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Entropy Minimization, Hilbert's Projective Metric, and Scaling Integral Kernels

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Suppose that (S, μ) and (T, ν) are given measure spaces with $\mu(S) < \infty$ and $\nu(T) < \infty$. If $k \in L^\infty(S \times T)$ is a nonnegative function and $\alpha \in L^1(S)$ and $\beta \in L^1(T)$ are positive almost everywhere, the so-called DAD problem (k, α, β) asks whether there exist $f \in L^1(S)$ and $g \in L^1(T)$, f and g positive almost everywhere, with

$$\int_T f(s)\, k(s, t)\, g(t)\, \nu(dt) = \alpha(s), \qquad \mu \text{ almost everywhere, and}$$

$$\int_S f(s)\, k(s, t)\, g(t)\, \mu(ds) = \beta(t), \qquad \nu \text{ almost everywhere.}$$

Such a pair (f, g), *if it exists, is called a solution of the* DAD problem (k, α, β). We present here essentially sharp conditions under which the DAD problem (k, α, β) has a solution. We also give results concerning the uniqueness (to within positive scalar multiples) of solutions (f, g), iterative schemes for approximating solutions, and continuous dependence of solutions on $(k\alpha, k\beta)$. Methods of proof involve a mixture of variational methods (entropy minimization) and fixed point theory; Hilbert's projective metric also plays a useful role. As corollaries of our results we obtain generalizations of a variety of earlier DAD theorems. We are also able to discuss limiting behaviour of sequences of matrix DAD problems, where the dimensions of the matrices approach infinity.

1. INTRODUCTION

Suppose that $A = (a_{ij})$ is a nonnegative $m \times n$ matrix (so $a_{ij} \geqslant 0$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$) and $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^n$ are vectors whose entries α_i and β_j are positive. The so-called matrix DAD problem for (A, α, β) asks whether there are vectors $f \in \mathbb{R}^m$ and $g \in \mathbb{R}^n$, whose entries f_i and g_j are positive, and which satisfy

$$\sum_{j=1}^{n} f_i a_{ij} g_j = \alpha_i \text{ for } 1 \leqslant i \leqslant m \qquad \text{and} \qquad \sum_{i=1}^{m} f_i a_{ij} g_j = \beta_j \text{ for } 1 \leqslant j \leqslant n.$$

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If f and g exist, we say the DAD problem (A, α, β) has a solution (f, g). Alternatively, the DAD problem (A, α, β) is equivalent to finding positive diagonal matrices $D_1 = \operatorname{diag}(f_1, f_2, ..., f_m)$ and $D_2 = \operatorname{diag}(g_1, g_2, ..., g_n)$ such that $D_1 A D_2$ has specified row sums α_i and column sums β_j; and this, of course, explains the terminology. Obviously, a necessary condition for a solution to exist is that

$$\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j,$$

but if $a_{ij} = 0$ for some i and j, this condition may not be sufficient.

If (f, g) is a solution of the DAD problem (A, α, β) and $\lambda > 0$, $(\lambda^{-1} f, \lambda g)$ is also a solution. We say that the solution (f, g) is unique if every other solution is of the form $(\lambda^{-1} f, \lambda g)$ for some $\lambda > 0$. Aside from finding conditions which insure existence and uniqueness of solutions to a DAD problem (A, α, β), it is clearly also important to determine convergent iteration schemes for finding a solution (f, g).

Matrix DAD problems have been studied for over 30 years. DAD problems arise in diverse applications, e.g., in statistics and telecommunications; sometimes, as in [30], a DAD problem appears in a slightly disguised, but equivalent, form. We refer the reader to the references at the end of this paper, and in [13, 15, 19, 27, 30, and 36], for some indication of applications and the extensive literature on the subject. If A is a square $n \times n$ matrix all of whose entries are strictly positive and $u \in \mathbb{R}^n$ denotes the vector all of whose components equal one, Sinkhorn [33] first proved that the DAD problem (A, u, u) has a unique solution, so there are unique (to within positive scalar multiples) positive diagonal matrices D_1 and D_2 with $D_1 A D_2$ doubly stochastic. Subsequently Sinkhorn and Knopp [34] and Brualdi *et al.* [7] independently gave necessary and sufficient conditions for the DAD problem (A, u, u) to have a solution when A is a square nonnegative matrix. This problem is far more subtle than the case $a_{ij} > 0$ for all i and j. Other work on the doubly stochastic case $(\alpha = \beta = u)$ can be found in [6, 11, 12, 16, 20, 29].

For a general matrix DAD problem (A, α, β), Menon and Schneider observed in [23] that there is a fairly obvious set of necessary conditions which must be satisfied for the DAD problem to have a solution. They proved that if these necessary conditions are satisfied, then the DAD problem (A, α, β) has a solution. As we note in Section 4 below, the Menon–Schneider result [20] can be used to recover the theorems of Sinkhorn and Knopp [34] and Brualdi *et al.* [7] for the doubly stochastic case.

We consider here the following generalization of matrix DAD problems: (S, μ) and (T, ν) are measure spaces with $\mu(S) < \infty$ and $\nu(T) < \infty$. We are given functions $k \in L^\infty(S \times T)$, $\alpha \in L^1(S)$, and $\beta \in L^1(T)$, with $k(s, t) \geqslant 0$, $\alpha(s) > 0$, and $\beta(t) > 0$ almost everywhere. We want functions $f \in L^1(S)$ and $g \in L^1(T)$ which are positive almost everywhere and satisfy

$$\begin{aligned} &\int_T f(s)\, k(s, t)\, g(t)\, \nu(dt) = \alpha(s) \qquad \mu \text{ almost everywhere, and} \\ &\int_S f(s)\, k(s, t)\, g(t)\, \mu(ds) = \beta(t), \qquad \nu \text{ almost everywhere.} \end{aligned} \tag{1.1}$$

We refer to the problem of finding solutions f and g of (1.1) as the *DAD* problem for (k, α, β) and call (f, g) a solution of the *DAD* problem. A solution (f, g) is called unique if every other solution is of the form $(\lambda^{-1} f, \lambda g)$ for some positive constant λ. Obviously, a necessary condition for the *DAD* problem (k, α, β) to have a solution is that

$$\int_S \alpha(s)\, \mu(ds) = \int_S \int_T f(s)\, k(s, t)\, g(t)\, \mu(ds)\, \nu(dt) = \int_T \beta(t)\, \nu(dt). \tag{1.2}$$

However, just as in the matrix case, if k vanishes on a set of positive measure, there may be other necessary conditions: see Definition 2.13 in Section 2 below and [20].

If there exists $\delta > 0$ such that $k(s, t) \geqslant \delta$ almost everywhere and if (1.2) is satisfied, Hilbert's projective metric can be used as in Section 4 of [27], not only to given a unique solution (f, g) of the *DAD* problem (k, α, β) but also to provide a geometrically convergent iteration scheme for finding f and g. (We note in passing that Hilbert's projective metric was independently used in matrix *DAD* problems by Franklin and Lorenz [15], although the applicability of Hilbert's projective metric was noted earlier in [25], pp. 233–234.) With somewhat less generality and with k bounded above and below by positive constants, related results can already be found in [16] and [35]. However, if k is only nonnegative (a case which is important for applications), the *DAD* problem (k, α, β) becomes much harder, and the literature in the non-matrix case is sparse. The Hilbert's projective metric approach is no longer directly available. When $S = T = [0, 1]$, $\mu = \nu =$ Lebesque measure, $\alpha = \beta = 1$, and $k(s, t)$ is a nonnegative continuous function such that $k(s, t) = k(t, s)$ for all s and t and $k(s, s) > 0$ for $0 \leqslant s \leqslant 1$, Nowosad [24] and later Karlin and Nirenberg [18] have shown that the *DAD* problem (k, α, β) has a solution (f, g) with $f = g$. If one removes the symmetry assumption that $k(s, t) = k(t, s)$, the Karlin–Nirenberg approach does not seem to generalize. A different method, using ideas of Csiszàr [11], was employed in Section 4 of [27] to prove that the *DAD* problem (k, α, β) still has a solution (f, g) in this case. In recent work, Borwein *et al.* [5] consider (as a special case) a continuous nonnegative function $k \in C([0,1] \times [0,1])$ with $k(s,s) > 0$ for $0 \leqslant s \leqslant 1$ and a function $\alpha \in L^1(S)$, α positive almost everywhere, and show that the *DAD* problem (k, α, α) has a unique solution (f, g) which can be approximated by a geometrically convergent iteration scheme.

In this paper we obtain a nearly best-possible theorem for the existence of a solution to the general *DAD* problem (k, α, β). We find necessary conditions for the solution of the *DAD* problem (k, α, β) and show that, with a few added mild technical assumptions, these necessary conditions are also sufficient: see Theorems 2.26, 2.30, and 3.9 below. Our results can be viewed as direct generalizations of the matrix results of Menon and Schneider [23]. We derive several new *DAD* theorems as consequences of our general results and we show that a range of earlier theorems also can be obtained from Theorem 2.30 or Theorem 3.9.

Solution of the *DAD* problem (k, α, β) is equivalent to finding fixed points of a certain map $F: \mathring{C}_1 \to \mathring{C}_1$, where $\mathring{C}_1$ denotes the interior of the cone of nonnegative functions in $L^\infty(S)$. The map F is order-preserving and homogeneous of degree one. A general theory of such maps is developed in [26] and can be used to obtain uniqueness (to within positive scalar multiples) of fixed points and geometrically convergent iteration schemes for finding fixed points. However, an irreducible analytical difficulty is always to prove the existence of a fixed point of F in the interior of a cone, and that is precisely what we do here for our special F.

This paper is long, so an outline may be in order. In Section 2 we use the well-known fact that finding a solution of the *DAD* problem (k, α, β) is equivalent to finding a fixed point in $\mathring{C}_1$ of a certain map $F: \mathring{C}_1 \to \mathring{C}_1$. Unlike the finite dimensional case in [22] and [23], we note in Section 2 that F does not in general extend continuously to C_1. Nevertheless, we are able in Theorem 2.26 and 2.30 to use fixed point arguments to obtain sharp results about the existence of solutions of the *DAD* problem (k, α, β). The major drawback is that we must assume that either $A_1: L^\infty(S) \to L^\infty(T)$ or $A_2: L^\infty(T) \to L^\infty(S)$ is a compact linear map, where A_1 and A_2 are defined by

$$(A_1 u)(t) = \int_S k(s,t)\, \alpha(s)\, u(s)\, \mu(ds) \text{ and}$$

$$(A_2 v)(s) = \int_T k(s,t)\, \beta(t)\, v(t)\, \nu(dt).$$

Both A_1 and A_2 will be compact if k is continuous, but one can easily give $k \in L^\infty(S \times T)$ for which neither A_1 nor A_2 is compact.

Section 3 treats the general case when neither A_1 nor A_2 is known to be compact. Here we use results of Csiszàr [11] as clarified and sharpened in [5]. Finding a solution of the *DAD* problem (k, α, β) is equivalent to minimizing an "entropy" functional subject to certain constraints. By combining Theorem 2.30 with results on entropy minimization [5], we prove a result (see Theorem 3.9) about the *DAD* problem (k, α, β) in which compactness of A_1 or A_2 no longer plays a direct role. The variational approach and the fixed point approach have been applied separately to *DAD* problems; here we combine the methods to obtain a complete answer.

There is a necessary condition which must be satisfied in order for the *DAD* problem (k, α, β) to have a solution. We call this the compatibility condition for (k, α, β) in Definition 2.13 below. To apply Theorem 2.30 or Theorem 3.9 it is necessary to verify the compatibility condition, and this may be nontrivial even in the matrix case. In Remark 3.14 and in Section 4 we present various cases in which the compatibility condition can be verified. Combining these results with Theorem 2.30 or Theorem 3.9 we obtain new existence theorems for *DAD* problems and generalize earlier results of Nowosad [24], Karlin and Nirenberg [18], Nussbaum [27, Section 4] and Borwein *et al.* [5, Section 5].

As already noted, once one proves an existence theorem for a *DAD* problem (k, α, β), theorems in Sections 2 and 3 of [26] provide information about uniqueness of solutions and convergent iteration schemes. This is indicated in Theorem 4.19 of Section 4. If a given *DAD* problem (k_0, α_0, β_0) has a solution (f_0, g_0), one can ask whether "nearby" *DAD* problems (k, α, β) necessarily have solutions (f, g) which vary smoothly with (k, α, β). With the aid of Theorem 4.19, an answer to this question is given in Theorem 5.9 of Section 5. Theorem 5.9 has a number of consequences. We particularly mention Remark 5.14, which discusses whether a sequence of matrix *DAD* problems (A^p, α^p, β^p) (where p is a superscript, A^p is an $m_p \times n_p$ matrix, and $\lim_{p \to \infty} m_p = \infty = \lim_{p \to \infty} n_p$) has solutions (f^p, g^p) which converge in an appropriate sense as $p \to \infty$. To answer this question rigorously one is necessarily led to our general framework, even in one is only interested in matrix *DAD* problems. The reader may also find interest in Remark 5.13, where a rapidly convergent numerical scheme for approximating solutions of *DAD* problems is suggested.

2. *DAD* Theorems when $k(s, t)\,\beta(t)$ Determines a Compact Linear Operator

We study in this section the *DAD* problem (1.1) when k is continuous or, more generally, when the integral kernel $k(s, t)\,\beta(t)$ defines a compact linear map from $L^\infty(T)$ to $L^\infty(S)$. We begin by establishing some standard notation and hypotheses. We always make the following assumptions about S and T:

HYPOTHESIS 2.1. *(S, μ) and (T, ν) are measure spaces with $\mu(S) < \infty$ and $\nu(T) < \infty$. Subsets of sets of measure zero in (S, μ) or (T, ν) are measurable.*

We frequently assume more about S and T:

HYPOTHESIS 2.2. *Hypothesis 2.1 is satisfied. Furthermore, S and T are compact metric spaces and μ (respectively, ν) is a regular Borel measure of full support on S (respectively, on T).*

By "full support on S" we mean that the μ measure of any nonempty open set in S is positive.

We always assume the following about the functions k, α, and β in (1.1):

HYPOTHESIS 2.3. *Hypothesis 2.1 is satisfied. The function k is an element of $L^\infty(S \times T)$, and k is nonnegative almost everywhere; $\alpha \in L^1(S)$ and $\beta \in L^1(T)$ and α and β are strictly positive almost everywhere.*

If Hypothesis 2.3 is satisfied, suppose that there exist functions $f \in L^1(S)$ and $g \in L^1(T)$ such that f and g are positive almost everywhere and

$$\int f(s)\, k(s, t)\, g(t)\, \nu(dt) = \alpha(s), \qquad \mu\text{—almost everywhere, and}$$

$$\int f(s)\, k(s, t)\, g(t)\, \mu(ds) = \beta(t), \qquad \nu\text{—almost everywhere.}$$

If f and g satisfy these conditions, we say that "the DAD problem (k, α, β) has a solution (f, g)" or "the DAD problem (1.1) has a solution (f, g)."

The arguments of this section do not directly apply to general functions $k \in L^\infty(S \times T)$. We consider DAD problems for general $k \in L^\infty(S \times T)$ in the next section. Here we restrict attention to those $k \in L^\infty(S \times T)$ for which the integral kernel $k\beta$ gives a compact linear map from $L^\infty(T)$ to $L^\infty(S)$. To describe this case more precisely we need some definitions.

DEFINITION 2.4. Assume Hypothesis 2.1. E and F are linear subspaces of $L^1(S \times T)$ defined by

$$E = \left\{ c \in L^1(S \times T) \colon \|c\|_E := \operatorname*{ess\,sup}_{t \in T} \left(\int |c(s, t)|\, \mu(ds) \right) < \infty \right\} \tag{2.5}$$

and

$$F = \left\{ c \in L^1(S \times T) \colon \|c\|_F := \operatorname*{ess\,sup}_{s \in S} \left(\int |c(s, t)|\, \nu(dt) \right) < \infty \right\}. \tag{2.6}$$

By using Fubini's theorem one can easily see that E and F are well-defined. We leave to the reader the exercise of showing that $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are Banach spaces.

We define linear subspaces V and W of E and F, respectively, by

$$V = \Big\{ c \in E \mid c(s, t) = \sum_{i=1}^{m} f_i(s)\, g_i(t) \text{ almost everywhere, where } m = m(c) < \infty \text{ and } f_i \in L^1(S) \text{ and } g_i \in L^\infty(T) \text{ for } 1 \leqslant i \leqslant m \Big\} \tag{2.7}$$

and

$$W = \Big\{ c \in E \mid c(s, t) = \sum_{i=1}^{m} f_i(s)\, g_i(t) \text{ almost everywhere, where } m = m(c) < \infty,\ f_i \in L^\infty(S), \text{ and } g_i \in L^1(T) \text{ for } 1 \leqslant i \leqslant m \Big\}. \tag{2.8}$$

DEFINITION 2.9. Assume Hypothesis 2.1. $L^1(S) \otimes L^\infty(T)$ denotes the closure in the norm $\|\cdot\|_E$ on E of the linear subspace V of E given by (2.7), and $L^\infty(S) \otimes L^1(T)$ denotes the closure in the norm $\|\cdot\|_F$ on F of the linear subspace W of F given by (2.8).

If $c \in L^1(S \times T)$, it is not hard to prove that $c \in L^\infty(S) \otimes L^1(T)$ if and only if there exists a set $N \subset S$ with $\mu(N) = 0$ such that $\{c(s, \cdot) \mid s \in S - N\}$ has compact closure in $L^1(T)$. A similar statement holds for $L^1(S) \otimes L^\infty(T)$. If $c \in W$ and $c(s, t) = \sum_{i=1}^{m} f_i(s)\, g_i(t)$ with $f_i \in L^\infty(S)$ and $g_i \in L^1(T)$, then c defines bounded linear operators $L_c: L^\infty(T) \to L^\infty(S)$ and $\Lambda_c: L^1(S) \to L^1(T)$ by

$$L_c(z) = \sum_{i=1}^{m} \left(\int g_i(t)\, z(t)\, \nu(dt) \right) f_i \text{ and}$$

$$\Lambda_c(y) = \sum_{i=1}^{m} \left(\int f_i(s)\, y(s)\, \mu(ds) \right) g_i.$$

It is not hard to show that $L_c = (\Lambda_c)^*$, the Banach space adjoint of Λ_c; and if $\|\Lambda_c\|$ and $\|L_c\|$ denote the operator norms of Λ_c and L_c as bounded linear operators,

$$\|c\|_F = \|L_c\| = \|\Lambda_c\|.$$

Thus the maps $c \to L_c$ and $c \to A_c$ are isometric imbeddings of W into Banach spaces of bounded linear operators and $L^\infty(S) \otimes L^1(T)$ is isometric to a closed linear subspace of compact linear operators in $B(L^\infty(T), L^\infty(S))$ and to a closed linear subspace in $B(L^1(S), L^1(T))$. (Here $B(X, Y)$ denotes the Banach space of bounded linear operators from a Banach space X to a Banach space Y.) A similar statement holds for $L^1(S) \otimes L^\infty(T)$. Since we shall not need most of these results, we omit the proofs.

Our immediate reason for introducing the spaces E, F, $L^1(S) \otimes L^\infty(T)$, and $L^\infty(T) \otimes L^1(S)$ is simple. First, if $c_1 \in E$, we can define a linear map $A_1 : L^\infty(S) \to L^\infty(T)$ by

$$(A_1 x)(t) = \int c_1(s, t)\, x(s)\, \mu(ds),$$

and we have

$$\|A_1 x\| = \operatorname*{ess\,sup}_{t \in T} |(A_1 x)(t)| \leqslant \operatorname*{ess\,sup}_{t \in T} \left(\int |c_1(s, t)|\, \|x\|\, \mu(ds) \right)$$
$$\leqslant \|c_1\|_E \|x\|,$$

which shows that $\|A_1\| \leqslant \|c_1\|_E$. The same argument shows that if $c_2 \in F$ and we define a linear map $A_2 : L^\infty(T) \to L^\infty(S)$ by

$$(A_2 y)(s) = \int c_2(s, t)\, y(t)\, v(dt),$$

then A_2 is a bounded linear operator and $\|A_2\| \leqslant \|c_2\|_F$.

