DENJOY-WOLFF THEOREMS FOR HILBERT'S AND THOMPSON'S METRIC SPACES

By

BAS LEMMENS,[1] BRIAN LINS, ROGER NUSSBAUM,[2] AND MARTEN WORTEL[3]

Abstract. We study the dynamics of fixed point free mappings on the interior of a normal, closed cone in a Banach space that are nonexpansive with respect to Hilbert's metric or Thompson's metric. We establish several Denjoy-Wolff type theorems which confirm conjectures by Karlsson and Nussbaum for an important class of nonexpansive mappings. We also extend and put into a broader perspective results by Gaubert and Vigeral concerning the linear escape rate of such nonexpansive mappings.

1 Introduction

The classical Denjoy-Wolff theorem asserts that all orbits of a fixed point free holomorphic mapping $f\colon \mathbb{D} \to \mathbb{D}$ on the open unit disc $\mathbb{D} \subseteq \mathbb{C}$ converge to a unique point $\eta \in \partial\mathbb{D}$. Since the inception of the theorem by Denjoy [14] and Wolff [51, 52] in the nineteen-twenties, a variety of extensions have been obtained; see, for example, [1, 8, 9, 10, 24, 46]. A detailed account of its history and an extensive list of references can be found in the recent survey articles [4, Appendices G and H], [26], and [45]. The problems considered in this paper originated in work by Beardon [6, 7] and Karlsson [25], who extended the Denjoy-Wolff theorem to fixed point free nonexpansive mappings on metric spaces that possess certain features of nonpositive curvature. Earlier studies of the Denjoy-Wolff theorem in the context of metric spaces can be found in [20, 19, 44].

A mapping $f\colon M \to M$ on a metric space (M, ρ) is called **nonexpansive** if

$$\rho(f(x), f(y)) \leq \rho(x, y) \text{ for all } x, y \in M.$$

Recall that each holomorphic self-mapping of the open unit disc $\mathbb{D} \subseteq \mathbb{C}$ is nonexpansive under the hyperbolic metric, by the Schwarz-Pick Lemma.

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Particularly interesting examples of metric spaces that possess features of non-positive curvature are Hilbert's metric spaces. Hilbert's metric spaces were introduced by Hilbert [22] and play a role in the solution of his fourth problem; see [42]. They are Finsler metric spaces that naturally generalize Klein's model of the real hyperbolic space. To define Hilbert's metric, let Σ be a convex set in a real vector space X such that for each $x \neq y$ in Σ, the straight line ℓ_{xy} through x and y has the property that $\ell_{xy} \cap \Sigma$ is a (relatively) open, bounded subset of ℓ_{xy}. On Σ, **Hilbert's metric** is given by

$$\delta(x, y) := \log\left(\frac{|x' - y|}{|x' - x|}\frac{|y' - x|}{|y' - y|}\right) \text{ for } x \neq y \in \Sigma, \tag{1}$$

where $x', y' \in \partial\Sigma$ are the end-points of the segment $\ell_{xy} \cap \Sigma$ such that x is between x' and y and y is between y' and x.

For finite dimensional Hilbert metric spaces, Karlsson and Nussbaum independently conjectured the following generalization of the Denjoy-Wolff theorem; see [26, 41].

Conjecture 1.1. If $f\colon \Sigma \to \Sigma$ is a fixed point free mapping on a finite dimensional Hilbert's metric space (Σ, δ), then there exists a convex set $\Omega \subseteq \partial\Sigma$ such that for each $x \in \Sigma$, all accumulation points $\omega(x; f)$ of the orbit $\mathcal{O}(x; f) := \{f^k(x)\colon k \geq 0\}$ lie in Ω.

In fact, Nussbaum conjectured that the same assertion holds in infinite dimensions under additional compactness conditions on f; see [41, Conjecture 4.21]. Note that if Σ is finite dimensional and its closure (in the usual topology) is strictly convex, then each convex subset of $\partial\Sigma$ reduces to a single point. Conjecture 1.1 was shown by Beardon [7] to hold in case Σ has a strictly convex closure, and by Lins [34] for polytopes. Further supporting evidence was obtained in [2, 28, 35, 41].

