ERRATA FOR

The Classification of the Finite Simple Groups

A.M.S. Surveys and Monographs 40

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The authors are grateful to the many people who have found errors, including Professors J. L. Alperin, K. Andersen, P. Flavell, N. Gill, R. Guralnick, the late C.-Y. Ho and J. E. Humphreys, M. W. Jacobsen, I. Korchagina, C. Ku, L. Morgan, C. Parker, K. Piterman, G. Seitz, P. Sin, G. Stroth, J. G. Thompson, and V. I. Zenkov.

ERRATA FOR NUMBER 1

The first three of these errata have been corrected in the second printing.

Pages 47,140,142: *The correct Background Reference is:*

[**Ca1**] R. W. Carter, *Simple Groups of Lie Type*, Wiley and Sons, London, 1972.

Page 142: *The correct Expository Reference is:*

[**Ca2**] R. W. Carter, *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*, Wiley–Interscience, London, 1985.

Pages 100, 102: *In Definitions 12.1 and 13.1, the group* $G_2(8)$ *should be removed from the set* $\mathcal{C}_3$ *and placed in* $\mathcal{T}_3$.

Page 12, Lines -10 to -9: ~~of significance only as part of the proof~~ significant only as part of the proof of

Page 20, Line -9: *At the end of Section 5, add the following paragraph:* The p-layer $L_{p'}(X)$ can alternatively be defined by $L_{p'}(X) = O^{p'}(E(X \bmod O_{p'}(X)))$. Here and elsewhere, if A is a function from groups to groups such that $A(G) \le G$ for every group G, and if $N \triangleleft G$, we write $A(G \mod N)$ to signify the full preimage in G of $A(G/N)$.

Page 33, Line -12: $\cancel{x_{-\alpha}(t^{-1})}$ $x_{-\alpha}(-t^{-1})$

Page 152, Line 14: *Add an entry to the glossary:* 23 K_y

ERRATA FOR NUMBER 2

Page 14, Line 4: ~~Set $\overline{X} = K/Z(E(X))$.~~ Set $Y = KZ(E(X))$ and $\overline{Y} = Y/Z(E(X))$, so that $\overline{Y} = \overline{K}$.

Page 15, Line 20: $\cancel{\langle E(C_X(E)) \mid |D : E| \le p \rangle}$ $\langle E(C_X(F)) \mid |D : F| \le p \rangle$

Page 15, Line 22: $\cancel{\left\langle E(C_X(E)) \,\middle|\, |B : E| \le \max\{p^n, p\} \text{ and } E \in \mathcal{W} \right\rangle}$ $\left\langle E(C_X(F)) \,\middle|\, |B : F| \le \max\{p^n, p\} \text{ and } F \in \mathcal{W} \right\rangle$

Page 18, Line 20: $\cancel{\pi = \{p\} \text{ or } \pi = 2'}$ $\pi' = \{p\}$ or $\pi' = 2'$

Page 24, Line -10: ~~first~~ second

Page 115, Line 6: ~~$J \cong SL_n(r^m)$, r odd~~ $J \cong SL_n(r^m)$, n and r odd

Page 117, Line -3: ~~$A \leq C_{P^g}(u)$~~ $R_1 \leq C_{P^g}(u)$

Page 117, Line -2: ~~$1 \neq R_1 \leq A \leq P \cap P^g \cap Y$~~ $1 \neq R_1 \leq P \cap P^g \cap Y$

Page 122, Line 19: *In Definition 21.1* ~~p'-subgroups~~ A-invariant p'-subgroups

Page 172, Line -17: *In Lemma 29.5, a hypothesis needs to be added. The following is adequate, following the first sentence:* Assume that there is a mapping $\phi : E \to D$ such that $\phi(i) \geq i$ for all $i \in E$, and whenever $i, j \in E$ with $i \leq j$, then $\phi(i) \leq \phi(j)$.

ERRATA FOR NUMBER 3

Page 18, Line 3: ~~$h_\alpha(t) = n_\alpha(t)n_\alpha(1)^{-1}$~~ $h_\alpha(t) = n_\alpha(1)^{-1}n_\alpha(t)$

Page 18, Line -3: ~~$h_{r_\beta(\alpha)}(c_{\alpha,\beta}t)$~~ $h_{r_\beta(\alpha)}(t)$

Page 36, Line -11: ~~$q(\sigma, \overline{K})$~~ $q(\overline{K}, \sigma)$

Page 37, Line 6: ~~${}^2G_2(2^{a+\frac{1}{2}})$~~ ${}^2G_2(3^{a+\frac{1}{2}})$

Page 43, Line 1: ~~$-a_{2m+1-i}$ for $1 \leq i \leq m$~~ $-a_{2m+2-i}$ for $1 \leq i \leq m+1$

Page 55, Line -11: *In the statement of Lemma 2.5.7:*
~~$C_{\mathrm{Aut}_1(\overline{K})}(K) = \langle\sigma\rangle$~~ $C_{\mathrm{Aut}_1(\overline{K})}(K, \sigma) = \langle\sigma\rangle$

Page 57, Line 18: *At the end of Definition 2.5.10, add:*
(a) (f) $\mathrm{Aut}_0(K) =$ image of $C_{\mathrm{Aut}_0(\overline{K})}(\sigma)$ in $\mathrm{Aut}(K)$.

Page 58, Line -10: ~~If $K \cong A_m(q)$, $D_{2m+1}(q)$~~ If $K \cong A_m(q)$ $(m > 1)$, $D_{2m+1}(q)$

Page 65, Line -2: ~~$J \subseteq \widehat{\Pi}$~~ $J \subseteq \widehat{\Pi}$, $J \neq \widehat{\Pi}$

Page 69, Line -1: ~~every e_{ab} by $-e_{-a,-b}$~~ every $t^m e_{ab}$ by $(-t)^m e_{-a,-b}$

Page 70, Line 14: ~~$x_{a_i+a_j}(t) = 1 + t(e_{i,-j} + e_{j,-i})$~~ $x_{a_i+a_j}(t) = 1 + t(e_{i,-j} + (-1)^{i+j}e_{j,-i})$

Page 70, Line 15: ~~$x_{-\alpha}(t) = x_\alpha(t)^A$~~ $x_{-\alpha}(t) = x_\alpha(t)^T$

Page 70, Line 19: ~~We may identify~~ Except for the case $D_2^+(2)$, we may identify

Page 70, Line 26: ~~e_{ab} by $-e_{-b,-a}$~~ e_{ab} by $-e_{-a,-b}$

Page 70, Line -11: *At the end of this paragraph, add:*
In the exceptional case $D_2^+(2)$, the group $O_4^+(2)$ is an extension of E_{3^2} by D_8, and the index of its commutator subgroup is 4. We define $\Omega_4^+(2)$ to be the kernel of the Dickson invariant in this case. Then $\Omega_4^+(2)$ is the direct product of two root $A_1(2)$-subgroups.

