x(x) \\ p_1 &= c_2 Z(x) \\ p_3 &= c_3 Z(x) \qquad \sharp \end{aligned}$$

If we restrict our solution to have symmetry

$$\begin{array}{rcll} u(x,y) & = & u(-x,y) & \text{even in } x \\ u(x,-y) & = & u(x,-y) & \text{even in } y \end{array}$$

then $\varphi(x,y) = Z(x)\cos\sqrt{\lambda_1}y$ is the only kernel.

Now rescaling the problem:

$$\Delta u + \lambda(-u + u^p) = 0 \ \mathbb{R}^{N-1} \times (0, 2\pi)$$

Applying the standard Crandall-Rabinowitz theory, Dancer (2001) proved that there exist solutions to (I) with

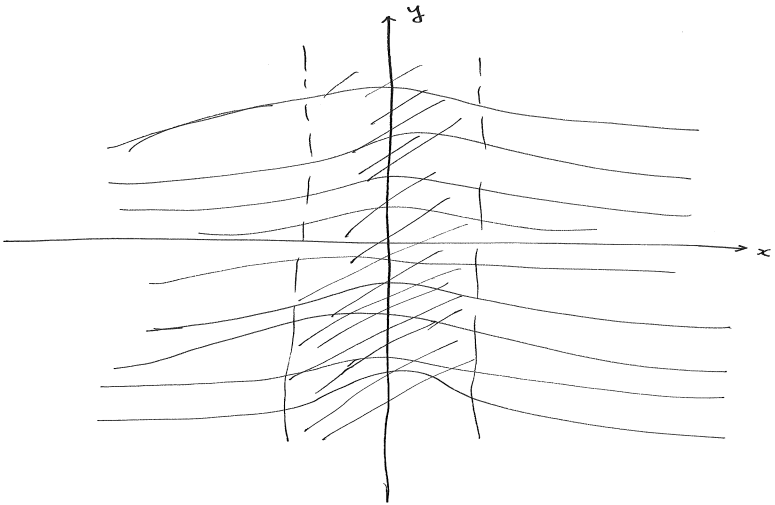
$$w_\delta(x,y) := w(x) + \delta Z(x)\cos\sqrt{\lambda_1}y + O(\delta^2 e^{-2|x|})$$

$$T = T_1 + O(\delta)$$

Dancer's Solutions

More generally, the Dancer solutions have two parameters

$$\begin{aligned} w_{\delta,\tau}(x,y) \quad := \quad & w(x) + \\ & \delta Z(x) \cos \sqrt{\lambda_1} y + \tau Z(x) \sin \sqrt{\lambda_1} y \\ & + O((\delta^2 + \tau^2) e^{-2|x|}) \end{aligned}$$



Solutions with Wriggled two-ends

Variational characterization of Dancer's solutions

Dancer's solution actually continues as $T > T_1$. Another way of obtaining Dancer's solutions is to consider the following problem in a strip:

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } \Sigma := \mathbb{R}^{N-1} \times (0, L), \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Sigma,\ u > 0,\ u \in H^1(\Sigma). \end{cases} \tag{5}$$

Here L is the parameter. We consider the so-called least energy solutions. More precisely, let

$$c(L) := \inf_{u \in H^1(\Sigma), u \not\equiv 0} \frac{\int_\Sigma (|\nabla u|^2 + u^2)}{\left(\int_\Sigma u^{p+1}\right)^{\frac{2}{p+1}}}. \tag{6}$$

- If $p < \frac{N+2}{N-2}$ when $N \geq 3$ and $p < +\infty$ when $N = 2$, then there exists a unique $L_* = \frac{\pi}{\sqrt{\lambda_1}}$ such that for $L \leq L_*$, $c(L)$ is attained by a trivial solution and for $L > L_*$, $c(L)$ is attained by a nontrivial non-one-dimensional solution (Dancer's solutions).
- There exist $L_2 \geq L_*$ such that the least energy solution is unique and nondegenerate for any $L \geq L_2$. As $L \to +\infty$, the least energy solution approaches the spike solution in $\mathbb{R}^N$.

Berestycki-Wei 2010

$N = 2, T = 2L$ gives Dancer's solution.

Summary

We have known so far the following three types of entire solutions

1. Spike $w(|x - x_0|)$

2. Front $w(x)$

3. Dancer's solutions-Wriggled Front $w_{\delta,\tau}(x, y)$

New Entire Solutions for (NLS)

We discuss three kinds of new entire solutions obtained since 2007

Solution I: Multiple $(2m)$ ends front solutions (del Pino-Kowalczyk-Pacard-Wei 2007)

Solution II: Triple-junction Y-shaped spike solutions (Malchiodi 2008)

Solution III: Front-Spike three end solutions (Santra-Wei 2010)

Solution I:Multiple Bump Line Solutions

We say that u, a solution of (NLS), is a multiple bump line with $2m$ ends if there exist $2m$ oriented half lines $\{\mathbf{a}_j \cdot \mathbf{x} + b_j = 0\}$, $j = 1, \ldots, 2m$ (for some choice of $\mathbf{a}_j \in \mathbb{R}^2$, $|\mathbf{a}_j| = 1$ and $b_j \in \mathbb{R}$) such that along these half lines and away from a compact set K containing the origin, the solution is asymptotic to $w_{\delta_j,\tau_j}(\mathbf{a}_j \cdot \mathbf{x} + b_j)$ for certain numbers δ_j, τ_j, $j = 1, \ldots, 2m$, that is there exist positive constants C, c such that:

$$\|u(\mathbf{x}) - \sum_{j=1}^{2m} w_{\delta_j,\tau_j}(\mathbf{a}_j \cdot \mathbf{x} + b_j)\|_{L^\infty(\mathbb{R}^2 \setminus K)} \leq Ce^{-c|\mathbf{x}|}. \qquad (7)$$

What we actually look for is a solution with a multiple bump line solution of of (NLS) whose asymptotic behavior is determined by m curves

$$\gamma_j = \{(x, z) \mid x = f_j(z)\}, \quad j = 1, \ldots, m,$$

$$f_1(z) \ll f_2(z) \ll \cdots \ll f_k(z)$$

which asymptotically resemble straight lines.

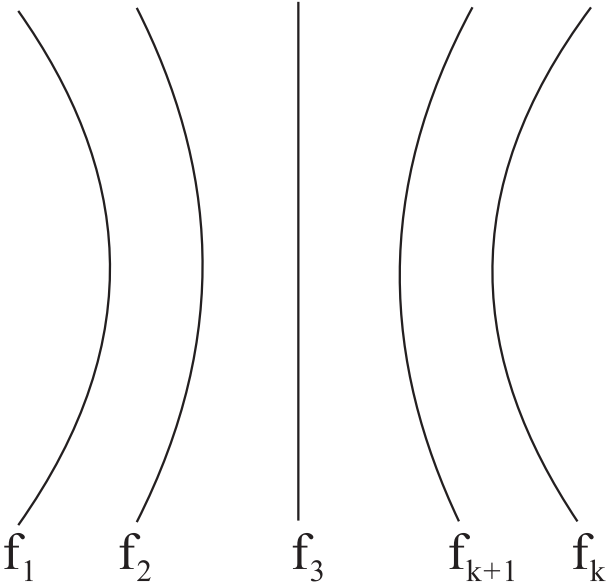


Figure: Multi-front solutions with even-ends.

Toda System

The functions f_j defining the curves γ_j are not arbitrary and turn out to be related to a second order system of differential equations, the Toda system, given by

$$c_p^2 f_j'' = e^{f_{j-1}-f_j} - e^{f_j - f_{j+1}} \quad \text{in } \mathbb{R}, \quad j = 1, \ldots, m \tag{8}$$

with the conventions $f_0 = -\infty$, $f_{m+1} = +\infty$.

Moser (1978), Kostants (1979): Toda system is integrable, i.e., all solutions can be written explicitly by $2m$ parameters.

Scaling Invariance

The Toda system has a special scaling property: We observe that if $\mathbf{f} = (f_1, \ldots, f_k)$ is a solution of this system, then function $\mathbf{f}_\varepsilon$ defined by

$$\mathbf{f}_\varepsilon = (f_{\varepsilon,1}, \ldots, f_{\varepsilon,k}), \quad f_{\varepsilon,j}(z) := f_j(\varepsilon z) + 2(j - \frac{m+1}{2}) \log \frac{1}{\varepsilon}, \tag{9}$$

is also a solution.

The functions f_j are asymptotically linear, namely the limits $\nu_j = f_j'(+\infty)$, exist and

$$\nu_1 < \nu_2 < \cdots < \nu_k, \quad \sum_{j=0}^{k} \nu_j = 0. \tag{10}$$

For the rescaled solutions, ε small

$$f_{\varepsilon,1}(z) \ll f_{\varepsilon,2}(z) \ll \cdots \ll f_{\varepsilon,k}(z), \qquad f_{\varepsilon,j}'(\pm\infty) = a_{\pm,j}\varepsilon,$$

and

$$f_{\varepsilon,j}(z) \;=\; a_{\pm,j}\varepsilon\, z + b_{\pm,j} + 2(j - \frac{k+1}{2}) \log \frac{1}{\varepsilon} + \mathcal{O}((\cosh z)^{-\vartheta\varepsilon}) \tag{11}$$

as $|z| \to +\infty$, for certain scalars $a_{\pm,j}, b_{\pm,j}$ and $\vartheta > 0$.

Deficiency Space

Let χ^+ (resp. χ^-) be a smooth cutoff function defined on $\mathbb{R}$ which is identically equal to 1 for $z > 1$ (resp. for $z < -1$) and identically equal to 0 for $z < -1$ (resp. for $z > 1$) and additionally $\chi^- + \chi^+ \equiv 1$. With these cutoff functions at hand, we define the 4 dimensional deficiency space

$$D := \mathrm{Span}\left\{\chi^\pm(z),\, z\,\chi^\pm(z)\right\}, \tag{12}$$

and, for all $\mu \in (0,1)$ and all $\theta \in \mathbb{R}$, we define the space $\mathcal{C}^{2,\mu}_\theta(\mathbb{R})$ of $\mathcal{C}^{2,\mu}$ functions h which satisfy

$$\|h\|_{\mathcal{C}^{2,\mu}_\theta(\mathbb{R})} := \|(\cosh z)^\theta\, h\|_{\mathcal{C}^{2,\mu}(\mathbb{R})} < \infty\,.$$

Theorem

(del Pino-Kowalczyk-Pacard-Wei 2007) Assume that $N = 2$ and $p > 2$. Given $m \geq 2$, for any sufficiently small number $\varepsilon > 0$, there exists a $4m$ parameter family of multiple bump line solutions of equation

$$\Delta u - u + u^p = 0 \text{ in } R^2 \quad (NLS)$$

with $2m$ ends. Their asymptotic profiles are determined by m curves

$$\gamma_{\varepsilon,j} = \{x = f_{\varepsilon,j}(z) + h_{\varepsilon,j}(\varepsilon z)\}.$$

Here $\mathbf{f}_\varepsilon$ is the rescaled solutions of Toda system. Functions $h_{\varepsilon,j} \in \mathcal{C}_\theta^{2,\mu}(\mathbb{R}) \oplus D$ representing small perturbations satisfy

$$\|h_{\varepsilon,j}\|_{\mathcal{C}_\theta^{2,\mu}(\mathbb{R}) \oplus D} \leq C\, \varepsilon^\kappa$$

with some constants $\theta, \kappa > 0$.

Remark 1: Moduli Spaces

Each bump line of this solution (represented by one curve $\gamma_{\alpha,j}$) consists of three parts: two Dancer ends and a middle "connector" which is a curved piece of the homoclinic inserted between the wriggling Dancer pieces. Each of the $2m$ Dancer ends depends on 2 free parameters. Each curve $\gamma_{\alpha,j}$ depends on 2 initial conditions for the Toda system. Thus in all there are $4m$ Dancer parameters and $2m$ initial conditions for the Toda system. This gives $6m$ parameters of which $2m$ Dancer parameters must be adjusted at the end. As a consequence we obtain $4m$ parameter family of solutions.

dimension of the multiple $2m$ end solutions: $4m$

Remark 2:Energy Point of View:

$$w'' - w + w^p = 0$$

$$E[u] = \tfrac{1}{2}\int_{\mathbb{R}} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}} u^2 - \frac{1}{p+1}\int_{\mathbb{R}} u^{p+1}$$

$$u \approx \sum_{j=1}^{k} w(x - a_j)$$

$$E[u] \approx KI[w] - \sum_{i\neq j} e^{-|a_i - a_j|} + \text{h.o.t.}$$

No stationary points

Now we extend it to $\mathbb{R}^2$

Choose $\quad u \approx \sum_{j=1}^{k} w(x - f_j(y))$

Then $E[u] = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + u^2) - \frac{1}{p+1}\int_{\mathbb{R}^2} u^{p+1}$

is not well-defined. However, we can define a renormalized energy

$$E[u] = \int_{-L}^{L}\int_{\mathbb{R}}[....]$$

$$\approx LkI[w] + c_L$$

$$+ \underbrace{\frac{1}{2}\sum_{j=1}^{k}|\dot{f}_j|^2 - \sum_{i \neq j} e^{-|f_i - f_j|}}_{I[f_1, ..., f_k]}$$

$I[f_1, ..., f_k]$ has stationary points $\Longleftrightarrow$ Toda system

Bridge

The scaling invariance of the Toda system

$$f_{\varepsilon,j}(z) := f_j(\varepsilon z) + 2(j - \frac{m+1}{2}) \log \frac{1}{\varepsilon}$$

provides a natural bridge between the NLS and Toda system.

$\Delta u + u - u^3 = 0$ in $\mathbb{R}^2$	$\longleftrightarrow$	Toda System in $\mathbb{R}^1$

Proof

1. Infinite-dimensional reduction method
2. Modulli space technique
3. theory on Toda system

I will explain this in Lecture 6—(the role of Jacobi-Toda system)

Solution II:

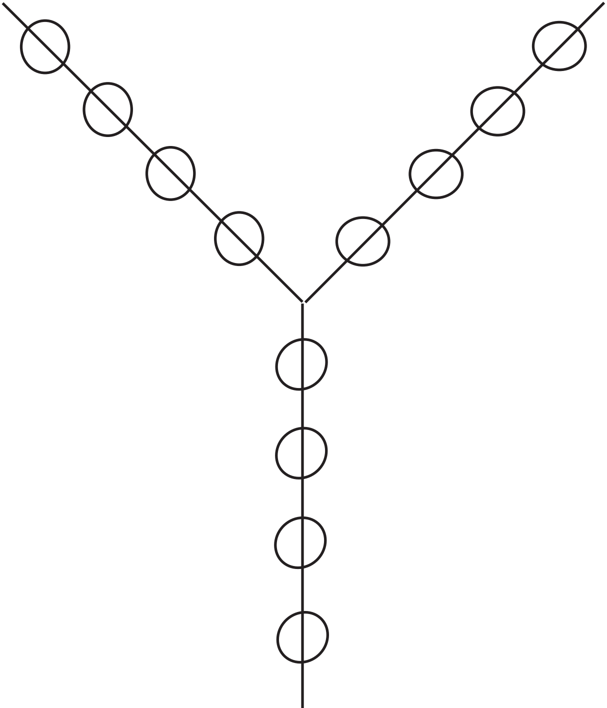
Malchiodi, 2008 constructed another new kind of solutions with three rays of spikes.

$$u(x,z) \approx \sum_{j=1}^{3} \sum_{i=1}^{+\infty} w((x,z) - iL\vec{l_j}) \tag{13}$$

where $\vec{l_j}, j = 1, 2, 3$ are three unit vectors satisfying some balancing conditions ($\mathbf{Y}$-shaped solutions). Here w is the unique solution to the two dimensional entire problem

$$\Delta w - w + w^p = 0, w = w(r), w \in H^1(\mathbb{R}^2). \tag{14}$$

Proof: finite dimensional reduction method (see Lecture Two)



Malchiodi's solution: Three lines meetings a common point, the angle between the lines must be large than $\frac{2\pi}{3}$.