Important examples of Hilbert metric nonexpansive mappings arising in mathematical analysis come from nonlinear mappings on cones. Let C be a closed cone with nonempty interior C° in a normed space X. Suppose that there exists a strictly positive linear functional $\varphi \in X^*$, i.e., $\varphi(x) > 0$ for all $x \in C \setminus \{0\}$, and let $\Sigma_\varphi^\circ := \{x \in C^\circ\colon \varphi(x) = 1\}$. If $f\colon C^\circ \to C^\circ$ preserves the partial ordering induced by C and is homogeneous (of degree 1), the mapping $g\colon \Sigma_\varphi^\circ \to \Sigma_\varphi^\circ$ given by

$$g(x) = \frac{f(x)}{\varphi(f(x))} \text{ for } x \in \Sigma_\varphi^\circ, \tag{2}$$

is nonexpansive under δ on Σ_φ°; see [31, Chapter 2]. Examples of such mappings $f\colon C^\circ \to C^\circ$ include reproduction-decimation operators in the analysis of diffusions on fractals [35, 38], dynamic programming operators in stochastic games (after a change of variables) [47], and mappings arising in nonlinear Perron-Frobenius theory [31].

Among other results, we establish the following Denjoy-Wolff type theorem for mappings g given in (2).

Theorem 1.2. *Let C be a closed cone with nonempty interior in a finite dimensional vector space and $\varphi \in X^*$ be a strictly positive functional. Suppose that $f\colon C^\circ \to C^\circ$ is an order-preserving homogeneous mapping with no fixed point in C° and partial spectral radius $r_{C^\circ}(f) = 1$. If there exists $x_0 \in C^\circ$ such that either*

(a) *$\mathcal{O}(x_0; f)$ has a compact closure in the norm topology, or,*

(b) *$\lim_{k\to\infty} \|f^k(x_0)\| = \infty$,*

then there exists a convex set $\Omega \subseteq \partial\Sigma_\varphi^\circ$ such that for each $x \in \Sigma_\varphi^\circ$, the accumulation points of $\mathcal{O}(x; g)$, where g is given by (2)*, lie in Ω.*

In fact, we prove a more general infinite dimensional version of this result; see Theorems 7.1 and 7.3. Unlike in finite dimensions, there need not exist a strictly positive linear functional $\varphi \in X^*$ if C is infinite dimensional; see [29, pp. 48–57]. In that case, we consider scalings of order-preserving homogeneous mappings $f\colon C^\circ \to C^\circ$ by using continuous homogeneous functions $q\colon C^\circ \to (0, \infty)$.

We conjecture that condition (a) or (b) in Theorem 1.2 always holds. In other words, we believe that there do not exist an order-preserving homogeneous mapping $f\colon C^\circ \to C^\circ$, with $r_{C^\circ}(f) = 1$, on the interior of a finite dimensional closed cone and a point $x_0 \in C^\circ$ such that $\mathcal{O}(x_0; f)$ is unbounded in the norm topology and $\mathcal{O}(x_0; f)$ has an accumulation point in ∂C. At present, we can only confirm this in case C is a polyhedral cone; see Theorem 7.4.

Thompson's metric [49] is closely related to Hilbert's metric and is defined on the interior of a closed cone C in a normed space X. For Thompson's metric we establish the following Denjoy-Wolff type theorem.

Theorem 1.3. *Let C be a closed cone with nonempty interior in a finite dimensional vector space X and $f\colon C^\circ \to C^\circ$ be a fixed point free mapping which is nonexpansive under Thompson's metric. If $\mathcal{O}(x_0; f)$ has compact closure in the norm topology for some $x_0 \in C^\circ$, then there exists a convex set $\Omega \subseteq \partial C$ such that for each $x \in C^\circ$, the accumulation points of $\mathcal{O}(x; f)$ lie in Ω.*

Again we establish a more general infinite dimensional version (see Theorem 3.2), which confirms [41, Conjecture 4.23] under the additional condition that there exists a pre-compact orbit in the norm topology.

