



C^m Positive Eigenvectors for Linear Operators Arising in the Computation of Hausdorff Dimension

Roger D. Nussbaum

Abstract. We consider a broad class of linear Perron–Frobenius operators $\Lambda : X \rightarrow X$, where X is a real Banach space of C^m functions. We prove the existence of a strictly positive C^m eigenvector v with eigenvalue $r = r(\Lambda) =$ the spectral radius of Λ . We prove (see Theorem 6.5 in Sect. 6 of this paper) that $r(\Lambda)$ is an algebraically simple eigenvalue and that, if $\sigma(\Lambda)$ denotes the spectrum of the complexification of Λ , $\sigma(\Lambda) \setminus \{r(\Lambda)\} \subseteq \{\zeta \in \mathbb{C} \mid |\zeta| \leq r_*\}$, where $r_* < r(\Lambda)$. Furthermore, if $u \in X$ is any strictly positive function, $(\frac{1}{r}\Lambda)^k(u) \rightarrow s_u v$ as $k \rightarrow \infty$, where $s_u > 0$ and convergence is in the norm topology on X . In applications to the computation of Hausdorff dimension, one is given a parametrized family $\Lambda_s, s > s_*$, of such operators and one wants to determine the (unique) value s_0 such that $r(\Lambda_{s_0}) = 1$. In another paper (Falk and Nussbaum in C^m Eigenfunctions of Perron–Frobenius operators and a new approach to numerical computation of Hausdorff dimension, submitted) we prove that explicit estimates on the partial derivatives of the positive eigenvector v_s of Λ_s can be obtained and that this information can be used to give rigorous, sharp upper and lower bounds for s_0 .

Mathematics Subject Classification. Primary: 47A53, 47A75, 47B65; Secondary: 28A78, 27A80.

Keywords. Measures of noncompactness, Essential spectrum, Radius of essential spectrum, Generalized Krein–Rutman theorem, C^m positive eigenvector.

1. Introduction

The motivation for this paper comes from the problem of finding rigorous, sharp estimates for the Hausdorff dimension of “invariant sets” for iterated function systems or for graph directed iterated function systems. We refer, for example to [9, 18, 19, 21–32, 40, 44] for definitions, background information

R. D. Nussbaum was partially supported by NSF DMS-1201328.

and discussions of some interesting examples. In [40] a general construction is given which associates to a graph directed iterated function system a family of “Perron–Frobenius” operators L_s defined naturally for $s > s_*$, where $s_* = 0$ if there are only finitely many functions in the graph directed iterated function system and L_{s_*} is defined in that case. A generalization of the Krein–Rutman theorem is used in [40] to prove that L_s has a strictly positive, Hölder continuous eigenvector v_s with eigenvalue $r_s > 0$, r_s equals the spectral radius of L_s , which we denote by $r(L_s)$, and $s \rightarrow r_s, s > s_*$, is a strictly decreasing, continuous positive function. Related results can be found in §5 and §6 of [38]. Under natural conditions on the graph directed iterated function system (so, in particular, the functions θ in the iterated function system must be “infinitesimal similitudes”), the Hausdorff dimension s_0 of the invariant set satisfies $s_0 = \inf\{s > s_* \mid r(L_s) < 1\}$ and usually $r(L_{s_0}) = 1$. The problem thus becomes one of efficient and rigorous estimation of the value of s_0 such that $r(L_{s_0}) = 1$; but despite this explicit formula, high order, rigorous approximation of s_0 is, in general, a nontrivial problem.

Typically, one can consider the linear maps L_s as bounded linear maps from a real Banach space X to itself; but many choices of X are possible, and in general, $\sigma(L_s)$ (by which we mean the spectrum of the complexification of L_s) depends sensitively on the choice of X , although our later theorems will usually imply that $r(L_s)$ is independent of X . In certain special cases X can be taken as a real Banach space of analytic functions (see our remarks below) and then the map $L_s : X \rightarrow X$ is almost always compact and possibly (see [19]) of trace class. In general, however, one cannot hope to find a real Banach space X of analytic functions such that L_s maps X to X ; and this motivates the theme of this paper, which is to study linear Perron–Frobenius operator or “transfer operators” in real Banach spaces of C^m functions.

Before proceeding further, it may be useful to give the reader some motivating examples. Let $J \subset \mathbb{R}$ be a closed, bounded interval, \mathcal{B} a finite index set and, for $b \in \mathcal{B}$, let $\theta_b : J \rightarrow J$ be a C^m map. Assume that there exists a constant $c < 1$ such that, for all $b \in \mathcal{B}$,

$$\sup_{x \in J} |\theta'_b(x)| \leq c.$$

Suppose, also, that $\theta'_b(x) \neq 0 \forall x \in J$ and $\forall b \in \mathcal{B}$. Define, for $s \geq 0$, a bounded linear map $L_{m,s} : Y_m := C^m(J) \rightarrow Y_m$

$$(L_{m,s}f)(x) = \sum_{b \in \mathcal{B}} |\theta'_b(x)|^s f(\theta_b(x)).$$

Then there exists a unique, compact, nonempty set K such that $K = \bigcup_{b \in \mathcal{B}} \theta_b(K)$; and Theorem 6.5 below implies that there exists a unique (to within scalar multiples) strictly positive eigenvector $v_s \in Y_m \setminus \{0\}$ of $L_{m,s}$ with eigenvalue $r(L_{m,s}) > 0$. If, for example, $\theta_b(K) \cap \theta_{b'}(K) = \emptyset$ for all $b, b' \in \mathcal{B}$ with $b \neq b'$, then s_0 , the Hausdorff dimension of K , equals the unique value of s for which $r(L_{m,s}) = 1$.

If there exists an open neighborhood $U \subset \mathbb{C}$ of K such that each map $\theta_b : K \rightarrow K$ extends to an analytic map $\tilde{\theta}_b : U \rightarrow \mathbb{C}$, then one can find a real

Banach space Z of analytic functions and a bounded linear map $\Lambda_s : Z \rightarrow Z$ such that $r(\Lambda_s) = r(L_{m,s})$. However, in general, this is not possible, and one is forced to work with $L_s : Y_m \rightarrow Y_m$.

Our next motivating example concerns an invariant set for a finite collection of analytic maps. Let $G \subset \mathbb{C}$ be a bounded, open connected set; we identify \mathbb{C} with \mathbb{R}^2 and $(x + iy)$ with (x, y) . Let \mathcal{B} be a finite index set and for $b \in \mathcal{B}$ let $\varphi_b : G \rightarrow G$ be an analytic map such that $\overline{\varphi_b(G)} \subset G$ and $(\frac{d}{dz})\varphi_b(z) \neq 0$ for all $z \in G$ and for all $b \in \mathcal{B}$. Let H be a bounded, open connected set such that $\overline{\varphi_b(G)} \subset H$ and $\bar{H} \subset G$ for all $b \in \mathcal{B}$. For each integer $m \geq 0$, let $Y_m := C^m(\bar{H})$ denote the real Banach space of continuous real-valued functions $f : H \rightarrow \mathbb{R}$ which have continuous partial derivatives of all orders less than or equal to m and all of whose partial derivatives of order less than or equal to m extend continuously to \bar{H} . For $s \geq 0$, define a Perron–Frobenius operator $L_{m,s} : Y_m \rightarrow Y_m$ by

$$(L_{m,s}f)(z) = \sum_{b \in \mathcal{B}} \left| \left(\frac{d}{dz} \right) \varphi_b(z) \right|^s f(\varphi_b(z)).$$

It is a consequence of Theorem 6.5 of this paper and the Caratheodory–Reiffen–Finsler metric (see §6 of [40]) that for $m \geq 1$, $L_{m,s}$ has a unique (to within scalar multiples), strictly positive eigenvector v_s with an algebraically simple eigenvalue $r(L_{m,s}) > 0$. Furthermore, if $\sigma(L_{m,s})$ denotes the spectrum $\sigma(\hat{L}_{m,s})$ of the complexification $\hat{L}_{m,s}$ of $L_{m,s}$, then there exists a number $\rho_{m,s} < r(L_{m,s})$ such that $|z| \leq \rho_{m,s}$ for all $z \in \sigma(\hat{L}_{m,s}) := \sigma(L_{m,s})$ with $z \neq r(L_{m,s})$.

Several points should be made about the above result. The spectral radius $r(L_{m,s})$ is independent of m for $m \geq 0$, but $\sigma(\hat{L}_{m,s})$ varies with m . The eigenvector v_s is C^∞ , but despite the analyticity of the functions $\varphi_b : G \rightarrow G$, it is not in general possible to study the operator $L_{m,s}$ in a real Banach space of analytic functions. There is a unique compact, nonempty set $K \subset H$ such that $K = \bigcup_{b \in \mathcal{B}} \theta_b(K)$; and under further assumptions, the Hausdorff dimension of K equals s_0 , where s_0 is the unique value of s for which $r(L_{m,s}) = 1$.

If $I_1 := \{m + ni \mid m \in \mathbb{N}, n \in \mathbb{Z}, i = \sqrt{-1}\}$, and if \mathcal{B} is a subset of I_1 , for $b \in \mathcal{B}$ define $\theta_b(z) = (\frac{1}{z+b})$. In this generality one can easily find a bounded, open connected set G and a compact set $D \subset G$ such that $\theta_b(G) \subset D$ for all $b \in \mathcal{B}$. If \mathcal{B} is a subset of \mathbb{N} , the positive integers, Jenkinson and Pollicott [19] have shown that L_s can be considered as a bounded linear map $L_s : X \rightarrow X$, where X is a real Banach space of analytic functions. Exploiting this fact and trace class arguments they obtain, at least if $|\mathcal{B}|$ is not large, very high order approximations to the unique value s_0 for which $r(L_{s_0}) = 1$. However, such an approach is not applicable if \mathcal{B} is not a subset of \mathbb{N} ; and if $\mathcal{B} \subset \mathbb{N}$ but $|\mathcal{B}|$ is large, the Jenkinson–Pollicott approach may not be optimal.

In ongoing joint work [14] with Professor Richard Falk we have taken a different viewpoint. We consider a general class of parametrized Perron–Frobenius linear operators $L_s : Y \rightarrow Y$, where Y is a Banach space of C^m real valued functions $f : \bar{H} \rightarrow \mathbb{R}$ and H is a bounded, open subset of \mathbb{R}^n .

Starting from the fact, which will be proved here, that L_s has a strictly positive C^m eigenvector v_s , we prove in [14] that one can obtain explicit bounds on certain partial derivatives of v_s and that this information can be used to improve greatly the accuracy of estimates for $r(L_s)$ and for s_0 , where s_0 is the unique value of s such that $r(L_s) = 1$.

We mention an example which is given in [30] and which illustrates the power of the approach outlined above. Let $\mathcal{B} = \{m + ni \mid m \in \mathbb{N}, n \in \mathbb{Z}, i = \sqrt{-1}\}$ and for $b = m + ni \in \mathcal{B}$, let $\varphi_b(z) = \frac{1}{z+b}$. If $G = \{z \in \mathbb{C} \mid |z - \frac{1}{2}| \leq \frac{1}{2}\}$ it is easy to show that $\varphi_b(\bar{G}) \subset \bar{G}$.

There is a naturally defined “invariant set” $J \subset \bar{G}$ such that $J = \bigcup_{b \in J} \varphi_b(J)$. Associated to the iterated function system $\{\varphi_b \mid b \in \mathcal{B}\}$ there is, for $s > 1$, a bounded linear operator $L_s : Y := C(\bar{G}) \rightarrow Y$, but we take $X = C^2(\bar{G})$ and view L_s as a bonded linear map $\Lambda_s : X := C^2(\bar{G}) \rightarrow X$ and note that L_s has a strictly positive eigenvector in X . The operator L_s is defined in this case by

$$(L_s(f))(z) = \sum_{b \in \mathcal{B}} \left| \left(\frac{1}{z + b} \right) \right|^{2s} f(\varphi_b(z)).$$

If $\dim_H(J)$ denotes the Hausdorff dimension of J , it is proved in [14] that $1.854 < \dim_H(J) < 1.857$. Mauldin and Urbanski proved in [30] that $1.2484 < \dim_H(J)$ and claimed (no details were given) that $\dim_H(J) < 1.9$. In his 2011 Rutgers University Ph.D. dissertation, Amit Priyadarshi proved that $1.787 < \dim_H(J)$.

We shall actually consider a class of “Perron–Frobenius operators” or “transfer operators”, Λ_s , more general than those which arise in the formulas for Hausdorff dimension. See Eq. (4.3) in Sect. 3 below. No simplification in the proofs is achieved by considering the less general case; and the general case has independent interest.

Indeed, there is a very large literature concerning “transfer operators”; see, for example, Baladi’s book [4] and the references there. We believe that the results in this paper may prove useful in contexts other than computation of Hausdorff dimension. We should also mention that the problem of analyzing $\{s \mid r(L_s) = 1\}$, where $\{L_s \mid s > 0\}$ is a parametrized family of linear operators, arises naturally in studying bifurcation of solutions of $F_s(x) = x$, where $\{F_s : s > 0\}$ is a parametrized family of nonlinear cone mappings. See [36].

The basic goal in this paper is (a) to prove the existence of a nonnegative C^m eigenvector v_s of Λ_s with eigenvalue r_s equal to the spectral radius of Λ_s , (b) to establish, under further assumptions, the existence of a strictly positive C^m eigenvector v_s and (c) to establish some basic facts about $\sigma(\Lambda_s)$, the spectrum of the complexification of Λ_s . Unlike most of the literature, e.g. [19], we use generalizations of the Krein–Rutman theorem [23] rather than thermodynamic formalism to study the problem.

A brief outline of this paper may be in order.

Section 2 reviews notation from [40] and proves some elementary results.

Section 3 has been included in an effort to make the paper self-contained. The reader who is familiar with basic facts about measures of noncompactness, the essential spectrum and the radius of the essential spectrum can skip most of Sect. 3. Note, however, Theorem 3.1, which generalizes the Krein–Rutman theorem and appears not to be well known.

In Sect. 4 the class of operators Λ of interest is introduced, and it is proved in Theorem 4.6 that Λ has a nonnegative C^m eigenvector with eigenvalue r equal to the spectral radius of Λ . Here $\Lambda : X \rightarrow X$, where X is a real Banach space of C^m functions as in Eq. (2.17). The key tool is Lemma 4.5, which proves that $\rho(\Lambda)$, the essential spectral radius of Λ , satisfies $\rho(\Lambda) < r$. This implies that $\sigma(\Lambda) \cap \{z \in \mathbb{C} \mid |z| > \rho(\Lambda)\}$ comprises only eigenvalues and that these eigenvalues are isolated points in $\sigma(\Lambda)$ and have finite algebraic multiplicity. (Here $\sigma(L)$ denotes the spectrum of $\bar{\Lambda}$, the complexification of Λ .)

In Sect. 5 it is proved (see Theorem 5.7) that if the assumptions in Sect. 4 are strengthened, Λ has a strictly positive C^m eigenvector v with eigenvalue r .

In Sect. 6, under a further strengthening of hypotheses in Sect. 4, it is proved in Theorem 6.5 that r is an algebraically simple eigenvalue of Λ with a strictly positive C^m eigenvector v and that there exists $r_* < r$ such that $\sigma(\Lambda) \setminus \{r\} \subseteq \{\zeta \in \mathbb{C} \mid |\zeta| \leq r_*\}$. Furthermore, for every $u \in X$, there exists $s_u \in \mathbb{R}$ such that $\lim_{k \rightarrow \infty} (\frac{1}{r}\Lambda)^k(u) = s_u v$ in the C^m norm topology on the Banach space X ; and necessarily $s_u > 0$ for a large subset of X .

Theorem 4.6 in Sect. 4, Theorem 5.7 in Sect. 5 and Theorem 6.5 in Sect. 6 are the main results of this paper; and, as we have noted, they play a crucial role in a sequel paper [14] which treats the rigorous approximation of the Hausdorff dimension of certain fractal sets.

With regard to eventual applications, we have included the case of graph directed iterated function systems. However, it must be admitted that the graph directed case leads to a considerable increase in notational complexity, although the underlying conceptual framework of the proof is the same as in the ordinary iterated function system case. For this reason, we have also included, immediately after the statements of the major theorems, corollaries which state these same theorems in the iterated function system case. Readers wishing to appreciate the flavor of our results may wish to start with these corollaries.

2. Notation, Definitions and Some Elementary Results

For the reader's convenience, we begin by recalling some definitions and notations from [40]. Throughout this paper $V := \{i \in \mathbb{N} : 1 \leq i \leq p\}$ and \mathcal{E} will denote finite sets. In the original construction of Mauldin and Williams [30] of a “graph-directed iterated function system”, V is the set of vertices and \mathcal{E} the set of edges of a directed multigraph. We shall consistently denote by Γ a given subset of $V \times \mathcal{E}$ and by $\alpha : \Gamma \rightarrow V$ a given map. For $i \in V$, we shall consistently denote sets Γ_i and \mathcal{E}_i by

$$\Gamma_i := \{(j, e) \in \Gamma \mid \alpha(j, e) = i\} \quad \text{and} \quad (2.1)$$

$$\mathcal{E}_i := \{e \in \mathcal{E} \mid (i, e) \in \Gamma\}. \quad (2.2)$$

We shall usually assume

(H1.1) For all $i \in V$, Γ_i is nonempty and \mathcal{E}_i is nonempty.