AO-Musso-Pacard-Wei 2012: the existence of positive solutions with any number and any directions of rays.

Solution III: Coexistence of Fronts and Spikes

Santra-Wei 2010: find entire solutions with coexistence of spikes and fronts

$$u(x,z) = w(x - f(z)) + \sum_{i=1}^{\infty} w((x,z) - \xi_i \vec{e}_1) \tag{15}$$

for suitable large $L > 0$ and ξ_i's are such that $\xi_1 - f(0) = L$ and

$$\xi_1 < \xi_2 < \cdots < \xi_i < \cdots$$

and satisfy $\xi_j = jL + O(1)$ for all $j \geq 1$.

Interaction of Spikes and Fronts

$$\begin{cases} f''(z) = \Psi_L(f,z) & \text{in } \mathbb{R} \\ \quad f(0) = 0, \quad f'(0) = 0, \end{cases} \tag{16}$$

where $\Psi_L(f,z)$ is a function measuring the interactions between bumps and fronts and asymptotically

$$\Psi_L(f,z) \sim ((f-L)^2 + z^2)^{-\frac{1}{2}} e^{-\sqrt{(f-L)^2+z^2}}$$

.

Theorem

(Santra-Wei 2010) Let $N = 2$*. For* $p > 2$ *and sufficiently large* $L > 0$*, (I) admits a one parameter family of positive solution satisfying*

$$\begin{cases} u_L(x,z) = u_L(x,-z) & \textit{for all } (x,z) \in \mathbb{R}^2 \\ u_L(x,z) \sim \left(w_\delta(x - f(z) - h_L(z), z) + \right. & \left. \sum_{i=1}^{\infty} U((x,z) - \xi_i \vec{e}_1) \right) \end{cases} \quad (17)$$

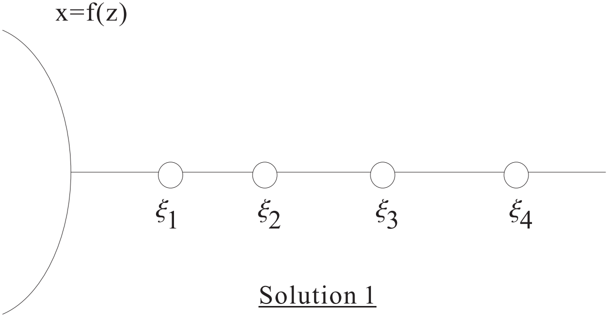


Figure: Front-Spike Solution

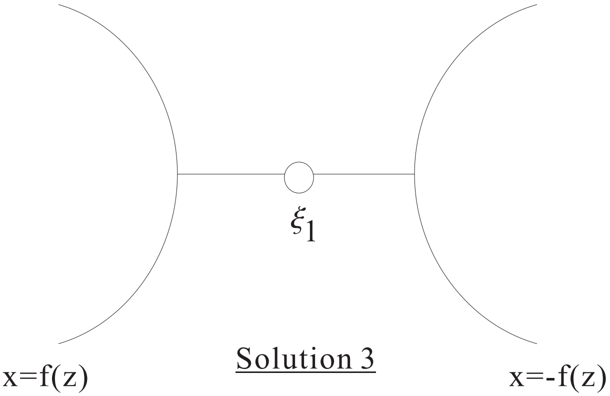


Figure: Front-Spike-Front Solutions

Entire Sign-Changing Finite Energy Solutions

$$\Delta u - u + u^3 = 0 \quad \text{in } \mathbb{R}^N \qquad (NLS).$$

Finite Energy

$$u \in H^1(\mathbb{R}^N) \implies \lim_{|x| \to +\infty} u(x) = 0$$

Obviously (NLS) is equivariant with respect to the action of the group of isometries of $\mathbb{R}^N$, it is henceforth natural to ask whether all finite energy solutions of (NLS) are radially symmetric.
Indeed this is case for positive solutions by the classical result of Gidas-Ni-Nirenberg 1981.
Therefore, nonradial solutions, if they exist, are necessarily sign-changing solutions.

Sign-Changing radial solutions

•• Berestycki-Lions 1983, Struwe 1984: the existence of infinitely many sign-changing radial solutions—variational methods

•• Conti-Merrizi-Terracini 2003, : the existence of infinitely many sign-changing radial solutions—parabolic flow method

•• T. Weth (thesis 2005):the existence of infinitely many sign-changing radial solutions—Nehari manifold method

•• Wei-Yao 2011: uniqueness (and nondegeneracy) of sign-changing radial solutions when $\frac{N+2}{N-2} - \varepsilon < p < \frac{N+2}{N-2}$

Open Question

existence of non-radial sign-changing solutions?

$$\begin{cases} \Delta U - U + |U|^{p-1}U = 0 \text{ in } \mathbb{R}^N \\ \lim_{|x|\to+\infty} U(x) = 0 \end{cases}$$

Sign-Changing Nonradial Solutions

•• Bartsch and Willem 1994: the existence of infinitely many sign-changing solutions in dimension $N = 4$ and $N \geq 6$.
The key idea is to look for solutions invariant under the action of $O(2) \times O(N-2) \subset O(N)$ to recover some compactness property.

$$\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{N-2m}$$

$$N = 4 \text{ or } N \geq 6$$

$$U(x^{'}, x^{''}, x^{'''}) = U(|x^{'}|, |x^{''}|, |x^{'''}|)$$

$$U(|x^{'}|, |x^{''}|, |x^{'''}|) = -U(|x^{''}|, |x^{'}|, |x^{'''}|)$$

variational method: Lijusternik-Schnirlman category theory

•• Lorca and Ubilla 2003: N=5

Low Dimensions

Open Question

Are there infinitely many sign-changing nonradial solutions in dimensions
$N=2, 3$?

Low Dimensions

Open Question

Are there infinitely many sign-changing nonradial solutions in dimensions
N=2, 3?

Note that $N = 2$ or 3 is the physically relevant dimension.
Klein-Gordon equation

$$\begin{cases} \Delta U - U + U^3 = 0 \text{ in } \mathbb{R}^2 \\ \lim_{|x| \to +\infty} U(x) = 0 \end{cases} \tag{18}$$

First Result

Musso-Pacard-Wei (2009) construct unbounded sequences of solutions of (NLS) in any dimensions $N \geq 2$.
The solutions we obtain are nonradial, have finite energy and are invariant under the action of $D_k \times O(N-2)$ where $D_k \subset O(2)$ is the dihedral group generated by the rotation of angle $2\pi/k$, for $k \geq 7$.
Moreover, these solutions are not invariant under the action of $O(2) \times O(N-2)$ and hence they are different from the solutions constructed by Bartsch-William, Lorca-Ubilla

Building Blocks

Our building block is the unique radially symmetric (in fact radially decreasing) positive solution of

$$\Delta w - w + w^p = 0, w > 0 \quad \text{in} \quad \mathcal{R}^N, w \in H^1(\mathbb{R}^N)$$

$$\lim_{r\to\infty} e^r \, r^{\frac{N-1}{2}} \, w = c_{N,p} > 0, \quad \text{and} \qquad \lim_{r\to\infty} \frac{w'}{w} = -1 \,, \qquad (19)$$

The function w together with its translations will constitute the building blocks of our construction.

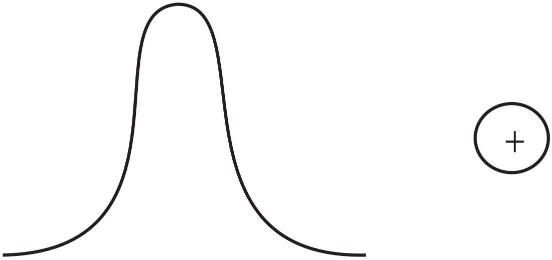
Nonradial Finite Energy Solutions

Theorem (Musso-Pacard-Wei 2009) Let $k \geq 7$ be a fixed integer. Then there exist $(m_i)_i$ and $(n_i)_i$, sequences of integers tending to $+\infty$, and u_i, a sequence of nonradial, sign-changing solutions of (NLS), such that

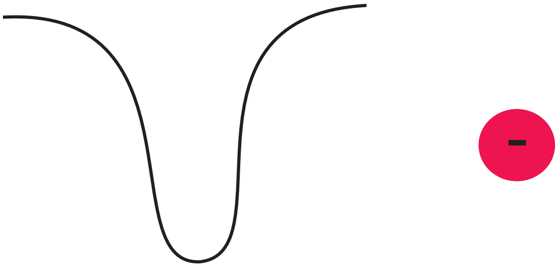
$$u_i = \sum_{i=1}^{m_i} w(x - z_i^+) - \sum_{j=1}^{n_i} w(x - z_j^-) + o(1)$$

Moreover, the solutions u_i are invariant under the action of $D_k \times O(N-2)$ but are not invariant under the action of $O(2) \times O(N-2)$.

Positive Bump: w



Negative Bump: w



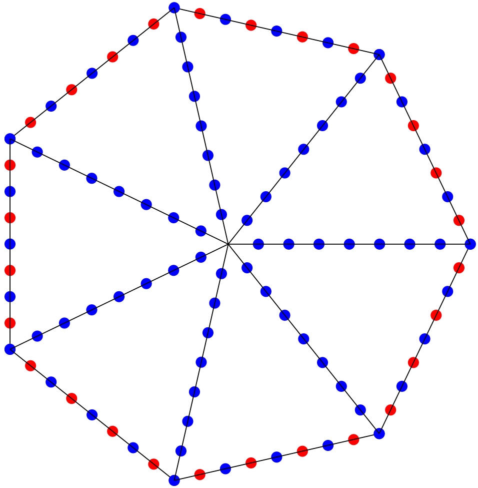


Figure: The location of the bumps. Here $k = 7, m = 8, n = 4$.

Let m be the number of positive bumps, n be the number of negative bumps, and l be the distance between positive bumps. Then the following relation hold:

$$\frac{2n-1}{m} = 2\sin\frac{\pi}{k}\left(1 - \ln\left(2\sin\frac{\pi}{k}\right)\ell^{-1} + O(\ell^{-2})\right). \tag{20}$$

$$l \approx \frac{\frac{2n-1}{m} - 2\sin\frac{\pi}{k}}{-\ln\left(2\sin\frac{\pi}{k}\right)} \tag{21}$$

$$m_i, n_i \to +\infty$$

$$l_i \to +\infty$$

Symmetry Class

The solutions constructed by Musso-Pacard-Wei are invariant under a large group of symmetries. More precisely, they will enjoy the following invariance :

$$u(x) = u(R\,x), \quad \text{for all} \quad R \in \{I_2\} \times O(N-2)\,, \tag{22}$$

also

$$u(R_k\,x) = u(x) \quad \text{and} \quad u(\Gamma\,x) = u(x)\,, \tag{23}$$

where $R_k \in O(2) \times \{I_{N-2}\}$ is the rotation of angle $2\pi/k$ in the (x_1, x_2)-plane and $\Gamma \in O(2) \times \{I_{N-2}\}$ is the symmetry with respect to the hyperplane $x_2 = 0$. Here I_n denotes the identity in $\mathbb{R}^n$.

The nonradial finite-energy solutions constructed by Bartsch-Willem has the symmetry

$$O(2) \times O(N-2)$$

The nonradial finite energy solutions in Theorem 1 has the symmetry

$$D_k \times O(N-2)$$

In view of these results, a natural question is the following:

Do all solutions of

$$(KG) \quad \Delta u - u + u^3 = 0 \text{ in } \mathbb{R}^2, u \in H^1(\mathbb{R}^2)$$

have a nontrivial group of symmetry ?

If we drop the finite energy assumption, then the answer is Yes. For example consider positive solutions

$$(KG) \;\; \Delta u - u + u^3 = 0 \text{ in } \mathbb{R}^2, u > 0$$

•• Malchiodi 2009: constructed positive (infinite energy) solutions of (KG) by perturbing a configuration of infinitely many copies of the positive solution w arranged along three rays meeting at a common point. The solutions he has constructed are bounded and possess No symmetry but they have infinite energy.

$$(KG) \quad \Delta u - u + u^3 = 0 \text{ in } \mathbb{R}^2, u \in H^1(\mathbb{R}^2)$$

Does the finite energy assumption or the decaying condition

$$\lim_{|x| \to +\infty} u(x) = 0$$

impose some kind of symmetry on the solutions?

Surprisingly, the answer to this question is negative. In fact, we prove the :

Theorem (Ao-Musso-Pacard-Wei 2012): *There exist infinitely many solutions of (KG) which have finite energy but whose group of symmetry reduces to the identity.*

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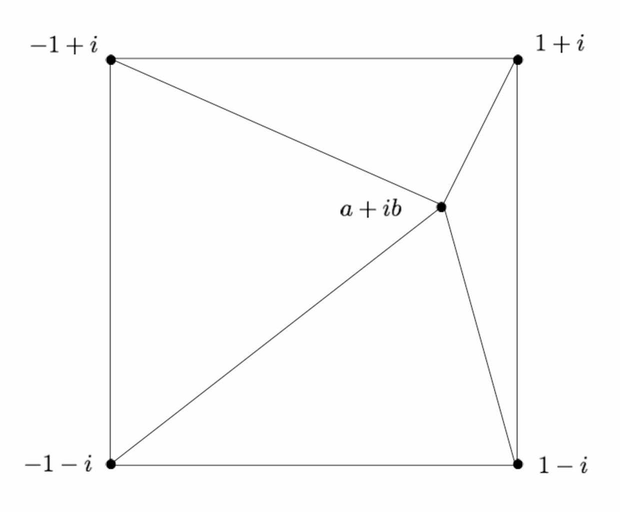
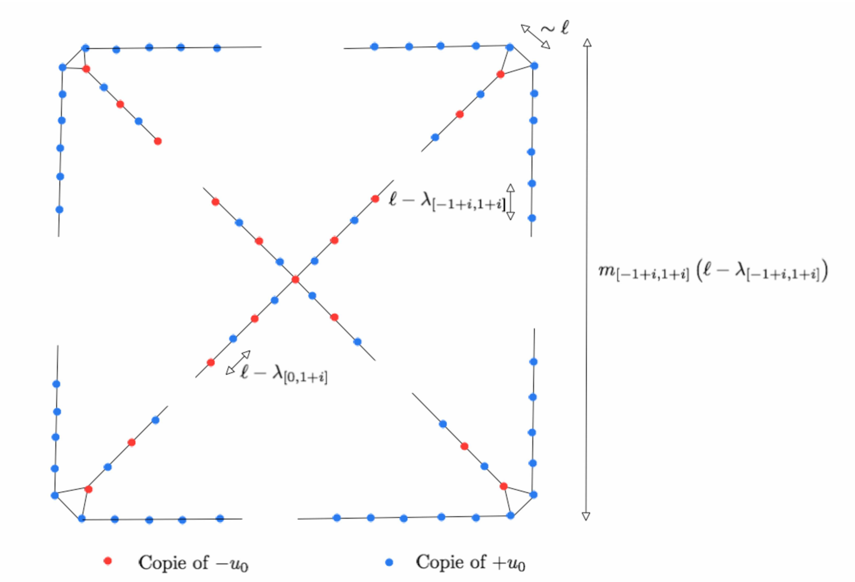
Lecture Two-Three: we will discuss these constructions, using finite dimensional reduction method



An example of nonsymmetric network.



