GEVREY CLASS REGULARITY FOR ANALYTIC DIFFERENTIAL-DELAY EQUATIONS

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Abstract This paper considers differential-delay equations of the form $x'(t) = p(t)\,x(t-1)$, where the coefficient function $p\colon \mathbb{R} \to \mathbb{C}$ is analytic and not bounded on any δ-neighborhood of the intervals $(-\infty, \gamma]$, $\gamma \in \mathbb{R}$. For these equations, we cannot apply the known results regarding the analyticity of the bounded solutions $x\colon\ (-\infty, \gamma] \to \mathbb{C}$. We prove Gevrey class regularity for such solutions.

Key words Delay equation, Analyticity, Gevrey class
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1. Introduction

The analyticity of globally defined bounded solutions of autonomous analytic delay equations was studied first in [6]. The result of [6] was generalized to the nonautonomous case in [4]. Paper [4] verifies that if $\gamma \in \mathbb{R}$, $x\colon\ (-\infty, \gamma] \to \mathbb{C}^n$ is a bounded, uniformly continuous solution of

$$x'(t) = f(t, x_t)$$

on $(-\infty, \gamma]$, and f is analytic and bounded on a δ-neighborhood of the set $\{(t, x_t) : t \in (-\infty, \gamma]\}$, then x is real analytic, i.e, there exists an open neighborhood V of $(-\infty, \gamma]$ and a complex analytic map $\hat{x} : V \to \mathbb{C}^n$ such that $\hat{x}|_{(-\infty,\gamma]} = x$. It is an interesting question whether the condition regarding the boundedness of f can be relaxed.

The result of [4] is not applicable to equations of the form

$$x'(t) = p(t)\,x(t-1) \tag{1}$$

if p is analytic but not bounded on any δ-neighborhood of $(-\infty, \gamma]$. Typical examples of such coefficient functions are $p(t) = e^{it^q}$ and $p(t) = \sin(t^q)$ with an integer $q \geq 2$.

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$|\eta'(t_0)| > 1$. Let $x_0 \in \mathbb{R}$ be given. The paper [3] gives a mild technical condition under which equation (2) with initial value $x(t_0) = x_0$ has no analytic solution in any open neighborhood of $t = t_0$. Using the results of [3], we easily show at the end of this paper that such solutions are not of Gevrey class q for any $q > 1$ either.

2. The proof of the Theorem

The proof of the theorem relies on two lemmas and estimates on the derivatives of the coefficient function p.

Recall that by the product rule,

$$(f_1(t) f_2(t))^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f_1^{(n-i)}(t) f_2^{(i)}(t)$$

for all $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, $f_1 \in C^n(\mathbb{R}, \mathbb{C})$, $f_2 \in C^n(\mathbb{R}, \mathbb{C})$ and $t \in \mathbb{R}$. Hence for any solution $x\colon \mathbb{R} \to \mathbb{C}$ of equation (1), $t \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \geq 1$,

$$x^{(n)}(t) = \sum_{i=0}^{n-1} \binom{n-1}{i} p^{(n-1-i)}(t)\, x^{(i)}(t-1). \tag{3}$$

We use this observation to express $x^{(n)}(t)$, $n \geq 1$, $t \in \mathbb{R}$, as a function of the values $x(t-k)$, $k \in \{1, \ldots, n\}$, and the derivatives of p at $t - l$, where $l \in \{0, \ldots, n-1\}$.

For all $n \geq 1$ and $1 \leq k \leq n$, let $\sum_{(n,k)}$ denote the sum taken over the elements of the set

$$S_{n,k} = \left\{(j_0, j_1, \ldots, j_k) \in \mathbb{N}^{k+1} : n = j_0 > j_1 > \ldots > j_k = 0\right\}. \tag{4}$$

Lemma 1. *Assume that $x\colon \mathbb{R} \to \mathbb{C}$ satisfies equation (1) on $\mathbb{R}$. Then for all $t \in \mathbb{R}$ and $n \geq 1$,*

$$x^{(n)}(t) = \sum_{k=1}^{n} q_{n,k}(t)\, x(t-k),$$

where

$$q_{n,k}(t) = n! \sum\nolimits_{(n,k)} \prod_{l=0}^{k-1} \frac{p^{(j_l - 1 - j_{l+1})}(t-l)}{j_l\,(j_l - 1 - j_{l+1})!} \tag{5}$$

for all $t \in \mathbb{R}$, $n \geq 1$ and $1 \leq k \leq n$.