In Section 4, we introduce a spectral radius $r_{C^\circ}(f)$ for order-preserving homogeneous mappings $f\colon C^\circ \to C^\circ$ and use it, not only to prove Theorems 7.1

and 7.3, but also to extend some results concerning the linear escape rate in [18]; see Theorem 4.6, Corollary 6.4 and Theorem 6.5. In Section 5, we study Funk and reverse-Funk horofunctions on the interiors of cones and characterize them for symmetric cones; see Theorem 5.6. We use the Funk and reverse-Funk horofunctions to establish a Wolff type theorem for order-preserving homogeneous mappings $f\colon C^\circ \to C^\circ$ (see Theorem 6.1), which plays a role in the proof of Theorem 7.3.

We will start by collecting some basic concepts in the next section.

2 Preliminaries

A convex subset C of a real vector space X is called a **cone** if $C \cap (-C) = \{0\}$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$. A cone C induces a partial ordering $\leq_C$ on X by

$$x \leq_C y \quad \text{if } y - x \in C.$$

Throughout this paper, we assume that C is a closed cone with nonempty interior, denoted C°, in a real Banach space $(X, \|\cdot\|)$. We often assume that C is **normal**, i.e., there exists a constant $\kappa \geq 0$ such that $\|x\| \leq \kappa\|y\|$ whenever $0 \leq_C x \leq_C y$.

For a closed cone C with nonempty interior in a Banach space $(X, \|\cdot\|)$ and $u \in C^\circ$, the **order unit norm**, $\|\cdot\|_u$ on X is defined by

$$\|x\|_u := \inf\{\lambda \geq 0\colon -\lambda u \leq_C x \leq_C \lambda u\}.$$

Note that C is a normal cone in $(X, \|\cdot\|_u)$ with normality constant $\kappa = 1$. Moreover, the order interval $[-u, u] := \{x \in X : -u \leq_C x \leq_C u\}$ (which is the unit ball in $\|\cdot\|_u$) is a neighborhood of 0 in the original topology, by [5, Lemma 2.5], and so the topology generated by $\|\cdot\|_u$ is coarser than the original topology. If C is normal in $(X, \|\cdot\|)$, the order unit norm $\|\cdot\|_u$ is equivalent with $\|\cdot\|$; see, for example, [5, Theorems 2.8 and 2.63].

A linear functional $\varphi\colon X \to \mathbb{R}$ is said to be **positive** if $\varphi(C) \subseteq [0, \infty)$; it is said to be **strictly positive** if $\varphi(x) > 0$ for all $x \in C$ with $x \neq 0$. Note that each positive functional on X is continuous with respect to $\|\cdot\|_u$, as $|\varphi(x)| \leq \varphi(u)$ for all $x \in X$ with $\|x\|_u \leq 1$. We denote the *dual cone* by C^*; so,

$$C^* := \{\varphi \in X^*\colon \varphi(C) \subseteq [0, \infty)\}.$$

Furthermore, we define

$$\Sigma_u^* := \{\varphi \in C^*\colon \varphi(u) = 1\}.$$

The following lemma collects some known facts concerning Σ_u^*. For the reader's convenience, we include the proofs.

Lemma 2.1. *Let C be a closed cone with nonempty interior in a Banach space X. For $u \in C^\circ$,*

(1) *$x \leq_C y$ if and only if $\varphi(x) \leq \varphi(y)$ for all $\varphi \in \Sigma_u^*$;*

(2) *for $x \in X$,*

$$\|x\|_u = \sup\{|\varphi(x)| \colon \varphi \in \Sigma_u^*\};$$

(3) *the set Σ_u^* is norm bounded by $1/d(u, X \setminus C)$, where*

$$d(u, X \setminus C) := \inf\{\|u - v\| \colon v \in X \setminus C\},$$

and Σ_u^* is weak compact. Moreover, if X is separable, then Σ_u^* is a weak* sequentially compact and there exists a strictly positive functional $\psi \in \Sigma_u^*$.*

Proof. To prove (1), note that if $x \not\leq_C y$, then $y - x \notin C$. In that case, there exist $\alpha \in \mathbb{R}$ and $\varphi \in X^*$ such that $\varphi(y - x) < \alpha$ and $\varphi(z) > \alpha$ for all $z \in C$, by the Hahn-Banach separation theorem. We can normalize φ such that $\varphi(u) = 1$. Also note that $0 = \varphi(0) > \alpha$, so $\varphi(y) < \varphi(x)$. As $\varphi(\lambda z) > \alpha$ for all $\lambda > 0$ and $z \in C$, we must have that $\varphi(z) \geq 0$ for all $z \in C$, and hence $\varphi \in \Sigma_u^*$. The opposite implication is trivial.

