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Boundary layer phenomena for differential-delay equations with state dependent time lags: II

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We study the limiting shape of solutions of the singularly perturbed differential-delay equation

$$\varepsilon \dot{x}(t) = f\big(x(t), x(t-r)\big), \quad r = r\big(x(t)\big) \tag{1}$$

as $\varepsilon \to 0$. More precisely, we take a sequence $x^n(t)$ of solutions of (1) for $\varepsilon = \varepsilon^n \to 0$, and consider the set $\Omega \subseteqq \mathbb{R}^2$ defined as the limit, in the Hausdorff sense, of the corresponding sequence of graphs $\Gamma^n \subseteqq \mathbb{R}^2$. Using the geometry of Ω, we make precise the sense in which points $(\tau^k, \xi^k) \in \Omega$ satisfy the difference equation

$$0 = f(\xi^k, \xi^{k-1}), \quad \tau^{k-1} = \tau^k - r(\xi^k), \tag{2}$$

thereby providing a means for determining Ω.

In the particular case that $x^n(t)$ is a sequence of slowly oscillating periodic solutions and f and r satisfy appropriate hypotheses, we use this theory with subtle scaling arguments to show that $\Omega \neq \mathbb{R} \times \{0\}$, or equivalently, for the sup norm,

$$\liminf_{n \to \infty} \|x^n\| > 0 . \tag{3}$$

Despite its pedestrian appearance, (3) is the crucial first step in determining Ω exactly.

We also prove generally that it the delay function $r(x)$ is not constant on any x-interval, and if $f(x,y)$ is piecewise monotone in its first argument, the set Ω is almost a graph. More precisely, for all but a countable set of $t \in \mathbb{R}$ the vertical slices $\Omega_t = \{(\tau, \xi) \in \Omega \mid \tau = t\}$ are single points. This is in marked contrast to the case of a constant delay, where it is possible for Ω_t to be a nontrivial interval for all real t.

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1. Introduction

This paper continues our study, initiated in [MP-Nu4], of singularly perturbed delay differential equations with state-dependent delays. Equations with history-dependent terms arise in many models in a variety of areas of science; see for example [Bé-Mack1], [Lo], [Lo-Mi1], [Lo-Mi2], [Lo-Mi3], [Ma], [Mack-Gl], [Mack-adH], [Mack-Mi1], [WC-La], and also the references in [MP-Nu1] and [MP-Nu4]. After a long period following the pioneering work of Driver [Dr1], [Dr2], [Dr3], [Dr4], [Dr-No], and later work of Nussbaum [Nu] and Alt [Al1], [Al2], there has recently been a resurgence of interest in models in which the delay itself depends on the state (solution) $x(t)$ of the system. While a great deal is known about the constant time-lag case, very few theoretical results are known about the case of such a state-dependent delay.

Our basic objective is to obtain detailed information about solutions of equations such as

$$\varepsilon \dot{x}(t) = f\big(x(t), x(t-r)\big), \quad r = r\big(x(t)\big) \tag{1.1}$$

for small $\varepsilon > 0$ through an analysis of the implicitly defined difference equation

$$0 = f(x^n, x^{n-1}), \quad t^n = t^{n-1} + r(x^n), \tag{1.2}$$

obtained as a formal limit of (1.1). While the difference equation (1.2) itself can itself possess a rich and complex structure, it is essentially a low-dimensional object amenable to a concrete analysis, which one hopes can shed light on the infinite-dimensional dynamical system generated by (1.1). Indeed, the introduction of the singular parameter ε provides a concrete mechanism with which one can analyze and understand the dynamics of such a differential-delay equation, just as an analysis of the Hopf bifurcation (see, for example [Fr-St], [St1], [St2]) does so in another region of parameter space. While the ultimate goal of a complete understanding of the attractor and its bifurcations will probably never be realized (and indeed, such a goal is still quite remote even for the much-studied Wright's equation

$$\dot{x}(t) = -\alpha x(t-1)\big(1 + x(t)\big);$$

see [Wr], [Hal-VL]), studies for various ranges of parameters provide at least a partial view of the complete picture.

The singular perturbation approach has been pursued by a number of authors (see [Iv1], [Iv2], [Iv-Sh], [La-Mi], [MP-Nu1], [MP-Nu2], [MP-Nu3], [P'e-Sh]) in the case of equations with a constant delay, where the equation

$$\varepsilon \dot{x}(t) = -x(t) + \tilde{f}\big(x(t-1)\big) \tag{1.3}$$

has been studied in relation to the difference equation

$$x^n = \tilde{f}(x^{n-1}). \tag{1.4}$$

The results of these studies clearly show that great care must be taken. For example, robust dynamical structures in (1.4) such as stable periodic orbits need not give rise to analogous structures in (1.3) even for small ε; see [MP-Nu2], [MP-Nu3]. On the other hand, many periodic solutions of (1.3) do exist and correspond in a natural way to periodic points of (1.4).

Slowly oscillating periodic solutions (SOPS's), namely solutions satisfying

$$\begin{aligned}
&x(0) = x(q^1) = x(q^2) = 0\,, \\
&x(t) > 0 \quad \text{for } 0 < t < q^1\,, \\
&x(t) < 0 \quad \text{for } q^1 < t < q^2\,, \\
&x(t + q^2) = x(t) \quad \text{for all } t \in \mathbb{R}
\end{aligned}$$

for quantities q^1, q^2 satisfying

$$q^1 > r(0) > 0 \quad \text{and} \quad q^2 - q^1 > r(0)$$

are the most important class of solutions of (1.1). Indeed, for reasons which remain incompletely understood (although results in this direction are found in [MP]), even in the constant time-lag case the SOPS's frequently exhibit strong global stability properties, at least numerically. If $\tilde{f}$ satisfies a negative feedback condition $x\tilde{f}(x) < 0$ for $x \neq 0$, and an instability condition $\tilde{f}'(0) < -1$ at the origin, along with technical conditions of smoothness and boundedness, then the existence of SOPS's of equation (1.3) for small ε was proved in [Had-To], [MP-Nu1]. Analogous existence results for equation (1.1), with

$$f(x,y) = -x + \tilde{f}(y)\,,$$

were proved in [MP-Nu4]. In [MP-Nu1] there was given a precise description, in terms of transition (boundary) layers, of the asymptotic shape of SOPS's of the constant delay problem (1.3) as $\varepsilon \to 0$.

The present paper achieves three main objectives. The first is to develop a general theory of so-called "limiting profiles" and transition layers for solutions of singularly perturbed delay differential equations. Beginning with any bounded sequence of (not necessarily periodic) solutions $x^n(t)$ of (1.1), for $\varepsilon = \varepsilon^n \to 0$, we take the limit $\Gamma^n \to \Omega$ of the graphs $\Gamma^n \subseteq \mathbb{R}^2$ of $x^n(\cdot)$ in the appropriate Hausdorff sense. The set Ω, termed the limiting profile, can contain vertical line segments and so need not be the graph of a function. Subsets $\Omega^{\pm} \subseteq \Omega$ corresponding to transition layers are identified, and the relation to the difference equation (1.2) is made precise. This theory, which is developed in Section 2, is presented in enough generality to include systems (so $x \in \mathbb{R}^N$) with multiple delays of the form

$$\begin{aligned}
&\varepsilon\dot{x}(t) = f\big(t, x(t), x(t - r_1), \ldots, x(t - r_M)\big)\,, \\
&r_k = r_k\big(t, x(t)\big), \quad 1 \leqq k \leqq M\,.
\end{aligned} \tag{1.5}$$

Such a degree of generalization is in fact needed in our later application of the theory to SOPS's of (1.1).

The second main objective of this paper is to prove, in a precise sense, that solutions of the state-dependent equation (1.1) with non-constant delay r generally are simpler than solutions of the same equation

$$\varepsilon \dot{x}(t) = f(x(t), x(t-1)) \tag{1.6}$$

with a constant delay. By "simpler" we mean that the limiting profile Ω of a state-dependent delay problem is almost a graph, in a sense described in Corollary 3.2. This notion is indeed paradoxical, as state-dependent problems are generally regarded as more technically complex than those with constant delay. A philosophical interpretation of our result is that the state-dependent problem (1.1) is more generic, or robust, than the problem (1.6) with constant delay. We believe this result, and the theory of Section 2, will in the long run provide a powerful tool for the analysis of solutions of the system (1.5) under quite general conditions.

Our third objective provides initial steps in the direction of such an analysis. We use the theory of Section 2 to describe, at least partially, the asymptotic shape of SOPS's of (1.1) as $\varepsilon \to 0$. Specifically, we show for any sequence $x^n(\cdot)$ of SOPS's that

$$\liminf_{n \to \infty} \|x^n\| > 0 \tag{1.7}$$

where $\|x^n\|$ denotes the sup norm of $x^n(\cdot)$, with $\varepsilon = \varepsilon^n \to 0$, and f in (1.1) satisfies conditions guaranteeing existence of such solutions as described above. The proof of the result (1.7) is by no means trivial; it employs delicate scaling arguments in which our theory of limiting profiles is used extensively. These are given in Section 5 and 6, following some modest generalization of existence and monotonicity results of [MP-Nu4] given in Section 4.

A complete description of the asymptotics of SOPS's of (1.1) is beyond the scope of this paper, but will be addressed in a subsequent paper [MP-Nu5]. Our results here, in particular (1.7), are the necessary first step to such a rigorous description. Nevertheless it is amusing to note that the equation (1.1) can easily be explored numerically. With a simple home computer and a program of no more than 20 lines of BASIC, the interested reader will observe a wide variety of limiting profiles. Moreover, much of the shape and structure of the limiting profile can often be discerned at moderate values of ε, that is, ε need not be taken too small before a pattern emerges. The simplest nontrivial example of a limiting profile is perhaps furnished by the choice

$$f(x, y) = -x - ky, \quad r(x) = 1 + cx$$

of nonlinearity and delay, with constants $k > 1$ and $c > 0$. In this case one observes a "sawtooth" limiting profile comprised first of the two line segments

$$\left\{ (t, x) \,\middle|\, t = 0 \text{ and } -\frac{1}{c} \leqq x \leqq \frac{k}{c} \right\}$$

and

$$\left\{(t,x)\,\middle|\,x=\frac{t-1}{c}\ \text{for}\ 0\leqq t\leqq k+1\right\},$$

which is then extended in the (t, x) plane to be periodic of period $k+1$ in t.

In a subsequent paper [MP-Nu 5] a broad class of equations, including the above sawtooth example, will be analyzed, and their limiting profiles rigorously determined. The inequality (1.7), to be established in the present paper, will provide a crucial element in determining such limiting profiles. We in fact expect that the proof of (1.7) given herein contains the bulk of the technical arguments needed for the complete analysis of (1.1) in [MP-Nu 5].

2. The limiting profile Ω

In this section we develop a geometric theory for describing the asymptotic shape of solutions of singularly perturbed delay differential equations. Typically, this description involves a finite dimensional map associated with the differential equation. In the simplest case of a scalar equation with a single constant delay, this map is one-dimensional. In the case of current interest, a scalar equation with a state-dependent delay, the map is two-dimensional.

We are interested in sequences

$$\varepsilon^n\dot{x}^n(t)=f^n(x^n(t),x^n(t-r^n)),\quad r^n=r^n(x^n(t))\,, \tag{2.1}$$

of delay differential equations, where

$$\begin{gathered} f^n:\mathbb{R}^2\to\mathbb{R}\quad\text{and}\quad r^n:\mathbb{R}\to\mathbb{R}\quad\text{are continuous}\,,\\ \varepsilon^n>0\quad\text{and}\quad\varepsilon^n\to 0\,. \end{gathered} \tag{2.2}$$

Here $x^n(\cdot)$ is a sequence of solutions satisfying (2.1) for $t\in\mathbb{R}$ with the boundedness condition

$$|x^n(t)|\leqq C=C(I)\quad\text{for all } n, \text{ and } t\in I\,, \tag{2.3}$$

for each compact interval $I\subseteq\mathbb{R}$, for some constant $C(I)$. In addition,

$$\lim_{n\to\infty} f^n(x,y)=f(x,y)\quad\text{and}\quad\lim_{n\to\infty} r^n(x)=r(x) \tag{2.4}$$

exist uniformly on compact subsets of $\mathbb{R}^2$ and $\mathbb{R}$ respectively, where $f:\mathbb{R}^2\to\mathbb{R}$ and $r:\mathbb{R}\to\mathbb{R}$ are continuous.

In studying systems of the above form, the sequence of solutions $x^n(\cdot)$ is assumed given, having been obtained beforehand. In specific cases of interest, of course, the nonlinearities f^n and r^n will satisfy additional properties (such as smoothness, feedback conditions for f^n, and nonnegativity of r^n), as will the solutions $x^n(\cdot)$. However, these properties are not needed for the general theory of this section, so are not assumed.

In fact, to develop some of the theory of this section, in particular to define the limiting profile Ω of a sequence $x^n(\cdot)$ and to establish some of its elementary properties, it is not necessary at all to assume that $x^n(\cdot)$ is the solution to a differential equation. All that is needed for much of this theory is that each $x^n: \mathbb{R} \to \mathbb{R}$ is a continuous function and that the boundedness property (2.3) holds. We take this as a basic assumption throughout this section.

Standing assumption. Throughout this section, unless stated otherwise, we assume that $x^n(\cdot)$ is a sequence of continuous functions $x^n: \mathbb{R} \to \mathbb{R}$, satisfying the bound (2.3) for each compact interval $I \subseteq \mathbb{R}$, for some constant $C(I)$.

We do eventually assume all of the conditions (2.1) through (2.4) later in this section, in order to fully develop the theory of limiting profiles. However, we shall explicitly note in this section any assumptions made beyond the above standing assumption, or any deviation from it.

We shall also show how our theory extends to more general classes of equations, the most important of which (for our present purposes) is to systems of equations. For simplicity of exposition, we present in full only the case of scalar x; at the end of this section we describe the easy modifications needed to consider the case of vector $x \in \mathbb{R}^N$, as well as generalizations in other directions, such as multiple delays.

Let $\Gamma^n \subseteq \mathbb{R}^2$ denote the graph

$$\Gamma^n = \{(t, x^n(t)) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$$

of the function $x^n(\cdot)$, and define the limit set $\Omega \subseteq \mathbb{R}^2$ of the sequence $x^n(\cdot)$ by

$$\Omega = \bigcap_{m=1}^{\infty} \overline{\bigcup_{n=m}^{\infty} \Gamma^n} = \{p \in \mathbb{R}^2 \mid \text{there exists } p^{n^i} \in \Gamma^{n^i} \text{ with } p^{n^i} \to p, \text{ for some subsequence } n^i \to \infty\}. \tag{2.5}$$

The set Ω, and its geometrical properties, are our central objects of study, and we shall refer to Ω as the *limiting profile* of the sequence $x^n(\cdot)$. Also define

$$\Omega_S = \{(\tau, \xi) \in \Omega \mid \tau \in S\} = \Omega \cap (S \times \mathbb{R})$$

for any subset $S \subseteq \mathbb{R}$, and denote simply

$$\Omega_\tau = \Omega_{\{\tau\}}$$

for any $\tau \in \mathbb{R}$.

Proposition 2.1. *If S is closed then Ω_S is closed. If S is compact then Ω_S is compact. If $S \neq \emptyset$ then $\Omega_S \neq \emptyset$, so in particular $\Omega_\tau \neq \emptyset$ for each $\tau \in \mathbb{R}$.*

Proof. We omit the elementary proof, save to note the uniform bound (2.3) is used. □

We recall here the Hausdorff metric on compact sets. Let (X, d) be a metric space, and let

$$\mathscr{K} = \{K \subseteq X \mid K \neq \emptyset \text{ is compact}\}$$

be the set of nonempty compact subsets of X. Define

$$\bar{d}(K^1, K^2) = \max\{\delta(K^1, K^2), \delta(K^2, K^1)\}$$

where

$$\delta(K^1, K^2) = \sup_{\xi \in K^1} d(\xi, K^2)$$

with $d(\xi, K^2)$ denoting the usual distance from a point to a set. Then $(\mathscr{K}, \bar{d})$ is a metric space. Moreover, $(\mathscr{K}, \bar{d})$ is compact if the space (X, d) is compact; see [Mi], Theorem 4.2 and [Vi].

Definition. The sequence of functions $x^n(\cdot)$ is called *regular* if there exist compact sets

$$K^1 \subseteq K^2 \subseteq \cdots \subseteq \mathbb{R}^2$$

with

$$\bigcup_{j=1}^{\infty} \mathring{K}^j = \mathbb{R}^2, \tag{2.6}$$

where $\mathring{K}^j$ denotes the interior of K^j, such that for each j the sequence $\{\Gamma^n \cap K^j\}_{n=1}^{\infty}$ converges in the Hausdorff metric. (We assume for each j that $\Gamma^n \cap K^j \neq \emptyset$ for all large n.)

Remark. If $x^n(\cdot)$ is a regular sequence, and K^j is as in the above definition, then denote

$$\Omega^j = \lim_{n \to \infty} \Gamma^n \cap K^j \tag{2.7}$$

as the limit in the Hausdorff metric. It is not in general the case that $\Omega^j = \Omega \cap K^j$. For example, with $x^n(t) = \arctan((t-2)/\varepsilon^n)$ and $K^j = [-j, j] \times [-j, j]$, the set $\Omega \cap K^2$ contains the line segment $\{2\} \times [-\pi/2, \pi/2]$, while Ω^2 contains $\{2\} \times [-\pi/2, 0]$ but does not contain $\{2\} \times (0, \pi/2]$. Quite generally, we have the following result.

Lemma 2.2. *Let $x^n(\cdot)$ be any sequence of functions, and let $K \subseteq \mathbb{R}^2$ be compact. Assume that either the limit $\lim_{n \to \infty} \Gamma^n \cap K = \tilde{\Omega}$ exists in the Hausdorff metric (with $\Gamma^n \cap K \neq \emptyset$ for large n), or else that $\Gamma^n \cap K = \emptyset$ for all large n. In the latter case set $\tilde{\Omega} = \emptyset$. Then*

$$\overline{\Omega \cap \mathring{K}} \subseteq \tilde{\Omega} \subseteq \Omega \cap K. \tag{2.8}$$

Proof. Observe that one only need prove $\Omega \cap \mathring{K} \subseteq \tilde{\Omega}$ to establish the first inclusion in (2.8), as $\tilde{\Omega}$ is closed. To do this, take any $p \in \Omega \cap \mathring{K}$. Then by the definition (2.5) of Ω, there exist points $p^{n^i} \in \Gamma^{n^i}$, for some subsequence n^i, with $p^{n^i} \to p$. Without loss $p^{n^i} \in \Gamma^{n^i} \cap K$ for each n^i, and so $d(p^{n^i}, \tilde{\Omega}) \to 0$. Thus $d(p, \tilde{\Omega}) = 0$, hence $p \in \tilde{\Omega}$ as $\tilde{\Omega}$ is closed.

Now assume $p \in \tilde{\Omega}$. Then as $d(p, \Gamma^n \cap K) \to 0$, there exist points $p^n \in \Gamma^n \cap K$ such that $p^n \to p$. Thus $p \in \Omega \cap K$ by the definition of Ω, and because K is closed. This establishes the second inclusion in (2.8). □

Proposition 2.3. *Every sequence $x^n(\cdot)$ of functions, as above, has a regular subsequence.*

Proof. Let $K^j = [-j,j] \times [-j,j]$. Then in light of the bound (2.3) there exists j_0 such that $\Gamma^n \cap K^j \neq \emptyset$ for all n if $j \geqq j_0$. For any fixed $j \geqq j_0$, each subsequence of $\{\Gamma^n \cap K^j\}_{n=1}^{\infty}$ has a further subsequence which converges in the Hausdorff metric; this follows from the compactness of the space

$$\mathscr{K}^j = \{K \subseteqq K^j \mid K \neq \emptyset \text{ is compact}\} .$$

An elementary diagonalization argument thus produces a single subsequence $n^i \to \infty$, such that for each $j \geqq j_0$ the sequence $\{\Gamma^{n^i} \cap K^j\}_{i=1}^{\infty}$ converges. This implies that $x^{n^i}(\cdot)$ is a regular subsequence, as required. □

Lemma 2.4. *Let $x^n(\cdot)$ be a regular sequence. Assume for some subsequence $n^i \to \infty$ that there exists $p^{n^i} \in \Gamma^{n^i}$ such that $p^{n^i} \to p \in \Omega$ for some p. Then there exists $p^n \in \Gamma^n$ for each $n \geqq 1$ such that $p^n \to p$, where $\{p^{n^i}\}$ is a subsequence of $\{p^n\}$. That is, the subsequence $\{p^{n^i}\}$ can be extended to a full sequence $\{p^n\}$ which still converges to the point p.*

Proof. By (2.6), there exists j such that $p \in \mathring{K}^j$, where $p \in \Omega$ is as in the statement of the lemma. Let this j be fixed for the remainder of the proof, and let the compact set $\Omega^j \subseteqq \mathbb{R}^2$ be defined as the limit (2.7) in the Hausdorff metric. Then, as

$$p^{n^i} \in \Gamma^{n^i} \cap \mathring{K}^j \subseteqq \Gamma^{n^i} \cap K^j$$

for all sufficiently large n^i, it follows that $p \in \Omega^j$. Therefore, (2.7) implies that one may choose for each $n \geqq 1$ a point $p^n \in \Gamma^n \cap K^j \subseteqq \Gamma^n$ such that $p^n \to p$; moreover, without loss p^n can be chosen so that $\{p^{n^i}\}$ is a subsequence of $\{p^n\}$. This completes the proof. □

Remark. Lemma 2.4 implies that

$$\Omega = \{p \in \mathbb{R}^2 \mid \text{there exists } p^n \in \Gamma^n \text{ with } p^n \to p\} \tag{2.9}$$

if $x^n(\cdot)$ is a regular sequence.

Proposition 2.5. *Let $x^n(\cdot)$ be a regular sequence, and let $I \subseteqq \mathbb{R}$ be an interval or a single point. Then the set Ω_I is connected. In particular, for each $\tau \in \mathbb{R}$*

$$\Omega_\tau = \{\tau\} \times [\underline{x}(\tau), \bar{x}(\tau)] \tag{2.10}$$

for some quantities $\underline{x}(\tau) \leqq \bar{x}(\tau)$.

Proof. First observe that it is sufficient to consider the case when I is a compact interval. This follows from the fact that any interval I can be written as a union $I = \bigcup_{j=1}^{\infty} I^j$ of nested compact intervals $I^1 \subseteqq I^2 \subseteqq \cdots$. Having established the connectedness of Ω_{I^j}, the connectedness of the nested union $\Omega_I = \bigcup_{j=1}^{\infty} \Omega_{I^j}$ is immediate.

We therefore assume $I = [\tau^1, \tau^2]$ is a compact interval (or a point). Fix

$$M = C([\tau^1 - 1, \tau^2 + 1]),$$

where C is the bound (2.3), and for each m let

$$\Gamma^{n,m} = \Gamma^n \cap \left(\left[\tau^1 - \frac{1}{m}, \tau^2 + \frac{1}{m} \right] \times [-M, M] \right).$$

Then there is a subsequence $\{n^i\}$ such that for each m the limit $\lim_{i \to \infty} \Gamma^{n^i,m} = \Omega^m$ exists. The set Ω^m so defined is compact and connected, being the limit of a sequence of compact connected sets. Moreover as $\Omega^1 \supseteqq \Omega^2 \supseteqq \cdots$, the intersection $\Omega^\infty = \bigcap_{m=1}^{\infty} \Omega^m$ is also connected.

Clearly $\Omega^m \subseteqq \Omega$ for each m, hence $\Omega^\infty \subseteqq \Omega$; in fact one sees $\Omega^\infty \subseteqq \Omega_I$. To complete the proof of the proposition we show $\Omega_I \subseteqq \Omega^\infty$. Let $p \in \Omega_I$; then (2.9), which follows from the regularity of the sequence $x^n(\cdot)$, implies there exist $p^n \in \Gamma^n$ with $p^n \to p$. For each m we have $p^n \in \Gamma^{n,m}$ for sufficiently large n, hence $p \in \Omega^m$. Thus $p \in \Omega^\infty$. □

Remark. The functions $\underline{x}$ and $\bar{x}$ defined by (2.10) are lower semi-continuous and upper semi-continuous respectively; this is equivalent to the fact that Ω is a closed set, given that each Ω_τ has the form (2.10).

With $\underline{x}$ and $\bar{x}$ given by (2.10), we shall denote

$$\underline{\bar{x}}(\tau) = [\underline{x}(\tau), \bar{x}(\tau)].$$

Thus $\underline{\bar{x}}$ is a set-valued function, and provided $x^n(\cdot)$ is regular we have

$$\Omega_\tau = \{\tau\} \times \underline{\bar{x}}(\tau).$$

Let us also denote

$$\underline{x}(\tau^0 \pm 0) = \liminf_{\tau \to \tau^0 \pm} \underline{x}(\tau), \quad \bar{x}(\tau^0 \pm 0) = \limsup_{\tau \to \tau^0 \pm} \bar{x}(\tau), \tag{2.11}$$

$$\underline{\bar{x}}(\tau^0 \pm 0) = [\underline{x}(\tau^0 \pm 0), \bar{x}(\tau^0 \pm 0)], \tag{2.12}$$

and

$$\Omega_{\tau^0 \pm 0} = \{\tau^0\} \times \underline{\bar{x}}(\tau^0 \pm 0). \tag{2.13}$$

Lemma 2.6. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a lower semi-continuous function which is bounded above on compact intervals. Then $\{\tau \in \mathbb{R} \mid \alpha$ is continuous at $\tau\}$ is a dense $\mathscr{G}_\delta$ set. The corresponding result for an upper semi-continuous function $\beta : \mathbb{R} \to \mathbb{R}$ which is bounded below on compact intervals also holds.*

Remark. Recall that a $\mathscr{G}_\delta$ set in a topological space X is a set $G \subseteq X$ which is a countable intersection $G = \bigcap_{m=1}^{\infty} G^m$ of open subsets $G^m \subseteq X$.

