GLOBAL STABILITY, TWO CONJECTURES AND MAPLE

Roger D. Nussbaum*

Summary

Consider the second order difference equation $u_{-1} > 0, u_0 > 0$ and $u_{n+1} = f(u_{n-1}, u_n)$ for $n \geq 0$, where either (a) $f(u,v) = \frac{u+pv}{u+qv}$ or (b) $f(u,v) = \frac{p+qv}{1+u}$. If $0 \leq q < p$ in case (a) or $p > 0$ and $q > 0$ in case (b), it has been conjectured (see [8]) that $\lim_{n\to\infty} u_n$ exists and equals L, where $L > 0$ and $L = f(L, L)$.

We prove this conjecture in case (a) and significantly extend the range of p and q for which it is known in case (b). In cases (a) and (b), these questions are equivalent to global stability of the fixed point (L, L) of the planar map $\Phi(u,v) = (v, f(u,v))$. For Φ as in case (a), we consider natural four dimensional extensions T of Φ^3 and S of Φ^2. For $0 \leq q < p$, we prove that (L, L, L, L) is a global stable fixed point of T, but we also describe precisely a range of parameters $0 \leq q < p$ for which S has at least three distinct fixed points in the positive orthant. We describe (Section 3) some general principles underlying our arguments. Symbolic calculations using Maple play a crucial role in our arguments in Section 4.

1. Introduction.

Recently, M. Kulenović [9] has informed the author of two interesting conjectures.

Conjecture 1.1. *Assume that* $0 \leq q < p$, *that* $u_{-1} > 0$ *and* $u_0 > 0$ *and that* $u_{n+1} = \frac{u_{n-1}+pu_n}{u_{n-1}+qu_n}$ *for* $n \geq 0$. *Then* $\lim_{n\to\infty} u_n = L := \left(\frac{1+p}{1+q}\right)$.

Conjecture 1.2. *Assume that* $0 < q, 0 < p, u_{-1} > 0$ *and* $u_0 > 0$ *and that* $u_{n+1} = \frac{p+qu_n}{1+u_{n-1}}$ *for* $n \geq 0$. *Then we have* $\lim_{n\to\infty} u_n = L$, *where* $L > 0$ *is the unique positive solution of* $L = \frac{p+qL}{1+L}$.

Conjecture 1.1 is Conjecture 6.10.5 on p.125 in [8] and Conjecture 1.2 is Conjecture 6.10.1 on p.124 in [8]. Despite their simple appearance, both conjectures have been open

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for several years. The conjectures arise as part of a program to understand the dynamics of nonlinear, second order difference equations of the form

$$x_{n+1} = g(x_{n-1}, x_n) := \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}. \tag{1.1}$$

Equivalently, one is interested in understanding the dynamics of iterates of a planar map G defined by

$$G(u,v) = (v, g(u,v)). \tag{1.2}$$

Simple changes of variable reduce eq. (1.1) to certain "normal forms." For example, if $\alpha = A = 0$, and (1) $BC \geq 0, \beta\gamma > 0, \gamma C > 0$ and $\beta C - B\gamma > 0$ or (2) $BC \geq 0, \beta\gamma > 0, \gamma C < 0$ and $\beta C - B\gamma < 0$, then the change of variable $x_n = \left(\frac{\gamma}{C}\right) y_n$ yields

$$y_{n+1} = \frac{y_{n-1} + py_n}{y_{n-1} + qy_n}, \tag{1.3}$$

where $0 \leq q < p$. In the case of eq. (1.3), the equivalent planar map Φ is given by

$$\Phi(u,v) = \left(v, \frac{u+pv}{u+qv}\right). \tag{1.4}$$

In Section 2 of this paper we shall prove Conjecture 1.1. In Section 6 we discuss Conjecture 1.2. We do not prove the full conjecture, but we extend significantly the range of parameters p and q for which it is known that $\lim_{n\to\infty} u_n = L$: see Theorem 6.1.

In Section 3 we discuss some general principles which underlie all the arguments in this paper. In particular it is useful to discuss maps which preserve a partial ordering induced by a (non-standard) cone in $\mathbb{R}^n$.

As discussed in Section 4, the map Φ in eq.(1.4) can be considered as mapping the set $W = \{(u,u,v,v) | u > 0, v > 0\} \subset \mathbb{R}^4$ into itself. With this identification we consider in Section 4 a map T (see eq. (4.10)) which takes int (K^4), the interior of the positive orthant in $\mathbb{R}^4$, into int (K^4) and is a natural extension of Φ^3. With L as in Conjecture 1.1, we prove that T has the point $(L,L,L,L) = \Lambda$ as a globally stable fixed point, i.e., $T^k(x) \to \Lambda$ for all $x \in$ int (K^4). This result generalizes Conjecture 1.1, but it is much more subtle. A crucial part of the argument involves using Maple to symbolically compute two polynomials in three variables with integral coefficients and to show that all coefficients are nonnegative and some are positive. Since the polynomials have several thousand terms and the coefficients are, in general, large, we know of no way of doing such a computation by hand. It would be interesting to find an argument which avoided the use of Maple.

In Section 5 we consider a map $S : \text{int}\,(K^4) \to \text{int}\,(K^4)$ which is a natural extension of Φ^2. The maps S and T both depend on parameters p and q as in Conjecture 1.1. In view of the positive results of Section 4, one might expect that for all $0 \leq q < p, \Lambda = (L, L, L, L)$ is a globally stable fixed point of S. However, in Theorem 5.1 we describe a wide range of p and q for which S has at least three distinct fixed points in int (K^4).

The essential idea of this paper is that some general theorems can, in combination with the symbolic computational power of Maple, yield results which are otherwise inaccessible. In a future paper we hope to show that this simple approach yields insights, for example, about other conjectures in [8].

2 Global Stability for $u_{n+1} = \frac{u_{n-1}+pu_n}{u_{n-1}+qu_n}$.

In this section we shall always assume that $0 \leq q < p$. We shall write, for $u > 0, v \geq 0$

$$f(u,v) := \frac{u+pv}{u+qv} \tag{2.1}$$

and

$$h(u,v) := f(v, f(u,v)) = \frac{(uv+qv^2+pu+p^2v)}{(uv+qv^2+qu+pqv)}. \tag{2.2}$$

In general, $(D_1g)(u,v)$ (respectively, $(D_2g)(u,v)$) will denote the partial derivative of a function g with respect to u (respectively, with respect to v). If we write $N(u,v) = uv + qv^2 + qu + pqv$, a calculation gives

$$(D_1h)(u,v) = -(p-q)^2v^2N(u,v)^{-2} < 0 \text{ and} \tag{2.3}$$

$$(D_2h)(u,v) = -(p-q)[(u+qv)^2 + q(p-q)v^2]N(u,v)^{-2} < 0. \tag{2.4}$$

Given $u_{-1} > 0$ and $u_0 > 0$ and f as in (2.1), we define

$$u_{n+1} = f(u_{n-1}, u_n), n \geq 0. \tag{2.5}$$

Kulenović and Ladas make the following conjecture.

Conjecture 2.1. *(See [8], Conjecture 6.10.5, p.125) If* $u_{-1} > 0, u_0 > 0$ *and* $p > q > 0$, *then* $\lim_{n\to\infty} u_n = L := (1+p)/(1+q)$.

If we define $\Phi(u,v) = (v, f(u,v))$, Conjecture 2.1 is equivalent to saying that $\lim_{n\to\infty} \Phi^k(u,v) = (L,L)$ whenever $u > 0$ and $v > 0$.

It is known (see Theorem 6.9.7, p.123 in [8]) that Conjecture 2.1 is true if $0 < q < p$ and $p \leq pq + 1 + 3q$.

In this section we shall prove the following theorem, which yields Conjecture 2.1:

Theorem 2.1. *Assume that $0 \le q < p$, that $u_{-1} > 0$ and $u_0 > 0$ and that u_n is defined by eq. (2.5) for $n \ge 0$. Then we have*

$$\lim_{n\to\infty} u_n = L := \frac{(1+p)}{(1+q)}. \tag{2.6}$$

We begin with a simple lemma. Note that $q = 0$ is allowed. If $q > 0$, one easily obtains that $1 \le u_n \le p/q$ for all $n \ge 1$.

Lemma 2.1. *Assume that $0 \le q < 1, u_{-1} > 0$ and $u_0 > 0$ and u_n is given by eq.(2.5). Then $u_n \ge 1$ for all $n \ge 1$ and, for $n \ge 4$,*

$$u_n \le \left(\frac{q+1+p+p^2}{q+1+q+qp}\right) = h(1,1). \tag{2.7}$$

If we define $L := \frac{(1+p)}{(1+q)} > 1, u_n \le L^2 - L + 1$ for all $n \ge 4$.

Proof. Obviously $f(u,v) > 1$ for $u > 0$ and $v > 0$, so $u_n \ge 1$ for $n \ge 1$. If $n \ge 4$, equations (2.3) and (2.4) imply that

$$u_n = h(u_{n-3}, u_{n-2}) \le h(1,1) \le \left(\frac{q+1+p+p^2}{q+1+q+qp}\right).$$

If we express the right hand side of (2.7) in terms of q and L, we obtain

$$u_n \le g(q,L) := \frac{L^2(q+1) - L + 1}{1 + Lq}$$

for $n \ge 4$. A calculation shows $(D_1 g)(q,L) < 0$ for $q \ge 0, L > 1$, so $g(q,L) \le g(0,L) = L^2 - L + 1$. □

Lemma 2.2. *Define $a_0 = 1$ and $b_0 = h(1,1)$ and for $k \ge 0$, define $a_{k+1} = h(b_k, b_k)$ and $b_{k+1} = h(a_k, a_k)$. If $u_0 > 0, u_{-1} > 0$ and u_n is defined by eq. (2.5), we have*

$$a_k \le u_n \le b_k \text{ for all } n \ge 4 + 3k. \tag{2.8}$$

Furthermore, we have $a_k \le a_{k+1} \le L := (1+p)/(1+q)$ and $L \le b_{k+1} \le b_k$ for all $k \ge 0$.

Proof. Lemma 2.1 implies that $a_0 \le u_n \le b_0$ for all $n \ge 4$. Assume, for some $k \ge 0$, that eq (2.8) holds. Then for $n \ge 4 + 3(k+3)$ we have

$$u_n = h(u_{n-3}, u_{n-2}).$$

Since $n-3 \geq 4+3k, a_k \leq u_{n-3} \leq b_k$ and $a_k \leq u_{n-2} \leq b_k$, so, using equations (2.3) and (2.4), we have

$$h(b_k, b_k) := a_{k+1} \leq u_n \leq b_{k+1} = h(a_k, a_k).$$

By induction, equation (2.8) holds for all $k \geq 0$.

We have $a_1 = h(b_0, b_0) > 1 = a_0$ and $b_1 = h(1,1) \leq b_0$. Also, we see that $1 < L = h(L,L) < h(1,1) = b_0$ and $a_1 = h(b_0, b_0) < h(L,L) = L$. Assume that for some $n \geq 0, a_n \leq a_{n+1} \leq L$ and $L \leq b_{n+1} \leq b_n$. Using equations (2.3) and (2.4) we see that

$$a_{n+1} = h(b_n, b_n) \leq h(b_{n+1}, b_{n+1}) = a_{n+2} \leq h(L,L) = L,$$

and an analogous argument shows that $L \leq b_{n+2} \leq b_{n+1}$. The lemma now follows by mathematical induction. □

Proof of Theorem 2.1. Let a_k and b_k be as defined in Lemma 2.2. Lemma 2.2 implies that $a_k \to a \leq L$ and $b_k \to b \geq L$. Because $a_{k+1} = h(b_k, b_k)$ and $b_{k+1} = h(a_k, a_k)$, the continuity of h implies that

$$a = h(b,b) \text{ and } b = h(a,a). \tag{2.9}$$

Thus it suffices to prove that if $x \geq L := \left(\frac{1+p}{1+q}\right)$ and

$$x = h(h(x,x), h(x,x)) \tag{2.10}$$

then $x = L$. However, writing $u = h(x,x) = f(x, f(x,x)) = f(x,L)$, we find that

$$h(h(x,x), h(x,x)) = f(u, f(u,u)) = f(u,L) = \frac{f(x,L)+pL}{f(x,L)+qL}. \tag{2.11}$$

so equation (2.10) has at most two distinct solutions. However, any solution x of $h(x,x) = x$ also solves equation (2.10), and $x = -p$ and $x = L$ solve $h(x,x) = x$. Thus equation (2.10) has no solution $x > L$. □

In the next section we shall present a useful abstract framework which generalizes the argument used to prove Theorem 2.1.