The blue and red dots correspond to the set of points in which positive/negative spikes are placed.

Summary

New (and old) Entire Solutions to

$$\Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N \qquad (NLS).$$

- Ground states (spikes)
- front solutions
- Dancer's solutions
- multiple $2m$ end solutions (del Pino-Kowalczyk-Pacard-Wei 2007)
- $Y-$shaped bump lines (Malchiodi 2007)
- Front-spike three end solutions (Santra-Wei 2010)
- Sign-Changing finite energy solutions (Musso-Pacard-Wei 2009, Ao-Musso-Pacard-Wei 2012)

PDE and Geometry

One of the striking features of all the existence results, which are purely PDE result, is that their counterparts can be found in geometric framework: the analogy between the theory of complete constant mean curvature surfaces in Euclidean 3-space and the study of entire solutions of I.

CMC surfaces in $\mathbb{R}^3$: mean curvature $=$Constant

Examples: spheres, cylinder

Delaunay Surfaces

Embedded constant mean curvature surfaces of revolution were found by Delaunay in the mid 19th century. They constitute a smooth one-parameter family of singly periodic surfaces D_τ, for $\tau \in (0, 1]$, which interpolate between the cylinder $D_1 = S^1(1) \times \mathbb{R}$ and the singular surface $D_0 := \lim_{\tau \to 0} D_\tau$, which is the union of an infinitely many spheres of radius $1/2$ centered at each of the points $(0, 0, n)$ as $n \in \mathbb{Z}$.

The Delaunay surface D_τ can be parametrized by

$$X_\tau(x,z) = (\varphi(z)\,\cos x, \varphi(z)\,\sin x, \psi(z)) \in D_\tau \subset \mathbb{R}^3,$$

for $(x,z) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$. Here the function φ is smooth solution of

$$(\varphi')^2 + \left(\frac{\varphi^2+\tau}{2}\right)^2 = \varphi^2,$$

and the function ψ is defined by

$$\psi' = \frac{\varphi^2+\tau}{2}.$$

When $\tau = 1$, the Delaunay surface is nothing but a right circular cylinder $D_1 = S^1(1) \times \mathbb{R}$, with the unit circle as the cross section. This cylinder is clearly invariant under the continuous group of vertical translations, in the same way that the single bump-line solution of (I) is invariant under a one parameter group of translations. It is then natural to agree on the correspondence between

$$\boxed{\begin{array}{c}\text{The cylinder}\\ D_1 = S^1 \times \mathbb{R}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The single bump-line}\\ (x,z) \longmapsto w(x)\end{array}}.$$

Inspection of the other end of the Delaunay family, namely when the parameter τ tends to 0, suggests the correspondence between

$$\boxed{\begin{array}{c}\text{The sphere}\\ S^1(1/2)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The radially symmetric solution}\\ (x,z) \longmapsto U(\sqrt{x^2+z^2})\end{array}}.$$

More generally, there is a natural correspondence between

$$\boxed{\begin{array}{c}\text{Delaunay surfaces}\\ D_\tau\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{Least energy solutions}\\ (x,z)\longmapsto w_L(x,z)\end{array}}.$$

$$\boxed{\begin{array}{c}\text{The Jacobi operator}\\ J_1=\frac{1}{\varphi^2}\,\left(\partial_x^2+\partial_z^2+1\right)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The linearized operator}\\ L=\partial_x^2+\partial_z^2-1+p\,w^{p-1}\end{array}}.$$

The ground state 1 of $\partial_x^2 + 1$	$\longleftrightarrow$	The first eigenfunction $Z(x)$ of $\partial_x^2 - 1 + p\, w^{p-1}$

,

Multiple-end solutions to (I) of DKPW	$\longleftrightarrow$	Connected Sums Cylinders of Mazzeo-Pacard, Pollack	,

$Y-$shaped solutions to (I) of Malchiodi	$\longleftrightarrow$	Noncompact Embedded CMC of Kapoulous

,

Front-Spike solutions to (I) of Santra-Wei	$\longleftrightarrow$	End-to-End Gluing of CMC of of Mazzeo-Pacard, Pollack

,

Sign-Changing finite energy solutions to (I) of Musso-Pacard-Wei	$\longleftrightarrow$	Immersed CMC surfaces of Kapoulous

,

However

$$\boxed{\begin{array}{c}\text{Toda}\\ \text{System}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{??}\\ \text{????}\end{array}},$$

We have established an intricate correspondence between the study of entire solutions of Nonlinear Schrodinger Equation and the theories of constant mean curvature surfaces (CMC):

Study of Entire Solutions of Nonlinear Schrodinger Equation $\Delta u - u + u^3 = 0$ in $\mathbb{R}^N$	$\longleftrightarrow$	Theory of Toda System or Constant Mean Curvature Surface (CMC)

Open Question I

In CMC theory, the Delaunay surfaces form a continum of solutions, depending on a parameter τ–the necksize, or the period L.

In NLS, we have found for the existence of nontrivial least energy solutions of period $L > L_1$ (Berestycki-Wei 2010)

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } \Sigma := \mathbb{R}^{N-1} \times (0, L), \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Sigma,\ u > 0,\ u \in H^1(\Sigma). \end{cases}$$

Question: Are these least energy solutions continuous in L?

Open Question II

In CMC theory, it has been proved by Koveraar-Kusner-Solomon (JDG 1989) that
The only embedded CMC surfaces in $\mathbb{R}^3$ with two ends are Delaunay surfaces

The corresponding PDE Conjecture is:
The only solutions to $\Delta u - u + u^p = 0$ in $\mathbb{R}^2$ that decays uniformly in y as $|x| \to +\infty$ are Dancer solutions, i.e. periodic in the other direction
This can be considered as NLS Gibbons' Conjecture

(Axially symmetry result by Gui-Malchiodi-Xu 2010)

Open Question III

$$\Delta u - u + u^3 = 0 \text{ in } \mathbb{R}^2$$

Solutions with fronts (one-dimensional concentrations) and spikes (zero-dimensional concentrations) are constructed.

$$\Delta u - u + u^p = 0 \text{ in } \mathbb{R}^N,\ N \geq 3$$

Question: Are there solutions concentrating on $k-$dimensional sets? What is the underlying geometry?

A natural guess is $k-$dimensional minimal manifolds in $\mathbb{R}^N$. But the proofs remain completely open.

Some Applications to real Physical Models

Application I: magnetic Ginzburg-Landau equations

The previous constructions for (NLS) can actually be applied to real physical problems.

Ginzburg-Landau energy functional

$$\mathcal{E}_{GL}(\psi, A) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_A \psi|^2 + (\nabla \times A)^2 + \frac{\lambda}{2}(|\psi|^2 - 1)^2, \qquad (24)$$

$$(mGL) \quad \begin{cases} -\Delta_A \psi + \lambda(|\psi|^2 - 1)\psi = 0 \\ \nabla \times \nabla \times A - \mathrm{Im}(\bar{\psi}\nabla_A \psi) = 0 \end{cases}$$

called the *magnetic Ginzburg-Landau* (mGL) equations.

Here $\lambda > 0$ is a constant depending on the material in question: when $\lambda < 1/2$ or $> 1/2$, the material is of type I or II superconductor, respectively; $\nabla_A = \nabla - iA$ is the covariant gradient, and $\Delta_A = \nabla_A \cdot \nabla_A$. For a vector field A, $\nabla \times A$ is the scalar $\partial_1 A_2 - \partial_2 A_1$ and for scalar ξ, $\nabla \times \xi$ is the vector $(-\partial_2 \xi, \partial_1 \xi)$.
We consider here configurations satisfying the superconducting boundary condition

$$|\psi(x)| \to 1 \qquad as \qquad |x| \to \infty.$$

Gauge-Invariance

Equations (mGL), in addition to being translationally and rotationally invariant, have translational and gauge symmetries: solutions are mapped to solutions under the transformations

$$\psi(x) \mapsto \psi(x-z), \qquad A(x) \mapsto A(x-z)$$

for any $z \in \mathbb{R}^2$, and

$$\psi \mapsto e^{i\gamma}\psi, \qquad A \mapsto A + \nabla\gamma$$

for twice differentiable $\gamma : \mathbb{R}^2 \to \mathbb{R}$.

To date, the only non-trivial, finite energy, rigorously known solutions to equations (mGL) on all of $\mathbb{R}^2$ are the radial and equivariant solutions of the form $u = (\psi^{(n)}, A^{(n)})$, with

$$\psi^{(n)}(x) = f_n(r)e^{in\theta} \qquad and \qquad A^{(n)}(x) = a_n(r)\nabla(n\theta) \tag{25}$$

called $n - vortices$. Here (r, θ) are the polar coordinates of the vector $x \in \mathbb{R}^2$ and $n = deg\, \psi_n$ is an integer.

$$\begin{cases} \frac{1}{r}\partial_r(r\partial_r f_n) - \frac{n^2 f_n}{r^2}(1 - a_n) + \frac{\lambda}{2} f_n(1 - |f_n|^2) = 0. \\ r\partial_r\left(\frac{\partial_r a_n}{r}\right) + f_n^2(1 - a_n) = 0. \end{cases}$$

One has the following information on the vortex profiles f_n and a_n: $0 < f_n < 1, 0 < a_n < 1$ on $(0, \infty)$; $f_n', a_n' > 0$; and $1 - f_n, 1 - a_n \to 0$ as $r \to \infty$ with exponential rates of decay. In fact,

$$
\begin{aligned}
f_n(r) &= 1 + O(e^{-m_\lambda r}) \quad \text{and} \\
a_n(r) &= 1 + O(e^{-r}) \quad \text{with} \\
m_\lambda &:= \text{min}(\sqrt{2\lambda}, 2).
\end{aligned}
$$

At the origin, $f_n \sim cr^n, a_n \sim dr^2$ ($c > 0, d > 0$ are constants) as $r \to 0$.

By Gauge-invariance, we also have the following families of solutions

$$\psi_{nz\gamma}(x) = e^{i\gamma(x)}\psi^{(n)}(x-z) \qquad A_{nz\gamma}(x) = A^{(n)}(x-z) + \nabla\gamma(x) \tag{26}$$

where n is an integer, $z \in \mathbb{R}^2$ and $\gamma : \mathbb{R}^2 \to \mathbb{R}$.

A result of the invariance is that the dimension of the kernel is $+\infty$:

$$\begin{pmatrix} \partial_{x_j}\psi \\ \partial_{x_j}A \end{pmatrix}, j = 1, 2, \begin{pmatrix} i\chi\psi \\ \nabla\chi \end{pmatrix}$$

Previous Results

Non-magnetic GL: $\Delta u + (1 - |u|^2)u = 0$

works of Bethuel-Brezis-Helen, FH Lin-Rivier, Serfaty, ...

In the case of the Ginzburg-Landau equation on *bounded* domains, non-radial *non-magnetic* solutions have been found by Bethuel-Brezis-Helen, Lin-Lin and non-radial *magnetic* solutions have been found by Sandier and Serfaty. This is due to the boundary forces which keep repelling vortices within the bounded domain.

In the case of the nonmagnetic Ginzburg-Landau equation on $\mathbb{R}^2$, Ovchinnikov and Sigal (2004) conjectured by numerical evidence that for the non-magnetic Ginzburg-Landau equations on the whole plane, non-radial solutions do exist.

Jaffre-Taubes Conjecture

In the famous book by Jaffre-Taubes

Vortices and Monopoles

they stated that the finite energy assumption is too rigid and conjectured

Conjecture The only stationary multi-vortex of degrees ± 1 for the magnetic Ginzburg-Landau equation is the radially symmetric profile.

Results: Finite-energy non-radial magnetic vortex solutions

Theorem (Ting-Wei 2011) Fix $\lambda > 1/2$ and an integer $k \geq 7$. There exists a sequence, $(u_i)_{i\geq 0} := (\psi_i, A_i)_{i\geq 0}$, of non-radial degree-changing solutions to (mGL) containing km_i vortices, $m_i \to \infty$, invariant under rotations by $\frac{2\pi}{k}$ (but not by rotations in $O(2)$ in general) and reflections in the $x_2 = 0$ line. Each u_i has finite-energy of the form

$$\mathcal{E}_{GL}(u_i) = km_i\mathcal{E}_{GL}(v^\bullet) + o(1) \quad \text{as } m_i \to \infty, \tag{27}$$

where $v^\bullet = v^{\pm 1}$ is the $+1$ or -1 degree vortex.

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analogous result of Musso-Pacard-Wei for (NLS)

Finite Energy Solutions Without Any Symmetry

Theorem (Ao–Pacard-Ting-Wei 2012) Fix $\lambda > 1/2$ and an integer $k \geq 7$. Let Γ be a balanced, flexible, closable network. Starting with this graph, there exists a sequence, $(u_i)_{i\geq 0} := (\psi_i, A_i)_{i\geq 0}$, of non-radial degree-changing solutions to (mGL).

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analogous result of Ao Musso-Pacard-Wei for (KG)

Key Elements of Proofs

1. The stability of degree ± 1 vortex to (mGL) was proved by Gustfason-Sigal (2004): The kernel conists exactly the translational modes and the gauge-part

$$\begin{pmatrix} \partial_{x_j}\psi \\ \partial_{x_j}A \end{pmatrix}, j=1,2, \begin{pmatrix} i\chi\psi \\ \nabla\chi \end{pmatrix}$$

2. The interaction between vortices is exponentially small.

3. A major problem is how to deal with the gauge-invariance kernel, which is infinite-dimensional.

Vortex Filaments and Magnetic Ginzburg-Landau system in $\mathbb{R}^N, N \geq 3$

There is an extension of magnetic Ginzburg-Landau system in higher dimensions $\mathbb{R}^N, N \geq 3$.
Edward Witten, From superconductors and four-manifolds to weak interactions, Bull. Amer. Math. Soc. 44 (2007), 361-391
We start by introducing the following the notations:

1. By $A = A_j dx^j$ we denote the magnetic field vector potential A. Thus A is a 1-form in $\mathbb{R}^N$, $N \geq 2$.
2. By $F = dA$ we denote the exterior derivative of A (2-from).
3. We introduce a 1 form $d_A = d - iA$, where d is the exterior differentiation.
4. For a k-form on $\mathbb{R}^N$ by $\star\omega$ we denote the Hodge star operation.