Proof. We consider only the case of the lower semi-continuous function α. Define $j : \mathbb{R} \to [0, \infty)$ by

$$j(\tau) = \left(\lim_{\varepsilon \to 0} \sup_{|t-\tau|<\varepsilon} \alpha(t) \right) - \alpha(\tau)$$

and observe that $j(\tau) = 0$ if and only if α is continuous at τ. Also observe that the function j is upper semi-continuous. (Both observations are easily proved using the lower semi-continuity of α.) Therefore, for each integer $m \geqq 1$ the set

$$F^m = \left\{ \tau \in \mathbb{R} \,\middle|\, j(\tau) \geqq \frac{1}{m} \right\}$$

is closed. To prove the lemma it is sufficient to prove for each m that the set F^m is nowhere dense; for then

$$\bigcap_{m=1}^{\infty} (\mathbb{R} \backslash F^m) = \{\tau \in \mathbb{R} \mid \alpha \text{ is continuous at } \tau\}$$

is a dense $\mathscr{G}_\delta$ set.

Suppose some F^m is not nowhere dense, and so contains an interval $[\tau^1, \tau^2] \subseteq F^m$ with $\tau^1 < \tau^2$. Then for each $\tau \in [\tau^1, \tau^2]$ and $\varepsilon > 0$, there exists τ' with $|\tau' - \tau| < \varepsilon$ and $\alpha(\tau') - \alpha(\tau) \geqq 1/(2m)$. Therefore, it is possible to choose a sequence $\tau^n \in (\tau^1, \tau^2)$ with $\lim_{n \to \infty} \alpha(\tau^n) = \infty$. But this contradicts the boundedness of α, and completes the proof. □

Proposition 2.7. *Suppose there exists a set $\hat{\Omega} \subseteq \mathbb{R}^2$ such that whenever $x^{n^i}(\cdot)$ is a regular subsequence of $x^n(\cdot)$, then $\Gamma^{n^i} \to \hat{\Omega}$ (in the sense of (2.5), but with n^i replacing n and $\hat{\Omega}$ replacing Ω, that is, $\hat{\Omega} = \bigcap_{m=1}^{\infty} \overline{\bigcup_{i=m}^{\infty} \Gamma^{n^i}}$). Then $x^n(\cdot)$ is in fact a regular sequence, and so $\Omega = \hat{\Omega}$.*

Proof. As there does exist a regular subsequence $x^{n^i}(\cdot)$ (by Proposition 2.3), the set $\hat{\Omega}$ enjoys property (2.10) for some locally bounded functions $\underline{x}, \bar{x} : \mathbb{R} \to \mathbb{R}$, which are lower and upper semi-continuous respectively. If $\sigma < \tau$ both belong to the set

$$G = \{t \in \mathbb{R} \mid \text{both } \underline{x}(\cdot) \text{ and } \bar{x}(\cdot) \text{ are continuous at } t\},$$

and if $M > C([\sigma, \tau])$ with C as in (2.3), then one easily sees the compact set $K = [\sigma, \tau] \times [-M, M]$ satisfies

$$\overline{\hat{\Omega} \cap \overset{\circ}{K}} = \hat{\Omega} \cap K \neq \emptyset . \tag{2.14}$$

Let $K \subseteq \mathbb{R}^2$ be any compact set satisfying (2.14); we claim that $\Gamma^n \cap K \neq \emptyset$ for large n, and that

$$\lim_{n \to \infty} \Gamma^n \cap K = \hat{\Omega} \cap K . \tag{2.15}$$

In order to prove this claim, it is enough to show that for any subsequence $\{n^i\}$, there exists a further subsequence $\{n^j\} \subseteq \{n^i\}$ such that $\Gamma^{n^j} \cap K \neq \emptyset$ for large j, and

$$\lim_{j \to \infty} \Gamma^{n^j} \cap K = \hat{\Omega} \cap K .$$

Therefore, let $\{n^i\}$ be a given subsequence. If $\Gamma^{n^i} \cap K \neq \emptyset$ for infinitely many i, then, as $\mathscr{K}$ is compact, there exists a subsequence $\{n^j\} \subseteq \{n^i\}$ such that $\Gamma^{n^j} \cap K \neq \emptyset$ for all large j, and $\lim_{j \to \infty} \Gamma^{n^j} \cap K = \tilde{\Omega}$ exists. But Lemma 2.2 and (2.14) imply that $\tilde{\Omega} = \hat{\Omega} \cap K$, as desired. If on the other hand $\Gamma^{n^i} \cap K = \emptyset$ for all large i then Lemma 2.2, with $\tilde{\Omega} = \emptyset$, implies $\hat{\Omega} \cap K = \emptyset$, a contradiction. This now establishes the claim (2.15).

To complete the proof of the proposition, observe that by Lemma 2.6 the set G is dense. Therefore there exist quantities σ^j, $\tau^j \in G$ with $\sigma^j < \tau^j$, and $M^j > C([\sigma^j, \tau^j])$, such that $\sigma^j \to -\infty$ and $\tau^j, M^j \to \infty$ as $j \to \infty$. Let $K^j = [\sigma^j, \tau^j] \times [-M^j, M^j]$. Then for each j the limit $\lim_{n \to \infty} \Gamma^n \cap K^j$ exists, and so the sequence $x^n(\cdot)$ is regular. □

Up to now we have only used the standing assumption, and have not needed to assume that the $x^n(\cdot)$ satisfy differential equations, or even that they are smooth. We now wish to identify and describe certain subsets $\Omega^{\pm}$ and Ω^* of Ω, and to this end we need the smoothness of $x^n(\cdot)$. The sequence of positive numbers $\varepsilon^n > 0$ also participates.

Assumption. For the remainder of this section, unless otherwise stated, we assume in addition to the standing assumption that $x^n: \mathbb{R} \to \mathbb{R}$ is C^1, and that $\varepsilon^n > 0$ is a given sequence of positive numbers. We also assume that $x^n(\cdot)$ is a regular sequence.

Although our principal interest is in singularly perturbed equations, in which $\varepsilon^n \to 0$, we do not assume this limit holds unless explicitly stated. Also, in light of Proposition 2.3, the assumption that $x^n(\cdot)$ is regular is not a significant restriction.

Define for each n the continuous function

$$\theta^n : \Gamma^n \to \mathbb{R}$$

on the graph Γ^n of $x^n(\cdot)$ by

$$\theta^n(p) = \varepsilon^n \dot{x}^n(t) \quad \text{where } p = (t, x^n(t)) .$$

Now define three subsets of Ω as follows:

$$\Omega^+ = \{p \in \Omega \mid \liminf_{n\to\infty} \theta^n(p^n) > 0 \text{ for every sequence } p^n \in \Gamma^n \text{ with } p^n \to p\},$$

$$\Omega^- = \{p \in \Omega \mid \limsup_{n\to\infty} \theta^n(p^n) < 0 \text{ for every sequence } p^n \in \Gamma^n \text{ with } p^n \to p\},$$

$$\Omega^* = \Omega \backslash (\Omega^+ \cup \Omega^-).$$

The sets Ω^+, Ω^-, and Ω^* are clearly pairwise disjoint, and their union is Ω.

Remark. Observe that the inequalities in the definition of Ω^+ and Ω^- are required to hold for every sequence $p^n \to p$, not merely for subsequences, or for a particular sequence. Consider for example the sequence of functions

$$x^n(t) = e^{-(t/\varepsilon^n)^2}, \tag{2.16}$$

with any $\varepsilon^n > 0$ satisfying $\varepsilon^n \to 0$. One sees that (2.16) is regular, and that the limiting profile

$$\Omega = (\mathbb{R} \times \{0\}) \cup (\{0\} \times [0,1])$$

contains a vertical "spike," namely $\{0\} \times [0,1]$. Points $p = (0, \xi)$ on this spike with $0 < \xi < 1$ have limiting sequences $p^n \to p$ for which $\theta^n(p^n)$ approaches a positive limit, as well as other sequences $q^n \to p$ for which $\theta^n(q^n)$ approaches a negative limit. Such p belong neither to Ω^+ nor to Ω^-. In fact, $\Omega^+ = \Omega^- = \emptyset$ and so $\Omega = \Omega^*$.

The following two theorems are concerned with the fundamental geometric properties of the three subsets $\Omega^\pm$ and Ω^*, and their relation to the limit functions f and r in (2.4). In Theorem 2.9 we assume the full set of conditions (2.1) through (2.4), although this is not needed in Theorem 2.8.

Theorem 2.8. (a) *If $p = (\tau, \xi) \in \Omega^+$, then there exists an open set $U \subseteq \mathbb{R}^2$ containing p, and $\delta > 0$, such that*

$$\Omega \cap U = \Omega^+ \cap U = \{\tau\} \times (\xi - \delta, \xi + \delta).$$

The same conclusion holds for $p \in \Omega^-$.

(b) *The sets Ω^+ and Ω^- are relatively open subsets of Ω. The set Ω^* is closed. For each $\tau \in \mathbb{R}$*

$$(\tau, \underline{x}(\tau)) \in \Omega^* \quad \text{and} \quad (\tau, \bar{x}(\tau)) \in \Omega^*.$$

(c) *Let $p = (\tau, \xi) \in \Omega^+$. Then the set $\Omega \backslash \{p\}$ has precisely two connected components, namely*

$$\begin{aligned} \Omega^L(p) &= \Omega_{(-\infty,\tau)} \cup (\{\tau\} \times [\underline{x}(\tau), \xi)), \\ \Omega^R(p) &= (\{\tau\} \times (\xi, \bar{x}(\tau)]) \cup \Omega_{(\tau,\infty)}. \end{aligned} \tag{2.17}$$

If $p = (\tau, \xi) \in \Omega^-$, then the set $\Omega \setminus \{p\}$ has precisely two connected components, namely

$$\Omega^L(p) = \Omega_{(-\infty,\tau)} \cup (\{\tau\} \times (\xi, \bar{x}(\tau)]),$$

$$\Omega^R(p) = (\{\tau\} \times [\underline{x}(\tau), \xi)) \cup \Omega_{(\tau,\infty)}.$$

Here $\underline{x}(\tau)$ and $\bar{x}(\tau)$ are as in (2.10).

Theorem 2.9. *Assume each $x^n(\cdot)$ satisfies a differential equation of the form* (2.1), *where $f^n: \mathbb{R}^2 \to \mathbb{R}$ and $r^n: \mathbb{R} \to \mathbb{R}$ are continuous, and $\varepsilon^n \to 0$, and where the limits* (2.4) *exist uniformly on compact subsets of $\mathbb{R}^2$ and $\mathbb{R}$ respectively, for some continuous $f: \mathbb{R}^2 \to \mathbb{R}$ and $r: \mathbb{R} \to \mathbb{R}$. Let $p = (\tau, \xi) \in \Omega$. Then there exists $q = (\sigma, \zeta) \in \Omega$ such that*

$$\sigma = \tau - r(\xi)$$

and such that

$$\pm f(\xi, \zeta) > 0 \quad \text{if } (\tau, \xi) \in \Omega^\pm, \tag{2.18}$$

while

$$f(\xi, \zeta) = 0 \quad \text{if } (\tau, \xi) \in \Omega^*.$$

Definition. Given a sequence of differential equations

$$\varepsilon^n \dot{x}(t) = f^n(x(t), x(t - r^n)), \quad r^n = r^n(x(t)), \tag{2.19}$$

where f^n, r^n, and ε^n are as in the statement of Theorem 2.9, we call the difference equation

$$0 = f(x(t), x(t - r)), \quad r = r(x(t))$$

the *formal limit* of (2.19).

Remark. Suppose for each $\xi \in \mathbb{R}$ that $f(\xi, \zeta)$ is strictly monotone as a function of ζ. Then the equation $f(\xi, \zeta) = 0$ has, for ξ in an open set $V \subseteq \mathbb{R}$, a unique solution $\zeta = g(\xi)$, where $g: V \to \mathbb{R}$ is continuous. (Here V is the set of $\xi \in \mathbb{R}$ such that $f(\xi, \zeta) = 0$ for some $\zeta \in \mathbb{R}$.) Define the function $\Phi: \mathbb{R} \times V \to \mathbb{R}^2$ by

$$\Phi(\tau, \xi) = (\tau - r(\xi), g(\xi)).$$

Then Theorem 2.9 implies, in particular, that

$$\Omega^* \subseteq \mathbb{R} \times V \quad \text{and} \quad \Phi(\Omega^*) \subseteq \Omega.$$

Remark. Assume for each $\xi \in \mathbb{R}$ that $f(\xi, \zeta)$ is strictly decreasing as a function of ζ. Then for each $p = (\tau, \xi) \in \Omega^+$, the point $\Phi(p)$ lies above some point $q \in \Omega$ in the sense that

$$\Phi(p) = (\sigma, g(\xi)) \quad \text{and} \quad q = (\sigma, \zeta) \quad \text{with} \quad g(\xi) > \zeta.$$

Similarly $\Phi(p)$ lies below some point $q \in \Omega$, for each $p \in \Omega^-$. (In case $\xi \notin V$, we make the natural interpretation $g(\xi) = \pm\infty$ when $\pm f(\xi, \zeta) > 0$ for all $\zeta \in \mathbb{R}$.) Roughly speaking,

$$\Phi(\Omega^+) \text{ lies above } \Omega, \quad \text{and} \quad \Phi(\Omega^-) \text{ lies below } \Omega.$$

In case $f(\xi, \zeta)$ is increasing in ζ, analogous conclusions hold.

Remark. Under quite general conditions (as proved in [MP-Nu4]) equation (2.1) possesses solutions $x^n(\cdot)$ for which the function

$$\eta^n(t) = t - r(x^n(t))$$

satisfies

$$\dot{\eta}^n(t) > 0 \quad \text{for all } t \tag{2.20}$$

(here we assume all delay functions $r^n(x) \equiv r(x)$ are the same, and are C^1). In this case, the locations of the sets Ω^+ and Ω^- are severely restricted. If $p = (\tau, \xi) \in \Omega^+$, for example, then necessarily $r'(\xi) \leqq 0$. If not, then with $p^n \to p$ as in the definition of Ω^+ we have

$$\theta^n(p^n) = \varepsilon^n \dot{x}^n(t^n) = \frac{\varepsilon^n(1 - \dot{\eta}(t^n))}{r'(x^n(t^n))} < \frac{\varepsilon^n}{r'(x^n(t^n))} \to 0,$$

hence $\limsup_{n \to \infty} \theta^n(p^n) \leqq 0$, a contradiction. In fact, the form of Ω^+, as a vertical open line segment in the plane as given in Theorem 2.8, ensures that $r'(\bar{\xi}) \leqq 0$ for all $\bar{\xi}$ near ξ, and hence

$$(\tau, \xi) \in \Omega^+ \quad \text{implies} \quad \xi \in \operatorname{int}\{x \in \mathbb{R} \mid r'(x) \leqq 0\};$$

similarly

$$(\tau, \xi) \in \Omega^- \quad \text{implies} \quad \xi \in \operatorname{int}\{x \in \mathbb{R} \mid r'(x) \geqq 0\}.$$

In case $r'(x) > 0$ for all x then $\Omega^+ = \emptyset$; and $r'(x) < 0$ for all x implies $\Omega^- = \emptyset$ (provided of course (2.20) holds).

Proof of Theorem 2.8. We consider only the case $p \in \Omega^+$, as the case $p \in \Omega^-$ is similar. Let $p = (\tau, \xi) \in \Omega^+$. We first claim there exists a neighborhood U of p, a quantity $\gamma > 0$, and an integer N such that if $p^n \in \Gamma^n \cap U$ for some $n \geqq N$, then $\theta^n(p^n) \geqq \gamma$. Indeed, we may choose

$$U = (\tau - \delta, \tau + \delta) \times (\xi - \delta, \xi + \delta) \tag{2.21}$$

for some $\delta > 0$. The proof of this claim is by contradiction: if the claim were false, then immediately one would obtain for some subsequence $n^i \to \infty$ points $p^{n^i} \in \Gamma^{n^i}$ with $p^{n^i} \to p$, and $\liminf_{n^i \to \infty} \theta^{n^i}(p^{n^i}) \leqq 0$. By Lemma 2.4 we could in fact assume this was a full sequence $p^n \in \Gamma^n$ with $p^n \to p$ and $\liminf_{n \to \infty} \theta^n(p^n) \leqq 0$; but this would imply $p \notin \Omega^+$, a contradiction. This now establishes the claim, with U given by (2.21) for some δ.

Recalling the definition of the function θ^n, we may restate the above claim as

$$\dot{x}^n(t) \geqq \frac{\gamma}{\varepsilon^n} \quad \text{whenever} \quad |t-\tau| < \delta, \quad |x^n(t) - \xi| < \delta, \quad \text{and} \quad n \geqq N.$$

We know moreover that there exist points

$$p^n = (t^n, x^n(t^n)) \to p = (\tau, \xi).$$

Therefore, for all large n there exist $\mu^n < t^n < v^n$ such that

$$x^n(t) \leqq \xi - \delta \quad \text{for } \tau - \delta \leqq t \leqq \mu^n, \tag{2.22}$$

$$|x^n(t) - \xi| \leqq \delta \quad \text{for } \mu^n \leqq t \leqq v^n, \tag{2.23}$$

$$x^n(t) \geqq \xi + \delta \quad \text{for } v^n \leqq t \leqq \tau + \delta, \tag{2.24}$$

and also

$$\dot{x}^n(t) \geqq \frac{\gamma}{\varepsilon^n} \quad \text{for } \mu^n \leqq t \leqq v^n, \tag{2.25}$$

with $v^n - \mu^n \leqq 2\delta\varepsilon^n/\gamma$; hence

$$\mu^n, v^n \to \tau. \tag{2.26}$$

Now define subsets of $\mathbb{R}^2$

$$\begin{aligned} \Lambda^L &= ((-\infty, \tau - \delta] \times \mathbb{R}) \cup ((-\infty, \tau) \times (-\infty, \xi - \delta]), \\ \Lambda^R &= ([\tau + \delta, \infty) \times \mathbb{R}) \cup ((\tau, \infty) \times [\xi + \delta, \infty)), \\ \Lambda^C &= \{\tau\} \times \mathbb{R}, \\ \Lambda &= \Lambda^L \cup \Lambda^C \cup \Lambda^R. \end{aligned}$$

Properties (2.22), (2.23), (2.24), and (2.26) immediately imply $\Omega \subseteqq \Lambda$, and hence

$$\Omega^+ \cap U \subseteqq \Omega \cap U \subseteqq \Lambda \cap U = \{\tau\} \times (\xi - \delta, \xi + \delta). \tag{2.27}$$

But properties (2.22) through (2.26), including (2.25), together imply that

$$\{\tau\} \times (\xi - \delta, \xi + \delta) \subseteqq \Omega^+, \tag{2.28}$$

so (2.27) and (2.28) together imply (a) in the statement of the theorem.

The claim (b) follows directly from (a) and from the definition of the set Ω^*.

To prove (c) first observe that

$$\begin{aligned}\Omega^L(p)\cup\Omega^R(p) &= \Omega_{\mathbb{R}\setminus\{\tau\}}\cup\left(\{\tau\}\times[\underline{x}(\tau),\xi)\right)\cup\left(\{\tau\}\times(\xi,\bar{x}(\tau)]\right)\\ &= \Omega\setminus\{p\}\subseteq\Lambda\setminus\{p\}.\end{aligned}$$

Below we shall prove that $\Omega^L(p)$ and $\Omega^R(p)$ are connected. This is sufficient to establish (c), since $\Lambda\setminus\{p\}$ contains exactly two connected components: one component of $\Lambda\setminus\{p\}$ contains Λ^L and hence contains $\Omega^L(p)$ (since $\Lambda^L\cap\Omega^L(p)\neq\emptyset$), and the other component of $\Lambda\setminus\{p\}$ contains $\Omega^R(p)$.

We now prove $\Omega^L(p)$ is connected, omitting the proof for $\Omega^R(p)$, which is similar. We first note that $\Omega^L(p)$ is the union (2.17) of the connected sets $\Omega_{(-\infty,\tau)}$ and $\{\tau\}\times[\underline{x}(\tau),\xi)$, hence is itself connected if we have

$$\overline{\Omega_{(-\infty,\tau)}}\cap\left(\{\tau\}\times[\underline{x}(\tau),\xi)\right)\neq\emptyset. \tag{2.29}$$

Take any sequence $(\tau^i,\xi^i)\in\Omega^L(p)$, with $\tau^i<\tau$, and $\tau^i\to\tau$; assume also (by taking a subsequence) that the limit $\xi^i\to\xi^\infty$ exists. Thus $(\tau,\xi^\infty)\in\overline{\Omega_{(-\infty,\tau)}}$. As $(\tau^i,\xi^i)\in\Lambda^L$, we have $\xi^i\leqq\xi-\delta$ for large i, hence $\xi^\infty\leqq\xi-\delta$. Of course $\xi^\infty\geqq\underline{x}(\tau)$. Therefore, (τ,ξ^∞) belongs to the intersection (2.29) as desired. This completes the proof. □

Proof of Theorem 2.9. Fix $p=(\tau,\xi)\in\Omega$, and consider any sequence

$$p^n=\left(t^n,x^n(t^n)\right)\in\Gamma^n$$

with $p^n\to p$. Then $s^n\to\sigma$ where we denote

$$s^n=t^n-r^n\left(x^n(t^n)\right)\quad\text{and}\quad\sigma=\tau-r(\xi).$$

The sequence $x^n(s^n)$ is bounded. Let $\{n^i\}$ be any subsequence such that $x^{n^i}(s^{n^i})$ converges and denote the limit $x^{n^i}(s^{n^i})\to\zeta$. Then the point $q=(\sigma,\zeta)$ belongs to Ω. Moreover, from the differential equation (2.1)

$$\theta^{n^i}(p^{n^i})=\varepsilon^{n^i}\dot{x}^{n^i}(t^{n^i})=f^{n^i}\left(x^{n^i}(t^{n^i}),x^{n^i}(s^{n^i})\right)\to f(\xi,\zeta). \tag{2.30}$$

Now if $p\in\Omega^+$, then the limit (2.30) is positive; similarly if $p\in\Omega^-$ then this limit is negative. This establishes (2.18).

Suppose then $p\in\Omega^*$. Since $p\notin\Omega^+$, we have $\liminf_{n\to\infty}\theta^n(p^{n+})\leqq 0$ for some sequence p^{n+} as above, and hence there is a subsequence for which the limit (2.30) satisfies $f(\xi,\zeta^+)\leqq 0$. Since also $p\notin\Omega^-$, there exists another sequence p^{n-}, with a subsequence whose limit satisfies $f(\xi,\zeta^-)\geqq 0$. Here both the points $q^+=(\sigma,\zeta^+)$ and $q^-=(\sigma,\zeta^-)$ belong to Ω, and have the same first coordinate $\sigma=\tau-r(\xi)$. Therefore both $\zeta^+,\zeta^-\in\Omega_\sigma$, and as Ω_σ is a connected set, there exists $\zeta\in\Omega_\sigma$ such that $f(\xi,\zeta)=0$. The point $q=(\sigma,\zeta)\in\Omega$ therefore satisfies the required conditions. This completes the proof. □

In a natural way the above results extend to nonautonomous equations, to equations with multiple delays, and to systems of equations, with the most technically difficult extension being that to systems. Consider, quite generally, a sequence of continuous functions $x^n : \mathbb{R} \to \mathbb{R}^N$ satisfying the bound (2.3). Using subscripts to denote the coordinates in $\mathbb{R}^N$ (so that $x^n(t) = (x_1^n(t), x_2^n(t), \ldots, x_N^n(t))$), consider separately the graph

$$\Gamma_j^n = \{(t, x_j^n(t)) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$$

of each coordinate function, and each corresponding limit set

$$\Omega_j = \bigcap_{m=1}^{\infty} \overline{\bigcup_{n=m}^{\infty} \Gamma_j^n}$$

as in (2.5), for $1 \leqq j \leqq N$. We define $x^n(\cdot)$ to be a regular sequence if $x_j^n(\cdot)$ is regular for each j, in the sense already defined.

Of course the obvious analogs of Propositions 2.1 and 2.5 for the sets $\Omega_{j,S}$ and $\Omega_{j,I}$ hold, and the analog of Proposition 2.3 is easily established. For the generalization of Proposition 2.7 one assumes the existence of sets $\hat{\Omega}_j \subseteqq \mathbb{R}^2$, for $1 \leqq j \leqq N$, such that $\Gamma_j^{n^i} \to \hat{\Omega}_j$ whenever $x^{n^i}(\cdot)$ is a regular subsequence of $x^n(\cdot)$, and concludes that $x^n(\cdot)$ is regular. Lemma 2.2 applied to each coordinate is used in proving this.

In defining the sets $\Omega_j^{\pm}$ (where the smoothness assumption following the proof of Proposition 2.7 is taken), only convergence of points p_j^n in the coordinate of interest (the j^{th}) is required. That is, we define for each j

$$\Omega_j^+ = \{p_j \in \Omega_j \mid \liminf_{n \to \infty} \theta_j^n(p_j^n) > 0 \text{ for every sequence } p_j^n \in \Gamma_j^n \text{ with } p_j^n \to p_j\}$$

and analogously define Ω_j^-, where θ_j^n is the obvious analog of θ^n before. Note in particular that no assumption is made about the convergence of the remaining coordinates p_k^n, for $k \notin j$, where

$$\begin{aligned} x^n(t^n) &= (x_1^n(t^n), x_2^n(t^n), \ldots, x_N^n(t^n)) \\ &= (p_1^n, p_2^n, \ldots, p_N^n) \in \Gamma_1^n \times \Gamma_2^n \times \cdots \times \Gamma_N^n \end{aligned} \tag{2.31}$$

for some t^n. With this definition Theorem 2.8 holds essentially as stated, except that Ω, Ω^+, $\Omega^L(p), \ldots$ are replaced by $\Omega_j, \Omega_j^+, \Omega_j^L(p_j), \ldots$, for each j. The proof of this theorem relies on the appropriate generalization of Lemma 2.4, in which one assumes, for some j, the convergence $p_j^{n^i} \to p_j \in \Omega_j$ for points $p_j^{n^i} \in \Gamma_j^{n^i}$, and concludes the existence of a sequence $p_j^n \in \Gamma_j^n$, of which $p_j^{n^i}$ is a subsequence, where $p_j^n \to p_j$. In this generalization of Lemma 2.4, no claims are made as to the convergence of the other coordinates p_k^n, with $k \neq j$.