As in [40], for a positive integer $\mu \geq 1$ and $i \in V$, we define

$$\begin{aligned} \Gamma^{(\mu)} = \{ & [(i_1, e_1), (i_2, e_2), \dots, (i_\mu, e_\mu)] : (i_j, e_j) \in \Gamma \text{ for } 1 \leq j \leq \mu \\ & \text{and } \alpha(i_{j+1}, e_{j+1}) = i_j \text{ for } 1 \leq j \leq \mu - 1\} \end{aligned} \quad (2.3)$$

and

$$\Gamma_i^{(\mu)} = \{[(i_1, e_1), (i_2, e_2), \dots, (i_\mu, e_\mu)] \in \Gamma^{(\mu)} : \alpha(i_1, e_1) = i\}. \quad (2.4)$$

Later, we shall need close “relatives” of the sets $\Gamma^{(\mu)}$ and $\Gamma_i^{(\mu)}$. If μ is a positive integer and $k \in V$, we define sets $\bar{\Gamma}^{(\mu)}$ and $\bar{\Gamma}_k^{(\mu)}$ by

$$\begin{aligned} \bar{\Gamma}^{(\mu)} = \{ & [(k_1, e_1), (k_2, e_2), \dots, (k_\mu, e_\mu)] : (k_j, e_j) \in \Gamma, 1 \leq j \leq \mu \\ & \text{and } k_{j+1} = \alpha(k_j, e_j) \text{ for } 1 \leq j \leq \mu - 1\} \end{aligned} \quad (2.5)$$

and

$$\bar{\Gamma}_k^{(\mu)} = \{[(k_1, e_1), (k_2, e_2), \dots, (k_\mu, e_\mu)] \in \bar{\Gamma}^{(\mu)} : k_1 = k\}. \quad (2.6)$$

If one is only interested in “iterated function systems” as opposed to “graph-directed iterated function systems”, most of the above notational complexity vanishes. In the iterated function system case, $V = \{1\}$, $\Gamma = \{1\} \times \mathcal{E}$ and $\alpha(1, e) = 1$ for all $e \in \mathcal{E}$. In essence, V no longer plays any role, and one can identify Γ with \mathcal{E} . With this identification, $\Gamma^{(\mu)} = \{[e_1, e_2, \dots, e_\mu] : e_j \in \mathcal{E} \text{ for } 1 \leq j \leq \mu\}$ and $\Gamma^{(\mu)} = \Gamma_1^{(\mu)} = \bar{\Gamma}^{(\mu)} = \bar{\Gamma}_1^{(\mu)}$. In our later work the reader should keep this simpler case in mind: the essential difficulties remain, but some of the notational complexities vanish.

Our work here will sometimes involve choosing a norm on \mathbb{R}^n , although the final theorems will be independent of the particular norm chosen. Recall that any two norms $\|\cdot\|$ and $|\cdot|$ on \mathbb{R}^n are equivalent, in the sense that there are positive constants a and b such that

$$a\|x\| \leq |x| \leq b\|x\| \quad \forall x \in \mathbb{R}^n.$$

Nevertheless, it will sometimes be convenient to use norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$, where

$$\|x\|_\infty := \max\{|x_i| : 1 \leq i \leq n\}, \quad x = (x_1, x_2, \dots, x_n) \quad (2.7)$$

and

$$\|x\|_1 := \sum_{i=1}^n |x_i|, \quad x = (x_1, x_2, \dots, x_n). \quad (2.8)$$

If $A = (a_{ij})$ is an $n \times n$ matrix with real entries, A defines a bounded linear map of \mathbb{R}^n to \mathbb{R}^n by $x \rightarrow Ax$, where $x \in \mathbb{R}^n$ is an $n \times 1$ column vector. It is known that

$$\|A\|_1 := \max\{\|Ax\|_1 : \|x\|_1 \leq 1\} = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| \tag{2.9}$$

and

$$\|A\|_\infty := \max\{\|Ax\|_\infty : \|x\|_\infty \leq 1\} = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| \tag{2.10}$$

For $i \in V, G_i$ will always denote a bounded open subset of \mathbb{R}^n for $1 \leq i \leq p := |V|$. As usual, $C(\bar{G}_i) := Y_i$ will denote the real Banach space of continuous maps $f_i : \bar{G}_i \rightarrow \mathbb{R}$, with $\|f_i\| := \max\{\|f_i(x)\| : x \in \bar{G}_i\}$. We shall always denote by Y the real Banach space $\prod_{j=1}^p Y_j$, so

$$Y := \{(f_1, f_2, \dots, f_p) : f_j \in C(\bar{G}_j) = Y_j \text{ for } 1 \leq j \leq p\}. \tag{2.11}$$

We shall define $\|(f_1, f_2, \dots, f_p)\|_Y$ by

$$\|(f_1, f_2, \dots, f_p)\|_Y := \max\{\|f_j\| : 1 \leq j \leq p\}. \tag{2.12}$$

We define $K \subset Y$ by

$$K := \{(f_1, f_2, \dots, f_p) \in Y \mid f_j(x) \geq 0 \forall x \in G_j, 1 \leq j \leq p\}. \tag{2.13}$$

Note that K is a ‘‘closed cone in Y ’’, i.e., K is closed and convex in $Y, \lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$, where $-K := \{-f : f \in K\}$.

If Z is a real Banach and $A : Z \rightarrow Z$ is a bounded linear operator, recall that standard functional analysis [10] tells us that $r(A)$, the spectral radius of A , is given by

$$r(A) := \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} = \inf_{n \geq 1} (\|A^n\|^{\frac{1}{n}}). \tag{2.14}$$

If we define $\hat{Z} = \{(u, v) : u, v \in Z\}$ and identify (u, v) with $u + iv, i = \sqrt{-1}, \hat{Z}$ becomes a complex vector space. If we define

$$\|(u, v)\| := \max\{\|u \cos \theta + v \sin \theta\| : 0 \leq \theta \leq 2\pi\}. \tag{2.15}$$

\hat{Z} becomes a complex Banach space with norm given by (2.15). The complex Banach space \hat{Z} is called the ‘‘complexification of Z ’’. The map A extends to a complex linear map $\hat{A} : \hat{Z} \rightarrow \hat{Z}$ if we define $\hat{A}(u + iv) = Au + iAv$; and one can check that $\|\hat{A}\| = \|A\|$ and $\|(\hat{A})^n\| = \|(\hat{A}^n)\| = \|A^n\|$, so $r(\hat{A}) = r(A)$. It now follows from standard functional analysis that

$$r(A) = r(\hat{A}) = \max\{|\lambda| \mid \lambda \in \sigma(\hat{A})\},$$

where $\sigma(\hat{A}) := \{\lambda \in \mathbb{C} \mid \lambda I - \hat{A} \text{ is not one-one and onto } \hat{Z}\}$ is the ‘‘spectrum of \hat{A} ’’ and I is the identity operator on \hat{Z} .

We shall need the following elementary lemma

Lemma 2.1. *Let Y and K be given by Eqs. (2.11) and (2.13) respectively and define $u = (u_1, u_2, \dots, u_p) \in K$ by $u_j(x) = 1$ for all $\bar{x} \in \bar{G}_j, 1 \leq j \leq p$. Suppose that $A : Y \rightarrow Y$ is a bounded linear map such that $A(K) \subset K$. Then $r(A)$, the spectral radius of A satisfies $r(A) = \lim_{k \rightarrow \infty} \|A^k(u)\|^{\frac{1}{k}}$.*

Proof. We know that $r(A) = \lim_{k \rightarrow \infty} \|A^k\|^{\frac{1}{k}}$. Thus it suffices to prove that $\|A^k\| = \|A^k(u)\|$. Since $\|u\| = 1$, we certainly have that

$$\|A^k(u)\| \leq \|A^k\| \|u\| = \|A^k\|.$$

On the other hand, if $f = (f_1, f_2, \dots, f_p) \in Y$ and $\|f\| \leq 1, -u_j(x) \leq f_j(x) \leq u_j(x)$ for all $x \in \bar{G}_j, 1 \leq j \leq p$, so $u - f \in K$ and $u + f \in K$. Since $A^k(K) \subset K, A^k(u - f) = A^k u - A^k f \in K$ and $A^k(u + f) = A^k u + A^k f \in K$. If $g, h \in Y$, we write $g \leq h$ if $h - g \in K$. If $-h \leq g$ and $g \leq h$, one can see that $h \in K$ and $\|g\| \leq \|h\|$. In our case, writing $g = A^k(f)$ and $h = A^k(u)$ we see that $-h \leq g$ and $g \leq h$, so $\|A^k(u)\| \geq \|A^k(f)\|$. It follows that $\|A^k(u)\| \geq \sup\{\|A^k(f)\| : \|f\| \leq 1\} = \|A^k\|$, so $\|A^k\| = \|A^k(u)\|$. \square

For a fixed positive integer m , suppose that $\psi : G_j \rightarrow \mathbb{R}$ is m times continuously differentiable. More precisely, if $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is any n -tuple of nonnegative integers (a “multi-index”) with $\sum_{i=1}^n \beta_i := \|\beta\|_1$, we write

$$(D^\beta \psi)(x) = (D_1^{\beta_1} D_2^{\beta_2} \dots D_n^{\beta_n} \psi)(x),$$

where $D_j := \frac{\partial}{\partial x_j}$. We assume that $x \rightarrow (D^\beta \psi)(x)$ is defined, continuous and bounded on G_j for all multi-indices β with $\|\beta\|_1 \leq m$. We shall say that $\psi \in C^m(\bar{G}_j)$ if and only if $x \rightarrow D^\beta \psi(x)$ is defined, continuous and bounded on G_j and $x \rightarrow D^\beta \psi(x)$ extends to a continuous function on \bar{G}_j for every multi-index β with $\|\beta\|_1 \leq m$. We shall always write $X_j := C^m(\bar{G}_j)$. If $\psi \in X_j$, we define $\|\psi\|_{X_j}$ by

$$\|\psi\|_{X_j} := \sup\{|D^\beta \psi(x)| : x \in G_j, \|\beta\|_1 \leq m, \beta \text{ a multi-index}\}. \tag{2.16}$$

It is known (and not hard to prove) that $C^m(\bar{G}_j) := X_j$ is a real Banach space.

For $p = |V|$, we shall always denote by X the real Banach space $\prod_{j=1}^p X_j$, so

$$X := \{(f_1, f_2, \dots, f_p) \mid f_j \in C^m(\bar{G}_j) := X_j, 1 \leq j \leq p\}. \tag{2.17}$$

For $f = (f_1, f_2, \dots, f_p) \in X$, we define

$$\|f\|_X = \max\{\|f_j\|_{X_j} : 1 \leq j \leq p\}. \tag{2.18}$$

If $A : Y \rightarrow Y$ is a bounded linear operator, where Y is given by Eq. (2.11), it may happen that $A(f) \in X$ for all $f \in X$, where X is given by Eq. (2.17). In this case one can define $B : X \rightarrow X$ by $B(f) = A(f)$ for $f \in X$, and the closed graph theorem implies that $B : X \rightarrow X$ is a bounded linear map of the Banach space X to X . In this situation, it is natural to ask whether $r(B) \geq r(A)$, where $r(B)$ (respectively, $r(A)$) denotes the spectral radius of B (respectively, A).

Lemma 2.2. *Let assumptions and notation be as in Lemma 2.1, so $A : Y \rightarrow Y$ is a bounded linear map and $A(K) \subset K$. Assume in addition that $A(f) \in X$ for all $f \in X$. Then A defines a bounded linear operator $B : X \rightarrow X$ by $B(f) = A(f)$ for $f \in X$ and $r(B) \geq r(A)$.*

Proof. As noted above, the closed graph theorem implies that $B : X \rightarrow X$ is a bounded linear operator. Lemma 2.1 implies that

$$r(A) = \lim_{k \rightarrow \infty} \|A^k(u)\|_k^{\frac{1}{k}},$$

where u is defined as in Lemma 2.1. Notice that $\|u\|_Y = \|u\|_X = 1$, while $\|f\|_X \geq \|f\|_Y$ for all $f \in X$. Thus we obtain

$$\|B^k\| \geq \|B^k(u)\|_X \geq \|B^k(u)\|_Y = \|A^k(u)\|_Y.$$

It follows that

$$r(B) = \lim_{k \rightarrow \infty} \|B^k\|_k^{\frac{1}{k}} \geq \lim_{k \rightarrow \infty} \|A^k(u)\|_Y^{\frac{1}{k}} = r(A),$$

which completes the proof. □

It will be convenient to assume that the boundaries of the bounded open sets $G_j \subset \mathbb{R}^n, 1 \leq j \leq p$, satisfy a mild regularity condition.

Definition 2.3. Let H be a bounded open subset of \mathbb{R}^n . We shall say that H is “mildly regular” if there exist numbers $\eta > 0$ and $M > 0$ such that whenever $x, y \in H$ and $\|x - y\|_1 < \eta$, there exists a Lipschitz map $\psi : [0, 1] \rightarrow H$ with $\psi(0) = x$ and $\psi(1) = y$ such that

$$\int_0^1 \|\psi'(t)\|_1 dt \leq M \|x - y\|_1. \tag{2.19}$$

Note that elementary real variables implies that each component of ψ in Eq. (2.19) is absolutely continuous, so $\psi'(t)$ exists Lebesgue almost everywhere in Eq. (2.19) and

$$\int_0^1 \psi'(t) dt = \psi(1) - \psi(0) = y - x.$$

Furthermore, if ψ can always be chosen in Definition 2.3 such that $|\psi(t_1) - \psi(t_2)|_1 \leq M|t_1 - t_2|$ for all $t_1, t_2 \in [0, 1]$, then Eq. (2.19) will automatically be satisfied.

If \bar{H} is a manifold with boundary and the coordinate charts are bi-Lipschitz, one can prove that H is mildly regular. We omit the simple proof, since usually for our examples of bounded open sets mild regularity will be obvious. Note, however, that mild regularity excludes simple examples like $H = \{(x, y) \mid -1 < x < 1, -1 < y < \sqrt{|x|}\}$. With greater care we could allow such examples but at the cost of technical complications which we prefer to avoid.

We shall almost always assume

(H1.2) For all $i \in V, |V| = p, G_i$ is a mildly regular, bounded open subset of \mathbb{R}^n .

Lemma 2.4. *Assume that H is a bounded, mildly regular open subset of \mathbb{R}^n and m is a positive integer. If S is a bounded subset of the Banach space $C^m(\bar{H})$ and β is a multi-index with $\|\beta\|_1 < m$, then $\{D^\beta h|h \in S\}$ is a bounded, equicontinuous family of functions in $C(\bar{H})$, so $\{D^\beta h|h \in S\}$ has compact closure in $C(\bar{H})$.*

Proof. Let η and M be as in Definition 2.3; and if $x, y \in H$ and $\|x - y\|_1 < \eta$, let $\psi : [0, 1] \rightarrow H$ as in Definition 2.3. If $h \in S$ and β is a multi-index with $\|\beta\|_1 < m$, we have

$$\begin{aligned} |D^\beta h(y) - D^\beta h(x)| &= \left| \int_0^1 \frac{d}{dt}(D^\beta h)(\psi(t))dt \right| \\ &= \left| \int_0^1 \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}(D^\beta h) \right) (\psi(t))\psi'_j(t)dt \right|, \end{aligned}$$

where $\psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_n(t))$ and for $1 \leq j \leq n$, $\psi'_j(t)$ exists on $[0, 1] \setminus E$, where E has Lebesgue measure 0. Since S is bounded in $C^m(\bar{H})$ and $\|\beta\|_1 < m$, there exists a constant C such that for all $u \in H$ and for $1 \leq j \leq n$,

$$\left| \left(\frac{\partial}{\partial x_j} D^\beta h \right) (u) \right| \leq C,$$

so we obtain

$$\begin{aligned} |D^\beta h(y) - (D^\beta h)(x)| &\leq C \int_0^1 \sum_{j=1}^n |\psi'_j(t)|dt \\ &= C \int_0^1 \|\psi'(t)\|_1 dt \leq MC\|x - y\|_1. \end{aligned}$$

Because $u \rightarrow (D^\beta h)(u)$ has a continuous extension to \bar{H} , the same estimate holds for all $x, y \in \bar{H}$. This proves that $\{D^\beta h|h \in S\}$ is a bounded, equicontinuous set in $C(\bar{H})$, which completes the proof. □

3. Measures of Noncompactness, the Essential Spectrum and Positive Linear Operators

Our purpose here is to review for the reader's convenience concepts and basic theorems which will play an essential role in the later sections of this paper.

If (Z, d) is a metric space with metric d and S is a bounded subset, K. Kuratowski [24] has defined $\alpha(S)$, the Kuratowski measure of noncompactness (or MNC) of S , by

$$\begin{aligned} \alpha(S) = \inf \left\{ \delta > 0 \mid S = \bigcup_{i=1}^n S_i \text{ for some } S_i \right. \\ \left. \text{with } \text{diam}(S_i) \leq \delta \text{ for } 1 \leq i \leq n < \infty \right\}. \end{aligned} \tag{3.1}$$

As usual, the diameter of a bounded set $T \subset Z$ is defined by

$$\text{diam}(T) = \sup\{d(x, y) \mid x, y \in T\}. \tag{3.2}$$

and, by definition, $T \subset (Z, d)$ is bounded if $\text{diam}(T) < \infty$.