Page 173, Lines 20–21: For $G = C_2(q)$, the entries in rows t_m and t'_m and in column $C_{C^*}(L^*)$ of Table 4.5.1 should be $\{q-1\}2$ and $\{q+1\}2$, respectively.

Page 173, Line 22: For $G = C_m(q)$, m even, the entry in row $t'_{m/2}$ and column $\mathrm{Out}_{C^{*o}}(L^*)$ of Table 4.5.1 should be 1.

Page 176, Line 4: ~~the extension of $\mathrm{Outdiag}(K)$ by Γ_K~~ $\mathrm{Outdiag}(K)\Gamma$

Page 176, Line 5: *Insert before "The image":* Here $\Gamma = \Gamma_K$ if K is untwisted, while if $K = {}^2\mathcal{L}(q)$, then K is the exponent 2 subgroup of Φ_K.//

Page 176, Line 10,23,25,27: ~~Γ_K~~ Γ

Page 181, Line -5: For $G = E_7(q)$, the entry in row t_4' and column $C_C(L)$ of Table 4.5.2 should be $(4, q+1)$

Page 211, Line 11: In Table 4.7.3B, the entry in row t_4 (with $q^2 \equiv -1$) and column $\mathrm{Out}_C(L)$ should be 1

Page 237, Line -3: ~~$P = P_T(P \cap K)$~~ $P_0 = P_T(P_0 \cap K)$

Page 237, Line -2: ~~P~~ P_0

Page 237, Line -1: ~~$b = \sum_{pm_0|i} n_i$~~ $b = \sum_{i=p^c m_0,\, c>0} n_i$

Page 261, Line -11: ~~Fi_{24}'~~ Fi_{23}

Page 275, Line 5 **of "SMALL REPRESENTATIONS":** ~~$\mathbf{F}_{5^2}$~~ $\mathbf{F}_{3^2}$

Page 279, Line -7: ~~$L_2(25)$~~ $L_2(25)\#2$

Page 288, Line -11: ~~$|M^\#|$~~ $|O_2(M)^\#|$

Page 290, Line 1: ~~$E(C(2A))$~~ $E(C(2B))$

Page 297, Line -18: ~~$K = Co_1$~~ $K = Co_0$

Page 299, Line 16: ~~lower bound for $P \times Q_8$ is 30~~ lower bound for a faithful complex representation of $P \times Q_8$ in which the involution of $Z(Q_8)$ acts as $-I$ is 30

Page 302, Line 12: ~~$E(C_K(z))$~~ $E(C_K(z_A))$

Page 302, Line 16: ~~becuase~~ because

Page 302, Line 19: ~~$|H|_3$~~ $|\overline{H}|_3$

Page 304, Line -13: ~~I is a homogeneous I-module~~ V_0 is a homogeneous I-module

Page 304, Line -8: ~~$z \in Q_0'$~~ $Z(J) \leq Q_0'$

Page 308, Line -17: ~~$B/Z(B) \cong 2\,{}^2E_6(2)$~~ $B/Z(B) \cong {}^2E_6(2)$

Page 309, Line -6: ~~$K \in \mathcal{K}$~~ $K \in \mathcal{K}$ and K is simple

Page 309, Line -1: ~~or $K \cong J_1$~~ or $K \cong {}^2G_2(3^{\frac{n}{2}})$, n odd, $n > 1$, or $K \cong J_1$

Page 314, Line 13: ~~$\dim(W_1)$~~ $\dim(W_i)$

Page 316, Line -2: ~~$V_q\alpha + V_q\beta$~~ $\mathbf{F}_q\alpha + \mathbf{F}_q\beta$

Page 316, Line -2: ~~$V_q\alpha$~~ $\mathbf{F}_q\alpha$

Page 317, Line 11: ~~$C_{(2\times 2)D_4(2)}(x)$~~ $C_{(2\times 2)D_4(2)}(x_1)$

Page 317, Line 16: ~~$|Sp_6(2)|$~~ $|Sp_6(2)|_2$

Page 319, Line 12: ~~Q is abelian~~ $\hat{Q}$ is abelian

Page 319, Line 13: ~~Q~~ $\hat{Q}$ *(twice)*

Page 319, Line 14: ~~Q is abelian~~ $\hat{Q}$ is abelian

Page 319, Line 15: $\cancel{Q}$ $\hat{Q}$ *(four times)*

Page 329, Line -12: $\cancel{(r^{2a}+\epsilon r^a+1)/3}$ $(r^{2a}+\epsilon r^a+1)/d$

Page 332, Line -3: *Theorem 6.5.5a misstates the structure of Borel subgroups of* ${}^2G_2(3^{\frac{n}{2}})$, $n>1$. *The assertion should be:*
(a) Borel subgroups of K, of order $q^3(q-1)$. If $B=UH$ is such a Borel subgroup, with $|U|=q^3$ and $|H|=q-1$, and if t is the involution of H, then $|C_U(t)|=q$, and the groups $O^2(B)$, $Z(U)H$ and $B/Z_2(U)$ are all Frobenius groups.

Page 333, Line -9: $\cancel{Z_2\times L_2(q^2)}$ $Z_2\times L_2(q)$

Page 338, Line 1: *Replace this line by:* We proceed in a sequence of lemmas.