Remark. With $N = 2$, consider the sequence $x^n(t) = (x_1^n(t), x_2^n(t))$ where

$$x_1^n(t) = \arctan(t/\varepsilon^n) \quad \text{and} \quad x_2^n(t) = \arctan((t - \tau^n)/\varepsilon^n)$$

where both $\varepsilon^n \to 0$ and $\tau^n \to 0$, but $|\tau^n/\varepsilon^n| \to \infty$ and $(-1)^n\tau^n > 0$. Then the sequence $x^n(\cdot)$ is regular, with $\Omega_1^+ = \Omega_2^+ = \{0\} \times (-\pi/2, \pi/2)$. Note, however, that if $p_1^n \in \Gamma_1^n$ is any sequence such that $p_1^n \to (0,0) \in \Omega_1^+$, then the corresponding sequence $p_2^n \in \Gamma_2^n$, as in (2.31), does not converge.

Remark. A philosophical difficulty with above approach to vector-valued functions $x^n : \mathbb{R} \to \mathbb{R}^N$ is that the analysis is wedded to a particular coordinate system in $\mathbb{R}^N$. One might be tempted to consider instead only the graph

$$\Gamma^n = \{(t, x^n(t)) \in \mathbb{R}^{N+1} \mid t \in \mathbb{R}\}$$

in $\mathbb{R}^{N+1}$, and to take its limit (or more precisely, the limits of intersections $\Gamma^n \cap K^j$, for appropriate sets $K^1 \subseteqq K^2 \subseteqq \cdots \subseteqq \mathbb{R}^{N+1}$) to obtain a set $\Omega \subseteqq \mathbb{R}^{N+1}$. With this approach one sees in particular in the example of the previous remark, that the limit of $\Gamma^n \cap K$ in the Hausdorff metric does not exist for any compact $K \subseteqq \mathbb{R}^3$ containing a sufficiently large neighborhood of the origin: the sequence $x^n(\cdot)$ ought not, therefore, to be considered regular. We have not followed this coordinate-free development, as it is by no means clear how to define analogs of the sets $\Omega^\pm$ and Ω^*, and what should be the generalizations of Theorems 2.8 and 2.9.

Suppose now that the functions $x^n : \mathbb{R} \to \mathbb{R}^N$ satisfy differential equations of the form

$$\begin{aligned} &\varepsilon^n \dot{x}^n(t) = f^n(t, x^n(t), x^n(t - r_1^n), \ldots, x^n(t - r_M^n)), \\ &r_k^n = r_k^n(t, x^n(t)), \quad 1 \leqq k \leqq M. \end{aligned} \tag{2.32}$$

Assume $\varepsilon^n > 0$ and $\varepsilon^n \to 0$, and that $x^n(\cdot)$ is regular and satisfies the bound (2.3). Also assume $f^n : \mathbb{R} \times \mathbb{R}^{N(M+1)} \to \mathbb{R}^N$ and $r_k^n : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are continuous, and where f^n and r_k^n (for each k) converge uniformly on compact subsets of $\mathbb{R} \times \mathbb{R}^{N(M+1)}$ and $\mathbb{R} \times \mathbb{R}^N$ respectively, to functions f and r_k. With this we may define the formal limit of (2.32) in the obvious way, generalizing the definition given earlier. A generalization of Theorem 2.9 can be given for the system (2.32), however, its statement and proof require somewhat more care than the generalizations of other results in this section. For this purpose denote

$$\bar{\underline{x}}(\tau) = [\underline{x}_1(\tau), \bar{x}_1(\tau)] \times [\underline{x}_2(\tau), \bar{x}_2(\tau)] \times \cdots \times [\underline{x}_N(\tau), \bar{x}_N(\tau)] \subseteqq \mathbb{R}^N$$

where $\Omega_{j,\tau} = \{\tau\} \times [\underline{x}_j(\tau), \bar{x}_j(\tau)]$ defines the functions $\underline{x}_j$ and $\bar{x}_j$.

Theorem 2.10. *Assume $x^n : \mathbb{R} \to \mathbb{R}^N$ satisfies (2.32), for some continuous*

$$f^n : \mathbb{R} \times \mathbb{R}^{N(M+1)} \to \mathbb{R}^N \quad \textit{and} \quad r_k^n : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R},$$

with f^n and r_k^n (for each k) converging uniformly on compact sets to functions f and r_k. Also assume $\varepsilon^n > 0$ with $\varepsilon^n \to 0$, and assume that the sequence $x^n(\cdot)$ is regular. Let $p_j = (\tau, \xi_j) \in \Omega_j$ for some j. Then there exist $\xi_k \in \mathbb{R}$, for $1 \leqq k \leqq N$ with $k \neq j$, such that

$$\xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \bar{\underline{x}}(\tau), \tag{2.33}$$

and for $1 \leqq k \leqq M$ there exist

$$(2.34) \qquad \zeta_k = (\zeta_{k,1}, \zeta_{k,2}, \ldots, \zeta_{k,N}) \in \bar{\underline{x}}(\sigma_k), \quad \textit{where } \sigma_k = \tau - r_k(\tau, \xi)$$

such that

$$(2.35) \qquad \pm f_j(\tau, \xi, \zeta_1, \zeta_2, \ldots, \zeta_M) > 0 \quad \textit{if } p_j \in \Omega_j^{\pm},$$

$$(2.36) \qquad f_j(\tau, \xi, \zeta_1, \zeta_2, \ldots, \zeta_M) = 0 \quad \textit{if } p_j \in \Omega_j^{*},$$

where $f = (f_1, f_2, \ldots, f_N)$.

Before presenting the proof of Theorem 2.10, we give the following technical lemma.

Lemma 2.11. *Let B and C be topological spaces, and assume B is connected. For each $b \in B$ let $\Gamma(b)$ be a nonempty, connected subset of C. Assume also that $\Gamma(b)$ varies upper-semicontinuously in b, that is, for each $b_0 \in B$ and each neighborhood $U \subseteqq C$ with $\Gamma(b_0) \subseteqq U$, there exists a neighborhood $b_0 \in V \subseteqq B$ such that $\Gamma(b) \subseteqq U$ whenever $b \in V$. Then the union*

$$S = \bigcup_{b \in B} \Gamma(b)$$

is a connected subset of C.

Proof. Assume S is not connected. Then there exist open sets $U_1, U_2 \subseteqq C$ such that $S_i = S \cap U_i \neq \emptyset$ for $i = 1, 2$, and $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. The connectedness of $\Gamma(b)$ implies that either $\Gamma(b) \cap U_1 = \emptyset$ or $\Gamma(b) \cap U_2 = \emptyset$ for each $b \in B$, hence $\Gamma(b) \subseteqq U_i$ for some $i = i(b)$. Let $B_i = \{b \in B \mid \Gamma(b) \subseteqq U_i\}$ for $i = 1, 2$. Then $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$. The upper-semicontinuity of $\Gamma(\cdot)$ implies that both B_1 and B_2 are open. Therefore, as B is connected, either $B_1 = \emptyset$ or $B_2 = \emptyset$. But this implies that $S \subseteqq U_i$ for some i, contradicting the fact that both $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. □

Proof of Theorem 2.10. With j and p_j as in the statement of the theorem, consider any sequence of points $p_j^n = (t^n, x_j^n(t^n)) \in \Gamma_j^n$ such that $p_j^n \to p_j$. One easily sees that for some subsequence $n^i \to \infty$, we have convergent sequences

$$x_k^{n^i}(t^{n^i}) \to \xi_k \in [\underline{x}_k(\tau), \bar{x}_k(\tau)] \quad \text{for } 1 \leqq k \leqq N \text{ with } k \neq j,$$

$$\left.\begin{aligned} t^{n^i} - r_k^{n^i}(t^{n^i}, x^{n^i}(t^{n^i})) &\to \tau - \tau_k(r, \xi) \\ x^{n^i}(t^{n^i} - r_k^{n^i}(t^{n^i}, x^{n^i}(t^{n^i}))) &\to \zeta_k \in \bar{\underline{x}}(\sigma_k) \end{aligned}\right\} \quad \text{for } 1 \leqq k \leqq M,$$

where (2.33) holds, and σ_k is given by (2.34). Moreover, we may assume without loss the limit

$$\theta_j^{n^i}(p_j^{n^i}) \to \theta_j \in \mathbb{R}$$

of $\varepsilon^{n^i} \dot{x}_j^{n^i}(t^{n^i})$ as $i \to \infty$, where in fact

$$f_j(\tau, \xi, \zeta_1, \zeta_2, \ldots, \zeta_M) = \theta_j,$$

which follows directly from the differential equation.

If $p_j \in \Omega_j^\pm$ then $\pm\theta_j > 0$ follows from the definition of $\Omega_j^\pm$. This establishes (2.35), and we are done.

If on the other hand $p_j \in \Omega_j^*$ then, as in the proof of Theorem 2.10, there exist two possible sequences $p_j^{n\pm}$. From this one proves existence of two points $\xi^\pm \in \bar{\underline{x}}(\tau)$, with $\xi_j^+ = \xi_j^- = \xi_j$, and of points $\zeta_k^\pm \in \bar{\underline{x}}(\sigma_k^\pm)$ with $\sigma_k^\pm = \tau - r_k(\tau, \xi^\pm)$, such that

$$\pm f_j(\tau, \xi^\pm, \zeta_1^\pm, \zeta_2^\pm, \ldots, \zeta_M^\pm) \geqq 0 . \tag{2.37}$$

Consider, for our fixed τ, the set

$$S_\tau = \{(\tau, \xi, \zeta_1, \zeta_2, \ldots, \zeta_M) \in \mathbb{R} \times \mathbb{R}^{N(M+1)} \mid \xi \in \bar{\underline{x}}(\tau) \text{ and } \zeta_k \in \bar{\underline{x}}(\sigma_k) \text{ where } \sigma_k = \tau - r_k(\tau, \xi), \text{ for } 1 \leqq k \leqq M\}$$

and the restriction $f_j : S_\tau \to \mathbb{R}$ of the continuous function f_j. In light of (2.37), all that is needed is to prove the connectedness of S_τ to establish the desired conclusion (2.36) of the theorem. But

$$S_\tau = \bigcup_{\xi \in \bar{\underline{x}}(\tau)} S_{\tau,\xi} \quad \text{where} \quad S_{\tau,\xi} = \{\tau\} \times \{\xi\} \times \prod_{k=1}^{M} \bar{\underline{x}}(\tau - r_k(\tau, \xi)) .$$

The semicontinuity properties of the functions $\underline{x}_k$ and $\bar{x}_k$ imply the connected set $S_{\tau,\xi}$ varies upper-semicontinuously as ξ varies throughout the connected set $\bar{\underline{x}}(\tau)$. By Lemma 2.11, therefore, the union S_τ is connected. □

3. Simplicity of Ω for non-constant delays

The results of this section say that under very general conditions – essentially that f should have only a bounded number of sign changes as a function of its first argument, and that the delay r should not be constant on any interval – the set Ω is almost like the graph of a function. Namely, at each $\tau \in \mathbb{R}$ the left- and right-hand limits of points in Ω, with $\tau^j \to \tau$, exist; moreover, for all but countably many values of τ the cross-section Ω_τ is a single point.

More generally, we obtain an upper bound on the number of oscillations a solution may have in a given region. This bound is expressed in terms of the range of the delay function r: the closer r is to a constant, the more oscillations a solution may have.

Before giving precise statements of our results, we define the number of sign changes sc of a function. Let $\alpha : I \to \mathbb{R}$, where I is an interval. We set

$$\operatorname*{sc}_{x \in I} \alpha(x) = \sup\{m \geqq 1 \mid \text{there exist } x^0 < x^1 < \cdots < x^m \text{ in } I \text{ such that } \alpha(x^{i-1})\,\alpha(x^i) < 0 \text{ for } 1 \leqq i \leqq m\} .$$

We have the following result.

Theorem 3.1. *Assume that* $x^n(\cdot)$ *is a regular sequence of solutions to the equations* (2.1), *where* (2.2) *holds, where* (2.3) *holds for some* $C(I)$, *for each compact interval* I, *and where the limits* (2.4) *hold uniformly on compact sets for some* f *and* r. *Also assume each* $r^n : \mathbb{R} \to \mathbb{R}$ *is locally lipschitz, and that there exists an integer* N *such that*

$$\operatorname*{sc}_{x \in \mathbb{R}} f^n(x,y) < N \quad \textit{for each } n \textit{ and } y \in \mathbb{R} .$$

Suppose for some $\tau^0 \in \mathbb{R}$ *that* Ω_{τ^0+0} *is a nontrivial interval, that is, as in the notation* (2.11), (2.12), *and* (2.13),

$$\begin{aligned} \Omega_{\tau^0+0} &= \{\tau^0\} \times \underline{\bar{x}}(\tau^0+0) \\ &= \{\tau^0\} \times [\underline{x}(\tau^0+0), \bar{x}(\tau^0+0)] , \end{aligned}$$

with the strict inequality

$$\underline{x}(\tau^0+0) < \bar{x}(\tau^0+0) .$$

Then the limit function r *is constant on the interval* $\underline{\bar{x}}(\tau^0+0)$. *Moreover, for each* $\xi \in \underline{\bar{x}}(\tau^0+0)$, *there exists* $\zeta \in \underline{\bar{x}}(\tau^0 - r^0 + 0)$, *such that* $f(\xi,\zeta) = 0$, *where* r^0 *is the constant value of* r *on* $\underline{\bar{x}}(\tau^0+0)$.

The analogous result holds for the set Ω_{τ^0-0} *obtained from the left-hand limit.*

The following corollary is a direct consequence of Theorem 3.1; we prove both these results later in this section.

Corollary 3.2. *Assume the hypotheses of Theorem* 3.1. *Also assume the function* r *is not constant on any interval. Then for every* $\tau^0 \in \mathbb{R}$ *the left-hand limits*

$$\lim_{\tau \to \tau^0-} \underline{x}(\tau) = \lim_{\tau \to \tau^0-} \bar{x}(\tau) \tag{3.1}$$

exist and are equal, and the right-hand limits

$$\lim_{\tau \to \tau^0+} \underline{x}(\tau) = \lim_{\tau \to \tau^0+} \bar{x}(\tau) \tag{3.2}$$

exist and are equal (although in general the left- and right-hand limits are different). Moreover, for each $k > 0$ *the set*

$$\{\tau^0 \in \mathbb{R} \mid \bar{x}(\tau^0) - \underline{x}(\tau^0) > k\} \tag{3.3}$$

contains no cluster point, and hence its intersection with any finite interval is a finite set. Thus for all but countably many values of τ^0 *the set* Ω_{τ^0} *is a single point, that is,*

$$\underline{x}(\tau^0) = \bar{x}(\tau^0) . \tag{3.4}$$

Remark. Theorem 3.1 and Corollary 3.2 fail dramatically in the case of a constant delay. Consider the equation

$$\varepsilon \dot{x}(t) = f(x(t), x(t-1)) \tag{3.5}$$

and suppose, for some (not necessarily small) $\varepsilon = \varepsilon^0 > 0$, that $x^0(\cdot)$ is a non-constant periodic solution of (3.5). (Many such solutions are known, for a wide variety of nonlinearities; typical are the slowly oscillating periodic solutions, which we discuss more fully in Section 4.) Let $T > 0$ denote the period of $x^0(\cdot)$, and for each $n \geqq 1$ define

$$x^n(t) = x^0(\mu^n t), \quad \text{where } \mu^n = 1 + nT.$$

As noted originally by K. Cooke, $x^n(\cdot)$ is a solution of (3.5) with $\varepsilon = \varepsilon^n$, where $\varepsilon^n = \varepsilon^0/\mu^n \to 0$. One sees easily that $x^n(\cdot)$ is a regular sequence, and that

$$\Omega = \mathbb{R} \times [\underline{x}^0, \bar{x}^0]$$

where $\underline{x}^0 < \bar{x}^0$ are given by

$$\underline{x}^0 = \min_{t \in \mathbb{R}} x^0(t), \quad \bar{x}^0 = \max_{t \in \mathbb{R}} x^0(t).$$

Remark. If Ω_{τ^0+0} is a nontrivial interval, and the other hypotheses of Theorem 3.1 hold, then $\Omega_{\tau^0-r^0+0}$ is also a nontrivial interval unless there exists ζ such that $f(\xi, \zeta) = 0$ for all $\xi \in \bar{\underline{x}}(r^0 + 0)$. Indeed, this result has powerful implications for the shape of Ω as the following corollary illustrates.

Corollary 3.3. *Assume the hypotheses of Theorem* 3.1, *and in addition that* $f(0,0) = 0$. *Also assume there exist positive quantities C and D such that whenever*

$$f(x^0, y^0) = f(x^1, y^1) = 0$$

for distinct points $(x^i, y^i) \in [-D, C] \times [-D, C]$, *then either* $x^0 \leqq x^1$ *and* $y^0 > y^1$, *or else* $x^0 \geqq x^1$ *and* $y^0 < y^1$; *and assume that whenever* $f(z^0, z^1) = f(z^1, z^2) = 0$ *with all* $z^i \in [-D, C]$, *then either* $z^0 \in \{-D, 0, C\}$ *or else* $|z^2| < |z^0|$. *Finally assume r is lipschitz and nonnegative in* $[-D, C]$, *and that for some* $\delta > 0$ *the function r is not constant in any subinterval of* $[0, \delta]$. *Then if* Ω *is the limiting profile of any regular sequence of solutions satisfying* $-D \leqq x^n(t) \leqq C$, *the conclusion of Corollary* 3.2 *holds.*

Remark. Among the nonlinearities f satisfying the hypotheses of Corollary 3.3 are the classic non-linearities

$$f(x, y) = -x + \tilde{f}(y),$$

where $\tilde{f}$ satisfies a negative feedback condition with an instability at the origin, has a unique period-two orbit, and is monotone in y.

Remark. Of particular interest in our result is the possibility that the delay r may be constant on $[-D, 0]$ and still satisfy the conditions of Corollary 3.3, provided only the local hypothesis in the interval $[0, \delta]$ holds. This is significant in the study of problems with piecewise linear f and r.

Theorem 3.1, which will be proved below along with Corollaries 3.2 and 3.3, is a consequence of the following general proposition. This result relates the number of oscillations of a solution over an interval with the range of the delay function r, and concerns a single equation

$$\dot{x}(t) = f(x(t), x(t-r)), \quad r = r(x(t)), \tag{3.6}$$

rather than a sequence of equations as above.

Proposition 3.4. *Assume $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous and $r: \mathbb{R} \to \mathbb{R}$ is locally lipschitz. Also assume there exists an integer N such that*

$$\operatorname*{sc}_{x \in \mathbb{R}} f(x, y) < N \quad \textit{for each } y \in \mathbb{R}. \tag{3.7}$$

Let $x(t)$ satisfy equation (3.6) for $t \in \mathbb{R}$, and suppose there exist quantities $a^0 < a^1$ and $t^0 < t^1 < \cdots < t^{N+1}$ such that

$$x(t^i) \leqq a^0 \quad \textit{for even } i, \quad \textit{and} \quad x(t^i) \geqq a^1 \quad \textit{for odd } i, \tag{3.8}$$

for $0 \leqq i \leqq N+1$. Then

$$\max_{[a^0, a^1]} r(\cdot) - \min_{[a^0, a^1]} r(\cdot) \leqq 3(t^{N+1} - t^0). \tag{3.9}$$

The same result (3.9) holds if in (3.8) the words even and odd are reversed.

Before proving Proposition 3.4, and then Theorem 3.1 and Corollaries 3.2 and 3.3, we need the following lemmas.

Lemma 3.5. *Let $w: [\alpha, \beta] \to \mathbb{R}$ be absolutely continuous. Then $w(Q)$ is measurable for each measurable set $Q \subseteqq [\alpha, \beta]$, and*

$$\operatorname{meas}(w(Q)) \leqq \int_Q |\dot{w}(t)| \, dt.$$

In particular, if

$$Q = \{t \in [\alpha, \beta] \mid \dot{w}(t) \textit{ exists and equals } 1\}$$

then

$$\operatorname{meas}(w(Q)) \leqq \operatorname{meas}(Q).$$

Proof. If Q is an interval the result is easily proved. If Q is a countable union of disjoint intervals then again the result is easy. In particular, if Q is a relatively open set in $[\alpha, \beta]$ then we have the result.

Now suppose Q is any measurable set. For each n there exists a (relatively) open set $G^n \subseteqq [\alpha, \beta]$ with

$$Q \subseteqq G^m \quad \text{and} \quad \operatorname{meas}(G^n \backslash Q) \leqq \frac{1}{n}. \tag{3.10}$$

We have

$$\text{meas}^*(w(Q)) \leqq \text{meas}(w(G^n)) \leqq \int_{G^n} |\dot{w}(t)|\,dt \tag{3.11}$$

where meas* denotes outer measure. From (3.10) and (3.11) we conclude that

$$\text{meas}^*(w(Q)) \leqq \int_Q |\dot{w}(t)|\,dt\,.$$

To complete the proof of the lemma we must show $w(Q)$ is measurable. For each n there exists a compact set K^n with

$$K^n \subseteqq Q \quad \text{and} \quad \text{meas}(Q\backslash K^n) \leqq \frac{1}{n}\,.$$

Applying (3.11) to $Q\backslash K^n$ gives

$$\text{meas}^*(w(Q)\backslash w(K^n)) \leqq \text{meas}^*(w(Q\backslash K^n)) \leqq \int_{Q\backslash K^n} |\dot{w}(t)|\,dt \to 0\,, \tag{3.12}$$

where the limit in (3.12) follows from the absolute continuity of the integral.

Now $\bigcup w(K^n)$ is a countable union of compact sets, hence is measurable. This union is a subset of $w(Q)$, and the difference $w(Q)\backslash\bigcup w(K^n)$ has outer measure zero, by (3.12). It follows then that $w(Q)$ is measurable. □

Remark. It is well known that there are continuous functions $w:[\alpha,\beta] \to \mathbb{R}$ and measurable sets $Q \subseteqq [\alpha,\beta]$ for which $w(Q)$ is not measurable.