3. Some general remarks about global stability of fixed points.

By a closed cone C (with vertex at 0) in a Banach space X we mean, as usual, a closed convex set $C \subset X$ such that (1) $tC \subset C$ for all $t \geq 0$ and (2) $C \cap (-C) = \{0\}$. A closed cone C induces a partial ordering $\leq_C$ on X by $x \leq_C y$ if and only if $y - x \in C$. A closed cone C is called "normal" if there exists a constant M such that whenever $0 \leq_C x \leq_C y, \|x\| \leq M\|y\|$. It is well known that any closed cone C in a finite dimensional

Banach space X is normal. If $X = \mathbb{R}^n$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \mathbb{R}^n$ satisfies $|\varepsilon_i| = 1$ for $1 \leq i \leq n$, one can define a closed cone $C = C_\varepsilon$ by

$$C := C_\varepsilon := \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | \varepsilon_i x_i \geq 0 \text{ for } 1 \leq i \leq n\}. \tag{3.1}$$

If $\varepsilon_i = 1$ for $1 \leq i \leq n$, we shall write K^n instead of C_ε, so

$$K^n := \{x \in \mathbb{R}^n | x_i \geq 0 \text{ for } 1 \leq i \leq n\}. \tag{3.2}$$

As usual, if G is a subset of a Banach space X and C is a closed cone in X, a map $T : G \to X$ will be called order-preserving (with respect to the partial ordering $\leq_C$ from C) if whenever $x, y \in G$ and $x \leq_C y$ it follows that $T(x) \leq_C T(y)$. In the special case of maps $T : G \subset \mathbb{R} \to \mathbb{R}$ we shall say that T is increasing (respectively, strictly increasing) if $T(x) \leq T(y)$ (respectively, $T(x) < T(y)$) whenever $x, y \in G$ and $x \leq y$ (respectively $x < y$). Of course T is decreasing (respectively, strictly decreasing) if $-T$ is increasing (respectively, strictly increasing).

The following theorem provides a useful abstract framework for studying global stability of fixed points.

Theorem 3.1. *Let C be a closed, normal cone in a Banach space X and let $T : G \subset X \to G$ be a continuous map. Make the following assumptions on T:*

(1) T is order-preserving with respect to the partial ordering from C.

(2) For every $x \in G$, the closure of $\{T^j(x) | j \geq 0\}$ is a compact subset of G.

(3) T has a unique fixed point x_* in G.

(4) For every $x \in G$ there exist y and z in G with $y \leq x \leq z, T(y) \geq y$ and $T(z) \leq z$. (Here, we write $\leq$ for $\leq_C$).

Then it follows that $\lim_{k\to\infty} T^k(x) = x_*$ for every $x \in G$.

Proof. Given $x \in G$, select $y \in G$ with $y \leq x$ and $Ty \geq y$. By property (1), we have $T^k(y) \leq T^{k+1}(y)$ and $T^k(y) \leq T^k(x)$ for all $k \geq 0$, and property (2) implies that M, the closure of $\{T^j(y) : j \geq 0\}$, is a compact subset of G. By compactness of M, there exists a subsequence $k_i \to \infty$ with $T^{k_i}(y) \to \eta \in G$. Since $T^{k_i}(y) \leq T^{k_j}(y)$ for $j \geq i, \eta \geq T^{k_i}(y)$ for all $i \geq 1$. Thus, if $j \geq k_i$, we have that

$$T^{k_i}(y) \leq T^j(y) \leq \eta.$$

Since $T^{k_i}(y) \to \eta$ and C is normal, it follows that $T^j(y) \to \eta$ as $j \to \infty$, and by continuity of $T, T(\eta) = \eta$. The same proof shows that if $x \leq z \in G$ and $Tz \leq z$, then $T^j(z) \geq T^j(x)$ for all $j \geq 1$ and $T^j(z) \to \zeta$ and $T(\zeta) = \zeta$. Thus we have proved that T has a fixed point in G, call it x_*, and by property (3), x_* is unique, so $\zeta = x_* = \eta$. Because $T^j(y) \leq T^j(x) \leq T^j(z)$ and $T^j(z) - T^j(y) \to 0$, the normality of C implies that $T^j(x) \to \zeta$. $\square$

Note that our proof shows that assumptions (1), (2) and (4) in Theorem 3.1 actually imply that T has a fixed point in G, so the point of assumption (3) in Theorem 3.1 is the uniqueness of the fixed point.

In certain applications, assumption (4) in Theorem 3.1 is too restrictive: G may always contain an element z as in assumption (4) of Theorem 3.1 but it may fail to contain y as in assumption (4), or vice-versa.

Theorem 3.2. *Let hypotheses and notation be as in Theorem 3.1, but replace assumption (4) by the following two asumptions:*

(4)′ for every $x \in G$, there exists $z \in G$ with $x \leq z$ and $T(z) \leq z$.

(5)′ If $x \in G$ and $x \leq x_*$, where x_* is the unique fixed point of T in G, then $x = x_*$.

Then it follows that $T^k(x) \to x_*$ for all $x \in G$.

Proof. Given $x \in G$, select $z \in G$ with $x \leq z$ and $T(z) \leq z$. The same argument as in Theorem 3.1 shows that $T^j(z) \to x_*$ as $j \to \infty$ and $T^j(x) \leq T^j(z)$ for all $j \geq 1$. We claim that $T^j(x) \to x_*$. If not, by using assumption (2) in Theorem 3.1, there exists a sequence $j_k \uparrow \infty$ such that $T^{j_k}(x) \to \xi \in G$ and $\xi \neq x_*$. However, $T^{j_k}(x) \leq T^{j_k}(z)$ and $T^{j_k}(z) \to x_*$, so $\xi \leq x_*$. Assumption (5)′ then implies that $\xi = x_*$, a contradiction. □

If H is a subset of a topological space Y, we shall use the notation $\text{int}(H)$ to denote the interior of H in Y. In our applications here, G will typically be a subset of $\text{int}(K^n) \subset \mathbb{R}^n = X$ and $C = C_\varepsilon$ will be as in equation (3.1). The following proposition gives an example of how the framework in Theorem 3.2 may arise.

Corollary 3.1. *Let Φ: int $(K^m) \to$ int (K^m) be a continuous map, with $\Phi(x) = (\varphi_1(x), \varphi_2(x), \cdots, \varphi_m(x))$ for $x \in$ int(K^m). Define $n = 2m$ and define $\Gamma \subset$ int (K^n) by*

$$\Gamma = \{z = (z_1, z_2, \cdots, z_n) \in \text{ int } (K^n) : z_{2i-1} \leq z_{2i} \text{ for } 1 \leq i \leq m\}.$$

For $z \in \Gamma$, define $B(z) \subset$ int (K^m) by

$$B(z) = \{x \in \text{ int } (K^m) : z_{2i-1} \leq x_i \leq z_{2i} \text{ for } 1 \leq i \leq m\}.$$

Define a closed cone C in $\mathbb{R}^n$ by

$$C = \{w = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n \,|(-1)^i w_i \geq 0 \text{ for } 1 \leq i \leq n\}.$$

and define $T(z) = (t_1(z), t_2(z), \cdots, t_n(z))$ for $z \in \Gamma$ by

$$t_{2i-1}(z) = \min\{\varphi_i(x) | x \in B(z)\}$$

and

$$t_{2i}(z) = \max\{\varphi_i(x) | x \in B(z)\}.$$

Then $T : \Gamma \to \Gamma$ is continuous and order-preserving with respect to the partial ordering $\leq_C$ induced by C. If, in addition, there exists $\zeta \in \Gamma$ such that $\Phi(B(\zeta)) \subset B(\zeta)$, then Φ has a fixed point $y_* = (L_1, L_2, \cdots, L_m) \in B(\zeta)$. Furthermore, if we define G by

$$G = \{z \in \Gamma | z \leq_C \zeta\},$$

then $T(G) \subset G, T(\zeta) \leq_C \zeta$ and G is a compact subset of $\mathbb{R}^n$. If we define $x_* = (L_1, L_1, L_2, L_2, \cdots, L_m, L_m) \in G$, then $T(x_*) = x_*$; and if $z \in G$ and $z \leq_C x_*$, then $z = x_*$. If T has only one fixed point x_* in G, then $\lim_{k\to\infty} T^k(z) = x_*$ for all $z \in G$ and $\lim_{k\to\infty} \Phi^k(x) = y_*$ for all $x \in B(\zeta)$.

Proof. We shall write $\leq$ instead of $\leq_C$. Notice that for $z, w \in \mathbb{R}^{2m}, z \leq w$ if and only $w_{2i-1} \leq z_{2i-1}$ for $1 \leq i \leq m$ and $z_{2i} \leq w_{2i}$ for $1 \leq i \leq m$. It follows easily that if $z, w \in \Gamma$ and $z \leq w$, then $B(z) \subset B(w)$; and this in turn implies that $t_{2i-1}(z) \geq t_{2i-1}(w)$ and $t_{2i}(z) \leq t_{2i}(w)$ for $1 \leq i \leq m$, so $T(z) \leq T(w)$ and T is order-preserving. Since $t_{2i-1}(z) \leq t_{2i}(z)$ for $1 \leq i \leq m$ and $z \in \Gamma$, we certainly have that $T(z) \in \Gamma$ when $z \in \Gamma$; and the continuity of T follows easily from the continuity of Φ.

For all $z \in \Gamma, B(z)$ is a compact, convex set, so if there exists $\zeta \in \Gamma$ with $\Phi(B(\zeta)) \subset B(\zeta)$, the Brouwer fixed point theorem implies that Φ has a fixed point $y_* = (L_1, L_2, \cdots, L_m) \in B(\zeta)$. Because we assume that $\Phi(B(\zeta)) \subset B(\zeta)$, we see that $t_{2i}(\zeta) \leq \zeta_{2i}$ and $t_{2i-1}(\zeta) \geq \zeta_{2i-1}$ for $1 \leq i \leq m$, which implies that $T(\zeta) \leq \zeta$. If $z \in G$, it follows that $T(z) \leq T(\zeta) \leq \zeta$, and this implies that $T(G) \subset G$. The reader can verify, that

$$G = \{z \in \mathbb{R}^{2m} | \zeta_{2i-1} \leq z_{2i-1} \leq z_{2i} \leq \zeta_{2i} \text{ for } 1 \leq i \leq m\},$$

so G is a compact, convex subset of $\mathbb{R}^n$.