To prove (2) note that it follows from (1) that for each $x \in X$,

$$\begin{aligned}\|x\|_u &= \inf\{\lambda \geq 0 \colon -\lambda u \leq_C x \leq_C \lambda u\} \\ &= \inf\{\lambda \geq 0 \colon -\lambda \leq \varphi(x) \leq \lambda \text{ for all } \varphi \in \Sigma_u^*\} \\ &= \sup\{|\varphi(x)| \colon \varphi \in \Sigma_u^*\}.\end{aligned}$$

To prove (3), we define for $x \in X$ the weak* continuous linear functional $\hat{x} \colon X^* \to \mathbb{R}$ by $\hat{x}(\varphi) = \varphi(x)$. So,

$$\Sigma_u^* = \left(\bigcap_{x \in C} \hat{x}^{-1}([0, \infty))\right) \cap \hat{u}^{-1}(\{1\}),$$

which is a weak* closed subset of X^*.

Let $r := d(u, X \setminus C) > 0$. If $\|z\| \leq r$ and $\varphi \in \Sigma_u^*$, then $u \pm z \in C$; and so $\varphi(u \pm z) \geq 0$, which yields

$$-1 = -\varphi(u) \leq \varphi(z) \leq \varphi(u) = 1.$$

Hence $|\varphi(z)| \leq 1$, and so $\|\varphi\| \leq 1/r$. Therefore, Σ_u^* is contained in a multiple of the unit ball of X^*, which is weak* compact by the Banach-Alaoglu Theorem, and so Σ_u^* is weak* compact.

It is well known that if X is separable, then bounded sets of X^* are weak* metrizable. In that case, Σ_u^* is sequentially compact, and hence Σ_u^* is separable. Let $(\varphi_k)_k$ be a dense sequence in Σ_u^*, and define $\varphi = \sum_{k\geq 1} 2^{-k}\varphi_k \in \Sigma_u^*$. For $x \in C$ with $x \neq 0$, $\|x\|_u > 0$; and hence by part (2), there exists $\sigma \in \Sigma_u^*$ with $\varepsilon := \sigma(x) > 0$. Consider the weak* neighborhood of σ

$$N_{\varepsilon,\sigma} := \{\varphi \in \Sigma_u^* \colon |(\varphi - \sigma)(x)| < \varepsilon\} = \{\varphi \in \Sigma_u^* \colon |\varphi(x) - \varepsilon| < \varepsilon\}.$$

As $(\varphi_k)_k$ is dense in Σ_u^*, there exists $\varphi_m \in N_{\varepsilon,\sigma}$, and hence $\varphi_m(x) > 0$. This implies that

$$\psi(x) := \sum_{k\geq 1} 2^{-k}\varphi_k(x) \geq 2^{-m}\varphi_m(x) > 0,$$

which shows that ψ is strictly positive. □

The partial ordering $\leq_C$ induces an equivalence relation $\sim_C$ on C by $x \sim_C y$ if there exist $0 < \alpha \leq \beta$ such that $\alpha y \leq_C x \leq_C \beta y$. The equivalence classes are called **parts** of C. It is easy to verify that C° is a part of C. Given $x \in X$ and $y \in C$, we let

$$M(x/y) := M(x/y; C) = \inf\{\beta \in \mathbb{R} \colon x \leq_C \beta y\}.$$

On C, **Thompson's metric** is defined by

$$d_C(x, y) := \begin{cases} \log(\max\{M(x/y), M(y/x)\}) & \text{for } x \sim_C y, \\ \infty & \text{otherwise.} \end{cases}$$

It was shown by Thompson [49] that d_C is a metric on each part of C, and its topology on C° coincides with the norm topology if C is a closed, normal cone in a Banach space.