Lemma 3.6. *Suppose for some $N \geqq 0$ there are quantities*

$$\alpha^0 < \beta^0 \leqq \alpha^1 < \beta^1 \leqq \cdots \leqq \alpha^N < \beta^N \quad \textit{and} \quad a^0 < a^1 \tag{3.13}$$

and functions

$$u^i:[\alpha^i,\beta^i] \to [a^0,a^1] \quad \textit{for } 0 \leqq i \leqq N, \qquad v:[a^0,a^1] \to \mathbb{R}\,, \tag{3.14}$$

such that each u^i is C^1, and v is lipschitz, and such that

$$u^i(\alpha^i) = a^{i \bmod 2} \quad \textit{and} \quad u^i(\beta^i) = a^{1-i \bmod 2} \tag{3.15}$$

and

$$\max_{[a^0,a^1]} v(\cdot) - \min_{[a^0,a^1]} v(\cdot) > \beta^N - \alpha^0 + 2\sum_{i=0}^{N}(\beta^i - \alpha^i)\,. \tag{3.16}$$

Then there exist quantities $\gamma^i \in [\alpha^i, \beta^i]$ *and* $c \in \mathbb{R}$ *such that for each* i

$$(-1)^i \dot{u}^i(\gamma) > 0\,, \tag{3.17}$$

$$\gamma^i - v(u^i(\gamma^i)) = c\,, \tag{3.18}$$

and such that either

$$u^0(\gamma^0) < u^1(\gamma^1) < \cdots < u^N(\gamma^N)\,, \tag{3.19}$$

or else

$$u^0(\gamma^0) > u^1(\gamma^1) > \cdots > u^N(\gamma^N)\,. \tag{3.20}$$

Proof. First define for each i

$$\mu^i(t) = \begin{cases} \sup\limits_{[\alpha^i, t]} u^i(\cdot) & \text{if } i \text{ is even,} \\ \inf\limits_{[\alpha^i, t]} u^i(\cdot) & \text{if } i \text{ is odd,} \end{cases} \tag{3.21}$$

and set

$$w^i(t) = t - v(u^i(t)), \quad \omega^i(t) = t - v(\mu^i(t))\,.$$

Also define sets

$$P^i = \{t \in [\alpha^i, \beta^i] \mid \mu^i(t) \neq u^i(t)\}\,,$$

$$Q^i = \{t \in [\alpha^i, \beta^i] \mid \dot{w}^i(t) \text{ exists and equals } 1\}\,,$$

$$R^i = \omega^i(P^i)\,,$$

$$S^i = w^i(Q^i)\,.$$

Note that P^i is open, that the function μ^i is constant on each connected component of P^i, and hence ω^i is a linear function with slope 1 on each such component. Therefore

$$\operatorname{meas}(R^i) \leqq \operatorname{meas}(P^i) \leqq \beta^i - \alpha^i\,. \tag{3.22}$$

One also has from Lemma 3.5, using the fact w^i is lipschitz, that

$$\operatorname{meas}(S^i) \leqq \operatorname{meas}(Q^i) \leqq \beta^i - \alpha^i\,. \tag{3.23}$$

Finally, define

$$b^0 = \min_{[a^0, a^1]} v(\cdot), \quad b^1 = \max_{[a^0, a^1]} v(\cdot)\,.$$

Now consider the set T defined as

$$T = [\beta^N - b^1, \alpha^0 - b^0] \setminus \left(\bigcup_{i=0}^N R^i\right) \setminus \left(\bigcup_{i=0}^N S^i\right).$$

We have

$$\begin{aligned}\operatorname{meas}(T) &\geqq b^1 - b^0 - (\beta^N - \alpha^0) - \sum_{i=0}^{N} \operatorname{meas}(R^i) - \sum_{i=0}^{N} \operatorname{meas}(S^i)\\ &\geqq b^1 - b^0 - (\beta^N - \alpha^0) - 2\sum_{i=0}^{N}(\beta^i - \alpha^i)\\ &> 0\end{aligned}$$

from (3.13), (3.16), (3.22), and (3.23), and hence $T \neq \emptyset$. Fix any $c \in T$, and also let $\tilde{a}^0, \tilde{a}^1 \in [a^0, a^1]$ be such that $v(\tilde{a}^i) = b^i$ for $i = 0, 1$. For definiteness assume $\tilde{a}^0 < \tilde{a}^1$ (in the case $\tilde{a}^0 > \tilde{a}^1$ one modifies the following proof appropriately), let $\tilde{\alpha}^i < \tilde{\beta}^i$ be such that $\alpha^i \leqq \tilde{\alpha}^i < \tilde{\beta}^i \leqq \beta^i$ with

$$\mu^i : [\tilde{\alpha}^i, \tilde{\beta}^i] \to [\tilde{a}^0, \tilde{a}^1], \tag{3.24}$$

and

$$\mu^i(\tilde{\alpha}^i) = \tilde{a}^{i \bmod 2}, \qquad \mu^i(\tilde{\beta}^i) = \tilde{a}^{1 - i \bmod 2}, \tag{3.25}$$

and note that

$$\omega^i(\tilde{\alpha}^i) = \tilde{\alpha}^i - b^{i \bmod 2}, \quad \omega^i(\tilde{\beta}^i) = \tilde{\beta}^i - b^{1 - i \bmod 2}. \tag{3.26}$$

Observe from (3.26) that c is in the range of ω^i on $[\tilde{\alpha}^i, \tilde{\beta}^i]$. Indeed, if i is even we have

$$\omega^i(\tilde{\beta}^i) \leqq \beta^N - b^1 \leqq c \leqq \alpha^0 - b^0 \leqq \omega^i(\tilde{\alpha}^i), \tag{3.27}$$

with a similar calculation for odd i. For each i let $\gamma^i \in [\tilde{\alpha}^i, \tilde{\beta}^i]$ be such that

$$\omega^i(\gamma^i) = c \tag{3.28}$$

and such that

$$\begin{aligned}&\omega^i(t) \neq c \quad \text{for } \tilde{\alpha}^i \leqq t < \gamma^i, \quad \text{if } i \text{ is even},\\ &\omega^i(t) \neq c \quad \text{for } \gamma^i < t \leqq \tilde{\beta}^i, \quad \text{if } i \text{ is odd}.\end{aligned} \tag{3.29}$$

We first claim that

$$\mu^0(\gamma^0) < \mu^1(\gamma^1) < \cdots < \mu^N(\gamma^N). \tag{3.30}$$

We in fact prove only $\mu^i(\gamma^i) < \mu^{i+1}(\gamma^{i+1})$ for even i, as the proof for odd i is similar. Suppose that $\mu^i(\gamma^i) \geqq \mu^{i+1}(\gamma^{i+1})$ for some even i. Then considering the function μ^i on the interval $[\tilde{\alpha}^i, \gamma^i]$, noting (3.24), (3.25), that

$$\mu^i(\tilde{\alpha}^i) = \tilde{a}^0 \leqq \mu^{i+1}(\gamma^{i+1}) \leqq \mu^i(\gamma^i),$$

implies that $\mu^i(\gamma^*) = \mu^{i+1}(\gamma^{i+1})$ for some $\gamma^* \in [\tilde{\alpha}^i, \gamma^i]$. Therefore

$$\begin{aligned}\omega^i(\gamma^*) &= \gamma^* - v(\mu^i(\gamma^*)) = \gamma^* - v(\mu^{i+1}(\gamma^{i+1})) \\ &= \gamma^* - \gamma^{i+1} + \omega^{i+1}(\gamma^{i+1}) = \gamma^* - \gamma^{i+1} + c < c\,.\end{aligned}$$

But $\omega^i(\tilde{\alpha}^i) \geqq c$ by (3.27), hence $\omega^i(\gamma^\#) = c$ for some $\gamma^\# \in [\tilde{\alpha}^i, \gamma^*) \subseteqq [\tilde{\alpha}^i, \gamma^i)$. This contradicts (3.29). Thus (3.30) is proved.

To complete the proof of the lemma, we note that the choice $c \notin R^i$ implies from (3.28) and the definition of R^i, that $\gamma^i \notin P^i$. Thus $\mu^i(\gamma^i) = u^i(\gamma^i)$, and therefore

$$(3.31) \qquad (-1)^i \dot{u}^i(\gamma^i) \geqq 0\,,$$

from (3.21). One also now sees that (3.18) and (3.19) hold, from (3.30).

All that is left, then, is to prove the strictness of the inequality (3.31). Suppose that $\dot{u}^i(\gamma^i) = 0$; then, as v is a lipschitz function, we have $\dot{w}^i(\gamma^i) = 1$. Thus $\gamma^i \in Q^i$, so $c = w^i(\gamma^i) \in S^i$. But this contradicts the choice of c in T, and so completes the proof. □

Proof of Proposition 3.4. This is a direct application of Lemma 3.6. First note there exist quantities α^i and β^i as in (3.13), with $t^i \in [\beta^{i-1}, \alpha^i]$ for $1 \leqq i \leqq N$, and $t^0 \leqq \alpha^0$ and $t^{N+1} \geqq \beta^N$, such that (3.14) and (3.15) hold, where we set

$$u^i(t) = x(t) \quad \text{for } \alpha^i \leqq t \leqq \beta^i\,.$$

Now suppose (3.9) fails; then, setting $v(x) \equiv r(x)$, we have

$$\begin{aligned}\max_{[a^0,a^1]} v(\cdot) - \min_{[a^0,a^1]} v(\cdot) &> 3(t^{N+1} - t^0) \geqq 3(\beta^N - \alpha^0) \\ &\geqq \beta^N - \alpha^0 + 2\sum_{i=0}^{N} (\beta^i - \alpha^i)\end{aligned}$$

as in (3.16). Let γ^i be as in Lemma 3.6. Then from (3.18)

$$\dot{x}(\gamma^i) = f(x(\gamma^i), x(\gamma^i - r(x(\gamma^i)))) = f(x(\gamma^i), x(c))\,,$$

hence from (3.17)

$$(3.32) \qquad (-1)^i f(x(\gamma^i), x(c)) > 0 \quad \text{for } 0 \leqq i \leqq N\,.$$

As either (3.19) or (3.20) hold, $x(\gamma^i)$ is a monotone sequence. Thus from (3.32)

$$\operatorname*{sc}_{z \in [a^0, a^1]} f(z, x(c)) \geqq N\,,$$

contradicting (3.7) and completing the proof. □

We can now prove the main result of this section.

Proof of Theorem 3.1. We prove only the results for right-hand limits. Let $a^0 < a^1$ be such that

$$\underline{x}(\tau^0 + 0) \leqq a^0 < a^1 \leqq \bar{x}(\tau^0 + 0)\,.$$

Fix $\delta > 0$. By assumption $x^n(\cdot)$ is a regular sequence, so there exists n^0 such that for each $n \geqq n^0$ there exist points

$$\tau^0 \leqq t^0 < t^1 < \cdots < t^{N+1} \leqq \tau^0 + \delta$$

such that (3.8) holds. Therefore, from Proposition 3.4,

$$\max_{[a^0, a^1]} r^n(\cdot) - \min_{[a^0, a^1]} r^n(\cdot) \leqq 3(t^{N+1} - t^0) \leqq 3\delta\,,$$

which implies the limit

$$\max_{[a^0, a^1]} r(\cdot) - \min_{[a^0, a^1]} r(\cdot) \leqq 3\delta\,.$$

As $\delta > 0$, and a^0 and a^1 are chosen arbitrarily as above, we conclude that r is constant on $\bar{\underline{x}}(\tau^0 + 0)$, as claimed.

Let r^0 denote the constant value of r on $\bar{\underline{x}}(\tau^0 + 0)$, and fix any $\xi \in \bar{\underline{x}}(\tau^0 + 0)$. We wish to produce ζ as in the statement of the theorem. In fact, to this end, it is enough to do this when ξ is in the interior of $\bar{\underline{x}}(\tau^0 + 0)$, that is

$$\underline{x}(\tau^0 + 0) < \xi < \bar{x}(\tau^0 + 0)\,. \tag{3.33}$$

We first observe that

$$\Omega \cap \big((\tau^0, \tau^0 + \delta) \times \{\xi\}\big) \neq \emptyset \tag{3.34}$$

for each positive δ. If (3.34) were false, then the connectedness of $\Omega_{(\tau^0, \tau^0 + \delta)}$ would imply that either

$$\Omega_{(\tau^0, \tau^0 + \delta)} \subseteqq (\tau^0, \tau^0 + \delta) \times (-\infty, \xi) \tag{3.35}$$

or else

$$\Omega_{(\tau^0, \tau^0 + \delta)} \subseteqq (\tau^0, \tau^0 + \delta) \times (\xi, \infty)\,. \tag{3.36}$$

In either case this would contradict (3.33) (as we recall the definitions (2.11)).

We next refine (3.34) and show that both

$$(\Omega^* \cup \Omega^+) \cap \big((\tau^0, \tau^0 + \delta) \times \{\xi\}\big) \neq \emptyset \tag{3.37}$$

and

$$(\Omega^* \cup \Omega^-) \cap ((\tau^0, \tau^0 + \delta) \times \{\xi\}) \neq \emptyset \tag{3.38}$$

hold. We prove only (3.38); if it is false for some δ, then in light of (3.34) and the structure of Ω as described in Theorem 2.8, there exist $\tau^1 < \tau^2$ in the interval $(\tau^0, \tau^0 + \delta)$ such that

$$\begin{gathered} (\tau^i, \xi) \in \Omega^+ \quad \text{for } i = 1, 2, \\ \Omega \cap ((\tau^1, \tau^2) \times \{\xi\}) = \emptyset. \end{gathered} \tag{3.39}$$

As above, (3.39) implies either (3.35) or (3.36), but with τ^1 and τ^2 replacing τ^0 and $\tau^0 + \delta$ in these formulas. For definiteness suppose

$$\Omega_{(\tau^1, \tau^2)} \subseteqq (\tau^1, \tau^2) \times (-\infty, \xi). \tag{3.40}$$

Denoting $p^i = (\tau^i, \xi)$, consider the connected set $\Omega^R(p^1)$ which is the union of two disjoint subsets as in (2.17) of Theorem 2.8. We shall show the intersection of the closures of the two subsets is empty:

$$(\{\tau^1\} \times [\xi, \bar{x}(\tau^1)]) \cap \overline{\Omega_{(\tau^1, \infty)}} = \emptyset. \tag{3.41}$$

This will provide a contradiction to the connectedness of $\Omega^R(p^1)$ and establishes (3.38). Indeed, (3.40) implies that the only possible point in the intersection in the left-hand side of equation (3.41) is the point (τ^1, ξ). However, (a) of Theorem 2.8 implies that $(\tau^1, \xi) \notin \overline{\Omega_{(\tau^1, \infty)}}$, and thus (3.38) holds. The proof of (3.37) is similar.

Now (3.37) and (3.38) together imply there exists a sequence $p^i = (\tau^i, \xi) \to (\tau^0, \xi)$, with $\tau^{i+1} < \tau^i$ for each i, and moreover, with $p^i \in \Omega^* \cup \Omega^+$ for even i, and $p^i \in \Omega^* \cup \Omega^-$ for odd i. Theorem 2.9 guarantees the existence of a sequence $q^i = (\tau^i - r^0, \tilde{\zeta}^i) \in \Omega$ such that

$$(-1)^i f(\xi, \tilde{\zeta}^i) \geqq 0. \tag{3.42}$$

It follows now from (3.42) and the connectedness of the set $\Omega_{[\tau^{i+1} - r^0, \tau^i - r^0]}$, that there exists a point

$$\tilde{q}^i = (\sigma^i, \zeta^i) \in \Omega_{[\tau^{i+1} - r^0, \tau^i - r^0]}$$

such that $f(\xi, \zeta^i) = 0$. Upon taking the limit of a subsequence of $\tilde{q}^i$, we obtain a point $q = (\tau^0 - r^0, \zeta) \in \Omega_{\tau^0 - r^0 + 0}$ for which $f(\xi, \zeta) = 0$. With this, the theorem is proved. □

We now prove the corollaries which follow Theorem 3.1.

Proof of Corollary 3.2. Certainly we have the existence of the limits (3.1) and (3.2) by Theorem 3.1. It follows immediately from this and the uniform bound (2.3) on solutions that the set (3.3) contains no cluster point, and hence that (3.4) holds for all but countably many τ^0. □

Proof of Corollary 3.3. We only prove (3.2), as the proof of (3.1) is similar, and the proofs of the remaining claims follow as in the proof of Corollary 3.2.

Suppose then (3.2) fails for some τ^0. Denote $I^0 = \underline{\bar{x}}(\tau^0 + 0)$ and let r^0 denote the constant value of r on I^0, as in Theorem 3.1. We attempt to define inductively

$$\tau^{i+1} = \tau^i - r^i, \quad I^{i+1} = \underline{\bar{x}}(\tau^i - r^i + 0),$$

and let r^{i+1} be the constant value of r on the nontrivial interval I^{i+1}. Indeed, that this is the case follows from the fact that for fixed $y \in [-D, C]$, one cannot have $f(x,y) = 0$ identically for x in any subinterval of $[-D, C]$.

Now take $\xi^0 \in I^0 \cap (-D, C)$ with $\xi^0 \neq 0$; then by Theorem 3.1 one has inductively the existence of $\xi^i \in I^i$ with $f(\xi^i, \xi^{i+1}) = 0$ for each i. We may assume moreover, by adjusting the choice of ξ^0 in the open interval $I^0 \cap (-D, C)$ if necessary, that $\xi^1 \in (-D, C)$. It now follows easily from our assumptions that $\xi^i \in (-D, C)$ and $\xi^i \xi^{i+1} < 0$ for each i, and also that $|\xi^{i+2}| < |\xi^i|$ for each i. Therefore the limits $\xi^{2i} \to \xi^*$ and $\xi^{2i+1} \to \xi^\#$ both exist, and

$$f(\xi^*, \xi^\#) = f(\xi^\#, \xi^*) = 0.$$

Necessarily then $\xi^* = \xi^\# = 0$, and we conclude that $\xi^j \in (0, \delta)$ for some j. Thus $I^j \cap (0, \delta)$ is a nontrivial interval on which r is constant; but this is a contradiction. □

4. Existence and monotonicity properties of solutions

In this section we consider results on existence and monotonicity of solutions in the spirit of those first obtained in [MP-Nu4]. In fact we shall make some modest extensions of the results in [MP-Nu4], in which we refine and sharpen the proofs to cover somewhat more general equations.

Throughout this section we assume $f: \mathbb{R}^2 \to \mathbb{R}$ and $r: \mathbb{R} \to \mathbb{R}$ are continuous. We are often concerned with solutions $x: \mathbb{R} \to \mathbb{R}$ of the equation

$$\dot{x}(t) = f\big(x(t), x(t-r)\big), \quad r = r\big(x(t)\big) \tag{4.1}$$

which are defined for all $t \in \mathbb{R}$; of particular interest are slowly oscillating periodic solutions, although our results often apply to more general solutions. At other times we assume $x(\cdot)$ is a solution only on some interval in $\mathbb{R}$; often it is the solution of an initial value problem.

Before stating our results we introduce some notation and terminology. We shall always denote

$$\eta(t) = t - r\big(x(t)\big). \tag{4.2}$$

If $I, J \subseteq \mathbb{R}$ are intervals with $I \subseteq J$, then we say that $x(\cdot)$ is a solution of (4.1) on (I, J) if $x: J \to \mathbb{R}$ is continuous, is C^1 on I, satisfies

$$\eta(t) \in J \quad \text{whenever } t \in I,$$

and is such that (4.1) holds for all $t \in I$. (If t is an endpoint of I, then by $\dot{x}(t)$ we always mean the appropriate one-sided derivative.)

The left- and right-hand upper Dini derivates of a function α are defined, respectively, as

$$D^{L+}\alpha(x) = \limsup_{\substack{h \to 0 \\ h > 0}} \frac{\alpha(x) - \alpha(x-h)}{h},$$

$$D^{R+}\alpha(x) = \limsup_{\substack{h \to 0 \\ h > 0}} \frac{\alpha(x+h) - \alpha(x)}{h}.$$

The lower Dini derivates $D^{L-}\alpha(x)$ and $D^{R-}\alpha(x)$ are defined by the same formulas except with lim inf in place of lim sup.

Now assume $f(\xi, \theta) = 0$ for some $(\xi, \theta) \in \mathbb{R}^2$. We say that f and r satisfy condition $E(\xi, \theta)$ if the following both hold:

(a) If $\zeta > \xi$ is such that $f(\zeta, \theta) \geqq 0$, and also $r(x) < r(\zeta)$ for all $x \in [\xi, \zeta)$, we have

$$D^{L-} r(\zeta) < \frac{1}{f(\zeta, \theta)}. \tag{4.3}$$

(b) If $\zeta < \xi$ is such that $f(\zeta, \theta) \leqq 0$, and also $r(x) < r(\zeta)$ for all $x \in (\zeta, \xi]$, we have

$$D^{R+} r(\zeta) > \frac{1}{f(\zeta, \theta)}.$$

If $f(\zeta, \theta) = 0$ in either (a) or (b) above, then we interpret

$$\frac{1}{f(\zeta, \theta)} = \begin{cases} \infty & \text{in (a)} \\ -\infty & \text{in (b)}. \end{cases} \tag{4.4}$$

Remark. Although the above condition seems rather technical, it holds for a wide variety of equations of interest. In particular, if f is strictly decreasing in its first argument then conditions (a) and (b) are satisfied (vacuously), so $E(\xi, \theta)$ holds whenever $f(\xi, \theta) = 0$. This is the case, for example, for the much-studied equation

$$\dot{x}(t) = -x(t) + \tilde{f}(x(t - r)).$$

If instead we have that $xf(x, 0) < 0$ for all $x \neq 0$, then $f(0, 0) = 0$, and $E(0, 0)$ holds vacuously, since the conditions $\zeta > 0$ and $f(\zeta, 0) \geqq 0$ in (a) can never simultaneously hold (and similarly with the corresponding conditions in (b)).

Finally, if we have that $xf(x, 0) \leqq 0$ for all $x \in \mathbb{R}$, then again $f(0, 0) = 0$. In this case $E(0, 0)$ holds provided also that r is locally lipschitz everywhere. Indeed, if $\zeta > 0$ is such that $f(\zeta, 0) \geqq 0$ as in (a), then necessarily $f(\zeta, 0) = 0$. Thus (4.3) holds with the interpretation (4.4), since $D^{L-} r(\zeta) < \infty$ by the lipschitz condition. One argues similarly for $\zeta < 0$.

Proposition 4.1. *Assume* $f(0,0)=0$, *and that* $E(0,0)$ *holds. With* $x(\cdot)$ *a solution on* (I,J) *for some intervals* $I \subseteq J$, *assume* $x(q^1)=x(q^2)=0$ *for some* $q^1 \in J$ *and* $q^2 \in I$ *satisfying*

$$q^2 - q^1 > r(0).$$

Then

$$\eta(t) > q^1 \quad \textit{for all } t \in [q^2, \infty) \cap I.$$

Proof. Assume the result is false. Then, as $\eta(q^2) = q^2 - r(0) > q^1$, it follows there exists $\tau \in (q^2, \infty) \cap I$ such that

$$\eta(t) > q^1 \quad \text{for all } t \in [q^2, \tau), \quad \eta(\tau) = q^1. \tag{4.5}$$

For $t \in [q^2, \tau)$ we therefore have

$$r(x(t)) = t - \eta(t) < \tau - \eta(\tau) = r(x(\tau)) \tag{4.6}$$

hence $x(t) \neq x(\tau)$. In particular, choosing $t = q^2$ shows that $x(\tau) \neq 0$; for definiteness assume this quantity is positive and denote it by

$$\zeta = x(\tau) > 0. \tag{4.7}$$

One concludes therefore from (4.6) and (4.7) that

$$r(x) < r(\zeta) \quad \text{for all } x \in [0, \zeta), \tag{4.8}$$

$$x(t) < x(\tau) \quad \text{for all } t \in [q^2, \tau); \tag{4.9}$$

moreover the differential equation (4.1), and (4.9) yield

$$\dot{x}(\tau) = f(\zeta, 0) \geqq 0. \tag{4.10}$$

Consider now $t = \tau - h$ for $h > 0$ sufficiently small; define $k > 0$ by $x(t) = x(\tau) - k$. Then from (4.5)

$$\begin{aligned} 0 < \frac{\eta(t) - \eta(\tau)}{h} &= \frac{t - \tau}{h} - \frac{r(x(t)) - r(x(\tau))}{h} \\ &= -1 - \left(\frac{r(\zeta - k) - r(\zeta)}{k}\right)\frac{k}{h} \\ &= -1 + \left(\frac{r(\zeta) - r(\zeta - k)}{k}\right)\left(\frac{x(\tau) - x(\tau - h)}{h}\right) \end{aligned} \tag{4.11}$$

and this implies, upon slightly rearranging the formula then letting $h \to 0$, that

$$D^{L-} r(\zeta) \geqq \frac{1}{\dot{x}(\tau)} = \frac{1}{f(\zeta, 0)}.$$

However, this violates the inequality (4.3) in condition $E(0,0)$, which holds in light of (4.7), (4.8), and (4.10). With this contradiction the proof is complete. □

Remark. The above result is particularly relevant to the study of slowly oscillating solutions, that is, solutions on $\mathbb{R}$ for which

$$q^2 - q^1 > r(0) > 0$$

for any $q^1 < q^2$ which satisfy $x(q^1) = x(q^2) = 0$. Typically the negative feedback condition

$$yf(0,y) < 0 \quad \text{for all } y \neq 0$$

is assumed when one considers slowly oscillating solutions.

We introduce here classes of slowly oscillating solutions which are periodic.

Definition. Assume $r(0) > 0$ and $yf(0,y) < 0$ for all $y \neq 0$. Let $m > 0$ be an integer. A *slowly oscillating* P_{2m} *solution* ($SOP_{2m}S$) of equation (4.1) is a slowly oscillating solution for which there exist quantities $\{q^n\}_{n=-\infty}^{\infty}$ such that for all n

$$\begin{aligned} q^{n+1} - q^n &> r(0)\,, \\ x(q^n) &= 0\,, \\ (-1)^n x(t) &> 0 \quad \text{for all } t \in (q^n, q^{n+1}) \\ x(t+T) &= x(t) \quad \text{for all } t \in \mathbb{R} \end{aligned}$$

where $T = q^{2m} - q^0$.

Remark. A slowly oscillating P_2 solution is usually called a *slowly oscillating periodic solution* (SOPS).

Remark. From the negative feedback condition one has $\dot{x}(q^n) \neq 0$ for all n, for an $SOP_{2m}S$.

Remark. If the assumptions of Proposition 4.1 hold then $\eta(t) > q^n$ for all $t \geqq q^{n+1}$, for an $SOP_{2m}S$.

Proposition 4.1 can be used to obtain monotonicity properties of periodic slowly oscillating solutions; the basic result is a sufficient condition for such solutions to possess Property M. Below we present a definition of Property M which, though equivalent to the definitions given in [MP-Nu 1] and [MP-Nu 4], is stated in a different and perhaps clearer fashion.

We first present some definitions which will be needed here, and also in later sections.

Definition. Let $\alpha : I \to \mathbb{R}$ where I is an interval. Then the *index of monotonicity* of α on I is defined to be

$$\text{(4.12)}\quad \operatorname*{mon}_{t\in I}\alpha(t)=\sup\{m\geqq 1\mid \text{there exist } t^0<t^1<\cdots<t^m \text{ with all } t^i\in I, \text{ such that } (\alpha(t^{i-1})-\alpha(t^i))(\alpha(t^i)-\alpha(t^{i+1}))<0 \text{ for } 1\leqq i\leqq m-1\}\,.$$

The index of monotonicity is either a positive integer or infinity; it is simply a count of the number of monotone pieces of α on I.

Definition. Let $\alpha:\mathbb{R}\to\mathbb{R}$, and let $C>0$ and $D>0$. We say that α possesses *Property* M *between* $-D$ *and* C if

(a) whenever $\alpha(q)=0$ then the derivative $\dot{\alpha}(q)\neq 0$ exists and is nonzero; and

(b) if $I\subseteqq\mathbb{R}$ is an open interval such that $\alpha(t)\neq 0$ and $\alpha(t)\in(-D,C)$ for $t\in I$ then

$$\text{(4.13)}\qquad \operatorname*{mon}_{t\in I}\alpha(t)\leqq 2\,;$$

also, if equality occurs in (4.13) then there exist $t^0<t^1<t^2$ in I with $|\alpha(t^1)|>|\alpha(t^i)|$, for $i=0,2$.

We present another useful definition.

Definition. A function $\alpha:\mathbb{R}\to\mathbb{R}$ satisfies *Property* MM if it satisfies Property M between $-D$ and C for every $C>0$ and $D>0$.

Remark. If $x(\cdot)$ is an $\mathrm{SOP}_{2m}\mathrm{S}$ solution which satisfies Property MM, and $T>0$ is its minimal period, then for any $\gamma\in\mathbb{R}$

$$2m\leqq \operatorname*{mon}_{t\in[\gamma,\gamma+T]} x(t)\leqq 2m+1\,.$$

The following generalization of the index of monotonicity will be useful later when applied to limiting profiles of solutions.