For $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$ define $S(x) = y \in \mathbb{R}^{2m}$ by

$$y = (x_1, x_1, x_2, x_2, \cdots, x_m, x_m)$$

and let $V = \{y \in \mathbb{R}^{2m} | y_{2i-1} = y_{2i} \text{ for } 1 \leq i \leq m\}$. Our definition of T immediately gives that $T(V \cap \text{ int } (K^n)) \subset V \cap \text{ int } (K^n)$ and

$$(S^{-1}TS)(x) = \Phi(x) \tag{3.3}$$

for all $x \in \text{int } (K^m)$, so Φ is conjugate to $T | V \cap \text{ int } (K^n)$. Equation (3.3) implies as a special case that x_* is a fixed point of T in G. If $z \in G$ and $z \leq x_*$, then $z_{2i-1} \geq L_i$ and $z_{2i} \leq L_i$ for $1 \leq i \leq m$; and since we must have that $z_{2i} \geq z_{2i-1}$, we conclude that $z_{2i-1} = L_i = z_{2i}$ for $1 \leq i \leq m$. If we now assume that T has a unique fixed point in G, all hypotheses of Theorem 3.2 are satisfied and $\lim_{k\to\infty} T^k(z) = x_*$ for all $z \in G$. The fact that $\lim_{k\to\infty} \Phi^k(x) = y_*$ for all $x \in B(\zeta)$ follows from equation (3.3). $\square$

Remark 3.1. If Φ: int $(K^2) \to$ int (K^2) is as in Section 2, we shall apply the framework of Corollary 3.1 to $\hat{\Phi} := \Phi^j$ rather then directly to Φ. In Section 4 we shall choose $j = 3$ and obtain a map T which extends naturally as an order-preserving map of int (K^4) to itself. In Section 5 we shall choose $j = 2$ and obtain a map S. We shall see that the global stability properties of the fixed points of the maps T and S are strikingly different.

Typically, a major difficulty in using Theorem 3.1 is verifying that T has exactly one fixed point in G. Although we shall not use it here, we mention a simple but useful criterion which utilizes topological degree and has been helpful in related problems. See [1-4, 10, 11, 12, 13] for discussions of topological degree. For simplicity we restrict attention to the finite dimensional case.

Proposition 3.1. *Let G be a bounded, open set in $\mathbb{R}^n$ and let $T : cl(G) \to \mathbb{R}^n$ be a continuous map such that $x \neq T(x)$ for all $x \in \partial G$. Let I denote the identity map and $I - T$ the map $x \to x - T(x)$. Assume that* $\deg(I - T, G, 0) = 1$, *where* $\deg(I - T, G, 0)$ *denotes the topological degree of $I - T$ on G. If $x_* \in G$ is any fixed point of T in G, assume that T is Fréchet differentiable at x_* with Jacobian matrix $T'(x_*), I - T'(x_*)$ is one-one and $sgn(\det(I - T'(x_*))) = 1$, where "det" denotes "determinant" and "sgn" denotes "sign". Then T has exactly one fixed point in G.*

Proof. By assumption and the implicit function theorem, the set of fixed points of T in G is compact and each fixed point is isolated. Thus T has finitely many fixed point, say $x_1, x_2, \cdots, x_m$. For each fixed point x_k there is an open neighborhood U_k of x_k such that x_k is the only fixed point of T in $cl(U_k)$ and $U_k \subset G$. The additivity property of the topological degree implies that

$$1 = \deg(I - T, G, 0) = \sum_{i=1}^{m} \deg(I - T, U_i, 0).$$

However, the properties of the topological degree also imply that

$$\deg(I - T, U_i, 0) = sgn(\det(I - T'(x_i))) = 1,$$

so

$$1 = \sum_{i=1}^{m} 1 = m,$$

and $m = 1$. $\square$

Remark 3.2. It is often easy to prove that $\deg(I-T, G, 0) = 1$. For example, if G is convex, $x \neq T(x)$ for $x \in \partial G$ and $T(\partial G) \subset cl(G)$, then $\deg(I - T, G, 0) = 1$. More generally, if there exists a continuous homotopy $T_s(x), 0 \leq s \leq 1$, with $T_0 = T, T_1(x) = y_* \in G$ for all $x \in cl(G)$ and $T_s(x) \neq x$ for all $x \in \partial G$ and for $0 \leq s \leq 1$, then $\deg(I - T_s, G, 0) = 1$ for $0 \leq s \leq 1$.

The argument which we have used in Section 2 and will use later in Section 6 does not quite fit the framework of Corollary 3.1, and it may be useful to abstract that argument. For simplicity we restrict attention to the two dimensional case.

We begin with some notation which will be used in Theorem 3.3 below.
If $g : \text{int } (K^2) \to (0, \infty)$ is a continuous map, we shall define $g_0(u, v) = v, g_1(u, v) = g(u, v)$ and

$$g_k(u, v) = g(g_{k-2}(u, v), g_{k-1}(u, v)) \tag{3.4}$$

for $k \geq 2$. If $u_{-1} > 0$ and $u_0 > 0, \langle u_k | k \geq -1 \rangle$ will denote the sequence given recursively for $k \geq 1$ by

$$u_k = g(u_{k-2}, u_{k-1}). \tag{3.5}$$

Clearly, we have for $k \geq 1$

$$u_k = g_k(u_{-1}, u_0). \tag{3.6}$$

If a and b are real numbers with $a \leq b$, we shall define

$$V(a, b) = \{(u, v) \in \mathbb{R}^2 | a \leq u \leq b \text{ and } a \leq v \leq b\}. \tag{3.7}$$

We shall denote by C (compare Corollary 3.1) the closed cone given by

$$C = \{(u, v) \in \mathbb{R}^2 | u \leq 0 \text{ and } v \geq 0\}. \tag{3.8}$$

Theorem 3.3. *Let g: int $(K^2) \to (0, \infty)$ be a continuous map and assume (1) there exists a unique $L \geq 0$ with $g(L, L) = L$. Asume also (2) there exist a positive integer m and positive reals $a_0 \leq L$ and $b_0 \geq L$ such that for all $(u, v) \in V(a_0, b_0)$ and all $k \geq m, g_k(u, v) \in V(a_0, b_0)$. Define G by*

$$G = \{(a, b) \in V(a_0, b_0) | a \leq L \text{ and } L \leq b\} \tag{3.9}$$

and for $(a, b) \in G$, define $T(a, b) = (t_1(a, b), t_2(a, b))$ by

$$t_1(a, b) = \min\{g_m(u, v) | (u, v) \in V(a, b)\} \tag{3.10}$$

and

$$t_2(a, b) = \max\{g_m(u, v) | (u, v) \in V(a, b)\}. \tag{3.11}$$

Then $T(G) \subset G$, T is a continuous map and T is order-preserving in the partial ordering $\leq_C$ induced by C. If T has a unique fixed point $x_* \in G$, then $x_* = (L, L)$ and for every

$x \in G$, $\lim_{k\to\infty} T^k(x) = x_*$. Furthermore, for every $(u_{-1}, u_0) \in V(a_0, b_0)$, $\lim_{k\to\infty} u_k = L$, where u_k is given by equation (3.5).

Proof. The proof that $T(G) \subset G, T$ is continuous and T is order-preserving is left to the reader (compare Corollary 3.1). Define $(a_k, b_k) = T^k(a_0, b_0)$ and note as in Section 2 that $a_k \le a_{k+1} \le L$ and $L \le b_{k+1} \le b_k$ for all $k \ge 0$. It follows that $(a_k, b_k) \to (a, b)$ and $T(a, b) = (a, b)$. Since we assume that T has a unique fixed point in G and since $T(L, L) = (L, L), (a, b) = (L, L)$. If $x \in G$ and $\le$ denotes the partial ordering induced by C, then we have

$$(L, L) \le x \le (a_0, b_0),$$

which implies that

$$T^k(L, L) = (L, L) \le T^k(x) \le (a_k, b_k)$$

for all $k \ge 1$. It follows that $T^k(x) \to (L, L)$ as $k \to \infty$. If $(u_{-1}, u_0) \in V(a_0, b_0)$ and u_j is given by equation (3.5), one can see as in Section 2 that for each $k \ge 1$ there is an integer $N(k)$ with $a_k \le u_j \le b_k$ for all $j \ge N(k)$, so $u_j \to L$ as $j \to \infty$. $\square$

4 Global stability for a four dimensional relative of $u_{n+1} = \frac{u_{n-1}+pu_n}{u_{n-1}+qu_n}$.

We continue to use the notation of Section 2; in particular, f and h are defined by equation (2.1) and equation (2.2), $\Phi : \text{int } (K^2) \to \text{int } (K^2)$ is defined by $\Phi(u, v) = (v, f(u, v))$ and $L = \frac{1+p}{1+q}$. Furthermore, we always assume in this section that $0 \le q < p$.

If we define j: $\text{int } (K^2) \to (0, \infty)$ by

$$j(u, v) := f(f(u, v), h(u, v)), \tag{4.1}$$

we see that

$$\Phi^3(u, v) = (h(u, v), j(u, v)). \tag{4.2}$$

If we write $U = f(u, v)$ and $V = h(u, v)$ and define $M(u, v)$ by

$$M(u, v) = (U + qV)^2(u + qv)^2(v + qU)^2, \tag{4.3}$$

then a calculation gives that

$$(D_1 j)(u, v) = [(p - q)^2 v(v^2 + 2qUv + pqU^2]/M(u, v) > 0 \tag{4.4}$$

and

$$(D_2 j)(u, v) = -(p - q)^2[(v + pU)(v + qU)u + U(u^2 + 2quv + pqv^2)]/M(u, v) < 0. \tag{4.5}$$

Equations (2.3) and (2.4) imply that $u \to h(u, v)$ and $v \to h(u, v)$ are strictly decreasing on $(0, \infty)$ for $u > 0$ and $v > 0$, and equations (4.4) and (4.5) imply that $u \to j(u, v)$ is strictly increasing on $(0, \infty)$ and $v \to j(u,v)$ is strictly decreasing on $(0, \infty)$ for $u > 0$ and $v > 0$.

We now define $\Psi(u,v) = \Phi^3(u,v)$ and define

$$G = \{z \in \text{ int } (K^4) : z_1 \le z_2 \text{ and } z_3 \le z_4\} \text{ and} \tag{4.6}$$
$$C = \{w = (w_1, w_2, w_3, w_4) | (-1)^{i-1} w_i \ge 0 \text{ for } 1 \le i \le 4\}$$

Given $z \in G$, we define $B(z) = \{(u,v) \in \text{int } (K^2) : z_1 \le u \le z_2, z_3 \le v \le z_4\}$ and, following Corollary 3.1, we define

$$t_1(z) = \min\{h(u,v) | (u,v) \in B(z)\}, t_2(z) = \max\{h(u,v) | (u,v) \in B(z)\} \tag{4.7}$$

and

$$t_3(z) = \min\{j(u,v) | (u,v) \in B(z)\}, t_4(z) = \max\{j(u,v) | (u,v) \in B(z)\}. \tag{4.8}$$

We define $T : G \to G$ by

$$T(z) = (t_1(z), t_2(z), t_3(z), t_4(z)). \tag{4.9}$$

Using the previously described monotonicity properties of h and j, it is easy to check that for $z = (z_1, z_2, z_3, z_4) \in G$,

$$T(z) = (h(z_2, z_4), h(z_1, z_3), j(z_1, z_4), j(z_2, z_3)). \tag{4.10}$$

Corollary 3.1 implies that T is order-preserving with respect to the partial ordering induced by the cone C in equation (4.6). Note that equation (4.10) actually defines T naturally as a map of int (K^4) to int (K^4); and the reader can verify directly that T: int $(K^4) \to$ int (K^4) given by (4.10) is order-preserving with respect to C.

If we can prove that for all $z \in G, T^k(z) \to (L, L, L, L,)$ as $k \to \infty$, equation (3.3) implies that for all $(u,v) \in$ int $(K^2), \Phi^k(u,v) \to (L, L)$. Thus a global stability result for the fixed point (L, L, L, L) of T will imply, as a special case, Theorem 2.1.

Our goal in this section is to prove the following global stability result.