Note that by Lemma 1,

$$q_{n,1}(t) = p^{(n-1)}(t) \qquad \text{and} \qquad q_{n,n}(t) = \prod_{l=0}^{n-1} p(t-l) \qquad \text{for } t \in \mathbb{R} \text{ and } n \geq 1.$$

Proof. It is clear that $x^{(n)}(t)$ exists for all $t \in \mathbb{R}$ and $n \geq 1$.

The proof goes by induction on n. By definition, $q_{11}(t) = p(t)$ for all real t, hence the assertion holds for all $t \in \mathbb{R}$ and $n = 1$. Let $n \geq 2$ and suppose the lemma holds for all $t \in \mathbb{R}$ and $i \in \mathbb{N}$ with $1 \leq i < n$. Then applying (3) and our induction

Let $x\colon \mathbb{R} \to \mathbb{C}$ be a solution of equation (1) *on $\mathbb{R}$ such that $|x(t)| \le M$ for all $t \le t_0$. Then*

$$\left|x^{(n)}(t)\right| \le M (2C)^n (|t| + n)^{(q-1)n} n! \qquad \textit{for all } t \le t_0 \textit{ and } n \in \mathbb{N}.$$

Proof. By assumption we have $|x(t)| \le M (2C)^0 (\max(|t|, 1))^{(q-1)0} 0!$ for all $t \le t_0$.

Fix $n \ge 1$ and $t \le t_0$. According to Lemma 1,

$$x^{(n)}(t) = \sum_{k=1}^{n} q_{n,k}(t)\, x(t-k),$$

where the coefficient functions $q_{n,k}$, $k \in \{1, \dots, n\}$, are defined by (4) and (5). The estimate (8) implies that

$$|q_{n,k}(t)| \le n! \sum\nolimits_{(n,k)} \prod_{l=0}^{k-1} \frac{C^{j_l - j_{l+1}} (\max(|t-l|, 1))^{(q-1)(j_l - 1 - j_{l+1})}}{j_l}$$

for all $1 \le k \le n$. Notice that

$$\max(|t-l|, 1) \le |t| + k \qquad \text{for any } k \ge 1 \text{ and } 0 \le l \le k-1.$$

Observe that

$$|S_{n,k}| = \binom{n-1}{k-1} \qquad \text{for all } 1 \le k \le n,$$

moreover,

$$\sum_{l=0}^{k-1} j_l - 1 - j_{l+1} = j_0 - j_k - k = n - k \qquad \text{and} \qquad \prod_{l=0}^{k-1} j_l \ge n (k-1)!$$

hold for all $1 \le k \le n$ and $(j_0, j_1, \dots, j_k) \in S_{n,k}$.

Hence

$$\begin{aligned}
|q_{n,k}(t)| &\le (n-1)! \sum\nolimits_{(n,k)} \frac{1}{(k-1)!} C^n (|t| + k)^{(q-1)(n-k)} \\
&= (n-1)!\, |S_{n,k}| \frac{1}{(k-1)!} C^n (|t| + k)^{(q-1)(n-k)} \\
&= (n-1)! \binom{n-1}{k-1} \frac{1}{(k-1)!} C^n (|t| + k)^{(q-1)(n-k)}
\end{aligned}$$

for all $1 \le k \le n$, and

(9)

$$\left|x^{(n)}(t)\right| \le \sum_{k=1}^{n} |q_{n,k}(t)|\, |x(t-k)| \le M C^n (n-1)! \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{(|t| + k)^{(q-1)(n-k)}}{(k-1)!}.$$

If we note that

$$\binom{n-1}{k-1} \frac{1}{(k-1)!} \le \binom{n-1}{k-1} \le 2^{n-1}$$

For each $u \in \mathbb{C}$,

$$u^q - t^q = \prod_{k=0}^{q-1} (u - \eta_k t), \qquad \text{where } \eta_k = e^{\frac{2\pi i k}{q}},\ k \in \{0, 1, \ldots, q-1\} \text{ and } i = \sqrt{-1}.$$

It follows that

$$e^{i(u^q - t^q)} = \sum_{j=0}^{\infty} \frac{(i(u^q - t^q))^j}{j!} = \sum_{j=0}^{\infty} \frac{i^j}{j!} \prod_{k=0}^{q-1} (u - \eta_k t)^j. \tag{12}$$

For each $j \geq 0$, define a set $R_{n,q,j}$ of q-tuples as

$$R_{n,q,j} = \left\{ (l_0, l_1, \ldots, l_{q-1}) \in \mathbb{N}^q : \sum_{k=0}^{q-1} l_k = n,\ l_0 = j,\ 0 \leq l_k \leq j \text{ for } 1 \leq k \leq q-1 \right\}.$$

Let $\sum^{(n,q,j)}$ denote the sum taken over the elements of $R_{n,q,j}$. Let D_t^n denote the n-fold differentiation with respect to t.