Definition. Let $F\subseteqq\mathbb{R}^2$ be any finite nonempty set, and let

$$\mathscr{C}(F)=\{\sigma=(\tau,\xi):\mathbb{R}\to\mathbb{R}^2\mid F\subseteqq\sigma(\mathbb{R}) \text{ and } \tau:\mathbb{R}\to\mathbb{R} \text{ is nondecreasing}\}\,.$$

Define

$$\operatorname{mon}F=\min_{(\tau,\xi)\in\mathscr{C}(F)}\ \operatorname*{mon}_{s\in\mathbb{R}}\xi(s)\,.$$

Definition. Let $G\subseteqq\mathbb{R}^2$ be any nonempty set. Define

$$\operatorname{mon}G=\sup_{\substack{\emptyset\neq F\subseteqq G,\\ F\text{ is finite}}}\operatorname{mon}F\,.$$

It is the case that for any regular sequence $x^n(\cdot)$, the value of $\operatorname{mon}\Omega$ is no larger than the limiting value (more precisely, the lim inf) of the index of monotonicity of the functions in the sequence. We prove this below in Proposition 4.5, following three technical lemmas.

Lemma 4.2. *Suppose $\tilde{\alpha}: I \to \mathbb{R}$ for some interval I. Let points $s^1 < s^2 < \cdots < s^k$ with each $s^i \in I$ be given, and denote $\tilde{x}^i = \tilde{\alpha}(s^i)$. Also let quantities $x^i \in \mathbb{R}$ be given, and suppose that*

$$x^i < x^j \quad \textit{implies} \quad \tilde{x}^i < \tilde{x}^j, \tag{4.14}$$

for $1 \leqq i, j \leqq k$ and $i \neq j$. Then there exists a function $\alpha: I \to \mathbb{R}$ such that $\alpha(s^i) = x^i$ for each i, and for which

$$\operatorname*{mon}_{s \in I} \alpha(s) \leqq \operatorname*{mon}_{s \in I} \tilde{\alpha}(s). \tag{4.15}$$

Proof. Let $\alpha: I \to \mathbb{R}$ denote the continuous function which is linear in each interval $[s^i, s^{i+1}]$ for $1 \leqq i \leqq k-1$, is constant on the interval $(-\infty, s^1] \cap I$ and also on the interval $[s^k, \infty) \cap I$, and takes the values $\alpha(s^i) = x^i$ for $1 \leqq i \leqq k$. Then it is easy to see that the supremum $m_0 = \operatorname*{mon}_{s \in I} \alpha(s)$ in the definition (4.12) (which without loss we assume is finite) can be achieved with a choice of points $\{t^0, t^1, \ldots, t^{m_0}\} \subseteqq \{s^1, s^2, \ldots, s^k\}$ taken from among the s^i, that is, $(\alpha(t^{i-1}) - \alpha(t^i))(\alpha(t^i) - \alpha(t^{i+1})) < 0$ for $1 \leqq i \leqq m_0 - 1$. But (4.14) now implies that $(\tilde{\alpha}(t^{i-1}) - \tilde{\alpha}(t^i))(\tilde{\alpha}(t^i) - \tilde{\alpha}(t^{i+1})) < 0$ for such i, and hence $m_0 \leqq \operatorname*{mon}_{s \in I} \tilde{\alpha}(s)$. From this we conclude (4.15).

Lemma 4.3. *Let $F \subseteqq \mathbb{R}^2$ be a finite set. Then there exists $\gamma > 0$ such that if $\tilde{F} \subseteqq \mathbb{R}^2$ is a finite set with the same number of elements as F, and if also $\bar{d}(F, \tilde{F}) < \gamma$ where $\bar{d}$ denotes the Hausdorff distance between sets, then*

$$\operatorname{mon} F \leqq \operatorname{mon} \tilde{F}.$$

Proof. One easily sees that there exists $\gamma > 0$ such that if $\tilde{F}$ is any set as in the statement of the lemma, then there exists a labeling of the points of F and of $\tilde{F}$ such that

$$F = \{p^1, p^2, \ldots, p^k\}, \quad p^i = (t^i, x^i), \quad \tilde{F} = \{\tilde{p}^1, \tilde{p}^2, \ldots, \tilde{p}^k\}, \quad \tilde{p}^i = (\tilde{t}^i, \tilde{x}^i),$$

and such that both

$$t^i < t^j \quad \text{implies} \quad \tilde{t}^i < \tilde{t}^j \tag{4.16}$$

holds for $1 \leqq i, j \leqq k$, and also that

$$x^i < x^j \quad \text{implies} \quad \tilde{x}^i < \tilde{x}^j \tag{4.17}$$

holds for $1 \leqq i, j \leqq k$. Now let $\tilde{\sigma} = (\tilde{\tau}, \tilde{\xi}) \in \mathscr{C}(\tilde{F})$ be such that

$$\operatorname{mon} \tilde{F} = \operatorname*{mon}_{s \in \mathbb{R}} \tilde{\xi}(s), \tag{4.18}$$

as in the definition of $\operatorname{mon}\tilde{F}$, and let $s^i \in \mathbb{R}$ be such that $\tilde{\sigma}(s^i) = \tilde{p}^i$. Observe that by relabeling the points $\tilde{p}^i$ and also the points p^i, we may assume without loss that $s^1 < s^2 < \cdots < s^k$, and still maintain the implications (4.16) and (4.17).

By Lemma 4.2, using the implications (4.16) and (4.17), there exist functions $\tau : \mathbb{R} \to \mathbb{R}$ and $\xi : \mathbb{R} \to \mathbb{R}$ such that $\tau(s^i) = t^i$ and $\xi(s^i) = x^i$ for each i, and such that

$$\operatorname*{mon}_{s \in \mathbb{R}} \tau(s) \leqq \operatorname*{mon}_{s \in \mathbb{R}} \tilde{\tau}(s), \quad \text{and} \quad \operatorname*{mon}_{s \in \mathbb{R}} \xi(s) \leqq \operatorname*{mon}_{s \in \mathbb{R}} \tilde{\xi}(s).$$

However, $\tilde{\tau}(\cdot)$ is nondecreasing since $(\tilde{\tau}, \tilde{\xi}) \in \mathscr{C}(\tilde{F})$, and hence $\operatorname*{mon}_{s \in \mathbb{R}} \tau(s) = \operatorname*{mon}_{s \in \mathbb{R}} \tilde{\tau}(s) = 1$. This implies that $\tau(\cdot)$ is either nondecreasing on $\mathbb{R}$, or is nonincreasing on $\mathbb{R}$. By replacing $\tau(s)$ and $\xi(s)$ with $\tau(-s)$ and $\xi(-s)$ if necessary, we may assume without loss that $\tau(\cdot)$ is also nondecreasing. Denoting $\sigma = (\tau, \xi)$, we see that since $\sigma(s^i) = p^i$ for each i, we have that $\sigma \in \mathscr{C}(F)$. Thus

$$\operatorname{mon} F \leqq \operatorname*{mon}_{s \in \mathbb{R}} \xi(s) \leqq \operatorname*{mon}_{s \in \mathbb{R}} \tilde{\xi}(s), \tag{4.19}$$

and (4.19) together with (4.18) imply the result. □

Lemma 4.4. *Let $\alpha : I \to \mathbb{R}$ for some interval $I \subseteqq \mathbb{R}$, and let $G \subseteqq \mathbb{R}^2$ denote the graph $G = \{(t, \alpha(t)) \mid t \in I\}$ of this function. Then*

$$\operatorname{mon} F \leqq \operatorname*{mon}_{t \in I} \alpha(t)$$

for any finite subset $F \subseteqq G$.

Remark. Associated to the function $\alpha : I \to \mathbb{R}$ in Lemma 4.4 are two quantities, $\operatorname*{mon}_{t \in I} \alpha(t)$ and $\operatorname{mon} G$, which are *a priori* different. An immediate consequence of Lemma 4.4 is the inequality

$$\operatorname{mon} G \leqq \operatorname*{mon}_{t \in I} \alpha(t). \tag{4.20}$$

Of course, one expects equality in (4.20). However, we do not need such a result, and so do not prove it here.

Proof. There exist points $s^1 < s^2 < \cdots < s^k$ with each $s^i \in I$ such that

$$F = \{(s^i, \alpha(s^i)) \mid 1 \leqq i \leqq k\}.$$

Define the function $\sigma = (\tau, \xi) : \mathbb{R} \to \mathbb{R}^2$ by setting $\tau(s) = s$ and $\xi(s) = \alpha(s)$ for $s \in [s^1, s^k]$, and taking the constant values $\sigma(s) = \sigma(s^1)$ for $s \leqq s^1$ and $\sigma(s) = \sigma(s^k)$ for $s \geqq s^k$. Then clearly $\sigma \in \mathscr{C}(F)$, and so $\operatorname{mon} F \leqq \operatorname*{mon}_{s \in \mathbb{R}} \xi(s)$, from the definition of $\operatorname{mon} F$. But one easily sees that

$$\operatorname*{mon}_{s \in \mathbb{R}} \xi(s) = \operatorname*{mon}_{s \in [s^1, s^k]} \xi(s) \leqq \operatorname*{mon}_{s \in I} \alpha(s),$$

and this implies the result. □

Proposition 4.5. *Suppose* $x^n : \mathbb{R} \to \mathbb{R}$ *is a sequence of continuous functions satisfying the bound* (2.3) *for some* $C(I)$, *for each interval* I, *and suppose in addition that the sequence* $x^n(\cdot)$ *is regular. Let* Ω *denote the limiting profile of* $x^n(\cdot)$, *and let* $I^n \subseteqq \mathbb{R}$ *be a sequence of intervals such that* $I^n \subseteqq I^{n+1}$ *and* $\bigcup I^n = \mathbb{R}$. *Then*

$$\text{mon}\,\Omega \leqq \liminf_{n\to\infty} \text{mon}_{t\in I^n} x^n(t)\,. \tag{4.21}$$

Proof. Let $m \geqq 1$ be any integer such that $m \leqq \text{mon}\,\Omega$. Then there exists a finite set $F \subseteqq \Omega$ such that $m \leqq \text{mon}\,F$. Let $G^n \subset \mathbb{R}^2$ denote the set $G^n = \{(t, x^n(t)) \mid t \in I^n\}$. Then for any finite subset $\tilde{F} \subseteqq G^n$ we have $\text{mon}\,\tilde{F} \leqq \text{mon}_{t\in I^n} x^n(t)$ by Lemma 4.4. Thus it is enough to find, for each sufficiently large n, a finite set $\tilde{F}^n \subseteqq G^n$ such that

$$\text{mon}\,F \leqq \text{mon}\,\tilde{F}^n\,, \tag{4.22}$$

in order to establish (4.21).

For every point $p \in F$ there exists a sequence $p^n \in G^n$ such that $p^n \to p$ as $n \to \infty$. This holds in particular because the sequence $x^n(\cdot)$ is regular. Thus for each n there exists a finite set $\tilde{F}^n \subseteqq G^n$ with the same number of elements as the set F, such that $\bar{d}(F, \tilde{F}^n) \to 0$. Thus (4.22) follows from Lemma 4.3 for large n, and this completes the proof of the proposition. □

We present one more definition before stating Theorem 4.6.

Definition. Let $p^i = (x^i, y^i)$ for $i = 1, 2$ be two points in the plane. We say

$$p^0 < p^1 \text{ in case } x^0 < x^1 \text{ and } y^0 \leqq y^1\,,$$

and

$$p^0 \ll p^1 \text{ in case } x^0 < x^1 \text{ and } y^0 < y^1\,.$$

The relations $>$ and $\gg$ are defined analogously.

Theorem 4.6. *Assume there exist quantities* $C > 0$ *and* $D > 0$ *such that*

(a) *if the point* $p^0 \in (-D, C) \times (-D, C)$ *satisfies* $f(p^0) = 0$, *and* $p^1 \in \mathbb{R}^2$ *satisfies* $\pm(p^1 - p^0) > 0$, *then* $\pm f(p^1) < 0$;

(b) *for each* $x \in (-D, C)$ *there exists* $y \in (-D, C)$ *such that* $f(x, y) = 0$;

(c) $yf(0, y) < 0$ *for all* $y \neq 0$;

(d) *if* $f(x^0, x^1) = f(x^1, x^2) = 0$ *and* $x^0 \in (-D, C)\backslash\{0\}$, *then* $|x^2| < |x^0|$; *and*

(e) r *is locally lipschitz, and* $r(0) > 0$.

Then every $\mathrm{SOP}_{2m}\mathrm{S}$ *of equation* (4.1), *for every* $m \geqq 1$, *satisfies Property* M *between* $-D$ *and* C.

Remark. If in Theorem 4.6 the function f is strictly monotone in its second argument (condition (a) only implies monotonicity which may not be strict), then for each $x \in (-D, C)$, the point $y \in (-D, C)$ in condition (b) is unique (see the remark following the statement of Theorem 2.9). Denoting $y = g(x)$, we see that condition (d) is equivalent to the condition $|g(g(x^0))| < |x^0|$ for all $x^0 \in (-D, C)\backslash\{0\}$.

We present a lemma before giving the proof of the above theorem.

Lemma 4.7. *Assume conditions* (a), (b), *and* (c) *as in the statement of Theorem* 4.6. *Then if* $f(x,y) = 0$ *we have either*

$$xy < 0 \quad or \quad x = y = 0\,. \tag{4.23}$$

Moreover if $x \in (-D, C)$ *then*

$$y \in (-D, C)\,. \tag{4.24}$$

If condition (d) *also holds, and if*

$$f(x, \tilde{y}) = f(y, z) = 0 \tag{4.25}$$

for quantities x, $\tilde{y}$, y, *and* z *satisfying*

$$x \in (-D, C) \quad and \quad |y| \leqq |\tilde{y}|, \quad with \quad xy \leqq 0\,, \tag{4.26}$$

then either $x = 0$ *or*

$$|z| < |x|\,. \tag{4.27}$$

Proof. Assume $f(x,y) = 0$. From (c) we have that $y = 0$ if $x = 0$; also, $f(0,0) = 0$, and so (4.23) easily follows from (a), with $p^0 = (0,0)$.

To prove (4.24) suppose $f(x,y) = 0$ with $x \in (-D, C)$, but $y \notin (-D, C)$; for definiteness assume $y \geqq C$. Fix any $x^0 \in (-D, x)$ and let $y^0 \in (-D, C)$ be such that $f(x^0, y^0) = 0$; such exists from (b). But now $(x^0, y^0) < (x, y)$, and so $f(x,y) < 0$ by (a). This is a contradiction.

Now assume (4.25) with (4.26) holding; also assume $x \neq 0$. Let $\tilde{z}$ be such that $f(\tilde{y}, \tilde{z}) = 0$; such exists from (b), as $\tilde{y} \in (-D, C)$ since $x \in (-D, C)$. Then (a) implies (with (4.25) and (4.26)) that $|z| \leqq |\tilde{z}|$, and hence

$$|z| \leqq |\tilde{z}| < |x| \tag{4.28}$$

with the strict inequality in (4.28) following from (d). This gives (4.27), as desired. □

Proof of Theorem 4.6. First observe that the hypotheses of Proposition 4.1 hold as $xf(x,0) < 0$ for all $x \neq 0$, by (a). Also observe that $xf(x,x) < 0$ for all $x \neq 0$.

Assume that $x(\cdot)$ is a slowly oscillating solution which possesses a bi-infinite set $\{q^n\}_{n=-\infty}^{\infty}$ of zeros, with $q^n < q^{n+1}$ and $(-1)^n x(t) > 0$ in (q^n, q^{n+1}). We claim that $\eta(t) < t$ for all $t \in \mathbb{R}$. Certainly this is true at each $t = q^n$, so assume there is a first time $\sigma \in (q^n, q^{n+1})$ at which $\eta(\sigma) = \sigma$, that is, $r(x(\sigma)) = 0$. Assume without loss that n is even, so $x(\sigma) > 0$; then $0 < x(t) < x(\sigma)$ for $t \in (q^n, \sigma)$ as σ is the first time at which $r(x(t)) = 0$, and so $\dot{x}(\sigma) \geqq 0$. But

$$\dot{x}(\sigma) = f(x(\sigma), x(\eta(\sigma))) = f(x(\sigma), x(\sigma)) < 0,$$

a contradiction. This proves the claim.

Next consider any τ such that $\dot{x}(\tau) = 0$; say $\tau \in (q^n, q^{n+1})$. Proposition 4.1 implies that $\eta(\tau) > q^{n-1}$, and $\eta(\tau) < q^{n+1}$ from above. Moreover, $f(x(\tau), x(\eta(\tau))) = 0$ from the differential equation, hence $x(\tau)$ and $x(\eta(\tau))$ have opposite signs by Lemma 4.7. Thus $\eta(\tau) \in (q^{n-1}, q^n)$.

Now assume $x(\cdot)$ is an $\mathrm{SOP}_{2m}\mathrm{S}$, as in the statement of the theorem. Following the proofs of Theorem 3.1 in [MP-Nu1] and Theorem 2.2 in [MP-Nu4], we consider the sets

$$\begin{aligned} T = \{\tau \in \mathbb{R} \mid \dot{x}(\tau) = 0, &\text{ with } \tau \in (q^n, q^{n+1}) \text{ for some } n, \\ &\text{such that there exist } \tau^1, \tau^2 \in (q^n, q^{n+1}) \text{ with} \\ &\tau^1 < \tau < \tau^2 \text{ and } |x(\tau)| < |x(\tau^i)| \text{ for } i = 1, 2\}, \end{aligned} \tag{4.29}$$

and

$$S = \{x(\tau) \mid \tau \in T\}.$$

As in [MP-Nu1], it is sufficient to prove $S \cap (-D, C) = \emptyset$ in order to conclude that Property M holds between $-D$ and C. In this direction, first note that

$$S \cap (-\delta, \delta) = \emptyset \tag{4.30}$$

for some $\delta > 0$, since $x(\cdot)$ is periodic and all its zeros are simple. We shall prove that for each

$$\xi \in S \cap (-D, C) \tag{4.31}$$

there exists

$$\zeta \in S \cap (-D, C) \tag{4.32}$$

such that $\xi\zeta < 0$ and

$$|\zeta| \leqq |\tilde{\zeta}| \text{ for some } \tilde{\zeta} \text{ such that } f(\xi, \tilde{\zeta}) = 0. \tag{4.33}$$

We first show that it is enough to find such ζ for each ξ in order to prove the theorem. Indeed, it follows from the existence of such ζ that beginning with any $\xi^0 \in S \cap (-D, C)$, one can construct a sequence $\xi^i \in S \cap (-D, C)$ such that $\xi^i \xi^{i+1} < 0$ and $|\xi^{i+1}| \leqq |\tilde{\xi}^{i+1}|$ for some $\tilde{\xi}^{i+1}$ satisfying $f(\xi^i, \tilde{\xi}^{i+1}) = 0$. Lemma 4.7 now implies $|\xi^{i+2}| < |\xi^i|$ for each i, and hence the limits $\xi^{2i} \to \xi^*$ and $\xi^{2i+1} \to \xi^\#$ exist as these sequences are monotone. The limits $\tilde{\xi}^{2i} \to \tilde{\xi}^*$ and $\tilde{\xi}^{2i+1} \to \tilde{\xi}^\#$ also exist, as these sequences too are monotone, by (a). Therefore, in the limit,

$$f(\xi^*, \tilde{\xi}^\#) = f(\xi^\#, \tilde{\xi}^*) = 0$$

with $|\xi^\#| \leqq |\tilde{\xi}^\#|$ and $\xi^* \xi^\# \leqq 0$, implying either $\xi^* = 0$ or $|\tilde{\xi}^*| < |\xi^*|$, by Lemma 4.7. But $|\xi^*| \leqq |\tilde{\xi}^*|$ by construction, and so necessarily $\xi^* = 0$. Similarly $\xi^\# = 0$, and so

$$\xi^i \to 0 \quad \text{as } i \to \infty . \tag{4.34}$$

But as $\xi^i \in S$, the limit (4.34) contradicts (4.30).

It follows therefore, that to complete the proof of the theorem, all that is needed is to show that (4.32) and (4.33) hold for some ζ, with $\xi\zeta < 0$, given ξ as in (4.31).

Assume then that ξ satisfies (4.31), say $\xi = x(\tau)$ where $\tau \in T$; for definiteness assume $x(\tau) > 0$. Let $q^n < \tau^1 < \tau < \tau^2 < q^{n+1}$ be as in the definition (4.29) of T. Without loss we may assume $x(\cdot)$ achieves its minimum in $[\tau^1, \tau^2]$ at the point τ, and that is the rightmost such point in that interval. For if not, we could replace τ with the rightmost point $\tau^0 \in (\tau^1, \tau^2)$ at which the minimum is assumed. By setting $\xi^0 = x(\tau^0)$, we would have that $\xi^0 \leqq \xi$ and hence that $|\tilde{\zeta}^0| \leqq |\tilde{\zeta}|$ for points $\tilde{\zeta}$ and $\tilde{\zeta}^0$ at which $f(\xi^0, \zeta) = 0$ and $f(\tilde{\xi}^0, \tilde{\zeta}) = 0$. Upon obtaining a point $\zeta^0 \in S \cap (-D, C)$ satisfying $\xi^0 \zeta^0 < 0$ and $|\zeta^0| \leqq |\tilde{\zeta}^0|$, we would see that $|\zeta^0| \leqq |\tilde{\zeta}|$ also holds, and we could thus set $\zeta = \zeta^0$ to obtain (4.33) exactly as stated for ξ.

We therefore assume without loss that

$$x(\tau) \leqq x(t) \quad \text{for } t \in [\tau^1, \tau^2] ,$$

with strict inequality at $t = \tau^1$ and for all $t \in (\tau, \tau^2]$, and attempt to find ζ and $\tilde{\zeta}$ as in (4.32) and (4.33).

We also claim that without loss we may assume

$$\dot{x}(\tau^i) \geqq 0 \quad \text{and} \quad x(\tau^i) \in (-D, C) \quad \text{for } i = 1, 2$$

and

$$\eta(\tau^1) < \eta(\tau) < \eta(\tau^2) .$$

For τ^2 this is easily seen: as $\dot{x}(\tau) = 0$, we have $\dot{\eta}(\tau) = 1$, hence $\eta(\tau^2) > \eta(\tau)$ for all $\tau^2 > \tau$ sufficiently near τ. And certainly there exist such point τ^2 with $\dot{x}(\tau^2) \geqq 0$, and with the strict inequality $x(\tau^2) > x(\tau)$.

More care is needed in choosing τ^1. First let $\tau^* \in (q^n, \tau)$ be a point where $x(\cdot)$ achieves its maximum in that interval; note that $x(\tau^*) > x(\tau)$. Consider any quantity ξ^* satisfying

$\xi < \xi^* < x(\tau^*)$, and let τ^1 be the leftmost point in (q^n, τ^*) for which $x(\tau^1) = \xi^*$. Then certainly $\dot{x}(\tau^1) \geqq 0$; and

$$\eta(\tau) - \eta(\tau^1) = \tau - \tau^1 - (r(\xi) - r(\xi^*)) \\ > \tau - \tau^* - (r(\xi) - r(\xi^*)). \tag{4.35}$$

It follows from (4.35) that if ξ^* is chosen sufficiently near ξ, then $\eta(\tau) > \eta(\tau^1)$, as desired.

We now have

$$f(x(\tau), x(\eta(\tau))) = \dot{x}(\tau) = 0, \quad f(x(\tau^i), x(\eta(\tau^i))) = \dot{x}(\tau^i) \geqq 0, \tag{4.36}$$

so property (a) and Lemma 4.7 imply that $x(\eta(\tau))$ and $x(\eta(\tau^i))$ are all negative, and

$$|x(\eta(\tau))| < |x(\eta(\tau^i))| \quad \text{for } i = 1, 2, \tag{4.37}$$

and

$$x(\eta(\tau)) \in (-D, C). \tag{4.38}$$

Necessarily $\eta(\tau^i) < q^n$ as $x(\eta(\tau^i)) < 0$. Upon observing that condition $E(0,0)$ holds (since $xf(x,0) < 0$ for all $x \neq 0$, by condition (a); see the remark preceding Proposition 4.1), we have from Proposition 4.1 that $\eta(\tau^i) > q^{n-1}$ and so $x(t) < 0$ for all

$$t \in [\eta(\tau^1), \eta(\tau^2)] \subseteqq (q^{n-1}, q^n).$$

Let $\tilde{\zeta} = x(\eta(\tau))$, let $\sigma \in (\eta(\tau^1), \eta(\tau^2))$ denote a point at which $|x(\cdot)|$ assumes its minimum in that interval, and set $\zeta = x(\sigma)$. Then certainly (4.32) follows from (4.37) and (4.38). Also, (4.33) holds as $f(\xi, \tilde{\zeta}) = 0$ by (4.36). This completes the proof of the theorem. □

Corollary 4.8. *Assume the hypotheses of Theorem* 4.6 *except that condition* (a) *need only hold for* $p^1 \in [-D, C] \times [-D, C]$, *condition* (c) *need only hold for* $y \in [-D, C] \setminus \{0\}$, *and* r *need only be lipschitz in* $[-D, C]$, *with* $r(0) > 0$. *Let* $x(\cdot)$ *be an* $\mathrm{SOP}_{2m}\mathrm{S}$ *which satisfies* $-D \leqq x(t) \leqq C$ *for all* t. *Then* $x(\cdot)$ *satisfies Property* MM.

Proof. We show $x(\cdot)$ satisfies Property M between $-D$ and C by suitably modifying f and r outside the sets $[-D, C] \times [-D, C]$ and $[-D, C]$ respectively so that Theorem 4.6 applies. Indeed, let

$$\tilde{f}(x, y) = f(\kappa(x), \kappa(y)) \quad \text{and} \quad \tilde{r}(x) = r(\kappa(x)) \tag{4.39}$$

where

$$\kappa(x) = \begin{cases} -D & \text{for } x < -D, \\ x & \text{for } -D \leqq x \leqq C, \\ C & \text{for } x > C. \end{cases} \tag{4.40}$$

Then $x(\cdot)$ satisfies (4.1) with $\tilde{f}$ and $\tilde{r}$ replacing f and r, and one easily sees that this modified system satisfies the conditions of Theorem 4.6. □

The following result is suited to solutions of small amplitude.