If $c_1 \in V$ (V as in (2.7)), the linear map A_1 has finite dimensional range and hence is compact. In general, if $c_1 \in L^1(S) \otimes L^\infty(T)$, there exists a sequence $c_{1n} \in V$ such that $\|c_{1n} - c_1\|_E \to 0$. If $A_{1n} : L^\infty(S) \to L^\infty(T)$ is the linear operator corresponding to c_{1n} and $A_1 : L^\infty(S) \to L^\infty(T)$ is the linear operator corresponding to c_1, we have

$$\|A_1 - A_{1n}\| \leqslant \|c_1 - c_{1n}\|_E.$$

This shows that A_1 is the limit in the operator norm topology of a sequence of compact linear operators, so A_1 is compact. The same argument shows that if $c_2 \in L^\infty(S) \otimes L^1(T)$, then the linear operator $A_2 : L^\infty(T) \to L^\infty(S)$ determined by c_2 is compact.

Of course $L^\infty(S \times T)$ is a linear subspace of both E and F. However, even when $S = T = [0, 1]$ and $\mu = v =$ Lebesque measure, it is not hard to show that $L^\infty(S \times T)$ is not contained in $L^\infty(S) \otimes L^1(T)$ or $L^1(S) \otimes L^\infty(T)$: see Remark 5.6 of [5], where a function $c \in L^\infty(S \times T)$ is given such that the corresponding operator $A_2 : L^\infty(T) \to L^\infty(S)$ is not compact.

Define a linear subspace U of $L^\infty(S\times T)$ by

$$U=\Big\{k\in L^\infty(S\times T): k(s,t)=\sum_{i=1}^{m} f_i(s)\,g_i(t) \text{ almost everywhere,}$$
$$\text{where } m=m(k)<\infty,\ f_i\in L^\infty(S),$$
$$\text{and } g_i\in L^\infty(T) \text{ for } 1\leqslant i\leqslant m\Big\}. \tag{2.10}$$

DEFINITION 2.11. Assume Hypothesis 2.1. $L^\infty(S)\otimes L^\infty(T)$ denotes the closure in the norm on $L^\infty(S\times T)$ of the linear subspace U (U as in (2.10)).

If $k\in L^\infty(S)\otimes L^\infty(T)$ and $B_2\colon L^1(T)\to L^\infty(S)$ and $B_1\colon L^1(S)\to L^\infty(T)$ are defined by

$$(B_1 v)(t)=\int k(s,t)\,v(s)\,\mu(ds) \qquad \text{and} \qquad (B_2 w)(s)=\int k(s,t)\,w(t)\,\nu(dt),$$

the same sort of argument used in discussing A_1 and A_2 shows that B_1 and B_2 are compact linear operators. If Hypothesis 2.2 is satisfied and $k\colon S\times T\to\mathbb{R}$ is continuous, it is known (and not hard to prove with a partition-of-unity argument) that $k\in L^\infty(S)\otimes L^\infty(T)$. Thus there are many examples of functions in $L^\infty(S)\otimes L^\infty(T)$. We leave the proofs of these facts to the reader.

If $k\in L^\infty(S)\otimes L^\infty(T)$, $\alpha\in L^1(S)$, and $\beta\in L^1(T)$, it is not hard to prove that $k\alpha\in L^1(S)\otimes L^\infty(T)$ and $k\beta\in L^\infty(S)\otimes L^1(T)$. However, one can give examples of functions k with $k\in L^\infty(S\times T)$, $k\in L^1(S)\otimes L^\infty(T)$, and $k\in L^\infty(S)\otimes L^1(T)$, but $k\notin L^\infty(S)\otimes L^\infty(T)$. For instance, if $S=T=[0,1]$ and $\mu=\nu=$ Lebesque measure and $k(s,t)=1$ for $s\geqslant t$ and $k(s,t)=0$ for $s<t$, k is such an example. The only tricky point to verify is that $k\notin L^\infty(S)\otimes L^\infty(T)$. However, this follows because the map $B_1\colon L^1(S)\to L^\infty(T)$ determined by k is *not* compact, as one can see by using the Ascoli–Arzela theorem. This observation and later applications are, in fact, our motivation for introducing $L^1(S)\otimes L^\infty(T)$ and $L^\infty(S)\otimes L^1(T)$.

There may be necessary conditions other than (1.2) which must be satisfied by (k,α,β) in order for the *DAD* problem (1.1) to have a solution. These conditions are well known in the matrix case (see [23]), and the arguments for necessary conditions carry over to our situation. Thus suppose that Hypothesis 2.3 is satisfied and that $f\in L^1(S)$ and $g\in L^1(T)$ are functions which are positive almost everywhere and satisfy (1.1). Define $\tilde{k}$ by

$$\tilde{k}(s,t)=f(s)\,k(s,t)\,g(t).$$

Assume that $I \subset S$ and $J \subset T$ are measurable sets of positive measure and that $k(s,t)=0$ for almost all $(s,t) \in I' \times J'$, where I' is the complement of I and J' the complement of J (in general, we will always write A' for the complement of a set A). Then, we have

$$\int_I \alpha(s)\,\mu(ds) = \int_I \int_T \tilde{k}(s,t)\,\nu(dt)\,\mu(ds)$$

$$= \int_I \int_J \tilde{k}(s,t)\,\nu(dt)\,\mu(ds) + \int_I \int_{J'} \tilde{k}(s,t)\,\nu(dt)\,\mu(ds),$$

and

$$\int_{J'} \beta(t)\,\nu(dt) = \int_{I'} \int_{J'} \tilde{k}(s,t)\,\nu(dt)\,\mu(ds) + \int_I \int_{J'} \tilde{k}(s,t)\,\nu(dt)\,\mu(ds)$$

$$= \int_I \int_{J'} \tilde{k}(s,t)\,\nu(dt)\,\mu(ds).$$

It follows that

$$\int_I \alpha(s)\,\mu(ds) \geqslant \int_{J'} \beta(t)\,\nu(dt) \tag{2.12}$$

and that strict inequality holds in (2.12) whenever $k \mid I \times J$ is positive on a subset of $I \times J$ of positive measure.

DEFINITION 2.13. Suppose that k, α, and β satisfy Hypothesis 2.3. We say that "(k, α, β) satisfies the compatibility condition" if

$$\int \alpha(s)\,\mu(ds) = \int \beta(t)\,\nu(dt),$$

and whenever $I \subset S$ and $J \subset T$ are measurable sets with $\mu(I) > 0$ and $\nu(J) > 0$ and $k(s,t)=0$ for almost all $(s,t) \in I' \times J'$, it follows that inequality (2.12) is satisfied and that strict inequality holds in (2.12) if k is positive on a subset of $I \times J$ of positive measure. If Hypothesis 2.2 and 2.3 are satisfied and k is continuous and

$$\int \alpha(s)\,\mu(ds) = \int \beta(t)\,\nu(dt),$$

we say that "(k, α, β) satisfies the weak compatibility condition" if, whenever $I \subset S$ and $J \subset T$ are nonempty open sets and $k(s,t)=0$ for all $(s,t) \in I' \times J'$, (2.12) is satisfied and strict inequality holds in (2.12) if $k(s_0, t_0) > 0$ for some $(s_0, t_0) \in I \times J$.

As we have noted, it is necessary that (k, α, β) satisfy the compatibility condition for (1.1) to have a solution.

Note that if (k, α, β) satisfies the compatibility condition and $k(s, t) = 0$ for almost all $(s, t) \in I' \times J'$ and almost all $(s, t) \in (I, J)$, then (taking $\tilde{I} = I'$ and $\tilde{J} = J'$) we obtain

$$\int_{I'} \alpha(s)\, \mu(ds) \geqslant \int_J \beta(t)\, \nu(dt).$$

Combining this with (2.12) and recalling that

$$\int_S \alpha(s)\, \mu(ds) = \int_T \beta(t)\, \nu(dt),$$

we find

$$\int_{I'} \alpha(s)\, \mu(ds) = \int_J \beta(t)\, \nu(dt) \qquad \text{and} \qquad \int_I \alpha(s)\, \mu(ds) = \int_{J'} \beta(t)\, \nu(dt).$$

We also need the analogue in our setting of the idea of an "indecomposable matrix."

DEFINITION 2.14. Suppose that S and T satisfy Hypothesis 2.1 (respectively, Hypothesis 2.2) and that $k \in L^{\infty}(S \times T)$ is a nonnegative function (respectively, $k \in C(S \times T)$ is a continuous, nonnegative function). We say that k is "indecomposable" (respectively, "weakly indecomposable") if whenever $I \subset S$ and $J \subset T$ are measurable (respectively, open) sets of positive measure with $k(s, t) = 0$ for almost all $(s, t) \in I' \times J'$ (respectively, $k(s, t) = 0$ for all $(s, t) \in I' \times J'$), it follows that there is a set $E \subset I \times J$ such that E has positive measure and $k(s, t) > 0$ for almost all $(s, t) \in E$.

Of course our definition here is a direct generalization of the definition of an $m \times n$ indecomposable nonnegative matrix.

To solve the DAD problem (1.1) in the matrix case, it is clearly necessary that the matrix have no zero row sums and no zero column sums. We shall use a condition on k which generalizes the matrix assumption.

HYPOTHESIS 2.15. *Hypothesis 2.1 is satisfied and $k \in L^{\infty}(S \times T)$ is a nonnegative function. There exists $\delta > 0$ with*

$$\int_S k(s, t)\, \mu(ds) \geqslant \delta, \qquad \nu \textit{ almost everywhere, and}$$

$$\int_T k(s, t)\, \nu(dt) \geqslant \delta, \qquad \mu \textit{ almost everywhere.}$$

Remark. If Hypothesis 2.15 is satisfied and $k(s,t)\leqslant M$ for almost all (s,t), Fubini's theorem implies that for μ almost all s, $k(s,t)\leqslant M$ for ν almost all t. Similarly, for ν almost all t, $k(s,t)\leqslant M$ for μ almost all s. For δ as in Hypothesis 2.15, select $c>0$ with

$$c\nu(T)\leqslant(\tfrac{1}{2})\delta \qquad \text{and} \qquad c\mu(S)\leqslant(\tfrac{1}{2})\delta.$$

For $s\in S$ and $t\in T$ define E_s and F_t by

$$E_s=\{t \mid k(s,t)\geqslant c\} \qquad \text{and} \qquad F_t=\{s \mid k(s,t)\geqslant c\}.$$

It follows that for almost all s,

$$\delta\leqslant\int_T k(s,t)\,\nu(dt)\leqslant M\nu(E_s)+c\nu(T-E_s)\leqslant M\nu(E_s)+\left(\frac{1}{2}\right)\delta,$$

so

$$\nu(E_s)\geqslant\delta_1=\left(\frac{2}{2M}\right).$$

Similarly, we find that

$$\mu(F_t)\geqslant\delta_1, \qquad \nu \text{ almost everywhere.}$$

There is one situation in which Hypothesis 2.15 is clearly satisfied.

LEMMA 2.16. *Suppose that Hypothesis* 2.2 *is satisfied and that* $k\in L^\infty(S\times T)$ *is nonnegative almost everywhere. For each* $s\in S$ *assume that there exist a positive constant* c_s, *an open neighborhood* U_s *of* s, *and a measurable set* $V_s\subset T$ *of positive measure with*

$$k(\sigma,\tau)\geqslant c_s \qquad \textit{for almost all} \quad (\sigma,\tau)\in U_s\times V_s.$$

Similarly, for each $t\in T$ *assume that there exist* $d_t>0$, *an open neighborhood* H_t *of* t, *and a measurable set* G_t *of positive measure with*

$$k(\sigma,\tau)\geqslant d_t \qquad \textit{for almost all} \quad (\sigma,\tau)\in G_t\times H_t.$$

Then k *satisfies Hypothesis* 2.15.

Proof. By compactness of S and T, we can find $s_i\in S$, $1\leqslant i\leqslant m$, and $t_j\in T$, $1\leqslant j\leqslant n$, such that

$$S=\bigcup_{i=1}^{m} U_{s_i} \qquad \text{and} \qquad T=\bigcup_{j=1}^{n} H_{t_j}.$$

If we define $c = \min\{c_i, d_j : 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ and $\eta = \min\{\nu(V_{s_i}), \mu(G_{t_j}) : 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$, we see that Hypothesis 2.15 is satisfied with $\delta = c\eta$. ∎

If k is continuous and nonnegative on $S \times T$ and Hypothesis 2.2 is satisfied, Lemma 2.16 implies that k satisfies Hypothesis 2.15 if and only if

$$\int k(s, t)\, \nu(dt) > 0 \text{ for all } s \qquad \text{and} \qquad \int k(s, t)\, \mu(ds) > 0 \text{ for all } t,$$

which is the exact analogue of the matrix condition.

Our approach in this section to solving (1.1) will be to reduce to an equivalent fixed point problem. Define $X = L^\infty(S)$ and $Y = L^\infty(T)$ and (throughout this paper) let C_1 be the cone of nonnegative functions in X and C_2 the cone of nonnegative functions in Y. As usual, $\mathring{C}_j$ denotes the interior of C_j. If Hypothesis 2.3 is satisfied, define linear operators $A_1 : X \to Y$ and $A_2 : Y \to X$ by

$$\begin{aligned} (A_1 v)(t) &= \int k(s, t)\, \alpha(s)\, v(s)\, \mu(ds) \qquad \text{and} \\ (A_2 w)(s) &= \int k(s, t)\, \beta(t)\, w(t)\, \nu(dt). \end{aligned} \tag{2.17}$$

It is easy to see that A_1 and A_2 are bounded linear operators and that

$$\|A_1\| \leqslant \|k\|_\infty \|\alpha\|_1 \qquad \text{and} \qquad \|A_2\| \leqslant \|k\|_\infty \|\beta\|_1.$$

Furthermore we see that $A_1(C_1) \subset C_2$ and $A_2(C_2) \subset C_1$. Similarly, we can define bounded linear operators $B_1 : L^1(S) \to L^\infty(T)$ and $B_2 : L^1(T) \to L^\infty(S)$ by

$$(B_1 f)(t) = \int k(s, t)\, f(s)\, \mu(ds) \qquad \text{and} \qquad (B_2 g)(s) = \int k(s, t)\, g(t)\, \nu(dt).$$

LEMMA 2.18. *Assume that Hypotheses 2.3 and 2.15 are satisfied. Then it follows that* $A_1(\mathring{C}_1) \subset \mathring{C}_2$ *and* $A_2(\mathring{C}_2) \subset \mathring{C}_1$, *where* A_j *is defined in* (2.17). *Furthermore, if* $f \in L^1(S)$ *is positive,* μ *a.e., and* $g \in L^1(T)$ *is positive,* ν *a.e., then* $B_1(f) \in \mathring{C}_2$ *and* $B_2(g) \in \mathring{C}_1$.

Proof. Because α and β are positive almost everywhere and $B_1(\alpha v) = A_1(v)$ and $B_2(\beta w) = A_2(w)$, it suffices to prove that $B_1(f) \in \mathring{C}_2$ if $f \in L^1(S)$ is positive almost everywhere and $B_2(g) \in \mathring{C}_2$ if $g \in L^1(T)$ is positive almost everywhere. We restrict attention to B_1, since the proof for B_2 is the same. Let $M = \|k\|_{L^\infty}$,

$$V = \{x \in L^\infty(S) : \|x\|_\infty \leqslant M\}, \text{ and}$$

$$V_1 = \left\{x \in V : \int x(s)\,\mu(ds) \geqslant \delta \text{ and } x(s) \geqslant 0,\ \mu \text{ almost everywhere}\right\},$$

where δ is a constant as in Hypothesis 2.15. The Banach–Alaoglu theorem implies that V is compact in the weak* topology, and since V_1 can be written as the intersection of weak* closed subsets of V, V_1 is closed in V and compact in the weak* topology. If $f \in L^1(S)$ is positive, μ a.e., then because $(L^1(S))^* = L^\infty(S))$, the map $\Phi\colon V_1 \to \mathbb{R}$,

$$\Phi(x) = \int x(s)\,f(s)\,\mu(ds),$$

is continuous in the weak* topology on V_1. Since Φ is positive on V_1, there exists $\kappa > 0$ with

$$\Phi(x) \geqslant \kappa \qquad \text{for all} \quad x \in V_1.$$

However, Hypothesis 2.15 implies that for almost all $t \in T$, $x(s) = k(s, t)$ yields an element of V_1, so

$$\int k(s, t)\,f(s)\,\mu(ds) \geqslant \kappa, \qquad \nu \text{ a.e.}$$

It follows easily that $B_1 f \in \mathring{C}_2$. ∎

We can define $J_k\colon \mathring{C}_k \to \mathring{C}_k$ $(k = 1, 2)$ by

$$(J_k z)(r) = \left(\frac{1}{z(r)}\right). \tag{2.19}$$

Assuming that Hypotheses 2.1, 2.3, and 2.15 are satisfied, we define $F\colon \mathring{C}_1 \to \mathring{C}_1$ by

$$F = A_2 J_2 A_1 J_1. \tag{2.20}$$

LEMMA 2.21. *Assume that Hypotheses* 2.3 *and* 2.15 *are satisfied and that* $\int \alpha(s)\,\mu(ds) = \int \beta(t)\,\nu(dt)$. *The DAD problem* (k, α, β) *has a solution* (f, g) *(see* (1.1)) *if and only if there exist* $x \in \mathring{C}_1$ *and* $\lambda > 0$ *with*

$$F(x) = \lambda x.$$

If $x \in \mathring{C}_1$ *and* $\lambda > 0$ *exist, then it is necessary that* $\lambda = 1$.

Proof. Suppose that $F(x) = \lambda x$ for some $x \in \mathring{C}_1$, and $\lambda > 0$. By Lemma 2.18, $y = A_1 J_1 x \in \mathring{C}_2$, so if we define f and g by

$$f(s) = \frac{\alpha(s)}{x(s)} \quad \text{and} \quad g(t) = \frac{\beta(t)}{y(t)},$$

$f \in L^1(S)$, $g \in L^1(T)$, and f and g are positive almost everywhere. The definition of y gives

$$\int g(t)\, k(s, t)\, f(s)\, \mu(ds) = \beta(t), \qquad \nu \text{ a.e.}$$

Because $F(x) = \lambda x$, we have that $A_2 J_2 y = \lambda x$ and

$$\int g(t)\, k(s, t)\, f(s)\, \nu(dt) = \lambda \alpha(s), \qquad \mu \text{ a.e.}$$

However, we also see that

$$\lambda \int \alpha(s)\, \mu(ds) = \iint g(t)\, k(s, t)\, f(s)\, \nu(dt)\, \mu(ds) = \int \beta(t)\, \nu(dt),$$

and we assume that

$$\int \alpha(s)\, \mu(ds) = \int \beta(t)\, \nu(dt),$$

so we must have $\lambda = 1$.