Theorem 4.1. *Assume that $0 \le p < q$ and let maps f, h and j be defined by equations (2.1), (2.2) and (4.1) respectively. Let T: int $(K^4) \to$ int (K^4) be defined by equation (4.10). Then for any z int (K^4) we have*

$$\lim_{k \to \infty} T^k(z) = (L, L, L, L),$$

where $L = (1+p)/(1+q)$.

Our strategy in proving Theorem 4.1 will be to show that the hypotheses of Theorem 3.1 are satisfied. We have already observed that T is order-preserving with respect to $\le_C$ (C as in equation (4.6)). The only real difficulty will be to prove that (L, L, L, L) is the only fixed point of T in int (K^4), and we shall prove this with the aid of a symbolic calculation by Maple.

We begin with an easy lemma.

Lemma 4.1. *Let $T\colon$ int $(K^4) \to$ int (K^4) be defined by equation (4.10) and assume that $0 \le q < p$. For any $z \in$ int (K^4), if $w = (w_1, w_2, w_3, w_4) = T^3(z)$, then*

$$1 \le w_i \le \left(\frac{1 + ph(1,1)}{1 + qh(1,1)}\right) \le 1 + ph(1,1), 1 \le i \le 4. \tag{4.11}$$

Proof. Let $\xi = T(z)$ and $\eta = T^2(z)$. Because $f(u,v) \ge 1$ for all $(u,v) \in$ int (K^2), we have that $1 \le \xi_i, 1 \le \eta_i$ and $1 \le w_i$ for $1 \le i \le 4$. The monotonicity properties of h imply that

$$\eta_1 = h(\xi_2, \xi_4) \le h(1,1) \text{ and } w_1 \le h(\eta_2, \eta_4) \le h(1,1),$$

and the same argument shows that

$$\eta_2 \le h(1,1) \text{ and } w_2 \le h(1,1).$$

Note that we have $h(1,1) = f(1, f(1,1)) \le f(1, f(1,L)) = f(1, h(1,1))$, because $1 \le L$. By definition of j,

$$w_3 = f(f(\eta_1, \eta_4), h(\eta_1, \eta_4)).$$

We know that $h(\eta_1, \eta_4) \le h(1,1)$ and $f(\eta_1, \eta_4) \ge 1$, so

$$w_3 \le f(1, h(1,1)) = \left(\frac{1 + ph(1,1)}{1 + qh(1,1)}\right).$$

The same argument gives the desired estimate for w_4. □

Lemma 4.2. *Assume that $0 \le q < p$, that C is as in equation (4.6) and $T\colon$ int $(K^4) \to$ int (K^4) is as in equation (4.10). If $x \in$ int (K^4), there exist $y \in$ int (K^4) and $z \in$ int (K^4) such that $y \le_C x \le_C z, y \le_C T(y)$ and $T(z) \le_C z$.*

Proof. For convenience we write $\le$ instead of $\le_C$. Given $x \in$ int $(K^4), y \le x$ is equivalent to $y_i \le x_i$ for $i = 1$ and $i = 3$ and $y_i \ge x_i$ for $i = 2$ and $i = 4$. If $y' = T(y), y' \ge y$ is equivalent to $h(y_2, y_4) \ge y_1, h(y_1, y_3) \le y_2, j(y_1, y_4) \ge y_3$ and $j(y_2, j_3) \le y_4$. Select $0 < y_1 \le \min(x_1, 1)$ and $0 < y_3 \le \min(x_3, 1)$. Because $h(u,v) \ge 1$ and $j(u,v) \ge 1$ for all $(u,v) \in$ int (K^2), we have $y_1' = h(y_2, y_4) \ge y_1$ and $y_3' = j(y_1, y_4) \ge y_3$, no matter how $y_2 > 0$ and $y_4 > 0$ are chosen. If we select $y_2 \ge \max(h(y_1, y_3), x_2)$, then $y_2 \ge y_2'$ and $y_2 \ge x_2$. Finally, if we select $y_4 \ge \max(j(y_2, y_3), x_4)$ we have arranged that $y_4' \le y_4$ and $y_4 \ge x_4$. With this choice of y we have shown that $y \le x$ and $y \le T(y)$. The proof of the existence of z is similar: Take $z_2 = \min(1, x_2), z_4 = \min(1, x_4), z_1 = \max(h(z_2, z_4), x_1)$ and $z_3 = \max(j(z_1, z_4), x_3)$. □

If T is as in equation (4.10) Lemmas 4.1 and 4.2 and our previous remark show that properties (1), (2) and (4) of Theorem 3.1 are satisfied. It remains to investigate whether (L, L, L, L) is the only fixed point of T in int (K^4).

Lemma 4.3. *Assume that $0 \leq q < p$, and T is defined by equation (4.10). Define F: int $(K^2) \to$ int (K^2) by $F(u,v) = (\psi_1(u,v), \psi_2(u,v))$, where*

$$\psi_1(u,v) = h(h(u,v), j(h(u,v),v)) \tag{4.12}$$

and

$$\psi_2(u,v) = j(u, j(h(u,v),v)). \tag{4.13}$$

Then (L,L,L,L) is the only fixed point of T in int (K^4) if and only if (L,L) is the only fixed point of F in int (K^2).

Proof. If $x \in \text{int }(K^4)$ and $T(x) = x$, we have $x_1 = h(x_2,x_4), x_2 = h(x_1,x_3), x_3 = j(x_1,x_4)$ and $x_4 = j(x_2,x_3)$. Expressing x_3 in terms of x_2 and x_4 we find that

$$x_1 = h(x_2,x_4), \text{ and } x_3 = j(h(x_2,x_4),x_4).$$

This gives

$$x_2 = h(h(x_2,x_4), j(h(x_2,x_4),x_4)) = \psi_1(x_2,x_4)$$

and

$$x_4 = j(x_2, j(h(x_2,x_4),x_4)) = \psi_2(x_2,x_4).$$

Writing $u = x_2$ and $v = x_4, F(u,v) = (u,v)$. Also, if $x \neq (L,L,L,L)$, we cannot have $x_2 = x_4 = L$, for if $x_2 = x_4 = L, x_1 = h(L,L) = L$ and $x_3 = j(x_1,x_4) = j(L,L) = L$.

Conversely, suppose that $F(u,v) = (u,v)$ for $(u,v) \in \text{int }(K^2)$. Defining $x_2 = u, x_4 = v, x_1 = h(u,v) = h(x_2,x_4)$ and $x_3 = j(x_1,x_4)$, the equation $F(u,v) = (u,v)$ implies that $x_2 = h(x_1,x_3)$ and $x_4 = h(x_2,x_3)$, so $T(x) = x$. If $(u,v) \neq (L,L)$, then we certainly have that $x \neq (L,L,L,L)$ □

Lemma 4.3 reduces a four dimensional problem to a more complicated two dimensional problem. Our next lemma makes a further reduction to two one dimensional problems.

Lemma 4.4. *Assume that $0 \leq q < p$ and that ψ_1, ψ_2 and $F$are as in Lemma 4.3. Then it follows that F is order-preserving in the partial ordering from K^2. Define maps $\theta_i : (0,\infty) \to (0,\infty), i = 1,2,$ by*

$$\theta_i(u) = \psi_i(u,u). \tag{4.14}$$

If $\theta_i(u) \neq u$ for $u > L$ and $i = 1$ and $i = 2$, then the map T defined by equation (4.10) has only the fixed point (L,L,L,L) in int (K^4).

Proof. The monotonicity properties of h and j (see equations (2.3), (2.4), (4.4) and (4.5)) easily imply that F is order-preserving in the partial ordering from K^2; details are left to the reader. In fact if $0 < u \leq u', 0 \leq v \leq v'$ and $(u,v) \neq (u',v')$, one sees that

$\psi_i(u,v) < \psi_i(u',v')$ for $i = 1,2$. It follows that θ_1 and θ_2 are strictly increasing maps on $(0,\infty)$.

If $z = (z_1, z_2, z_3, z_4) \in int(K^4)$ is a fixed point of T, note that $\xi = (z_2, z_1, z_4, z_3)$ is a fixed point of T. Suppose we can prove that whenever $z = (z_1, z_2, z_3, z_4)$ is a fixed point of T, then $z_2 \leq L$ and $z_4 \leq L$, where $L := \frac{1+p}{1+q}$. Since $\xi = (z_2, z_1, z_4, z_3)$ is also a fixed point of T, it follows that $z_1 \leq L$ and $z_3 \leq L$. However, $z_1 = h(z_2, z_4)$; and because $z_2 \leq L$ and $z_4 \leq L$,

$$z_1 = h(z_2, z_4) \geq h(L, L) = L, \tag{4.15}$$

with strict inequality in (4.15) if $z_2 < L$ or $z_4 < L$. Thus we must have $z_2 = z_4 = L$. Because $z_2 = L = h(z_1, z_3)$ and $z_1 \leq L$ and $z_3 \leq L$, the same argument shows that we must have $z_1 = z_3 = L$, so $z = (L, L, L, L)$.

Before continuing, it is convenient to make some preliminary observations. A calculation shows that $(D_1 f)(u,v) < 0$ and $(D_2 f)(u,v) > 0$ for all $(u,v) \in \text{int}\ (K^2)$, so $u \to f(u,v)$ is strictly decreasing and $v \to f(u,v)$ is strictly increasing (always assuming $0 \leq q < p$). Because $f(u,v) > 1$ for all $(u,v) \in \text{int}\ (K^2)$, we also have $h(u,v) > 1$ and $j(u,v) > 1$. It follows that for all $u > 0$,

$$1 \leq \theta_1(u) = h(h(u,u), j(h(u,u), u)) \leq h(1,1). \tag{4.16}$$

If $u \geq \alpha > 0$, we claim also that there is a constant $M = M(\alpha)$ with

$$1 \leq \theta_2(u) \leq M(\alpha). \tag{4.17}$$

The monotonicity properties of h give, for $u \geq \alpha > 0$,

$$1 \leq h(u,u) \leq h(\alpha, \alpha),$$

and the monotonicity properties of j then imply that

$$1 \leq j(h(u,u), u) := V \leq j(h(\alpha,\alpha), \alpha) := M_1(\alpha). \tag{4.18}$$

We deduce from eqution (4.18) that

$$1 \leq f(u, V) = 1 + \frac{(p-q)V}{u + qV} \leq 1 + \frac{(p-q)M_1(\alpha)}{\alpha} := M_2(\alpha) \tag{4.19}$$

and

$$1 \leq h(u, V) = f(V, f(u, V)) \leq f(1, M_2(\alpha)) := M_3(\alpha). \tag{4.20}$$

We deduce from equation (4.20) that

$$\theta_2(u) = j(u,V) = f(f(u,V), h(u,V)) \le f(1, M_3(\alpha)) := M(\alpha). \tag{4.21}$$

We now return to the main thread of the argument. We have shown that it suffices to prove that whenever $z = (z_1, z_2, z_3, z_4)$ is a fixed point of T, then $z_2 \le L$ and $z_4 \le L$. We write $u_* = z_2$ and $v_* = z_4$, and we recall that Lemma 4.3 implies that $u_* = \psi_1(u_*, v_*)$ and $v_* = \psi_2(u_*, v_*)$. There are two possibilities: (a) $u_* \le v_*$ and (b) $v_* \le u_*$. In case (a) we see that

$$u_* \le v_* = \psi_2(u_*, v_*) \le \psi_2(v_*, v_*) = \theta_2(v_*).$$

A simple induction shows that $\theta_2^k(v_*) \le \theta_2^{k+1}(v_*)$ for $k \ge 0$; and using equation (4.17), we conclude that $\theta_2^k(v_*) \le M(v_*) < \infty$ for all $k \ge 0$. It follows that $\theta_2^k(v_*) \to v \ge v_*$ and $\theta_2(v) = v$. By assumption, $v \le L$, so we must have $u_* \le v_* \le L$. The proof in case (b) is analogous and is left to the reader. □

It remains to prove that the equation $\theta_i(x) = x$ has no solution $x > L$ for $i = 1$ or $i = 2$. This appears to be a difficult calculus question. We shall write $p = q + r$ and $x = L(1+z)$, where $L = \frac{1+p}{1+q}$, and we shall reduce the question to whether certain polynomials in the variables q, r and z and with *integral* coefficients are positive for all positive values of q, r and z. Although there are several thousand terms in the polynomials in question, with the aid of a symbolic calculation using Maple 10, we can compute all the integral coefficients and show that all integral coefficients are nonnegative. We emphasize that the procedure using Maple is exact, since it computes only polynomials with integral coefficients.