Corollary 4.9. *Suppose that*

$$\operatorname{sgn} f(x,y) = \operatorname{sgn}(-x + h(y))$$

for all (x,y) *in some neighborhood of* $(0,0)$ *for some (not necessarily strictly) monotone decreasing function* h *which also satisfies*

$$yh(y) < 0 \quad \text{and} \quad |h(h(y))| > |y| \quad \text{for } y \neq 0,$$

for all y *near* 0. *Assume also that* r *is lipschitz in some neighborhood of the origin, and that* $r(0) > 0$. *Then there exists* $\delta > 0$ *such that any* $\mathrm{SOP}_{2m}\mathrm{S}$ *satisfying* $|x(t)| \leqq \delta$ *for all* $t \in \mathbb{R}$ *possesses Property* MM.

Proof. Choose any $C > 0$ sufficiently small, and set $D = -h(C)$. Note that

$$h(-D) = h(h(C)) > C.$$

One easily checks now that the hypotheses of Corollary 4.8 hold. In particular (b) holds as $(-D, C) \subseteqq h((-D, C))$, so that for each $x \in (-D, C)$ there exists $y \in (-D, C)$ with $x = h(y)$, hence $f(x,y) = 0$. Thus it is enough to choose $\delta \leqq \min\{C, D\}$. □

Remark. The following result was proved in a less general form in [MP-Nu4] as Theorem 2.1; we include it here for completeness although it will not be needed for our current work. We present also Corollary 4.11, which is in some sense the analog of Proposition 4.1 for critical points rather than zeros of a solution.

Proposition 4.10. *Assume condition* $E(\xi, \theta)$ *holds whenever* $f(\xi, \theta) = 0$. *Also assume the function* r *is locally lipschitz. Then, with* $x(\cdot)$ *a solution on some* (I, J) *we have*

$$\eta(t) > \eta(\tau) \quad \text{for all } t \in (\tau, \infty) \cap I \tag{4.41}$$

whenever $\tau \in I$ *and* $\dot{x}(\tau) = 0$.

Proof. We first claim that if $\dot{x}(\tau) = 0$ then the function η is strictly increasing in some neighborhood of τ in the interval I. To see this, let $K > 0$ be a local lipschitz constant for r, in a neighborhood of the point $x(\tau)$, and consider points $t^1 < t^2$, both in I, in a sufficiently small neighborhood of τ, so that $|\dot{x}(t)| \leqq 1/(2K)$ for all t between t^1 and t^2. Then

$$\begin{aligned} \eta(t^2) - \eta(t^1) &= t^2 - t^1 - (r(x(t^2)) - r(x(t^1))) \\ &\geqq t^2 - t^1 - K|x(t^2) - x(t^1)| \geqq t^2 - t^1 - |t^2 - t^1|/2 > 0 \end{aligned}$$

as required. This proves the claim.

We next note that, as in the proof of Theorem 2.1 in [MP-Nu4], in order to prove the proposition it is sufficient to prove only

$$\eta(t) > \eta(\tau) \quad \text{for all } t \in (\tau, \sigma]$$

where $\dot{x}(\tau) = 0$ and $\sigma > \tau$ is any point in I such that $\dot{x}(t) \neq 0$ for all $t \in (\tau, \sigma)$. We prove this by contradiction, assuming for some $\sigma > \tau$ that

$$\eta(t) > \eta(\tau) \quad \text{for all } t \in (\tau, \sigma), \quad \eta(\sigma) = \eta(\tau),$$

and for definiteness

$$\dot{x}(t) > 0 \quad \text{for all } t \in (\tau, \sigma). \tag{4.42}$$

The remainder of the proof now closely follows that of Proposition 4.1. Denote $\xi = x(\tau)$ and $\zeta = x(\sigma)$, and let $\theta = x(\eta(\tau))$. Then $\dot{x}(\tau) = f(\xi, \theta) = 0$. From (4.42) we conclude that $\zeta > \xi$ and $\dot{x}(\sigma) = f(\zeta, \theta) \geqq 0$. An estimate similar to (4.6) allows us to conclude

$$r(x) < r(\zeta) \quad \text{for all } x \in [\xi, \zeta).$$

Finally, by letting $t = \sigma - h$ for small $h > 0$ and $x(t) = x(\sigma) - k$ (where $k > 0$), and considering the quotient

$$\frac{\eta(t) - \eta(\sigma)}{h} > 0$$

as in (4.11), we conclude that

$$D^{L-} r(\zeta) \geqq \frac{1}{f(\zeta, \theta)}.$$

This however contradicts condition $E(\xi, \theta)$, and completes the proof. □

Corollary 4.11. *Assume that*

$$\pm f(p^1) < 0 \text{ whenever } \pm(p^1 - p^0) > 0 \text{ and } f(p^0) = 0, \tag{4.43}$$

and that $yf(0, y) < 0$ *for all* $y \neq 0$ *(so in particular* $f(0,0) = 0$*). Also assume the function* r *is locally lipschitz. Let* $x(\cdot)$ *be a solution on some* (I, J) *such that*

$$0 < |x(t)| < |x(\tau)| \quad \text{for } t \in (\tau, q^0),$$

and such that $|x(\cdot)|$ *is nonincreasing in* (τ, q^0) *with* $\dot{x}(\tau) = 0$*, where* $[\tau, q^0) \subseteqq I$*, with either* $q^0 \in I$ *and* $x(q^0) = 0$*, or* $q^0 = \infty$*. Then there exists* $q^* \in J$ *such that* $q^* > \eta(\tau)$ *and*

$$0 < |x(t)| < |x(\eta(\tau))| \quad \text{for } t \in (\eta(\tau), q^*), \tag{4.44}$$

and

$$x(q^*) = 0. \tag{4.45}$$

If the second inequality in the hypothesis (4.43) *is changed from* $>$ *to* $\gg$, *but we still assume that* $xf(x,0) < 0$ *for all* $x \neq 0$, *then the second inequality in the conclusion* (4.44) *is changed from* $<$ *to* $\leqq$.

Remark. Corollary 4.11 is of particular interest when $x(\cdot)$ is a solution possessing Property MM. Suppose in fact that $x(\cdot)$ is a slowly oscillating solution having Property MM; let $\{q^n\}$ denote the successive zeros of this solution, and let $\tau^n \in (q^n, q^{n+1})$ denote the rightmost point in (q^n, q^{n+1}) at which $|x(\cdot)|$ attains its maximum in (q^n, q^{n+1}). Then under the conditions (4.43) we have $\eta(\tau^n) \geqq \tau^{n-1}$; in fact, the conditions of Proposition 4.10 always hold, and so

$$\eta(t) > \eta(\tau^n) \geqq \tau^{n-1} \quad \text{for all } t > \tau^n .$$

Proof. We first note that the hypotheses of Proposition 4.10 hold, and hence so does the conclusion (4.41).

Next, assuming for definiteness that $x(\tau) > 0$ and noting that $x(\eta(\tau)) < 0$, since $f(x(\tau), x(\eta(\tau))) = 0$, we show that

$$x(\eta(t)) > x(\eta(\tau)) \quad \text{for all } t \in (\tau, q^0) . \tag{4.46}$$

Indeed, if (4.46) failed at some point, then as $x(t) < x(\tau)$, it would follow that

$$\dot{x}(t) = f(x(t), x(\eta(t))) > f(x(\tau), x(\eta(\tau))) = \dot{x}(\tau) = 0 ,$$

a contradiction.

We finally note that if $q^0 < \infty$, then

$$0 \geqq \dot{x}(q^0) = f(0, x(\eta(q^0)))$$

hence $x(\eta(q^0)) \geqq 0$, and it follows immediately from (4.41) and (4.46) that there exists $q^* \in J$ such that (4.44) and (4.45) hold. If on the other hand $q^0 = \infty$, then $x(t)$ and hence $x(\eta(t))$ are positive for all large t. Again there exists q^* as in the statement of the corollary, as desired.

The proof of the final sentence in the statement of the corollary follows the same lines as above, except that the inequality in (4.46) is not strict. □

We now examine the question of existence of SOPS's, again in the spirit of [MP-Nu 4]. We reintroduce the parameter $\varepsilon > 0$ and consider

$$\varepsilon\dot{x}(t) = f(x(t), x(t-r)), \quad r = r(x(t)) . \tag{4.47}$$

For the rest of this section ε is best regarded as a bifurcation parameter rather than a singular parameter, as we consider a range of values ε, not necessarily small.

We shall first need to consider the existence and uniqueness questions for solutions of initial value problems of (4.47). In fact we must consider the system

(4.48) $$\varepsilon \dot{x}(t) = f^{\varrho}(x(t), x(t-r^{\varrho}))$$

with a parameter $\varrho \in [0,1]$ which serves to make a homotopy to a constant delay problem. Here f^{ϱ} and r^{ϱ} are as follows. To define f^{ϱ} first let

$$\mu: [-D, C] \times [-D, C] \to [0,1]$$

be a lipschitz function satisfying

$$\mu(x,y) = 0 \quad \text{if and only if} \quad x \in \{-D, C\} \text{ and } xf(x,y) \geqq 0,$$

and

$$\mu(x,y) = 1 \quad \text{for all } (x,y) \text{ near } (0,0).$$

Also let $\check{f}: [-D, C] \to [-D, C]$ be a C^1 function satisfying $y\check{f}(y) < 0$ for all

$$y \in [-D, C] \setminus \{0\},$$

and $\check{f}'(0) = -B$, and set $\hat{f}(x,y) = -Ax + \check{f}(y)$, where A and B are positive constants to be given later. Let

(4.49) $$f^{\varrho}(x,y) = \begin{cases} (1 - 2\varrho + 2\varrho\mu(x,y))f(x,y) & \text{if } \varrho \in [0, 1/2], \\ (2 - 2\varrho)\mu(x,y)f(x,y) + (2\varrho - 1)\hat{f}(x,y) & \text{if } \varrho \in [1/2, 1]. \end{cases}$$

The delay r^{ϱ} is defined as

(4.50) $$r^{\varrho} = \begin{cases} r(x(t)) & \text{if } \varrho \in [0, 1/2], \\ (2 - 2\varrho)r(x(t)) + (2\varrho - 1)r(0) & \text{if } \varrho \in [1/2, 1]. \end{cases}$$

We first present the following result, whose proof, at least for uniqueness, is not entirely standard. We note the quantities C and D need not be positive, although this will surely be the case later. We denote the space of lipschitz functions

$$L_K[-R, 0] = \{\varphi \in C[-R, 0] \mid -D \leqq \varphi(\theta) \leqq C \text{ for all } \theta \in [-R, 0], \text{ and } \operatorname{lip}(\varphi) \leqq K\}$$

for any $K > 0$, where $\operatorname{lip}(\varphi)$ denotes the lipschitz constant of φ. Endowed with the usual metric of $C[-R, 0]$, the set $L_K[-R, 0]$ is a compact space.

Proposition 4.12. *Let $-D < C$, assume R satisfies*

(4.51) $$0 \leqq r(x) \leqq R \quad \text{if } x \in [-D, C],$$

and f satisfies

(4.52) $$f(C, y) \leqq 0 \quad \text{if } y \in \mathscr{S}_C, \quad f(-D, y) \geqq 0 \quad \text{if } y \in \mathscr{S}_{-D},$$

where

$$\mathscr{S}_x = \begin{cases} [-D, C] & \text{if } r(x) > 0, \\ \{x\} & \text{if } r(x) = 0. \end{cases}$$

Then if $\varphi \in C[-R,0]$, *with* $-D \leqq \varphi(\theta) \leqq C$ *for all* $\theta \in [-R,\theta]$, *there exists a solution* $x(\cdot)$ *of* (4.48) *on* (I,J) *with* $I=[0,\infty)$ *and* $J=[-R,\infty)$, *and with* $x|_{[-R,0]}=\varphi$ *satisfying*

$$-D \leqq x(t) \leqq C \quad \textit{for all } t \geqq 0\,. \tag{4.53}$$

If f and r are locally lipschitz and $\varphi \in L_K[-R,0]$ *for some* $K>0$ *then this solution is unique in the class of solutions* (4.53), *and moreover varies continuously as a function of*

$$(t,\varphi,\varepsilon,\varrho) \in [0,\infty) \times L_K[-R,0] \times (0,\infty) \times [0,1]\,.$$

Proof. It is enough to prove the existence and uniqueness in the case $\varrho=0$, that is, for the original f and r; this is because the assumptions (4.51) and (4.52) hold true for f^ϱ and r^ϱ as well, due to the form (4.49), (4.50) of the homotopy.

To prove existence we note that if f and r are replaced with $\tilde{f}$ and $\tilde{r}$ as given in (4.39) and (4.40), then the initial value problem for this modified system possesses a unique solution $\tilde{x}(t)$ for $t \geqq 0$ by the existence theory in [Hal-VL]. It is a simple matter, using the inequalities (4.52), to show that $\tilde{x}(\cdot)$ in fact satisfies (4.53), hence $x(t)=\tilde{x}(t)$ is a solution of (4.47).

To prove the claim of uniqueness assume $\bar{x}(t)$ is another solution of (4.47) with the same initial condition φ, also satisfying (4.53). Without loss assume K is also a lipschitz constant for $\varepsilon^{-1}f$ (in each argument) and r in $[-D,C]$, and as well an upper bound for $|\varepsilon^{-1}f(x,y)|$ for $x,y \in [-D,C]$ and hence a lipschitz constant for $x(\cdot)$ and $\bar{x}(\cdot)$. Let $y(t)=|x(t)-\bar{x}(t)|$ and let η be as in (4.2). Then

$$\begin{aligned}
\dot{y}(t) &= \varepsilon^{-1}|f(x(t),x(t-r(x(t))))-f(\bar{x}(t),\bar{x}(t-r(\bar{x}(t))))| \\
&\leqq Ky(t)+K|x(t-r(x(t)))-\bar{x}(t-r(\bar{x}(t)))| \\
&\leqq Ky(t)+K|x(t-r(x(t)))-\bar{x}(t-r(x(t)))| \\
&\quad +K|\bar{x}(t-r(x(t)))-\bar{x}(t-r(\bar{x}(t)))| \\
&= Ky(t)+Ky(\eta(t))+K|\bar{x}(t-r(x(t)))-\bar{x}(t-r(\bar{x}(t)))| \\
&\leqq Ky(t)+Ky(\eta(t))+K^3y(t)\,,
\end{aligned}$$

hence with $z(t)=\max\limits_{[0,t]} y(\cdot)$ we have

$$\begin{aligned}
y(t) &\leqq \int_0^t (K^3+K)y(s)+Ky(\eta(s))\,ds \\
&\leqq (K^3+2K)\int_0^t z(s)\,ds \leqq (K^3+2K)\int_0^\tau z(s)\,ds
\end{aligned}$$

provided $\tau \geqq t$. Maximizing $y(t)$ above over $[0,\tau]$ yields

$$z(\tau) \leqq (K^3+2K)\int_0^\tau z(s)\,ds$$

which implies $z(\tau) = 0$ for all $\tau \geqq 0$, as desired. This proves uniqueness.

The final claim of continuous dependence is proved in a standard fashion by considering sequences $(\varphi^n, \varepsilon^n, \varrho^n) \to (\varphi, \varepsilon, \varrho)$ in $L_K[-R, 0] \times (0, \infty) \times [0, 1]$, and obtaining by Ascoli's theorem a subsequence $x^{n^i}(t) \to x(t)$ converging uniformly on compact sets to a limiting solution. Using the uniqueness of the limiting solution, one shows in fact $x^n(t) \to x(t)$ compact-uniformly for the full sequence; this implies continuous dependence. □

The next result is needed to define the appropriate cone map as in [MP-Nu4]. For this purpose we denote

$$P_K[-R, 0] = \{\varphi \in L_K[-R, 0] \mid \varphi(0) = 0 \text{ and } \varphi(\theta) \geqq 0 \text{ for all } \theta \in [-R, 0]\},$$

$$U_K[-R, 0] = \{\varphi \in P_K[-R, 0] \mid \varphi(\theta) > 0 \text{ for some } \theta \in [-r(0), 0]\}.$$

Proposition 4.13. *Assume the functions f and r are locally lipschitz, and that* $0 < \varepsilon \leqq \varepsilon^*$ *for some quantity* ε^*. *Also assume there exist positive quantities C, D, and R such that*

(a) *r satisfies the inequalities* (4.51), *with* $r(0) > 0$;

(b) *f satisfies the inequalities* (4.52);

(c) $yf(0, y) < 0$ *for all* $y \in [-D, C] \backslash \{0\}$; *and*

(d) $xf(x, y) < 0$ *if* x, y $[-D, C]$ *and* $xy > 0$.

Let $\varphi \in P_K[-R, 0]$ *for some* $K > 0$, *let* $\varrho \in [0, 1]$, *and let* $x(\cdot)$ *denote the unique solution of* (4.48) *through* φ *satisfying* (4.53).

If $\varphi \notin U_K[-R, 0]$ *then* $x(t) = 0$ *for all* $t \geqq 0$. *If on the other hand* $\varphi \in U_K[-R, 0]$, *then there exist* $q^0 \in [0, r(0))$ *and* $q^1 \in (q^0 + r(0), \infty]$, *and also* $q^2 \in (q^1 + r(0), \infty]$ *if* $q^1 < \infty$, *such that*

(4.54) $$x(t) = 0 \quad \text{for } 0 \leqq t \leqq q^0,$$

(4.55) $$x(t) < 0 \quad \text{for } q^0 < t < q^1,$$

(4.56) $$x(t) > 0 \quad \text{for } q^1 < t < q^2, \quad \text{if } q^1 < \infty,$$

and also $\dot{x}(q^1) > 0$ *and* $\dot{x}(q^2) < 0$ *provided* q^1, *respectively* q^2, *are finite. Also*

(4.57) $$\dot{x}(t) > 0 \quad \text{for } q^0 + R < t < q^1,$$

(4.58) $$\dot{x}(t) < 0 \quad \text{for } q^1 + R < t < q^2,$$

provided $q^0 + R < q^1$, *respectively* $q^1 + R < q^2$, *and the relevant quantities are finite.*

Finally, there exists a continuous function $\gamma : [2R, \infty) \to (0, \infty)$, *with* $\gamma(t) \to 0$ *as* $t \to \infty$, *such that for* $i = 0$ *or* 1

$$(4.59) \qquad |x(t)| \leqq \gamma(t - q^i) \quad \textit{for } q^i + 2R < t < q^{i+1}$$

provided $q^i + 2R < q^{i+1}$. *The function* γ *depends on* ε^*, *but is independent of* ε, K, φ, *and* ϱ.

Proof. First observe that conditions (a) through (d) hold for f^ϱ and r^ϱ, for $0 \leqq \varrho \leqq 1$, by the form (4.49), (4.50) of the homotopy.

If $\varphi \notin U_K[-R, 0]$, that is, φ vanishes identically on $[-r(0), 0]$, then the uniqueness of solutions in Proposition 4.12 ensures $x(t) = 0$ on $[0, \infty)$. If on the other hand $\varphi \in U_K[-R, 0]$, then either φ vanishes identically on $[-r(0), -r(0) + q^0]$ for some $q^0 \in [0, r(0))$ but for no larger quantity $\bar{q}^0 > q^0$, or else $\varphi(-r(0)) > 0$; in the latter case set $q^0 = 0$. Then an easy argument using the uniqueness in Proposition 4.12 and the negative feedback condition (c) shows (4.54) holds for this q^0, but not for any $\bar{q}^0 > q^0$.

Let $q^* \in [q^0, q^0 + r(0)]$ be any point in that interval such that $x(q^*) = 0$. We claim this implies that

$$(4.60) \qquad x(t) \leqq 0 \quad \text{for } t > q^* \text{ sufficiently near } q^*$$

if $q^* < q^0 + r(0)$, and also that

$$(4.61) \qquad x(t) \geqq 0 \quad \text{for } t < q^* \text{ sufficiently near } q^*$$

if $q^* > q^0$. The claims (4.60) and (4.61) follow directly from the observation that $y = x(t)$ satisfies the ordinary differential equation $\dot{y} = g(t, y)$, where

$$g(t, y) = f^\varrho(y, x(t - r^\varrho(y)))$$

satisfies

$$g(t, 0) = f^\varrho(0, x(t - r^\varrho(0))) \leqq 0 \quad \text{for } q^0 \leqq t \leqq q^0 + r(0),$$

from the negative feedback condition (c). Having now established (4.60) and (4.61), one sees immediately that (4.60) implies $x(t) \leqq 0$ for all $t \in [q^0, q^0 + r(0)]$. This fact, together with (4.61), implies that if $x(q^*) = 0$ for some $q^* \in (q^0, q^0 + r(0)]$, then $x(t) = 0$ for $t < q^*$ sufficiently near q^*, and hence for all $t \in [q^0, q^*]$. However, this would contradict the choice of q^0. Thus we conclude that $x(\cdot)$ does not vanish anywhere in $(q^0, q^0 + r(0)]$, that is, $x(t) < 0$ for all $t \in (q^0, q^0 + r(0)]$. This now yields (4.55) for some $q^1 \in (q^0 + r(0), \infty]$, as claimed. This inequality, and (c), imply $\dot{x}(q^1) > 0$ if $q^1 < \infty$.

The proofs of (4.56) and the simplicity of the zero q^2 if $q^2 < \infty$ follow similar lines.

The monotonicity properties (4.57) and (4.58) follow immediately from (d), and from these in turn we conclude that if $q^0 + 2R < q^1$ then $x(t - r^\varrho) < x(t)$ for $t \in [q^0 + 2R, q^1)$.

Define

$$g(y) = \min_{0 \leq \varrho \leq 1} \min_{-D \leq z \leq y} f^{\varrho}(y, z)$$

for $-D \leqq y \leqq 0$; then $g(y) > 0$ for $-D \leqq y < 0$ and $g(0) = 0$. We have now

$$\dot{x}(t) = \varepsilon^{-1} f^{\varrho}(x(t), x(t - r^{\varrho})) \geqq \varepsilon^{-1} g(x(t)) \geqq (\varepsilon^*)^{-1} g(x(t))$$

for $t \in [q^0 + 2R, q^1)$, and hence $x(t) \geqq \gamma_-(t - q^0)$ for such t, where $\gamma_- : [2R, \infty) \to (-\infty, 0)$ is the unique solution of

$$\dot{\gamma}_- = (\varepsilon^*)^{-1} g(\gamma_-), \quad \gamma_-(2R) = -D.$$

Clearly $\gamma_-(t) \to 0$ as $t \to \infty$. In a similar fashion one has an upper bound $x(t) \leqq \gamma_+(t - q^1)$ in $[q^1 + 2R, q^2)$, with $\gamma_+(t) \to 0$ as $t \to \infty$. Upon setting $\gamma(t) = \max\{|\gamma_-(t)|, \gamma_+(t)\}$, we obtain the uniform decay estimate (4.59). This completes the proof of the proposition. □

In the same fashion as in [MP-Nu4], the existence question of SOPS's may be converted into an equivalent fixed point problem. Fix any $\varepsilon^* > 0$ and consider only $\varepsilon \geqq \varepsilon^*$. (We work here in the setting of Proposition 4.13 with the homotopy of Proposition 4.12.) For such ε the quantity

$$K = (\varepsilon^*)^{-1} \max_{x, y \in [-D, C]} |f(x, y)| \tag{4.62}$$

is a lipschitz constant (at least in the range of t where the differential equation (4.47) holds) for any solution with range in $[-D, C]$. We define a map

$$\mathscr{T}: P_K[-R, 0] \times [\varepsilon^*, \infty) \times [0, 1] \to P_K[-R, 0] \tag{4.63}$$

by setting

$$\mathscr{T}(\varphi, \varepsilon, \varrho) = \psi$$

as follows. Let $x(t)$ denote the solution of (4.48) through φ satisfying (4.53). If $\varphi \in U_K[-R, 0]$ and $q^1 < \infty$ and $q^2 < \infty$, then set

$$\psi(\theta) = \begin{cases} x(q^2 + \theta) & \text{for } \theta \in [-R, 0] \cap [q^1 - q^2, 0], \\ 0 & \text{for } \theta \in [-R, 0] \cap (-\infty, q^1 - q^2]; \end{cases}$$

otherwise let ψ be the zero function. Then just as in [MP-Nu4] the map $\mathscr{T}$ is continuous, and there is a one-to-one correspondence between nontrivial fixed points φ of $\mathscr{T}$ and SOPS's of (4.48). In particular, the uniform decay estimate (4.59) is used to prove continuity of $\mathscr{T}$ at those φ for which $q^1 = \infty$ or $q^2 = \infty$. Also, $E(0, 0)$ holds by the remark preceding Proposition 4.1, since $xf(x, 0) \leqq 0$ by (d) in the statement of Proposition 4.13, and r is locally lipschitz. We note this fact since as in [MP-Nu4], Proposition 4.1 is needed to prove the correspondence between nontrivial fixed points of $\mathscr{T}$ and SOPS's.

The next lemma shows that any bifurcation of an SOPS from zero must occur at the value $\varepsilon = \varepsilon^0$ corresponding to an SOPS of the linearized equation.