Note that $\eta_0 = 1$ and $\eta_k \neq 1$ if $1 \leq k \leq q-1$. This observation and the product rule for higher order derivatives together give that

$$\begin{aligned} D_u^n \prod_{k=0}^{q-1} (u - \eta_k t)^j \,|_{u=t} &= \sum\nolimits^{(n,q,j)} \frac{n!}{l_0! l_1! \ldots l_{q-1}!} \prod_{k=0}^{q-1} \frac{j!}{(j - l_k)!} (t - \eta_k t)^{j - l_k} \\ &= n! t^{qj-n} \sum\nolimits^{(n,q,j)} \prod_{k=0}^{q-1} \binom{j}{l_k} (1 - \eta_k)^{j - l_k}. \end{aligned}$$

As $l_k \leq j$ for all $0 \leq k \leq q-1$, we see that $n \leq qj$. The above sum is nonempty if and only if

$$\frac{n}{q} \leq j \leq n.$$

Substituting into equation (12), we deduce that

$$\begin{aligned} D_u^n e^{iu^q}|_{u=t} &= e^{it^q} D_u^n e^{i(u^q - t^q)}|_{u=t} \\ &= e^{it^q} \sum_{\frac{n}{q} \leq j \leq n} \frac{i^j}{j!} n! t^{qj-n} \sum\nolimits^{(n,q,j)} \prod_{k=0}^{q-1} \binom{j}{l_k} (1 - \eta_k)^{j - l_k}. \end{aligned}$$

Actually we eventually shall need a formula for $D_t^n e^{i\alpha t^q}$, where $\alpha \in \mathbb{R}$ is a constant. However, such a formula follows easily from the above formula for $D_t^n e^{it^q}$. Select $\beta \in \mathbb{C}$ such that $\beta^q = \alpha$ and write $u = \beta t$. Then

$$D_t^n e^{i(\beta t)^q} = \beta^n D_u^n e^{iu^q}|_{u=\beta t}.$$

Our estimates imply that

$$\left|D_t^n e^{i\alpha t^q}\right| \le \sum_{\frac{n}{q}\le j\le n} |\alpha|^j \frac{n!}{j!\,(n-j)!}\,(n-j)!\,(\max(|t|,1))^{(q-1)n}\,2^{2(q-1)j+q-2}$$

$$\le 2^{q-2}\,(\max(|t|,1))^{(q-1)n}\,(n-j_*)! \sum_{\frac{n}{q}\le j\le n} \binom{n}{j} \left(|\alpha|\,2^{2(q-1)}\right)^j,$$

where j_* denotes the smallest positive integer j such that $n/q \le j \le n$. By the binomial theorem,

$$\sum_{\frac{n}{q}\le j\le n} \binom{n}{j} \left(|\alpha|\,2^{2(q-1)}\right)^j \le \sum_{0\le j\le n} \binom{n}{j} \left(|\alpha|\,2^{2(q-1)}\right)^j = \left(1+|\alpha|\,2^{2(q-1)}\right)^n,$$

so

$$\left|D_t^n e^{i\alpha t^q}\right| \le 2^{q-2}\,(\max(|t|,1))^{(q-1)n}\left(1+|\alpha|\,4^{(q-1)}\right)^n (n-j_*)!. \tag{13}$$

We conclude that there exists a constant $C \ge 1$ such that for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\left|D_t^n e^{i\alpha t^q}\right| \le C^{n+1}\,(\max(|t|,1))^{(q-1)n}\,(n-j_*)! \le C^{n+1}\,(\max(|t|,1))^{(q-1)n}\,n!.$$

As a consequence we can verify the Theorem.

Proof of Theorem. Let $p(t) = \sum_{m\in F} A_m e^{im\omega t^q}$ **for all** $t \in \mathbb{R}$, where F is a finite set of integers, $A_m \in \mathbb{C}$ for $m \in F$, $\omega > 0$, $A_0 = 0$ and $q \ge 2$ is an integer. Our calculations above show that p satisfies inequality (8) in Lemma 2. The boundedness of x on intervals of the form $(-\infty, t_0]$ is clear because $\lim_{t\to-\infty} x(t)$ exists and is finite. If one applies Lemma 2 and uses Stirling's formula, the theorem follows. □

Remark. In fact, with the aid of the advanced calculus form of Stirling's formula, one can replace $(n-j_*)!$ in (13) with $(n!)^{1-1/q}$.