Conversely, suppose that $f \in L^1(S)$ and $g \in L^1(T)$ are positive almost everywhere and solve (1.1). If we define x and y by

$$x(s) = \int k(s, t)\, g(t)\, \nu(dt) \quad \text{and} \quad y(t) = \int k(s, t)\, f(s)\, \mu(ds),$$

we obtain from (1.1) that

$$f(s) = \frac{\alpha(s)}{x(s)} \text{ and } g(t) = \frac{\beta(t)}{y(t)} \qquad \text{almost everywhere.}$$

Lemma 2.18 implies that $x \in \mathring{C}_1$ and $y \in \mathring{C}_2$. Thus, for almost all $t \in T$, we have

$$\begin{aligned}(A_1 J_1 x)(t) &= \int k(s, t) \left(\frac{\alpha(s)}{x(s)}\right) \mu(ds) = \int k(s, t)\, f(s)\, \mu(ds) \\ &= \frac{\beta(t)}{g(t)} = y(t).\end{aligned}$$

Similarly, for almost all $s \in T$, we have

$$(A_2 J_2 y)(s) = \int k(s, t) \frac{\beta(t)}{y(t)} \nu(dt) = \int k(s, t)\, g(t)\, \nu(dt)$$
$$= \frac{\alpha(s)}{f(s)} = x(s),$$

so $F(x) = x$. ∎

Note that the proof of Lemma 2.21 also shows that if the DAD problem (k, α, β) has a solution (f, g), then $f = \alpha/x$ and $g = \beta/y$ for some $x \in \mathring{C}_1$ and $y \in \mathring{C}_2$.

Of course the idea (at least in the matrix case) that solving (1.1) is equivalent to finding a fixed point of a certain operator F goes back to the earliest work on DAD problems: see [20] and [22] for example. In addition, in the matrix case Menon and Schneider [23] have observed that F extends continuously to C_1. This observation has been extended to a larger class of maps $\mathcal{M}$ in [27, Sect. 2]. In general, however, it is not hard to see that F may not extend continuously to all of C_1. To see this, take $S = T = [0, 1]$ with $\mu = \nu =$ Lebesque measure. Take $\alpha = \beta = 1$ and $k = 1$ on $S \times T$. If we assume that F extends continuously to C_1 and if $x \in C_1$ and $x(s) = 0$ for s in a set of positive measure, it is easy to see that $F(x) = 0$. Thus, if $\varepsilon_n > 0$ and $\lim_{n \to \infty} \varepsilon_n = 0$ and if we define $x_n \in C_1$ by

$$x_n(s) = \begin{cases} \sqrt{s - \varepsilon_n}, & {}_n \leqslant s \leqslant 1 \\ 0, & 0 \leqslant s \leqslant \varepsilon_n \end{cases}$$

we must have $F(x_n) = 0$. On the other hand, if $x(s) = \sqrt{s}$, it is not hard to see that

$$F(x) = c = \left(\int_0^1 \left(\frac{1}{\sqrt{s}} \right) ds \right)^{-1} > 0.$$

Since $\lim_{n \to \infty} x_n = x$ in C_1, we see that F cannot be continuous.

Our strategy now will be to replace F by a nicer operator F_ε, $\varepsilon > 0$, prove that F_ε has an eigenvector $x_\varepsilon \in \mathring{C}_1$, and show (under appropriate assumptions) that there exists a sequence $\varepsilon_j \to 0^+$ such that $x_{\varepsilon_j} \to x \in \mathring{C}_1$ in the $L^\infty(S)$ norm. Proving convergence of x_{ε_j} to a point in $\mathring{C}_1$ will be the difficult point.

Before proceeding further we need to recall some facts about Hilbert's projective metric. We refer to Sections 1 and 2 of [26] or to [8] for more details and references to the literature. Recall that a cone C in a Banach space X is a closed convex subset of X such that (a) $tC \subset C$ for all $t \geqslant 0$ and (b) $x \in C - \{0\}$ implies that $-x \notin C$. A cone C induces a partial ordering by $x \leqslant y$ if $y - x \in C$. A cone C is called "normal" if there exists a constant M such that $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant M \|y\|$. If $u \in C - \{0\}$, we shall denote by C_u the set

$$C_u = \{x \in C \mid \exists a > 0 \text{ and } b > 0 \text{ such that } au \leqslant x \leqslant bu\}.$$

If C has nonempty interior $\mathring{C}$ and $u \in \mathring{C}$, note that

$$C_u = \mathring{C}.$$

If $u \in C - \{0\}$ and $x, y \in C_u$, we define

$$M(y/x) = \inf\{b > 0 \mid y \leqslant bx\} \text{ and}$$

$$m(y/x) = \sup\{a > 0 \mid ax \leqslant y\}.$$

If $b = M(y/x)$ and $a = m(y/x)$, we define $d(x, y)$, Hilbert's projective metric, by

$$d(y, x) = \log\left(\frac{b}{a}\right).$$

If C is a normal cone, $u \in C - \{0\}$, ψ is a continuous linear functional which is strictly positive on C_u, and $\Sigma = \{x \in C_u \mid \psi(x) = 1\}$, it is known that d restricted to $\Sigma \times \Sigma$ makes (Σ, d) a complete metric space and the d topology on Σ is the same as the norm topology. (See Sect. 1 of [26] for references to the literature.)

Next suppose that C is a cone in a Banach space X and D is a cone in a Banach space Y. Suppose that $L: X \to Y$ is a bounded linear map and that, for $u \in C - \{0\}$ and $v \in D - \{0\}$,

$$L(C_u) \subset D_v.$$

Let d_1 denote Hilbert's projective metric on C_u and d_2 Hilbert's projective metric on D_v and define

$$\Delta(L) = \sup\{d_2(Lx, Ly) : x, y \in C_u\}.$$

If $\Delta(L) < \infty$, and if we define

$$\kappa = \tanh(\tfrac{1}{4}\Delta(L)) < 1,$$

where tanh denotes the hyperbolic tangent, Birkhoff [2, 3] has proved that for all $x, y \in C_u$,

$$d_2(Lx, Ly) \leqslant \kappa\, d_1(x, y).$$

The inequality holds with $\kappa = 1$ even if $\Delta(L) = \infty$. Summaries of closely related material and further references to the literature can found in [26, pp. 42–45].

If Hypotheses 2.3 and 2.15 are satisfied and $\varepsilon > 0$, define $A_{2,\varepsilon}: L^{\infty}(T) \to L^{\infty}(S)$ by

$$(A_{2,\varepsilon}w)(s) = \int [k(s,t) + \varepsilon]\, \beta(t)\, w(t)\, \nu(dt) \tag{2.22}$$

and define $F_\varepsilon: \mathring{C}_1 \to \mathring{C}_1$ by

$$F_\varepsilon = A_{2,\varepsilon} J_2 A_1 J_1. \tag{2.23}$$

LEMMA 2.24. *Assume that Hypotheses 2.3 and 2.15 are satisfied and that*

$$\int \alpha(s)\, \mu(ds) = \int \beta(t)\, \nu(dt).$$

If $\varepsilon > 0$ and F_ε is given by (2.23), F_ε has an eigenvector $x_\varepsilon \in \mathring{C}_1$,

$$F_\varepsilon(x_\varepsilon) = \lambda_\varepsilon x_\varepsilon,$$

and $\lambda_\varepsilon > 1$. The eigenvector x_ε is unique to within scalar multiples. If $\psi: \mathring{C}_1 \to \mathbb{R}^+$ is defined by

$$\psi(x) = \int x(s)\, \mu(ds)$$

and if, for $v \in \mathring{C}_1$, v_n is defined by

$$v_n = \frac{F_\varepsilon^n(v)}{\psi(F_\varepsilon^n(v))},$$

then v_n converges geometrically in norm to the unique eigenvector x_ε of F_ε such that $\psi(x_\varepsilon) = 1$ and *$x_\varepsilon \in \mathring{C}_1$.*

Proof. If, for $v \in C_2 - \{0\}$, we define

$$\delta = \int \beta(t)\, v(t)\, \nu(dt) > 0$$

and if $M = \|k + \varepsilon\|_\infty$, it is clear that

$$\varepsilon\delta \leqslant (A_{2,\varepsilon}v)(s) \leqslant M\delta, \qquad \mu \text{ almost everywhere.}$$

If u denotes the function in $\mathring{C}_1$ identically equal to one, we have (where d_j denotes Hilbert's projective metric on $\mathring{C}_j$)

$$d_1(A_{2,\varepsilon}v, \delta u) = d_1(A_{2,\varepsilon}v, u) \leqslant \log(M/\varepsilon).$$

It follows by the triangle inequality for d_1 that for any $v_1, v_2 \in C_2 - \{0\}$

$$d_1(A_{2,\varepsilon}v_1, A_{2,\varepsilon}v_2) \leqslant 2\log\left(\frac{M}{\varepsilon}\right) \qquad \text{and}$$

$$\Delta(A_{2,\varepsilon}) \leqslant 2\log\left(\frac{M}{\varepsilon}\right).$$

The previously mentioned result of Birkhoff [2, 3] implies that

$$d_1(A_{2,\varepsilon}v, A_{2,\varepsilon}w) \leqslant \kappa\, d_2(v, w) \qquad \text{for all} \quad v, w \in \mathring{C}_2,$$

$$\kappa = \tanh(\tfrac{1}{4}\Delta(A_{2,\varepsilon})) < 1.$$

It is easy to see that for all $x, y \in \mathring{C}_k$,

$$d_k(J_k x, J_k y) = d_k(x, y),$$

and we have already noted that

$$d_2(A_1 x, A_1 y) \leqslant d_1(x, y) \qquad \text{for all} \quad x, y \in \mathring{C}_1.$$

It follows that for all $x, y \in \mathring{C}_1$,

$$d_1(F_\varepsilon x, F_\varepsilon y) \leqslant \kappa\, d_1(x, y).$$

If we define $\Sigma = \{x \in \mathring{C}_1 : \psi(x) = 1\}$ and define $\Phi_\varepsilon : \Sigma \to \Sigma$ by

$$\Phi_\varepsilon(x) = \frac{F_\varepsilon(x)}{\psi(F_\varepsilon(x))},$$

then the basic properties of d_1 imply that

$$d_1(\Phi_\varepsilon(x), \Phi_\varepsilon(y)) \leqslant \kappa\, d_1(x, y) \qquad \text{for all} \quad x, y \in \Sigma.$$

Because (Σ, d_1) is a complete metric space (and because F_ε is homogeneous of degree one), the contraction mapping principle implies that for any $v \in \Sigma$ (and hence for all $v \in \mathring{C}_1$)

$$\lim_{n \to \infty} \Phi_\varepsilon^n(v) = \lim_{n \to \infty} \left(\frac{F_\varepsilon^n(v)}{\psi(F_\varepsilon^n(v))}\right) = x_\varepsilon,$$

where x_ε is the unique fixed point of Φ_ε in Σ. It follows that

$$F_\varepsilon(x_\varepsilon) = \lambda_\varepsilon x_\varepsilon, \qquad \lambda_\varepsilon = \psi(F_\varepsilon(x_\varepsilon)).$$

The uniqueness (to within scalar multiplies) of an eigenvector of F_ε in $\mathring{C}_1$ follows from the above remarks and the homogeneity of F_ε.

It remains to prove that $\lambda_\varepsilon > 1$. If we define $f_\varepsilon(s)$ and $g_\varepsilon(t)$ by

$$f_\varepsilon(s) = \frac{\alpha(s)}{x_\varepsilon(s)}, \qquad y_\varepsilon = (A_1 J_1)(x_\varepsilon), \qquad \text{and} \qquad g_\varepsilon(t) = \frac{\beta(t)}{y_\varepsilon(t)},$$

the argument in Lemma 2.21 shows that

$$\int g_\varepsilon(t)\, k(s, t)\, f_\varepsilon(s)\, \mu(ds) = \beta(t), \qquad \nu \text{ a.e.},$$

and

$$\int g_\varepsilon(t)(k(s, t) + \varepsilon)\, f_\varepsilon(s)\, \nu(dt) = \lambda_\varepsilon \alpha(s), \qquad \mu \text{ a.e.}$$

It follows that

$$\lambda_\varepsilon \int \alpha(s)\, \mu(ds) = \iint g_\varepsilon(t)(k(s, t) + \varepsilon)\, f_\varepsilon(s)\, \nu(dt)\, \mu(ds)$$

$$> \iint g_\varepsilon(t)\, k(s, t)\, f_\varepsilon(s)\, \nu(dt)\, \mu(ds) = \int \beta(t)\, \nu(dt),$$

so

$$\lambda_\varepsilon > \left(\frac{\int \beta(t)\, \nu(dt)}{\int \alpha(s)\, \mu(ds)} \right) = 1. \quad \blacksquare$$

Remark 2.25. It is important to note that, under the hypothesis of Lemma 2.24 and for $0 < \varepsilon \leqslant 1$, there exists a constant M, independent of ε, with

$$\lambda_\varepsilon \leqslant M.$$

To see this, observe that by the homogeneity of F_ε, we can choose $x_\varepsilon \in \mathring{C}_1$ with $\|x_\varepsilon\|_\infty = 1$ and

$$F_\varepsilon(x_\varepsilon) = \lambda_\varepsilon x_\varepsilon.$$

If $y_\varepsilon = (A_1 J_1)(x_\varepsilon)$ we have

$$y_\varepsilon(t) = \int k(s, t) \frac{\alpha(s)}{x_\varepsilon(s)} \mu(ds) \geqslant \int k(s, t)\, \alpha(s)\, \mu(ds), \qquad \nu \text{ a.e.}$$

Lemma 2.18 implies that $A_1(\mathring{C}_1) \subset \mathring{C}_2$, so there exists a positive constant κ

$$\int k(s, t)\, \alpha(s)\, \mu(ds) \geqslant \kappa, \qquad \nu \text{ a.e.}$$

It follows that

$$\lambda_\varepsilon x_\varepsilon(s) \leqslant \int (k(s, t) + \varepsilon) \left(\frac{\beta(t)}{\kappa}\right) \nu(dt), \qquad \mu \text{ a.e.}$$

There exists a constant M, independent of ε for $0 < \varepsilon \leqslant 1$, with

$$\int (k(s, t) + \varepsilon) \left(\frac{\beta(t)}{\kappa}\right) \nu(dt) \leqslant M, \qquad \mu \text{ a.e.}$$

Since $\|x_\varepsilon\|_\infty = 1$, it follows that $\lambda_\varepsilon \leqslant M$ for $0 < \varepsilon \leqslant 1$.

Note that the above argument also shows that

$$(y_\varepsilon(t))^{-1} \leqslant \kappa^{-1} < \infty, \qquad \nu \text{ almost everywhere.}$$

We are now in a position to prove our first theorem.

THEOREM 2.26. *Assume that Hypotheses* 2.3 *and* 2.15 *are satisfied. Suppose also that k is indecomposable* (*Definition* 2.14) *and that* (k, α, β) *satisfies the compatibility condition* (*Definition* 2.13). *If* $A_2: L^\infty(T) \to L^\infty(S)$ *and* $A_1: L^\infty(S) \to L^\infty(T)$ *are defined by* (2.17), *assume that either* A_2 *or* A_1 *is compact.* (*Recall that* A_1 *is compact if* $k\alpha \in L^1(S) \otimes L^\infty(T)$, *as in Definition* 2.9, *and* A_2 *is compact if* $k\beta \in L^\infty(S) \otimes L^1(T)$. *In particular* A_1 *and* A_2 *are compact if* $k \in L^\infty(S) \otimes L^\infty(T)$ *or if Hypothesis* 2.2 *is satisfied and k is continuous.*) *Then there exist* $x \in \mathring{C}_1$ (*the interior of the cone of nonnegative functions in* $L^\infty(S)$) *and* $y \in \mathring{C}_2$ (*the interior of the cone of nonnegative functions in* $L^\infty(T)$) *so that if* $f(s) = \alpha(s)/x(s)$ *and* $g(t) = \beta(t)/y(t)$, (f, g) *is a solution of the DAD problem* (k, α, β).

Proof. By symmetry in the roles of S and T, we can assume that A_2 is compact. If $k\beta \in L^\infty(S) \otimes L^1(T)$, we have already remarked that A_2 is compact. If $k \in L^\infty(S) \otimes L^\infty(T)$ or if Hypothesis 2.2 is satisfied and k is continuous, we have also already noted that $k\beta \in L^\infty(S) \otimes L^1(T)$, so A_2 is compact in this case too.

Let F_ε, $\varepsilon > 0$, be defined by (2.23), and by Lemma 2.24 select $\lambda_\varepsilon > 1$ and $x_\varepsilon \in \mathring{C}_1$, $\|x_\varepsilon\|_\infty = 1$, with

$$F_\varepsilon(x_\varepsilon) = \lambda_\varepsilon x_\varepsilon.$$

We already know (see Remark 2.25) that there exist positive constants M and κ, independent of ε for $0<\varepsilon\leqslant 1$, with

$$\lambda_\varepsilon \leqslant M \qquad \text{and} \qquad \|J_2(y_\varepsilon)\|_\infty \leqslant \kappa^{-1},$$

where $y_\varepsilon = A_1 J_1 x_\varepsilon$. Since

$$F_\varepsilon(x_\varepsilon) = A_{2,\varepsilon} J_2(y_\varepsilon) = \lambda_\varepsilon x_\varepsilon$$

and $J_2(y_\varepsilon)$ is bounded in $L^\infty(T)$, the compactness of A_2 and the boundedness of λ_ε imply that there exists a sequence $\varepsilon_j \downarrow 0$, $x \in C_1$, and $\lambda \geqslant 1$ with

$$\lim_{j\to\infty} |\lambda_{\varepsilon_j} - \lambda| = 0 \qquad \text{and} \qquad \lim_{j\to\infty} \|x_{\varepsilon_j} - x\|_\infty = 0.$$

Furthermore, because $z_{\varepsilon_j} = J_2(y_{\varepsilon_j})$ is bounded in $L^\infty(T)$, by taking a further subsequence, which we continue to label ε_j, we can assume that z_{ε_j} converges to z in the weak* topology on $L^\infty(T)$,

$$z_{\varepsilon_j} \xrightarrow[w^*]{} z.$$

It is easy to see that $z(t)\geqslant 0$, ν a.e. For notational convenience, we write $x_j = x_{\varepsilon_j}$, $y_j = y_{\varepsilon_j}$, $z_j = z_{\varepsilon_j}$, and $\lambda_j = \lambda_{\varepsilon_j}$.

For each $s\in S$, $k(s,\cdot)\,\beta(\cdot)$ is a function in $L^1(T)$, so

$$\lambda x(s) = \lim_{j\to\infty} \lambda_j x_j(s) = \lim_{j\to\infty} \int (k(s,t)+\varepsilon_j)\,\beta(t)\,z_j(t)\,\nu(dt)$$

$$= \int k(s,t)\,\beta(t)\,z(t)\,\nu(dt), \qquad \mu \text{ a.e.}$$

Here we have used the fact that

$$\lim_{j\to\infty} \int \varepsilon_j \beta(t)\,z_j(t)\,\nu(dt) = 0$$

because $\|z_j\|$ is bounded uniformly in j.

We define measurable sets I and J by

$$I=\{s\in S : x(s)>0\} \qquad \text{and} \qquad J=\{t\in T : z(t)=0\}.$$

(Of course I and J are only defined to within sets of measure zero.) We know that $\mu(I)>0$, because $\|x\|_\infty = 1$. If $\nu(J)=0$, z is positive almost everywhere and Lemma 2.18 implies that $x\in \mathring{C}_1$, where

$$\lambda x(s) = \int k(s,t)\,\beta(t)\,z(t)\,\nu(dt), \qquad \mu \text{ a.e.}$$

However, if $x \in \mathring{C}_1$, the fact that x_j approaches x in $L^\infty(S)$ norm implies that

$$\lim_{j \to \infty} \|J_1(x_j) - J_1(x)\|_\infty = 0.$$

Using this, it is easy to argue that

$$\lambda x = \lim_{j \to \infty} F_{\varepsilon_j}(x_j) = F(x),$$

and Lemma 2.21 then yields our theorem.