Lemma 4.14. *Assume f and r are locally, lipschitz and satisfy conditions* (a) *through* (d) *of Proposition* 4.13 *for some positive quantities C, D, and R. Also assume f is differentiable at* (0,0), *with*

$$Df(0,0) = (-A, -B), \quad \textit{with } 0 < A < B, \tag{4.64}$$

and set

$$\varepsilon^0 = \frac{r(0)(B^2 - A^2)^{1/2}}{\arccos(-A/B)} \tag{4.65}$$

where $\pi/2 < \arccos(-A/B) < \pi$. *Fix* $\varepsilon^* < \varepsilon^0$, *let K be as in* (4.62), *and set*

$$\mathscr{C} = \{(\varphi, \varepsilon, \varrho) \in P_K[-R, 0] \times [\varepsilon^*, \infty) \times [0,1] \mid \mathscr{T}(\varphi, \varepsilon, \varrho) = \varphi \text{ and } \varphi \not\equiv 0\}.$$

Then

$$\bar{\mathscr{C}} \subseteq \mathscr{C} \cup (\{0\} \times \{\varepsilon^0\} \times [0,1]).$$

Proof. We clearly have $\bar{\mathscr{C}} \subseteq \mathscr{C} \cup (\{0\} \times [\varepsilon^*, \infty) \times [0,1])$. Take $(\varphi^n, \varepsilon^n, \varrho^n) \in \mathscr{C}$ with $(\varphi^n, \varepsilon^n, \varrho^n) \to (0, \varepsilon^*, \varrho^*)$ for some ε^* and ϱ^*. We must show that $\varepsilon^* = \varepsilon^0$. As described above, we have a corresponding sequence $x^n(t)$ of SOPS's of (4.48), with $(\varepsilon, \varrho) = (\varepsilon^n, \varrho^n)$. The continuous dependence results of Proposition 4.12 and the monotonicity properties (4.57), (4.58) imply that

$$\|x^n\| \to 0 \quad \text{where } \|x^n\| = \sup_{t \in \mathbb{R}} \|x^n(t)\|.$$

Let $\gamma^n \in \mathbb{R}$ denote a point where the maximum

$$|x^n(\gamma^n)| = \|x^n\|$$

is achieved, and denote $\alpha^n = \|x^n\|$. Without loss assume $x^n(\gamma^n) > 0$, and set

$$w^n(t) = (\alpha^n)^{-1} x^n(t + \gamma^n),$$

so that $|w^n(t)| \leqq 1$ for all t, and $w^n(0) = 1$. Also note that as $x^n(\cdot)$ is slowly oscillating, there exists an interval I^n of length greater than $r^\varrho(0) = r(0)$, and containing the origin, such that $x^n(t) > 0$ for all $t \in I^n$.

The scaled function $w^n(\cdot)$ satisfies the equation

$$\varepsilon^n \dot{w}^n(t) = f_{\alpha^n}^{\varrho^n}(w^n(t), w^n(t - r_{\alpha^n}^{\varrho^n})), \quad r_{\alpha^n}^{\varrho^n} = r_{\alpha^n}^{\varrho^n}(w^n(t)) \tag{4.66}$$

where

$$f_\alpha^\varrho(x, y) = \begin{cases} \alpha^{-1} f^\varrho(\alpha x, \alpha y) & \text{for } \alpha > 0, \\ -Ax - By & \text{for } \alpha = 0, \end{cases}$$

$$r_\alpha^\varrho(x) = r^\varrho(\alpha x) \qquad \text{for all } \alpha,$$

where we note the derivatives A and B of f^ϱ at the origin are independent of ϱ. An easy application of Ascoli's theorem to (4.66) produces a solution $w(\cdot)$ satisfying the limiting equation

$$\varepsilon^* \dot{w}(t) = -Aw(t) - Bw(t - r(0)) \tag{4.67}$$

for $t \in \mathbb{R}$, and such that $|w(t)| \leqq 1$ for all t, with $w(0) = 1$ (so $w(\cdot)$ is nontrivial), and also such that

$$w(t) \geqq 0 \quad \text{for all } t \in I^*$$

for some interval I^* of length greater than or equal to $r(0)$. It now follows from standard arguments and well-known results about linear autonomous equations (see, for example [Be-Co], [MP-Nu1]) that the characteristic equation of equation (4.67) possesses a purely imaginary root, with imaginary part in the interval $(0, \pi/r(0))$. This in turn implies that $\varepsilon^* = \varepsilon^0$, and completes the proof. □

We now state the basic existence theorem for SOPS's; this result directly generalizes Theorem 1.1 of [MP-Nu4].

Theorem 4.15. *Assume the hypotheses and notation of Lemma* 4.14. *Consider* SOPS'*s* $x(\cdot)$ *of* (4.47), *normalized so that*

$$x(0) = 0 \quad \textit{and} \quad \dot{x}(0) < 0, \tag{4.68}$$

and let

$$\mathscr{S} = \{(\varphi, \varepsilon) \in C[-R, 0] \times (0, \infty) \mid \textit{there exists an } \text{SOPS } x(\cdot) \textit{ of } (4.47) \textit{ satisfying } (5.53) \textit{ and } (4.68), \textit{ with } x|_{[-R,0]} = \varphi\}.$$

Then

$$\bar{\mathscr{S}} = \mathscr{S} \cup \{(0, \varepsilon^0)\},$$

and for each $\varepsilon < \varepsilon^0$ *there exists* $(\varphi, \varepsilon) \in \mathscr{S}$ *on the connected component of* $\bar{\mathscr{S}}$ *containing* $(0, \varepsilon^0)$.

Proof. With ε^0 as in (4.64) and (4.65), fix $\varepsilon^* < \varepsilon^0$ and let K be as in (4.62). Also, for each $\varepsilon \neq \varepsilon^0$ in $[\varepsilon^*, \infty)$ let $\delta = \delta(\varepsilon) > 0$ be such that the map $\mathscr{T}(\cdot, \varepsilon, \varrho)$ in (4.63) has no nontrivial fixed points in the closed ball $\bar{B}_\delta \subseteqq P_K[-R, 0]$, where B_δ is the open ball of radius δ centered at zero in $P_K[-R, 0]$. The existence of such δ follows from Lemma 4.14.

Now, as in [MP-Nu4], Theorem 1.1, a more or less standard application of the fixed-point index to the map $\mathscr{T}(\cdot, \varepsilon, 0)$ at the homotopy value $\varrho = 0$ shows there exists a connected set

$$\mathscr{S}^{\varepsilon^*} \subseteqq \{(\varphi, \varepsilon) \in P_K[-R, 0] \times [\varepsilon^*, \infty) \mid \mathscr{T}(\varphi, \varepsilon, 0) = \varphi \text{ and } \varphi \not\equiv 0\}$$

of fixed points such that $(0, \varepsilon^0) \in \bar{\mathscr{S}}^{\varepsilon^*}$, and such that for each $\varepsilon \in [\varepsilon^*, \varepsilon^0)$ there exists some $(\varphi, \varepsilon) \in \mathscr{S}^{\varepsilon^*}$. Indeed, the existence of the set $\mathscr{S}^{\varepsilon^*}$ follows directly once one computes the fixed-point index

$$(4.69) \qquad i(\mathscr{T}(\cdot,\varepsilon,0),B_\delta)=\begin{cases}0 & \text{for } \varepsilon^* \leqq \varepsilon < \varepsilon\,,\\ 1 & \text{for } \varepsilon > \varepsilon^0\,,\end{cases}$$

and recalls that $i(\mathscr{T}(\cdot,\varepsilon,0),P_K[-R,0])=1$.

The formula (4.69) follows directly from the homotopy property of the index, which implies that

$$i(\mathscr{T}(\cdot,\varepsilon,\varrho),B_\delta)=i(\mathscr{T}(\cdot,\varepsilon,0),B_\delta) \quad \text{for } 0 \leqq \varrho \leqq 1$$

provided $\varepsilon \neq \varepsilon^0$, and from the fact that one knows from [MP-Nu4], Lemma 1.7, the value of the index

$$i(\mathscr{T}(\cdot,\varepsilon,1),B_\delta)=\begin{cases}0 & \text{for } \varepsilon^* \leqq \varepsilon < \varepsilon\,,\\ 1 & \text{for } \varepsilon > \varepsilon^0\,,\end{cases}$$

at $\varrho = 1$.

To complete the proof of the theorem, one considers the union

$$\tilde{\mathscr{S}} = \bigcup_{0<\varepsilon^*<\varepsilon^0} \mathscr{S}^{\varepsilon^*} \subseteqq C[-R,0]\times(0,\infty)\,,$$

observing that the closure of $\tilde{\mathscr{S}}$ is connected and contains the bifurcation point $(0,\varepsilon^0)$. Each point in $\tilde{\mathscr{S}}$ corresponds, in a one-to-one fashion, with an SOPS, and this yields the set $\mathscr{S}$ as in the statement of the theorem. This completes the proof. □

5. A lower bound for $\|x\|$: part I

In this and the next section we obtain a uniform lower bound

$$\|x\| \geqq \kappa > 0$$

for all SOPS's of the equation

$$(5.1) \qquad \varepsilon\dot{x}(t)=f(x(t),x(t-r)), \quad r=r(x(t))\,,$$

for sufficiently small $\varepsilon > 0$. In particular, the quantity κ does not depend on (small) ε. We denote here

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)|\,,$$

and of course impose appropriate conditions on f and r.

We always assume $f:\mathbb{R}^2 \to \mathbb{R}$ and $r:\mathbb{R}\to\mathbb{R}$ are continuous. In addition, the following hypotheses will be needed for our lemmas and Theorem 5.1 below. We shall always state precisely which hypotheses are assumed at any time.

H1. The conditions on f and r as in the statement of Corollary 4.9 hold.

H2′. The function f is differentiable at $(0,0)$, with derivative $Df(0,0)=(-A,-B)$, where $A>0$ and $B>0$; we denote $k=B/A$.

H2. Hypothesis H2′ holds, and in addition, $A < B$.

H3′. Hypothesis H2 holds, and in addition

$$r(x) - r(0) = O(|x|^{\nu}) \quad \text{as } x \to 0$$

for some real number $\nu \geqq 1$.

H3. Hypothesis H3′ holds, together with

$$\lim_{x \to 0\pm} \frac{r(x) - r(0)}{|x|^{\nu}} = Q^{\pm}$$

where either $Q^{+} > k^{-\nu}|Q^{-}|$ or $Q^{-} > k^{-\nu}|Q^{+}|$.

Remark. If f is C^1 and r is lipschitz, both in some neighborhood of the origin, then H1 and H2 hold if and only if $f(0,0) = 0$ and $r(0) > 0$, and $0 < A < B$ where

$$Df(0,0) = (-A, -B).$$

Remark. Observe the conditions on the delay r in Hypothesis H3 include

$$r(x) = r(0) + Qx^{\nu}$$

where either ν is an odd integer and $Q \neq 0$, or else ν is even and $Q > 0$. It does not include, however, the case of ν even and $Q < 0$. Although we believe Theorem 5.1 is valid without any restrictions on $Q^{\pm}$, other than $(Q^{+}, Q^{-}) \neq (0,0)$, our arguments do not extend to all cases. We hope to return to this point in a future paper.

Our goal in this and the next section is to prove the following result.

Theorem 5.1. *Assume Hypotheses* H1 *and* H3. *Then there exist* $\varepsilon_0 > 0$ *and* $\kappa > 0$ *such that if* $0 < \varepsilon \leqq \varepsilon_0$ *then all* SOPS's *of equation* (5.1) *satisfy* $\|x\| \geqq \kappa$.

The key to the proof of Theorem 5.1 is an appropriate scaling argument. Let $x(\cdot)$ be any SOPS with (minimal) period $T > 0$, and define $\sigma > 0$ by

$$T = 2(r(0) + \sigma). \tag{5.2}$$

Choose quantities $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha > 0$ and $\beta > 0$, and define functions

$$y(\tau) = \alpha^{-1}x(\gamma - \beta\tau), \quad z(\tau) = \alpha^{-1}x(\gamma + r(0) + \sigma - \beta\tau). \tag{5.3}$$

One now easily verifies that

$$\begin{aligned} \varepsilon\beta^{-1}\dot{y}(\tau) &= -f_{\alpha}(y(\tau), z(\tau - \sigma\beta^{-1} + \alpha^{\nu}\beta^{-1}q_{\alpha}(y(\tau)))), \\ \varepsilon\beta^{-1}\dot{z}(\tau) &= -f_{\alpha}(z(\tau), y(\tau - \sigma\beta^{-1} + \alpha^{\nu}\beta^{-1}q_{\alpha}(z(\tau)))), \end{aligned} \tag{5.4}$$

where the function f_α and q_α are defined by

$$f_\alpha(x, y) = \alpha^{-1} f(\alpha x, \alpha y), \quad q_\alpha(x) = \alpha^{-\nu}(r(\alpha x) - r(0)),$$

with ν fixed (and to be chosen later as in Hypothesis H 3′). Note the existence of the limits

$$f_0(x, y) = -Ax - By,$$

$$q_0(x) = \begin{cases} Q^+ x^\nu, & x \geqq 0, \\ Q^- |x|^\nu, & x \leqq 0, \end{cases}$$

of the above functions as $\alpha \to 0$, provided the appropriate hypotheses hold.

We prove Theorem 5.1 by first assuming there exists a sequence $x^n(\cdot)$ of SOPS's of equation (5.1) with $\varepsilon = \varepsilon^n$, where

$$\varepsilon^n > 0 \quad \text{and} \quad \varepsilon^n \to 0, \tag{5.5}$$

and

$$\|x^n\| \to 0, \tag{5.6}$$

and then obtaining a contradiction. Our strategy is to use the scaled equations (5.4), with particular choices of $\alpha = \alpha^n$, $\beta = \beta^n$, and $\gamma = \gamma^n$, to obtain our contradiction. We always choose

$$\alpha^n = \|x^n\| \tag{5.7}$$

to scale the variable x. We choose γ^n as any time at which $|x^n(t)|$ achieves its maximum. By passing to a subsequence, we may assume without loss that all $x^n(\gamma^n)$ have the same sign. In fact, we may assume without loss that all these quantities are positive and so

$$x^n(\gamma^n) = \|x^n\|, \tag{5.8}$$

by making the change of variables $x \to -x$ in equation (5.1) if necessary.

The choice of β^n, which scales time, is more complicated. Three cases must be considered; these are outlined in Table 5.1. Note that the three results proved are mutually contradictory, thereby establishing Theorem 5.1 as desired. In Cases I and II, which are considered in this section, the theory of limiting profiles as developed in Section 2 is used as the scaled system (5.4) is singularly perturbed. Case III yields a regularly perturbed system which is handled in Section 6 by different methods.

Case	Result proved	Where proved	Choice of β
I	$\sigma = O(\varepsilon + \alpha^\nu)$	Lemma 5.3	σ
II	$\alpha^\nu = O(\varepsilon)$	Lemma 5.4	α^ν
III	$\varepsilon = o(\sigma + \alpha^\nu)$	Lemma 6.1	ε

Table 5.1

We therefore make the following standing assumption.

Standing assumption. For the remainder of this section, and all of Section 6, we assume that $f: \mathbb{R}^2 \to \mathbb{R}$ and $r: \mathbb{R} \to \mathbb{R}$ are continuous with $yf(0,y) < 0$ for all $y \neq 0$ near 0 and with $r(0) > 0$. We also assume that for some sequence ε^n satisfying (5.5), we have SOPS's $x^n(\cdot)$ to equation (5.1) with $\varepsilon = \varepsilon^n$, and that the limit (5.6) holds. We denote α^n as in (5.7), we assume (5.8) holds for some γ^n, and we let $T = T^n$ denote the (minimal) period of $x^n(\cdot)$, with $\sigma = \sigma^n$ given by (5.2).

The Hypotheses H1 through H3, above, will not be assumed unless explicitly stated, and this is always in addition to the standing assumption.

Let us denote for convenience the function

$$r_\alpha(x) = r(\alpha x),$$

with the constant

$$r_0 = r_0(x) = r(0).$$

Before proceeding with the three cases of Table 5.1 we make some preliminary observations. Let $y^n(\tau)$ and $z^n(\tau)$ be as in (5.3), with the above choices of α^n and γ^n, and with $\beta^n \to 0$ yet to be given. Then clearly

$$\text{(5.9)} \qquad \begin{aligned} &|y^n(\tau)| \leqq 1 \text{ and } |z^n(\tau)| \leqq 1 \quad \text{for all } \tau \in \mathbb{R}, \\ &y^n(0) = 1. \end{aligned}$$

If $y^n(\cdot)$ and $z^n(\cdot)$ moreover are regular sequences, let Ω_1 and Ω_2 respectively denote their limiting profiles; denote

$$\begin{aligned} \Omega_{1,\tau} &= \{\tau\} \times \bar{\underline{y}}(\tau), \quad \bar{\underline{y}}(\tau) = [\underline{y}(\tau), \bar{y}(\tau)], \\ \Omega_{2,\tau} &= \{\tau\} \times \bar{\underline{z}}(\tau), \quad \bar{\underline{z}}(\tau) = [\underline{z}(\tau), \bar{z}(\tau)], \end{aligned}$$

following Section 2. Then $\Omega_i \subseteqq \mathbb{R} \times [-1, 1]$, and

$$\bar{y}(0) = 1$$

from (5.9).

If also Hypothesis H1 holds, then $x^n(\cdot)$ enjoys Property MM by Corollary 4.9. In this case, with

$$\begin{aligned} I^n &= [-T^n/4\beta^n, T^n/4\beta^n], \\ J_1^n &= [\gamma^n - T^n/4, \gamma^n + T^n/4], \\ J_2^n &= [\gamma^n + T^n/4, \gamma^n + 3T^n/4], \\ J^n &= J_1^n \cup J_2^n, \end{aligned}$$

one easily checks that

$$\text{(5.10)} \qquad \begin{aligned} \operatorname*{mon}_{\tau \in I^n} y^n(\tau) + \operatorname*{mon}_{\tau \in I^n} z^n(\tau) &= \operatorname*{mon}_{t \in J_1^n} x^n(t) + \operatorname*{mon}_{t \in J_2^n} x^n(t) \\ &\leqq 1 + \operatorname*{mon}_{t \in J^n} x^n(t) \leqq 1 + 3 = 4. \end{aligned}$$

Therefore, as $\beta^n \to 0$, we have using Proposition 4.5

$$\operatorname{mon}\Omega_1 + \operatorname{mon}\Omega_2 \leqq 4\,. \tag{5.11}$$

The finiteness of $\operatorname{mon}\Omega_1$ implies the existence and equality of the left-hand limits

$$\lim_{\tau\to\tau^0-} \underline{y}(\tau) = \lim_{\tau\to\tau^0-} \bar{y}(\tau)\,, \tag{5.12}$$

and also the right-hand limits

$$\lim_{\tau\to\tau^0+} \underline{y}(\tau) = \lim_{\tau\to\tau^0+} \bar{y}(\tau)\,, \tag{5.13}$$

for every τ^0. Indeed, if the inequality in, say, (5.12) failed for some τ^0, then there would exist points $p^i = (\tau^i, \xi^i) \in \Omega_1$, with $\tau^1 < \tau^2 < \cdots \to \tau^0$ and $(-1)^i(\xi^{i+1} - \xi^i) > 0$ for each $i \geqq 1$. For the finite subset $F = \{p^1, p^2, \ldots, p^m\}$ of Ω_1 one has $\operatorname{mon} F = m - 1$, implying $\operatorname{mon}\Omega_1 = \infty$, a contradiction. It now follows from the fact that (5.12) and (5.13) hold for every τ^0, that for each integer $m \geqq 1$ the set

$$\{\tau \in \mathbb{R} \mid \bar{y}(\tau) - \underline{y}(\tau) > 1/m\}$$

contains no cluster point. From this we conclude that $\underline{y}(\tau) = \bar{y}(\tau)$ for all but countably many τ. One also has existence and equality of the limits at infinity

$$y(\infty) = \lim_{\tau\to\infty} \underline{y}(\tau) = \lim_{\tau\to\infty} \bar{y}(\tau)\,,$$

and

$$y(-\infty) = \lim_{\tau\to-\infty} \underline{y}(\tau) = \lim_{\tau\to-\infty} \bar{y}(\tau)\,,$$

again from the finiteness of $\operatorname{mon}\Omega_1$. The corresponding statements for $\underline{z}(\tau)$ and $\bar{z}(\tau)$ also hold.

We need one other result before considering the three cases of Table 5.1.

Lemma 5.2. *Assume Hypothesis* H2′. *Then* $\lim_{n\to\infty} T^n = 2r_0$, *and hence* $\lim_{n\to\infty} \sigma^n = 0$.

Proof. Set $w^n(t) = (\alpha^n)^{-1} x^n(t + \gamma^n)$; then $w^n(0) = 1$ and $|w^n(t)| \leqq 1$ for all t. Assume also, by passing to a subsequence if necessary, that $w^n(\cdot)$ is a regular sequence, and let $\Omega \subseteqq \mathbb{R}^2$ denote its limiting profile in the sense of Section 2. In that spirit denote

$$\Omega_\tau = \{\tau\} \times \underline{\bar{w}}(\tau), \quad \underline{\bar{w}}(\tau) = [\underline{w}(\tau), \bar{w}(\tau)]\,,$$

for each $\tau \in \mathbb{R}$. Note here that

$$\bar{w}(0) = 1\,. \tag{5.14}$$

One sees that $w^n(\cdot)$ satisfies the equation

$$\varepsilon^n \dot{w}^n(t) = f_{\alpha^n}\big(w^n(t), w^n(t - r_{\alpha^n})\big), \quad r_{\alpha^n} = r_{\alpha^n}\big(w^n(t)\big), \tag{5.15}$$

and that Theorem 2.9 applies to this $w^n(\cdot)$ and the equations (5.15). In particular, we have the limits

$$\lim_{n \to \infty} f_{\alpha^n}(\xi, \zeta) = f_0(\xi, \zeta) = -A\xi - B\zeta, \quad \lim_{n \to \infty} r_{\alpha^n}(\xi) = r_0(\xi) = r_0,$$

and therefore the formal limit of (5.14) is the equation

$$0 = -Aw(t) - Bw(t - r_0).$$

The rigorous interpretation of this limit is of course given by Theorem 2.9.

Suppose that $\xi = \bar{w}(\tau) > 0$ for some τ. By Theorem 2.8 we have $(\tau, \xi) \in \Omega^*$, so by Theorem 2.9 there exists $(\sigma, \zeta) = (\tau - r_0, \zeta) \in \Omega$ satisfying $A\xi + B\zeta = 0$. Thus

$$\underline{w}(\tau - r_0) \leqq \zeta = -k^{-1}\xi = -k^{-1}\bar{w}(\tau).$$

Similarly, if $\underline{w}(\tau) < 0$ for some τ then one has $\bar{w}(\tau - r_0) \geqq -k^{-1}\underline{w}(\tau)$. These observations together with (5.14) imply that

$$\begin{aligned} &\bar{w}(-mr_0) \geqq k^{-m} > 0 \quad \text{for even } m \geqq 0, \\ &\underline{w}(-mr_0) \leqq -k^{-m} < 0 \quad \text{for odd } m > 0. \end{aligned} \tag{5.16}$$

Now (5.16) and the definition of Ω as the limit of the graphs of the solutions $w^n(\cdot)$ imply the following fact: given $m > 0$ and any real number $\theta > 0$, then for all large n the solutions $w^n(\cdot)$ and hence $x^n(\gamma^n + \cdot)$ possess at least m zeros in $[-mr_0 - \theta, \theta]$. If m is odd, then this interval contains at least $(m-1)/2$ periods of $w^n(\cdot)$, hence

$$\left(\frac{m-1}{2}\right) T^n \leqq r_0 m + 2\theta. \tag{5.17}$$

Dividing (5.17) by $(m-1)/2$, letting $m \to \infty$, and using the fact that θ can be taken arbitrarily small, implies easily that $\limsup_{n \to \infty} T^n \leqq 2r_0$. But $T^n > 2r_0$ for all n as $x^n(\cdot)$ is slowly oscillating; this gives the desired result. □

With the aid of Lemma 5.2 we can prove Case I.

Lemma 5.3. *Assume Hypotheses* H1 *and* H3′. *Then* $\sigma^n = O\big(\varepsilon^n + (\alpha^n)^\nu\big)$.

Proof. We assume the result is false and seek a contradiction. By taking a subsequence if necessary, we have

$$\lim_{n \to \infty} \frac{\varepsilon^n}{\sigma^n} = \lim_{n \to \infty} \frac{(\alpha^n)^\nu}{\sigma^n} = 0. \tag{5.18}$$

Let $\beta^n = \sigma^n$, define $y^n(\tau)$ and $z^n(\tau)$ by the formulas (5.3) with appropriate superscripts n, and assume by taking a further subsequence if necessary that $y^n(\cdot)$ and $z^n(\cdot)$ are regular sequences with limiting profiles Ω_1 and Ω_2. In light of (5.18) and the choice of β^n, the formal limit of the system (5.4) as $n \to \infty$ is

$$\begin{aligned} 0 &= Ay(\tau) + Bz(\tau - 1)\,, \\ 0 &= Az(\tau) + By(\tau - 1)\,, \end{aligned} \tag{5.19}$$

with a rigorous interpretation of (5.19) given by Theorem 2.10.

We use (5.19) first to prove that

$$\underline{y}(\tau), \bar{y}(\tau), \underline{z}(\tau), \bar{z}(\tau) \to 0 \quad \text{as } \tau \to \pm\infty\,. \tag{5.20}$$

The fact that these limits exist follows directly from the piecewise monotonicity of $y^n(0)$ and $z^n(\cdot)$, and the fact that $\beta^n = \sigma^n \to 0$ in the scaling (5.3), by Lemma 5.2. For simplicity consider only the limits at ∞; denote these by $y(\infty)$ and $z(\infty)$. If either $\xi = \underline{y}(\tau)$ or $\xi = \bar{y}(\tau)$, then $(\tau, \xi) \in \Omega_1^*$, so by Theorem 2.10 applied to the first equation in (5.19) there exists $\zeta \in \underline{\bar{z}}(\tau - 1)$ such that $A\xi + B\zeta = 0$. As $\tau \to \infty$ we have $\xi \to y(\infty)$ and $\zeta \to z(\infty)$, hence $Ay(\infty) + Bz(\infty) = 0$. Similarly, the second equation in (5.19) yields $Az(\infty) + By(\infty) = 0$; as $0 < A < B$, we conclude $y(\infty) = z(\infty) = 0$, as desired. This establishes (5.20).