Kuratowski observed that:

(A1) if (Z, d) is a complete metric space and $S \subset Z$ is bounded, $\alpha(S) = 0$ if and only if \bar{S} is compact.

Property (A1) explains the terminology “measure of noncompactness”.

It is easy to verify the following properties, which are valid for general metric spaces (Z, d) :

(A2) $\alpha(S) \leq \alpha(T)$ for all bounded sets $S \subset T \subset Z$

(A3) $\alpha(S \cup \{x_0\}) = \alpha(S)$ for all bounded sets $S \subset Z$ and for all $x_0 \in Z$

(A4) $\alpha(\bar{S}) = \alpha(S)$ for all bounded sets $S \subset Z$.

If $(Z, \|\cdot\|)$ is a normed linear space (over \mathbb{R} or \mathbb{C}) and S and T are bounded subsets of Z , we shall denote by $\text{co}(S)$ the convex hull of S , i.e., the smallest convex set containing S , and we shall write

$$S + T = \{s + t \mid s \in S, t \in T\} \quad \text{and} \quad \lambda S = \{\lambda s \mid s \in S\},$$

where λ is an arbitrary scalar. If the metric d on the normed linear space Z is given by $d(x, y) = \|x - y\|$, G. Darbo [11] observed that the Kuratowski MNC satisfies the following extremely useful properties:

(A5) $\alpha(\text{co}(S)) = \alpha(S)$ for all bounded sets in Z

(A6) $\alpha(S + T) \leq \alpha(S) + \alpha(T)$ for all bounded sets $S, T \subset Z$

(A7) $\alpha(\lambda S) = |\lambda|\alpha(S)$ for all bounded sets $S \subset Z$ and all scalars λ .

Properties (A1)–(A7) (but particularly (A5)–(A7)) make the Kuratowski MNC a useful tool in functional analysis and in fixed point theory, and fixed point theory was Darbo’s original application in [11]. Note, for example, that an application of (A1), (A4) and (A5) yields a classical theorem of Mazur (see [33] or [10], pg.180): if S is a compact subset of a Banach space Z , then the closure of $\text{co}(S)$, is compact.

Although we shall not exploit it, the Kuratowski MNC also satisfies the so-called “set-additivity property”, namely

(A8) $\alpha(S \cup T) = \max(\alpha(S), \alpha(T))$ for all bounded sets $S, T \subset Z$.

Property (A8) is true in general metric spaces (Z, d) and gives (A2) and (A3) as special cases.

If $(Z, \|\cdot\|)$ is a Banach space (real or complex), let $\mathcal{B}(Z)$ denote the set of all bounded subsets of Z . A map $\beta : \mathcal{B}(Z) \rightarrow [0, \infty)$ is called a homogeneous measure of noncompactness or homogeneous MNC in [28, 29] if β satisfies properties (A1)–(A7), with β replacing α in those formulas. If β and γ are homogeneous MNC’s on a Banach space Z , we call β and γ equivalent if there exist positive constants a and b such that

$$a\beta(S) \leq \gamma(S) \leq b\beta(S) \quad \text{for all bounded } S \subset Z. \tag{3.4}$$

It is proved in [28,29] that there exist inequivalent homogeneous MNC's on many Banach spaces Z . We refer the reader to [1-3,5,27] for further information about general measures of noncompactness.

If Z is a Banach space (over \mathbb{R} or \mathbb{C}) and $B : Z \rightarrow Z$ is a bounded linear map, we define $N(B) = \{x \in Z | B(x) = 0\}$, the null space of B , and $R(B) = \{B(x) | x \in Z\}$, the range of B . We also consider $Z/R(B)$, the vector space of equivalence classes $[x]$, where $x \sim y$ iff $x - y \in R(B)$. The operator B is called "Fredholm" if $\dim(N(B)) < \infty, \dim(Z/R(B)) := \text{codim}(R(B)) < \infty$ and $R(B)$ is closed; and by definition $i(B) = \dim(N(B)) - \text{codim}(R(B))$ is the index of the Fredholm operator B . If $R(B)$ is closed and either (a) $\dim(N(B)) < \infty$ or (b) $\text{codim}(R(B)) < \infty$, B is called "semi-Fredholm". These concepts are actually defined for closed, densely defined linear operators $T : D(T) \subset Z \rightarrow Z$. We refer to Kato's book [20] for a detailed discussion of Fredholm and semi-Fredholm operators and their properties.

If Z is an infinite dimensional, complex Banach space and $A : Z \rightarrow Z$ is a bounded linear operator there are several inequivalent definitions of the so-called "essential spectrum of A ". (These definitions also apply when $A : D(A) \subset Z \rightarrow Z$ is closed and densely defined, but we shall only consider the case that A is a bounded operator.) F. E. Browder [8] defines $\text{ess}(A)$ to be the set of complex λ such that (a) λ is an accumulation point of $\sigma(A)$ or (b) $R(\lambda I - A)$ is not closed or (c) $\bigcup_{j \geq 1} N((\lambda I - A)^j)$ is not finite dimensional. Recall that $\lambda \in \mathbb{C}$ is called an eigenvalue of A if $N(\lambda I - A) \neq \{0\}$ and λ is said to be of finite algebraic multiplicity m if $m = \dim(\bigcup_{j \geq 1} N((\lambda I - A)^j)) < \infty$, so case (c) above amounts to saying that λ is an eigenvalue of A which does not have finite algebraic multiplicity. One can also define

$$\text{ess}(A) = \{\lambda \in \mathbb{C} | \lambda I - A \text{ is not Fredholm of index } 0\}.$$

F. Wolf [46] defines $\text{ess}(A) = \{\lambda \in \mathbb{C} | \lambda I - A \text{ is not Fredholm}\}$ and T. Kato defines $\text{ess}(A) = \{\lambda \in \mathbb{C} | \lambda I - A \text{ is not semi-Fredholm}\}$.

By using classical results of Gohberg and Krein [15] and theorems in [20] concerning semi-Fredholm operators, one can prove that, if Z is an infinite dimensional, complex Banach space, $\text{ess}(A)$ is a nonempty subset of $\sigma(A)$, the spectrum of A . Furthermore, even though the various definitions of $\text{ess}(A)$ are inequivalent,

$$\rho(A) := \sup\{|\lambda| : \lambda \in \text{ess}(A)\}. \tag{3.5}$$

gives the same number for any of the previous definitions of $\text{ess}(A)$. The number $\rho(A)$ is called the "radius of the essential spectrum of A ". Note that $\text{ess}(A)$ is empty if $\dim(Z) < \infty$, and in this case we define $\rho(A) = 0$.

If $\lambda \in \sigma(A)$ and $|\lambda| > \rho(A)$, it is known that $\lambda I - A$ is Fredholm of index 0, λ is an isolated point of $\sigma(A)$ and λ is an eigenvalue of A of finite algebraic multiplicity. Thus, if $\rho(A) < r(A)$ one obtains nontrivial information concerning $\sigma(A) \cap \{\lambda \in \mathbb{C} | \rho(A) < |\lambda| \leq r(A)\}$.

If Z is a infinite dimensional complex Banach space, $A : Z \rightarrow Z$ is a bounded linear map and α denotes the Kuratowski MNC, it follows from

results in [34] that

$$\rho(A) = \limsup_{k \rightarrow \infty} (\alpha(A^k(B_1)))^{\frac{1}{k}}, \tag{3.6}$$

where, $B_1 := \{x \in Z \mid \|x\| \leq 1\}$. If β is a homogeneous MNC equivalent to α , one easily obtains from (3.6) that

$$\rho(A) = \limsup_{k \rightarrow \infty} (\beta(A^k(B_1)))^{\frac{1}{k}}. \tag{3.7}$$

It is proved in [29] that Eq. (3.7) is valid for any homogeneous MNC β on X , even if β is not equivalent to α . However, it is also proved in [29] that various other formulas in [34] do not generalize in a straightforward manner to homogeneous MNC's β which are not equivalent to α . Note that Eqs. (3.6) and (3.7) give $\rho(A) = 0$ when Z is finite dimensional.

In our applications, Z will be a real Banach space and $A : Z \rightarrow Z$ a bounded linear map. Recall that $\hat{Z} := \{(u, v) \mid u, v \in Z\}$ is the complexification of Z , that we identify (u, v) with $u + iv, i = \sqrt{-1}$ and that $\hat{A}(u + iv) := Au + iAv$ defines a bounded, complex linear map of $\hat{Z} \rightarrow \hat{Z}$. If $u + iv \in \hat{Z}$, we define $\Re(u + iv) = u$ and if $\hat{S} \subset \hat{Z}$ we define $\Re(\hat{S}) = \{\Re(\hat{z}) \mid \hat{z} \in \hat{S}\}$; and if $\|\cdot\|$ denotes the norm on \hat{Z} and $|\cdot|$ denotes the norm on Z

$$\|\hat{z}\| := \sup_{0 \leq \theta \leq 2\pi} |\Re(e^{-i\theta}\hat{z})|. \tag{3.8}$$

If β is a homogeneous MNC on Z , it is observed in [29] that one can define a homogeneous MNC $\hat{\beta}$ on \hat{Z} by defining, for \hat{S} a bounded subset of \hat{Z} ,

$$\hat{\beta}(\hat{S}) = \sup_{0 \leq \theta \leq 2\pi} \beta(\Re(e^{-i\theta}\hat{S})). \tag{3.9}$$

If α is the Kuratowski MNC on Z , it is proved in Proposition 11 of [29] that $\hat{\alpha}$ given by Eq. (3.9) gives the Kuratowski MNC on \hat{Z} . By using Eq. (3.9) it is not hard to show that

$$\hat{\beta}(\hat{A}^m(\hat{B}_1)) = \beta(A^m(B_1)), \tag{3.10}$$

where $\hat{B}_1 := \{\hat{z} \in \hat{Z} \mid \|\hat{z}\| \leq 1\}$ and $B_1 := \{z \in Z \mid |z| \leq 1\}$. It follows that

$$\rho(\hat{A}) = \limsup_{m \rightarrow \infty} (\beta(A^m(B_1)))^{\frac{1}{m}} = \limsup_{m \rightarrow \infty} (\alpha(A^m(B_1)))^{\frac{1}{m}}. \tag{3.11}$$

We shall write $\rho(A) := \rho(\hat{A})$ and call $\rho(A)$ the essential spectral radius of A .

We also need to recall an old generalization [37] of the classical Krein–Rutman theorem [23]. We refer the reader to [7, 27, 37–39, 42, 43, 45] for some of the many related results. If Z is a real Banach space, $C \subset Z$ is called a closed cone (with vertex at 0) if C is a closed convex set, $\{tx \mid x \in C\} \subset C$ for all $t \geq 0$ and $C \cap (-C) = \{0\}$, where $-C := \{-x \mid x \in C\}$. A closed cone C induces a partial order \leq on Z by $x \leq y$ iff $y - x \in C$.

The cone C is called “normal” if there exists a constant M such that $\|x\| \leq M\|y\|$ whenever $x, y \in C$ and $x \leq y$. The cone C is “reproducing”

if $Z = \{x - y \mid x, y \in C\}$ and C is “total” if $Z = \text{closure}\{x - y \mid x, y \in C\}$. If Z is infinite dimensional, it may easily happen that C is total but not reproducing. If Z^* is the dual Banach space of Z and C is a closed, total cone in Z , then $C^* := \{\theta \in C^* \mid \theta(u) \geq 0 \forall u \in C\}$ is a closed cone in Z^* .

If C is a total, closed cone in a real Banach space Z and $A : Z \rightarrow Z$ is bounded, compact linear map such that $A(C) \subset C$ and $r(A) > 0$, where $r(A)$ denotes the spectral radius of A , the classical Krein–Rutman theorem (see [23]) implies that there exists $u \in C \setminus \{0\}$ with

$$A(u) = r(A)u.$$

Furthermore, if $C^* = \{\theta \in Z^* \mid \theta(u) \geq 0 \forall u \in C\}$ and $A^* : Z^* \rightarrow Z^*$ denotes the Banach space adjoint of A , $A^*(C^*) \subset C^*$ and there exists $\theta \in C^* \setminus \{0\}$ with

$$A^*(\theta) = r(A)\theta.$$

Notice that because A is assumed compact with $r(A) > 0$, $\rho(A) = 0 < r(A)$, where $\rho(A)$ denotes the essential spectral radius of A .

The following theorem is proved in Corollary 2.2 of [37] and is a direct generalization of the Krein–Rutman theorem.

Theorem 3.1. (See Corollary 2.2. in [37]). *Let Z be a real Banach space and let $A : Z \rightarrow Z$ be a bounded linear map such that $\rho(A) < r(A)$, where $\rho(A)$ denotes the essential spectral radius of A (see Eq. (3.11)) and $r(A)$ denotes the spectral radius of A . Assume that C is a closed, total cone in Z and $A(C) \subset C$. Then there exist $v \in C \setminus \{0\}$ and $\theta \in C^* \setminus \{0\}$ such that*

$$A(v) = r(A)v \quad \text{and} \quad A^*(\theta) = r(A)\theta. \tag{3.12}$$

4. The Existence of C^m Nonnegative Eigenvectors

In this section we shall use the notation of Sect. 2, so $G_j, 1 \leq j \leq p = |V|$ will denote bounded open subsets of \mathbb{R}^n , m is a fixed positive integer and Y and X are real Banach spaces as in Eqs. (2.11) and (2.17) and $K \subset Y$ is a closed cone as in Eq. (2.13). We shall define a bounded linear operator $\Lambda : X \rightarrow X$ such that $\Lambda(K \cap X) \subset K \cap X$; Λ is of a type sometimes called a “Perron–Frobenius operator”. The key difficulty will be to prove that $\rho(\Lambda) < r(\Lambda)$.

For the reader’s convenience we list here hypotheses and notation which we shall use in this section.

We continue to use the notation of Sect. 2.

We assume that H1.1 and H1.2 are satisfied. We shall also need the following additional assumptions.

(H4.1) For each $(j, e) \in \Gamma, b_{(j,e)} \in C^m(\bar{G}_j) := X_j$ where $m \geq 1$. For all $x \in \bar{G}_j, b_{(j,e)}(x) \geq 0$ and for all $x \in \bar{G}_j$,

$$\sum_{e \in \mathcal{E}_j} b_{(j,e)}(x) > 0,$$

where \mathcal{E}_j is as in Eq. (2.2).

If H is a bounded open subset of \mathbb{R}^n and $\theta : H \rightarrow \mathbb{R}^n$,

$$\theta(x) = (\theta_1(x), \theta_2(x), \dots, \theta_n(x)),$$

we shall say that $\theta \in C^m(\bar{H})$ if $\theta_j \in C^m(\bar{H})$ for $1 \leq j \leq n$.

(H4.2) For each $(j, e) \in \Gamma, \theta_{(j,e)} : G_j \rightarrow \mathbb{R}^n$ and $\theta_{(j,e)} \in C^m(\bar{G}_j)$, where $m \geq 1$. Furthermore, $\theta_{(j,e)}(G_j) \subset G_{\alpha(j,e)}$, where $\alpha : \Gamma \rightarrow V$ is as in Sect. 2.

Assuming (H1.1), (H1.2), (H3.1) and (H3.2) we define a bounded linear map $L : Y \rightarrow Y$ (Y as in Eq. (2.11)) by $L(f_2, f_2, \dots, f_n) = (g_1, g_2, \dots, g_n)$, where

$$g_j(x) := (L(f)(x))_j := \sum_{(j,e) \in \Gamma} b_{(j,e)}(x) f_{\alpha(j,e)}(\theta_{(j,e)}(x)). \tag{4.1}$$

If $f = (f_1, f_2, \dots, f_n) \in X$ (X as in Eq. (2.17)), then $L(f) \in X$. Since $L : Y \rightarrow Y$ is a bounded linear operator, as previously noted, L defines a bounded linear map $\Lambda : X \rightarrow X$ by $\Lambda(f) = L(f)$ for $f \in X$.

We shall need to consider the μ -th iterates L^μ and Λ^μ of L and Λ . Following notation in Section 3 of [40], let $\bar{\Gamma}^{(\mu)}$ and $\bar{\Gamma}_j^{(\mu)}$ be given by Eqs. (2.5) and (2.6). Given $[(j_1, e_1), (j_2, e_2), \dots, (j_\mu, e_\mu)] \in \bar{\Gamma}^{(\mu)}$, we shall write $J := (j_1, j_2, \dots, j_\mu), E := (e_1, e_2, \dots, e_\mu)$; and we shall use (J, E) to denote $[(j_1, e_1), (j_2, e_2), \dots, (j_\mu, e_\mu)]$. We have already defined $b_{(J,E)}(x) := b_{(j_1, e_1)}(x)$ and $\theta_{(J,E)}(x) = \theta_{(j_1, e_1)}(x)$ for $\mu = 1, x \in G_{j_1}$. Arguing inductively, if for some $\mu > 1$ we have defined $b_{(J', E')}$ and $\theta_{(J', E')}$ for $J' = (j'_1, j'_2, \dots, j'_{\mu-1}), E' = (e'_1, e'_2, \dots, e'_{\mu-1})$ and $(J', E') \in \bar{\Gamma}^{(\mu-1)}$, then for $j_\mu = \alpha(j'_{\mu-1}, e'_{\mu-1})$ and e_μ such that $(j_\mu, e_\mu) \in \Gamma, J := (J', j_\mu)$ and $E := (E', e_\mu)$, define for $x \in \bar{G}_{j'_1}$

$$b_{(J,E)}(x) = b_{(J', E')}(x) b_{(j_\mu, e_\mu)}(\theta_{(J', E')}(x)) \tag{4.2}$$

and

$$\theta_{(J,E)}(x) = \theta_{(j_\mu, e_\mu)}(\theta_{(J', E')}(x)). \tag{4.3}$$

This defines $b_{(J,E)}$ and $\theta_{(J,E)}$ inductively for all $(J, E) \in \bar{\Gamma}^{(\mu)}$.