Thus we assume that $\nu(J) > 0$ and try to obtain a contradiction. We also know that $\nu(J') > 0$, since otherwise $x(s) = 0$ almost everywhere. We claim that

$$\int_I \alpha(s)\, \mu(ds) > \int_{J'} \beta(t)\, \nu(dt). \tag{2.27}$$

If $\mu(I') = 0$, we have

$$\int_I \alpha(s)\, \mu(ds) = \int_S \alpha(s)\, \mu(ds) = \int_T \beta(t)\, \nu(dt) > \int_{J'} \beta(t)\, \nu(dt).$$

Thus, for the purposes of proving (2.27), we can assume that $\mu(I') > 0$. For almost all $s \in I'$, we have

$$0 = \lambda x(s) = \int_{J'} k(s, t)\, \beta(t)\, z(t)\, \nu(dt).$$

Since $z(t) > 0$ for almost all $t \in J'$, it follows that for almost all $s \in I'$, $k(s, t) = 0$ for almost all $t \in J'$. By using Fubini's theorem, we conclude that $k(s, t) = 0$ for almost all $(s, t) \in I' \times J'$. Since k is indecomposable, it follows that $k \mid I \times J$ is positive on a set of positive measure. The compatibility condition for (k, α, β) now implies that (2.27) is satisfied.

For almost all $t \in J'$, we have that

$$\begin{aligned} y_j(t) &= \int_S k(s, t) \left(\frac{\alpha(s)}{x_j(s)}\right) \mu(ds) \\ &= \int_I k(s, t) \left(\frac{\alpha(s)}{x_j(s)}\right) \mu(ds), \end{aligned}$$

because k vanishes almost everywhere on $I' \times J'$. It follows that

$$\int_{J'} \beta(t)\, \nu(dt) = \int_{J'} \int_I \left(\frac{\beta(t)}{y_j(t)}\right) k(s, t) \left(\frac{\alpha(s)}{x_j(s)}\right) \mu(ds)\, \nu(dt). \tag{2.28}$$

Let χ_J denote the characteristic function of J. Because z_j is bounded in norm uniformly in j and $z_j \xrightarrow[w^*]{} z$, we have

$$\begin{aligned}\lim_{j\to\infty} \int_T (k(s,t)+\varepsilon_j)\,\chi_J(t)\,\beta(t)\,z_j(t)\,\nu(dt) &= \lim_{j\to\infty} \int_T k(s,t)\,\chi_J(t)\,\beta(t)\,z_j(t)\,\nu(dt)\\ &= \int_T k(s,t)\,\chi_J(t)\,\beta(t)\,z(t)\,\nu(dt)=0.\end{aligned}$$

(Recall that z vanishes almost everywhere on J.) It follows that for almost all $s\in I$,

$$\begin{aligned}\lambda x(s) &= \lim_{j\to\infty} \int_T k(s,t)\,\chi_J(t)\,\beta(t)\,z_j(t)\,\nu(dt)\\ &\quad+\lim_{j\to\infty}\int_T k(s,t)(1-\chi_J(t))\,\beta(t)\,z_j(t)\,\nu(dt)\\ &=\lim_{j\to\infty}\int_{J'} k(s,t)\,\beta(t)\,z_j(t)\,\nu(dt).\end{aligned}$$

For almost all $s\in I$, it follows that

$$\lambda\alpha(s)=\lim_{j\to\infty}\left(\frac{\alpha(s)}{x_j(s)}\right)\int_{J'} k(s,t)\left(\frac{\beta(t)}{y_j(t)}\right)\nu(dt).$$

From (2.28) and Fatou's lemma we obtain

$$\begin{aligned}\lambda\int_I \alpha(s)\,ds &= \int_I \lim_{j\to\infty}\left[\left(\frac{\alpha(s)}{x_j(s)}\right)\int_{J'} k(s,t)\left(\frac{\beta(t)}{y_j(t)}\right)\nu(dt)\right]\mu(ds)\\ &\leqslant \varliminf_{j\to\infty}\int_I\int_{J'}\frac{\alpha(s)}{x_j(s)}\,k(s,t)\,\frac{\beta(t)}{y_j(t)}\,\nu(dt)\,\mu(ds)\\ &=\int_{J'}\beta(t)\,\nu(dt).\end{aligned}$$

Since $\lambda\geqslant 1$, this contradicts (2.27). It follows that our assumption that $\nu(J)>0$ is wrong, and the theorem is proved. ∎

As we see in Section 4, the condition of indecomposability may be important. However, we now prove that for the pure question of existence which is treated in Theorem 2.26, indecomposability is irrelevant. Our strategy is to show that k can be written as a finite direct sum of kernels which are indecomposable. This will follow by exploiting Hypothesis 2.15.

LEMMA 2.29. *Assume that* (S, μ) *and* (T, ν) *are finite measure spaces and that* $k \in L^{\infty}(S \times T)$ *is a nonnegative function which satisfies Hypothesis* 2.15. *Then there exist pairwise disjoint measurable sets* $S_i \subset S$, $1 \leqslant i \leqslant m$, *and pairwise disjoint measurable sets* $T_i \subset T$, $1 \leqslant i \leqslant m$, *such that* $\mu(S_i) > 0$ *and* $\nu(T_i) > 0$ *for* $1 \leqslant i \leqslant m$ *and*

(1) $S = \bigcup_{i=1}^{m} S_i$ *and* $T = \bigcup_{i=1}^{m} T_i$,

(2) $k \mid S_i \times T_i$ *is indecomposable for* $1 \leqslant i \leqslant m$, *and*

(3) $k(s, t) = 0$ *for almost all* $(s, t) \in (S \times T - \bigcup_{i=1}^{m} S_i \times T_i)$.

Proof. For $s \in S$ and $t \in T$ and $c > 0$ define E_s and F_t by

$$E_s = \{t \mid k(s, t) \geqslant c\} \qquad \text{and} \qquad F_t = \{s \mid k(s, t) \geqslant c\}.$$

As noted in the remark following Hypothesis 2.15, there exist $c > 0$ and $\delta_1 > 0$ with

$$\nu(E_s) \geqslant \delta_1 \text{ and } \mu(F_t) \geqslant \delta_1 \qquad \text{almost everywhere.}$$

Now suppose that k is not indecomposable, so there exist measurable sets $I \subset S$ and $J \subset T$ with $\mu(I) > 0$, $\nu(J) > 0$, and

$$k(s, t) = 0 \qquad \text{for almost all} \quad (s, t) \in (I \times J) \cup (I' \times J').$$

It follows from Hypothesis 2.15 that $\mu(I') > 0$ and $\nu(J') > 0$. Furthermore, if $s \in I'$ we see that $E_s \subset J$, so

$$\nu(J) \geqslant \nu(E_s) \geqslant \delta_1.$$

Similarly, if $s \in I$, $E_s \subset J'$, so we have

$$\nu(J') \geqslant \nu(E_s) \geqslant \delta_1.$$

The same sort of argument shows that

$$\mu(I) \geqslant \delta_1 \qquad \text{and} \qquad \mu(I') \geqslant \delta_1.$$

Define, for the moment, $S_1 = I'$, $S_2 = I$, $T_1 = J$, and $T_2 = J'$ and consider $k \mid S_1 \times T_1$. If $k \mid S_1 \times T_1$ and $k \mid S_2 \times T_2$ are indecomposable, we are done. If not and if $k \mid S_1 \times T_1$ is not indecomposable, the same argument used above shows that we can find pairwise disjoint subsets $\tilde{S}_1$ and $\tilde{S}_2$ of S_1 and $\tilde{T}_1$ and $\tilde{T}_2$ of T_2 (all sets of positive measure) such that

$$S_1 = \tilde{S}_1 \cup \tilde{S}_2 \qquad \text{and} \qquad T_1 = \tilde{T}_1 \cup \tilde{T}_2$$

and

$$k(s, t) = 0 \qquad \text{for almost all } (s, t) \in (S_1 \times T_1) - (\tilde{S}_1 \times \tilde{T}_1) \cup (\tilde{S}_2 \times \tilde{T}_2).$$

The crucial observation is that our previous argument shows that

$$\mu(\tilde{S}_1) \geqslant \delta_1, \qquad \mu(\tilde{S}_2) \geqslant \delta_1, \qquad \nu(\tilde{T}_1) \geqslant \delta_1, \qquad \text{and} \qquad \nu(\tilde{T}_2) \geqslant \delta_1,$$

where δ_1 is the *same* constant as before. If we change our notation, we have shown that

$$S = \bigcup_{i=1}^{m} S_i \qquad \text{and} \qquad T = \bigcup_{i=1}^{m} T_i,$$

where $m \geqslant 3$, the sets S_i are pairwise disjoint, the sets T_i are pairwise disjoint,

$$\mu(S_i) \geqslant \delta_1 \text{ and } \nu(T_i) \geqslant \delta_1 \qquad \text{for} \quad 1 \leqslant i \leqslant m,$$

and

$$k(s, t) = 0 \qquad \text{for almost all} \quad (s, t) \notin \bigcup_{i=1}^{m} (S_i \times T_i).$$

If $k \mid S_i \times T_i$ is indecomposable for $1 \leqslant i \leqslant m$, we are done. Otherwise, we can repeat the procedure and increase m by one. However, our construction issues that at any step in the procedure

$$\mu(S) = \mu\left(\bigcup_{i=1}^{m} S_i\right) = \sum_{i=1}^{m} \mu(S_i) \geqslant m\delta_1,$$

where δ_1 is independent of m. It follows that

$$m \leqslant (\delta_1)^{-1} \mu(S),$$

and at some point we must have $k \mid S_i \times T_i$ indecomposable for $1 \leqslant i \leqslant m$. ▮

As a direct consequence of Lemma 2.29 we see that the assumption of indecomposability in Theorem 2.26 is unnecessary.

THEOREM 2.30. *Let assumptions be as Theorem 2.26 except do not assume that k is indecomposable. Then (see (1.1)) the DAD problem (k, α, β) has a solution (f, g). If (f, g) is a solution of the DAD problem (k, α, β), there are positive constants c_1 and c_2 (dependent on f and g) with*

$$c_1 \leqslant \frac{\alpha(s)}{f(s)} \leqslant c_2 \text{ and } c_1 \leqslant \frac{\beta(t)}{g(t)} \leqslant c_2 \qquad \text{almost everywhere.}$$

Proof. The final assertion of Theorem 2.30 follows from Lemma 2.18, so it suffices to prove f and g exist.

By Lemma 2.29 there exist pairwise disjoint, measurable sets $S_i \subset S$, $1 \leqslant i \leqslant m$, and pairwise disjoint, measurable sets $T_i \subset T$, $1 \leqslant i \leqslant m$, with $\mu(S_i) > 0$ and $v(T_i) > 0$ for $1 \leqslant i \leqslant m$, $S = \bigcup_{i=1}^m S_i$, $T = \bigcup_{i=1}^m T_i$, $k(s, t) = 0$ for almost all $(s, t) \notin \bigcup_{i=1}^m (S_i \times T_i)$ and $k_i = k \mid (S_i \times T_i)$ indecomposable for $1 \leqslant i \leqslant m$. The reader can verify that all the conditions of Theorem 2.26 are satisfied for k_i, $\alpha_i = \alpha \mid S_i$ and $\beta_i = \beta \mid T_i$. Thus there exist functions $f_i \in L^1(S_i)$ and $g_i \in L^1(T_i)$ with f_i and g_i positive almost everywhere on S_i and T_i, respectively, and

$$\int_{T_i} f_i(s)\, k(s, t)\, g_i(t)\, v(dt) = \alpha(s), \qquad \mu \text{ a.e. on } S_i, \text{ and}$$

$$\int_{S_i} f_i(s)\, k(s, t)\, g_i(t)\, \mu(ds) = \beta(t), \qquad v \text{ a.e. on } T_i.$$

If we define $f(s) = f_i(s)$ for $s \in S_i$ and $1 \leqslant i \leqslant m$, and $g(t) = g_i(t)$ for $t \in T_i$ and $1 \leqslant i \leqslant m$, one can check that f and g give a solution of (1.1). ∎

The main difficulty in applying Theorem 2.30 is to verify that (k, α, β) satisfies the compatibility condition. If k has the same zero set as a kernel $\tilde{k}$ and the DAD problem $(\tilde{k}, \alpha, \beta)$ has a solution, this difficulty is circumvented.

COROLLARY 2.31. *Assume that (k, α, β) satisfies Hypotheses* 2.3 *and* 2.15 *and that either A_1 or A_2 is a compact linear map, where A_1 and A_2 are defined by* (2.17). *Assume that $\tilde{k} \in L^\infty(S \times T)$, $(\tilde{k}, \alpha, \beta)$ satisfies Hypothesis* 2.3, *and that the DAD problem $(\tilde{k}, \alpha, \beta)$ has a solution. If, for almost all $(s, t) \in S \times T$, $\tilde{k}(s, t) = 0$ if and only if $k(s, t) = 0$, then the DAD problem (k, α, β) satisfies the compatibility condition and has a solution (f, g) as in Theorem* 2.30.

Proof. Because the DAD problem $(\tilde{k}, \alpha, \beta)$ is assumed to have a solution, $(\tilde{k}, \alpha, \beta)$ must satisfy the (necessary) compatibility condition (Definition 2.13). It follows that (k, α, β) satisfies the compatibility condition, and Theorem 2.30 implies that the DAD problem (k, α, β) has a solution. ∎

Theorem 2.30 is applicable when k is continuous, but we want to prove that it suffices to assume that (k, α, β) satisfies the weak compatibility condition and that α/f and β/g are positive continuous functions.

LEMMA 2.32. *Assume Hypotheses* 2.2, 2.3, *and* 2.15 *and that $\int_S \alpha(s)\, \mu(ds) = \int_T \beta(t)\, v(dt)$. Assume that k is continuous. If (k, α, β) satisfies* *the weak compatibility condition, then* (k, α, β) *satisfies the compatibility condition. If* k *is weakly indecomposable,* k *is indecomposable.*

Proof. Assume first that (k, α, β) satisfies the weak compatibility condition. Suppose that $I \subset S$ and $J \subset T$ are measurable sets of positive measure and that $k \mid I' \times J' = 0$ almost everywhere. (Here A' denotes the complement of a set A and $\bar{A}$ its closure). We have to prove that

$$\int_I \alpha(s)\, \mu(s) \geq \int_{J'} \beta(t)\, \nu(dt), \tag{2.33}$$

with strict inequality if k is positive on a subset of $I \times J$ of positive measure.

Our first claim is that, by modifying I and J on sets of measure zero, we can assume $k(s, t) = 0$ for all $(s, t) \in I' \times J'$. We can suppose that I' and J' have positive measure or it is immediate that strict inequality holds in (2.33). By Fubini's theorem, there exists a set $N \subset J'$ of measure zero such that for $t \in J' - N$, $k(s, t) = 0$ for almost all $s \in I'$. Because $J' - N$ is a separable metric space, select a countable dense set of points t_j, $j \geq 1$, in $J' - N$. For each $j \geq 1$, let $E_j \supset I$ be a measurable set with $\mu(I' - E_j') = 0$ and $k(s, t_j) = 0$ for all $s \in E_j'$. If we define $E = \bigcap_{j \geq 1} E_j$ it follows that $E \supset I$, $\mu(I' - E') = 0$, and $k(s, t_j) = 0$ for all $s \in E'$ and $j \geq 1$. If we define $F' = J' - N$, we conclude from the continuity of k that $k(s, t) = 0$ for all $(s, t) \in E' \times F'$, $E \supset I$, $F \supset J$, $\mu(E - I) = 0$, and $\nu(F - J) = 0$. Using this, we see that to prove (2.33) it suffices to prove

$$\int_E \alpha(s)\, \mu(ds) \geq \int_{F'} \beta(t)\, \nu(dt),$$

with strict inequality if k is positive on a set of positive measure in $E \times F$.

The continuity of k implies that $k(s, t) = 0$ for all $(s, t) \in \overline{(E')} \times \overline{(F')}$. If $G = \overline{(E')}'$ and $H = \overline{(F')}'$, G and H are open sets, $G \subset E$, $H \subset F$, and $k(s, t) = 0$ for all $(s, t) \in G' \times H'$. If G or H is empty, we contradict Hypothesis 2.15. Thus G and H are nonempty open sets, and the weak compatibility condition implies that

$$\int_G \alpha(s)\, \mu(ds) \geq \int_{H'} \beta(t)\, \nu(dt),$$

with strict inequality holding if $k(s_0, t_0) > 0$ for some $(s_0, t_0) \in G \times H$. It follows from the above inequality and the fact that $G \subset E$ and $H \subset F$ that

$$\int_E \alpha(s)\, \mu(ds) \geq \int_{F'} \beta(t)\, \nu(dt)$$

and that strict inequality holds if $\mu(E-G)>0$ or $\mu(F-H)>0$ or $k(s_0, t_0)>0$ for some $(s_0, t_0) \in G \times H$. Thus we have strict inequality unless $\mu(E-G)=0$, $\mu(F-H)=0$, and $k(s, t)=0$ for all $(s, t) \in G \times H$. However, the latter conditions imply that $k(s, t)=0$ for almost all $(s, t) \in E \times F$. Thus (k, α, β) satisfies the compatibility condition.

Suppose now that k is weakly indecomposable and that I, J, E, F, G, and H are as above. To prove k indecomposable, we must prove that k is positive almost everywhere on a set of positive measure in $I \times J$, and this is equivalent to proving that k is positive almost everywhere on a set of positive measure in $E \times F$. We know that G and H are nonempty open sets and that $k(s, t)=0$ for all $(s, t) \in G' \times H'$. Weak indecomposability implies that $k(s_0, t_0)>0$ for some $(s_0, t_0) \in G \times H$, and continuity of k implies that there exists an open neighborhood U of (s_0, t_0) with $U \subset G \times H \subset E \times F$ and $k(s, t)>0$ for all $(s, t) \in U$. Since U has positive measure, we have proved k indecomposable. ∎

We can now give a stronger version of Theorem 2.30 for k continuous.

THEOREM 2.34. *Assume Hypothesis* 2.2 *and* 2.3 *and suppose that* $k: S \times T \to \mathbb{R}$ *is continuous and that* (k, α, β) *satisfies the weak compatibility condition* (*Definition* 2.13). *Assume that*

$$\int k(s, t)\, \nu(dt)>0 \text{ for all } s \qquad \text{and} \qquad \int k(s, t)\, \mu(ds)>0 \text{ for all } t.$$

Then there exist positive, continuous functions $x \in C(S)$ *and* $y \in C(T)$ *such that if* $f(s)=\alpha(s)/x(s)$ *and* $g(t)=\beta(t)/y(t)$, (f, g) *gives a solution of the DAD problem* (k, α, β) (*see* (1.1)).

Proof. Lemma 2.32 implies that (k, α, β) satisfies the compatibility condition. It is an easy continuity and compactness argument to prove that Hypothesis 2.15 is satisfied. Thus Theorem 2.30 and Lemma 2.21 imply that (for F as in (2.20)) there exists $x \in \mathring{C}_1$, the interior of the cone of non-negative functions in $L^\infty(S)$, with $F(x)=x$. If A_1 and A_2 are as in (2.17), one can see from the continuity of k and Lemma 2.18 that $A_1(\mathring{C}_1) \subset (C(T) \cap \mathring{C}_2)$ and $A_2(\mathring{C}_2) \subset (C(S) \cap \mathring{C}_1)$. It follows that x is continuous and positive and that $y=(A_1 J_1)(x)$ is continuous and positive. The theorem now follows from Lemma 2.21. ∎

3. *DAD* THEOREMS FOR GENERAL $k \in L^\infty(S \times T)$

The arguments of the previous section depend strongly on the assumption that either A_2 or A_1 is a compact linear map, where A_2 and A_1 are defined as in (2.17). Actually, if hypotheses are as in Theorem 2.26 or 2.30, but A_2 and A_1 are not assumed compact, one can still prove Theorem 2.30 if one can prove (in the notation of Theorem 2.26) that there exist $\kappa_1 > 0$ and $\kappa_2 > 0$ with

$$\kappa_1 \leqslant x_{\varepsilon_j}(s) \leqslant \kappa_2 \tag{3.1}$$

for almost all s and all $j \geqslant 1$. However, proving (3.1) directly appears difficult.