1. We define the magnetic field $B = \star F$. By definition $|B|^2 = \star F \wedge \star F$. In general, when ω is a k-from we agree that $|\omega|^2 = \star\omega \wedge \star\omega$.
2. We introduce a scalar complex field $u\colon \mathbb{R}^N \to \mathbb{C}$.
3. The magnetic Ginzburg-Landau energy functional is defined by:

$$\mathcal{E}_{GL}(B, u) = \int_{\mathbb{R}^N} \{\frac{1}{2}|B|^2 + \frac{1}{2}|d_A u|^2 + \frac{\lambda}{8}(1 - |u|^2)^2\}\, dx. \tag{28}$$

Here $\lambda > 0$ is a parameter.

The Euler-Lagrange equation of this system takes form:

$$\begin{aligned} \star d \star F + \frac{i}{2} \star \star \{ \bar{u} d_A u - u \overline{d_A u} \} &= 0, \\ \star d_A \star d_A u + \frac{\lambda}{2} u (1 - |u|^2) &= 0. \end{aligned} \tag{29}$$

Gauge invariance

The fundamental property of the Ginburg-Landau system, which has very important consequences for us, is its invariance with respect to the gauge transformation:

$$(A, u) \longmapsto (A + \nabla\chi, ue^{i\chi}), \quad \chi \colon \mathbb{R}^N \to \mathbb{R}, \tag{30}$$

where with some abuse of notation we identify $\nabla\chi$ with a 1-form $\partial_{x_j}\chi dx^j$.

Note that the generator of the gauge transformation is:

$$\mathfrak{g}_\chi \equiv (\nabla\chi, i\chi u), \quad \chi\colon \mathbb{R}^N \to \mathbb{R}. \tag{31}$$

Of course (mGL) is also invariant with respect to translations of $\mathbb{R}^N$:

$$(A(x), u(x)) \longmapsto (A(x+v), u(x+v)), \quad v \in \mathbb{R}^N. \tag{32}$$

The generator of this group is:

$$Z_j \equiv (\partial_{x_j} A, \partial_{x_j} u), \quad j = 1, \ldots, N. \tag{33}$$

These two invariances play a crucial role in the theory of the linearized Ginzburg-Landau operator.

Coulomb Gauge

Let us now mention one important choice of gauge, called the *Coulomb gauge*, which is obtained by taking ξ such that

$$d\star(A+\nabla\xi) = d\star A + \Delta\xi = \operatorname{div} A + \Delta\xi = 0.$$

Denoting $\mathtt{A} = A + \nabla\xi$, $\mathtt{u} = e^{i\xi}u$ we get the Ginzburg-Landau system in the Coulomb gauge:

$$\begin{aligned} (-1)^N \Delta\mathtt{A} + \star\star\frac{i}{2}\{\overline{\mathtt{u}}d_{\mathtt{A}}\mathtt{u} - \mathtt{u}\overline{d_{\mathtt{A}}\mathtt{u}}\} &= 0, \\ \star d_{\mathtt{A}}\star d_{\mathtt{A}}\mathtt{u} + \frac{\lambda}{2}\mathtt{u}(1-|\mathtt{u}|^2) &= 0. \end{aligned} \tag{34}$$

Note that the first equation can be written as

$$\Delta A + \operatorname{Im}\left(\overline{u}(\nabla - iA)u\right) = 0, \quad A = (A_1, \ldots, A_N),$$

and the second is:

$$(\nabla - iA)\cdot(\nabla - iA)u + \frac{\lambda}{2}\mathtt{u}(1-|\mathtt{u}|^2) = 0.$$

Pohozaev identity

Let $(,)$ be a solution of the Ginzburg-Landau system in the Coulomb gauge. Let $X = X^j\partial_{x_j}$ be a vector field in $\mathbb{R}^N$. Then it holds:

$$
\begin{aligned}
&\operatorname{div} \sum_k \{\frac{1}{2}|\nabla_k|^2 X - X(_k)\nabla_k\} + \operatorname{div}\{(\frac{1}{2}|d_|^2 \\
&+ \frac{\lambda}{8}(1-||^2)^2)X - \frac{1}{2}X()\overline{d} - \frac{1}{2}\overline{X()}d_\}\\
&\quad = \frac{1}{2}\sum_k |\nabla_k|^2 \operatorname{div} X - \frac{1}{2}\sum_k(\partial_{x_l}X^m + \partial_{x_m}X^l)\partial_{x_l}k\partial_{x_m}k \qquad (35)\\
&\qquad + \{\frac{1}{2}|d_|^2 + \frac{\lambda}{8}(1-||^2)^2\}\operatorname{div} X\\
&- \frac{1}{2}\{\partial_{x_j}\overline{(d_)l} + \partial_{x_j}(d_)l\}\partial_{x_l}X^j.
\end{aligned}
$$

The balance of forces for the two dimensional vortices

Let us assume that we have a solution (A, u) of the Ginzburg-Landau system in $\mathbb{R}^2$ with k vortices located at points $\{z_j\}_{j=1,\dots k} \subset \mathbb{R}^2$. The degrees of these vortices will be denoted by n_j, $j = 1, \dots k$ and the total degree by $n = \sum_j n_j$.
A consequence of Pohozaev identity is

$$\sum_j n_j z_j \cdot X = 0, \tag{36}$$

for all constant vector fields X in $\mathbb{R}^2$, which is the balance of forces formula for the k vortex solution.

Toda system and Vortex Filaments

Theorem (Kowalczyk-Wei 2012) Let $\lambda > 1/2$. For any $m \geq 1$ there exists $\varepsilon_m > 0$ such that for any $\varepsilon \in (0, \varepsilon_m]$ there exist: a set $\Lambda^\varepsilon_{4m} = \{\lambda^\varepsilon_1, \ldots, \lambda^\varepsilon_{4m}\}$ of $4m$ affine-lines with dihedral symmetry, and a solution $(u_\varepsilon, A_\varepsilon)$ of the magnetic Ginzburg-Landau system with $4m+1$ vortex lines such that the following holds:

$$(u_\varepsilon, A_\varepsilon) - (A^{(n_0)} + \sum_j A^{(n_j)}_{\lambda^\varepsilon_j}, u^{(n_0)} \prod_j u^{(n_j)}_{\lambda^\varepsilon_j})(x) \to 0, |x| \to \infty, n_j = (-1)^j$$

Above $(A^{(n_0)}, u^{(n_0)})$ is the standard vortex line of degree $n_0 = \pm 1$ centered at the x_3-axis.
In addition, if by e^ε_j we denote the direction vectors of the half-lines λ^ε_j then

$$\mathsf{e}^\varepsilon_j \cdot (0,0,1) = \mathcal{O}(\varepsilon), \quad \varepsilon \to 0.$$

Toda system in $\mathbb{R}^3$

Let $\vec{f_j}$ be a curve in $\mathbb{R}^3$. The Toda system, governing the dynamics of vortex filaments, is described by the following (nonlocal) ODE

$$\vec{f_j}'' = \partial_\nu (n_i n_j \sum_{i \neq j} e^{-d(\vec{f_i}, \vec{f_j})})$$

The Toda system is an expression of the balance of the normal forces resulting from the interactions. The tangential forces are balanced because of the Pohozhaev identity.

When we have dihedral symmetry or reflectional symmetry this system can be reduced to the one-dimensional Toda system.

$$f_j'' = e^{-(f_j - f_{j-1})} - e^{-(f_{j+1} - f_j)}$$

Other Physical Models

- Chern-Simons-Higgs

$$\mathcal{E}_{CSH}(\psi, A) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_A \psi|^2 + \frac{\mu^2}{4}\frac{(\nabla \times A)^2}{|\psi|^2} + \frac{\lambda}{2}|\psi|^2(|\psi|^2 - 1)^2.$$

  Self-dual case: Yang, Caffarelli-Yang
  Non-self-dual case: radial case Chen-Spirn
  Infinitely many nonradial solutions Ao-Pacard-Ting-Wei 2013

- Yang-Mills-Higgs

$$\mathcal{E}_{YMH}(\psi, A) = \frac{1}{2}\int_{\mathbb{R}^3} \{|F_A|^2 + |d_A\psi|^2 + \frac{\lambda}{4}(|\psi|^2 - 1)^2\}.$$

- Gauged Harmonic Map
- Kadomtsev−−Petviashvili I equation

Lecture 4: Entire Solutions of Allen-Cahn Equation: An Overview

Rutgers University

FRG on Nonlinear PDEs

May 14, 2013

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Allen-Cahn Equation

In lectures 4 to 6, we consider entire solutions of the Allen-Cahn equation

(AC) $$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N$$

Euler-Lagrange equation for the energy functional

$$J(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int (1-u^2)^2$$

$u = +1$ and $u = -1$ are global minimizers of the energy representing, in the gradient theory of phase transitions, two distinct phases of a material.

1. Background

Equation (AC) arises in the gradient theory of phase transitions by Cahn-Hilliard and Allen-Cahn, in connection with the energy functional in bounded domains Ω
$\mathsf{J}_\varepsilon(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon}\int_\Omega (1-u^2)^2, \quad \frac{1}{|\Omega|}\int_\Omega u = m\mathbf{e}$
whose Euler-Lagrange equation corresponds precisely to
$\varepsilon^2 \Delta u + u - u^3 = \lambda$ in Ω, $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega\mathbf{e}$

- Function $F(u) = \frac{1}{4}(1-u^2)^2$ has two minima of equal depth (double well potential).

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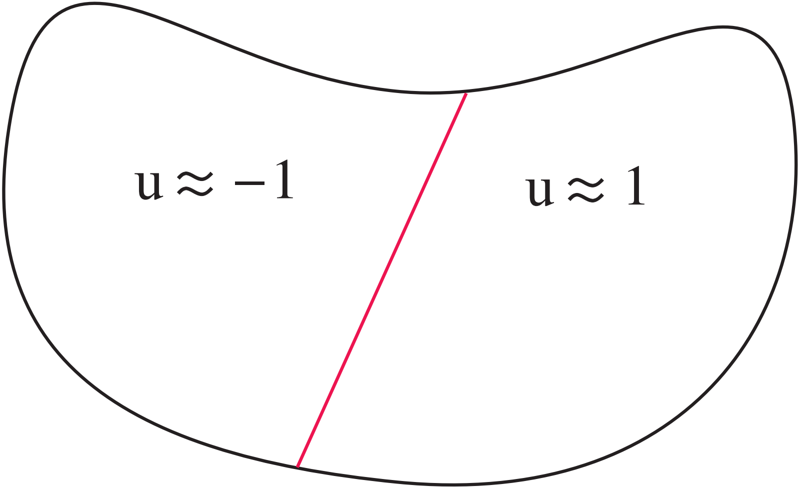
$\varepsilon^2 \Delta u + u - u^3 = \lambda$ in Ω, $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega\mathbf{e}$

- Function $F(u) = \frac{1}{4}(1-u^2)^2$ has two minima of equal depth (double well potential).
- $u = -1$ and $u = +1$ represent two phases
- The gradient term penalizes sharp transition between the phases (phase transitions).

Phase Separation



Γ-Convergence

The theory of Γ-convergence developed in the 70s and 80s, showed a deep connection between this problem and the theory of minimal surfaces,

- •• Modica-Mortola 77
- •• Modica 87
- •• Kohn-Sternberg 89

In fact, the functional J_ε converges in suitable sense as $\varepsilon \to 0$ to the perimeter functional of the limiting interface between the stable *phases* $u = 1$ and $u = -1$, so that roughly speaking, interfaces of local minimizers of J_ε are expected to approach minimal surfaces.

$$J_\varepsilon[u] \sim c_0 \text{Perimeter of } \{u = 0\}$$

Rigorous Results Caffarelli-Cordoba 95, 2006, Roger-Tonegawa 2008

Minimal Surfaces

critical points of $J_\varepsilon[u] \sim$ critical point of the Perimeter

$\sim$ mean curvature $= 0$

Minimal Surfaces are surfaces with zero mean curvature

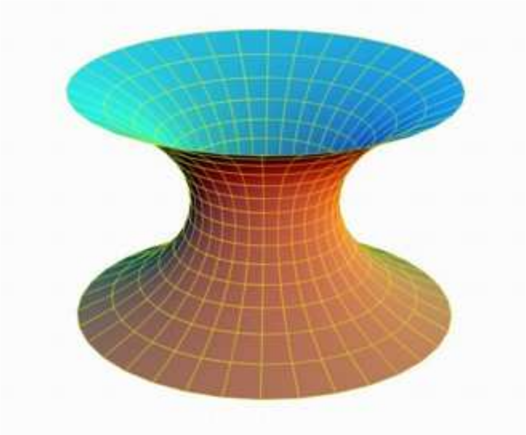
$$H = 0$$

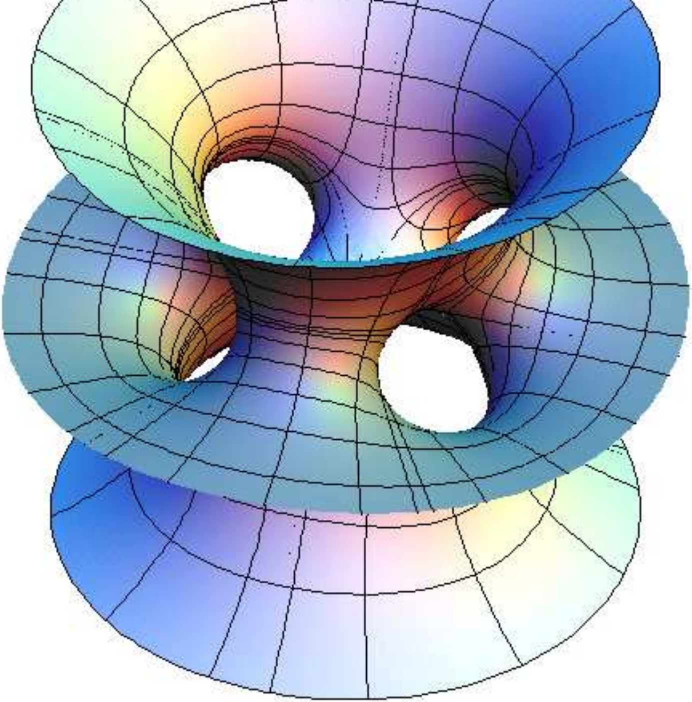
Typical examples of minimal surfaces

•• hyperplanes $x_N = 0$

•• catenoid

•• Costa surface





Bernstein Conjecture for Minimal Graphs

If minimal surfaces are graphs, they are called minimal graphs.

Minimal graphs: $x_N = F(x_1, ..., x_{N-1})$ where F satisfies

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}$$

For example $F(x') = a \cdot x' + b, \nabla F = a$, an affine function, is a minimal graph in any dimension (its graph is a hyperplane).