It is proved in Section 3 of [40] that for $f \in Y$ and $x \in \bar{G}_j$,

$$(L^\mu(f))_j(x) = \sum b_{(J,E)}(x) f_{\alpha(j_\mu, e_\mu)}(\theta_{(J,E)}(x)), \tag{4.4}$$

where the summation in Eq. (4.6) is taken over all $J = (j_1, j_2, \dots, j_\mu), E = (e_1, e_2, \dots, e_\mu)$ with $(J, E) \in \bar{\Gamma}_j^{(\mu)}$ and $j = j_1$. Obviously, if $f \in X$, Eq. (4.4) holds if we substitute Λ for L .

It remains to make a crucial assumption on the maps $\theta_{(J,E)}$ for $(J, E) \in \bar{\Gamma}^\mu$. Assume that \mathbb{R}^n is given the norm $\| \cdot \|_1$. If j_1 denotes the first coordinate of J and $x \in G_{j_1}$, let $D\theta_{(J,E)}(x)$ denote the Jacobian matrix of $\theta_{(J,E)}$, so $D\theta_{(J,E)}(x)$ defines a linear map of $(\mathbb{R}^n, \| \cdot \|_1)$ to $(\mathbb{R}^n, \| \cdot \|_1)$. We let $\|D\theta_{(J,E)}(x)\|_1$ denote the norm of this linear map, so using Eq. (2.9) we see that

$$\|D\theta_{(J,E)}(x)\|_1 = \max_{1 \leq j \leq n} \left(\sum_{i=1}^n \left| \frac{\partial \theta_{(J,E)i}(x)}{\partial x_j} \right| \right), \tag{4.5}$$

where $\theta_{(J,E)}(x) = (\theta_{(J,E)1}(x), \theta_{(J,E)2}(x), \dots, \theta_{(J,E)n}(x))$.

We assume

(H4.3) There exists a constant $M_1 > 0$ and a constant $c, 0 \leq c < 1$, such that for all $(J, E) \in \bar{\Gamma}^\mu$, all $\mu \geq 1$ and all x in the domain of $\theta_{(J,E)}$,

$$\|D\theta_{(J,E)}(x)\|_1 \leq M_1 c^\mu, \tag{4.6}$$

where $\|D\theta_{(J,E)}(x)\|_1$ is given by Eq. (4.5).

Remark 4.1. We could have taken a different norm $|\cdot|$ on \mathbb{R}^n . Then the norm of the linear map $D\theta_{(J,E)}(x) : (\mathbb{R}^n, |\cdot|) \rightarrow (\mathbb{R}^n, |\cdot|), |D\theta_{(J,E)}(x)|$ would not be given by Eq. (4.5). However, since all norms on finite dimensional vector spaces are equivalent, one can prove that there is a constant M_2 such that for all $\mu \geq 1$, all $(J, E) \in \bar{\Gamma}^\mu$ and all x in the domain of $\theta_{(J,E)}$, $|D\theta_{(J,E)}(x)| \leq M_2 c^\mu$.

In other words, assumption (H4.3) is actually independent of the norm on \mathbb{R}^n , after modification of the constant M_1 .

We begin with an elementary lemma.

Lemma 4.2. *Let $(Z_j, \|\cdot\|_j), 1 \leq j \leq p$, be Banach spaces over the same scalar field (\mathbb{R} or \mathbb{C}) and let $Z = \prod_{j=1}^p Z_j = \{(f_1, f_2, \dots, f_p) : f_j \in Z_j \text{ for } 1 \leq j \leq p\}$. For $f = (f_1, f_2, \dots, f_p) \in Z$, define $\|f\| = \max\{\|f_j\|_j : 1 \leq j \leq p\}$ and note that Z is a Banach space. Define projections $Q_j : Z \rightarrow Z_j$ by $Q_j((f_1, f_2, \dots, f_p)) = f_j$ for $1 \leq j \leq p$. Let α denote the Kuratowski MNC on $(Z, \|\cdot\|)$ and let α_j denote the Kuratowski MNC on $(Z_j, \|\cdot\|_j)$.*

If S is a bounded subset of Z

$$\alpha(S) = \max\{\alpha_j(Q_j(S)) : 1 \leq j \leq p\}. \tag{4.7}$$

Proof. The reader can verify that $\|Q_j\| = 1$, so $\alpha_j(Q_j(S)) \leq \alpha(S)$ for $1 \leq j \leq p$ and the right hand side of Eq. (4.7) is less than or equal to the left hand side.

Conversely, let $d = \max_{1 \leq j \leq p} \alpha_j(Q_j(S))$. If $\varepsilon > 0$ and $1 \leq j \leq p$ it follows from the definition of α_j that $Q_j(S) = \bigcup_{i=1}^{k_j} T_{i,j}$, where $k_j < \infty$ and $\|f_j - g_j\|_j \leq d + \varepsilon$ whenever $f_j, g_j \in T_{i,j}, 1 \leq i \leq k_j$. Consider all p -tuples $I = (i_1, i_2, \dots, i_p)$ with $1 \leq i_j \leq k_j$ for $1 \leq j \leq p$. For each such p -tuple I , define

$$S_I = \{f = (f_1, f_2, \dots, f_p) \mid f_j \in T_{i_j,j} \text{ for } 1 \leq j \leq p\}.$$

If $f \in S_I$ and $g \in S_I, \|f_j - g_j\|_j \leq d + \varepsilon$ for $1 \leq j \leq p$, so $\|f - g\| \leq d + \varepsilon$. It follows that the diameter of S_I in $(Z, \|\cdot\|)$ is less than or equal to $d + \varepsilon$. If \mathcal{I} denotes the finite collection of such p -tuples I , our construction shows that

$$S = \bigcup_{I \in \mathcal{I}} S_I,$$

So $\alpha(S) \leq d + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\alpha(S) \leq d$ and Eq. (4.7) holds. \square

We shall apply Lemma 4.2 to X in Eq. (2.17) and $X_j := C^m(\bar{G}_j), 1 \leq j \leq p$. For the remainder of this section, α will denote the Kuratowski MNC on X and α_j will denote the Kuratowski MNC on $X_j = C^m(\bar{G}_j)$. As in Lemma 4.2, if $f = (f_1, f_2, \dots, f_p) \in X$ and $Q_j(f) := f_j \in X_j$, then Lemma 4.2 implies that for any bounded subset of X ,

$$\alpha(S) = \max\{\alpha_j(Q_j(S)) : 1 \leq j \leq p\}.$$

If Y and $Y_j := C(\bar{G}_j), 1 \leq j \leq p$, are as in Eq. (2.11), ν will denote the Kuratowski MNC on Y and ν_j the Kuratowski MNC on Y_j . If $f = (f_1, f_2, \dots, f_p) \in Y$, we shall abuse notation slightly and write $Q_j(f) = f_j$. If T is a bounded subset of Y , Lemma 4.2 implies that

$$\nu(T) = \max\{\nu_j(Q_j(T)) : 1 \leq j \leq p\}.$$

Lemma 4.3. *Let $G_j, 1 \leq j \leq p$, be bounded, open subsets of \mathbb{R}^n and assume that H1.2 holds. If S is a bounded set in $X_j := C^m(\bar{G}_j)$, then*

$$\alpha_j(S) = \max_{\|\beta\|_1 \leq m} \nu_j(\{D^\beta f : f \in S\}) = \max_{\|\beta\|_1 = m} \nu_j(\{D^\beta f : f \in S\}) \quad (4.8)$$

where β in Eq. (4.8) denotes a multi-index.

Proof. By Lemma 2.4 and Property (A1) of ν_j (see Sect. 3), if $\|\beta\|_1 < m, \nu_j(\{D^\beta f : f \in S\}) = 0$. The remainder of the proof follows by the same argument used in Lemma 4.2, but we provide the details for completeness. For each multi-index β with $\|\beta\|_1 \leq m$, define $\pi_\beta : C^m(\bar{G}_j) := X_j \rightarrow C(\bar{G}_j) := Y_j$ by $f \rightarrow D^\beta f$. By definition of the norms on X_j and $Y_j, \|\pi_\beta\| \leq 1$, and it follows that

$$\alpha_j(S) \geq \max_{\|\beta\|_1 \leq m} \nu_j(\pi_\beta(S)) := d.$$

Select $\varepsilon > 0$. By definition of ν_j , for each β with $\|\beta\|_1 \leq m$, there exists a positive integer k_β and sets $T_{i,\beta}, 1 \leq i \leq k_\beta$, with

$$\pi_\beta(S) = \bigcup_{i=1}^{k_\beta} T_{i,\beta}$$

and diameter $(T_{i,\beta}) < d + \varepsilon$, where the diameter is taken in the metric on Y_j . There are N multi-indices β with $\|\beta\|_1 \leq m, N < \infty$; and we label these multi-indices $\beta^1, \beta^2, \dots, \beta^j, \dots, \beta^N$ and write $k_j := k_{\beta^j}$. Let \mathcal{I} denote the finite collection of all N -tuples $I = (i_1, i_2, \dots, i_N)$, where $1 \leq i_j \leq k_j$ for $1 \leq j \leq N$. If $I \in \mathcal{I}$, define $T_I = \{f \in S \mid \pi_{\beta^j}(f) \in T_{i_j, \beta^j}\}$, where $I = (i_1, i_2, \dots, i_N)$. One can check that

$$S = \bigcup_{I \in \mathcal{I}} T_I$$

and that the diameter of T_I in the metric on X_j is less than $d + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows that $\alpha_j(S) \leq d$ and completes the proof. \square

Lemma 4.4. *Assume Hypotheses (H1.1), (H1.2), (H4.1) and (H4.2).*

Then we have that $r(\Lambda) \geq r(L) > 0$ and

$$\|L^\mu\| = \max_{1 \leq j_1 \leq p} \left(\sup_{x \in G_{j_1}} \left(\sum_{(J,E) \in \bar{\Gamma}_{j_1}^{(\mu)}} b_{(J,E)}(x) \right) \right). \tag{4.9}$$

Proof. Equation (4.9) follows from Lemma 2.1, Eq. (4.4) and the assumption in (H4.1) that $b_{(j,e)}(x) \geq 0$ for all $x \in G_j$. The inequality $r(\Lambda) \geq r(L)$ follows from Lemma 2.1.

To prove that $r(\Lambda) > 0$, it suffices to prove that $r(L) > 0$. (H3.1) implies that there is a positive constant $\delta > 0$ such that, for all $x \in G_j$ and $1 \leq j \leq p$

$$\sum_{e \in \mathcal{E}_j} b_{(j,e)}(x) \geq \delta.$$

In the notation of Lemma 2.1, this implies that

$$L(u) \geq \delta u,$$

which implies that for all $\mu \geq 1$

$$L^\mu(u) \geq \delta^\mu u,$$

so $\|L^\mu\| \geq \delta^\mu$ and $r(L) = \lim_{\mu \rightarrow \infty} \|L^\mu\|^{(\frac{1}{\mu})} \geq \delta$. □

Our next lemma provides a crucial tool for all our subsequent results.

Lemma 4.5. *Assume that hypotheses (H1.1), (H1.2) and (H4.1)–(H4.3) are satisfied and that L and Λ are given by Eq. (4.1). If B is the unit ball in X , μ is a positive integer and M_1 and c are as in (H4.3), α denotes the Kuratowski MNC on X and m is as in (H4.1),*

$$\alpha(\Lambda^\mu(B)) \leq 2(nM_1)^m c^{m\mu} \|L^\mu\|. \tag{4.10}$$

Proof. Fix $\mu \geq 1$. By Lemma 4.2 and Eq. (4.4), there exists $j, 1 \leq j \leq p$, such that

$$\alpha(\Lambda^\mu(B)) = \alpha_j \left(\left\{ \sum_{(J,E) \in \bar{\Gamma}_j^{(\mu)}} b_{(J,E)}(\cdot) f_{\alpha(j_\mu, e_\mu)}(\theta_{(J,E)}(\cdot)) \mid f \in B \right\} \right),$$

where $f = (f_1, f_2, \dots, f_p)$, $J = (j_1, j_2, \dots, j_\mu)$ and $E = (e_1, e_2, \dots, e_\mu)$ and $j_1 = j$.

By Lemma 4.2, we obtain

$$\alpha(\Lambda^\mu(B)) = \max_{\|\beta\|_1=m} \nu_j \left(\left\{ D^\beta \left(\sum_{(J,E) \in \bar{\Gamma}_j^{(\mu)}} b_{(J,E)}(\cdot) f_{\alpha(j_\mu, e_\mu)}(\theta_{(J,E)}(\cdot)) \mid f \in B \right) \right\} \right),$$

where β is a multi-index. If $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ is a multi-index, we shall write $\gamma \leq \beta$ if $\gamma_i \leq \beta_i$ for $1 \leq i \leq n$, and a calculation gives for $f \in B$ that

$$\begin{aligned}
 & D^\beta \left(\sum_{(J,E) \in \bar{\Gamma}_j^{(\mu)}} b_{(J,E)}(x) f_{\alpha(j_\mu, e_\mu)}(\theta_{(J,E)}(x)) \right) \\
 &= \sum_{\gamma \leq \beta, \gamma \neq 0} c_{\beta, \gamma} \left(\sum_{(J,E) \in \bar{\Gamma}_j^{(\mu)}} (D^\gamma b_{(J,E)})(x) (D^{\beta-\gamma} f_{\alpha(j_\mu, e_\mu)} \circ \theta_{(J,E)})(x) \right) \\
 &\quad + \sum_{(J,E) \in \bar{\Gamma}_j^{(\mu)}} b_{(J,E)}(x) (D^\beta f_{\alpha(j_\mu, e_\mu)} \circ \theta_{(J,E)})(x)
 \end{aligned}$$

where $c_{\beta, \gamma} = \beta! / (\gamma! (\beta - \gamma)!)$.

For notational convenience, for a fixed multi-index β with $\|\beta\|_1 = m$, we write

$$\begin{aligned}
 S &= \left\{ D^\beta \left(\sum_{(J,E) \in \bar{\Gamma}_j^{(\mu)}} b_{(J,E)}(\cdot) f_{\alpha(j_\mu, e_\mu)}(\theta_{(J,E)}(\cdot)) \right) \mid f \in B \right\}, \\
 T &= \left\{ \sum_{\gamma \leq \beta, \gamma \neq 0} c_{\beta, \gamma} \left(\sum_{(J,E) \in \bar{\Gamma}_j^{(\mu)}} D^\gamma b_{(J,E)} \right) (\cdot) (D^{\beta-\gamma} f_{\alpha(j_\mu, e_\mu)} \circ \theta_{(J,E)})(\cdot) \mid f \in B \right\},
 \end{aligned}$$

and

$$S_1 = \left\{ \sum_{(J,E) \in \bar{\Gamma}_j^{(\mu)}} b_{(J,E)}(\cdot) (D^\beta f_{\alpha(j_\mu, e_\mu)} \circ \theta_{(J,E)})(\cdot) \mid f \in B \right\}. \tag{4.11}$$

Lemma 2.4 implies that $\{D^\delta(f_{\alpha(j_\mu, e_\mu)} \circ \theta_{(J,E)}) \mid \|\delta\| < m, (J, E) \in \bar{\Gamma}_j^{(\mu)}, f \in B\}$ is bounded and equicontinuous, and it follows that T is a bounded, equicontinuous family of functions in $Y_j = C(\bar{G}_j)$, so T has compact closure in Y_j . Property (A1) of the Kuratowski MNC implies that $\nu_j(T) = 0$. Because $S \subset S_1 + T$, property (A6) implies that $\nu_j(S) \leq \nu_j(S_1 + T) \leq \nu_j(S_1)$; and since $S_1 \subset S$, we must have that $\nu_j(S) = \nu_j(S_1)$.

For a fixed $(J, E) \in \bar{\Gamma}_j^{(\mu)}$, write $u(x) = u = \theta_{(J,E)}(x)$, so $u_k(x) = \theta_{(J,E)k}(x)$. We need to evaluate $D^\beta f_{\alpha(j_\mu, e_\mu)}(u)$ by repeated applications of the chain rule. For notational convenience, write $g = f_{\alpha(j_\mu, e_\mu)}$. The first application of the chain rule gives

$$\sum_{k_1=1}^n \frac{\partial g}{\partial u_{k_1}} \frac{\partial u_{k_1}}{\partial x_{p_1}} = \frac{\partial}{\partial x_{p_1}} g(u), \tag{4.12}$$

where $p_1, 1 \leq p_1 \leq n$, is the smallest positive integer p such that $\beta_p > 0$.