It is not hard to show that for general $k \in L^{\infty}(S \times T)$, neither A_2 nor A_1 need be a compact map: see Remark 5.6 in [5]. Thus there is a gap between the results of Section 2 and the case of general $k \in L^{\infty}(S \times T)$. To handle the general case we shall combine some results on "entropy minimization" from [5] with theorems from the previous section. We could also use theorems from a paper by Csiszàr [11], but the underlying optimization problem (which we prefer to emphasize) is hidden in [11] and some of the derivations seem obscure (see for example, paragraph 2, p. 154, in [11] and remarks in [4]).

We begin by describing results from [5] which we need. Suppose that (S, μ) and (T, ν) are finite measure spaces and that $\tilde{k} \in L^1(S \times T, \mu \times \nu)$ is a nonnegative function. Assume that $\alpha \in L^1(S, \mu)$ and $\beta \in L^1(T, \nu)$ are positive almost everywhere and

$$\int_S \alpha(s)\, \mu(ds) = \int_T \beta(t)\, \nu(dt).$$

We obtain a measure σ on $S \times T$ from $\tilde{k}$ by

$$\sigma(E) = \int_E \int \tilde{k}(s, t)\, \mu(ds)\, \nu(dt), \tag{3.2}$$

and we write $d\sigma = \tilde{k}(s, t)\, \mu(ds)\, \nu(dt)$. We are interested in a slight generalization of our previous DAD problem: Do there exist maps $f\colon S \to \mathbb{R}$ and $g\colon T \to \mathbb{R}$, positive almost everywhere, such that if $u(s, t) = f(s)\, g(t)$, then $u \in L^1(S \times T, \sigma)$ and

$$\begin{aligned} &\int u(s, t)\, \tilde{k}(s, t)\, \nu(dt) = \alpha(s), \qquad \mu \text{ a.e., and} \\ &\int u(s, t)\, \tilde{k}(s, t)\, \mu(ds) = \beta(t), \qquad \nu \text{ a.e.} \end{aligned} \tag{3.3}$$

If f and g as above exist we say that (f, g) is a generalized solution of the DAD problem (k, α, β). Note, however, that even if Hypothesis 2.3 is satisfied and (f, g) is a generalized solution of the DAD problem (k, α, β), it does not a priori follow that f is μ-measurable or $f \in L^1(S, \mu)$. Such facts must be derived from further assumptions.

Define a convex continuous function φ: $[0, \infty) \to \mathbb{R}$ by $\varphi(0) = 0$ and

$$\varphi(u) = u \log(u) - u.$$

If $u \in L^1(S \times T, \sigma)$ and $u(s, t) \geqslant 0$, σ a.e., define

$$I_\varphi(u) = \int_{S \times T} \int \varphi(u(s, t)) \tilde{k}(s, t) \mu(ds) \nu(dt). \tag{3.4}$$

Of, course, we may have $I_\varphi(u) = \infty$. Define K to be the set of $u \in L^1(S \times T, \sigma)$ with $u(s, t) \geqslant 0$, σ a.e., $I_\varphi(u) < \infty$ and

$$\begin{aligned} \int u(s, t) \tilde{k}(s, t) \nu(dt) &= \alpha(s), \qquad \mu \text{ a.e., and} \\ \int u(s, t) \tilde{k}(s, t) \mu(ds) &= \beta(t), \qquad \nu \text{ a.e.} \end{aligned} \tag{3.5}$$

If $u \in K$, we say that u is "feasible." If K is nonempty, it is not hard to show that K is convex and there exists a unique $u_0 \in K$ such that

$$I_\varphi(u_0) = \inf_{u \in K} I_\varphi(u).$$

See Corollary 2.2 in [5]. If there exists $u \in K$ with $u(s, t) > 0$, σ a.e., then it is proved in Section 3 of [5] that $u_0(s, t) > 0$, σ a.e., and that there exist functions $f(s)$ and $g(t)$, with $f(s) > 0$, μ a.e., $g(t) > 0$, ν a.e., and

$$u_0(s, t) \tilde{k}(s, t) = f(s) g(t) \tilde{k}(s, t), \qquad \mu \times \nu \text{ a.e.}$$

Thus, if there exists $u \in K$ with $u > 0$, σ a.e., one obtains a generalized solution of the DAD problem (k, α, β) (see (3.3)). However, it is not even immediately apparent that f is μ-measurable and g is ν-measurable.

If A and B are subsets of $S \times T$, define

$$A \,\Delta B = (A - B) \cup (B - A).$$

Say that $A \subset S \times T$ is a measurable rectangle if there exist measurable sets $I \subset (S, \mu)$ and $J \subset (T, \nu)$ such that

$$(\mu \times \nu)(A \,\Delta(I \times J)) = 0.$$

If $\tilde{k}$ is as before, define Ω by

$$\tilde{\Omega} = \{(s, t) : \tilde{k}(s, t) > 0\},$$

so $\tilde{\Omega}$ is defined to within a set of measure zero. If $\tilde{\Omega}$ is a countable union of measurable rectangles, one can prove as in [5] that the function f is μ-measurable and g is ν-measurable.

The previous remarks reduce finding a generalized solution of the *DAD* problem (k, α, β) (see (3.3)) to the problem of finding $u \in K$ with $u > 0$, σ a.e. We approach this problem by using Theorem 2.30 from Section 2.

For technical reasons it will be convenient to give a strengthened version of Hypothesis 2.15.

HYPOTHESIS 3.6. *(S, μ) and (T, ν) are finite measure spaces and $k \in L^\infty(S \times T)$ is nonnegative $(\mu \times \nu)$ almost everywhere. There exist a constant $c > 0$ and measurable sets $E_i \subset S$ and $F_i \subset T$ for $1 \leqslant i \leqslant m$ and $G_j \subset S$ and $H_j \subset T$ for $1 \leqslant j \leqslant n$ such that*

(1) *$\mu(E_i)\, \nu(F_i) > 0$ for $1 \leqslant i \leqslant m$ and $\mu(G_j)\, \nu(H_j) > 0$ for $1 \leqslant j \leqslant n$,*

(2) *$\mu(S - \bigcup_{i=1}^m E_i) = 0$ and $\nu(T - \bigcup_{j=1}^n H_j) = 0$, and*

(3) *$k(s, t) \geqslant c$ for almost all $(s, t) \in \bigcup_{i=1}^m (E_i \times F_i)$ and almost all $(s, t) \in \bigcup_{j=1}^n (G_j \times H_j)$*

LEMMA 3.7. *Assume Hypothesis 2.3 and suppose that the DAD problem (k, α, β) has a generalized solution (f, g). If $\Omega := \{(s, t) \in S \times T \mid k(s, t) > 0\}$, suppose that there is a sequence of measurable rectangles $I_p \times J_p \subset \Omega$, $p \geqslant 1$, with $\mu(S - \bigcup_{p=1}^\infty I_p) = 0$ and $\nu(T - \bigcup_{p=1}^\infty J_p) = 0$. Then f is μ-measurable and g is ν-measurable. If Hypothesis 3.6 is satisfied, then $f \in L^1(S, \mu)$ and $g \in L^1(T, \nu)$, and there exist positive constants c_1 and c_2 with*

$$c_1 \leqslant f(s)/\alpha(s) \leqslant c_2,\ \mu \text{ a.e.}, \qquad c_1 \leqslant (g(t)/\beta(t)) \leqslant c_2,\ \nu \text{ a.e.}$$

Proof. Define $u(s, t) = f(s)\, g(t)$, so we know $uk \in L^1(S \times T, \mu \times \nu)$. Our assumptions imply that $uk \mid (I_p \times J_p)$ is integrable, so Fubini's theorem implies that $u(\cdot, t)\, k(\cdot, t) \mid I_p$ and $k(\cdot, t) \mid I_p$ are μ-measurable for ν almost all $t \in J_p$. Because g is positive almost everywhere on J_p and k is positive almost everywhere on $I_p \times J_p$, there exists (assuming, as we can, that $\nu(J_p) > 0$) $t_p \in J_p$ with $g(t_p) > 0$, $k(s, t_p) > 0$ for almost all $s \in I_p$, $u(\cdot, t_p)\, k(\cdot, t_p) \mid I_p$ is measurable, and $k(\cdot, t_p) \mid I_p$ is measurable. It follows that

$$f \mid I_p = (1/g(t_p))(1/k(\cdot, t_p))(f(\cdot)\, g(t_p)\, k(\cdot, t_p)) \mid I_p$$

is measurable. It follows that f is measurable on S, and the proof that g is measurable is analogous.

If Hypothesis 3.6 is satisfied, we immediately obtain sets I_p and J_p as above and f and g are measurable. If we use the notation of Hypothesis 3.6 we find that for almost all $s \in E_i$, $1 \leqslant i \leqslant m$,

$$cf(s) \int_{F_i} g(t)\, \nu(dt) \leqslant f(s) \int_S k(s, t)\, g(t)\, \nu(dt) = \alpha(s).$$

We know that f and g are measurable and positive almost everywhere, so the above inequality implies that

$$0 < d_i := \int_{F_i} g(t)\, \nu(dt) < \infty \text{ and}$$

$$f(s) \leqslant (cd_i)^{-1}\, \alpha(s) \text{ for almost all } s \in F_i.$$

We conclude from Hypothesis 3.6 that there is a constant $c_2 = \max_{1 \leqslant i \leqslant m} (cd_i)^{-1}$ with $f(s) \leqslant c_2 \alpha(s)$, μ a.e. By symmetry in the roles of f and g, the same argument also gives (possibly increasing c_2) $g(t) \leqslant c_2 \beta(t)$, ν a.e. It follows that $f \in L^1(S, \mu)$, $g \in L^1(T, \nu)$, and

$$f(s)\, \|k\|_\infty\, \|g\|_1 \geqslant \alpha(s), \qquad \mu \text{ a.e., and}$$

$$g(t)\, \|k\|_\infty\, \|f\|_1 \geqslant \beta(t), \qquad \nu \text{ a.e.}$$

We deduce from the above inequalities that there exists $c_1 > 0$ with

$$f(s) \geqslant c_1 \alpha(s),\ \mu \text{ a.e.,} \qquad \text{and} \qquad g(t) \geqslant c_1 \beta(t),\ \nu \text{ a.e.} \quad \blacksquare$$

If (k, α, β) satisfies Hypothesis 2.3 and the *DAD* problem (k, α, β) has a generalized solution (f, g), suppose that $\Omega = \{(s, t) \mid k(s, t) > 0\}$ actually equals a countable union of measurable rectangles $I_p \times J_p$, $1 \leqslant p < \infty$. Then it follows that $\mu(S - \bigcup_{p=1}^{\infty} I_p) = 0$ and $\nu(T - \bigcup_{p=1}^{\infty} J_p) = 0$. To see this let $E = \{s \in S \mid s \notin \bigcup_{p=1}^{\infty} I_p\}$ and note that $k = 0$ a.e. on $E \times T$. Because

$$0 = \int_E \int_T u(s, t)\, k(s, t)\, \nu(dt)\, \mu(ds) = \int_E \alpha(s)\, \mu(ds)$$

and $\alpha > 0$ a.e., it follows that $\mu(E) = 0$.

We can now substantially weaken the assumption in Theorem 2.30 that, for A_1 and A_2 as in (2.17), A_1 or A_2 induces a compact linear operator. To do this we introduce a technical condition which we later show is easily verified.

HYPOTHESIS 3.8. *Hypothesis* 2.3 *is satisfied by* (k, α, β). *There exists a nonnegative function* $k_1 \in L^\infty(S \times T)$ *with the following properties*: (1) *if* $G_1 : L^\infty(S) \to L^\infty(T)$ *and* $G_2 : L^\infty(T) \to L^\infty(S)$ *are defined by*

$$(G_1 v)(t) = \int k_1(s, t)\, \alpha(s)\, v(s)\, \mu(ds) \text{ and}$$

$$(G_2 w)(s) = \int k_1(s, t)\, \beta(t)\, w(t)\, \nu(dt),$$

G_1 *or* G_2 *is a compact linear operator.* (2) $k_1(s, t) > 0$ *for almost all* $(s, t) \in \Omega$, *where* $\Omega := \{(s, t) \mid k(s, t) > 0\}$. (3) *There exists a constant* M *such that* $k_1(s, t) \leqslant Mk(s, t)$ *for almost all* $(s, t) \in S \times T$.

Of course property (1) in Hypothesis 3.8 is satisfied if $k_1 \alpha \in L^1(S) \otimes L^\infty(T)$ or $k_1 \beta \in L^\infty(S) \otimes L^1(T)$.

THEOREM 3.9. *Assume Hypotheses* 3.6 *and* 3.8 *and suppose that* (k, α, β) *satisfies the compatibility condition* (*Definition* 2.13). *Then the DAD problem* (k, α, β) *has a solution* (f, g) (*see* (1.1)). *There exist positive constants with*

$$c_1 \alpha(s) \leqslant f(s) \leqslant c_2 \alpha(s),\ \mu \text{ a.e.}, \qquad \text{and} \qquad c_1 \beta(t) \leqslant g(t) \leqslant c_2 \beta(t),\ \nu \text{ a.e.}$$

Proof. If notation is as in Hypothesis 3.6, let χ_R denote the characteristic function of $R = \bigcup_{i=1}^m (E_i \times F_i) \cup \bigcup_{j=1}^n (G_j \times H_j)$ and define $k_2 = k_1 + c\chi_R$. Since $\chi_R \in L^\infty(S) \otimes L^\infty(T)$, we know that the kernel $\chi_R \alpha$ defines a compact linear map from $L^\infty(S)$ to $L^\infty(T)$ and $\chi_R \beta$ gives a compact linear map from $L^\infty(T)$ to $L^\infty(S)$. Thus $k_2 \alpha$ defines a compact linear map from $L^\infty(S)$ to $L^\infty(T)$ or $k_2 \beta$ defines a compact linear map from $L^\infty(T)$ to $L^\infty(S)$. One can also see that $k_2 > 0$ almost everywhere on Ω and $k_2 \leqslant (M+1)k$, almost everywhere. In addition, k_2 satisfies Hypothesis 3.6 and hence Hypothesis 2.15. Because $\{(s, t) \mid k_2(s, t) > 0\} = \Omega$ and because (k, α, β) satisfies the compatibility condition, (k_2, α, β) satisfies the compatibility condition. Theorem 2.30 thus implies that there exist functions $x \in \mathring{C}_1$ (the interior of the cone of nonnegative functions in $L^\infty(S)$) and $y \in \mathring{C}_2$ with

$$\int (\alpha(s)/x(s))\, k_2(s, t)(\beta(t)/y(t))\, \nu(dt) = \alpha(s), \qquad \mu \text{ a.e., and}$$

$$\int (\alpha(s)/x(s))\, k_2(s, t)(\beta(t))/y(t))\, \mu(ds) = \beta(t), \qquad \nu \text{ a.e.}$$

We now apply our previously mentioned results on entropy minimization. Define $\tilde{k}$ and u by

$$\tilde{k}(s,t)=\alpha(s)\,k(s,t)\,\beta(t) \qquad \text{and}$$

$$u(s,t)=\left(\frac{1}{x(s)}\right)\left(\frac{k_2(s,t)}{k(s,t)}\right)\left(\frac{1}{y(t)}\right)\leqslant (M+1)\left(\frac{1}{x(s)}\right)\left(\frac{1}{y(t)}\right).$$

Note that $\Omega=\{(s,t):\tilde{k}(s,t)>0\}$, $u\in L^{\infty}(S\times T,\mu\times\nu)$, $\tilde{k}\in L^1(S\times T,\mu\times\nu)$, $u>0$ on Ω, and

$$\int u(s,t)\,\tilde{k}(s,t)\,\nu(dt)=\alpha(s), \qquad \mu \text{ a.e. and}$$

$$\int u(s,t)\,\tilde{k}(s,t)\,\mu(ds)=\beta(t), \qquad \nu \text{ a.e.}$$

If $\varphi(u)=u\log u-u$ for $u>0$, $\varphi(0)=0$, we also see that

$$I_{\varphi}(u)=\int \varphi(u(s,t))\,\tilde{k}(s,t)\,\mu(ds)\,\nu(dt)<\infty.$$

It follows from the entropy minimization results that there exist functions $\tilde{f}\colon S\to R$ and $\tilde{g}\colon T\to \mathbb{R}$, positive almost everywhere, with $\tilde{f}\tilde{k}\tilde{g}\in L^1((S\times T),\mu\times\nu)$ and

$$\int \tilde{f}(s)\,\tilde{k}(s,t)\,\tilde{g}(t)\,\nu(dt)=\alpha(s), \qquad \mu \text{ a.e.,}$$

$$\int \tilde{f}(s)\,\tilde{k}(s,t)\,\tilde{g}(t)\,\mu(ds)=\beta(t), \qquad \nu \text{ a.e.}$$

If we define $f=\alpha\tilde{f}$ and $g=\beta\tilde{g}$, we find that $fkg\in L^1(S\times T,\mu\times\nu)$, and f and g are positive almost everywhere, and

$$\int f(s)\,k(s,t)\,g(t)\,\nu(dt)=\alpha(s), \qquad \mu \text{ a.e.}$$

$$\int f(s)\,k(s,t)\,g(t)\,\mu(ds)=\beta(t), \qquad \nu \text{ a.e.}$$

Lemma 3.7 now implies that f and g are measurable and satisfy the conditions of Theorem 3.9. ∎

The interest of Theorem 3.9 lies in the fact that simple assumptions imply Hypothesis 3.8.

HYPOTHESIS 3.10. *Hypothesis* 2.3 *is satisfied. There exist a sequence of measurable sets* $E_j \subset S \times T$, $j \geqslant 1$, *and a sequence of positive reals* ε_j *with* $\lim_{j \to \infty} \varepsilon_j = 0$ *such that* (1) $k(s, t) > \varepsilon_j$ *for almost all* $(s, t) \in E_j$, (2) E_j *is a countable union of measurable rectangles, and* (3) $\Omega := \{(s, t) \mid k(s, t) > 0\} = \bigcup_{j=1}^{\infty} E_j$.