Page 338, Line 10: *Replace this line by:* We set $Y=K_1X$, so that $X\le O_{r'}(Y)$, and next prove:

Page 345, Line 11: $\cancel{\Gamma_{E_2,*-1}(K)}$ $\Gamma'_{E_2,*-1}(K)$

Page 345, Line 12: $\cancel{\Gamma_{E_2,*-1}(U)\le\Gamma_{E_2,*-1}(K)}$ $\Gamma'_{E_2,*-1}(U)\le\Gamma'_{E_2,*-1}(K)$

Page 345, Line 13: $\cancel{\Gamma_{E_2,*-1}(K)}$ $\Gamma'_{E_2,*-1}(K)$

Page 354, Line 14: *In the proof of Theorem 7.3.3, we omitted here a reduction to the case that* $m_p(E)=2$. *This reduction is needed to justify the assertion in line 15 that* $\Gamma=\Gamma_{E,*-1}(K)$. *Thus, the following paragraph should be inserted before "We set":* We first reduce the proof to the case $m_p(E)=2$. Indeed, if the theorem holds in that case, then to complete the proof we must argue that if a noncyclic elementary abelian p-group E acts faithfully on K in such a way that one of the conclusions of 7.3.3 is satisfied by each $F\in\mathcal{E}_2(E)$, then E itself satisfies that same conclusion. This is accomplished by a few observations in the various cases. In case 7.3.3c, $\mathrm{Out}(K)$ has order 3 by 2.5.12, so $m_2(\mathrm{Aut}(K))=m_2(K)=3$ and the desired conclusion is obvious. In cases 7.3.3ehijkl, as well as the case $K={}^2A_2(2)$ of 7.3.3a, it is immediate from 4.10.3 and 2.5.12 that $m_p(\mathrm{Aut}(K))=2$, with $\mathrm{Out}(K)$ a p'-group in case (e) and $m_p(K)=1$ in cases (h) and (i). Thus the desired conclusions hold in these cases as well. In the remaining cases, it suffices to assume that $m_p(E)=3$ and derive a contradiction. In cases 7.3.3df, $\mathrm{Out}(K)$ is a p'-group by 2.5.12, and 4.10.3ae implies that $m_p(K)=3$ and that every element of K of order p lies in a conjugate of E. But in these cases of 7.3.3 it is stipulated that certain conjugacy classes of K of order p do not meet E, contradiction. In cases 7.3.3bg, we consider the character of E on the natural K-module, which (since $p\ne r$) lifts to a complex character χ. The conditions of cases (b) and (g) force $\chi(x)=-1$ for each $x\in E^\#$. As $(\chi,1_E)$ is an integer, $\chi(1)\equiv -1 \mod p^3$. However, $\chi(1)=5$ or 8, with $p=2$ or 3, respectively, a contradiction. Finally, the only remaining case is that 7.3.3a holds and E acts on $K\cong L_p^\epsilon(q)$ like a subgroup $E^*\le GL_p^\epsilon(q)$, and the preimage F^* in E^* of any $F\in\mathcal{E}_2(E)$ satisfies $(F^*)'=\Omega_1(Z(K))$.

But then $(E^*)' = \Omega_1(Z(K))$, and so $C_{E^*}(x)$ is a maximal subgroup of E^* for all $x \in E^* - Z(E^*)$. Choosing such an element x and using $m_p(E) = 3$, we find $y \in E^*$ such that $\langle x, y\rangle$ is abelian and has a noncyclic image in E, a final contradiction accomplishing our reduction.

Page 354, Line 22: ~~p'-subgoup~~ p'-subgroup

Page 357, Line -8: ~~its Lie components.~~ its Lie components. (See also Definitions 4.2.2 and 4.9.3, and Proposition 4.9.4.)

Page 358, Line -2: $\cancel{{}^2F_4(2^{\frac{1}{2}})}$ ${}^2F_4(2^{\frac{1}{2}})'$

Page 364, Line -15: ~~Then~~ Then if we define $\Gamma^{r'}_{\hat{E},*-1}(\hat{K})$ to be the subgroup of $\hat{K}$ generated by all r-elements centralizing some subgroup of $\hat{E}$ of index 2, we have

Page 365, Line -16: $\cancel{t_2^{(3)} \text{ and } t_2^{(4)}}$ t_2'' and t_2'''

Page 381, Line -4: $\cancel{\Psi_{ij}}$ Ω_{ij}

Page 381, Line -3: $\cancel{\Psi_{ij} \cup \Omega_0}$ $\Omega_{ij} \cup \Omega_0$

Page 382, Line 3: $\cancel{\Gamma_{E,*-r}(K)}$ $\Gamma'_{E,*-r}(K)$

Page 382, Line 5: $\cancel{A_{\Psi_{ij}}}$ $A_{\Phi_{ij}}$

Page 382, Line 8: $\cancel{\text{then } \Psi_{ij} = \Phi_{ij};}$ then

Page 382, Line 9: $\cancel{4 = |\Phi_{ij}| = |\Psi_{ij}| = |\Omega_{ij}| + |\Omega_0|}$ $4 = |\Phi_{ij}| = |\Omega_{ij}| + |\Omega_0|$

Page 382, Line 15: $\cancel{O^2(A_{\Phi_{ij}})}$ $O^2(A_{\Phi_{ij}})$

Page 383, Line -7: Add the conclusion (a') $(p, K) = (3, L_2(8))$ or $(5, {}^2B_2(2^{\frac{5}{2}}))$;

Page 384, Line 1: ~~proved in [**GL1**,24–1]~~ proved in [**GL1**,24–1, 24–4] (the latter reference to be applied to $X \times Z_p$ if $m_p(K) = 1$)

Page 384, Line 21: ~~eight~~ nine

Page 384, Line 24: Add the conclusion (a') $\Gamma_{Q,1}(K) = \Gamma_{P,1}(K)$ is a Frobenius group of order $p^2(p-1)$ with $Q \cong Z_{p^2}$;

Page 384, Line -13: Add the sentence: If 7.6.1a' holds, then $Q \cong Z_{p^2}$ and for every $g \in P - Q$ of order p, $C_K(g) \cong L_2(2)$ or ${}^2B_2(2^{\frac{2}{2}})$ is p-closed, so $\Gamma_{P,1}(K) \le N_K(\Omega_1(Q))$, which is a Frobenius group as claimed (see 6.5.1, 6.5.4).