Bernstein Conjecture:any minimal hypersurface in $\mathbb{R}^N$, which is also a graph of a function of $N - 1$ variables, must be a hyperplane.

Some history on Bernstein Conjecture

- Bernstein (1910) $N = 3$.
- Flemining (1962) and De Giorgi (1965) reduced the problem to showing the non-existence of a minimal cone in dimension one less (extension to $N = 4$).
- Almgren ($N = 5$, 1966) and Simons ($N \leq 8$ 1968).
- In $\mathbb{R}^8$ there is a minimal cone (Simons' cone 1968):

$$\mathcal{C} = \{u = v\}, \quad u = \sqrt{x_1^2 + \cdots + x_4^2}, \;\; v = \sqrt{x_5^2 + \cdots + x_8^2}.$$

- Bombieri, De Giorgi, Giusti (1969) found an analytic minimal graph that is not a hyperplane for $N \geq 9$. They looked for a solution of the minimal surface equation in the form $x_9 = F(u, v)$, and found an F such that $F(u, v) = -F(v, u)$.
- Miranda's program: given a minimal cone one can construct a minimal graph.
- Simon (1988) showed a key estimate to complete this program and obtained further examples (Ferus-Karcher, Lawson).

The Bombieri-De Giorgi-Giusti minimal graph:

Explicit construction by super and sub-solutions. $N = 9$:

$$\cdot \left(\frac{F}{\sqrt{1 + |F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8.$$

$$F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}, \quad (,) \mapsto F(u,v), \quad u = ||,\ v = ||.$$

In addition, $F(u,v) > 0$ for $v > u$ and

$$F(u,v) = -F(v,u).$$

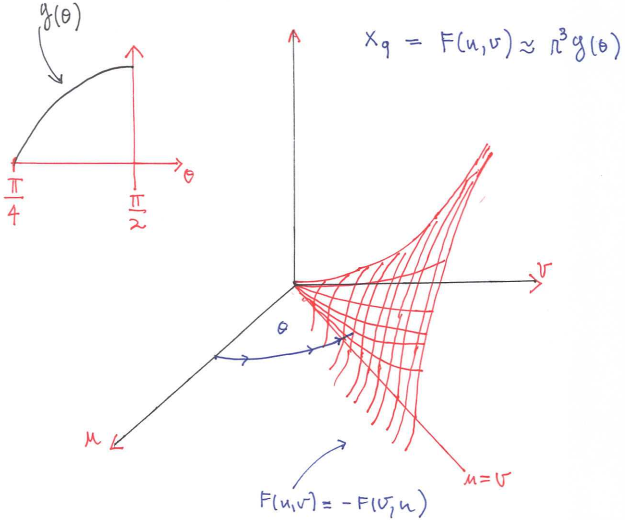
Asymptotic behavior for BDG surface in polar coordinates $u = r\cos\theta,\ v = r\sin\theta$ (del Pino, Kowalczyk, Wei (2008)):

$$\boxed{F(u,v) = r^3 g(\theta) + O(r^{-\sigma}) \quad \text{as } r \to +\infty}$$

where

$$\boxed{g(\theta) > 0, \quad \theta \in (\frac{\pi}{4}, \frac{\pi}{2}), \quad g(\frac{\pi}{4}) = 0 = g'(\frac{\pi}{2}).}$$

$$21\ \text{g}\ \sin^3 2\theta \frac{}{\sqrt{9g^2+g'^2}\ +\ \left(\frac{g'\sin^3 2\theta}{\sqrt{9g^2+g'^2}}\right)'=0\left(\frac{\pi}{4},\frac{\pi}{2}\right)\mathbf{e}}$$



$g(\theta)$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
θ
$X_q = F(u,v) \approx r^3 g(\theta)$
v
u
θ
$u = v$
$F(u,v) = -F(v,u)$

2. De Giorgi's Conjecture

Associated with Allen-Cahn equation is the famous De Giorgi's Conjecture.
De Giorgi's Conjecture concerns solutions of (AC) that connect these two values. They represent states in which the two phases coexist.

Solutions that "connect" the values -1 and +1 along some direction, say x_N:

$$\lim_{x_N \to -\infty} u(x', x_N) = -1, \quad \lim_{x_N \to +\infty} u(x', x_N) = +1, \text{ for all } x' \in \mathbb{R}^{N-1}$$

The case $N = 1$. The function

$$w(t) := \tanh\left(\frac{t}{\sqrt{2}}\right)$$

connects monotonically -1 and $+1$ and solves

$$w'' + w - w^3 = 0, \quad w(\pm\infty) = \pm 1, \quad w' > 0.$$

Canonical Example

For any $p, \nu \in \mathbb{R}^N$, $|\nu| = 1, \nu_N > 0$, the functions

$$u(x) := w((x - p) \cdot \nu)$$

solve equation (AC) and connects -1 and $+1$ along x_N.

De Giorgi's conjecture (1978): *Let u be a bounded solution of equation*

(AC) $$\Delta u + u - u^3 = 0 \quad \textit{in } \mathbb{R}^N,$$

which is monotone in one direction, say $_{x_N} u > 0$. Then, **at least** *when $N \leq 8$, there exist p, ν such that*

$$u(x) = w(\,(x - p) \cdot \nu).$$

This statement is equivalent to:

At least when $N \leq 8$, all level sets of u, $[u = \lambda]$ must be hyperplanes.

Parallel statement of Bernstein Conjecture for minimal graphs.
Minimal graphs: $x_N = F(x_1, ..., x_{N-1})$ where F satisfies

$$H = \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}$$

For example $F(x') = a \cdot x' + b, \nabla F = a$, an affine function, is a minimal graph in any dimension (its graph is a hyperplane).

Bernstein Conjecture:any minimal graph in $\mathbb{R}^N$ must be a hyperplane.
True for $N = 3$: Bernstein (1910), $N = 4$ De Giorgi (1965), $N = 4$ Fleming (1962), $N = 5$ Almgren (1966), $N \leq 8$ Simons (1968). False for $N \geq 9$: Bombieri-De Giorgi-Giusti found a counterexample (1969)

History of De Giorgi's conjecture

- True when $N = 2$, Ghossoub and Gui (1998) .

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$$\lim_{x_N \to \pm\infty} u(x^{'}, x_N) = \pm 1$$

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- False when $N \geq 9$, del Pino-Kowalczyk-Wei (2011)

Main Ideas of Proofs

Two Main Ingredients of Our Proofs:

I.detailed analysis of minimal surfaces, in particular the asymptotic behavior of such minimal surfaces, the global estimates for its derivatives, analysis of Jacobi operator and Jacobi fields

$$\Delta_\Gamma h + |A|^2 h = g$$

II. infinite-dimensional Liapunov-Schmidt reduction method developed in

del Pino-Kowalczyk-Wei 2007, (for compact geodesics)

del Pino-Kowalczyk-Pacard-Wei 2009 (for the whole R^N)
Lecture 5/6.

Gluing Method: Infinite-dimensional Liapunov-Schmidt Reduction Method

The basic idea is to generalize the finite-dimensional Liapunov-Schmidt reduction method to infinite-dimensional Liapunov-Schmidt reduction.

In finite-dimensional reduction method, one moves the points (which is a finite-dimensional space) in order to find the true solution.

In infinite-dimensional reduction method, we move curves or surfaces (which are infinite-dimensional space).

3. Beyond De Giorgi Conjecture–Stable Solutions

The assumption of monotonicity in one direction for the solution u in De Giorgi conjecture implies a form of stability.
In general, given a bounded solution u to the semilinear elliptic equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N.$$

We say that u is stable if

$$\int_{\mathbb{R}^N} (|\nabla \varphi|^2 - f^{'}(u)\varphi^2) \geq 0, \ \forall \varphi \in C_0^\infty(\mathbb{R}^N)$$

u is stable if and only if there exists a positive function $h > 0$ such that

$$\Delta h + f^{'}(u)h = 0 \text{ in } \mathbb{R}^N, h > 0 \text{ in } \mathbb{R}^N$$

Stability Conjecture

Stability Conjecture: Let u be a bounded stable solution of equation

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N.$$

Then the level sets $\{u = \lambda\}$ are all hyperplanes.

- Ambrosio-Cabre: $N = 2$, Stability Conjecture is true
- Pacard-Wei: $N = 8$, Stability Conjecture is False

Theorem

(Pacard-Wei 2009) Let $N = 8$. Then there exists a stable and bounded solution to (AC) such that its level set is not hyperplane.

A Dictionary for Allen-Cahn Equation

Bernstein Conjecture	$\longrightarrow$	De Giorgi Conjecture
Bernstein's Proof $N = 2$	$\longrightarrow$	Ghoussoub-Gui's Proof: $N = 2$
Almgren's Proof $N = 3$	$\longrightarrow$	Ambrosio-Cabre's Proof: $N = 3$
Simon's Proof $4 \leq N \leq 8$	$\longrightarrow$	Savin's Proof: $4 \leq N \leq 8$
BDG's Counterexample $N \geq 9$	$\longrightarrow$	DKW's Counterexample $N \geq 9$
Foliation of Simon's Cone	$\longrightarrow$	Pacard-Wei's Stable Solution

4. Beyond De Giorgi Conjecture–Finite Morse Index Solution

After stable solutions, next we study solutions which are not too unstable–Finite Morse Index Solutions.

Morse index of a solution u of (AC), $m(u)$: roughly, the number of negative eigenvalues of the linearized operator, namely those of the problem

$$\Delta\varphi + (1 - 3u^2)\varphi + \lambda\varphi = 0 \quad \varphi \in L^\infty(\mathbb{R}^N).$$

Finite Morse Index: $m(u) < +\infty$. Easy to check: $m(u) < +\infty$ if and only if there exists a compact set K such that

$$\int_{\mathbb{R}^N} (|\nabla\varphi|^2 - f'(u)\varphi^2) \geq 0, \ \forall\varphi \in C_0^\infty(\mathbb{R}^N \backslash K)$$

In other words, finite Morse index implies that u stable outside a finite region.

4.1: Finite Morse Index Solutions of Allen-Cahn Equation in $\mathbb{R}^2$

(AC) $$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^2$$

As we have discussed before, the only stable solution to (AC) in $\mathbb{R}^2$ is given by

$$u = \tanh\left(\frac{(x-p)\cdot\nu}{\sqrt{2}}\right)$$

Next we consider finite-Morse index solutions.

4.1: Finite Morse Index Solutions of Allen-Cahn Equation in $\mathbb{R}^2$

$$\text{(AC)} \qquad \Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^2$$

As we have discussed before, the only stable solution to (AC) in $\mathbb{R}^2$ is given by

$$u = \tanh\left(\frac{(x-p)\cdot\nu}{\sqrt{2}}\right)$$

Next we consider finite-Morse index solutions.
Multiple-end solutions are finite Morse index solutions.

$k-$end solutions

We say that u, a solution of (AC), is a multiple ends with k ends if there exist k oriented half lines $\{\mathbf{a}_j \cdot \mathbf{x} + b_j = 0\}$, $j = 1, \ldots, k$ (for some choice of $\mathbf{a}_j \in \mathbb{R}^2$, $|\mathbf{a}_j| = 1$ and $b_j \in \mathbb{R}$) such that along these half lines and away from a compact set K containing the origin, the solution is asymptotic to $w(\mathbf{a}_j \cdot \mathbf{x} + b_j)$, that is there exist positive constants C, c such that:

$$\|u(\mathbf{x}) - \sum_{j=1}^{k} w(\mathbf{a}_j \cdot \mathbf{x} + b_j)\|_{L^\infty(\mathbb{R}^2 \setminus K)} \leq Ce^{-c|\mathbf{x}|}.$$

Similar definitions as in the Nonlinear Schrödinger Equation

Balancing Formula:

$$\sum_{j=1}^{k} \mathbf{a}_j = 0$$

k

Existence of $2m$-ends Solutions and Toda System

Theorem

(del Pino-Kowalczyk-Pacard-Wei 2008) Given any solution $f_1 < f_2 < ... < f_K$ to the (integrable) Toda System:

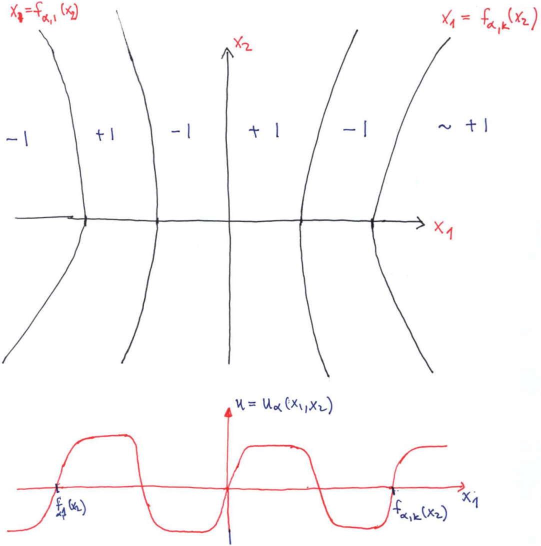
$$f_j'' = e^{f_{j-1}-f_j} - e^{f_j - f_{j+1}}, j = 1, ..., K, f_0 = -\infty, f_{K+1} = +\infty,$$

there exists a solution to the Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^2$$

with the following property:

$$u(x, y) \sim \sum_{j=1}^{K} (-1)^j w(x - f_j(\alpha y)) + (-1)^{K-1}$$



x1 = fα,1 (x2)
x1 = fα,k(x2)
x2
-1
+1
-1
+1
-1
~ +1
x1
u = uα (x1,x2)
x1
fα1(x2)
fα,k(x2)

A close look at $4-$end Solutions I: Saddle Solution

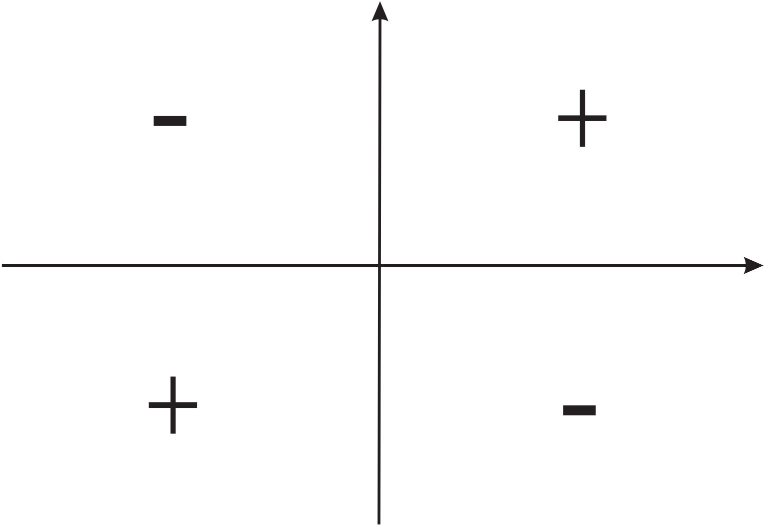
(AC) $$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^2$$

• Dang, Fife, Peletier (1992). The cross saddle solution:
$u(x_1, x_2) > 0$ for $x_1, x_2 > 0$,

$$u(x_1, x_2) = -u(-x_1, x_2) = -u(x_1, -x_2).$$

Nodal set two lines (4 ends). Super-subsolutions in first quadrant.