In general, let $p_j, 1 \leq p_j \leq n$, be the integers p for which $\beta_p > 0$. Assume that $p_1 \leq p_2 \leq \dots \leq p_m$ and p_j is repeated with multiplicity β_j . Thus, we shall write $p_1 = p_2 = p_3$ if $\beta_{p_1} = 3$. By using Lemma 2.4 one can see that in repeated applications of the chain rule, starting with (4.12), the only terms which will affect the value of $\nu_j(S_1)$ are terms in which partial differentiation

has been applied to $g(u)$ m times. Other terms will lead to sets with compact closure in Y_j , like the set T before. After m applications of the chain rule (corresponding to $|\beta| = m$) and after only the terms in which partial differentiation has been applied to g m times are retained, we obtain

$$\Phi(f; (J, E))(x) := \sum_{k_m=1}^n \sum_{k_{m-1}=1}^n \cdots \sum_{k_1=1}^n \left(\frac{\partial^m g}{\partial u_{k_m} \partial u_{k_{m-1}} \cdots \partial u_{k_1}} \right) (u) \left(\frac{\partial u_{k_m}(x)}{\partial x_{p_m}} \right) \left(\frac{\partial u_{k_{m-1}}(x)}{\partial x_{p_{m-1}}} \right) \cdots \left(\frac{\partial u_{k_1}(x)}{\partial x_{p_1}} \right). \tag{4.13}$$

This gives

$$\nu_j(S_1) = \nu_j \left(\left\{ \sum_{(J,E) \in \Gamma_j^{(\mu)}} b_{(J,E)}(\cdot) \Phi(f; (J, E))(\cdot) : f \in B \right\} \right). \tag{4.14}$$

Because $f \in B$, we have, for the term $(\frac{\partial^m g}{\partial u_{k_m} \partial u_{k_{m-1}} \cdots \partial u_{k_1}})(u)$ in Eq. (4.13) the estimate

$$\left| \left(\frac{\partial^m g}{\partial u_{k_m} \partial u_{k_{m-1}} \cdots \partial u_{k_1}} \right) (u) \right| \leq 1. \tag{4.15}$$

By using (H3.3) and Eqs. (4.5) and (4.6), we obtain the estimate

$$\left| \frac{\partial u_{k_i}(x)}{\partial x_{p_i}} \right| \leq M_1 c^\mu \tag{4.16}$$

for the terms $\frac{\partial u_{k_i}(x)}{\partial x_{p_i}}$ in Eq. (4.13). Since there are n^m terms in the summation in Eq. (4.13), we conclude that for all $x \in G_j$

$$|\Phi(f; (J, E))(x)| \leq n^m (M_1 c^\mu)^m = (nM_1)^m c^{\mu m}. \tag{4.17}$$

It follows that the set $\{\sum_{(J,E) \in \Gamma_j^{(\mu)}} b_{(J,E)}(\cdot) \Phi(f; (J, E))(\cdot) : f \in B\}$ is contained in a ball of radius $C_j (nM_1)^m c^{\mu m}$, where $C_j := \sup\{x \in G_j | \sum_{(J,E) \in \Gamma_j^{(\mu)}} b_{(J,E)}(x)\}$. By Lemma 4.3, $C_j \leq \|L^\mu\|$, so the diameter of the ball is less than or equal to $2\|L^\mu\|(nM_1)^m c^{\mu m}$ and

$$\nu_j(S) = \nu_j(S_1) \leq 2\|L^\mu\|(nM_1)^m c^{\mu m}. \tag{4.18}$$

Using Eqs. (4.18) and (4.11), we obtain Eq. (4.10). □

With these preliminaries we can easily obtain a theorem which will play a crucial role in our further work.

Theorem 4.6. *Assume that hypotheses and notation are as Sect. 2. In addition assume that hypotheses (H4.1)–(H4.3), which are stated at the beginning of this section, are satisfied and that m is a positive integer as in (H4.1). Let Y, K and X be as defined in Eqs. (2.11), (2.13) and (2.17) and let $L : Y \rightarrow Y$ and $\Lambda : X \rightarrow X$ be the bounded linear operators defined by Eq. (4.1) at the beginning of this section. If $\rho(\Lambda)$ denotes the essential spectral radius of*

$\Lambda, r(\Lambda)$ the spectral radius of Λ and $r(L)$ the spectral radius of L , we have for $0 \leq c < 1$ as in (H4.3) (see Eq. (4.6)) that

$$\rho(\Lambda) \leq c^m r(L) < r(L) = r(\Lambda). \tag{4.19}$$

Furthermore, there exists $v \in (K \cap X) \setminus \{0\}$ such that

$$\Lambda(v) = r(\Lambda)v.$$

If C is any closed, total cone in X such that $\Lambda(C) \subset C$, there exists $w \in C \setminus \{0\}$ such that $\Lambda(w) = r(\Lambda)w$.

Proof. Let $B := \{f \in X \mid \|f\| \leq 1\}$. By Eq. (3.11) in Sect. 3 we have

$$\rho(\Lambda) = \limsup_{\mu \rightarrow \infty} (\alpha(\Lambda^\mu(B)))^{\frac{1}{\mu}},$$

where α denotes the Kuratowski MNC on X . However, Lemma 4.5 implies that

$$\begin{aligned} \limsup_{\mu \rightarrow \infty} (\alpha(\Lambda^\mu(B)))^{\frac{1}{\mu}} &\leq \limsup_{\mu \rightarrow \infty} (2(nM_1)^m c^{m\mu} \|L^\mu\|)^{\frac{1}{\mu}} \\ &= c^m r(L). \end{aligned}$$

Since Lemma 4.4 implies that $r(L) > 0$ and $r(L) \leq r(\Lambda), \rho(\Lambda) < r(L) \leq r(\Lambda)$. It is easy to check that $C_1 := K \cap X$ is a closed, reproducing cone in X and $\Lambda(C_1) \subset C_1$, so Theorem 3.1 in Sect. 3 implies that there exists $v \in C_1 \setminus \{0\}$ with $\Lambda(v) = r(\Lambda)v$. Since $v \in K \setminus \{0\} \subset Y, L(v) = r(\Lambda)v$, which implies that $r(\Lambda) = r(L)$. The final statement of Theorem 4.6 follows immediately from Theorem 3.1. □

Remark 4.7. If $\hat{\Lambda} : \hat{X} \rightarrow \hat{X}$ denotes the complexification of Λ and \hat{X} the complexification of X , the results described in Sect. 3 imply that $\rho(\hat{\Lambda}) = \rho(\Lambda)$ and $r(\hat{\Lambda}) = r(\Lambda)$. In particular if $\lambda \in \sigma(\hat{\Lambda})$ and $|\lambda| > \rho(\Lambda), \lambda$ is an isolated point in $\sigma(\hat{\Lambda})$ and λ is an eigenvalue of finite algebraic multiplicity. It follows that, writing $r := r(\Lambda), \dim(\bigcup_{p \geq 1} N((rI - \Lambda)^p)) < \infty$.

Remark 4.8. Suppose in Theorem 4.6 we replace (H4.1) by a weakened version: (H4.1)': for each $(j, e) \in \Gamma, b_{(j,e)} \in C^m(\tilde{G}_j) := X_j$ and $b_{(j,e)}(x) \geq 0$ for all $x \in \tilde{G}_j$.

The argument in Lemma 4.4 shows that if (H1.1), (H1.2) and (H4.1)′ hold, then Eq. (4.9) is satisfied and $r(\Lambda) \geq r(L) \geq 0$. If (H1.1), (H1.2), (H4.1)′, (H4.2) and (H4.3) are satisfied, the proof of Lemma 4.5 still is valid and proves Eq. (4.10). If (H1.1), (H1.2), (H4.1)′, (H4.2) and (H4.3) are satisfied and if, in addition $r(L) > 0$, the proof of Theorem 4.6 remains valid and the conclusions of Theorem 4.6 still hold. If, however, $r(L) = 0$, it must be true that $r(\Lambda) = 0$. For if $r(\Lambda) > 0$, we would find that $\rho(\Lambda) < r(\Lambda)$, which would imply that Λ has an eigenvector in $K \cap X$ with eigenvalue $r(\Lambda) > 0$. But this would imply that L has an eigenvalue $r(\Lambda) > 0$, which would contradict $r(L) = 0$.

Our next result is an immediate consequence of Theorem 4.6, but it takes a simpler form because we are working in the iterated function case.

Corollary 4.9. *Let G be a bounded, open, mildly regular subset of \mathbb{R}^n . Let \mathcal{E} be a finite index set and m a positive integer.*

In addition make the following assumptions:

- (A1) *For each $e \in \mathcal{E}$, $b_e : G \rightarrow \mathbb{R}$ is a nonnegative function, $b_e \in C^m(\bar{G})$ and $\sum_{e \in \mathcal{E}} b_e(x) > 0$ for all $x \in \bar{G}$.*
- (A2) *For each $e \in \mathcal{E}$ and for $1 \leq j \leq n$, $\theta_e : G \rightarrow G$ and $\theta_{e,j} \in C^m(\bar{G})$, where $\theta_e(x) = (\theta_{e,1}(x), \theta_{e,2}(x), \dots, \theta_{e,n}(x))$.*
- (A3) *For each positive integer $\mu \geq 1$, define $\mathcal{E}_\mu := \{\omega = (e_1, e_2, \dots, e_\mu) \mid e_j \in \mathcal{E}, 1 \leq j \leq \mu\}$. For $\omega := (e_1, e_2, \dots, e_\mu) \in \mathcal{E}_\mu$, define $\theta_\omega = \theta_{e_\mu} \cdot \theta_{e_{\mu-1}} \dots \theta_{e_1}$. Assume that there exists constants M_1 and c , with $c < 1$, such that or all $x \in \bar{G}$, for all $\omega \in \mathcal{E}_\mu$ and for all $\mu \geq 1$,*

$$\|D\theta_\omega(x)\| \leq M_1 c^\mu.$$

Let Y denote the Banach space $C(\bar{G})$ and X the Banach space $C^m(\bar{G})$. Define a bounded linear operator $L : Y \rightarrow Y$ by

$$(Lf)(x) = \sum_{e \in \mathcal{E}} b_e(x) f(\theta_e(x)) \tag{4.20}$$

and define a bounded linear operator $\Lambda : X \rightarrow X$ by the same formula. If $\rho(\Lambda)$ denotes the essential spectral radius of Λ and $r(\Lambda)$ (respectively, $r(L)$) denotes the spectral radius of Λ (respectively of L), then

$$\rho(\Lambda) \leq c^m r(\Lambda) \quad \text{and} \quad 0 < r(\Lambda) = r(L) := r. \tag{4.21}$$

If K denotes the set of nonnegative functions in Y , there exists $v \in (K \cap X) \setminus \{0\}$ with $\Lambda(v) = rv$; and if C is any closed, total cone in X with $\Lambda(C) \subset C$, there exists $w \in C \setminus \{0\}$ with $\Lambda(w) = rw$. If $\sigma(\Lambda) \subseteq \mathbb{C}$ denotes the spectrum of the complexification $\hat{\Lambda}$ of Λ and if $z \in \sigma(\Lambda)$ and $|z| > \rho(\Lambda)$, then z is an isolated point of $\sigma(\Lambda)$ and z is an eigenvalue of $\hat{\Lambda}$ of finite algebraic multiplicity.

5. The Existence of C^m Strictly Positive Eigenvectors

If $v = (v_1, v_2, \dots, v_p) \in K \cap X$ is the eigenvector ensured by Theorem 4.6, we know that $v \neq 0$ and that $v_j(x) \geq 0$ for all $x \in G_j$. In the application to the computation of Hausdorff dimension (see [14, 40]) we need to know that $v_j(x) > 0$ for all $x \in \bar{G}_j$ and for $1 \leq j \leq p$. However, to obtain this strict positivity we shall need stronger assumptions on $b_{(j,e)}$ and $\theta_{(j,e)}$ for $(j, e) \in \Gamma$. We shall gather below several hypotheses which, together with our earlier assumptions, are sufficient to imply the strict positivity of the eigenvector v in Theorem 4.6. We shall always assume the hypotheses (see (H1.1) and (H1.2)) and notation of Sect. 2. In addition, we list here, for the reader's convenience, additional hypotheses which will be used in this section. Recall that hypotheses (H4.1)–(H4.3) are stated at the beginning of Sect. 4.

(H5.1) For all $(j, e) \in \Gamma$, $b_{(j,e)} \in C^m(\bar{G}_j) = X_j$ and $b_{(j,e)}(x) > 0$ for all $x \in \bar{G}_j$.

(H5.2) For $1 \leq j \leq p = |V|$, let $G_j \subset \mathbb{R}^n$ be bounded open sets as in Sect. 2. Assume that there exists a constant M such that if $1 \leq j \leq p$ and if $x, y \in G_j$, there exists a Lipschitz map $\psi : [0, 1] \rightarrow G_j$ with $\psi(0) = x, \psi(1) = y$ and $\int_0^1 \|\psi'(t)\| dt \leq M\|x - y\|$, where $\|\cdot\|$ is some fixed norm on \mathbb{R}^n .

(H5.3) Assume that (H1.1), (H1.2) and (H4.2) hold. In addition, assume that there exists an integer $\mu \geq 1$ and a constant κ with $0 \leq \kappa < 1$, such that for $1 \leq j \leq p = |V|$, for all $(J, E) \in \bar{\Gamma}_j^{(\mu)}$ and all $x, y \in \tilde{G}_j$

$$\|\theta_{(J,E)}(x) - \theta_{(J,E)}(y)\| \leq \kappa\|x - y\|.$$

We shall also need a condition which is directly analogous to the assumption of irreducibility for a $p \times p$ matrix M with nonnegative entries.

(H5.4) Let notation be as in Sect. 2. For any pair of integers $i, k \in V$, assume that there exists an integer $\nu = \nu(i, k) \geq 1$ and a sequence $(j_s, e_s) \in \Gamma$ for $1 \leq s \leq \nu$ with $\alpha(j_s, e_s) = j_{s+1}$ for $1 \leq s < \nu, j_1 = i$ and $\alpha(j_\nu, e_\nu) = k$.

Remark 5.1. Recall that a $p \times p$ matrix M with nonnegative entries m_{ij} is called “irreducible” if $p = 1$ or if $p > 1$ and for each pair of integers (i, k) with $1 \leq i \leq p$ and $1 \leq k \leq p$ there exists a positive integer $\nu = \nu(i, k)$ such that the (i, k) entry of M^ν is positive. The matrix M is called “primitive” if there exists a positive integer ν such that all entries of M^ν are positive.

If V, \mathcal{E}, Γ and α are as in §1 and (H1.1) is satisfied, define a nonnegative, $p \times p$ matrix $A = (a_{ik})$ by $a_{ik} > 0$ if there exists $e \in \mathcal{E}$ with $(i, e) \in \Gamma$ and $\alpha(i, e) = k$ and $a_{ik} = 0$ if there does not exist $e \in \mathcal{E}$ with $(i, e) \in \Gamma$ and $\alpha(i, e) = k$. We shall call A a “ $p \times p$ nonnegative matrix associated with (Γ, α, V) ”.

The reader can verify that (H5.4) is satisfied if and only if a $p \times p$ nonnegative matrix A associated with (Γ, α, V) is irreducible. The integer $\nu(i, k)$ in (H4.4) can be chosen to be independent of (i, k) if and only if A is primitive. If A is primitive, the graph directed iterated function system is called “strongly connected” in Definition 4.7 in [40].

Lemma 5.2. *Assume that (H1.1), (H1.2) and (H5.1) hold, where as usual we assume that $m \geq 1$ in (H4.1). Then for all $(j, e) \in \Gamma, x \rightarrow \log(b_{(j,e)}(x))$ is a Lipschitz map from \tilde{G}_j to \mathbb{R} .*

Proof. By the argument used in Lemma 2.4, there is a constant M_2 such that for all $x, y \in \tilde{G}_j$ with $\|x - y\|_1 \leq \eta, |b_{(j,e)}(x) - b_{(j,e)}(y)| \leq M_2\|x - y\|_1$. (H5.1) and the continuity of $b_{(j,e)}$ on \tilde{G}_j imply that there is a positive number δ such that $b_{(j,e)}(x) \geq \delta$ for all $(j, e) \in \Gamma$ and all $x \in \tilde{G}_j$. By the mean value theorem, for all $x, y \in \tilde{G}_j$ with $\|x - y\|_1 < \eta, |\log(b_{(j,e)}(x)) - \log(b_{(j,e)}(y))| \leq M_2(\frac{1}{\delta})\|x - y\|_1$.

If M_3 is chosen so that $|\log(b_{(j,e)}(x))| \leq M_3$ for all \tilde{G}_j , then if $x, y \in \tilde{G}_j$ and $\|x - y\|_1 \geq \eta,$

$$|\log(b_{(j,e)}(x)) - \log(b_{(j,e)}(y))| \leq M_3 \leq \left(\frac{M_3}{\eta}\right) \|x - y\|_1.$$

It follows that for all $x, y \in \bar{G}_j$,

$$|\log(b_{(j,e)}(x)) - \log(b_{(j,e)}(y))| \leq \max\left(\frac{M_3}{\eta}, \frac{M_2}{\delta}\right) \|x - y\|_1.$$

□

Note that since all norms on \mathbb{R}^n are equivalent, under the hypotheses of Lemma 5.2 the map $x \in G_j \rightarrow \log(b_{(j,e)}(x))$ is Lipschitz with respect to any norm on \mathbb{R}^n .