Lemma 4.5. *Assume that $0 \le q < p$ and let $\theta_i(x), x > 0, i = 1, 2$, be defined by equations (4.12)-(4.14). Then $\theta_i(x) \ne x$ for $x > L := \left(\frac{1+p}{1+q}\right)$ and for $i = 1, 2$.*

Proof. We define $p = q + r$, so $r > 0$, and we write $x = L(1 + z)$. We shall associate to θ_i a polynomial $w_i = w_i(q, r, z)$ with integer coefficients such that $\theta_i(x) \ne x$ for all $x > L$ if and only if $w_i > 0$ for all $q \ge 0, r > 0, z > 0$. The polynomials w_i can be computed with the aid of Maple, and it turns out that all the integer coefficients are nonnegative and some are positive, a much stronger result than we need.

We construct w_1 and w_2 in stages:

$$h(x,x) = f(x,L) = \frac{x + pL}{x + qL} = \frac{1 + p + z}{1 + q + z} := \frac{u_1}{v_1} \tag{4.22}$$

$$u_1 = 1 + p + z \text{ and } v_1 = 1 + q + z. \tag{4.23}$$

Note that u_1 and v_1 are polynomials with integer coefficients in q, r and z. Next we have

$$f(h(x,x), x) = \frac{h(x,x) + px}{h(x,x) + qx} = \frac{(1+q)u_1 + p(1+p)(1+z)v_1}{(1+q)u_1 + q(1+p)(1+z)v_1} := \frac{u_2}{v_2}, \tag{4.24}$$

where

$$u_2 := (1+q)u_1 + p(1+p)(1+z)v_1 \text{ and } v_2 := (1+q)u_1 + q(1+p)(1+z)v_1 \tag{4.25}$$

Again u_2 and v_2 are polynomials with integer coefficients in the variables q, r, z.

$$h(h(x,x),x) = f(x, f(h(x,x),x)) = \frac{(1+p)(1+z)v_2 + p(1+q)u_2}{(1+p)(1+z)v_2 + q(1+q)u_2} := \frac{u_3}{v_3}, \tag{4.26}$$

where

$$u_3 = (1+p)(1+z)v_2 + p(1+q)u_2 \text{ and } v_3 = (1+p)(1+z)v_2 + q(1+q)u_2. \tag{4.27}$$

By definition of j we have

$$j(h(x,x),x) = f\left(\frac{u_2}{v_2}, \frac{u_3}{v_3}\right) = \frac{u_2v_3 + pu_3v_2}{u_2v_3 + qu_3v_2} = \frac{u_4}{v_4}, \tag{4.28}$$

where

$$u_4 = u_2v_3 + pu_3v_2 \text{ and } v_4 = u_2v_3 + qu_3v_2. \tag{4.29}$$

Using equation (4.22) and (4.28) we obtain that

$$f(h(x,x), j(h(x,x),x)) = f\left(\frac{u_1}{v_1}, \frac{u_4}{v_4}\right) = \frac{u_1v_4 + pu_4v_1}{u_1v_4 + qu_4v_1} := \frac{u_5}{v_5}, \tag{4.30}$$

where

$$u_5 = u_1v_4 + pu_4v_1 \text{ and } v_5 = u_1v_4 + qu_4v_1. \tag{4.31}$$

Using the definition of h we obtain that

$$\theta_1(x) = f\left(\frac{u_4}{v_4}, \frac{u_5}{v_5}\right) = \frac{u_6}{v_6}, \tag{4.32}$$

where

$$u_6 = u_4v_5 + pu_5v_4 \text{ and } v_6 = u_4v_5 + qu_5v_4. \tag{4.33}$$

It follows that $\theta_1(x) = x$ for some $x > L$ if and only if

$$w_1 := (1+p)(1+z)v_6 - (1+q)u_6 = 0 \tag{4.34}$$

for some $z > 0$. Our construction insures that u_j and $v_j, 1 \le j \le 6$, and w_1 are polynomials in the variables q, r and z with integer coefficients.

To proceed analogously for $\theta_2(x)$ we write

$$f(x, j(h(x,x),x)) = f\left(x, \frac{u_4}{v_4}\right) = \frac{(1+p)(1+z)v_4 + p(1+q)u_4}{(1+p)(1+z)v_4 + q(1+q)u_4} := \frac{U_5}{V_5}, \tag{4.35}$$

where

$$U_5 = (1+p)(1+z)v_4 + p(1+q)u_4 \text{ and } V_5 = (1+p)(1+z)v_4 + q(1+q)u_4. \tag{4.36}$$

It then follows that

$$h(x, j(h(x,x),x)) = f\left(\frac{u_4}{v_4}, \frac{U_5}{V_5}\right) := \frac{U_6}{V_6}, \tag{4.37}$$

where

$$U_6 = u_4V_5 + pU_5v_4 \text{ and } V_6 = u_4V_5 + qU_5v_4. \tag{4.38}$$

It follows fron the definition of j that

$$\theta_2(x) = f\left(\frac{U_5}{V_5}, \frac{U_6}{V_6}\right) := \frac{U_7}{V_7}, \tag{4.39}$$

where

$$U_7 = U_5V_6 + pU_6V_5 \text{ and } V_7 = U_5V_6 + qU_6V_5. \tag{4.40}$$

If we define w_2 by

$$w_2 = (1+p)(1+z)V_7 - (1+q)U_7, \tag{4.41}$$

w_2 is a polynomial in the variables q, r and z, w_2 has integer coefficients and $\theta_2(x) = x$ for some $x > L$ if and only if $w_2 = 0$ for some $z > 0$.

Using the above sequence of steps it is easy to write a Maple program which computes the polynomials w_1 and w_2 and verifies that all the integer coefficients are nonnegative and that, even if q is set equal to zero, some coefficients are positive. A simple Maple program which accomplishes this is given in Appendix A. □

Proof of Theorem 4.1. We have already noted that, for C as in equation (4.6), T is order-preserving with respect to $\le_C$. Lemma 4.1 proves that property 2 of Theorem 3.1 is satisfied, and Lemma 4.2 shows that property 4 of Theorem 3.1 is satisfied, Lemmas 4.3-4.5 prove that T has a unique fixed point in int (K^4). Theorem 4.1 now follows from Theorem 3.1. □

Remark 4.1. If $\Phi : \text{int } (K^2) \to \text{int } (K^2)$ is given by $\Phi(u,v) = (v, f(u,v))$ and if $H = \{x \in \text{int } (K^4) : x_1 = x_2 \text{ and } x_3 = x_4\}$, we have already noted in the proof of Corollary 3.1 that $T(H) \subset H$, that H can be identified with int (K^2) and that $T|H$ is conjugate to Φ^3.

5. Another four dimensional relative of $u_{n+1} = \frac{u_{n-1}+pu_n}{u_{n-1}+qu_n}$.

In this section we always assume at least that $0 \leq q \leq p$ and $p > 0; f$ and h will denote the functions in equations (2.1) and (2.2) and $\Phi(u,v) := (v, f(u,v))$, so $\Phi^2(u,v) = (f(u,v), h(u,v))$.

If one applies the construction in Corollary 3.1 to Φ^2, one obtains a map $S : \text{int } (K^4) \to \text{int } (K^4)$ defined by

$$S(x_1,x_2,x_3,x_4) = (y_1,y_2,y_3,y_4) = (f(x_2,x_3), f(x_1,x_4), h(x_2,x_4), h(x_1,x_3)). \tag{5.1}$$

Since the map T in Section 4 was obtained by applying the construction in Corollary 3.1 to Φ^3 and since we have proved that the fixed point $\Lambda := (L,L,L,L), L := \left(\frac{1+p}{1+q}\right)$, satisfies $T^k(x) \to \Lambda$ for all $x \in \text{int } (K^4)$ whenever $0 \leq q < p$ one might hope that the same theorem is true for S. Our goal in this section is to prove that this hope is false, a failure which suggests the delicacy of such results. Specifically we shall prove the following theorem:

Theorem 5.1. *Assume that $0 \leq q < p$ and that S is defined by equation (5.1). The equation $1 - 2t + 2t^2 - 2t^3 = 0$ has a unique real root t_*, and t_* is approximately equal to .647798871. If $\frac{p-q}{(1+p)(1+q)} > t_*, S$ has at least three distinct fixed points in int (K^4).*

We shall not study here the question of when the fixed point $\Lambda = (L,L,L,L)$ of S is globally stable, but we make the following conjecture.

Conjecture 5.1. *If $0 \leq q \leq p$, $p > 0$ and $\frac{(p-q)}{(1+p)(1+q)} \leq t_*$, then for every $x \in$ int $(K^4), S^k(x) \to \Gamma = (L,L,L,L)$ as $k \to \infty$.*

We shall view elements x of $\mathbb{R}^n$ as column vectors, but we shall abuse notation and write $x = (x_1, x_2, \cdots, x_n)$. If A is an $n \times n$ real matrix, A induces a linear map of R^n to $\mathbb{R}^n$ in the usual way by $x \to Ax = y$.

The results of this section are suggested by an analysis of the eigenvalues of the Jacobian matrix of S at $\Lambda = (L,L,L,L)$.

Lemma 5.1. *Assume that $0 \leq q \leq p$, $p > 0, L := \frac{1+p}{1+q}$ and $\Lambda = (L,L,L,L)$. If $S'(\Lambda)$ denotes the Jacobian matrix of S at Λ, then*

$$S'(\Lambda) = \begin{bmatrix} 0 & -\lambda & \lambda & 0 \\ -\lambda & 0 & 0 & \lambda \\ 0 & -\lambda^2 & 0 & -\lambda(1-\lambda) \\ -\lambda^2 & 0 & -\lambda(1-\lambda) & 0 \end{bmatrix} := M(\lambda), \tag{5.2}$$

where $\lambda := \frac{(p-q)}{(1+p)(1+q)}$. If two dimensional subspaces $V \subset \mathbb{R}^4$ and $W \subset \mathbb{R}^4$ are defined by

$$V = \{x \in \mathbb{R}^4 | x_1 = -x_2 \text{ and } x_3 = -x_4\} \tag{5.3}$$

and

$$W = \{x \in \mathbb{R}^4 | x_1 = x_2 \text{ and } x_3 = x_4\}, \tag{5.4}$$

then $M(\lambda)(V) \subset V$ *and* $M(\lambda)(W) \subset W$. *If we define* 2×2 *matrices* $M_1(\lambda)$ *and* $M_2(\lambda)$ *by*

$$M_1(\lambda) = \begin{bmatrix} \lambda & \lambda \\ \lambda^2 & \lambda(1-\lambda) \end{bmatrix} \text{ and } M_2(\lambda) = \begin{bmatrix} -\lambda & \lambda \\ -\lambda^2 & -\lambda(1-\lambda) \end{bmatrix}, \tag{5.5}$$

every eigenvalue z *of* $M(\lambda)$ *is an eigenvalue of* $M_1(\lambda)$ *or* $M_2(\lambda)$; *and if* z *is an eigenvalue of* $M_1(\lambda)$ *or* $M_2(\lambda), z$ *is an eigenvalue of* $M(\lambda)$. *Every eigenvalue* z *of* $M_2(\lambda)$ *satisfies* $|z| < 1$. *The equation* $1 - 2t + 2t^2 - 2t^3 = 0$ *has a unique real root* t_* *approximately equal to .6477988713; and if* $0 \leq \lambda < t_*$, *every eigenvalue* z *of* $M_1(\lambda)$ *satisfies* $|z| < 1$. *If* $\lambda > t_*, M_1(\lambda)$ *has two real eigenvalues* z_1 *and* z_2 *with* $-1 < z_1 < 1 < z_2$.