Two orthogonal lines



cross-solution

$4-$end Solutions II: Two Parallel Lines

Theorem (del Pino, Kowalczyk, Pacard, Wei (2008))

If f satisfies

$$\frac{\sqrt{2}}{24} f''(z) = e^{-2\sqrt{2} f(z)}, \quad f'(0) = 0,$$

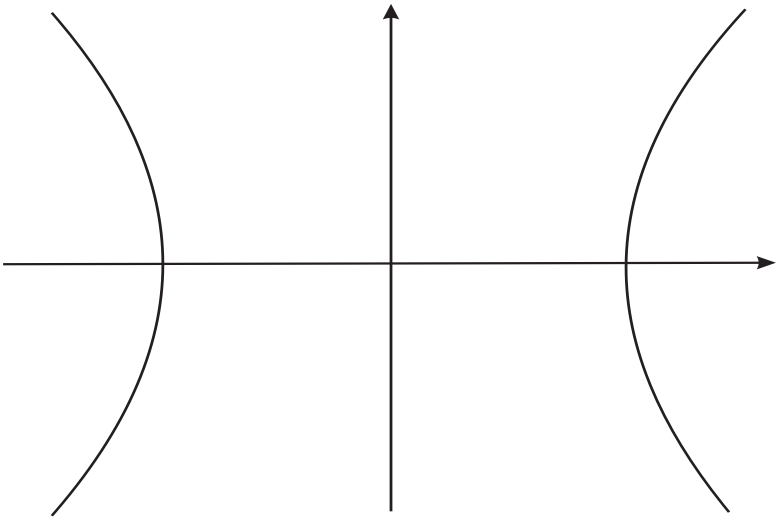
and $f_\varepsilon(z) := \sqrt{2}\log\frac{1}{\varepsilon} + f(\varepsilon z)$, then there exists a solution u_ε to (AC) in $\mathbb{R}^2$ with

$$\boxed{u_\varepsilon(x_1, x_2) = w(x_1 + f_\varepsilon(x_2)\,) + w(x_1 - f_\varepsilon(x_2)\, - 1\, + o(1)}$$

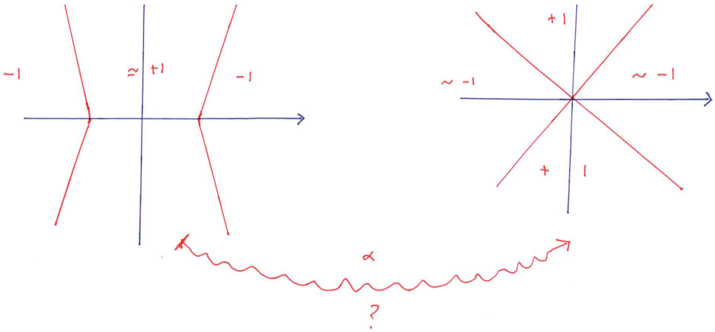
as $\varepsilon \to 0^+$. Here $w(s) = \tanh(s/\sqrt{2})$.

This solution has 2 transition lines.

$$f(z) = A|z| + B + o(1) \quad \text{as } z \to \pm\infty.$$



Two-end solution



2-line transition layer and 4 end saddle: Do they connect?

Gui 2011: proved that any four-end solution is even symmetric with respect to two orthogonal lines.

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Kowalczyk-Liu-Pacard (2012):Modulo rigid motions they constitute a one parameter family which is diffeomorphic to $\mathbb{R}$. The solution can be parametrized by the angle s, u_s.

Consequently, given any two lines, one find a solution to Allen-Cahn whose zero-level set approaches these two lines.

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Question:

Given any k lines in $\mathbb{R}^2$, $k \geq 3$, can one find a solution to Allen-Cahn whose zero-level set approaches these k lines?

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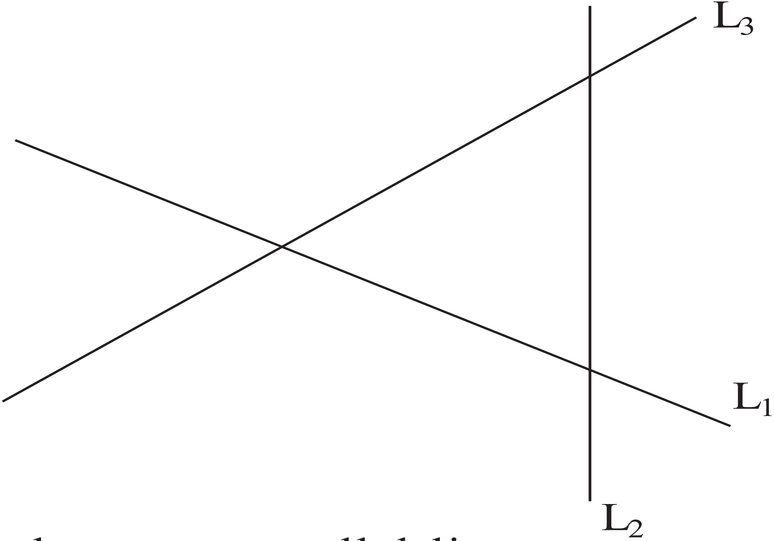
Question:

Given any k lines in $\mathbb{R}^2$, $k \geq 3$, can one find a solution to Allen-Cahn whose zero-level set approaches these k lines?

Answer: Yes, generic for $k > 2$.



three-non-parallel-lines

Theorem

(Kowalczyk-Pacard-Liu-Wei 2012) Let Σ be a set of k straight lines, $k > 2$. Suppose any two lines in Σ are not parallel and the intersection of any three lines are empty. Also suppose that the angle between any two of these lines is not equal to some exceptional values $\theta_i, i = 1, ..., n$. Then there exists a family of $2k$-end solutions u_ε to (AC) such that the nodal sets of the functions $u_\varepsilon\left(\frac{z}{\varepsilon}\right)$ converge to Σ on any compact set of $\mathbb{R}^2$, as $\varepsilon \to 0$.

Ingredients of proofs: moduli spaces theory and Cauchy data matching

5. Finite Morse Index Solutions in $\mathbb{R}^3$

We now consider three dimensional case:

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^3.$$

Finite Morse index solutions in $\mathbb{R}^3$ are generated by

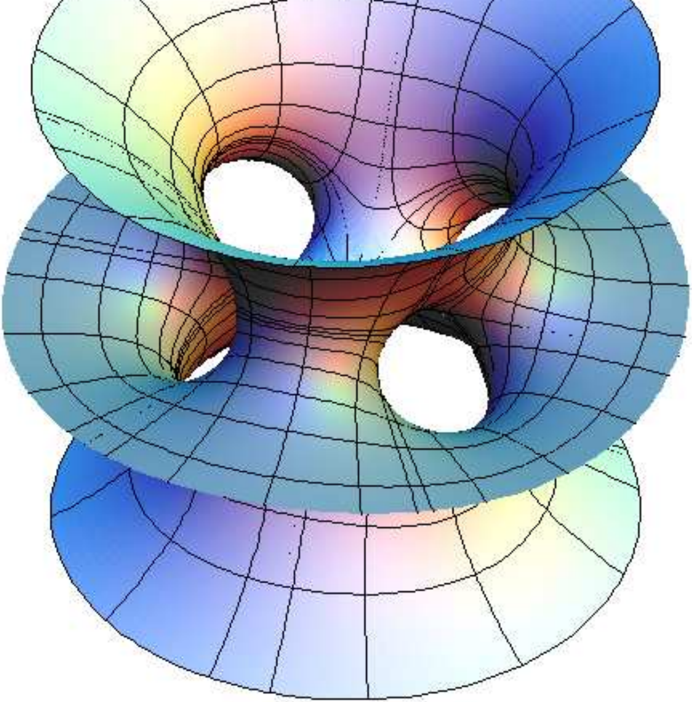
- embedded minimal surfaces with finite total curvature in $\mathbb{R}^3$
- the Toda system in $\mathbb{R}^2$

Embedded minimal surfaces of finite total curvature

The theory of embedded, minimal surfaces of finite total curvature in $\mathbb{R}^3$, has reached a notable development in the last 25 years. For more than a century, only two examples of such surfaces were known: the plane and the catenoid. The first nontrivial example was found in 1981 by C. Costa. The Costa surface is a genus one minimal surface, complete and properly embedded, which outside a large ball has exactly three components (its *ends*), two of which are asymptotically catenoids with the same axis and opposite directions, the third one asymptotic to a plane perpendicular to that axis. The complete proof of embeddedness is due to Hoffman and Meeks (1990).

Costa-Hoffman-Meeks Minimal Surfaces

Hoffman and Meeks also generalized notably Costa's example by exhibiting a class of three-end, embedded minimal surface, with the same look as Costa's far away, but with an array of tunnels that provides arbitrary genus $k \geq 1$. This is known as the Costa-Hoffman-Meeks surface with genus k.



Many other examples of multiple-end embedded minimal surfaces have been found.

Kapoulous (1997)

Traizet (2002)

etc.

In general all these surfaces look like parallel planes, slightly perturbed at their ends by asymptotically logarithmic corrections with a certain number of catenoidal links connecting their adjacent sheets.

Osserman, Schoen, ...: All ends are either catenoid or planes

Main results

Let M be a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature .

For $x = (x_1, x_2, x_3) = (x', x_3) \in \mathbb{R}^3$, we denote

$$r = r(x) = |(x_1, x_2)| = \sqrt{x_1^2 + x_2^2}.$$

After a suitable rotation of the coordinate axes, outside the infinite cylinder $r < R_0$ with sufficiently large radius R_0, then M decomposes into a finite number m of unbounded components $M_1, \ldots, M_m$, its *ends*.

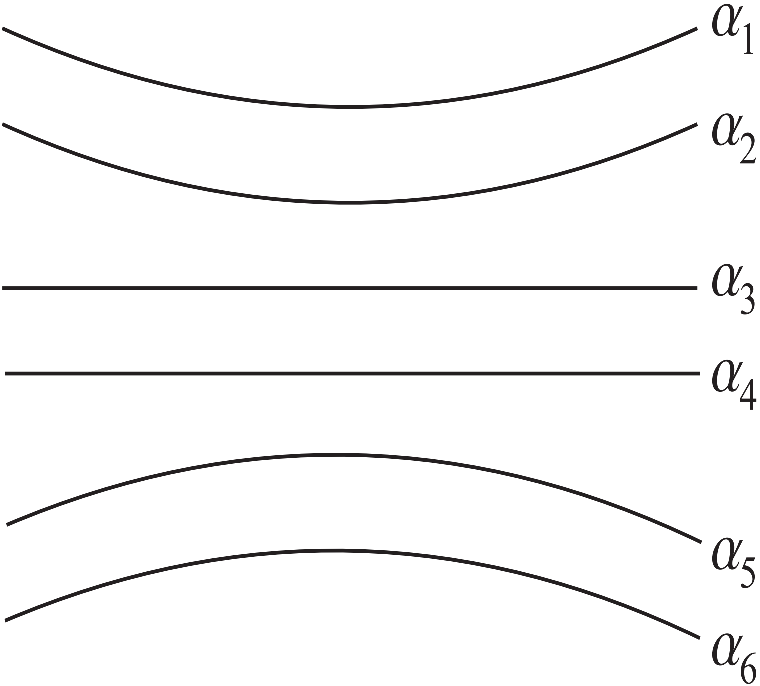
Osserman, R. Schoen: asymptotically each end of M_k either resembles a plane or a catenoid.

$$M_k = \{\, y \in \mathbb{R}^3 \ / \ r(y) > R_0,\ y_3 = F_k(y')\,\}$$

where F_k is a smooth function which can be expanded as $\mathsf{F}_k(y') = a_k \log r + b_k + b_{ik}\frac{y_i}{r^2} + O(r^{-3})$ as $r \to +\infty, \mathbf{e} for certain constants \mathsf{a}_k$, b_k, b_{ik},

$$a_1 \le a_2 \le \ldots \le a_m \ , \qquad \sum_{k=1}^{m} a_k \ = \ 0 \, . \qquad (37)$$

This last condition is the so-called Balancing Formula.



α_1
α_2
α_3
α_4
α_5
α_6

Jacobi Operator

Let us consider the Jacobi operator of M (h) $:=$ $\Delta_M h + |A|^2 h \mathbf{e} where$—A—$^2 = -2K$ is the Euclidean norm of the second fundamental form of M.
A smooth function $z(y)$ defined on M is called a Jacobi field if $(z) = 0$. Rigid motions of the surface induce naturally some bounded Jacobi fields: Associated with respectively translations along coordinates axes and rotation around the x_3-axis, are the functions

$$z_1(y) = \nu(y) \cdot e_i, \quad y \in M, \quad i = 1, 2, 3,$$

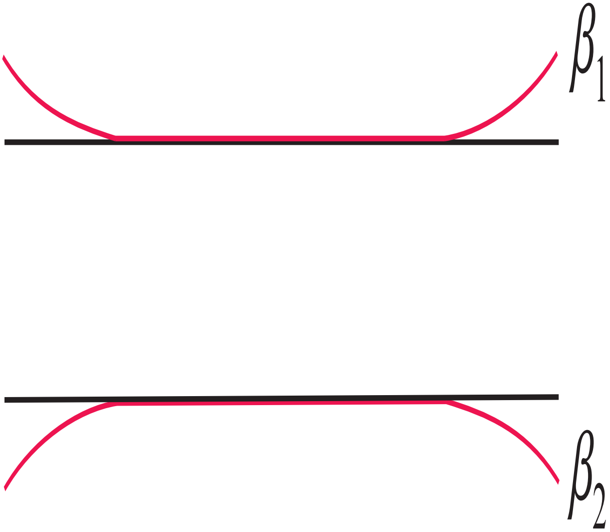
z$_4(y) = (-y_2, y_1, 0) \cdot \nu(y), \quad y \in M.\mathbf{e}$

We assume that M is non-degenerate in the sense that these functions are actually *all* the bounded Jacobi fields, namely $\{$ z$\in L^\infty(M)$ / $(z) = 0\}$ $=$ $\mathrm{span}\{z_1, z_2, z_3, z_4\}$ .e$Wedenoteinwhatfollowsby$J$thedimension$($\leq$ $4)$ of the above vector space. Note that for a catenoid, $z_{04} = 0$ so that $J = 3$.

This assumption, expected to be generic for this class of surfaces, is known in some important cases, most notably the catenoid and the Costa-Hoffmann-Meeks surface.
Nayatani 1994 (genus $k \leq 37$)
Morabito 2008 (genus $k \geq 38$)

Adjustment of Ends

If the two ends are parallel, say $a_{k+1} = a_k$, we need to adjust the ends by logarithmic growth β.