An argument similar to that in Lemma 5.2 shows that for all $(j, e) \in \Gamma$, the map $\theta_{(j,e)} : G_j \rightarrow G_{\alpha(j,e)}$ is Lipschitz. Since all norms on \mathbb{R}^n are equivalent, this statement is independent of which particular norm on \mathbb{R}^n is chosen.

Lemma 5.3. *Assume that (H1.1), (H1.2) and (H4.2) hold. Then for all $(j, e) \in \Gamma$, the map $x \in G_j \rightarrow \theta_{(j,e)}(x) \in G_{\alpha(j,e)}$ is Lipschitz.*

Proof. By assumption H4.2, there exists a constant M_1 such that $\|D\theta_{(j,e)}(x)\|_1 \leq M_1$ for all $x \in G_j$. By (H1.2), G_j is mildly regular, so there exist positive constants η and M as in Definition 2.3. If $x, y \in G_j$ and $\|x - y\|_1 < \eta$, there exists a Lipschitz map $\psi : [0, 1] \rightarrow G_j$ with $\psi(0) = x$ and $\psi(1) = y$ such that

$$\int_0^1 \|\psi'(t)\|_1 dt \leq M \|x - y\|_1.$$

We can assume that ψ is written as a column vector.

We have

$$\begin{aligned} \|\theta_{(j,e)}(y) - \theta_{(j,e)}(x)\|_1 &= \left\| \int_0^1 \frac{d}{dt} \theta_{(j,e)}(\psi(t)) dt \right\|_1 \\ &\leq \int_0^1 \|D\theta_{(j,e)}(\psi(t))\|_1 \|\psi'(t)\|_1 dt \\ &\leq M_1 \int_0^1 \|\psi'(t)\|_1 dt \leq M_1 M \|x - y\|_1. \end{aligned}$$

This gives the desired estimate if $\|x - y\|_1 < \eta$. If $\|x - y\|_1 \geq \eta$, the desired estimate follows by the same sort of argument used in Lemma 5.2. □

Remark 5.4. Before proceeding further it will be useful to make a few observations about (H3.3). By (H3.3) and Remark 4.1, (H3.3) is equivalent to the assumption that there exists $M > 0$ and $c, 0 < c < 1$, such that

$$\|D\theta_{(J,E)}(x)\| \leq M c^\mu$$

for all $(J, E) \in \bar{\Gamma}_j^{(\mu)}$, all $\mu \geq 1$ and all $x \in G_j, 1 \leq j \leq p = |V|$. Here $\|\cdot\|$ is a norm on \mathbb{R}^n and

$$\|D\theta_{(J,E)}(x)\| := \sup\{\|(D\theta_{(J,E)}(x))y\| : y \in \mathbb{R}^n, \|y\| \leq 1\}.$$

Choosing a different norm on \mathbb{R}^n only changes the constant M .

If, for a given norm, there exists $\mu_0 \geq 1$ such that

$$\|D\theta_{(J,E)}(x)\| \leq \kappa < 1 \tag{5.1}$$

for all $(J, E) \in \bar{\Gamma}_j^{(\mu_0)}$, all $x \in G_j$ and all j with $1 \leq j \leq p = |V|$, define $c = \kappa^{(\frac{1}{\mu_0})} < 1$, so $\|D\theta_{(J,E)}(x)\| \leq c^{\mu_0}$. It follows by the chain rule that for all positive integers t , $\|D\theta_{(J,E)}(x)\| \leq c^{(t\mu_0)}$ for all $(J, E) \in \bar{\Gamma}_j^{(t\mu_0)}$, all $x \in G_j$ and all j with $1 \leq j \leq p = |V|$. With the help of Lemma 5.3 it then follows easily that there exists a constant M_2 such that for all $(J, E) \in \bar{\Gamma}_j^{(\mu)}$, all $\mu \geq 1$, all j with $1 \leq j \leq p = |V|$ and all $x \in G_j$, $\|D\theta_{(j,E)}(x)\| \leq M_2 c^\mu$.

Thus, assuming (H1.1), (H1.2) and (H4.2), assumption (H4.3) is equivalent to the assumption that Eq. (5.1) is satisfied for some positive integer μ_0 and some κ with $0 \leq \kappa < 1$.

If, for a given $\|\cdot\|$ on \mathbb{R}^n , there exists a positive integer μ_0 and $\kappa, 0 \leq \kappa < 1$, such that

$$\|\theta_{(J,E)}(x) - \theta_{(J,E)}(y)\| \leq \kappa \|x - y\| \tag{5.2}$$

for all $(J, E) \in \bar{\Gamma}_j^{(\mu_0)}$, $1 \leq j \leq p = |V|$ and all $x, y \in G_j$, it is easy to see that Eq. (5.1) is satisfied, so, with the aid of (H1.1), (H1.2) and (H4.2), we obtain (H4.3).

We are interested in the converse: if we assume (H1.1), (H1.2), (H4.2) and (H4.3), can we obtain Eq. (5.2) for some $\mu_0 \geq 1$ and some κ with $0 \leq \kappa < 1$? This is not true, but if we strengthen (H1.2), we shall see that we can obtain Eq. (5.2) from (H1.1), (H4.2), (H4.3) and the strengthened version of (H1.2).

Lemma 5.5. *Let notation be as in Sect. 2 and assume that hypotheses (H1.1), (H4.2), (H4.3) and (H5.2) are satisfied. Then there exists a positive integer μ and a constant κ with $0 \leq \kappa < 1$, such that for $1 \leq j \leq p = |V|$, Eq. (5.2) is satisfied for all $(J, E) \in \bar{\Gamma}_j^{(\mu)}$ and all $x, y \in G_j$.*

Proof. If $x, y \in G_j$ select a Lipschitz map $\psi : [0, 1] \rightarrow G_j$ such that $\psi(0) = x, \psi(1) = y$ and $\int_0^1 \|\psi'(t)\| dt \leq M \|x - y\|$, where M is a constant as in (H5.2). If ν is a positive integer and $(J, E) \in \bar{\Gamma}_j^{(\nu)}$,

$$\begin{aligned} \|\theta_{(J,E)}(y) - \theta_{(J,E)}(x)\| &= \left\| \int_0^1 D\theta_{(J,E)}(\psi(t))\psi'(t) dt \right\| \\ &\leq \int_0^1 \|D\theta_{(J,E)}(\psi(t))(\psi'(t))\| dt. \end{aligned}$$

By (H4.3) we see that there exist constants $M_1 > 0$ and $c_1, 0 \leq c_1 < 1$, such that $\|D\theta_{(J,E)}(u)\| \leq M_1 c_1^\nu$ for all $(J, E) \in \bar{\Gamma}_j^{(\nu)}$, $u \in G_j$, $1 \leq j \leq p$ and $\nu \geq 1$. It follows from (H5.2) that

$$\|\theta_{(J,E)}(y) - \theta_{(J,E)}(x)\| \leq M_1 c_1^\nu \int_0^1 \|\psi'(t)\| dt \leq M_1 M c_1^\nu \|y - x\|.$$

Choose $\nu := \mu$ so large that $M_1 M c'_1 := \kappa < 1$ to obtain the conclusion of Lemma 5.5. □

The reader can check that (H5.2) remains true, but with a different M , if the norm $\|\cdot\|$ is replaced by a different norm. Similarly, Lemma 5.5 remains true, but with a different μ and a different κ , if the norm in Lemma 5.5 is replaced by a different norm.

Henceforth we shall always need to know that Eq. (5.2) is satisfied, so, rather than using Lemma 4.4, we shall directly make the assumption that Eq. (5.2) holds.

The key idea now will be to define a special subcone of the closed cone $K_j \cap X_j$ in X_j , where K_j is the set of nonnegative functions in $C(\bar{G}_j)$ and $X_j = C^m(\bar{G}_j)$, $1 \leq j \leq p = |V|$. Variants of the subcone we shall consider have already been used in [9], in Sections 5 and 6 of [38], in [40] and in Section 2.2 of [25].

If M is a positive real, H is a bounded, open, mildly regular subset of \mathbb{R}^n and m is a nonnegative integer, we define a closed cone $K(H; M, m) \subset C^m(\bar{H})$ by

$$K(H; M, m) = \{h \in C^m(\bar{H}) : h(x) \leq h(y) \exp(M\|x - y\|) \forall x, y \in \bar{H}\} \tag{5.3}$$

The definition of $K(H; M, m)$ depends on the particular norm $\|\cdot\|$ on \mathbb{R}^n which is used, but we do not indicate this dependence in the notation. The reader can easily verify that if $h \in K(H; M, m)$, then $h(x) \geq 0 \forall x \in \bar{H}$; and if $h \in K(H, M, m)$ is not identically zero, then $h(x) > 0$ for all $x \in \bar{H}$.

Lemma 5.6. *Assume that H is a bounded, open, mildly regular subset of \mathbb{R}^n , that $M > 0$ and that m is a nonnegative integer. If $K(H; M, m)$ is defined by Eq. (5.3) and $m \geq 1$, then $K(H; M, m)$ is a closed, reproducing cone in $C^m(\bar{H})$. If $m = 0$, $K(H; M, 0)$ is a closed, total cone in $C(\bar{H})$, but it is not reproducing.*

Proof. We leave to the reader the exercise of proving that $K(H; M, m)$ is a closed cone in $C^m(\bar{H})$. Note also that if $h \in C^m(\bar{H})$, $h \in K(H; M, m)$ if and only if $h(x) = 0$ for all $x \in \bar{H}$ or $h(x) > 0$ for all $x \in \bar{H}$ and, for all $x, y \in \bar{H}$,

$$|\log(h(x)) - \log(h(y))| \leq M\|x - y\|, \tag{5.4}$$

where \log denotes the natural logarithm.

To prove that $K(H; M, m)$ is reproducing in $C^m(\bar{H})$ if $m \geq 1$, observe that if $g \in C^m(\bar{H})$ and C is any positive constant, then $g(x) = (g(x) + C) - C$. Since any positive constant is an element of $K(H; M, m)$, it suffices to prove that $x \rightarrow g(x) + C$ defines an element of $K(H; M, m)$ for C large. Thus, it suffices to prove that if $h(x) := g(x) + C$ is large, Eq. (5.4) is satisfied for all $x, y \in \bar{H}$.

Because $m \geq 1$, the argument used in Lemma 2.4 shows that there exists a constant M_1 such that, for all $x, y \in \bar{H}$,

$$|g(x) - g(y)| \leq M_1\|x - y\|.$$

If N is any positive integer, we can choose $C > 0$ such that $C + g(x) \geq N$ for all $x \in \bar{H}$. It follows by the mean value theorem that

$$|\log(C + g(x)) - \log(C + g(y))| \leq \left(\frac{1}{N}\right) |g(x) - g(y)| \leq \left(\frac{M_1}{N}\right) \|x - y\|.$$

If N is chosen so that $\frac{M_1}{N} \leq M$, it follows that the map $x \rightarrow C + g(x)$ is an element of $K(H; M, m)$, so $K(H; M, m)$ is reproducing in $C^m(\bar{H})$.

It remains to consider the case $m = 0$. For any $f \in C(\bar{H})$ and any $\varepsilon > 0$, it is known that there exists $h \in C^1(\bar{H})$ with $|f(x) - h(x)| < \varepsilon$ for all $x \in \bar{H}$. Since $K(H; M, 1)$ is reproducing in $C^1(\bar{H})$, it follows that $K(H; M, 0) \supset K(H; M, 1)$ is total in $C(\bar{H})$.

If $h \in K(H; M, 0)$ and h is not identically zero, h satisfies Eq. (5.4) and $h(x) > 0$ for all $x \in \bar{H}$. It follows that $x \rightarrow h(x)$ is Lipschitzian if $h \in K(H; M, 0)$. This implies that if $f = h_1 - h_2$, where $h_1, h_2 \in K(H; M, 0)$, f is Lipschitzian. Since not all elements of $C(\bar{H})$ are Lipschitzian, $K(H; M, 0)$ is not reproducing in $C(\bar{H})$. \square

We shall use the notation of Sect. 2, so for $1 \leq j \leq p = |V|$, G_j is a bounded, mildly regular open subset of \mathbb{R}^n , $Y_j = C(\bar{G}_j)$, m is a positive integer, $X_j = C^m(\bar{G}_j)$ and Y, K and X are given by Eqs. (2.11), (2.13) and (2.17) respectively. If M is a positive constant and m is a positive integer as above, we shall write for $1 \leq j \leq p = |V|$ (compare Eq. (5.3))

$$K_j(M, m) := K(G_j; M, m). \tag{5.5}$$

We define $K(M, m) \subset X$ by

$$K(M, m) = \{f = (f_1, f_2, \dots, f_p) \in X \mid f_j \in K_j(M, m) \text{ for } 1 \leq j \leq p\}. \tag{5.6}$$

It follows from Lemma 5.6 that $K(M, m)$ is a closed, reproducing cone in X .

Theorem 5.7. *Assume that hypotheses H5.1 and H5.3 are satisfied, that X is defined by Eq. (2.17), that $L : Y \rightarrow Y$ and $\Lambda : X \rightarrow X$ are defined by Eq. (4.3) and that $K(M, m)$ is given by Eq. (5.6). Then there exists $M > 0$, a positive integer μ and $v := (v_1, v_2, \dots, v_p) \in K(M, m) \setminus \{0\}$ such that $\Lambda^\mu(K(M, m)) \subset K(M, m)$ and $\Lambda^\mu(v) = r^\mu v$, where $r = r(\Lambda) = r(L)$ denotes the spectral radius of Λ . There exists $w := (w_1, w_2, \dots, w_p) \in (K \cap X) \setminus \{0\}$ with $\Lambda(w) = rw$. If (H5.4) is also satisfied, $w_j(x) > 0$ for all $x \in \bar{G}_j$ and for $1 \leq j \leq p = |V|$, and there exists $M' > 0$ with $w \in K(M', m) \setminus \{0\}$.*

Proof. Let $\kappa, 0 < \kappa < 1$, and $\mu \geq 1$ be as in (H5.3). By using (H5.1) and (H5.3) and Lemmas 5.2 and 5.3, one can show that there exists a constant $M_0 > 0$ such that $b_{(J,E)} \in K_j(M_0, m)$ for all $(J, E) \in \bar{\Gamma}_j^{(\mu)}, 1 \leq j \leq p$. If $M \geq (\frac{M_0}{1-\kappa})$, we claim that $\Lambda^\mu(K(M, m)) \subset K(M, m)$. To see this, suppose that $f = (f_1, f_2, \dots, f_p) \in K(M, m)$. For $j \in V$ and $x \in G_j$ we have

$$(\Lambda^\mu(f))_j(x) = \sum_{(J,E) \in \bar{\Gamma}_j^{(\mu)}} b_{(J,E)}(x) f_{\alpha(j_\mu, e_\mu)}(\theta_{(J,E)}(x)).$$

Because $K_j(M, m)$ is a cone, it suffices to prove that $b_{(J,E)}(\cdot)f_{\alpha(j_\mu, e_\mu)}(\theta_{(J,E)}(\cdot)) \in K_j(M, m)$ for all $(J, E) \in \bar{\Gamma}_j^{(\mu)}$. For $x, y \in \bar{G}_j$ we have

$$b_{(J,E)}(x)f_{\alpha(j_\mu, e_\mu)}(\theta_{(J,E)}(x)) \leq \Psi_1 \Psi_2$$

where we define Ψ_1 and Ψ_2 by

$$\Psi_1 := [b_{(J,E)}(y) \exp(M_0(\|x - y\|))]$$

and

$$\Psi_2 := f_{\alpha(j_\mu, e_\mu)}(\theta_{(J,E)}(y)) \exp(M\|\theta_{(J,E)}(x) - \theta_{(J,E)}(y)\|).$$

If we define $\Phi(z)$ for $z \in G_{j_1}$ by

$$\Phi(z) := b_{(J,E)}(z)f_{\alpha(j_\mu, e_\mu)}(\theta_{(J,E)}(z)).$$

we conclude that

$$\Phi(x) \leq \Phi(y) \exp(M_0 + \kappa M\|x - y\|).$$

Since we assume that $M \geq (\frac{M_0}{1-\kappa})$, $M_0 + \kappa M \leq M$, and the above calculation shows that $(\Lambda^\mu)(K(M, m)) \subset K(M, m)$.