Page 385, Line 1: $\cancel{SL_2(5) = 2A_1(4),}$ $SL_2(5) = 2A_1(4)$, $(2)^2B_2(2^{\frac{3}{2}})$,

Page 385, Line 19: Add the condition: p divides $|K|$

Page 387, Line 15: ~~But~~ Since p divides $|K|$, it also divides $|C_K(x)|$, which embeds in $\mathrm{Inndiag}(L_1)$ by 4.9.1b. Hence p divides $|L_1|$. But

Page 396, Line -11: ~~K locally k-balanced~~ K is locally k-balanced

Page 399, Line -15: ~~irreducibly on P~~ irreducibly on $\Omega_1(P)$

Page 402, Line -4: ~~Theorem 7.8.1~~ Proposition 7.8.1

ERRATA FOR NUMBER 4

ERRATA FOR NUMBER 5

Page 3, Line -14: *In the definition of $\mathcal{K}^{(7)*}$, second line*

~~$\{A_4^\epsilon(q) \mid \epsilon = 1$ or q odd$\}$~~ $\{A_4^\epsilon(q) \mid \epsilon = 1$ or $q \notin \{2,4\}\}$.

Page 11, Line 7: ~~if and only if~~ provided that

The converse is true under the extra assumption $x \in Z(Q)$.

Page 11, Line 15: ~~The converse is trivial.~~

Page 22, Line 17: ~~$L^o_{2'}(C_L(z))$~~ $L^o_{p'}(C_L(z))$

ERRATA FOR NUMBER 6

Page 464, Line 16: ~~$\Phi(P)$~~ $\Phi(Z)$

ERRATA FOR NUMBER 7

Page 20, Lines 17 to 19: *Replace Lemma 1.5 with the following weaker lemma, proved in Lemma 2.8 of Chapter 2 of Volume 6.*

Lemma. If T is a 2-group and $m_2(T) \geq 5$, then T is connected and possesses a normal subgroup isomorphic to E_{2^3}.

Note that Lemma 1.5, as printed in Volume 7, is the full strength of MacWilliams' theorem, which we do not prove, and which is not one of our assumed Background Results. As we therefore cannot quote the full strength, we avoid its use by providing the corrigenda below for pages 91ff. and 338.

Page 91, Line 11: *All of Section 8 after the proof of Lemma 8.7 should be deleted and replaced by the following. In the replacement, references are made to certain portions of the old material, using the original numbering.*

Lemma 8.14 (Alperin). *Let T be a 2-group. Let A be a normal abelian subgroup of T maximal such that $A = \Omega_2(A)$. Then $A = \Omega_2(C_T(A))$.*

To prove Alperin's lemma, we introduce the following terminology. If T is a p-group, say that T **is of class** 2^- if and only if $\Phi(T) \leq Z(T)$. Obviously, class 2^- implies class 2.

Lemma 8.15. *If T is a 2-group of class 2^-, then $\Omega_2(T)$ has exponent dividing 4.*

Proof. Let $x, y \in T$ with $x^4 = y^4 = 1$. Then $x^2, y^2, [x,y] \in Z(T)$, so

$$(xy)^4 = (x^2y^2[x,y])^2 = x^4y^4[x,y]^2 = [x^2,y] = 1.$$

$\square$

Lemma 8.16. *Let T be a 2-group. Let Q be a normal subgroup of T containing an abelian subgroup $E = \Omega_2(E) \triangleleft T$. If there is an abelian subgroup $Y = \Omega_2(Y) \leq Q$ such that $|Y : E| = 2$, then there is such a subgroup which is normal in T.*

Proof. The proof is by induction on $|T| + |Q|$. Let $Q_1 = \langle Y^T \rangle \lhd T$. Then $Y \leq Q_1 \leq Q$ and if $Q_1 < Q$, we are done by induction. So assume that $Q = \langle Y^T \rangle$. We may assume that $Y \neq T$, so $Q < T$. By induction there is $U = \Omega_2(U) \lhd Q$ with U abelian and $|U : E| = 2$. Hence $U \leq V$, where $V/E = \Omega_1(Z(C_Q(E)/E))$. In particular $\Omega_2(V) \geq U > E$. Notice that $V \lhd T$ and V is of class 2^-. Hence $U = \Omega_2(U) \leq \Omega_2(V) \lhd T$. Choose any $X \lhd T$ such that $E < X \leq \Omega_2(V)$. By the preceding lemma, $\Omega_2(V)$ and X have exponent dividing 4. The proof is complete. $\square$

Lemma 8.14 follows immediately from Lemma 8.16, with $Q = T$.

Lemma 8.17. *Let T be a 2-group. Then T is connected under either of the following conditions:*

(a) *$m_2(T) \geq 5$; or*
(b) *$T \geq E = Q_1Q_2Q_3$ with $[Q_i, Q_j] = 1$ for all $i \neq j$ and $Q_i \cong Q_8$ for all $i = 1, 2, 3$.*

Proof. By [**III**$_2$; 1.5], (a) is sufficient, so assume that $m_2(T) \leq 4$ and E exists as in (b). Since T is not of maximal class, there exists $U \lhd T$ with $U \cong E_{2^2}$ and $U \neq C_T(U)$. Let $U \leq A \leq T$ with A maximal with respect to the properties that A is a normal abelian subgroup of T and $A = \Omega_2(A)$. By Alperin's Lemma 8.14, $A = \Omega_2(C_T(A))$. In particular, $C_E(A) = E \cap A$. If $m_2(A) > 2$, then T is connected. Hence we may assume that $m_2(A) = 2$, whence $U = \Omega_1(A)$.

If $|A| = 8$, then a Sylow 2-subgroup of $Aut(A)$ has order at most 8. Then $|E : E \cap A| \leq 8$ and $|E \cap A| \leq 8$, whence $|E| \leq 2^6$, contrary to assumption. So, we may assume that $|A| = 16$, whence $A \cong Z_4 \times Z_4$. Let $S \in Syl_2(Aut(A))$ and $S_0 = C_S(U) = C_S(A/U) \lhd S$. Then $S_0 \cong Hom(A/U, U) \cong E_{2^4}$ and $S/S_0 \cong Z_2$. In particular, S contains no copy of Q_8. Hence, $\Phi(E) \leq C_E(A) = E \cap A$ and $Aut_E(A)$ is therefore elementary abelian.

Suppose first that E is extraspecial with $|E| = 2^7$. Then $Z(E) \leq A$. If $E \cap A$ contains a cyclic subgroup B of order 4, then since $B \lhd E$, $|Aut_E(A)| \leq 8$. and so $A \leq E$, which is absurd. Therefore $E \cap A \leq U$ and so $E \cap A = U$ and $Aut_E(A) \cong 2^5$. But S is not elementary abelian, a contradiction.