It is obvious that $(n-j_*)! = 1$ for $n = 1, 2$ because $q \ge 2$. Thus we can assume that $n \ge 3$.

Note that $n/q \le j_* < n/q + 1$, so if we choose $q_* > 0$ such that $j_* = n/q_*$, then

$$\frac{1}{q} \le \frac{1}{q_*} < \frac{1}{q} + \frac{1}{n} \qquad \text{and} \qquad n - \frac{n}{q} \ge n - \frac{n}{q_*} > n - \frac{n}{q} - 1.$$

Also, since $n \ge 3$ and $q \ge 2$, it is true that $n - n/q > 1$.

By Stirling's formula,

$$(n-j_*)! = \frac{\left(n\left(1-\frac{1}{q_*}\right)\right)^{n\left(1-\frac{1}{q_*}\right)}}{e^{n\left(1-\frac{1}{q_*}\right)}} \sqrt{2\pi n\left(1-\frac{1}{q_*}\right)} \exp\left(\frac{\lambda\left(n\left(1-\frac{1}{q_*}\right)\right)}{12n\left(1-\frac{1}{q_*}\right)}\right).$$

We know from Theorem 4.2 in [3] that the finite limit $\lim_{n\to\infty} w_n = w_\infty$ exists, and if $w_\infty \neq 0$, then equation (14) with initial value $y(0) = y_0$ has no analytic solution in any neighborhood of $\tilde{t} = 0$. It is clear that the corresponding solutions of (2) are also nonanalytic.

We claim that if $w_\infty \neq 0$, then the solutions y of (14) with $y(0) = y_0$ do not belong to any Gevrey class in any neighborhood of $\tilde{t} = 0$ either. We can prove this by contradiction. Suppose that the solution y is of Gevrey class q with some $q > 1$ in a neighborhood of $\tilde{t} = 0$. Then there exists a constant $C > 0$ such that

$$\left|y^{(n)}(0)\right| \leq C^{n+1}(n!)^q \qquad \text{for all } n \in \mathbb{N}.$$

As $w_\infty \neq 0$, there exists a constant $\underline{w} > 0$ such that $|w_n| > \underline{w}$ for all large n. Then for such n, the definition of w_n and Stirling's formula together yield that

$$\left|\lambda^{\frac{n(n-1)}{2}}\beta_0^n\right|\underline{w} < \left|y^{(n)}(0)\right| \leq C^{n+1}(n!)^q < C^{n+1}\left(\frac{n}{e}\right)^{qn}(2\pi n)^{\frac{q}{2}}e^{\frac{q}{12n}}.$$

Taking the n^{th} root, we obtain that

$$\frac{\lambda^{\frac{n-1}{2}}}{n^q} \leq \frac{C^{1+\frac{1}{n}}(2\pi n)^{\frac{q}{2n}}e^{\frac{q}{12n^2}}}{e^q|\beta_0|\underline{w}^{\frac{1}{n}}} \qquad \text{for all large } n. \tag{16}$$

Applying L'Hôpital's rule repeatedly, we see that

$$\lim_{n\to\infty}\frac{\lambda^{\frac{n-1}{2}}}{n^q} = \infty.$$

This is a contradiction as the right hand side of (16) is bounded.

According to [7], the composite of Gevrey functions is of Gevrey class again. This implies that if x is a solution of (2) corresponding to a solution y of (14) with $w_\infty \neq 0$, then x cannot be of Gevrey class q in any neighborhood of $t = t_0$ for any $q > 1$ either.

References

[1] **Gevrey, Maurice, Sur la nature analytique des solutions des équations aux dérivées partielles. Premier mémoire, Annales Scientifiques de l'École Normale Supérieure 3 (1918), no. 35, 129–190.**

[2] **Krisztin, Tibor, Analyticity of solutions of differential equations with a threshold delay. Recent Advances in Delay Differential Equations (F. Hartung and M. Pituk eds.) Springer Proceedings in Mathematics & Statistics 94 (2014), 173–180.**

[3] Mallet-Paret, John, Nussbaum, Roger D., Analyticity and nonanalyticity of solutions of delay-differential equations. SIAM J. Math. Anal. 46 (2014), no. 4, 2468–2500.

[4] Mallet-Paret, John, Nussbaum, Roger D., Analytic solutions of delay-differential equations, in preparation.

[5] Mallet-Paret, John, Nussbaum, Roger D., Asymptotic homogenization of differential-delay equations and a question of analyticity, in preparation.