Proof. The formula for $S'(\Lambda)$ follows by a simple calculation, which we leave to the reader. Note that our assumptions on p and q insure that $0 \leq \lambda < 1$. One can also easily verify that $M(\lambda)(V) \subset V$ and $M(\lambda)(W) \subset W$. If $x = (u, -u, v, -v) \in V, M(\lambda)x = y = (u', -u', v', -v')$, where

$$M_1(\lambda) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u' \\ v' \end{pmatrix}, \tag{5.6}$$

and a similar formula holds if $x \in W$ and $M_1(\lambda)$ is replaced by $M_2(\lambda)$ in equation (5.6). The assertions about the realtionship between eigenvalues of $M(\lambda)$ and eigenvalues of $M_1(\lambda)$ and $M_2(\lambda)$ now follow easily.

If a and b are real numbers, recall the elementary result that all solutions z of

$$z^2 + az + b = 0$$

satisfy $|z| < 1$ if and only if

$$|a| < 1 + b \text{ and } b < 1. \tag{5.7}$$

The eigenvalues z of $M_2(\lambda)$ are solutions of

$$z^2 + (2\lambda - \lambda^2)z + \lambda^2 = 0, \tag{5.8}$$

and using equation (5.7) and the fact that $0 \leq \lambda < 1$, we se that all solutions z of eqaution (5.8) satisfy $|z| < 1$. The eigenvalues z of $M_1(\lambda)$ satisfy

$$z^2 - (2\lambda - \lambda^2)z + (\lambda^2 - 2\lambda^3) = 0 \tag{5.9}$$

Because $0 \leq \lambda < 1, \lambda^2 - 2\lambda^3 < 1$ and so equation (5.7) implies that all roots z of equation (5.9) satisfy $|z| < 1$ if and only if

$$1 - 2\lambda + 2\lambda^2 - 2\lambda^3 > 0. \tag{5.10}$$

If we define $g(t)$ by

$$g(t) = 1 - 2t + 2t^2 - 2t^3,$$

one has

$$g'(t) = -6[(t - \frac{1}{3})^2 + \frac{2}{9}] < 0,$$

and since $g(0) = 1$ and $g(1) = -1, g(t) = 0$ has exactly one real root t_* and $0 < t_* < 1$. It follows that all roots z of equation (5.9) satsify $|z| < 1$ if and only if $\lambda < t_*$. It is easy to estimate t_* as in the statement of Lemma 5.1. The roots of equation (5.9) are

$$z_1 = \frac{2\lambda - \lambda^2 - \sqrt{4\lambda^3 + \lambda^4}}{2} \text{ and } z_2 = \frac{2\lambda - \lambda^2 + \sqrt{4\lambda^3 + \lambda^4}}{2},$$

so both roots are real and distinct if $\lambda > 0$ and $|z_2| > |z_1|$ if $\lambda > 0$. It follows that $z_2 > 1$ for $\lambda > t_*$. Since $|z_1 z_2| = |\lambda^2 - 2\lambda^3| < 1$, we must have that $|z_1| < 1$. □

Remark 5.1. The same analysis can be applied to the Jacobian matrix $T'(\Lambda)$ for $\Lambda = (L, L, L, L)$ and T as in Section 4, but in that case one finds that all eigenvalues z of $T'(\Lambda)$ satisfy $|z| < 1$ if $0 \leq q < p$.

For technical reasons, we also need to prove that if $0 \leq q \leq p$ and $p > 0, S^3(\text{ int } (K^4))$ is contained in a compact, convex subset of int (K^4).

Lemma 5.2. *Assume that $0 \leq q \leq p$ and $p > 0$ and that S is given by equation (5.1). If $x \in int(K^4)$ and $w = (w_1, w_2, w_3, w_4) = S^3(x)$, we have $1 \leq w_i \leq 1 + p + p^2 + p^3$ for $i = 1, 2$, and $1 \leq w_i \leq 1 + p + p^2$ for $i = 3, 4$.*

Proof. We write $y = S(x)$ and $z = S(y)$, so $w = S(z)$. Because $f(u, v) \geq 1$ for all $u > 0$ and $v > 0$, $y_i \geq 1$ for $1 \leq i \leq 4$. The same argument implies that $z_i \geq 1$ and $w_i \geq 1$ for $1 \leq i \leq 4$. The monotonicity properties of h imply that

$$z_3 = h(y_2, y_4) \leq h(1, 1) \text{ and } z_4 = h(y_1, y_3) \leq h(1, 1),$$

and the same argument also gives that

$$w_3 \leq h(1, 1) \text{ and } w_4 \leq h(1, 1).$$

A calculation gives that

$$h(1, 1) = 1 + \frac{(p - q)(1 + p)}{1 + 2q + pq} \leq 1 + p(1 + p) = 1 + p + p^2.$$

The monotonicity properties of f now imply that

$$w_1 = f(z_2, z_3) \leq f(1, h(1,1)) = 1 + \frac{(p-q)h(1,1)}{1+qh(1,1)} \leq 1 + ph(1,1) \leq 1 + p + p^2 + p^3,$$

and the same argument gives $w_2 \leq 1 + p + p^2 + p^3$. □

At this point we need again to use the topological degree: see [1-4, 10, 11, 12, 14]. Recall that if G is a bounded open subset of $\mathbb{R}^n$ and $\Psi : cl(G) \to \mathbb{R}^n$ is a continuous map such that $\Psi(x) \neq a$ for all $x \in \partial G$, one can assign an integer $\deg(\Psi, G, a)$, called the topological degree of Ψ on G with respect to a. Roughly speaking, $\deg(\Psi, G, a)$ is an algebraic count of the number of solutions $x \in G$ of $\Psi(x) = a$.

Lemma 5.3. *Assume that $0 \leq q \leq p$ and $p > 0$ and let S be defined by equation (5.1). Assume that $0 < r_1 < 1$ and $r_2 > 1 + p + p^2 + p^3$ and define $G = \{x \in \mathbb{R}^4 | r_1 < x_i < r_2$ for $1 \leq i \leq 4\}$. Let I denote the identity map, so $I - S$ denotes the map $x \to x - S(x)$. Then it follows that*

$$\deg(I - S, G, 0) = 1. \tag{5.11}$$

Proof. Notice that the map S actually depends on $p > 0$ and q, $0 \leq q \leq p$. We shall view p as fixed, allow q to vary with $0 \leq q \leq p$, and write $S_q(x)$ instead of $S(x)$ to indicate the dependence of S on q. If $x = S_q(x)$ for some $x \in \mathrm{int}(K^4)$, then $x = S_q^3(x)$ and Lemma 5.2. implies that $1 \leq x_i \leq 1 + p + p^2 + p^3$ for $1 \leq i \leq 4$. It follows that all fixed points of S_q in $\mathrm{int}(K^4)$ lie in a compact set contained in G and $S_q(x) \neq x$ for $0 \leq q \leq p$ and $x \in \partial G$. The homotopy property of the topological degree implies that $\deg(I - S_q, G, 0)$ is defined and constant for $0 \leq q \leq p$. However, if $q = p$, $S_q(x) = (1,1,1,1)$ for all $x \in \mathrm{int}(K^4)$; and since $(1,1,1,1) \in G$, for $0 \leq q \leq p$ we have

$$\deg(I - S_p, G, 0) = 1 = \deg(I - S_q, G, 0),$$

which completes the proof. □

Lemma 5.4. *Assume that $0 \leq q \leq p, p > 0$ and S is given by equation (5.1). If W is given by equation (5.4), then $S(W \cap$ int $(K^4)) \subset W \cap$ int (K^4). If $x \in W \cap$ int (K^4) and $S(x) = x$, it follows that $x = (L, L, L, L)$, where $L = \left(\frac{1+p}{1+q}\right)$.*

Proof. It is straightforward to see that $S(W \cap \mathrm{int}(K^4)) \subset W \cap \mathrm{int}(K^4)$. If $x = (u, u, v, v)$, where $u > 0$ and $v > 0$, and if $S(x) = x$, then equation (5.1) gives

$$u = f(u, v) \tag{5.12}$$

and

$$v = h(u, v) = f(v, f(u, v)) = f(v, u). \tag{5.13}$$

If $u = v$, we find that $u = f(u,u) = L = v$, and we are done, so we assume, by way of contradiction, that $u \neq v$. Equation (5.12) gives

$$u^2 + quv = u + pv, \tag{5.14}$$

and equation (5.13) gives

$$v^2 + quv = v + pu. \tag{5.15}$$

Subtracting equation (5.15) from (5.14) and dividing by $u - v$ we obtain

$$u + v = 1 - p. \tag{5.16}$$

However, $f(u,v) \geq 1$ and $f(v,u) \geq 1$, so $u + v \geq 2$, which contradicts equation (5.16) □

With the aid of Lemmas 5.1 - 5.4, Theorem 5.1 now follows by a simple argument using the topological degree.

Proof of Theorem 5.1. Let $S'(\Lambda)$ denotes the Jacobian matrix for S at $\Lambda = (L, L, L, L)$ and assume that $\frac{(p-q)}{(1+p)(1+q)} > t_*$, where t_* is the unique real root of $1 - 2t + 2t^2 - 2t^3 = 0$ which is guaranteed by Lemma 5.1. Lemma 5.1 implies that $S'(\Lambda)$ has one real eigenvalue $z_2 > 1$ and all other eigenvalues z of $S'(\Lambda)$ satisfy $|z| < 1$. If we express the determinant of $I - S'(\Lambda), \det(I - S'(\Lambda))$, in terms of the eigenvalues of $S'(\Lambda)$, it follows that $I - S'(\Lambda)$ is invertible and

$$sgn(\det(I - S'(\Lambda))) = -1.$$

The implicit function theorem implies that Λ is an isolated fixed point of S, so there exist $\varepsilon > 0$ such that if

$$B_\varepsilon = \{x \in \mathbb{R}^4 \,\big|\, \|x - \Lambda\| < \varepsilon\},$$

then Λ is the only fixed point of S in $cl(B_\varepsilon)$, the closure of B_ε. Elementary properties of the topological degree imply that

$$\deg(I - S, B_\varepsilon, 0) = -1.$$

If G is defined as in Lemma 5.3, we can take $\varepsilon > 0$ so small that $cl(B_\varepsilon) \subset G$, and if $H_\varepsilon := G \setminus B_\varepsilon$, the additivity property of the topological degree implies that

$$\deg(I - S, B_\varepsilon, 0) + \deg(I - S, H_\varepsilon, 0) = \deg(I - S, G, 0).$$

Lemma 5.3 implies that $\deg(I - S, G, 0) = 1$, so

$$\deg(I - S, H_\varepsilon, 0) = 2. \tag{5.17}$$

It follows from equation (5.7) that S has a fixed point in H_ε, but *a priori*, we cannot assert that S has at least two fixed points in H_ε. Notice, however, that if $x = (x_1, x_2, x_3, x_4)$ is a fixed point of S then $y = (x_2, x_1, x_4, x_3)$ is also a fixed point of S and $y \neq \Lambda$. Lemma 5.4 implies that $y \neq x$, so S has at least three distinct fixed points. □

Remark 5.1. Define $D = \{x \in \mathbb{R}^4 | x_1 \geq L, x_2 \leq L, x_3 \geq L \text{ and } x_4 \leq L\}$ so D is a closed cone with vertex at $\Lambda = (L, L, L, L)$. One can verify that $S(D \cap \text{ int}(K^4)) \subset D \cap \text{int}(K^4)$. If, for $\varepsilon > 0, V_\varepsilon = \{x \in D \big| \|x - \Lambda\| < \varepsilon\}$ one can use the so-called "fixed point index" (see [1,4,12] for expositions.) Standard arguments show that, for ε small and $\frac{(p-q)}{(1+p)(1+q)} > t_*, i_D(S, V_\varepsilon) = 0$. On the other hand, one shows that for G as in Lemma 5.3 $i_D(S, G \cap D) = 1$, so S has a fixed point x in $(G \cap D) \setminus V_\varepsilon$. In fact, if one freezes q with $\left(\frac{1}{1+q}\right) > t_*$ and one views $p \geq q$ as a parameter, abstract global bifurcation theorems as in [12] and [13] are applicable.