Therefore $|\Phi(E)| > 2$, whence $U \leq Z(E)$ and so $Aut_E(A) \leq S_0$. Hence, $|E : A \cap E| \leq 2^4$, and so $|E \cap A| \geq 2^4$. Thus $A \leq E$ and $Aut_E(A) = S_0$. We may assume without loss that $\Phi(Q_2) \neq \Phi(Q_1) \neq \Phi(Q_3)$. Suppose that $E = Q_1 \times Q_2Q_3$. Let $a \in A - U$. Since $[a, E] = [a, Aut_E(A)] = U$, it follows that a projects onto an element of Q_1 of order 4. Then $A/Z(Q_2)$ projects isomorphically onto Q_1, contrary to the fact that A is abelian. Therefore, $E = Q_1Q_2Q_3$ with $Z(E)$ a four-group and every element of $\mathcal{E}_1(Z(E))$ has the form $\langle z_i \rangle = Z(Q_i)$ for a unique $i = 1, 2, 3$. Any element of $E - U$ has the form $x = x_1x_2x_3$ with $x_i \in Q_i$. The set of indices for which $x_i \notin \langle z_i \rangle$ is uniquely determined and called the support of x. Elements with support of cardinality 3 are involutions. Therefore A must be generated by two elements x, y with support of the same cardinality (1 or 2). But if this cardinality is 1, A is obviously not self-centralizing. So the cardinality is 2, say overlapping in $\{1\}$. Then, since $[x, y] = 1$, we may assume that $x_1 = y_1 \in C_E(A) - A$, a final contradiction. $\square$

LEMMA 8.18. *The following conditions hold:*

(a) *R_0 is the commuting product of $r - s$ quaternion groups;*
(b) *If $r - s \geq 2$, then R is neither cyclic nor of maximal class; and*
(c) *If $r - s \geq 3$ or $m_2(R) \geq 5$, then R and all of its overgroups are connected.*

PROOF. As $R_0 \in \mathrm{Syl}_2(M_0)$, $R_i := R_0 \cap M_i \in \mathrm{Syl}_2(M_i)$ is a quaternion group for all $i = 1, \ldots r$. Then (a) and (b) are obvious, and (c) follows directly from Lemma 8.17. $\square$

LEMMA 8.19 (cf. Lemma 8.9). *Suppose that there is a four-group $D \leq R$ such that the pumpup of I in $C_G(d)$ is trivial for all $d \in D^\#$. Then some 2-overgroup of R is not connected and the following conditions hold:*

(a) *$r - s \leq 2$ and $s \geq 2$;*
(b) *$\widetilde{I} \not\cong L_3^\pm(h)$ or $SL_2(h)$ for any $h > 3$; and*
(c) *Suppose $b \notin IC(t, I)$. Then $s = 2$, $I/O_{2'}(I) \cong HSpin_8^+(q)$ or $Spin_n^\pm(q)$ for some $n \in \{6, 7, 8\}$, and $M_1M_2/O_{2'}(M_1M_2) \cong SL_2(q_1) * SL_2(q_2)$. Moreover $Z \leq R$, $Z^*(M_1M_2) \leq Z^*(I)$ and either*

$$m_2(Z) = 3 \text{ with } \widetilde{M_3}\widetilde{M_4} = \widetilde{M_3} \times \widetilde{M_4},$$

or

$$\widetilde{M} = (\widetilde{M_1} * \widetilde{M_2}) \times (\widetilde{M_3} * \widetilde{M_4}).$$

PROOF. The first three paragraphs of Lemma 8.9 show that some 2-overgroup of R is not connected, whence by Lemma 8.18, $m_2(R) \leq 4$ and $r - s \leq 2$. As $r \geq 4$, (a) holds. If $\widetilde{I} \cong L_3^\pm(h)$ or $SL_2(h)$, then $s = 1$, contrary to (a), proving (b). Finally, suppose that $b \notin IC(t, I)$. As $s \geq 2$, the first sentence of (c) holds by [**III**$_{11}$;13.9]. Then $Z \leq R$. If $m_2(Z) = 3$ and $\widetilde{M_3}\widetilde{M_4}$ has a center of order 2, then $m_2(M) \geq m_2(M_3M_4) + 2 = 5$, contrary to the fact that some 2-overgroup of R is not connected. Hence, the second statement of (c) holds by (8F2). $\square$

LEMMA 8.20 (identical to Lemma 8.10). *The following conditions hold:*

(a) *$I/O_{2'}(I) \notin \mathcal{S}$;*
(b) *b normalizes I; and*
(c) *For any involution $u \in C_R(b)$, (u, I_u) is a trivial or vertical pumpup of (t, I).*

PROOF. The proof is identical to that of Lemma 8.10. $\square$

Now we sharpen our choice of the configuration $(b, t, J_1, \ldots, J_r)$ satisfying (8F). We assume, as we may, that we have chosen our configuration so that in addition,

(8M)
(1) $|I/O_{2'}(I)|$ is as large as possible;
(2) Subject to (1), $|C(t,I)|_2$ is as large as possible; and
(3) Subject to (1) and (2), $t \in Z \cap I$, if possible.

Note that $(8M3)$ is realized if and only if $|Z^*(I)|$ is even.

Lemma 8.21 (cf. Lemma 8.11a). *The following conditions hold:*

(a) *Let $z \in \mathcal{I}_2(Z \cap R)$. Then I_z is a trivial pumpup of I;*
(b) *$s \geq 2$; and*
(c) *$I/O_{2'}(I) \in \mathcal{G}_2^6$.*

Proof. The proof of (a) is identical to that of Lemma 8.11a. Now suppose that $s = 1$. Then $Z \leq J_1R$. Hence, if either $m_2(Z) \geq 3$ or $Z \cap J_1 \leq R$, then $m_2(Z \cap R) \geq 2$. Then by (a) and Lemma 8.19a, $r - s \leq 2$ and so $r \leq 3$, a contradiction. Thus we may assume that $m_2(Z) = 2$ and $Z \cap J_1 \not\leq R$. Then $\overline{J} = \overline{J_1} \times \overline{J_2 \cdots J_r}$ with $Z \cap J_2 \cdots J_r = 1$. But then $\overline{J}$ cannot contain H_0 as in (8F2), a final contradiction, proving (b). Since $I/O_{2'}(I) \in \mathcal{G}_2^6 \cup \mathcal{G}_2^7$ by Lemma 8.20a, (c) follows immediately from (b). □

Lemma 8.22 (cf. Lemma 8.11b). *Either $b \in IC(t,I)$, or $t \in Z \cap I$ and $Z \leq R$.*

Proof. Suppose that $b \notin IC(t,I)$. Then by Lemma 8.19c, $Z^*(M_1M_2) \leq Z^*(I)$ and $Z \leq R$. By our choice in (8M), $t \in Z \cap I$. □

We then modify the proof of Lemma 8.12.