Let β be a vector of given m real numbers with
$\beta = (\beta_1, \ldots, \beta_m), \qquad \sum_{i=1}^{m} \beta_i = 0$ .e
The parameters β must satisfy an additional constraint. It is clear that if two ends are parallel, say $a_{k+1} = a_k$, we need at least that $\beta_{k+1} - \beta_k \geq 0$, for otherwise the ends would eventually intersect. Our further condition on these numbers is that these ends in fact diverge at a sufficiently fast rate. We require
$\beta_{k+1} - \beta_k > 4 \quad \text{if} \quad a_{k+1} = a_k$ .e

Theorem 5.1. (del Pino-Kowalczyk-Wei 2009) Let $N = 3$ and M be a minimal surface embedded, complete with finite total curvature which is nondegenerate. Then, given β satisfying relations (158) and (158), there exists a bounded solution u_ε of Allen-Cahn equation, defined for all sufficiently small ε, such that $\mathrm{u}_\varepsilon(x) = w(z-q(y))+O(\varepsilon)\, x = y+z\nu(\varepsilon y), |z-q(y)| < \frac{\delta}{\varepsilon}, \mathbf{e} where the function \mathrm{q} satisfies q(y) = (-1)^k\beta_k \log|\varepsilon y'| + O(1) \quad y \in (-1,1)$, the level set $[u_\varepsilon = \lambda]$ is an embedded surface that decomposes for all sufficiently small ε into m disjoint components (ends) outside a bounded set. The k-th end lies at $O(1)$ distance from the graph

$$y_3 = \varepsilon^{-1} F_k(\varepsilon y) + \beta_k \log|\varepsilon y'|.$$

Next we discuss the connection of the Morse index of the solutions of Theorem 5.1 and the index of the minimal surface M, $i(M)$, which has a similar definition relative to the quadratic form for the Jacobi operator: The number $i(M)$ is the largest dimension for a vector spaced E of compactly supported smooth functions in M with

$$\int_M |k|^2 \, dV - \int_M |A|^2 k^2 \, dV \; < 0 k \in E \setminus \{0\}.$$

For complete, embedded surfaces, finite index is equivalent to finite total curvature (Gulliver, 1986). Thus, for our surface M, $i(M)$ is indeed finite.

Theorem 5.2. Let u_ε the solution given by Theorem 5.2. Then for all sufficiently small ε we have

$$m(u_\varepsilon) \;=\; i(M).$$

Besides, the solution is non-degenerate, in the sense that any bounded solution of

$$\Delta\varphi + f'(u_\varepsilon)\varphi = 0 \quad \text{ in } \mathbb{R}^3$$

must be a linear combination of the functions $Z_i,\ i = 1, 2, 3, 4$ defined as

$$Z_i = \partial_i u_\varepsilon, \quad i = 1, 2, 3, \quad Z_4 = -x_2\partial_1 u_\varepsilon + x_{12} u_\varepsilon.$$

Finite Morse Index Solutions

In the Costa-Hoffmann-Meeks surface it is known that $i(M) = 2l - 1$ where l is the genus of M. (Natayani, Morabito). As a Corollary, we have the following

Corollary: For each positive odd integer k, there exists a solution to the Allen-Cahn equation in $\mathbb{R}^3$ with Morse index k.

Moduli Space of Solutions

The solutions in Theorem 5.1 depends, for fixed ε, on m parameters. Taking into account the constraint $\sum_{j=1}^{m} \beta_j = 0$ and the rescaling parameter ε, this gives $m-1$ independent parameters corresponding to logarithmic twisting of the ends of the level sets. Thus, besides the trivial rigid motions of the solution, translation along the coordinates axes, and rotation about the x_3 axis, this family of solutions depends exactly on $m-1$ "independent" parameters. Our second result is that the bounded kernel of the linearization of equation (AC) about one of these solutions is made up exactly of the generators of the rigid motions, so that in some sense the solutions found are L^{∞}-isolated, and the set of bounded solutions nearby is actually $m-1+J$-dimensional.

Corollary. The set of the solutions constructed in Theorem 5.1 form a $m-1+J$-dimensional analytic space

A result parallel to this one, in which the moduli space of the minimal surface M is described by a similar number of parameters, has been obtained by Perez and Ros (1996)

Family of Morse index 1 solutions

The catenoid gives Morse index 1 solution to (AC)

Theorem

(Agudelo-del Pino-Wei 2013)
There exists an axially symmetric solution with nodal set made up of two components $\Gamma_\pm$ which are graphs of two functions

$$\varphi_\pm(r) \sim \pm 2\log(1+\varepsilon r) \pm \log\frac{1}{\varepsilon}$$

as $r \to +\infty$. This solution has Morse index 1.

role of Liouville Equation

$$\Delta f + e^f = 0 \text{ in } \mathbb{R}^2$$

Towards a classification of finite Morse index solutions

Understanding bounded, entire solutions of nonlinear elliptic equations in $\mathbb{R}^N$ is a problem that has always been at the center of PDE research. This is the context of various classical results in PDE literature like the Gidas-Ni-Nirenberg theorems on radial symmetry of one-signed solutions, Liouville type theorems, or the achievements around De Giorgi conjecture. In those results, the geometry of level sets of the solutions turns out to be a posteriori very simple (planes or spheres). More challenging seems to be the problem of classifying solutions with finite Morse index, in a model as simple as the Allen-Cahn equation. While the solutions predicted by Theorem 1 are generated in an asymptotic setting, it seems plausible that they contain germs of generality, in view of parallel facts in the theory of minimal surfaces. In particular we believe that the following two statements hold true for a bounded solution u to Allen-Cahn equation in $\mathbb{R}^3$.

Open Question One

(1) If u has finite Morse index and $\nabla u(x) \neq 0$ outside a bounded set, then each level set of u must have outside a large ball a finite number of components, each of them asymptotic to either a plane or to a catenoid. After a rotation of the coordinate system, all these components are graphs of functions of the same two variables.

Analogue results for minimal surfaces: Osserman, Schoen

Open Question Two

(2) If u has Morse index equal to one. Then u must be axially symmetric, namely after a rotation and a translation, u is radially symmetric in two of its variables. Its level sets have two ends, both of them catenoidal.

Analogue results for minimal surfaces: Schoen

Open Question Three

An interesting question is the classification of Morse index 1 solutions.

Question 3.1: Are the two family of the solutions, one with single catenoid ends, another one generated by Liouville equation

$$\Delta u + e^u = 0 \text{ in } \mathbb{R}^2$$

connected?

The main problem is the compactness of solutions with catenoid ends

$$a \log |x| + b$$

Open Question Four

The Morse index 1 solution is generated by Liouville equation in $\mathbb{R}^2$: ($\Delta f + e^f = 0$ in $\mathbb{R}^2$) which is a special case of Toda system in $\mathbb{R}^2$

$$\begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} = 0 \quad \text{in } \mathbb{R}^2, \\ \Delta u_2 + 2e^{u_2} - e^{u_1} - e^{u_3} = 0 \quad \text{in } \mathbb{R}^2, \\ ..... \\ \Delta u_k + 2e^{u_k} - e^{u_{k-1}} - e^{u_{k+1}} = 0 \quad \text{in } \mathbb{R}^2, \\ .... \\ \Delta u_N + 2e^{u_N} - e^{u_{N-1}} = 0 \quad \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_i} < +\infty, \quad i = 1, ..., N \end{cases}$$

A complete classification is given by Lin-Wei-Ye (2012)

Conjecture: All solutions of Toda system can be emedded into solutions to (AC) with finite Morse index.

6. Beyond De Giorgi's Conjecture–Solutions with Infinite Morse Index

Next we study solutions with infinite Morse index

$$m(u) = +\infty$$

First we consider the two-dimensional case. Recall that given any finite Toda system

$$c_* q_i'' = e^{\sqrt{2}(q_{i-1}-q_i)} - e^{\sqrt{2}(q_i - q_{i+1})}, i = 1, ..., k, q_0 = -\infty, q_{k+1} = +\infty$$

there is a corresponding finite Morse index solution to (AC).
How about infinite Toda lattice equation?

$$c_* q_i'' = e^{\sqrt{2}(q_{i-1}-q_i)} - e^{\sqrt{2}(q_i - q_{i+1})}, i \in Z. \tag{38}$$

It admits the following set of one traveling soliton solution

$$q_i(t) = \frac{1}{\sqrt{2}} S_k(i - \alpha t), i \in Z. \tag{39}$$

$$S_k(t) = \ln \frac{\cosh k\left(t - \frac{1}{2}\right)}{\cosh k\left(t + \frac{1}{2}\right)}.$$

scaling

By the scaling symmetry admitted by the Toda lattice, the following family of functions, which has ε as its parameter, are also solutions to the infinite Toda system

$$q_{i,\varepsilon}(t) := q_i(\varepsilon t) - \sqrt{2}i \ln \varepsilon, i \in Z. \tag{40}$$

Theorem (Kowalczyk-Liu-Wei 2013) For each $\varepsilon > 0$ small, there exists a singly periodic solution u_ε to the Allen-Cahn equation in $\mathbb{R}^2$, whose zero level set is close to the one-soliton solution $q_{i,\varepsilon}$ of the infinite Toda lattice:

$$c_* q_i'' = e^{\sqrt{2}(q_{i-1}-q_i)} - e^{\sqrt{2}(q_i - q_{i+1})}, i \in Z. \tag{41}$$

Moreover, it satisfies

$$u_\varepsilon(z) = u_\varepsilon(-z),$$
$$u_\varepsilon(z) = -u_\varepsilon(z + e_\varepsilon),$$

for suitable vector $e_\varepsilon \in \mathbb{R}^2$ depending on ε. The Morse index of u_ε is $+\infty$.

Solutions with Infinite Morse Index in $\mathbb{R}^3$

Theorem

(Agudelo-del Pino-Wei 2013)
There exists an axially symmetric solution with nodal sets Γ_1, Γ_2 made up of two components diverging logarithmically from a largely dilated catenoid, $\varepsilon^{-1}\Gamma_0$, one inside, the other outside. graphs for $r > \frac{1}{\varepsilon}$ of functions

$$\varphi_1(r) \sim 4\varepsilon^{-1}\log(r\varepsilon) + 2\log r, \varphi_1(r) \sim 4\varepsilon^{-1}\log(r\varepsilon) - 2\log r$$

This solution has Morse index growing to infinity with $\varepsilon \to 0$

The role of Jacobi-Toda system

$$\Delta_\Gamma f_i + |A|^2 f_i = e^{f_{i-1}-f_i} - e^{f_i - f_{i+1}}$$

7. Beyond de Giorgi Conjecture: Parabolic De Giorgi Conjecture

Next, we move on to the parabolic Allen-Cahn equation
parabolic Allen-Cahn equation (parabolic AC)

$$u_t = \Delta u + u - u^3 \quad \text{in } \mathbb{R}^N \times \mathbb{R} \quad .$$

Parabolic De Giorgi Conjecture:
Consider eternal solutions of parabolic Allen-Cahn equation

$$u_t = \Delta u + f(u), (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{42}$$

Assuming their monotonicity in the x_N direction:

$$\partial_{x_N} u > 0, \quad \lim_{x_N \to \pm\infty} u(x', x_N, t) = \pm 1, \quad t \in \mathbb{R}$$

then u is one-dimensional.

This conjecture is false even in dimension $N = 2$.

- In 2007 Chen, Guo, Hamel, Ninomiya, Roquejoffre showed the existence of solutions to (42) of the form $u(x', x_N - ct) = U(r, x_{N+1} - ct)$, $r = |x'|$, $N \geq 1$. Functions U have paraboloid-like profiles of their nodal sets Γ.
- U satisfies the traveling wave Allen-Cahn

$$(AC)_{TW} \qquad \Delta u + u - u^3 + cu_{x_N} = 0 \quad \text{in } \mathbb{R}^N.$$

- the asymptotic profiles of the fronts are given:

$$\lim_{\substack{x_N \to +\infty \\ (x', x_N) \in \Gamma}} \frac{r^2}{2x_N} = \frac{N-1}{c}, \quad \text{if } N > 1.$$

- When $N = 1$ the ends of the fronts become asymptotically parallel.

Traveling Wave De Giorgi Conjecture

Let u be a bounded solution of equation

$$(AC)_{TW} \qquad \Delta u + u - u^3 + c u_{x_N} = 0 \quad \text{in } \mathbb{R}^N.$$

which satisfies

$$_{x_N} u > 0$$

Then, u must be axially symmetric in x'.

Parabolic Allen-Cahn Equation and Mean Curvature Flow

Consider the mean curvature flow for a graph $x_{N+1} = F(x)$:

$$\frac{\partial F}{\partial t} = \sqrt{1 + |\nabla F|^2}\nabla(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}}) \tag{43}$$

Mean Curvature Solitons: Graphs which are translated by the mean curvature (MC) flow with constant velocity (say 1) in a fixed direction satisfy:

$$\nabla(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}}) = \frac{1}{\sqrt{1 + |\nabla F|^2}}. \tag{44}$$

Rotationally symmetric eternal solution to the MC flow

Altschuler-Wu, Clutterbuck-Schnurer-Schulze (CVPDE 2003).
There exists a unique radially symmetric solution F of (44):

$$F(r) = \frac{r^2}{2(N-1)} - \log r + 1 + O(r^{-1}), \quad r \gg 1. \tag{45}$$

The first term in this asymptotic behavior coincides with the asymptotic behavior of the nodal set of solutions to (AC-TW) found by Chen, Guo, Hamel, Ninomiya, Roquejoffre.

Bernstein Type Conjecture for MC Solitons

Let F be a solution of

$$\nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right)=\frac{1}{\sqrt{1+|\nabla F|^2}} \text{ in } \mathbb{R}^N. \quad (46)$$

Then F is rotationally or cylindrically symmetric.

A natural critical dimension seems to be $N = 8$. However

X-J Wang , Annals Math., showed that the existence of non-radial eternal convex graphs when $N \geq 3$.

Revised Traveling Wave De Giorgi Conjecture

Let u be a bounded solution of equation

$$(AC)_{TW} \qquad \Delta u + u - u^3 + cu_{x_N} = 0 \quad \text{in } \mathbb{R}^N.$$

which satisfies

$$_{x_N} u > 0$$

Then, u must be axially symmetric in x', at least when $N \geq 3$.