Since (H5.1) and (H5.3) imply the hypotheses of Theorem 4.6, $r(L) = r(\Lambda)$ and $\rho(\Lambda) \leq c^m r(\Lambda) < \nu(\Lambda)$, where $c = \kappa^{(\frac{1}{\mu})} < 1$. The formula for the spectral radius implies that $r(\Lambda^\mu) = r(L^\mu) = (r(\Lambda))^\mu$. Using formulas in [34] or results from [35], one can prove that $\rho(\Lambda^\mu) = (\rho(\Lambda))^\mu$; and in any event, Eq. (3.7) implies that $\rho(\Lambda^\mu) \leq (\rho(\Lambda))^\mu$. Thus $\rho(\Lambda^\mu) < r(\Lambda^\mu) = r(L^\mu)$ and $L^\mu(K(M, m)) \subset K(M, m)$. Since $K(M, m)$ is a closed, total cone in X , Theorem 3.1 implies that there exists $v = (v_1, v_2, \dots, v_p) \in K(M, m) \setminus \{0\}$ with $\Lambda^\mu(v) = r^\mu v$ and $r = r(\Lambda)$.

If we define $\Lambda_r : X \rightarrow X$ by $\Lambda_r(f) = (\frac{1}{r})\Lambda(f)$, we find that $\Lambda_r^\mu(v) = v$; and if we define $w = (w_1, w_2, \dots, w_p) \in X$ by

$$w = \sum_{s=0}^{\mu-1} (\Lambda_r)^s(v), \tag{5.7}$$

it is easy to check that $\Lambda_r(w) = w$. Also, if t is a positive integer, one can check that

$$tw = \sum_{s=0}^{t\mu-1} (\Lambda_r)^s(v). \tag{5.8}$$

If w is defined by Eq. (5.7) and if we assume that (H5.4) also holds, we claim that $w_j(x) > 0$ for all $x \in \bar{G}_j$ and $1 \leq j \leq p$. Because $v \in K(M, m) \setminus \{0\}$, there exists $k, 1 \leq k \leq p$, with $v_k(x) > 0$ for all $x \in \bar{G}_k$. By (H5.4), for each j with $1 \leq j \leq p, j \neq k$, there exists a positive integer $\nu = \nu(j, k)$ and a sequence $(j_i, e_i) \in \Gamma, 1 \leq i \leq \nu$, with $j_1 = j, \alpha(j_i, e_i) = j_{i+1}$ for $1 \leq i < \nu$ and $\alpha(j_\nu, e_\nu) = k$. If $\nu = \nu(j, k)$, then for all $x \in \bar{G}_j$ we have

$$((\Lambda_r^\nu)(v))_j(x) = \left(\frac{1}{r}\right)^\nu \sum_{(J,E) \in \bar{\Gamma}_j^{(\nu)}} b_{(J,E)}(x)v_{\alpha(j_\nu, e_\nu)}(\theta_{(J,E)}(x)).$$

By assumption, there exists $(J, E) \in \bar{\Gamma}_j^{(\nu)}$ with $\alpha(j_\nu, e_\nu) = k$ and $v_k(\theta_{(J,E)}(x)) > 0$ for all $x \in \bar{G}_j$. It follows that for all $x \in \bar{G}_j$,

$$((\Lambda_r^\nu(v))_j(x) > 0.$$

If we take an integer $t \geq 1$ such that $t\mu - 1 \geq \nu(j, k)$ for all $j \in V \setminus \{k\}$, it follows from Eq. (5.8) that $w_j(x) > 0$ for all $x \in \bar{G}_j$ and for all $j \in V$.

Because $w_j(x) > 0$ for all $x \in \bar{G}_j$ and $w_j \in X_j$, the same argument used in Lemma 5.2 shows that $x \rightarrow \log(w_j(x))$ is Lipschitz on \bar{G}_j , which implies that there exists $M' \geq M$ such that $w_j \in K_j(M', m)$ for $1 \leq j \leq p$. \square

Our next result is an immediate corollary of Theorem 5.7.

Corollary 5.8. *Let G be a bounded, open mildly regular subset of \mathbb{R}^n . Let \mathcal{E} be a finite index set and m a positive integer and for positive integers μ , let \mathcal{E}_μ be as defined in Corollary 4.9. Assume hypotheses (A1) and (A2) in Corollary 4.9 but strengthen (A1) by assuming that $b_e(x) > 0$ for all $e \in \mathcal{E}$ and all $x \in \bar{G}$. Assume also that there exists an integer $\mu \geq 1$ and κ with $\kappa < 1$ such that*

$$\|\theta_\omega(x) - \theta_\omega(y)\| \leq \kappa \|x - y\|$$

for all $x, y \in \bar{G}$ and all $\omega \in \mathcal{E}_\mu$.

Then all conclusions of Corollary 4.9 are satisfied. In addition there exists $v \in (K \cap X) \setminus \{0\}$ such that $\Lambda(v) = rv$, where $r = r(\Lambda) > 0$ and $v(x) > 0$ for all $x \in \bar{G}$.

6. $r(\Lambda)$ is an Algebraically Simple Eigenvalue

In this section we shall need some results concerning “ u_0 -positivity” for “positive linear operators”. Theorem 6.1 below gives the information which we shall need. A proof of Theorem 6.1 can be found in [21] or [22].

It is worth remarking that Theorem 6.1 can also be derived in a few pages from the so-called Birkhoff–Hopf theorem for positive linear operators, although we shall not give a derivation here. An exposition and generalization of the results of Birkhoff [6] and Hopf [16, 17] (see also Samelson [41]) can be found in the articles [12] and [13] and the appendix of [26], and the latter sources give further references to the literature.

Theorem 6.1. (See [21] and [22]). *Let C be a closed, reproducing cone in a real Banach space Z . Let $A : Z \rightarrow Z$ be a bounded linear operator such that $A(C) \subset C$ and assume that there exists $v \in C \setminus \{0\}$ and $r > 0$ with $A(v) = rv$. Let \leq denote the partial ordering on Z induced by C , so $x \leq y \Leftrightarrow y - x \in C$. Assume (this is the u_0 -positivity of A) that there exists $u_0 \in C \setminus \{0\}$ with the following property: For every $x \in C \setminus \{0\}$ there exists an integer $m(x) \geq 1$ and positive reals $a(x)$ and $b(x)$ such that either (i) $A^{m(x)}(x) = 0$ or (ii) $a(x)u_0 \leq A^{m(x)}(x) \leq b(x)u_0$. Then r is algebraically simple as an eigenvalue of A ; and if $\zeta \in \mathbb{C}$ is an eigenvalue of \hat{A} where \hat{A} denotes the complexification of A , and $\zeta \neq r$, then $|\zeta| < r$. If $A(w) = \lambda w$ for some $w \in C \setminus \{0\}$, then $\lambda = r$ and w is a scalar multiple of v .*

Remark 6.2. If A and Z are as in Theorem 6.1, let \hat{Z} denote the complexification of Z and $\hat{A} : \hat{Z} \rightarrow \hat{Z}$ the complexification of A . Note that Theorem 6.1 only applies to eigenvalues of \hat{A} . If $\sigma(\hat{A})$ denotes the spectrum of \hat{A} , it is a priori possible that there exists $\zeta \in \sigma(\hat{A})$ with $|\zeta| = r$ and $\zeta \neq r$ or that there exists $\zeta \in \sigma(\hat{A})$ with $|\zeta| > r$. However, if C in Theorem 6.1, is also normal, one can prove that $|\zeta| \leq r$ for all $\zeta \in \sigma(\hat{A})$.

Corollary 6.3. *Let C be a closed, reproducing cone in a real Banach space Z . Let $A : Z \rightarrow Z$ be a bounded linear operator such that $A(C) \subset C$ and let \hat{A} denote the complexification of A and $\sigma(\hat{A})$ the spectrum of \hat{A} . Assume that $\rho(A) < r(A)$, where $r(A)$ denotes the spectral radius of A and $\rho(A)$ the essential spectral radius of A . Assume that there exists $u_0 \in C \setminus \{0\}$ which satisfies the u_0 -positivity property in Theorem 6.1. Then there exists $v \in C \setminus \{0\}$ such that $A(v) = rv, r := r(A)$; and r is algebraically simple as an eigenvalue of A and r is an isolated point of $\sigma(\hat{A})$. There exists $r_1 < r$ such that if $\zeta \in \sigma(\hat{A}) \setminus \{r\}$ we have $|\zeta| \leq r_1$.*

Proof. By Theorem 3.1, there exists $v \in C \setminus \{0\}$ with $A(v) = rv$ and $r = r(A)$. By the properties of the essential spectral radius and the assumption that $\rho(A) < r(A)$, we know that for each $\zeta \in \sigma(\hat{A})$ with $|\zeta| > \rho(A), \zeta$ is an isolated point of $\sigma(\hat{A}), \zeta$ is an eigenvalue of \hat{A} and ζ has finite algebraic multiplicity as an eigenvalue of \hat{A} . The conclusions of Corollary 6.3 now follow from Theorem 6.1. □

Corollary 6.4. *Let hypotheses and notation be as in Corollary 6.3 and define $B(z) = (\frac{1}{r})A(z)$ for $z \in Z$. Then for every $z \in Z$, there exists $s_z \in \mathbb{R}$ such that*

$$\lim_{k \rightarrow \infty} B^k(z) = s_z v, \tag{6.1}$$

where $Av = rv, r = r(A), v \in C \setminus \{0\}$ and convergence is in the norm topology on Z . If $z \in C$ and there exist an integer $N \geq 0$ and positive reals $a = a_z$ and $b = b_z$ such that $av \leq A^N(z) \leq bv$, then $s_z > 0$.

Proof. Corollary 6.3 implies that there exists $r_1 < r$ such that $|\zeta| \leq r_1$ for all $\zeta \in \sigma(\hat{A}) \setminus \{r\}$, where \hat{A} denotes the complexification of A . Furthermore, r is an eigenvalue of A of algebraic multiplicity one. Under these conditions, Eq. (6.1) is a standard result which can be obtained by using spectral projections for \hat{A} . If $z \in C$ and $a > 0, b > 0$ and N are as in the statement of Corollary 6.4, $a'v \leq B^N z \leq b'v$, where $a' = r^{-N}a$ and $b' = r^{-N}b$. Because $B(v) = v$ and $B(C) \subset C$, it follows that $a'v \leq B^k(z) \leq b'v$ for all $k \geq N$, which implies that $a' \leq s_z \leq b'$. □

It remains to add an assumption which will allow us to verify the hypotheses of Corollaries 6.3 and 6.4. We shall show that the following strengthening of (H5.4) is sufficient. As usual, notation is as in Sect. 2. Note that (H5.1)–(H5.4) are stated at the beginning of Sect. 5, as is Remark 5.1.

(H6.1) There exists a positive integer ν such that for any pair of integers $i, k \in V$, there exists a sequence $(i_j, e_j) \in \Gamma, 1 \leq j \leq \nu$ with $\alpha(i_j, e_j) = i_{j+1}$ for $1 \leq j < \nu, i_1 = i$ and $\alpha(i_\nu, e_\nu) = k$.

Note (see Remark 5.1) that (H6.1) is trivially true if $p = |V| = 1$; and if $p > 1$, (H5.1) is true if and only if a $p \times p$ nonnegative matrix A associated with (Γ, α, V) is primitive.

Theorem 6.5. *Assume that hypotheses (H5.1), (H5.3) and (H6.1) are satisfied, that X and Λ are defined by Eqs. (2.17) and (4.3) respectively, that $\sigma(\Lambda) := \sigma(\hat{\Lambda})$ denotes the spectrum of $\hat{\Lambda}$, the complexification of Λ , and that, for $M > 0$ and $m \geq 1$ as in Eq. (2.17), $K(M, m)$ is given by Eq. (5.3). If $\rho(\Lambda), r(\Lambda)$ and $r(L)$ denote, respectively, the essential spectral radius of Λ , the spectral radius of Λ and the spectral radius of L and μ and $\kappa < 1$ are as in (H5.3), we have*

$$\rho(\Lambda) \leq \kappa^{\left(\frac{m}{\mu}\right)} r(\Lambda) < r(\Lambda) = r(L) := r.$$

There exist $M' > 0$ and $w = (w_1, w_2, \dots, w_p) \in K(M', m) \setminus \{0\}$ such that $w_j(x) > 0$ for all $x \in \bar{G}_j, 1 \leq j \leq p = |V|$, and

$$\Lambda(w) = rw.$$

There exists $r_1 < r$ such that if $\zeta \in \sigma(\hat{\Lambda}) \setminus \{r\}$, then $|\zeta| \leq r_1$; and r is an isolated point of $\sigma(\hat{\Lambda})$ and an eigenvalue of algebraic multiplicity 1. If $u \in X$, there exists a real number s_u such that

$$\lim_{k \rightarrow \infty} \left(\left(\frac{1}{r} \right) \Lambda \right)^k (u) = s_u w.$$

If $u \in K(M, m) \setminus \{0\}$ for some $M > 0$, then $s_u > 0$.

Proof. The first part of Theorem 6.5, up to the existence of w , follows directly from Theorems 4.6 and 5.7.

Let ν be as in (H6.1) and let A be a $p \times p$ nonnegative matrix associated with (Γ, α, V) . It follows (see Remark 5.1) that all entries of A^ν are positive. (H1.1) implies that no row of A is the zero vector, so A^{ν_1} has all positive entries for all integers $\nu_1 \geq \nu$; and (H6.1) is satisfied for any $\nu_1 \geq \nu$.

If μ and $\kappa, 0 \leq \kappa < 1$, are as in hypothesis (H5.3) and t is a positive integer, $\theta_{(J,E)}$ is a Lipschitz map (with respect to the norm $\|\cdot\|$ in (H5.3)) on \bar{G}_j with Lipschitz constant $\text{lip}(\theta_{(J,E)}) \leq \kappa^t$ for all $(J, E) \in \bar{\Gamma}_j^{(t\mu)}, 1 \leq j \leq p$. By Lemma 5.2 there exists a constant $C_1 \geq 1$ such that $\text{lip}(\theta_{(j,e)}) \leq C_1$ for all $(j, e) \in \Gamma$. If $t\mu \leq \nu_1 < (t + 1)\mu$, it follows that for all $(J, E) \in \bar{\Gamma}_j^{(\nu_1)}$ and $1 \leq j \leq p$ we have the estimate

$$\text{lip}(\theta_{(J,E)}) \leq C_1^{\mu-1} \kappa^t.$$

Using this estimate, one can see that there is an integer μ_0 such that for all $\nu_1 \geq \mu_0$ and all $(J, E) \in \bar{\Gamma}^{(\nu_1)}$,

$$\text{lip}(\theta_{(J,E)}) \leq \kappa < 1.$$

For μ_0, ν and κ as above, take $\nu_1 \geq \max(\nu, \mu_0)$. We already know that there exists $M' > 0$ and $w = (w_1, w_2, \dots, w_p) \in K(M', m)$ with $\Lambda(w) = rw$ and $w_j(x) > 0$ for all $x \in \bar{G}_j$. As in the proof of Theorem 5.7, there exists a constant $M_0 > 0$ such that $b_{(J,E)} \in K_j(M_0, m)$ for all $(J, E) \in \bar{\Gamma}_j^{(\nu_1)}, 1 \leq j \leq p$. If $M > \kappa M + M_0 := M_1$, the argument in the proof of Theorem 5.7 shows that $\Lambda^{\nu_1}(K(M, m)) \subset K(M_1, m)$. By increasing M , we can also arrange that $M_1 := \kappa M + M_0 > M'$. By using (H6.1) we see that if $u = (u_1, u_2, \dots, u_p) \in K(M, m) \setminus \{0\}$, then $\Lambda^{\nu_1}(u) := y = (y_1, y_2, \dots, y_p)$ satisfies $y \in K(M_1, m)$ and $y_j(x) > 0$ for all $x \in \bar{G}_j$ and for all $j, 1 \leq j \leq p$.

At this point we want to apply Corollaries 6.3 and 6.4 to the operator $\Lambda^{\nu_1} : K(M, m) \rightarrow K(M, m)$. Here w will take the place of u_0 in Theorem 6.1. For $u \in K(M, m) \setminus \{0\}$ and $y := \Lambda^{\nu_1}(u)$, we have to prove that there exist positive constants $a = a_u$ and $b = b_u$ with

$$aw \leq y \leq bw. \tag{6.2}$$

There is a subtlety, however. In the notation of Theorem 6.1, $Z = X$ and the closed, reproducing cone $C = K(M, m)$. Thus the partial ordering \leq in Eq. (6.2) is the partial ordering induced by $K(M, m)$, and the positive constants a and b in Eq. (6.2) must satisfy $bw - y \in K(M, m)$ and $y - aw \in K(M, m)$. The fact that $\Lambda^{\nu_1}(K(M, m)) \subseteq K(M_1, m)$, where $M' \leq M_1 < M$ will play a crucial role in the argument. Similar issues arise in Section 2.2 of [25].