Lemma 8.23 (cf. Lemma 8.12). *If all pumpups of (t,I) are trivial, then (t,I) is 2-terminal in G.*

Proof. By definition of 2-terminality [$\mathbf{I}_G$, 6.26], it is enough to show that for any $z \in \Omega_1(Z(R))^{\#}$, we have $R \in Syl_2(C(z,I_z))$. By Lemma 8.22, either $t \in Z \cap I$ or $b \in IC(t,I)$. Let $R \leq R^* \in Syl_2(C(z,I_z))$. Then R^* centralizes $R \cap Z^*(I_z)$. In the first case, $t \in Z^*(I_z)$, and so $R^* = R$, as desired. So we may assume that $b \in IC(t,I)$. Then our choice of R guarantees that $b \in RC_S(R)$. But $RC_S(R)$ centralizes $z \in \Omega_1(Z(R))$ and contains a Sylow 2-subgroup of M. Hence, z normalizes $M_1, \ldots, M_r$ and then centralizes $M/O_{2'}(M)$ by [$\mathbf{III}_{11}$, 6.3e]. Thus we have $z \in \mathcal{I}_2(C(t,I)) \cap C_G(M/O_{2'}(M)) \cap C_G(b)$. Then by Lemma 8.6a, $(b, z, I_z, J_1, \ldots, J_r)$ satisfies (8F), and the desired conclusion follows by the maximal choice in (8M). □

Now choose, as we may by [$\mathbf{I}_G$; 6.10], a 2-terminal long pumpup (t^*, I^*) of (t,I). By Lemmas 8.4 and 8.3, $m_2(C(t^*, I^*)) = 1$. By Lemma 8.21c, $I/O_{2'}(I) \in \mathcal{G}_2^6$, so by [$\mathbf{III}_{11}$;1.2], $I^*/O_{2'}(I^*) \in \mathcal{G}_2^6$. In particular, $(t^*, I^*) \in \mathcal{J}_2(G)$. But $(x,K) \in \mathcal{J}_2^*(G)$. By the definition of this term and by the pumpup-monotonicity of $\mathcal{F}$ [$\mathbf{III}_7$, 3.2], [$\mathbf{III}_{11}$, 12.3e],

$$\mathcal{F}(K) \geq \mathcal{F}(I^*/O_{2'}(I^*)) \geq \mathcal{F}(I/O_{2'}(I)).$$

Thus, by Lemma 8.2ac,

(8N) $\mathcal{F}(I/O_{2'}(I)) \le (q^9, A)$ or (q^4, BC), according as $K/Z(K) \cong L_4^{\pm}(q)$ or $PSp_4(q)$.

The possible isomorphism types of $I/O_{2'}(I)$ are further restricted by an additional condition. Namely, b is restricted by Lemma 8.22, and $\widetilde{M_1} \lhd\lhd C_{\widetilde{I}}(b)$.

We can now apply [$\mathbf{III}_{11}$, 12.7, 13.9] to obtain (cf. (8J)):

(8O)
(1) $s = 2$; and
(2) One of the following holds:
  (a) $Z \le R$, and $I/O_{2'}(I) \cong \mathrm{Spin}_6^{\pm}(q)$ or $\Omega_8^{+}(q^{\frac{1}{2}})$;
  (b) $I/Z^*(I) \cong PSp_4(q_1)$ or $G_2(q_1)$, $q_1 \ge q$, or $L_4^{\pm}(q)$.

Note that $I/O_{2'}(I) \not\cong Spin_7(q)$, for otherwise, $\mathcal{F}(K) < \mathcal{F}(I/O_{2'}(I))$.

Lemma 8.24 (cf. Lemma 8.13). *(t, I) has a nontrivial pumpup in G.*

Proof. As $s = 2$, we have $r - s \ge 2$, and so $m_2(C(t, I)) > 1$. Hence, (t, I) is not 2-terminal in G by Lemma 8.3, and so Lemma 8.23 yields this lemma. $\square$

Now we let $\mathcal{I}_2^v(R)$ be the set of involutions $u \in R$ for which I_u is a vertical pumpup of I. We argue that

(8P)
$$\mathcal{I}_2^v(R) \ne \emptyset.$$

Suppose the contrary. Using Lemmas 8.20 and 8.19 instead of 8.10 and 8.9, we conclude, as in the four lines following (8L), that R is not connected. Hence, $r - s = 2$. Let $u \in \mathcal{I}_2(R)$ with I_u a nontrivial pumpup of I, as guaranteed by Lemma 8.24, and chosen so that $R_1 := C_R(u)$ has maximal order. As $u \notin \mathcal{I}_2^v(R)$, I_u is a diagonal pumpup of I. Thus R_1 has a subgroup R_u of index at most 2 centralizing $I_u/O_{2'}(I_u)$.

Suppose there is a four-group $E \le C_R(\langle u, b\rangle)$. Since we are assuming that (8P) fails, (e, I_e) is a trivial pumpup of (t, I) for all $e \in E^{\#}$ by Lemma 8.20c. Then (u, I_u) is a trivial pumpup of (t, I) by [$\mathbf{III}_{11}$, 17.2], as in the second paragraph in the proof of Lemma 8.9, a contradiction. Hence, $m_2(C_R(\langle u, b\rangle)) = 1$. As $C_R(b) \ge R_0$, we certainly have $u \notin Z(R)$. Thus $Z(R) \le R_1$ with $Z(R) \cap R_u = 1$, whence $R_1 = Z(R) \times R_u$ with $|Z(R)| = 2$. As $R_1 < R$, we may choose $a \in N_R(R_1) - R_1$ with $a^2 \in R_1$. Our maximal choice of u implies that a centralizes no involution $w \in Z(R_u)$, since otherwise $C_R(w) \ge R_1\langle a\rangle$, contrary to the choice of u. Since a leaves $\Phi(R_1) = \Phi(R_u)$ invariant, we conclude that $\Phi(R_1) = 1$ and so R_1 is elementary abelian. Also, a leaves $R_u \cap R_u^a$ invariant, whence $|R_u| = 2$ and $|C_R(u)| = 4$. But then R is dihedral or semidihedral by [$\mathbf{I}_G$; 10.24], contrary to $R_0 \le R$ with $r - s = 2$, proving (8P).