8. Beyond De Giorgi Conjecture: Allen-Cahn Systems

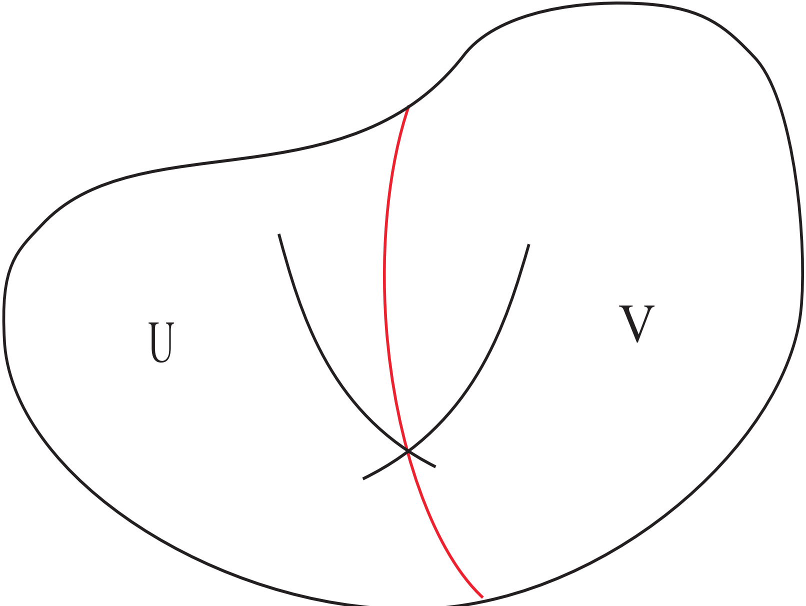
Consider the following Bose-Einstein competition system

$$\begin{cases} -\Delta u + \alpha u^3 + \Lambda v^2 u = \lambda_1 u & \text{in } \Omega, \\ -\Delta v + \beta v^3 + \Lambda u^2 v = \lambda_2 v & \text{in } \Omega, \\ u > 0, \quad v > 0 & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{47}$$

Asymptotic Behavior when $\Lambda \to +\infty$

•• Wei and Weth (2007): $u_{j,\Lambda}$ is uniformly equicontinuous
•• Noris-Tavares-Terracini-Verzini (2009): uniform Hölder convergence.

Phase Separation



Lotelra-Volteria system:

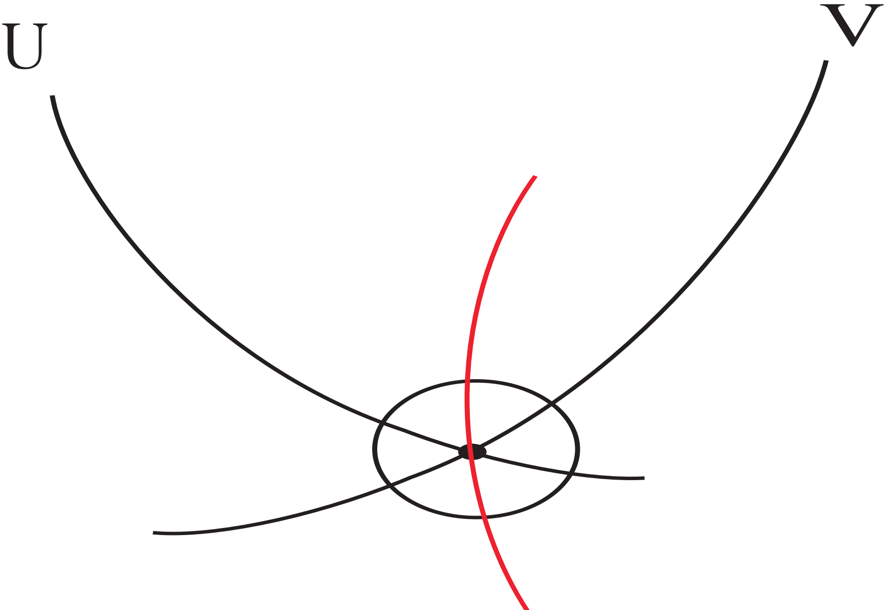
$$\begin{cases} -\Delta u_i = u_i u_i - \sum_{\substack{j=1 \\ j\neq i}}^{k} \alpha_{ij} u_j u_i & \text{in } \Omega, \\ \quad u_1, \ldots, u_k > 0 & \text{in } \Omega, \\ \quad u_1 = \cdots = u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Dancer and Du (2000), Conti, Terracini and Verzini 2007) and Caffarelli and Lin (2009)
Limiting Elliptic System

$$\Delta U = UV,\ \Delta V = VU$$

$$\Delta(U - V) = 0$$

Phase Separation



Limiting System

Let $x_\Lambda \in \Omega$ be a point where u_Λ and v_Λ meet, i.e., $u_\Lambda(x_\Lambda) = v_\Lambda(x_\Lambda) = m_\Lambda$.
Then as $\Lambda \to +\infty, x_\Lambda \to x_0 \in \{u_0 = v_0 = 0\}$. Suppose $x_0 \in \Omega$ and we do the following scaling

$$\tilde{u}_\Lambda(y) = \frac{1}{m_\Lambda} u_\Lambda(m_\Lambda y + x_\Lambda), \tilde{v}_\Lambda(y) = \frac{1}{m_\Lambda} v_\Lambda(m_\Lambda y + x_\Lambda). \quad (48)$$

Then $(\tilde{u}_\Lambda, \tilde{v}_\Lambda)$ satisfies

$$-\Delta \tilde{u}_\Lambda + m_\Lambda^4 \alpha \tilde{u}_\Lambda^3 + m_\Lambda \Lambda^4 \tilde{v}_\Lambda^2 \tilde{u}_\Lambda = m_\Lambda^2 \lambda_1 \tilde{u}_\Lambda \text{ in } \Omega_\Lambda, \quad (49)$$

$$-\Delta \tilde{v}_\Lambda + m_\Lambda^4 \beta \tilde{v}^3 + m_\Lambda \Lambda^4 \tilde{u}_\Lambda^2 \tilde{v}_\Lambda = m_\Lambda^2 \lambda_1 \tilde{v}_\Lambda \text{ in } \Omega_\Lambda. \quad (50)$$

where $\Omega_\Lambda = \frac{\Omega - x_\Lambda}{m_\Lambda}$.

Limiting Elliptic System

Letting $\Lambda \to +\infty$ and assuming that

$$m_\Lambda^4 \Lambda \to C_0 > 0 \tag{51}$$

we derive formally the following system of equations (after rescaling)

$$\boxed{\Delta U = UV^2,\ \Delta V = VU^2,\ U, V \geq 0,\ \text{in } \mathbb{R}^N.} \tag{52}$$

Limiting System

$$\begin{cases} \Delta U = UV^2 \quad \text{in } \mathbb{R}^N, \\ \\ \Delta V = VU^2 \quad \text{in } \mathbb{R}^N \\ \\ U > 0, V > 0 \end{cases}$$

Complete Analysis in One-Dimensional Case

First, we completely classify the one-dimensional solutions, using the method of moving planes.

Theorem (Berestycki-Lin-Wei-Zhao 2009) Let $N = 1$ and (U, V) be a solution to (52).

(1) (Symmetry) There exists $x_0 \in \mathbb{R}$ such that

$$V(y - x_0) = U(x_0 - y). \tag{53}$$

(2) (Asymptotic behavior) The following alternatives hold: either

$$\begin{cases} U(-\infty) = 0,\ U'(-\infty) = 0,\ U' > 0,\ U'(\infty) = \sqrt{T_\infty}, \\ V(\infty) = 0,\ V'(\infty) = 0,\ V' < 0,\ V'(-\infty) = -\sqrt{T_\infty}, \end{cases} \tag{54}$$

or

$$\begin{cases} U(\infty) = 0,\ U'(\infty) = 0,\ U' < 0,\ U'(-\infty) = -\sqrt{T_\infty}, \\ V(-\infty) = 0,\ V'(-\infty) = 0,\ V' > 0,\ V'(\infty) = \sqrt{T_\infty}. \end{cases} \tag{55}$$

(continued)

(3) (Nondegeneracy) (U, V) *is nondegenerate, i.e., the solution to the linearized equation*

$$\varphi^{''} = V^2\varphi + 2UV\psi, \ \psi^{''} = U^2\psi + 2UV\varphi, \ \varphi, \psi \text{ are bounded} \quad (56)$$

is given by $(\varphi, \psi) = c(U^{'}, V^{'})$.

(4) (Uniqueness) There exists a unique solution (U, V)*, up to translation and scaling*

(Berestycki-Terracini-Wang-Wei 2012)

New De Giorgi Conjecture

New De Giorgi Conjecture: Under the following monotone condition

$$\frac{\partial U}{\partial y_N} > 0, \ \frac{\partial V}{\partial y_N} < 0, \tag{57}$$

the solutions to the system

$$\Delta U = UV^2, \Delta V = U^2 V, U, V > 0 \textit{ in } \mathbb{R}^N$$

are one-dimensional.

Farina (2013): If $|U| + |V|$ has polynomial growth, then

$$\frac{\partial U}{\partial y_N} > 0 \implies \frac{\partial V}{\partial y_N} < 0$$

New Stability Conjecture

For a solution (U, V) to the limiting system, we say it is stable if

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 + |\nabla\psi|^2 + \int_{\mathbb{R}^N} V^2\varphi^2 + U^2\psi^2 + 4UV\varphi\psi \geq 0, \quad (58)$$

for any compactly supported smooth functions φ, ψ.

New Stability Conjecture

For a solution (U, V) to the limiting system, we say it is stable if

$$\int_{\mathbb{R}^N} |\nabla \varphi|^2 + |\nabla \psi|^2 + \int_{\mathbb{R}^N} V^2 \varphi^2 + U^2 \psi^2 + 4UV\varphi\psi \geq 0, \quad (58)$$

for any compactly supported smooth functions φ, ψ.
New Stability Conjecture: The stable solutions to the system

$$\Delta U = UV^2, \Delta V = U^2 V, U, V > 0 \text{ in } \mathbb{R}^N$$

$$are$$

$$one - dimensional.$$

Results

•• Berestycki-Lin-Wei-Zhao (2011): Let $N = 2$. De Giorgi Conjecture holds if U and V has linear growth.

•• Berestycki-Lin-Wei-Zhao (2011): The one-dimensional solution is stable.

•• Berestycki-Terracini-Wang-Wei (2012): Let $N = 2$. Stability Conjecture holds if U and V has linear growth.

Saddle Solutions: Entire Solutions with Polynomial Growth

$$\Delta U = UV^2, \ \Delta V = VU^2, \ \text{in } \mathbb{R}^N$$

For Allen-Cahn equation, u is bounded between -1 and $+1$. On the other hand, the system (52) has unbounded one-dimensional solutions with linear growth. Is there a growth estimate for (52)

$$U(x) + V(x) = O(|x|)???$$

Saddle Solutions: Entire Solutions with Polynomial Growth

$$\Delta U = UV^2,\ \Delta V = VU^2,\ \text{in } \mathbb{R}^N$$

For Allen-Cahn equation, u is bounded between -1 and $+1$. On the other hand, the system (52) has unbounded one-dimensional solutions with linear growth. Is there a growth estimate for (52)

$$U(x) + V(x) = O(|x|)???$$

Answer: NO
Result: For any harmonic function with polynomial growth, there exists a solution (U, V) with

$$U(x) + V(x) \sim |x|^d$$

For any $d \geq 1$, define the harmonic polynomial Φ as

$$\Phi := \mathrm{Re}(z^d).$$

Note that Φ has some reflectional symmetry. That is, take its d nodal lines $L_1, \cdots, L_d$ and denote the corresponding reflection with respect to these lines as $T_1, \cdots, T_d$, then

$$\Phi(T_i z) = -\Phi(z). \tag{59}$$

Theorem (Berestycki-Terracini-Wang-Wei 2012):There exists a solution (u, v) to the problem (52), satisfying

1. $u - v > 0$ in $\{\Phi > 0\}$ and $u - v < 0$ in $\{\Phi < 0\}$;
2. $u \geq \Phi^+$ and $v \geq \Phi^-$;
3. $\forall i = 1, \cdots, d$, $u(T_i z) = v(z)$;
4. $\forall r > 0$,

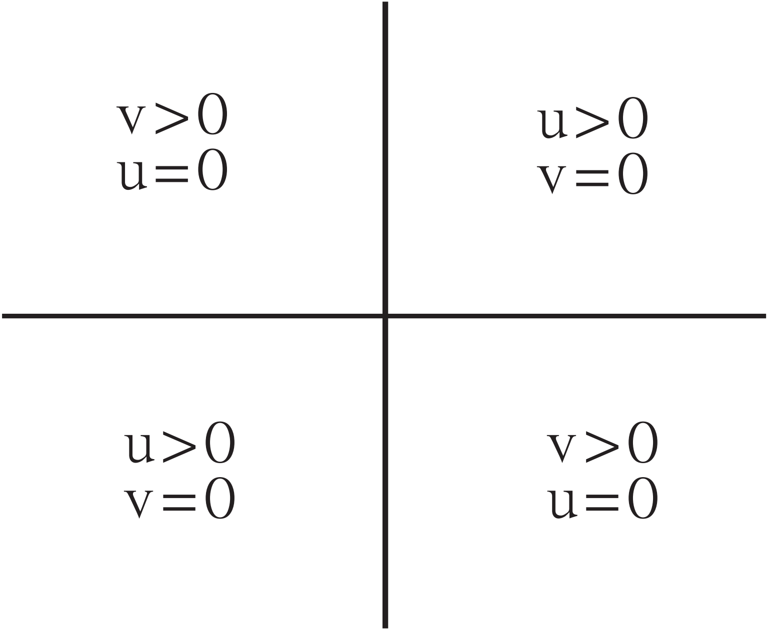
$$N(r) := \frac{r \int_{B_r(0)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2}{\int_{\partial B_r(0)} u^2 + v^2} \leq d$$

   and in fact the Almgren frequency exponent

$$N(r) \to d \text{ as } r \to +\infty$$

5. $\exists C > 0$, for all $z \in \mathbb{R}^2$,

$$u(z) \leq \Phi^+(z) + C(1 + |z|)^{\frac{d-1}{2}}, v(z) \leq \Phi^-(z) + C(1 + |z|)^{\frac{d-1}{2}}.$$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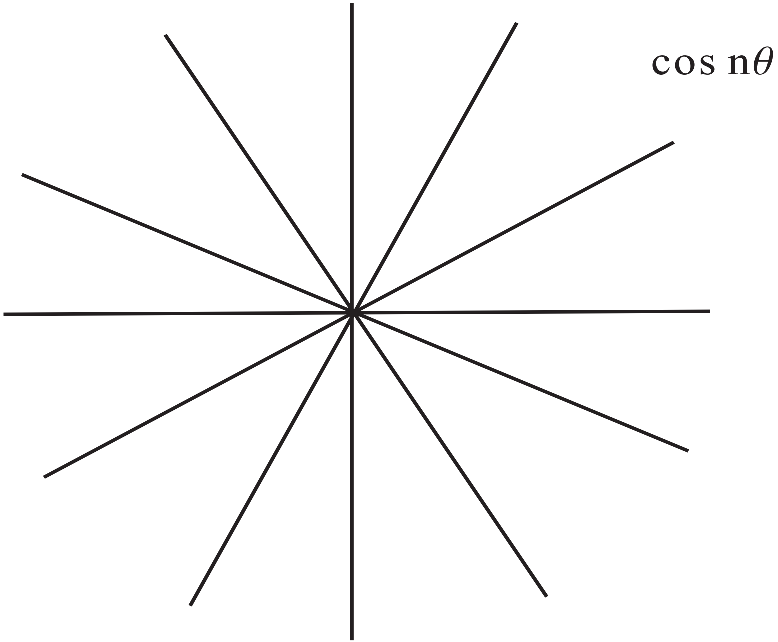
u=0
u>0
v=0
u>0
v=0
v>0
u=0


cos nθ