LEMMA 3.11. *If Hypothesis* 3.10 *is satisfied, Hypothesis* 3.8 *is satisfied.*

Proof. We must construct a function k_1 as in Hypothesis 3.8. By our previous remarks, if $k_1 \in L^\infty(S) \otimes L^\infty(T)$ the linear maps G_1 and G_2 in (3.8) will both be compact, so it suffices to find a nonnegative $k_1 \in L^\infty(S) \otimes L^\infty(T)$ which satisfies conditions (2) and (3) in Hypothesis 3.8. By modifying E_j on a set of measure zero, we can write $E_j = \bigcup_{p=1}^{\infty} (I_{jp} \times J_{jp})$, where I_{jp} is μ-measurable and J_{jp} is ν-measurable for $p \in \mathbb{N}$. We know that $k(s, t) > \varepsilon_j$ for almost all $(s, t) \in E_j$, and Ω differs from $\bigcup_{j=1}^{\infty} E_j$ by a set of measure zero. Let χ_{jp} denote the characteristic function of $(I_{jp} \times J_{jp})$. For fixed j define

$$f_j(s, t) = \sum_{p=1}^{\infty} \varepsilon_j 2^{-p} \chi_{jp}(s, t).$$

If we define f_{jn} by

$$f_{jn} = \sum_{p=1}^{n} \varepsilon_j 2^{-p} \chi_{jp},$$

we see that $f_{jn} \in L^\infty(S) \otimes L^\infty(T)$ and $\|f_{jn} - f_j\|_\infty \to 0$, so $f_j \in L^\infty(S) \otimes L^\infty(T)$. By construction we have

$$f_j(s, t) > 0 \text{ for } (s, t) \in E_j \qquad \text{and} \qquad f_j(s, t) = 0 \text{ for } (s, t) \notin E_j.$$

Since $k(s, t) > \varepsilon_j$ almost everywhere on E_j we see that $f_j \leqslant k$ almost everywhere. If we define k_1 by

$$k_1 = \sum_{j=1}^{\infty} 2^{-j} f_j,$$

we see that $k_1 \in L^\infty(S) \otimes L^\infty(T)$ (because $\sum_{j=1}^{n} 2^{-j} f_j \in L^\infty(S) \otimes L^\infty(T)$ and $\lim_{n \to \infty} \|k_1 - \sum_{j=1}^{n} 2^{-j} f_j\|_\infty = 0$). We also know $k_1 \geqslant 2^{-j} f_j > 0$ on E_j, so $k_1 > 0$ on $\bigcup_{j=1}^{\infty} E_j$ and $k_1 > 0$ almost everywhere on Ω. Finally, we have

$$k_1 = \sum_{j=1}^{\infty} 2^{-j} f_j \leqslant \sum_{j=1}^{\infty} 2^{-j} k = k \qquad \text{almost everywhere,}$$

so Hypothesis 3.8 is satisfied. ∎

Remark 3.12. If Hypothesis 2.3 is satisfied and k is a nonnegative function in $L^\infty(S)\otimes L^\infty(T)$, then Hypothesis 3.10 is satisfied. For reasons of length, we omit the proof.

COROLLARY 3.13. *Assume Hypothesis* 3.6 *and* 3.10 *and suppose that* (k, α, β) *satisfies the compatibility condition* (*Definition* 2.13). *Then the DAD problem* (k, α, β) *has a solution* (f, g) (*see* (1.1)).

Proof. Lemma 3.11 implies that Hypothesis 3.8 is satisfied, so Corollary 3.13 follows from Theorem 3.9. ∎

It may be that Hypothesis 3.8 is an artifact of our argument and that Theorem 3.9 remains true without Hypothesis 3.8 and with Hypothesis 2.15 replacing Hypothesis 3.6.

Remark 3.14. Suppose that (k, α, β) satisfies Hypothesis 3.6 and 3.8. In addition suppose that there is a nonnegative function $\tilde{k}\in L^\infty(S\times T)$ such that $\tilde{k}(s, t)>0$ if and only if $k(s, t)>0$. If the *DAD* problem $(\tilde{k}, \alpha, \beta)$ has a solution, then the *DAD* problem (k, α, β) has a solution. The argument is the same as in Corollary 2.31.

4. VERIFYING THE COMPATIBILITY CONDITION FOR (k, α, β)

It remains for us to give concrete conditions under which the hypotheses of Theorems 2.30, 2.33, or 3.9 are satisfied and the *DAD* problem (k, α, β) has a solution. This problem reduces to finding conditions which imply that (k, α, β) satisfies the compatibility condition (Definition (2.13)). Even in the matrix case this may be nontrivial. Corollary 2.31 provides some information but we prefer a more constructive approach. We consider several cases. If $S=T$, $\mu=\nu$, and $\alpha=\beta$, we give natural hypotheses which imply the compatibility condition. Our results imply classical theorems about scaling nonnegative matrices to doubly stochastic matrices (see [7] and [34]), yield a theorem of Nowosad [24] and Karlin and Nirenberg [18], and also give a generalization and different proof of the theorems in [5] and [27]. We also show that for certain $k\in L^\infty(S\times T)$, verifying the compatibility condition for (k, α, β) is equivalent to verifying the compatibility condition for an associated matrix *DAD* problem. In this way we obtain direct generalizations to an infinite dimensional framework of matrix *DAD* theorems.

We begin with some useful generalities.

PROPOSITION 4.1. *Assume Hypothesis* 2.3 *and suppose that* (k, α, β) *satisfies the compatibility condition and* k *is indecomposable* (*Definition* 2.14). *Assume that* $\tilde{k}\in L^\infty(S\times T)$ *is nonnegative and that, for almost all*

$(s, t) \in S \times T$, $\tilde{k}(s, t) = 0$ *implies that* $k(s, t) = 0$. *Then* $(\tilde{k}, \alpha, \beta)$ *satisfies the compatibility condition and* $\tilde{k}$ *is indecomposable.*

Proof. Suppose that $I \subset S$ and $J \subset T$ are measurable sets of positive measure and that $\tilde{k} \mid I' \times J' = 0$ almost everywhere. Our assumptions imply that $k \mid I' \times J' = 0$ almost everywhere. Since k is indecomposable, k is positive on a set $E \subset I \times J$ of positive measure, so $\tilde{k}$ is positive on E and $\tilde{k}$ is indecomposable. Because (k, α, β) satisfies the compatibility condition for (1.1), it follows that

$$\int_I \alpha(s)\, \mu(ds) > \int_{J'} \beta(t)\, \nu(dt),$$

and $(\tilde{k}, \alpha, \beta)$ satisfies the compatibility condition. ∎

Proposition 4.1 is false if k is not indecomposable, even for 2×2 matrices, so it is important to determine when k is indecomposable. Assuming Hypothesis 2.3, define nonnegative functions $c_m: S \times S \to \mathbb{R}$, $m \geqslant 1$, by

$$c_1(s, \sigma) = \int_T k(s, t)\, k(\sigma, t)\, \beta(t)\, \nu(dt) \text{ and} \tag{4.2}$$

$$c_m(r, s) = \int_S c_1(r, \sigma)\, c_{m-1}(\sigma, s)\, \alpha(\sigma)\, \mu(d\sigma), \qquad m > 1. \tag{4.3}$$

If A_1 and A_2 are defined by (2.17), one can check that

$$((A_2 A_1)^m (w))(s) = \int_S c_m(s, \sigma)\, \alpha(\sigma)\, w(\sigma)\, \mu(d\sigma). \tag{4.4}$$

PROPOSITION 4.5. *Assume Hypotheses* 2.3 *and* 2.15 *and let* c_j *be defined by* (4.3). *Assume that there exists* $m \geqslant 1$ *such that* $c_m(r, s) > 0$ *for almost all* $(r, s) \in S \times S$. *Then* k *is indecomposable.*

Proof. Assume, by way of contradiction, that there exist measurable sets $I \subset S$ and $J \subset T$ of positive measure with $k \mid I \times J = 0$ and $k \mid I' \times J' = 0$. If $(s, \sigma) \in I \times I'$ or $(s, \sigma) \in I' \times I$ we obtain from (4.2) that

$$c_1(s, \sigma) = \int_J k(s, t)\, k(\sigma, t)\, \beta(t)\, \nu(dt) + \int_{J'} k(s, t)\, k(\sigma, t)\, \beta(t)\, \nu(dt) = 0.$$

Assume, by way of induction, that

$$c_j \mid I' \times I = 0 \text{ and } c_j \mid I \times I' = 0 \qquad \text{for} \quad 1 \leqslant j < n.$$

Using (4.3) and integrating over I and I' separately, we find that $c_n \mid I' \times I = 0$ and $c_n \mid I \times I' = 0$. It follows that $c_n \mid I' \times I = 0$ almost everywhere for all n. This contradicts the assumption that $c_m(s, t) > 0$ a.e., unless $\mu(I') = 0$. But if $\mu(I') = 0$, we contradict Hypothesis 2.15. ∎

From Proposition 4.5 we see that it is important to have conditions which insure that $c_m > 0$ a.e. on $S \times S$. It is easy to give such conditions; we mention two results which are proved in Lemmas 5.10 and 5.12 of [5] or can be proved by the reader.

PROPOSITION 4.6 (See Lemma 5.10 in [5]). *Assume Hypotheses* 2.2 *and* 2.3 *and let* c_m *be defined by* (4.3). *Assume the following condition*:

> *There exists a positive integer* m *so that for any two points* $r = s_0 \in S$ *and* $s = s_m \in S$ *there are nonempty open sets* G_i, $0 \leqslant i \leqslant m$, *with* $r \in G_0$, $s \in G_m$, *and* $c_1(u, v) \geqslant \delta(r, s) > 0$ *for almost all* $(u, v) \in G_i \times G_{i+1}$, $0 \leqslant i < m$. (4.7)

Then there exists δ *such that* $c_m \geqslant \delta$ *almost everywhere on* $S \times S$.

PROPOSITION 4.8 (See Lemma 5.12 in [5]). *Assume Hypotheses* 2.2 *and* 2.3 *and suppose that* S *is connected. For every* $s \in S$ *assume there exists an open neighborhood* U_s *of* s, *a set* $V_s \subset T$ *of positive measure, and* $\delta_s > 0$ *with* $k > \delta_s$ *almost everywhere on* $U_s \times V_s$. *Then there exists* $m \geqslant 1$ *such that condition* (4.7) *is satisfied, so* $c_m \geqslant \delta$ *a.e. on* $S \times S$.

The crucial step in the proof of Proposition 4.8 is actually the following result.

PROPOSITION 4.8′ (See Lemma 5.12 in [5]). *Let* S *be a compact, connected metric space and* μ *a regular Borel measure of full support on* S. *Assume that* g_1 *is a nonnegative function in* $L^\infty(S \times S)$ *and that for each* $s \in S$ *there exists an open neighborhood* U_s *of* s *and* $\delta_s > 0$ *so that* $g_1(s, t) \geqslant \delta_s$ *a.e. on* $U_s \times U_s$. *If* g_n *is defined inductively by*

$$g_n(s, t) = \int g_1(s, r)\, g_{n-1}(r, t)\, \mu(dr),$$

then there exists $m \geqslant 1$ *and* $\delta > 0$ *so that* $g_m \geqslant \delta$ *almost everywhere on* $S \times S$.

If, in Proposition 4.6, we weaken condition (4.7) by assuming only that $c_1(u, v) > 0$ a.e. on $G_i \times G_{i+1}$ for $0 \leqslant i < m$, we conclude that $c_m > 0$ a.e. on $S \times S$. Similarly, if we weaken Proposition 4.8 by allowing $\delta_s = 0$, we find that there is an integer m with $c_m > 0$ a.e. on $S \times S$.

If we exploit the idea of indecomposability and use Propositions 4.1 and 4.5, we can generalize our basic existence result, Theorem 3.9, concerning solutions of the *DAD* problem (1.1).

THEOREM 4.9. *Assume that* (k, α, β) *satisfies Hypotheses* 2.3 *and* 3.6, *that* (k, α, β) *satisfies the compatibility condition* (*see Definition* 2.13), *and that* k *satisfies Hypothesis* 3.8 *or* 3.10. *If* c_j *is defined by* (4.3) *for* $j \geqslant 1$, *assume that there exists* m *with* $c_m > 0$ *a.e. on* $S \times S$. *Suppose* $\tilde{k} \in L^\infty(S \times T)$ *is a non-negative function, that* $\tilde{k}$ *satisfies Hypothesis* 3.6 *and Hypothesis* 3.8 *or* 3.10, *and that for almost all* $(s, t) \in S \times T$, $\tilde{k}(s, t) = 0$ *implies that* $k(s, t) = 0$. *Then the DAD problem* $(\tilde{k}, \alpha, \beta)$ *has a solution* $(\tilde{f}, \tilde{g})$.

Proof. Proposition 4.5 implies that k is indecomposable, and Proposition 4.1 implies that $\tilde{k}$ is indecomposable and $(\tilde{k}, \alpha, \beta)$ satisfies the compatibility condition. The conclusion now follows from Theorem 3.9. ∎

We now consider the case that $S = T$ and $\alpha = \beta$. For convenience we isolate an assumption which will play an important role.

HYPOTHESIS 4.10. *Hypothesis* 2.3 *holds,* $(S, \mu) = (T, \nu)$, *and* $\alpha = \beta$. *If* $I \subset S$ *is any measurable set such that* $k \mid I \times I' = 0$ *a.e., then* $k \mid I' \times I = 0$ *a.e., where* I' *denotes the complement of* I.

THEOREM 4.11. *Assume Hypothesis* 4.10 *and Hypothesis* 3.8 *or* 3.10. *Assume that there exist* $\delta > 0$ *and measurable sets of positive measure* $G_i \subset S$, *for* $1 \leqslant i \leqslant m$, *with* $S = \bigcup_{i=1}^m G_i$ and $k \mid G_i \times G_i \geqslant \delta$ *a.e. Then the DAD problem* (k, α, α) *has a solution* (f, g). *There exist positive constants* κ_1 *and* κ_2 *with* $\kappa_1 \alpha(s) \leqslant f(s) \leqslant \kappa_2 \alpha(s)$ *and* $\kappa_1 \alpha(t) \leqslant g(t) \leqslant \kappa_2 \alpha(t)$ *almost everywhere.*

Proof. Our assumptions imply that all hypotheses of Theorem 3.9 are satisfied except possibly the compatibility condition for (k, α, α). To verify compatibility, assume that $I \subset S$ and $J \subset S = T$ are measurable sets of positive measure with $k \mid I' \times J' = 0$ a.e. We have

$$(I' \times J') \cap (G_i \times G_i) = (I' \cap G_i) \times (J' \cap G_i) := U_i$$

and $k \geqslant \delta$ a.e. on U_i and $k = 0$ a.e. on U_i, so $\mu(U_i) = 0$. It follows that for $1 \leqslant i \leqslant m$,

$$\mu(I' \cap G_i) = 0 \qquad \text{or} \qquad \mu(J' \cap G_i) = 0. \tag{4.12}$$

Our assumptions on G_i give

$$I' \cap J' = \left(\bigcup_i G_i \cap I' \right) \cap \left(\bigcup_j G_j \cap J' \right) = \bigcup_{i,j} (G_i \cap G_j \cap I' \cap J')$$

and (4.12) implies

$$\mu(G_i \cap G_j \cap I' \cap J') = 0 \qquad \text{for all } i, j.$$

We conclude that $\mu(I' \cap J') = 0$ and

$$J' \subset I \cup N_1 \qquad \text{and} \qquad I' \subset J \cup N_2,$$

where N_1 and N_2 have measure zero. If $\mu(I - J') > 0$, it follows that

$$\int_I \alpha(s)\, \mu(ds) > \int_{J'} \beta(t)\, \mu(dt) = \int_{J'} \alpha(s)\, \mu(ds).$$

If $\mu(I - J') = 0$, we obtain

$$\int_I \alpha(s)\, \mu(ds) = \int_{J'} \beta(t)\, \mu(dt) = \int_I \alpha(s)\, \mu(ds).$$

However, in this case $k \mid I' \times I = 0$ a.e., so Hypothesis 4.10 implies that $k \mid I \times J = k \mid I \times I' = 0$, a.e. Thus the compatibility condition for (k, α, α) is satisfied. ∎

It remains to give conditions which imply Hypothesis 4.10. If Hypothesis 2.1 is satisfied, $(S, \mu) = (T, \nu)$ and $k \in L^\infty(S \times S)$ define $d_j \in L^\infty(S \times S)$, $j \geqslant 1$, by

$$d_1(r, s) = \int k(r, t)\, k(t, s)\, \mu(dt) \tag{4.12}$$

$$d_n(r, s) = \int d_1(r, t)\, d_{n-1}(t, s)\, \mu(dt), \qquad n > 1. \tag{4.13}$$

COROLLARY 4.14. *Assume Hypothesis 3.8 or 3.10, and assume Hypothesis 2.3 holds, with $(S, \mu) = (T, \nu)$ and $\alpha = \beta$. Assume that there exist $\delta > 0$ and measurable sets $G_i \subset S$ for $1 \leqslant i \leqslant n$, with $S = \bigcup_{i=1}^n G_i$ and $k \mid G_i \times G_i \geqslant \delta$ a.e., for $1 \leqslant i \leqslant n$. Assume there exists $m \geqslant 1$ such that $d_m > 0$ a.e. on $S \times S$, where d_m is given by (4.13). Then the DAD problem (k, α, α) has a solution (f, g) and there exist $\kappa_1 > 0$, $\kappa_2 > 0$, with $\kappa_1 \alpha(s) \leqslant f(s) \leqslant \kappa_2 \alpha(s)$ and $\kappa_1 \alpha(t) \leqslant g(t) \leqslant \kappa_2 \alpha(t)$ almost everywhere. If Hypothesis 2.2 is satisfied and k is continuous, there exist positive continuous functions x and y in $C(S)$ with $f = \alpha/x$ and $g = \alpha/y$.*

Proof. By Theorems 2.34, 3.9, and 4.11, it suffices to verify Hypothesis 4.10. Thus it suffices to show that if $I \subset S$ is measurable and $\mu(I) > 0$ and $\mu(I') > 0$, then $k \mid I \times I'$ is positive on a set of positive measure.

If not, there exists $I \subset S$ with $\mu(I) > 0$, and $k \mid I \times I' = 0$ a.e. For almost all $(r, s) \in I \times I'$ we see that

$$d_1(r, s) = \int_I k(r, t)\, k(t, s)\, \mu(dt) + \int_{I'} k(r, t)\, k(t, s)\, \mu(dt) = 0.$$

If we assume by induction that $d_j \mid I \times I' = 0$ a.e. for $1 \leqslant j < n$, we conclude by the same argument used for d_1 that $d_n \mid I \times I' = 0$ a.e. For $n = m$ this gives a contradiction. ▌

COROLLARY 4.15. *Assume Hypothesis* 3.8 *or* 3.10 *and suppose that Hypotheses* 2.2 *and* 2.3 *hold, with* $(S, \mu) = (T, \nu)$ *and* $\alpha = \beta$. *Assume that* S *is connected. Assume that there exist* $\delta > 0$ *and nonempty open sets* $G_i \subset S$ *for* $1 \leqslant i \leqslant m$ *with* $S = \bigcup_{i=1}^m G_i$ *and* $k(s, t) \geqslant \delta$ *for almost all* $(s, t) \in (G_i \times G_i)$, $1 \leqslant i \leqslant m$. *Then the DAD problem* (k, α, α) *has a solution* (f, g). *If* k *is continuous, there exist positive continuous functions* x *and* y *with* $f = \alpha/x$ *and* $g = \alpha/y$.

Proof. By Corollary 4.14, it suffices to prove that $d_m > 0$ a.e. on $S \times S$ for some $m \geqslant 1$. For almost all $(r, s) \in G_i \times G_i$ we obtain from (4.12) that

$$d_1(r, s) \geqslant \int_{G_i} k(r, t)\, k(t, s)\, \mu(dt) \geqslant \delta^2 \mu(G_i).$$

It follows from Proposition 4.8′ that there exists $\eta > 0$ and $m \geqslant 1$ with $d_m(r, s) \geqslant \eta$ a.e. on $S \times S$. ▌

Corollary 4.14 generalizes Theorem 5.16 in [5], where a more restrictive condition than $d_m > 0$ a.e. on $S \times S$ is given. Corollary 4.15 is essentially Corollary 5.17 in [5].

If we take $S = T = [0, 1]$, $\mu = \nu =$ Lebesque measure, $\alpha = \beta$, and k is a nonnegative continuous function with $k(s, s) > 0$ for $0 \leqslant s \leqslant 1$, we find, by applying Corollary 4.15, that the corresponding DAD problem (k, α, α) has a solution (f, g) with $f = \alpha/x$ and $g = \beta/y$ and x and y positive continuous functions. If one assumes in addition that $\alpha = \beta = 1$, this result was proved in Section 4 of [27], but the argument in [27] does not extend to general α.