Finally, we choose $u \in \mathcal{I}_2^v(R)$ with $|C_R(u)|$ maximal. Then $I/O_{2'}(I) \in \mathcal{G}_2^6$ by Lemma 8.24b. Moreover, (u, I_u) has a 2-terminal long pumpup (v, I_1) by [$\mathbf{I}_G$; 6.10] and then

$$\mathcal{F}(I/O_{2'}(I)) \le \mathcal{F}(I_u/O_{2'}(I_u)) \le \mathcal{F}(I_1/O_{2'}(I_1)) \le \mathcal{F}(K),$$

since $(x,K) \in \mathcal{J}_2^*(G)$. As $s = 2$ and $\mathcal{F}(I/O_{2'}(I)) \le (q^9, A)$, we have that one of the following holds:

$$\text{(8Q)} \quad \begin{array}{l} (1)\ I/Z^*(I) \cong PSp_4(q_1) \text{ or } G_2(q_1) \text{ for some } q_1 \ge q,\ L_4(q) \text{ or } U_4(q); \\ \quad\ \text{or} \\ (2)\ I/O_{2'}(I) \cong \Omega_8^+(q^{\frac{1}{2}}), \end{array}$$

with $t \in Z(I)$ in the second case.

Now (u, I_u) is a vertical pumpup of (t, I) and $\mathcal{F}(I_u/O_{2'}(I_u)) \le \mathcal{F}(K)$. As $t \in Z(I)$, we cannot have $(t,I) < (u, I_u)$ with $I/O_{2'}(I) \cong \Omega_8^+(q^{\frac{1}{2}})$ and $I_u/O_{2'}(I_u) \cong Spin_9(q^{\frac{1}{2}})$. Hence $(8Q1)$ holds. Then by [**III**$_{11}$, 12.6b], the only possibilities for $(I/O_{2'}(I), I_u/O_{2'}(I_u))$ satisfy

$$C_{Aut(I_u/O_{2'}(I_u))}(I \cap I_u/I \cap O_{2'}(I_u)) \cong Z_2.$$

Hence if we set $R_u = C_R(u)$, then $R_u = \langle t \rangle \times R_t$, where $R_1 = C(u, I_u) \cap R$. Furthermore, by [**III**$_{11}$, 12.6b], the pumpup of I_u in $C_G(u')$ is trivial for all $u' \in \mathcal{I}_2(R_1)$, which implies that $I_{u'}/O_{2'}(I_{u'}) \cong I_u/O_{2'}(I_u)$. By Lemma 8.18b, R is neither cyclic nor of maximal class, so it follows by [**III**$_2$; 1.16] and the maximal choice of u that $u \in Z(R)$. Thus $R_u = R$ and $|R/R_1| = 2$. Hence any involution in a quaternion subgroup of R lies in R_1. Hence, in particular, if $z \in \mathcal{I}_2(R \cap M_3)$, then (z, I_z) is a nontrivial pumpup of (t, I). However this contradicts Lemma 8.21, completing the proof of Proposition 8.1.

Taken together, Propositions 2.3, 2.6, 3.1, 4.1, 6.1, and 8.1 imply Theorem 2.

Page 142, Lines 21 to 23: *Delete the last two sentences of the proof of Lemma 14.27. Replace with the following:*

Let $X = N\mathrm{Aut}_K(E)$. Write $\mathrm{Aut}_L(E) = \langle t, t' \rangle$, where t and t' are transpositions in $\mathrm{Aut}_K(E) \cong \Sigma_5$, and reflections on E. Note that $C_E(\langle t, t'\rangle) = D$. By [**III**$_{11}$, 22.6] (see below), with the role of R there played by N, there is $s \in \mathcal{I}_2(X)$, also a reflection on E, such that $X_1 := \langle t, t', s\rangle$ is a faithful extension of $Z_4 \times Z_4$ by Σ_3, and t, t', and s are X_1-conjugate. As s is a reflection on E, $D_1 := C_E(X_1) = C_D(s) \ne 1$; let $d \in D_1^{\#}$. By all the cases of Proposition 12.1 ruled out so far, the pumpup L_d of L in $C_G(d)$ is either a level pumpup of $L \cong SL_3(q^2)$, or $L_d \cong A_5^{\eta}(q)$, $q = 4^n$, $\eta = (-1)^{n+1}$. Hence by [**III**$_{11}$, 22.7] (see the erratum for page 339 below), $\mathrm{Aut}_{L_d}(E)$ contains no copy of X_1. But $X_1 = \langle t^{X_1} \rangle$, with $t \in \mathrm{Aut}_L(E)$, so $X_1 \le \mathrm{Aut}_{L_d}(E)$. This contradiction completes the proof of the lemma.

Page 338, Lines 8 to 21: *Replace these lines with the following:*

In particular, $X_1 \in \mathcal{L}ie(r)$. Let $L = O^{r'}(C_{X_1}(x))$, a central product of groups in $\mathcal{L}ie(r)$. We may assume that $L = A_1(r^k)^u$ for some k, for otherwise $m_2(L) > 1$ and we are done. Using [**I**$_A$; 4.5.1, 4.5.2], and our assumption that $m_2(X_1) \ge 3$, we are reduced to the following cases for X_1, L, and the conjugacy class of x in the notation of [**I**$_A$; 4.5.1]: $(X_1, L, x) = (A_3^{\pm}(q)^u, A_1(q^2), t_2')$, $(B_2(q)^u, B_1(q), t_1 \text{ or } t_1')$, $(B_2(q)^u, A_1(q^2), t_2')$.