Note that as $\bar{y}(0) = 1$, we have $-k^{-1} \in \underline{\bar{z}}(-1)$ from (5.19), hence

$$\underline{z}(-1) < 0\,. \tag{5.21}$$

Thus neither Ω_1 nor Ω_2 is the trivial set $\mathbb{R} \times \{0\}$; this fact, with the limits (5.20) and bound (5.11) on the index of monotonicity, imply easily that

$$\operatorname{mon} \Omega_1 = \operatorname{mon} \Omega_2 = 2\,, \tag{5.22}$$

$$0 \leqq \bar{y}(\tau^1) \leqq \underline{y}(\tau^2) \quad \text{if } \tau^1 < \tau^2 < 0 \text{ or } 0 < \tau^2 < \tau^1\,, \tag{5.23}$$

and that there exists $\mu \in \mathbb{R}$ such that

$$\bar{z}(\tau^2) \leqq \underline{z}(\tau^1) \leqq 0 \quad \text{if } \tau^1 < \tau^2 < \mu \text{ or } \mu < \tau^2 < \tau^1\,. \tag{5.24}$$

Moreover

$$\underline{y}(\tau) \geqq 0 \quad \text{and} \quad \bar{z}(\tau) \leqq 0 \quad \text{for all } \tau \in \mathbb{R}\,, \tag{5.25}$$

and hence if $(\tau, 0) \in \Omega_i$ for some τ and i, then $(\tau, 0) \in \Omega_i^*$.

Two possibilities now pesent themselves: either

$$\underline{y}(\tau) > 0 \quad \text{for all } \tau > \tau^* \tag{5.26}$$

for some τ^*, or else

$$(5.27) \qquad (\tau^*, \infty) \times \{0\} \subseteqq \Omega_1$$

for some τ^*. We show each of these possibilities leads to a contradiction.

Suppose (5.26) holds. Then from the first equation of (5.19), for any τ there exists $\zeta \in \bar{z}(\tau - 1)$ such that $A\underline{y}(\tau) + B\zeta = 0$, and so $\bar{z}(\tau - 1) \geqq \zeta = -k^{-1}\underline{y}(\tau)$. The second equation of (5.19) gives $\underline{y}(\tau - 2) \leqq -k^{-1}\bar{z}(\tau - 1)$, hence

$$(5.28) \qquad \underline{y}(\tau - 2) \leqq k^{-2}\underline{y}(\tau) < \underline{y}(\tau) \leqq \bar{y}(\tau)$$

by (5.26), provided $\tau \geqq \tau^*$. But (5.28) contradicts (5.23) if $\tau > 2$. This proves (5.26) is impossible.

Suppose (5.27) holds. Then $(\tau, 0) \in \Omega_1^*$ for each $\tau > \tau^*$, so from (5.19) we have $(\tau - 1, 0) \in \Omega_2$, hence $(\tau - 1, 0) \in \Omega_2^*$, for such τ. Continuing this argument shows that $\mathbb{R} \times \{0\} \subseteqq \Omega_i$ for $i = 1, 2$, and in light of (5.23), (5.24), and (5.25) we have for all $\tau \in \mathbb{R}$

$$(5.29) \qquad \underline{y}(\tau) = \bar{y}(\tau) = 0, \quad \text{except } \underline{y}(0) = 0 < \bar{y}(0),$$

$$(5.30) \qquad \underline{z}(\tau) = \bar{z}(\tau) = 0, \quad \text{except } \underline{z}(\mu) < 0 = \bar{z}(\mu).$$

But $\underline{z}(-1) < 0$ by (5.21), and so $\bar{y}(-2) > 0$ by the second equation of (5.19). This contradicts (5.29), and so proves (5.27) is impossible. With this final contradiction the proof of the lemma is complete. □

Let us now prove Case II.

Lemma 5.4. *Assume Hypotheses* H1 *and* H3. *Then* $(\alpha^n)^\nu = O(\varepsilon^n)$.

Proof. We assume

$$(5.31) \qquad \lim_{n \to \infty} \frac{\varepsilon^n}{(\alpha^n)^\nu} = 0$$

and seek a contradiction. Let $y^n(\cdot), z^n(\cdot)$, and Ω_i be as in the proof of Lemma 5.3, but with $\beta^n = (\alpha^n)^\nu$. Also assume by taking a subsequence if necessary that

$$\lim_{n \to \infty} \frac{\sigma^n}{\beta^n} = \sigma^0 \geqq 0$$

exists; that this can be done follows from Lemma 5.3 and (5.31). Observe now the formal limit of (5.4) is

$$(5.32) \qquad \begin{aligned} 0 &= Ay(\tau) + Bz(\tau - \sigma^0 + q_0(y(\tau))), \\ 0 &= Az(\tau) + By(\tau - \sigma^0 + q_0(z(\tau))), \end{aligned}$$

instead of (5.19).

Just as in the proof of Lemma 5.3, one can establish the limits (5.20) and monotonicity properties (5.22), (5.23), (5.24), and (5.25) of Ω_i. Again, one of two possibilities, (5.26) or (5.27), occurs. One also sees that neither Ω_1 nor Ω_2 is the trivial set $\mathbb{R} \times \{0\}$, and while $\bar{y}(0) = 1$ still holds, we may not have equation (5.21).

Consider now the following general construction: take any $\tau^0 \in \mathbb{R}$ and define inductively

$$\xi^m = \begin{cases} \underline{y}(\tau^m) & \text{for } m \text{ even}, \\ \bar{z}(\tau^m) & \text{for } m \text{ odd}, \end{cases} \tag{5.33}$$

$$\tau^{m+1} = \tau^m - \sigma^0 + q_0(\xi^m) \tag{5.34}$$

for $m \geqq 0$. For m even the first equation of (5.32) implies, with $\tau = \tau^m$, that $A\xi^m + B\zeta = 0$ for some $\zeta \in \underline{\bar{z}}(\tau^{m+1})$, hence $\xi^{m+1} = \bar{z}(\tau^{m+1}) \geqq \zeta = -k^{-1}\xi^m$. Similarly, $\xi^{m+1} \leqq -k^{-1}\xi^m$ for m odd. One concludes that

$$(-1)^m \xi^m \geqq 0, \tag{5.35}$$

$$|\xi^m| \leqq k^{-m} |\xi^0|, \tag{5.36}$$

for all m.

As in the proof of Lemma 5.3, we show both (5.26) and (5.27) lead to contradictions. The arguments here, however, differ depending on whether $\sigma^0 > 0$ or $\sigma^0 = 0$; indeed, one sees the full hypotheses of the lemma are not needed in the case $\sigma^0 > 0$. In any case though, we assume below, for definiteness, the inequality

$$Q^+ > k^{-\nu} |Q^-| \tag{5.37}$$

of Hypothesis H 3.

Suppose (5.26) holds and $\sigma^0 > 0$. For the above sequence (5.33), (5.34) we have for the second term

$$\begin{aligned} \tau^2 &= \tau^0 - 2\sigma^0 + q_0(\xi^0) + q_0(\xi^1) \\ &= \tau^0 - 2\sigma^0 + Q^+(\xi^0)^\nu + Q^-|\xi^1|^\nu \\ &\leqq \tau^0 - 2\sigma^0 + (Q^+ + k^{-\nu}|Q^-|)(\underline{y}(\tau^0))^\nu < \tau^0 \end{aligned}$$

provided τ^0 is sufficiently large, since $\bar{y}(\tau) \to 0$ as $\tau \to \infty$. But from (5.35), (5.36), for τ^0 large we have

$$\underline{y}(\tau^2) = \xi^2 \leqq k^{-2} \underline{y}(\tau^0) < \underline{y}(\tau^0) \leqq \bar{y}(\tau^0) \tag{5.38}$$

as $\underline{y}(\tau^0) > 0$, and this contradicts the monotonicity (5.23).

Suppose (5.27) holds and $\sigma^0 > 0$. As in the proof of Lemma 5.3 one concludes (5.29) and (5.30). Each point of the spike $(0, \xi)$, with $0 \leqq \xi \leqq \bar{y}(0)$, belongs to Ω_1^*, by (c) of

Theorem 2.8. The first equation of (5.32), with $\tau = 0$, implies that $-k^{-1}\xi \in \underline{\bar{z}}(-\sigma^0 + q_0(\xi))$ for all such ξ, hence

$$\underline{z}(-\sigma^0 + q_0(\xi)) < 0 \quad \text{for } 0 < \xi \leqq \bar{y}(0).$$

But q_0 is not constant in $[0, \bar{y}(0)]$ as $Q^+ > 0$, so $\underline{z}$ is strictly negative on an interval. This contradicts (5.30).

Suppose $\sigma^0 = 0$. We show (5.26) cannot hold, and in fact show that

$$\bar{\underline{y}}(\tau) = \{0\} \quad \text{for all large } |\tau|. \tag{5.39}$$

For any τ^0 one has from (5.34), (5.36), and the formula for q_0, that

$$|\tau^{m+1} - \tau^m| \leqq Q|\xi^m|^\nu \leqq Qk^{-\nu m}$$

hence

$$|\tau^m - \tau^0| \leqq \frac{Q}{1 - k^{-\nu}},$$

where $Q \max\{Q^+, |Q^-|\}$. Therefore $\{\tau^m\}$ is a Cauchy sequence with limit $\tau^m \to \tau^\infty$. As $\xi^m \to 0$, we conclude

$$(\tau^\infty, 0) \in \Omega_1 \quad \text{for some } \tau^\infty \text{ with } |\tau^\infty - \tau^0| \leqq \frac{Q}{1 - k^{-\nu}}. \tag{5.40}$$

As τ^0 is arbitrary, one concludes (5.39) from (5.40) and from the monotonicity (5.23).

We have now (5.39), with $\sigma^0 = 0$. Let

$$\tau^* = \sup\{\tau \in \mathbb{R} \,|\, \bar{y}(\sigma) = 0 \text{ for all } \sigma < \tau\}.$$

Then $\bar{\underline{y}}(\tau) = \{0\}$ for $\tau < \tau^*$, and $\underline{y}(\tau^*) = 0$, but either $\bar{y}(\tau^*) = 0$ or $\bar{y}(\tau^*) > 0$ may hold. Suppose $\bar{y}(\tau^*) = 0$. The monotonicity property (5.23) implies that $\tau^* < 0$, and that $\underline{y}(\tau) > 0$ for $\tau^* < \tau < 0$. Choose τ^0 strictly between τ^* and 0; then

$$\begin{aligned} \tau^2 &= \tau^0 + q_0(\xi^0) + q_0(\xi^1) \\ &= \tau^0 + Q^+(\xi^0)^\nu + Q^-|\xi^1|^\nu \\ &\geqq \tau^0 + (Q^+ - k^{-\nu}|Q^-|)(\underline{y}(\tau^0))^\nu > \tau^0 \end{aligned}$$

from (5.37). Moreover, if τ^0 is sufficiently near τ^* then also $\tau^2 < 0$, since $\underline{y}(\tau^0) \to \bar{y}(\tau^*) = 0$ as $\tau^0 \to \tau^*$, by (5.23). But as before (5.38) holds, and this contradicts (5.23) since $\tau^0 < \tau^2 < 0$.

Therefore $\bar{y}(\tau^*) > 0$. Part (c) of Theorem 2.8 implies that

$$\Omega_{1,\tau^*} \subseteqq \Omega_1^* \cup \Omega_1^+.$$

Therefore if $0 \leqq \xi \leqq \bar{y}(\tau^*)$ we have from (5.32) that $0 \leqq A\xi + B\zeta$ for some $\zeta \in \underline{\bar{z}}(\tau^* + q_0(\xi))$; without loss

$$\zeta = \bar{z}(\tau^* + q_0(\xi)) \geqq -k^{-1}\xi .$$

Denote $\sigma = \tau^* + q_0(\xi)$. Then again from (5.32), with $\tau = \sigma$,

$$\underline{y}(\sigma + q_0(\zeta)) \leqq -k^{-1}\zeta \leqq k^{-2}\xi . \tag{5.41}$$

Observe that if $\xi > 0$ then

$$\begin{aligned} \sigma + q_0(\zeta) &= \tau^* + q_0(\xi) + q_0(\zeta) \\ &= \tau^* + Q^+\xi^\nu + Q^-|\zeta|^\nu \\ &\geqq \tau^* + (Q^+ - k^{-\nu}|Q^-|)\xi^\nu > \tau^* \end{aligned} \tag{5.42}$$

from (5.37), and that $\sigma + q_0(\zeta) \to \tau^*$ as $\xi \to 0$. This limit, and (5.41) and (5.42), imply the right-hand limit $\underline{y}(\tau^* + 0) = 0$; and this in turn implies that $\tau^* < 0$ is impossible, since $\underline{y}(\tau) \geqq \bar{y}(\tau^*) > 0$ for all $\tau \in (\tau^*, 0)$, by monotonicity (5.23). Thus $\tau^* = 0$. Moreover, we see that $\underline{y}(\tau) = \bar{y}(\tau) = 0$ for all $\tau > 0$, again using the monotonicity (5.23), and the right-hand limit $\underline{y}(\tau^* + 0) = 0$, with $\tau^* = 0$. Thus (5.29) holds, and so $\Omega_1 = \Omega_1^*$ by (c) of Theorem 2.8. Therefore

$$(\tau + q_0(\xi), -k^{-1}\xi) \in \Omega_2 \quad \text{for each } (\tau, \xi) \in \Omega_1$$

by (5.32). In particular $(\tau, 0) \in \Omega_1$ and hence $(\tau, 0) \in \Omega_2$ and

$$\bar{z}(\tau) = 0 \quad \text{for each } \tau; \tag{5.43}$$

and also $(0, \xi) \in \Omega_1$ hence $(q_0(\xi), -k^{-1}\xi) \in \Omega_2$ and

$$\underline{z}(q_0(\xi)) < 0 \quad \text{for } 0 < \xi \leqq \bar{y}(0) = 1 . \tag{5.44}$$

But (5.43) and (5.44) imply that $\underline{z}(\tau) < \bar{z}(\tau)$ for τ in some interval, namely for

$$\tau = q_0(\xi) = Q^+\xi^\nu \in (0, Q^+] .$$

This contradicts the monotonicity (5.24), and completes the proof of the lemma. □

6. A lower bound for $\|x\|$: part II

We complete our proof of Theorem 5.1 by treating Case III of Table 5.1. Recall that the standing assumption of the previous section continues to hold, including the existence of a sequence $x^n(\cdot)$ of SOPS's satisfying (5.5) and (5.6), with scaling parameters α^n and γ^n satisfying (5.7) and (5.8). Here we choose $\beta^n = \varepsilon^n$. Observe that with this choice, the scaled system (5.4) is not singularly perturbed as it was in Cases I and II, but rather has a regular limit.

We begin with a technical lemma about constant coefficient linear functional differential equations.

Lemma 6.1. *Consider the scalar equation*

$$\dot{w}(t) = L(w_t) \tag{6.1}$$

where, following the notation of [Hal-VL], *we assume* $L: C[-R, 0] \to \mathbb{R}$ *is a bounded linear functional. Assume there exists a nontrivial solution* $w(\cdot)$ *of* (6.1) *which is bounded and nonnegative for* $t \leqq 0$. *Then there exists a root* λ *of the characteristic equation*

$$\lambda - L(e^{\lambda \cdot}) = 0 \tag{6.2}$$

with $\lambda \in [0, \infty)$.

Proof. Necessarily $w(t)$ is a finite linear combination of the form

$$w(t) = \sum p^k(t) e^{\lambda^k t} \tag{6.3}$$

where each p^k is a polynomial, and each λ^k is a root of (6.2) with $\operatorname{Re} \lambda^k \geqq 0$. Assume (6.2) has no solution in $[0, \infty)$. Then (6.3) can be written

$$w(t) = \sum_k \sum_j c^{k,j}(t)^j e^{\mu^k t} \sin(\nu^k t + \theta^{k,j}) \tag{6.4}$$

with real coefficients $c^{k,j}$, with $\mu^k \geqq 0$ and $\nu^k > 0$, such that the quantities $\mu^k + i\nu^k$ are all distinct, and where $(t)^j$ denotes the j^{th} power of t. Let μ^* denote the minimum of the values μ^k, and let N denote the maximum degree of the polynomials in (6.4) for such k, that is, the maximum of those j for which $c^{k,j} \neq 0$ for those k with $\mu^k = \mu^*$. We have then

$$e^{-\mu^* t}(t)^{-N} w(t) = q(t) + O(t^{-1}) \quad \text{as } t \to -\infty \tag{6.5}$$

for some nontrivial quasiperiodic function q which is a linear combination of terms $\sin(\nu^k t + \theta^{k,j})$. In particular q is nonconstant and has mean value zero

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} q(t)\, dt = 0,$$

so there exists a sequence $t^n \to -\infty$ such that $q(t^n) \to q^\infty < 0$. But this contradicts the fact that the left-hand side of (6.5) is nonnegative, so completes the proof. □

Lemma 6.2. *Assume Hypotheses* H1 *and* H3. *Then* $\varepsilon^n = o(\sigma^n + (\alpha^n)^\nu)$.

Proof. Assume the result is false, so that both ratios σ^n / ε^n and $(\alpha^n)^\nu / \varepsilon^n$ remain bounded as $n \to \infty$. Without loss we assume the limits

$$\lim_{n \to \infty} \frac{\sigma^n}{\varepsilon^n} = \sigma^0 \geqq 0, \quad \lim_{n \to \infty} \frac{(\alpha^n)^\nu}{\varepsilon^n} = \alpha^0 \geqq 0$$

exist, by taking a subsequence if necessary. Set $\beta^n = \varepsilon^n$, and define $y^n(\tau)$ and $z^n(\tau)$ by (5.3). Then

(6.6) $$|y^n(\tau)|, |z^n(\tau)| \leqq 1 \quad \text{for all } \tau,$$

(6.7) $$y^n(0) = 1$$

and moreover for some $K > 0$ we have

(6.8) $$|\dot{y}^n(\tau)|, |\dot{z}^n(\tau)| \leqq K \quad \text{for all } \tau \text{ and } n.$$

The bound (6.8) follows immediately from the differential equations (5.1) and implies, with Ascoli's theorem, the existence of limits

$$\lim_{n \to \infty} \big(y^n(\tau), z^n(\tau)\big) = \big(y(\tau), z(\tau)\big)$$

uniformly on compact τ-intervals, after possibly passing to a further subsequence. The functions $y(\cdot)$ and $z(\cdot)$ so obtained are in fact C^1 and satisfy the system

(6.9) $$\begin{aligned} \dot{y}(\tau) &= Ay(\tau) + Bz\big(\tau - \sigma^0 + \alpha^0 q_0(y(\tau))\big), \\ \dot{z}(\tau) &= Az(\tau) + By\big(\tau - \sigma^0 + \alpha^0 q_0(z(\tau))\big) \end{aligned}$$

obtained as the limit of (5.4). One establishes this in a standard fashion by writing (5.4) in integrated form, taking the limit $n \to \infty$, then differentiating to obtain (6.9). The bounds (6.6), and equation (6.7), imply

(6.10) $$|y(\tau)|, |z(\tau)| \leqq 1 \quad \text{for all } \tau, \quad y(0) = 1,$$

for the limiting solution. Moreover, one obtains

(6.11) $$\operatorname*{mon}_{\tau \in \mathbb{R}} y(\tau) + \operatorname*{mon}_{\tau \in \mathbb{R}} z(\tau) \leqq 4$$

from (5.10). From (6.11) one concludes the existence of limits $y(\pm\infty)$ and $z(\pm\infty)$. Upon letting $\tau \to \infty$ in (6.9), one sees that $\dot{y}(\tau)$ and $\dot{z}(\tau)$ also approach limits, which must therefore be 0. Consequently both $Ay(\infty) + Bz(\infty) = 0$ and $Az(\infty) + By(\infty) = 0$, and this implies $y(\infty) = z(\infty) = 0$. The corresponding result at $-\infty$ also holds; therefore

(6.12) $$y(\pm\infty) = z(\pm\infty) = 0.$$

Upon noting that $y(0) = 1$ is the maximum of $y(\cdot)$, and then setting $\tau = 0$ in the first equation of (6.9), one obtains $0 = A + Bz(\sigma)$ with $\sigma = -\sigma^0 + \alpha^0 q_0(y(0))$. Thus $z(\sigma) < 0$. From this, from (6.11), and from (6.12), it follows that

(6.13) $$\begin{gathered} \operatorname*{mon}_{\tau \in \mathbb{R}} y(\tau) = \operatorname*{mon}_{\tau \in \mathbb{R}} z(\tau) = 2, \\ y(\tau) \geqq 0 \quad \text{and} \quad \tau \dot{y}(\tau) \leqq 0 \quad \text{for all } \tau, \end{gathered}$$

and that there exists $\mu \in \mathbb{R}$ such that

$$z(\tau) \leqq 0 \quad \text{and} \quad (\tau - \mu)\dot{z}(\tau) \geqq 0 \quad \text{for all } \tau . \tag{6.14}$$

To prove the lemma we obtain a contradiction, assuming the existence of a nontrivial solution of (6.9) satisfying (6.13) and (6.14). We first show the equation

$$\lambda = A - Be^{-\lambda\sigma^0} \tag{6.15}$$

possesses a root satisfying

$$\lambda \in (0, \infty)$$

under such conditions. To this end, write (6.9) as

$$\begin{aligned} \dot{y}(\tau) &= (A + \psi^1(\tau))y(\tau) + Bz(\tau - \sigma^0), \\ \dot{z}(\tau) &= (A + \psi^2(\tau))z(\tau) + By(\tau - \sigma^0), \end{aligned} \tag{6.16}$$

where

$$\begin{aligned} \psi^1(\tau) &= \frac{B(z(\tau - \sigma^0 + \alpha^0 q_0(y(\tau))) - z(\tau - \sigma^0))}{y(\tau)} \\ &= \frac{B\alpha^0 q_0(y(\tau))}{y(\tau)} \int_0^1 \dot{z}(\tau - \sigma^0 + \theta\alpha^0 q_0(y(\tau)))\, d\theta \end{aligned}$$

and $\psi^2(\tau)$, defined similarly, are continuous and satisfy

$$\lim_{\tau \to \pm\infty} \psi^i(\tau) = 0 \tag{6.17}$$

since both $\dot{y}(\tau), \dot{z}(\tau) \to 0$ as $\tau \to \pm\infty$. Set

$$u(\tau) = |y(\tau)| + |z(\tau)| = y(\tau) - z(\tau),$$

observing that $u(\tau) \to 0$ monotonically for large $|\tau|$. Take any sequence $\tau^n \to -\infty$ and let

$$v^n(\tau) = \frac{u(\tau + \tau^n)}{u(\tau^n)};$$

then for large n

$$0 \leqq v^n(\tau) \leqq 1 \quad \text{for all } \tau \leqq 0, \tag{6.18}$$

$$v^n(0) = 1, \tag{6.19}$$

and moreover $v^n(\cdot)$ is equicontinuous on $(-\infty, 0]$ since from (6.16) and (6.17)

$$\dot{v}^n(\tau) = Av^n(\tau) - Bv^n(\tau - \sigma^0) + o(1) \quad \text{as } n \to \infty$$

uniformly for $\tau \leqq 0$. Upon passing to a subsequence one obtains in the limit a solution $v(\cdot)$ to

$$\dot{v}(\tau) = Av(\tau) - Bv(\tau - \sigma^0)$$

for $\tau \leqq 0$. This solution is nonnegative, bounded, and nontrivial for $\tau \leqq 0$ by (6.18), (6.19), so by Lemma 6.1 the characteristic equation (6.15) possesses a root $\lambda \in [0, \infty)$. In fact one sees $\lambda \in (0, \infty)$ as claimed as $A \neq B$.

Having established the existence of such λ, we complete the proof of the lemma. Set

$$w(\tau) = -u(\tau) + Be^{-\lambda\sigma^0} \int_{\tau-\sigma^0}^{\tau} e^{\lambda(\tau-s)} u(s)\, ds$$

and observe, by a short calculation using (6.15) and (6.16), that

$$\dot{w}(\tau) - \lambda w(\tau) = -\psi^1(\tau) y(\tau) + \psi^2(\tau) z(\tau). \tag{6.20}$$

Also note that in view of the monotonicity properties (6.13), (6.14), we have for large τ

$$\begin{aligned} w(\tau) &\geqq -u(\tau) + Be^{-\lambda\sigma^0} \int_{\tau-\sigma^0}^{\tau} e^{\lambda(\tau-s)} u(\tau)\, ds \\ &= \left(-1 + \frac{B(1-e^{-\lambda\sigma^0})}{\lambda}\right) u(\tau) = \left(\frac{B-A}{\lambda}\right) u(\tau) \end{aligned} \tag{6.21}$$

and so from (6.20) and (6.21)

$$\dot{w}(\tau) - \lambda w(\tau) \geqq -\psi(\tau) u(\tau) \geqq -\left(\frac{\psi(\tau)\lambda}{B-A}\right) w(\tau) \geqq -\frac{\lambda w(\tau)}{2}$$

for large τ, where $\psi(\tau) = \max\{\psi^1(\tau), \psi^2(\tau)\}$ approaches zero as $\tau \to \infty$. But now the differential inequality $\dot{w}(\tau) \geqq \lambda w(\tau)/2$ so obtained, with the boundedness and nonnegativity (6.21) of $w(\tau)$, implies that $w(\tau) = 0$ for all large τ. With (6.21) it follows that $u(\tau) = 0$, hence $y(\tau) = z(\tau) = 0$ for all large τ. And as $(y(\tau), z(\tau))$ satisfies the system (6.16), with $B \neq 0$, one concludes that $y(\tau) = z(\tau) = 0$ for all $\tau \in \mathbb{R}$. But this contradicts (6.10), and so completes the proof. □

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