Assume Hypothesis 2.3 and 2.15 and suppose that

$$\int \alpha(s)\, \mu(ds) = \int \beta(t)\, \nu(dt). \tag{4.16}$$

Let C_1 (respectively, C_2) denote the cone of nonnegative functions in $L^\infty(S)$ (respectively, $L^\infty(T)$) with interior $\mathring{C}_1$ and let F be defined by (2.20).

Recall (Lemma 2.21) that the DAD problem (1.1) has a solution if and only if F has a fixed point in $\mathring{C}_1$. Define a linear functional ψ on $L^\infty(S)$ by

$$\psi(x) = \int x(s)\, \mu(ds). \tag{4.17}$$

Define $G: \mathring{C}_1 \to \mathring{C}_1$ by

$$G(x) = F(x)/\psi(F(x)). \tag{4.18}$$

The homogeneity of F implies that if F has a fixed point in $\mathring{C}_1$, then F has a fixed point x_0 with $\psi(x_0) = 1$ and $G(x_0) = x_0$.

Once one proves existence of fixed points of F in $\mathring{C}_1$ as in Theorem 3.9 and Lemma 2.21, there are powerful theorems (see [26]) for proving the uniqueness of fixed points (to within scalar multiples) and for approximating fixed points. The following result is Theorem 5.3 in [5]. Theorem 5.3 in [5] is stated slightly less generally but the same argument applies. It is obtained by using Theorems 2.7 and 3.2 in [26] (see pp. 78 and 93), linear theory outlined in Theorem 2.4 and Remark 2.4 on p. 44 in [26], and an argument on pp. 50–52 in [26].

THEOREM 4.19 (Theorem 5.3 in [5]). *Assume Hypotheses* 2.3 *and* 2.15 *and suppose* $\int_S \alpha(s)\, \mu(ds) = \int_T \beta(t)\, \nu(dt)$. *If* F *is defined by* (2.20) *and* C_1, ψ *and* G *are as above, assume that the DAD problem* (k, α, β) *has a solution, so there exists* $x_0 \in \mathring{C}_1$ *with*

$$x_0 = F(x_0) = G(x_0), \qquad \psi(x_0) = 1.$$

If c_n *is defined by* (4.3) *assume that there exists* $\delta > 0$ *and* $m \geqslant 1$ *with* $c_m(r, s) \geqslant \delta$ *a.e. on* $S \times S$. *Then every fixed point of* F *in* $\mathring{C}_1$ *is a positive scalar multiple of* x_0. *If the DAD problem* (k, α, β) *has solutions* (f_0, g_0) *and* (f_1, g_1), *there is a positive constant* $\mu > 0$ *with* $f_1 = \mu f_0$ *and* $g_1 = \mu^{-1} g_0$. *If* $x \in \mathring{C}_1$, *there exists* $\lambda(x) > 0$ *with*

$$\lim_{j \to \infty} \|F^j(x) - \lambda(x)\, x_0\|_\infty = 0.$$

Also, one has that for each $x \in \mathring{C}_1$,

$$\lim_{j \to \infty} \|G^j(x) - x_0\|_\infty = 0,$$

and the convergence is geometric. If $Y = \{x \in L^\infty(S) : \psi(x) = 0\}$, $I - G'(x_0) \mid Y$ *is one–one and onto* Y, *where* $G'(x_0)$ *denotes the Fréchet derivative of* G *at* x_0 *and* I *is the identity map.*

Remark 4.20. Let notation and assumptions be as in Theorem 4.19, select any functions $\xi_0 \in \mathring{C}_1$ and $\eta_0 \in \mathring{C}_2$, and define $k_0(s, t)$ by

$$k_0(s, t) = \left(\frac{\alpha(s)}{\xi_0(s)}\right) k(s, t) \left(\frac{\beta(t)}{\eta_0(t)}\right).$$

Multiply $k_0(s, t)$ by $\gamma_0(t)$ such that $\int \gamma_0(t)\, k_0(s, t)\, \mu(ds) = \beta(t)$ a.e., and then multiply $\gamma_0(t)\, k_0(s, t)$ by $\delta_0(s)$ such that $\int \delta_0(s)\, \gamma_0(t)\, k_0(s, t)\, \nu(dt) = \alpha(s)$ a.e. Thus we alternately rescale k_0 so that first its "columns" integrate to β and then its "rows" integrate to α. If we define k_1 by $k_1(s, t) = \gamma_0(t)\, \delta_0(s)\, k_0(s, t)$ we find

$$k_1(s, t) = \frac{\alpha(s)}{\xi_1(s)}\, k(s, t)\, \frac{\beta(t)}{\eta_1(t)},$$

where $\eta_1 = A_1(J_1 \xi_0)$ and $\xi_1 = F(\xi_0)$. If we find $k_m(s, t)$ by this procedure, and $k_m(s, t) = (\alpha(s)/\xi_m(s))\; k(s, t)(\beta(t)/\eta_m(t))$ select $\gamma_m(t)$ such that $\int \gamma_m(t)\, k_m(s, t)\, \mu(ds) = \beta(t)$ a.e. and $\delta_m(s)$ so that $\int \delta_m(s)\, \gamma_m(t)\, k_m(s, t)\, \nu(dt) = \alpha(s)$. It is easy to see that

$$k_{m+1}(s, t) := \delta_m(s)\, \gamma_m(t)\, k_m(t) = \left(\frac{\alpha(s)}{\xi_{m+1}(s)}\right) k(s, t) \left(\frac{\beta(t)}{\eta_{m+1}(t)}\right),$$

where $\eta_{m+1} = A_1(J_1 \xi_m)$ and $\xi_{m+1} = F(\xi_m)$. It follows that $\eta_{m+1} = A_1(J_1 \xi_m)$ and $\xi_{m+1} = F^{m+1}(\xi_0)$, so Theorem 4.19 implies that there exist $x \in \mathring{C}_1$ and $y \in \mathring{C}_2$ with

$$\lim_{m \to \infty} \|\xi_m - x\|_{L^\infty(S)} = 0 \qquad \text{and} \qquad \lim_{m \to \infty} \|\eta_m - y\|_{L^\infty(T)} = 0.$$

Thus we see that the procedure of alternating making "rows" and "columns" of the kernel integrate to α and β respectively gives a convergent scheme: $(\alpha\beta)^{-1} k_m$ converges in $L^\infty(S \times T)$ to an integral kernel $c \in L^\infty(S \times T)$, where

$$c(s, t) = \left(\frac{1}{x(s)}\right) k(s, t) \left(\frac{1}{y(t)}\right) \qquad \text{for some } x \in \mathring{C}_1 \text{ and } y \in \mathring{C}_2,$$

and $\int \alpha(s)\, c(s, t)\, \beta(t)\, \nu(dt) = \alpha(s)$ a.e. and $\int \alpha(s)\, c(s, t)\, \beta(t)\, \mu(ds) = \beta(t)$ a.e. This observation generalizes earlier results of Sinkhorn [34] and Sinkhorn and Knopp [35].

With the aid of Theorem 4.19 one can prove that for k symmetric and $\alpha = \beta$ there is a solution of the DAD problem with $f = g$.

COROLLARY 4.21. *Assume Hypothesis* 2.3 *and Hypothesis* 3.8 *or* 3.10. *Suppose that* $(S, \mu) = (T, \nu)$, $\alpha = \beta$, *and* $k(s, t) = k(t, s)$ *almost everywhere on* $S \times S$. *Assume that there exist* $\delta > 0$ *and* G_i *as in Theorem* 4.11. *Assume that if* c_j *is defined as in* (4.3), *there exists* $\delta_1 > 0$ *and* $m \geqslant 1$ *so that* $c_m(s, t) \geqslant \delta_1$ *almost everywhere. Then the DAD problem* (k, α, α) *has a unique solution* (f, f). *If* k *is continuous, there exists a positive continuous function* $x \in C(S)$ *with* $f = \alpha/x$.

Proof. Because k is symmetric, Hypothesis 4.10 is automatically satisfied and Theorem 4.11 implies that the *DAD* problem (k, α, α) has a solution (f_1, g_1). The symmetry of k implies that (g_1, f_1) is also a solution of the *DAD* problem (k, α, α). Because $c_m \geqslant \delta_1$ almost everywhere, Theorem 4.19 implies that f_1 and g_1 are unique to within scalar multiples and there exists $\mu > 0$ with $g_1 = \mu f_1$. If we define $f = \sqrt{\mu} f_1$, we have the desired result. The existence of x when k is continuous is Theorem 2.34. ∎

COROLLARY 4.22. *Assume Hypotheses* 2.2 *and* 2.3 *and suppose that* $(S, \mu) = (T, \nu)$, $\alpha = \beta$, k *is continuous,* $k(s, t) = k(t, s)$ *for all* $(s, t) \in S \times S$, *and* $k(s, s) > 0$ *for all* $s \in S$. *Assume that* S *is connected. Then the DAD problem* (k, α, α) *has a unique solution* (f, f) *and there exists a positive continuous function* $x \in C(S)$ *with* $f = \alpha/x$.

Proof. Because k is symmetric, the same argument given in the proof of Corollary 4.15 (using Proposition 4.8′) shows that there exists $\delta_1 > 0$ and $m \geqslant 1$ so $c_m(s, t) \geqslant \delta_1$ almost everywhere. Corollary 4.15 implies that the *DAD* problem (k, α, α) has a solution (f_1, g_1), and uniqueness and the fact that one can take $f_1 = g_1$ follows as in Corollary 4.21. ∎

In the special case $S = T = [0, 1]$, $\mu =$ Lebesgue measure, and $\alpha = \beta = 1$, Corollary 4.22 implies a theorem of Nowosad [24]. A more elegant proof of a generalization of Nowosad's theorem was given by Karkin and Nirenberg [18].

Remark 4.23. Theorem 2.33 and the results of this section give all the classical *DAD* theorems for matrices. We briefly describe the connection. If S and T are finite sets with m and n elements, respectively, and Hypotheses 2.2 and 2.3 hold, k can be considered an $m \times n$ matrix and (k, α, β) has a solution (f, g) if and only if there are positive diagonal matrices $D_1 = \text{diag}(f_1, f_2, ..., f_m)$ and $D_2 = \text{diag}(g_1, g_2, ..., g_n)$ and the ith row of $D_1 k D_2$ sums to α_i and the jth column of $D_1 k D_2$ sums to β_j. For c_j as in (4.3), the condition that $c_p(r, s) > 0$ for all $(r, s) \in S \times S$ and some $p \geqslant 1$ is just the assumption that $(kk^*)^p$ has all positive entries. If $S = T$ and d_j is given by (4.13), $d_p(r, s) > 0$ for all $(r, s) \in S \times S$ and some $p \geqslant 1$ if and only if k^{2p} has all positive entries.

Classical DAD theorems consider the special case that $S=T=\{i\in\mathbb{N}: 1\leqslant i\leqslant n\}$ and $\alpha=\beta=1$, so k is an $n\times n$ nonnegative matrix. A nonnegative square matrix k is called "irreducible" if e^k has all positive entries. It is known (and easily proved) that if k has a positive main diagonal (so $k_{ii}>0$ for $1\leqslant i\leqslant n$) and is irreducible, then there exists $m\geqslant 1$ such that k^{2m} and $(kk^*)^m$ have all positive entries. It follows from Corollary 4.14 and Theorem 4.20 that if k is a nonnegative, irreducible square matrix with positive main diagonal, the DAD problem (k, α, α) has a solution (f, g) which is unique in the sense of Theorem 4.20. In particular there are square positive diagonal matrices D_1 and D_2 which are unique to within scalar multiples and are such that all rows and columns of D_1kD_2 sum to one.

A nonnegative square matrix k is called "fully indecomposable" if there exist permutation matrices P and Q such that PkQ is irreducible and has a positive main diagonal; see Lemma 2.3 in [7]. In general, if k is a nonnegative square matrix and the DAD problem $(k, 1, 1)$ has a solution, then for any permutation matrices P and Q, the DAD problem $(PkQ, 1, 1)$ has a solution. For if $(k, 1, 1)$ has a solution, there are positive diagonal matrices D_1 and D_2 such that all rows and columns of D_1kD_2 sum to one; and $PD_1P^{-1}:=\tilde{D}_1$ and $Q^{-1}D_2Q:=\tilde{D}_2$ are positive diagonal matrices and all rows and columns of $\tilde{D}_1(PkQ)\tilde{D}_2=P(D_1kD_2)Q$ sum to one. By using these remarks we see that if k is a nonnegative square matrix and k is a direct sum of fully indecomposable matrices, there exist positive diagonal matrices D_1 and D_2 such that all rows and columns of D_1kD_2 sum to one. This is the classical DAD theorem.

The previous results of this section mostly treat the case $\alpha=\beta$. We now describe some results which, when combined with Propositions 4.1 and 4.5, allow one to establish the compatibility condition for (k, α, β) by reducing to an equivalent matrix case.

HYPOTHESIS 4.24. *Hypothesis 2.3 is satisfied. There exist measurable sets of positive measure* $S_i\subset S$, $1\leqslant i\leqslant m$, *and* $T_j\subset T$, $1\leqslant j\leqslant n$, *with* $S=\bigcup_{i=1}^m S_i$, $T=\bigcup_{j=1}^n T_j$, $\mu(S_i\cap S_l)=0$ *for all* $i\neq l$ *and* $\nu(T_j\cap T_l)=0$ *for all* $j\neq l$. *For each* (i, j) *with* $1\leqslant i\leqslant m$ *and* $1\leqslant j\leqslant n$, *either* $k(s, t)>0$ *for almost all* $(s, t)\in S_i\times T_j$ *or* $k(s, t)=0$ *for almost all* $(s, t)\in S_i\times T_j$.

Assuming Hypothesis 4.24 holds, define $\hat{S}=\{i\in\mathbb{N}\mid 1\leqslant i\leqslant m\}$ *and* $\hat{T}=\{j\in\mathbb{N}\mid 1\leqslant j\leqslant n\}$, and let $\hat{\mu}$ and $\hat{\nu}$ be the counting measures on $\hat{S}$ and $\hat{T}$. Define $\hat{k}\in L^\infty(S\times T)$ by

$$\hat{k}(i, j)=\begin{cases}0, & \text{if } k(s, t)=0 \text{ a.e. on } S_i\times T_j\\ 1, & \text{if } k(s, t)>0 \text{ a.e. on } S_i\times T_j.\end{cases} \tag{4.25}$$

Obviously $\hat{k}$ can be identified with an $m \times n$ nonnegative matrix $\hat{k}$ whose $i-j$ entry is $\hat{k}(i, j)$. Define $\hat{\alpha} \in L^1(\hat{S})$ and $\beta \in L^1(\hat{T})$ by

$$\hat{\alpha}(i) = \int_{S_i} \alpha(s)\, \mu(ds) \qquad \text{and} \qquad \hat{\beta}(j) = \int_{T_j} \beta(t)\, \nu(dt), \tag{4.26}$$

so that $\hat{\alpha}$ and $\hat{\beta}$ can be identified with vectors. It makes sense to ask if $(\hat{k}, \hat{\alpha}, \hat{\beta})$ satisfies the compatibility condition (Definition 2.13). The matrix *DAD* problem $(\hat{k}, \hat{\alpha}, \hat{\beta})$ satisfies the compatibility condition if and only if

$$\sum_{i=1}^{m} \hat{\alpha}(i) = \sum_{j=1}^{n} \hat{\beta}(j) \tag{4.27}$$

and for every nonempty set $\hat{I} \subset S$ and nonempty $\hat{J} \subset \hat{T}$ with $\hat{k} \mid (\hat{I})' \times (\hat{J})' = 0$,

$$\sum_{i \in \hat{I}} \hat{\alpha}(i) \geqslant \sum_{j \in (\hat{J})'} \hat{\beta}(j) \tag{4.28}$$

with strict inequality holding in (4.28) if $\tilde{k}(i, j) > 0$ for some $(i, j) \in \hat{I} \times \hat{J}$.

PROPOSITION 4.29. *Assume Hypothesis* 4.24 *for* (k, α, β) *and let* $\hat{k}$, $\hat{\alpha}$, *and* $\hat{\beta}$ *be defined by* (4.25) *and* (4.26). *Then* (k, α, β) *satisfies the compatibility condition if and only if the matrix DAD problem* $(\hat{k}, \hat{\alpha}, \hat{\beta})$ *satisfies the compatibility condition, and* k *is indecomposable if and only if* $\hat{k}$ *is indecomposable. If* $\tilde{k} \in L^\infty(S \times T)$ *is nonnegative and* $\tilde{k}(s, t) = 0$ *implies* $k(s, t) = 0$ *for almost all* (s, t), *then if* $\hat{k}$ *is indecomposable and* $(\hat{k}, \hat{\alpha}, \hat{\beta})$ *satisfies the compatibility condition,* $(\tilde{k}, \alpha, \beta)$ *satisfies the compatibility condition and* $\tilde{k}$ *is indecomposable.*

Proof. If (k, α, β) satisfies the compatibility condition (or is indecomposable) it is easy to see that $(\hat{k}, \hat{\alpha}, \hat{\beta})$ satisfies the compatibility condition (or is indecomposable). To prove the other implication, assume that $(\hat{k}, \hat{\alpha}, \hat{\beta})$ satisfies the compatibility condition. Let $I \subset S$ and $J \subset T$ be measurable sets of positive measure such that $k \mid I' \times J' = 0$ a.e. Define $\hat{I}_2$ and $\hat{J}_2$ by

$$\hat{I}_2 = \{ i \in \hat{S} \mid \mu(S_i \cap I') > 0 \} \qquad \text{and}$$
$$\hat{J}_2 = \{ j \in \hat{T} \mid \nu(T_j \cap J') > 0 \}.$$

By using Hypothesis 4.24 we see that

$$I' \times J' \subset \Big(\bigcup_{i \in \hat{I}_2,\, j \in \hat{J}_2} (S_i \times T_j) \Big) \cup N, \qquad (\mu \times \nu)(N) = 0.$$

Also, if $i \in \hat{I}_2$ and $j \in \hat{J}_2$, $k(s, t) = 0$ a.e. on $(S_i \cap I') \times (T_j \times J')$, which is a set of positive measure, so Hypothesis 4.24 implies $k \mid S_i \times T_j = 0$ a.e. and $\hat{k}(i, j) = 0$. Define $\hat{I}_1 = (\hat{I}_2)'$, $\hat{J}_1 = (\hat{J}_2)'$,

$$I_1 = \bigcup_{i \in \hat{I}_1} S_i, \qquad \text{and} \qquad J_1 = \bigcup_{j \in \hat{J}_1} T_j,$$

so $I \supset I_1$ and $J \supset J_1$. Because $\hat{k} \mid (\hat{I}_1)' \times (\hat{J}_1)' = 0$, the compatibility condition for $(\hat{k}, \hat{\alpha}, \hat{\beta})$ gives

$$\sum_{i \in \hat{I}_1} \hat{\alpha}(i) = \int_{I_1} \alpha(s)\, \mu(ds) \geqslant \sum_{j \in (\hat{J}_1)'} \hat{\beta}(j) = \int_{J_1'} \beta(t)\, v(dt), \tag{4.30}$$

and strict inequality holds in (4.30) if k is positive on a set of positive measure in $I_1 \times J_1$ (so $\hat{k}(i, j) > 0$ for some $(i, j) \in \hat{I}_1 \times \hat{J}_1$). It follows that if $\mu(I - I_1) > 0$ or $v(J - J_1) > 0$ or k is positive on a set of positive measure in $I_1 \times J_1$, then

$$\int_I \alpha(s)\, \mu(ds) > \int_{J'} \beta(t)\, v(dt).$$

If none of these possibilities occurs, $k \mid I \times J = 0$ a.e. and (4.30) gives

$$\int_I \alpha(s)\, \mu(ds) \geqslant \int_{J'} \beta(t)\, v(dt).$$

This shows that (k, α, β) satisfies the compatibility condition.