To prove Eq. (6.2), first select positive numbers σ and τ such that $\sigma \leq w_j(x) \leq \tau$ and $\sigma \leq y_j(x) \leq \tau$ for all $x \in \bar{G}_j, 1 \leq j \leq p$. It suffices to prove that there exists $b > 0$ such that $y \leq bw$. The argument that $w \leq a^{-1}y$ for some $a > 0$ is completely symmetrical, with the roles of w and y reversed and a^{-1} taking the role of b . As a first step to proving the existence of b , we choose b so that $b\sigma > \tau$, which implies that $bw_j(x) > y_j(x)$ for all $x \in \bar{G}_j, 1 \leq j \leq p$. We know that $bw - y \in K(M, m)$ if and only if the map $x \rightarrow \log(bw_j(x) - y_j(x)) := \varphi_j(x), x \in \bar{G}_j$, is Lipschitzian with $\text{lip}(\varphi_j) \leq M$ for $1 \leq j \leq p$. If we define $\psi_j(x) = \log(b) + \log(w_j(x))$ for $x \in \bar{G}_j, \text{lip}(\psi_j) \leq M_1$ for $1 \leq j \leq p$; and we can write, for $x \in \bar{G}_j$,

$$\varphi_j(x) = \psi_j(x) + \log\left(1 - \frac{y_j(x)}{bw_j(x)}\right) := \psi_j(x) + g_j(x).$$

Thus it suffices to prove that for b large enough and $1 \leq j \leq p, \text{lip}(g_j) \leq M - M_1$. Since $1 - \frac{y_j(x)}{bw_j(x)} \geq (1 - \frac{\tau}{b\sigma})$ for $x \in \bar{G}_j$, we obtain from the mean value theorem that for $x, z \in \bar{G}_j$,

$$\begin{aligned} |g_j(x) - g_j(z)| &= \left| \log\left(1 - \frac{y_j(x)}{bw_j(x)}\right) - \log\left(1 - \frac{y_j(z)}{bw_j(z)}\right) \right| \\ &\leq \left(1 - \frac{\tau}{b\sigma}\right)^{-1} \left(\frac{1}{b}\right) \left| \frac{y_j(x)}{w_j(x)} - \frac{y_j(z)}{w_j(z)} \right|. \end{aligned}$$

Because $x \rightarrow \log(y_j(x))$ and $x \rightarrow \log(w_j(x))$ are Lipschitzian with Lipschitz constant M_1 and because $\log(y_j(x)) \leq \log(\tau)$ and $\log(w_j(x)) \leq \log(\tau)$ for

$x \in \bar{G}_j$, we obtain from the mean value theorem that $\text{lip}(y_j) \leq \tau M_1$ and $\text{lip}(w_j) \leq \tau M_1$ for $1 \leq j \leq p$. It follows that

$$\begin{aligned} \left| \frac{y_j(x)}{w_j(x)} - \frac{y_j(z)}{w_j(z)} \right| &\leq \frac{|y_j(x) - y_j(z)|w_j(z) + y_j(z)|w_j(z) - w_j(x)|}{w_j(x)w_j(z)} \\ &\leq (\tau M_1) \left[\frac{1}{\sigma} + \frac{\tau}{\sigma^2} \right] \|x - z\|_1, \end{aligned}$$

which implies that

$$\text{lip}(g_j) \leq \left(\frac{1}{b - (\tau/\sigma)} \right) (\tau M_1) \left[\frac{1}{\sigma} + \frac{\tau}{\sigma^2} \right].$$

It follows that for all b sufficiently large, $\text{lip}(g_j) \leq M - M_1$ and $bw - y \in K(M, m)$.

Applying Corollaries 6.3 and 6.4 to the operator Λ^{ν_1} , we conclude that for $r = r(\Lambda)$, there exists $r_1 < r$ such that if $z \in \sigma(\hat{\Lambda}^{\nu_1}) \setminus \{r^{\nu_1}\}$, then $|z| \leq r_1^{\nu_1}$ and r^{ν_1} is an eigenvalue of Λ^{ν_1} of algebraic multiplicity one. Furthermore, for every $u \in X$ there exists a number $s_u \in \mathbb{R}$ such that

$$\lim_{k \rightarrow \infty} B^k(u) = s_u w, \tag{6.3}$$

where $B := (\frac{1}{r^{\nu_1}})\Lambda^{\nu_1}$ and $s_u > 0$ if $u \in K(M, m)$ for some $M > 0$. Applying $(\frac{1}{r}\Lambda)^j$ to Eq. (6.3) for $0 \leq j < \nu_1$ and noting that $((\frac{1}{r}\Lambda)(w)) = w$, we conclude that for every $u \in X$,

$$\lim_{k \rightarrow \infty} \left(\left(\frac{1}{r} \right) \Lambda \right)^k (u) = s_u w, \tag{6.4}$$

and $s_u > 0$ if $u \in K(M, m)$ for some $M > 0$.

Because $\sigma(\hat{\Lambda}^{\nu_1}) = \{\zeta^{\nu_1} \mid \zeta \in \sigma(\hat{\Lambda})\}$, for $\zeta \in \sigma(\hat{\Lambda})$ we must have $|\zeta| \leq r_1$ unless $\zeta^{\nu_1} = r^{\nu_1}$. However, if $\zeta^{\nu_1} = r^{\nu_1}$ and $\zeta \neq r$, then because $\rho(\Lambda) < r(\Lambda)$, ζ is an eigenvalue of $\hat{\Lambda}$ and $\hat{\Lambda}v = \zeta v$ for some $v \in \hat{X}$, the complexification of X . It follows that $\hat{\Lambda}^{\nu_1}(v) = r^{\nu_1}v$ and v is not a scalar multiple of w . However, this contradicts the fact that r^{ν_1} is an eigenvalue of $\hat{\Lambda}^{\nu_1}$ of algebraic multiplicity one. Thus, if $\zeta \in \sigma(\Lambda)$ and $|\zeta| = r$, $\zeta = r$.

It remains to prove that r is an eigenvalue of Λ of algebraic multiplicity one. Any eigenvector of Λ with eigenvalue r must be a multiple of w , for otherwise we contradict the fact that r^{ν_1} is an eigenvalue of Λ^{ν_1} of algebraic multiplicity one. Therefore, to prove that r is an eigenvalue of Λ of algebraic multiplicity one, it suffices to prove that there does not exist $u \in X$ with

$$ru - \Lambda u = w.$$

However, if such a u exists, one can see that

$$\left(\left(\frac{1}{r} \right) \Lambda \right)^k (u) = u - \left(\frac{k}{r} \right) w,$$

which contradicts Eq. (6.4). Thus such a u does not exist and r has algebraic multiplicity one. □

The following result is an immediate corollary of Theorem 6.5.

Corollary 6.6. *Let G be a bounded, open mildly regular subset of \mathbb{R}^n . Let \mathcal{E} be a finite index set and let m be a positive integer. Assume*

- (B1) *For each $e \in \mathcal{E}$, $b_e \in C^m(\bar{G})$ and $b_e(x) > 0$ for all $x \in \bar{G}$.*
- (B2) *For each $e \in \mathcal{E}$, $\theta_e : G \rightarrow G$ and $\theta_e \in C^m(\bar{G})$.*
- (B3) *For each positive integer μ , $\mathcal{E}_\mu := \{\omega = (e_1, e_2, \dots, e_\mu) \mid e_j \in \mathcal{E} \text{ for } 1 \leq j \leq \mu\}$, and if $\omega = (e_1, e_2, \dots, e_\mu) \in \mathcal{E}_\mu$, $\theta_\omega(x) := (\theta_{e_\mu} \cdot \theta_{e_{\mu-1}} \dots \theta_{e_1})(x)$. There exists a positive integer μ and a constant $\kappa < 1$ such that for all $\omega \in \mathcal{E}_\mu$ and all $x, y \in \bar{G}$,*

$$\|\theta_\omega(x) - \theta_\omega(y)\| \leq \kappa \|x - y\|.$$

Let $Y = C_{\mathbb{R}}(\bar{G})$ and $X = C_{\mathbb{R}}^m(\bar{G})$ and define a bounded linear map $L : Y \rightarrow Y$ by

$$(Lf)(x) = \sum_{e \in \mathcal{E}} b_e(x) f(\theta_e(x))$$

a bounded linear map $\Lambda : X \rightarrow X$ by $\Lambda(f) = L(f)$ for $f \in X$. If $r(L)$ (respectively, $r(\Lambda)$) denotes the spectral radius of L (respectively, Λ) and $\rho(\Lambda)$ denotes the essential spectral radius of Λ

$$\rho(\Lambda) \leq (\kappa^{\frac{m}{\mu}})r(\Lambda) < r(\Lambda) \quad \text{and} \quad r(\Lambda) = r(L) := r.$$

There exists $v \in X \setminus \{0\}$ such that $v(x) > 0 \forall x \in \bar{G}$ and $\Lambda(v) = rv$. If $\hat{\Lambda}$ denotes the complexification of Λ and $\sigma(\Lambda) := \sigma(\hat{\Lambda})$ denotes the spectrum of $\hat{\Lambda}$, there exists $r_1 < r$ such that

$$\sigma(\hat{\Lambda}) \setminus \{r\} \subseteq \{z \in \mathbb{C} \mid |z| \leq r_1\}$$

and r is an algebraically simple eigenvalue of $\hat{\Lambda}$. If $u \in X$ and $u(x) > 0$ for all $x \in \bar{G}$, there exists $s = s_\mu > 0$ such that

$$\lim_{k \rightarrow \infty} \left(\frac{1}{r}\right)^k \Lambda^k(u) = s_u v,$$

where the convergence is in the norm topology on $X = C_{\mathbb{R}}^m(\bar{G})$.

References

- [1] Akhmerov, R.R., Kamenskij, M.I., Potapov, A.S., Rodkina, A.E., Sadovskij, B.N.: Measures of noncompactness and condensing operators (in Russian). Nauka, Novosibirsk (1986) [English Translation: Birkhäuser Verlag, Basel (1992)]
- [2] Appell, J.: Measures of noncompactness, condensing operators and fixed points: an application-oriented survey. Fixed Point Theory **6**, 157–229 (2005)
- [3] Ayerbe Toledano, J.M., Dominguez Benavides, T., López Acedo, G.: Measures of Noncompactness in Metric Fixed Point Theory. Birkhäuser, Basel (1997)

- [4] Baladi, V.: Positive Transfer Operators and Decay of Correlations. World Scientific Publishing Co. Pte. Ltd, Singapore (2000)
- [5] Banás, J., Goebel, K.: Measures of Noncompactness in Banach Spaces. M. Dekker, New York (1980)
- [6] Birkhoff, G.: Extensions of Jentzsch's theorem. *Trans. Am. Math. Soc.* **85**, 219–227 (1957)
- [7] Bonsall, F.F.: Linear operators in complete positive cones. *Proc. Lond. Math. Soc.* **8**, 53–75 (1958)
- [8] Browder, F.E.: On the spectral theory of elliptic differential operators. *Math. Ann.* **142**, 22–130 (1961)
- [9] Bumby, R.T.: Hausdorff dimensions of Cantor sets. *J. Reine Angew. Math.* **331**, 192–206 (1982)
- [10] Conway, J.B.: A Course in Functional Analysis, 2nd edn. Springer, Berlin (1990)
- [11] Darbo, G.: Punti Uniti in Trasformazioni a Condominio Non Compatto. *Rend. Sem. Mat. Univ. Padova* **24**, 84–92 (1955)
- [12] Eveson, S.P., Nussbaum, R.D.: An elementary proof of the Birkhoff–Hopf theorem. *Math. Proc. Camb. Philos. Soc.* **117**, 31–55 (1995)
- [13] Eveson, S.P., Nussbaum, R.D.: Applications of the Birkhoff–Hopf theorem to the spectral theory of positive linear operators. *Math. Proc. Camb. Philos. Soc.* **117**, 491–512 (1995)
- [14] Falk, R., Nussbaum, R.D.: C^m Eigenfunctions of Perron–Frobenius operators and a new approach to numerical computation of Hausdorff dimension (submitted)
- [15] Gohberg, I., Krein, M.G.: The basic propositions on defect numbers, root numbers and indices of linear operators. *Amer. Math. Soc. Transl. Ser.* **2**(13), 185–264 (1960)
- [16] Hopf, E.: An inequality for positive integral operators. *J. Math. Mech.* **12**, 683–692 (1963)
- [17] Hopf, E.: Remarks on my paper, “An inequality for positive integral operators”. *J. Math. Mech.* **12**, 889–892 (1963)
- [18] Hutchinson, J.E.: Fractals and self-similarity. *Indiana Univ. Math. J.* **30**, 731–747 (1981)
- [19] Jenkinson, O., Pollicott, M.: Computing the dimension of dynamically defined sets: E_2 and bounded continued fractions. *Ergod. Theory Dyn. Syst.* **21**, 1429–1445 (2001)
- [20] Kato, T.: Perturbation Theory for Linear Operators. Springer, New York (1966)
- [21] Krasnosel'skii, M.A.: Positive Solutions of Operator Equations. P. Noordhoff Ltd., Groningen (1964)
- [22] Krasnosel'skii, M.A., Lifschits, J.A., Sobolev, A.V.: Positive Linear Systems: The Method of Positive Operators. Sigma Series Applied Mathematics, vol. 5. Heldermann Verlag, Berlin (1989)
- [23] Kreĭn, M.G., Rutman, M.A.: Linear Operators Leaving Invariant a Cone in a Banach Space (in Russian). *Uspekhi Mat. Nauk.* **31**(23), 3–95 (1948) [English translation in *Am. Math. Soc. Trans.* **26** (1950)]
- [24] Kuratowski, K.: Sur les Espaces Complets. *Fund. Math.* **15**, 301–309 (1930)

- [25] Lemmens, B., Nussbaum, R.D.: Birkhoff's version of Hilbert's metric and applications, Chapter 10. In: Papadopoulos, A., Troyanov, M. (eds.) *Handbook of Hilbert Geometry*. European Mathematical Society Publishing House, Zürich (2015)
- [26] Lemmens, B., Nussbaum, R.D.: *Nonlinear Perron–Frobenius Theory*. Cambridge Tracts in Mathematics, vol. 189. Cambridge University Press, Cambridge (2012)
- [27] Mallet-Paret, J., Nussbaum, R.D.: Generalizing the Krein–Rutman Theorem, Measures of Noncompactness and the Fixed Point Index. *J. Fixed Point Theory Appl.* **7**, 103–143 (2010)
- [28] Mallet-Paret, J., Nussbaum, R.D.: Inequivalent measures of noncompactness and the radius of the essential spectrum. *Proc. Am. Math. Soc.* **139**, 917–930 (2011)
- [29] Mallet-Paret, J., Nussbaum, R.D.: Inequivalent measures of noncompactness. *Annali di Matematica Pura Ed Applicata* **190**, 453–488 (2011)
- [30] Mauldin, R.D., Urbanski, M.: Dimensions and measures in infinite iterated function systems. *Proc. Lond. Math. Soc.* **3**, 105–154 (1996)
- [31] Mauldin, R.D., Urbanski, M.: *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*. Cambridge Tracts in Mathematics, vol. 148. Cambridge University Press, Cambridge (2003)
- [32] Mauldin, R.D., Williams, S.C.: Hausdorff dimension in graph directed constructions. *Trans. Am. Math. Soc.* **309**, 811–829 (1988)
- [33] Mazur, S.: Über die Kleinste Konvexe Menge, Die Gegebene Kompakte Menge enthält. *Stud. Math.* **2**, 7–9 (1930)
- [34] Nussbaum, R.D.: The radius of the essential spectrum. *Duke J. Math.* **38**, 473–478 (1970)
- [35] Nussbaum, R.D.: Spectral mapping theorems and perturbation theorems for Browder's essential spectrum. *Trans. Am. Math. Soc.* **150**, 445–455 (1970)
- [36] Nussbaum, R.D.: A periodicity threshold theorem for some nonlinear integral equations. *SIAM J. Math. Anal.* **9**, 356–376 (1978)
- [37] Nussbaum, R.D.: *Eigenvectors of Nonlinear Positive Operators and the Linear Krein–Rutman Theorem, Fixed Point Theory*. Lectures Notes in Mathematics, vol. 886, pp. 309–330. Springer, Berlin (1981)
- [38] Nussbaum, R.D.: Periodic points of positive linear operators and Perron–Frobenius operators. *Integral Equ. Oper. Theory* **39**, 41–97 (2001)
- [39] Nussbaum, R.D., Walsh, B.J.: Approximation by polynomials with nonnegative coefficients and the spectral theory of positive linear operators. *Trans. Am. Math. Soc.* **350**, 2367–2391 (1998)
- [40] Nussbaum, R.D., Priyadarshi, A., Verduyn Lunel, S.: Positive operators and Hausdorff dimension of invariant sets. *Trans. Am. Math. Soc.* **364**, 1029–1066 (2012)
- [41] Samelson, H.: On the Perron–Frobenius theorem. *Mich. Math. J.* **4**, 57–59 (1957)
- [42] Schaefer, H.H.: Halbgeordnete Lokal Konvexe Vektorräume II. *Math. Ann.* **138**, 259–286 (1959)
- [43] Schaefer, H.H., Wolff, M.P.: *Topological Vector Spaces*, 2nd edn. Springer, New York (1999)

- [44] Schief, A.: Self-similar sets in complete metric spaces. *Proc. Am. Math. Soc.* **124**, 481–490 (1996)
- [45] Toland, J.F.: Self-adjoint operators and cones. *J. Lond. Math. Soc.* **53**, 167–183 (1996)
- [46] Wolf, F.: On the essential spectrum of partial differential boundary problems. *Commun. Pure Appl. Math.* **12**, 211–228 (1959)

Roger D. Nussbaum (✉)
Department of Mathematics
Rutgers, The State University
Hill Center, 110 Fellinghuysen Road
Piscataway, NJ 08854-8019, USA
e-mail: nussbaum@math.rutgers.edu

Received: April 1, 2015.

Revised: December 9, 2015.