6. Global stability for $u_{n+1} = \frac{p+qu_n}{1+u_{n-1}}$.

In this section we change notation, and for $(u, v) \in K^2$ we define $f(u, v)$ by

$$f(u, v) := \frac{p + qv}{1 + u} \tag{6.1}$$

and

$$h(u, v) := f(v, f(u, v)) = \frac{p(1 + u) + q(p + qv)}{(1 + u)(1 + v)}. \tag{6.2}$$

We shall always assume that $p > 0$ and $q > 0$. Sometimes it will be convenient to write $f(u, v) = f_1(u, v), h(u, v) = f_2(u, v)$ and for $j > 2$,

$$f_j(u, v) = f(f_{j-2}(u, v), f_{j-1}(u, v)). \tag{6.3}$$

If $p > 0, q > 0, u_{-1} > 0$ and $u_0 > 0$ and

$$u_{n+1} = f(u_{n-1}, u_n), n \geq 0, \tag{6.4}$$

it has long been conjectured (see [5], [6] and Conjecture 6.10.1 on p. 124 of [8]) that

$$\lim_{n\to\infty} u_n = L := \frac{1}{2}(q - 1) + \frac{1}{2}\sqrt{(q - 1)^2 + 4p}. \tag{6.5}$$

The constant L in equation (6.5) denotes the unique nonnegative solution of $f(L, L) = L$. We shall call this conjecture "the global stability conjecture for equation (6.4)." A simple argument (see Theorem 6.3.3, p. 81, in [8]) proves the global stability conjecture if $0 \leq q < 1$. Results in [5] prove the conjecture when $p < q$; see, also, Theorem 6.3.3 in [8]. Theorem 3.4.3 in [6] proves the conjecture for $q \leq p \leq 2(q + 1)$ and $q \geq 1$.

In this section we shall present a unified approach which generalizes the above results, although it does not yield the full conjecture. Our goal is to prove the following theorem.

Theorem 6.1. *Assume either that* $0 < q \leq 1$ *and* $p > 0$ *or that* $q > 1$ *and*

$$0 < p \leq 2q + \left(\frac{4q^2}{(q-1)^2} \right). \tag{6.6}$$

Then if $u_{-1} > 0, u_0 > 0$ *and* $u_n, n \geq 1$, *is defined by equation (6.4),* $\lim_{n\to\infty} u_n = L$, *where* L *is as in equation (6.5).*

Notice that the right hand side of equation (6.6) is always greater than $2(q+2)$, and for $q-1$ of moderate size it may be substantially larger than $2(q+2)$. For example, if $q = 2$, previous theorems allow $0 \leq p \leq 6$, while Theorem 6.1 allows $0 \leq p \leq 20$.

Because L in equation (6.5) plays an important role in our arguments, it is convenient to take a parametric representation which puts L in a simple form. We write

$$L = q + s. \tag{6.7}$$

and note that, becuase $p > 0, s$ satisfies

$$s > -\min(1, q). \tag{6.8}$$

The reader can use equations (6.5) and (6.7) to verify that

$$p = (q+s)(1+s). \tag{6.9}$$

We shall sometimes use q and s as parameters, rather than p and q.

A simple calculation yields, for $u \geq 0$ and $v \geq 0$,

$$(D_1 f)(u,v) = \frac{-(p+qv)}{(1+u)^2} < 0 \text{ and } (D_2 f)(u,v) = \frac{q}{1+u} > 0. \tag{6.10}$$

One also obtains that

$$(D_1 h)(u,v) = -\frac{q(p+qv)}{(1+v)(1+u)^2} < 0 \tag{6.11}$$

and

$$(D_2 h)(u,v) = \left[\frac{1}{(1+u)(1+v)^2} \right] \left[-p(1+u) + q^2 - pq \right]. \tag{6.12}$$

If $p \geq (q^2/q+1)$, equation (6.12) implies that $(D_2 h)(u,v) < 0$ for all $u \geq 0$ and $v \geq 0$, but in general the sign of $D_2 h(u,v)$ depends on u.

For the reader's convenience we include the proof of the following elementary lemma.

Lemma 6.1. *Assume that $p > 0$ and $q > 0$ and that for $u_{-1} > 0$ and $u_0 > 0, u_n, n \geq 1$, is defined by equation (6.4). Then for all $n \geq 2$ we have*

$$u_n \leq p + q \max(p, q) := b_0. \tag{6.13}$$

For all $n \geq 5$ we have

$$u_n \geq \frac{p + q \min(p, q)}{1 + b_0} := a_0. \tag{6.14}$$

Proof. Because $u \to h(u, v)$ is decreasing for $u \geq 0$ and $v \geq 0$, we see that

$$h(u, v) \leq h(0, v) = \frac{p}{1 + v} + \frac{q(p + qv)}{1 + v} \leq p + \frac{q \max(p, q)(1 + v)}{1 + v} = p + q \max(p, q). \tag{6.15}$$

Since $u_n = h(u_{n-3}, u_{n-2})$ for $n \geq 2$, we deduce equation (6.13) from equation (6.15).

For $n \geq 5$, we know that $u_{n-3} \leq b_0$ and $u_{n-2} \leq b_0$, so

$$\begin{aligned} u_n = h(u_{n-3}, u_{n-2}) \geq h(b_0, u_{n-2}) &= \frac{p}{1 + v} + \frac{2(p + qv)}{(1 + b_0)(1 + v)} \\ &\geq \frac{p}{(1 + b_0)} + \frac{q \min(p, q)}{1 + b_0}, \end{aligned}$$

which establishes equation (6.14) □

We now argue roughly as in Section 2.

Lemma 6.2. For h as in equation (6.2), L as in equation (6.5) and for $0 < a \leq L \leq b$, define functions $\theta_1(a, b)$ and $\theta_2(a, b)$ by

$$\theta_1(a, b) = \min\{h(u, v) : a \leq u \leq b, a \leq v \leq b\}. \tag{6.16}$$

and

$$\theta_2(a, b) = \max\{h(u, v) : a \leq u \leq b, a \leq v \leq b\}. \tag{6.17}$$

Then we have that

$$\theta_1(a, b) = \min\{h(b, a), h(b, b)\} \tag{6.18}$$

and

$$\theta_2(a, b) = \max\{h(a, a), h(a, b)\}. \tag{6.19}$$

If $q^2 - p - pq - pa \leq 0$, $\theta_1(a, b) = h(b, b)$ and $\theta_2(a, b) = h(a, a)$. If $q^2 - p - pq - pa > 0$ and $q^2 - p - pq - pb \leq 0$, $\theta_1(a, b) = h(b, b)$ and $\theta_2(a, b) = h(a, b)$. If $q^2 - p - pq - pa > 0$ and $q^2 - p - pq - pb > 0$, $\theta_1(a, b) = h(b, a)$ and $\theta_2(a, b) = h(a, b)$. It is always the case that $\theta_1(a, b) \leq L \leq \theta_2(a, b)$.

Proof. Equations (6.18) and (6.19) and the other assertions of the lemma follow directly from equation (6.11) and (6.12). Because $a \leq L \leq b$ and $h(L, L) = L$, we also see that $\theta_1(a, b) \leq L \leq \theta_2(a, b)$. □

Lemma 6.3. *Let a_0 and b_0 be as in Lemma 6.1 and θ_1 and θ_2 as in Lemma 6.2. For $k \geq 1$ define a_k and b_k inductively by $a_k = \theta_1(a_{k-1}, b_{k-1})$ and $b_k = \theta_2(a_{k-1}, b_{k-1})$. Then we have $b_n \geq b_{n+1} \geq L$ and $a_n \leq a_{n+1} \leq L$ for all $n \geq 0$. If $u_{-1} > 0, u_0 > 0$ and u_n is defined by equation (6.4), then $a_k \leq u_j \leq b_k$ for all $j \geq 5 + 3k$.*

Proof. The proof of Lemma 6.1 actually shows that for all $u \geq 0, v \geq 0$, we have $h(u, v) \leq b_0$. Also, the proof of Lemma 6.1 shows that $h(u, v) \geq a_0$ for all u, v with $0 \leq u \leq b_0$ and $0 \leq v \leq b_0$. It follows that $a_1 = \theta_1(a_0, b_0) \geq a_0$ and $b_1 = \theta_2(a_0, b_0) \leq b_0$. Since $h(L, L) = L \leq b_0$ and $h(L, L) = L \geq a_0$, Lemma 6.2 implies that $b_1 \geq L \geq a_1$.

We now argue by induction and assume that $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq L$ and $L \leq b_n \leq b_{n-1} \leq \cdots \leq b_1 \leq b_0$ for some $n \geq 1$. By definition of θ_1 and $\theta_2, \theta_1(a_n, b_n) \geq \theta_1(a_{n-1}, b_{n-1})$ because $[a_{n-1}, b_{n-1}] \supset [a_n, b_n]$ and similarly $\theta_2(a_n, b_n) \leq \theta_2(a_{n-1}, b_{n-1})$ because $[a_{n-1}, b_{n-1}] \supset [a_n, b_n]$. It follows that $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$, and Lemma 6.2 implies that $a_{n+1} \leq L \leq b_{n+1}$. Thus we have proved the first part of Lemma 6.3 by mathematical induction.

Lemma 6.1 implies that $a_0 \leq u_j \leq b_0$ for all $j \geq 5$. We argue by induction and assume that for some $k \geq 0$ we have proved that $a_k \leq u_j \leq b_k$ for all $j \geq 5+3k$. If $j \geq 5+3(k+1)$, we can write $u_j = h(u_{j-3}, u_{j-2})$ and $a_k \leq u_{j-3} \leq b_k$ and $a_k \leq u_{j-2} \leq b_k$. By definition of θ_1 and θ_2, it follows that

$$\theta_1(a_k, b_k) = a_{k+1} \leq u_j \leq b_{k+1} = \theta_2(a_k, b_k),$$

so the second part of Lemma 6.3 also follows by mathematical induction. □

Just as in Section 2, if a_n and b_n are as in Lemma 6.3 we see that $\lim_{n\to\infty} a_n = a \leq L$, $\lim_{n\to\infty} b_n = b \geq L$ and

$$\theta_1(a, b) = a \text{ and } \theta_2(a, b) = b. \tag{6.20}$$

Furthermore, Lemma 6.3 implies that if $u_{-1} > 0$ and $u_0 > 0$, and u_n is given by equation (6.4) for $n \geq 1$, then

$$a \leq \liminf_{n\to\infty} u_n \text{ and } \limsup_{n\to\infty} u_n \leq b. \tag{6.21}$$

If we can prove that $(a, b) = (L, L)$ is the only solution (a, b) of equation (6.20) with $0 < a \leq L \leq b$, equation (6.21) implies that $\lim_{n\to\infty} u_n = L$. In our next lemma we analyze equation (6.20).