If $\hat{k}$ is indecomposable, $\hat{k}(i, j) > 0$ for some $(i, j) \in \hat{I}_1 \times \hat{J}_1$, and Hypothesis 4.24 implies that k is positive almost everywhere on $S_i \times T_j \subset I \times J$ and k is decomposable.

The final statement of Proposition 4.29 follows immediately from Proposition 4.1. ▮

5. Approximation of Solutions and Dependence of Solutions on (k, α, β)

Theorems 2.30, 2.33, and 3.9 (and the results of Section 4 concerning the compatibility condition for *DAD* problems) provide an essentially complete answer for the question of existence of solutions of the *DAD* problem (k, α, β). Thus we now assume that a *DAD* problem (k_0, α_0, β_0) has a solution (f_0, g_0), and we show how Theorem 4.20 and ideas from Sections 2 and 3 of [26] can be used to obtain existence and continuous dependence of solutions for nearly *DAD* problems (k, α, β). Our techniques also give rapidly convergent iteration schemes for finding solutions of *DAD* problems and provide information about matrix DAD problems for large matrices.

We begin by recalling some notation. Let E and F denote Banach spaces as in Eqs. (2.5) and (2.6). Assume Hypothesis 2.1 and denote by $\mathring{C}_1$ (respectively, $\mathring{C}_2$) the interior of the cone C_1 (respectively, C_2) of non-negative functions in $L^\infty(S)$ (respectively, $L^\infty(T)$). Define $J_i\colon \mathring{C}_i \to \mathring{C}_i$, $i = 1, 2$, by

$$(J_i x)(r) = (x(r))^{-1}. \tag{5.1}$$

Define $A_1\colon L^\infty(S) \times E \to L^\infty(T)$ and $A_2\colon L^\infty(T) \times F \to L^\infty(S)$ by

$$(A_1(u, h_1))(t) = \int h_1(s, t)\, u(s)\, \mu(ds) \tag{5.2}$$

and

$$(A_2(v, h_2))(s) = \int h_2(s, t)\, v(t)\, \nu(dt). \tag{5.3}$$

We leave to the reader the verification that A_1 and A_2 are continuously differentiable maps. Define sets $U \subset E$ and $V \subset F$ by

$$U = \Big\{ h \in E \mid h(s, t) \geqslant 0 \text{ a.e. and there exists } \delta = \delta_h > 0 \text{ so that}$$

$$\int h(s, t)\, \mu(ds) \geqslant \delta \text{ for almost all } t \Big\}. \tag{5.4}$$

and

$$V = \Big\{ h \in F \mid h(s, t) \geqslant 0 \text{ a.e. and there exists } \delta = \delta_h > 0 \text{ with}$$

$$\int h(s, t)\, \nu(dt) \geqslant \delta \text{ for almost all } s \Big\}. \tag{5.5}$$

Recall (see Lemma 2.18) that if (k, α, β) satisfies Hypotheses 2.3 and 2.15 then $k\alpha \in U$ and $k\beta \in V$. If $u \in \mathring{C}_1$ and $h_1 \in U$, one can see that $A_1(u, h_1) \in \mathring{C}_2$; and if $v \in \mathring{C}_2$ and $h_2 \in V$, one has $A_2(v, h_2) \in \mathring{C}_1$. Furthermore, if $u_0 \in \mathring{C}_1$ and $h_0 \in U$, there exists $\theta > 0$ such that $A_1(u, h) \in \mathring{C}_2$ for all $u \in L^\infty(S)$ and $h \in E$ with $\|u - u_0\| < \theta$ and $\|h - h_0\|_E < \theta$. A similar statement holds for $A_2(v, h)$. We leave the elementary verifications of these facts to the reader.

Define maps $\Phi_1: \mathring{C}_1 \times E \to L^\infty(T)$ and $\Phi_2: \mathring{C}_2 \times F \to L^\infty(S)$ by

$$\Phi_1(u, h_1) = A_1(J_1 u, h_1) \quad \text{and} \quad \Phi_2(v, h_2) = A_2(J_2 v, h_2). \tag{5.6}$$

If $(u, h) \in \mathring{C}_1 \times U$, then $\Phi_1(u, h) \in \mathring{C}_2$. Also, given $(u_0, h_0) \in \mathring{C}_1 \times U$, there exists $\theta > 0$ so that $\Phi_1(u, h) \in \mathring{C}_2$ for all $u \in \mathring{C}_1$ with $\|u - u_0\| < \theta$ and all $h \in E$ with $\|h - h_0\|_E < \theta$. Similar statements hold for Φ_2.

If $(u, h_1, h_2) \in \mathring{C}_1 \times U \times V$, define $\Phi: \mathring{C}_1 \times U \times V \to \mathring{C}_1$ by

$$\Phi(u; h_1, h_2) = \Phi_2(\Phi_1(u, h_1), h_2). \tag{5.7}$$

For any $(u_0, f_0, g_0) \in \mathring{C}_1 \times U \times V$, there exists $\theta = \theta(u_0, f_0, g_0) > 0$ such that $\Phi_1(u, f) \in \mathring{C}_2$ and $\Phi(u; f, g) \in \mathring{C}_1$ if $\|u - u_0\| < \theta$, $\|f - f_0\| < \theta$ and $\|g - g_0\| < \theta$. Thus Φ can be defined on an open neighborhood of $\mathring{C}_1 \times U \times V$ and still map into $\mathring{C}_1$. Also, by using the fact that A_1, A_2, J_1, and J_2 are C^1 maps and that the composition of C^1 maps is C^1, one can prove that Φ is defined and C^1 on an open neighborhood of $\mathring{C}_1 \times U \times V$. In fact, Φ is C^∞ on an open neighborhood of $\mathring{C}_1 \times U \times V$. If ψ is the linear functional defined by equation (4.18) (i.e., integration over S), we define $G: \mathring{C}_1 \times U \times V \to \mathring{C}_1$ by

$$G(u, h_1, h_2) = \Phi(u; h_1, h_2)/\psi(\Phi(u; h_1, h_2)). \tag{5.8}$$

The map G extends to an open neighborhood of $\mathring{C}_1 \times U \times V$ and is continuously differentiable there.

With these preliminaries we can state our main result.

THEOREM 5.9. *Assume that* (k_0, α_0, β_0) *satisfies Hypotheses* 2.3 *and* 2.15 *and that the DAD problem* (k_0, α_0, β_0) *has a solution* (f_0, g_0), *so (for G as in* (5.8)) *there exists* $x_0 \in \mathring{C}_1$ *with* $G(x_0; k_0\alpha_0, k_0\beta_0) = x_0$. *If* c_n *is defined by Eqs.* (4.2) *and* (4.3) (*with* (k_0, α_0, β_0) *replacing* (k, α, β) *in* (4.2) *and* (4.3)), *assume that there exist* $m \geq 1$ *and* $\delta > 0$ *with* $c_m(s, r) \geq \delta$ *almost everywhere. Then there exist* $\theta > 0$ *and a* C^1 *function* x,

$$x: \{(h_1, h_2) \in E \times F \mid \|h_1 - k_0\alpha_0\|_E < \theta \text{ and } \|h_2 - k_0\beta_0\|_F < \theta\} \to \mathring{C}_1,$$

with

$$G(x(h_1, h_2); h_1, h_2) = x(h_1, h_2) \quad \text{and} \quad x(k_0\alpha_0, k_0\beta_0) = x_0.$$

In particular, if $k \in L^\infty(S \times T)$, $\alpha \in L^1(S)$, $\beta \in L^1(T)$, $\|k\alpha - k_0\alpha_0\|_E < \theta$, $\|k\beta - k_0\beta_0\|_F < \theta$, and $\int \alpha\mu(ds) = \int \beta\nu(dt)$, then for $x = x(k\alpha, k\beta) \in \mathring{C}_1$, $y = \Phi_1(J_1 x, k\alpha) \in C_2$, $f = \alpha/x \in L^1(S)$, and $g = \beta/y \in L^1(T)$ we have a solution of the *DAD* problem (k, α, β):

$$\int f(s)\, k(s, t)\, g(t)\, \nu(dt) = \alpha(s) \text{ a.e.} \quad \text{and} \quad \int f(s)\, k(s, t)\, g(t)\, \mu(ds) = \beta(t) \text{ a.e.}$$

Proof. By our previous remarks, there exists $\theta_0 > 0$ such that if $W = \{(x, h_1, h_2) \in \mathring{C}_1 \times E \times F \mid \|x - x_0\| < \theta_0,\ \|h_1 - k_0\alpha_0\|_E < \theta_0$, and $\|h_2 - k_0\beta_0\|_F < \theta_0\}$, then G is defined and C^1 on W. If ψ is given by (4.18) define

$$Y_0 = \{z \in L^\infty(S) \mid \psi(z) = 0\}$$

and

$$W_0 = \{(z, h_1, h_2) \in Y_0 \times E \times F \mid \|z\| < \theta_0,\ \|h_1 - k_0\alpha_0\|_E < \theta_0 \text{ and } \|h_2 - k_0\beta_0\|_F < \theta_0\}.$$

Define $H\colon W_0 \to Y_0$ by

$$H(v; h_1, h_2) = x_0 + v - G(x_0 + v; h_1, h_2),$$

so H is a C^1 map. Theorem 4.20 implies that the Fréchet derivative of the map $v \to H(v; k_0\alpha_0, k_0\beta_0)$ at $v = 0$ is one–one and onto Y_0. It follows from the implicit function theorem that there exist $\theta > 0$ and a C^1 function v,

$$v\colon \{(h_1, h_2) \in E \times F \mid \|h_1 - k_0\alpha_0\|_E < \theta,\ \|h_2 - k_0\beta_0\|_F < \theta\} \to Y_0,$$

with $v(k_0\alpha_0, k_0\beta_0) = 0$ and $H(v(h_1, h_2); h_1, h_2) = 0$. If we define $x(h_1, h_2) = x_0 + v(h_1, h_2)$, we have the desired function.

The final part of Theorem 5.9 follows by the same argument used in Lemma 2.21, and the proof is left to the reader. ∎

Remark 5.10. Note that it may happen in Theorem 5.9 that k, α, or β is negative on a set of positive measure.

Remark 5.11. Let k_0, α_0, β_0, and x_0 be as in Theorem 5.9, assume $c_m \geqslant \delta$ a.e., and define $k_\varepsilon(s, t) = k_0(s, t) + \varepsilon$ for $\varepsilon > 0$. The same argument used in Lemma 2.24 shows that for each $\varepsilon > 0$ the equation $x = G(x; k_\varepsilon\alpha_0, k_\varepsilon\beta_0)$ has a unique solution $x_\varepsilon \in \mathring{C}_1$. Theorem 5.9 implies that the map $\varepsilon \to x_\varepsilon$ is continuous for $\varepsilon > 0$. It follows that there exist positive constants κ_1 and κ_2 such that for $0 < \varepsilon \leqslant 1$

$$\kappa_1 \leqslant x_\varepsilon(s) \leqslant \kappa_2 \qquad \text{almost everywhere.} \tag{5.12}$$

The reader will recall (see the remarks at the beginning of Section 3) that the estimate (5.12) would provide a direct proof of Theorem 3.9. If the hypotheses of Theorem 3.9 are satisfied and $c_m(r, s) \geqslant \delta > 0$ a.e. (for c_j as in (4.3) and some $m \geqslant 1$), Theorems 3.9 and 5.9 imply that (5.12) is satisfied. It would be interesting to find a direct proof of (5.12) with explicit estimates for κ_1 and κ_2, but, when k is not bounded below by a positive constant, the only such results of which we are aware require $k(s, t) = k(t, s)$ a.e. and $\alpha = \beta$: see [5] and [18].

Remark 5.13. Theorem 5.9 suggests a numerical procedure for solving the *DAD* problem (k, α, β) —assuming the problem has a solution. Suppose that Hypotheses 2.3 and 2.15 hold for (k, α, β) and that there exist $m \geqslant 1$ and $\delta > 0$ with $c_m(r, s) \geqslant \delta$ a.e. For $h_1 = k\alpha$ and $h_2 = k\beta$ and G as in (5.8), define $\Gamma: \mathring{C}_1 \to \mathring{C}_1$ by

$$\Gamma(u) = G(u;\, h_1, h_2).$$

Take $u \in \mathring{C}_1$. If $\xi \in \mathring{C}_1$ is the unique fixed point of Γ, we know that $\Gamma^j(u)$ converges at a geometric rate of convergence to ξ: see Section 2 of [26]. For j large, define $u_1 = \Gamma^j(u)$. The choice of j would depend on some kind of stopping condition, perhaps $\|\Gamma^j(u) - \Gamma^{j+1}(u)\| < \varepsilon$, where ε is given. Define a map $\Lambda: Y_0 \to Y_0$ by

$$\Lambda(v) = u_1 + v - \Gamma(u_1 + v).$$

We know that $\Lambda'(\xi - u_1)$ is one–one and onto Y_0, so if u_1 is close enough to ξ, Newton's fixed point iteration method will give a sequence of points v_p, $v_1 = 0$,

$$v_{p+1} = v_p - \Lambda'(v_p)(\Lambda(v_p))$$

which will converge at a quadratic rate to v_∞, $u_1 + v_\infty = \xi$.

Remark 5.14. Theorem 5.9 can also be used to discuss the limiting behaviour of a sequence of matrix *DAD* problems. Indeed, this is a possible motivation for studying general *DAD* problems (k, α, β), even if one is only interested in matrix *DAD* problems. Thus let A^p be an $m_p \times n_p$ nonnegative matrix with entries $a_{ij}^p \geqslant 0$. The letter "p" denotes a superscript here. Let α^p denote an $m_p \times 1$ column vector with positive entries α_i^p and β^p a $1 \times n_p$ row vector with positive entries β_j^p. Assume that

$$\sum_{i=1}^{m_p} \alpha_i^p = \sum_{j=1}^{n_p} \beta_j^p.$$

We are interested in the limiting behaviour of the matrix *DAD* problems (A^p, α^p, β^p): Do solutions exist for all large p and do these solutions converge in an appropriate sense?

Let $S = T = [0, 1]$ and $\mu = \nu =$ Lebesque measure on $[0, 1]$. For $p \geqslant 1$, define intervals I_i^p, $1 \leqslant i \leqslant m_p$, and J_j^p, $1 \leqslant j \leqslant n_p$, by

$$I_i^p = [(i-1)/m_p,\, i/m_p) \qquad \text{and} \qquad J_j^p = [(j-1)/n_p,\, j/n_p). \tag{5.15}$$

We abuse notation slightly and define $\alpha^p \in L^1(S)$ and $\beta^p \in L^1(T)$ by

$$\alpha^p(s) = \alpha_i^p/n_p \qquad \text{for} \quad s \in I_i^p,\ 1 \leqslant i \leqslant m_p, \tag{5.16}$$

and

$$\beta^p(t)=\beta_j^p/m_p \qquad \text{for} \quad t\in J_j^p,\ 1\leqslant j\leqslant n_p. \tag{5.17}$$

Define $k^p\in L^\infty(S\times T)$ by

$$k^p(s,t)=a_{ij}^p \qquad \text{for} \quad (s,t)\in I_i^p\times J_j^p. \tag{5.18}$$

One can easily check that

$$\int_S \alpha^p(s)\,\mu(ds)=(m_pn_p)^{-1}\sum_{i=1}^{m_p}\alpha_i^p=(m_pn_p)^{-1}\sum_{j=1}^{n_p}\beta_j^p=\int_T \beta^p(t)\,\nu(dt).$$

If the DAD problem (A^p,α^p,β^p) has a solution, there are positive reals, f_i^p, $1\leqslant i\leqslant m_p$, and g_j^p, $1\leqslant j\leqslant n_p$, with

$$\sum_{j=1}^{n_p} f_i^pa_{ij}^pg_j^p=\alpha_i^p,\ 1\leqslant i\leqslant m_p, \quad \text{and} \quad \sum_{i=1}^{m_p} f_i^pa_{ij}^pg_j^p=\beta_j^p,\ 1\leqslant j\leqslant n_p. \tag{5.19}$$

If $f^p(s)=f_i^p$ on I_i^p and $g^p(t)=g_j^p$ on J_j^p, one can check that (f^p,g^p) is a solution of the DAD problem (k^p,α^p,β^p). Conversely, if the DAD problem (k^p,α^p,β^p) has a solution (f^p,g^p), one can see that f^p is constant on each interval I_i^p, $1\leqslant i\leqslant m_p$, and g^p is constant on each interval J_j^p. It follows that if $f_i^p=f^p(s)$ on I_i^p and $g_j^p=g^p(s)$ on J_j^p, we obtain a solution of (5.19) and the DAD problem (A^p,α^p,β^p). Thus the DAD problems (k^p,α^p,β^p) and (A^p,α^p,β^p) are equivalent.

Now suppose that there exist $\alpha\in L^1(S)$, α positive a.e., $\beta\in L^1(T)$, $\beta>0$ a.e., and $k\in L^\infty(S\times T)$, such that

$$\lim_{p\to\infty}\|k^p\alpha^p-k\alpha\|_E=0 \qquad \text{and} \qquad \lim_{p\to\infty}\|k^p\beta^p-k\beta\|_F=0,$$

where $\|\cdot\|_E$ and $\|\cdot\|_F$ are given by (2.5) and (2.6). If (k,α,β) satisfies the hypotheses of Theorem 5.9, the DAD problem (k,α,β) has a solution $(\alpha/x,\beta/y)$, where $x,y\in \mathring{C}_1$. Furthermore, the DAD problem (k^p,α^p,β^p) has a solution $(\alpha^p/x^p,\beta^p/y^p)$ for all p sufficiently large and

$$\lim_{p\to\infty}\|x^p-x\|_{L^\infty}=0=\lim_{p\to\infty}\|y^p-y\|_{L^\infty}.$$

Note that (k,α,β) will satisfy the conditions of Theorem 5.9 if, for example, $k\in C([0,1]\times[0,1])$, $k\geqslant 0$, $k(s,s)>0$ for $0\leqslant s\leqslant 1$, $\alpha=\beta\in L^1([0,1])$ and $\alpha>0$ a.e.

Conversely, suppose that $k \in C([0,1]\times[0,1])$ is nonnegative and $k(s,s)>0$ for $0 \leqslant p \leqslant 1$. Assume that $\alpha=\beta \in L^1([0,1])$ and $\alpha>0$ a.e. Define $m_p=n_p=p$, $E^p_{ij}=I^p_i \times J^p_j$ for $1 \leqslant i, j \leqslant p$, and

$$a^p_{ij}=p^2 \int_{E^p_{ij}} k(s,t)\,ds\,dt \qquad \text{and} \qquad \alpha^p_i = p \int_{I^p_i} \alpha(s)\,ds = \beta^p_i.$$

If $k^p=a^p_{ij}$ on E^p_{ij} and $\alpha^p=\alpha^p_i$ on I^p_i, one can prove (we omit the proof) that

$$\lim_{p\to\infty} \|k^p\alpha^p - k\alpha\|_E = 0 \qquad \text{and} \qquad \lim_{p\to\infty} \|k^p\alpha^p - k\alpha\|_F = 0.$$

It follows from Theorem 5.9 that the solution of the *DAD* problem $(k^p, \alpha^p, \alpha^p)$ approaches the solution of (k, α, α) as $p \to \infty$.

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