Page 339, Lines -15 to -1: *Delete Lemma 22.5 and replace it with the following two lemmas:*

Lemma 22.6. *Let $V = E_{5^4}$ and $X \leq Aut(V) \cong GL_4(5)$. Suppose that $R = O_2(X)$ is of symplectic type and $w(R) \geq 2$. Suppose that $H \leq X$ with $H \cong \Sigma_5$. Let $t \in H$ be a transposition which is a reflection in X, and let $t' \in t^H$ be such that $W_0 := \langle t, t' \rangle \cong \Sigma_3$. Then there exists a reflection $s \in X$ such that $\langle W_0, s \rangle = D_0 W_0$ with $F^*(D_0 W_0) = D_0 \cong Z_4 \times Z_4$ and $s \in t^{D_0 W_0}$.*

Proof. Since $w(R) \geq 2$, R (or any extraspecial subgroup of R of width 2) is absolutely irreducible on V, so $C_X(R) = Z(R)$ is cyclic of order dividing 4. Let $H_0 \leq H$ with $\langle t, t' \rangle \leq H_0 \cong \Sigma_4$ and let $H_1 = \langle t^{H_0 R} \rangle \geq \langle t, t' \rangle$. Let $v = tt' \in \mathcal{I}_3(H_0)$.

Suppose first that $[R, v] \cong Q_8$. Therefore $H' := [H, H]$ has a unique nontrivial module on $R/\Phi(R)$, and it is the natural A_5 permutation module, which is projective. Hence $[R, H'] \cong Q_8 * D_8$ and $[R, H'] \triangleleft RH'$. Then $[R, H']$ has exactly 5 E_{2^2}-subgroups, and they are permuted transitively by H'. As a result, one of them, say U, is normalized by $O^2(H_0)$ and hence centralized by $O_2(H_0)$. Then $E_{2^4} \cong UO_2(H_0) \leq [X, X] \leq SL(V)$, a contradiction as $m_2(SL(V)) = 3$.

Therefore $[R, v]$ has width 2. As $v \in H_1 \triangleleft H_0 R$, $[R, v] \leq H_1$. If $O_2(H_1)$ were of symplectic type, then $[O_2(H_1), v]$ would be extraspecial and equal $[R, v] O_2(H_0)$. But then $R = Z(R)[R, v]$ would centralize $O_2(H_0)$, a contradiction. Therefore $O_2(H_1)$ is not of symplectic type. Hence by P. Hall's theorem, there is a noncyclic elementary abelian E char $O_2(H_1)$, whence $E \triangleleft H_1$. Now $H_1 \geq H_0[R, v]$ so $|H_1|_2 \geq 2^8$. Since H_1 is irreducible on V, it is indecomposable on V. Hence by [$\mathbf{III}_{17}$, 1.4] H_1 is monomial on V, and is writable as $H_1 = F\Sigma$, where $\Sigma \cong \Sigma_4$ permutes the four subspaces in a frame $\mathcal{F}$ of V naturally, and F is diagonal with respect to $\mathcal{F}$. Since $|H_1|_2 \geq 2^8$, F has exponent 4. If $|H_1|_2 = 2^8$, then $Z(R) \cap F \cong Z_2$, and $F/\Omega_1(F) \cong E_{2^2}$. But this is impossible as the natural permutation module of Σ_4 has a unique minimal submodule and it is a trivial module. Therefore $|H_1|_2 > 2^8$, whence $Z(R) \cong Z_4$ and $Z(R) \leq H_1$. Then $|R[F, v]/\Omega_1(R[F, v])| \geq 2^3$ so $|H_1|_2 = 2^9$ and $F \cong Z_4 \times Z_4 \times Z_4$.

Finally, $F_0 := [F, v] \cong Z_4 \times Z_4$ is W_0-invariant as $\langle v \rangle \triangleleft W_0$. Moreover, $F_0 W_0 / \Omega_1(F_0) \cong \Sigma_4$ is generated by the images of t, t', and a further F_0-conjugate s of t. Hence $F_0 W_0 = \langle t, t', s \rangle$ and the proof is complete. $\square$

Lemma 22.7. *Let $L = L_3(16^n)$. Suppose that $M \in \mathcal{K}_5 \cap \mathcal{C}hev(2)$ and either $M = L$ or $L \uparrow_5 M$. In the latter case assume that $m_5(M) = 3$ and either $q(M) = q(L)$ or $M \cong A_5^{\eta}(4^n)$, $\eta = (-1)^{n+1}$. Assume also that $\mathcal{F}(M) \leq (16^{16n}, A)$. Let $P \in \mathcal{E}_3^5(M)$. Then $Aut_M(P)$ does not contain a faithful extension of $Z_4 \times Z_4$ by Σ_3.*

Proof. If $q(M) = q(L)$, then since $\mathcal{F}(M) \leq (16^{16n}, A)$, M has untwisted Lie rank at most 3 or $M/Z(M) \cong L_5(16^n)$. Hence $\mathrm{Aut}_M(P)$ is a Weyl group of type A_4, A_3, C_3, or A_2. It is then clear that $\mathrm{Aut}_M(P)$

does not contain a copy of $Z_4 \times Z_4$. Finally, suppose that $M \cong A_5^{\eta}(4^n)$, $\eta = (-1)^{n+1}$. Again $\mathrm{Aut}_M(P) \cong W(C_3)$ so the lemma is proved. $\square$

ERRATA FOR NUMBER 8

Page 541, Line 18: Add the sentence: (Here $\epsilon = \pm 1$, and in the unitary case, "diagonalizable" means with respect to an orthonormal basis.)

Page 541, Line 19: $\cancel{SL_n(q)}$ $SL_n^{\epsilon}(q)$

Page 541, Line 20: $\cancel{L_n(q) \le X \le PGL_n(q)}$ $L_n^{\epsilon}(q) \le X \le PGL_n^{\epsilon}(q)$

Page 541, Line 25: $\cancel{PGL_n(q)/L_n(q)}$ $PGL_n^{\epsilon}(q)/L_n^{\epsilon}(q)$

ERRATA FOR NUMBER 9

Page 312, Line -4: Add the condition $q \notin \{2, 8\}$.

Page 344, Line -11: $\cancel{K = L_4(3)}$ $K = L_4^{\pm}(3)$

Page 357, Line -1: Add the sentence: In the final assertion, (b) or (c) holds or K is a quotient of $\Omega_6^{\pm}(3)$, and the assertion is easily checked.

Page 358, Line 21: ~~and$K$~~ and K

Page 375, Line -8: $\cancel{vr = 2}$ $r = 2$

Page 436, Line 4: ~~invering~~ inverting

Page 444, Line 7: ~~resuult~~ result

Page 457, Line 8: ~~respectiive~~ respective

Page 505, Line -6: Add the condition $q \notin \{2, 8\}$.

Page 508, Line -10: $\cancel{L_4(8), L_5(8),}$ $L_4(8)$ or $L_5(8)$.

Page 512, Line 22: ~~so X~~ so x