Lemma 6.4. *Assume that $0 < q \leq 1$ and $p > 0$ or that $q > 1$ and p satisfies equation (6.6). Let L be as in equation (6.5) and θ_1 and θ_2 be functions as in Lemma 6.2. If $0 < a \leq L \leq b$ and $\theta_1(a, b) = a$ and $\theta_2(a, b) = b$, then $a = L = b$.*

Proof. As we have noted if $L = 1+s, s > -\min(1, q)$ and $p = (q+s)(1+s)$. Furthermore, for $q > 1$, the reader can verify that $p > 0$ satisfies inequality (6.6) if and only if

$$-1 < s \leq \left(\frac{q+1}{q-1}\right). \tag{6.22}$$

We assume, by way of contradiction, that equation (6.20) has a solution (a, b) with $0 < a \leq L \leq b$ and $a < b$. Using Lemma 6.2, we distinguish three cases: (1) $q^2 - p - pq - pa \leq 0$,
(2) $q^2 - p - pq - pa > 0$ and $q^2 - p - pq - pb \leq 0$ and
(3) $q^2 - p - pq - pa > 0$ and $q^2 - p - pq - pb > 0$.

Case 1. In this case Lemma 6.2 implies that $\theta_1(a, b) = h(b, b)$ and $\theta_2(a, b) = h(a, a)$. Using equation (6.2) for h we find that

$$(p + qp) + (p + q^2)a = (1 + a)^2 b \tag{6.23}$$

and

$$(p + qp) + (p + q^2)b = (1 + b)^2 a. \tag{6.24}$$

Subtracting equation (6.24) from equation (6.23) and dividing by $(a - b)$ (since $a < b$), we obtain that

$$b = \frac{p + q^2 + 1}{a}. \tag{6.25}$$

Substituting from equation (6.25) for b in equation (6.23) and simplifying yields a quadratic equation for a:

$$a^2 + a[p + 2q^2 + 2 - pq] + (p + q^2 + 1) = 0. \tag{6.26}$$

If $0 < q \leq 1$, the coefficients of a in equation (6.26) are all postive, so equation (6.26) has no positive solution a. Thus we can assume $q > 1$. Using equation (6.24) and equation (6.25), note that $b > a$ also solves equation (6.26), so solving equation (6.26) yields

$$a = -\frac{B}{2} - \frac{\sqrt{R}}{2}, \tag{6.27}$$

where

$$B = p + 2q^2 + 2 - pq \text{ and } R = B^2 - 4(p + q^2 + 1). \tag{6.28}$$

If $B \geq 0$, equation (6.27) implies that the real part of a is not positive, which contradicts our assumption that $a > 0$. Thus we must have $B < 0$ or, equivalently,

$$p > \frac{2(q^2 + 1)}{q - 1} = 2(q + 1) + \frac{4}{q - 1}; \tag{6.29}$$

otherwise we have obtained a contradiction. Equation (6.29) implies that $s > 1$, where $p = (q + s)(1 + s)$. A calculation (use Maple) shows that

$$R = ((q - 1)s - (q + 1))((q - 1)s + q(q + 1))(s^2 + (q + 1)s - q), \tag{6.30}$$

so $R < 0$ if $1 < s < \frac{q+1}{q-1}$ and $R = 0$ if $s = \frac{q+1}{q-1}$. If $R < 0$, we already have a contradiction, and if $R = 0, a = b = -B/2$, which is again a contradiction.

Case 2. In Case 2, Lemma 6.2 implies that $\theta_1(a,b) = h(b,b)$ and $\theta_2(a,b) = h(a,b)$. Arguing as in Case 1 we find that

$$p(1+b) + q(p+qb) = a(1+b)^2 \tag{6.31}$$

and

$$p(1+a) + q(p+qb) = b(1+a)(1+b). \tag{6.32}$$

Subtracting equation (6.32) from equation (6.31) yields

$$p(b-a) = -(1+b)(b-a),$$

which implies that $1 + b = -p$, a contradiction.

Case 3. In case 3, Lemma 6.2 implies that $\theta_1(a,b) = h(b,a)$ and $\theta_2(a,b) = h(a,b)$. Thus we find that

$$(p+qp) + pa + q^2b = (1+a)(1+b)b \tag{6.33}$$

and

$$(p+qp) + pb + q^2a = (1+a)(1+b)a. \tag{6.34}$$

Subtracting equation (6.34) from equation (6.33) gives

$$q^2 - p = (1+a)(1+b) \text{ and } b = \frac{q^2-p-1-a}{1+a}. \tag{6.35}$$

Substituting from equation (6.35) in equation (6.33) and simplifying yields

$$a^2 + (q+1)a + (q^2+q-p) = 0. \tag{6.36}$$

If $p \geq q^2+q$, we are in case 1, so we can assume that $q^2+q-p > 0$, in which case equation (6.36) clearly has no solution $a > 0$. □

Proof of Theorem 6.1. By our previous remark it suffices to prove that under the given assumption, equation (6.20) has no solution (a,b) with $0 < a \leq L \leq b$ and $a < b$. However, this is the content of Lemma 6.4. □

Remark 6.1. Given $k \geq 2$, one can define, for $0 < a \leq L \leq b$,

$$\psi_1(a,b) = \min\{f_k(u,v) : a \leq u \leq b \text{ and } a \leq v \leq b\}$$

and

$$\psi_2(a,b) = \max\{f_k(u,v) : a \le u \le b \text{ and } a \le v \le b\}.$$

Suppose that $0 < q \le p$ and k are such that if

$$\psi_1(a,b) = a \text{ and } \psi_2(a,b) = b,$$

then $a = b = L$. For this p and q and for $u_{-1} \ge 0, u_0 \ge 0$ and u_j given by equation (6.4), it then follows by our previous arguments that $\lim_{j\to\infty} u_j = L$. One might assume that as k increases, the results obtained in this way automatically improve, however this is **not** the case. For example, the results obtained by choosing $k = 3$ are worse than than those obtained by taking $k = 2$.

Some insight can be obtained by a linear analysis at the point (L, L). To obtain positive results for a given k, it is necessary that the map $(a,b) \to (\psi_1(a,b), \psi_2(a,b)) := \Psi(a,b)$ be locally stable at (L, L). One can prove that a necessary condition for the local stability of Ψ at (L, L) is that

$$|D_1 f_k(L,L)| + |D_2 f_k(L,L)| \le 1, \tag{6.37}$$

and strict inequality in equation (6.37) is a sufficient condition for local stability. If we write $\alpha_k = D_1 f_k(L,L)$ and $\beta_k = D_2 f_k(L,L)$, so (using the parametrization $L = q + s$ and $p = (q+s)(1+s)$ for $s \ge 0$) $\alpha_1 = -\left(\frac{q+s}{q+s+1}\right)$ and $\beta_1 = \frac{q}{q+s+1}$, one can prove that, for $k \ge 1$,

$$M_k := \begin{bmatrix} \alpha_{k-1} & \alpha_k \\ \beta_{k-1} & \beta_k \end{bmatrix} = M_1^k = \begin{bmatrix} 0 & \alpha_1 \\ 1 & \beta_1 \end{bmatrix}^k. \tag{6.38}$$

Here we make the convention that $\alpha_0 = 0$ and $\beta_0 = 1$.

The eigenvalues of M_1 are

$$z = \frac{q}{2(q+s+1)} \pm \frac{i}{2(q+s+1)}\sqrt{4(q+s)(q+s+1) - q^2}, \tag{6.39}$$

so $|z|^2 = \frac{q+s}{q+s+1}$. Notice that if $s = o(q)$ and q is large, z is approximately equal to $|z|\exp(i\pi/3)$. For z as in equation (6.37), one can calculate that equation (6.37) is satisfied if and only if

$$\frac{1}{|Im(z)|}[|\alpha_1||Im(z^k)| + |Im(z^{k+1})|] \le 1. \tag{6.40}$$

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Appendix A.

We describe below a list of Maple 10 instructions which implements the sequence of steps in Lemma 4.5 and computes the polynomial w_1 in equation (4.34). The polynomial w_1 in the variables q, r and z has several thousand terms with integer coefficients, so it is important to note that, after w_1 has been put in appropriate form, the instruction min(coeffs (w_1)) computes the minimum of these coefficients and obviates the need to print the full polynomial. One can, of course, replace any or all colons by semicolons below to have Maple print out u_j, v_j and w_1. We denote by W_1 below the polynomial w_1 evaluated at $q = 0$. Maple will verify that the minimum coefficient of w_1 and of W_1 is 1.

1. $p := q + r$:
2. $L := (1 + p)/(1 + q)$:
3. $x := L * (1 + z)$:
4. $u1 := 1 + p + z$:
5. $v1 := 1 + q + z$:
6. $u2 := (1 + q) * u1 + p * (1 + p) * (1 + z) * v1$:
7. $v2 := u2 + (q - p) * (1 + p) * (1 + z) * v1$:
8. $u3 := (1 + p) * (1 + z) * v2 + p * (1 + q) * u2$:
9. $v3 := u3 + (q - p) * (1 + q) * u2$:
10. $u3 :=$ normal $(u3)$:
11. $v3 :=$ normal $(v3)$:
12. $u4 := u2 * v3 + p * u3 * v2$
13. $v4 := u4 + (q - p) * u3 * v2$:
14. $u4 :=$ normal $(u4)$:
15. $v4 :=$ normal $(v4)$:
16. $u5 := u1 * v4 + p * u4 * v1$:
17. $v5 := u5 + (q - p) * u4 * v1$:
18. $u5 :=$ normal $(u5)$:
19. $v5 :=$ normal $(v5)$:
20. $u6 := u4 * v5 + p * u5 * v4$:
21. $v6 := u6 + (q - p) * u5 * v4$:
22. $u6 :=$ normal $(u6)$:
23. $v6 :=$ normal $(v6)$:
24. $w1 := (1 + p) * (1 + z) * v6 - (1 + q) * u6$:
25. $w1 :=$ normal $(w1)$:
26. $w1 :=$ expand $(w1)$:
27. min(coefficients $(w1)$);
28. $W1 :=$ eval $(w1, q = 0)$:
29. $W1 :=$ normal $(W1)$:
30. $W1 :=$ expand $(W1)$:
31. min(coefficients $(W1)$);

To obtain the polynomial w_2 in equation (4.41), we follow instructions 1-15 above and then replace instructions 16-31 by the instructions below. We denote by W_2 below the polynomial w_2 evaluated at $q = 0$. Again, Maple will verify that the minimum coefficient of w_2 and of W_2 is 1.

16. $U5 := (1 + p) * (1 + z) * v4 + p * (1 + q) * u4 :$
17. $V5 := U5 + (q - p) * (1 + q) * u4 :$
18. $U5 :=$ normal $(U5) :$
19. $V5 :=$ normal $(V5) :$
20. $U6 := u4 * V5 + p * U5 * v4 :$
21. $V6 := U6 + (q - p) * U5 * v4 :$
22. $U6 :=$ normal $(U6) :$
23. $V6 :=$ normal $(V6) :$
24. $U7 := U5 * V6 + p * U6 * V5 :$
25. $V7 := U7 + (q - p) * U6 * V5 :$
26. $U7 :=$ normal $(U7) :$
27. $V7 :=$ normal $(V7) :$
28. $w2 := (1 + p) * (1 + z) * V7 - (1 + q) * U7 :$
29. $w2 :=$ normal $(w2) :$
30. $w2 :=$ expand $(w2) :$
31. min(coefficients $(w2)$);
32. $W2 :=$ eval $(w2, q = 0) :$
33. $W2 :=$ normal $(W2) :$
34. $W2 :=$ expand $(W2) :$
35. min(coefficients $(W2)$);

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Mathematics Department, Rutgers University, Hill Center, Busch Campus, Piscataway, NJ 08854-8019

E-mail address: nussbaum@math.rutgers.edu