Also on C, **Hilbert's (projective) metric** is defined by

$$\delta_C(x, y) := \begin{cases} \log(M(x/y)M(y/x)) & \text{for } x \sim_C y, \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\delta_C(\mu x, \nu y) = \delta_C(x, y)$ for all $\mu, \nu > 0$ and $x \sim_C y$. It is known that δ_C is a metric between pairs of rays in each part of C if C is closed; see [31, Chapter 2]. Moreover, if there exists a strictly positive linear functional $\varphi \in X^*$, then δ_C coincides with Hilbert's metric δ given in (1) on $\Sigma_\varphi^\circ = \{x \in C^\circ \colon \varphi(x) = 1\}$. In finite dimensional spaces, the set Σ_φ° is bounded in the norm topology, but it may be unbounded in infinite dimensional normed spaces. In this paper, we work with δ_C rather than δ, and consider δ_C on subsets $\Sigma \subseteq C^\circ$ with the property that for each $y \in C^\circ$, there exists a unique $\lambda > 0$ such that $\lambda y \in \Sigma$.

The following basic lemma is useful.

Lemma 2.2. *Let C be a closed cone with nonempty interior in a Banach space X, and $u \in C^\circ$. For each $x \in X$ and $y \in C^\circ$,*

$$M\left(\frac{x}{y}\right) = \sup_{\varphi \in \Sigma_u^*} \frac{\varphi(x)}{\varphi(y)},$$

and the supremum is attained. Moreover, $(x, y) \mapsto M(x/y)$ is a continuous map from $X \times C^\circ$ into $\mathbb{R}$.

Proof. By Lemma 2.1(1),

$$\begin{aligned} M(x/y) &= \inf\{\beta \in \mathbb{R} \colon x \leq_C \beta y\} \\ &= \inf\{\beta \in \mathbb{R} \colon \varphi(x) \leq \beta\varphi(y) \text{ for all } \varphi \in \Sigma_u^*\} \\ &= \sup\{\varphi(x)/\varphi(y) \colon \varphi \in \Sigma_u^*\}. \end{aligned}$$

The supremum is attained by weak* compactness of Σ_u^*.

For the second statement, recall that the $\|\cdot\|_u$-topology is coarser than the $\|\cdot\|$-topology on X; so we may assume that X is equipped with $\|\cdot\|_u$. By Lemma 2.1, the map $x \mapsto \hat{x}$ is an isometric order isomorphism from $(X, \|\cdot\|_u)$ into $(C(\Sigma_u^*), \|\cdot\|_\infty)$, where $\hat{x}(\varphi) := \varphi(x)$ and Σ_u^* is equipped with the weak* topology. The continuity statement now follows from the fact that the map is the composition of the continuous maps

$$(x, y) \mapsto (\hat{x}, \hat{y}) \mapsto \left(\hat{x}, \frac{1}{\hat{y}}\right) \mapsto \frac{\hat{x}}{\hat{y}} \mapsto \sup_{\varphi \in \Sigma_u^*} \frac{\hat{x}}{\hat{y}}(\varphi) = \sup_{\varphi \in \Sigma_u^*} \frac{\varphi(x)}{\varphi(y)} = M\left(\frac{x}{y}\right).$$

□

3 A Denjoy-Wolff theorem for Thompson's metric

In this section, we prove a Denjoy-Wolff theorem for fixed point free Thompson's metric nonexpansive mappings $f \colon C^\circ \to C^\circ$, where one of the orbits of f has a compact closure in the norm topology. We denote the set of accumulation points of $\mathcal{O}(x; f)$ in C by $\omega(x; f)$. As we are allowing infinite dimensional cones, some care must be taken to ensure that all accumulation points of the orbits of fixed point free nonexpansive mapping lie in ∂C. Indeed, in [15] Edelstein gave an example of a fixed point free nonexpansive mapping $f \colon H \to H$ on a separable Hilbert space H such that $\mathcal{O}(0; f)$ is unbounded in norm but has 0 as an accumulation point; see also [48]. To exclude such situations, we assume that the nonexpansive mapping satisfies the following property.

Definition 3.1. Let C be a normal, closed cone with nonempty interior in a Banach space $(X, \|\cdot\|)$, and let $f\colon C^\circ \to C^\circ$ be a continuous mapping. We say that f has the **fixed point property on C° with respect to** d_C if f has a fixed point in D for each bounded, convex, closed subset D of (C°, d_C) with $f(D) \subseteq D$.

Of course, if X is finite dimensional, then every continuous mapping $f\colon C^\circ \to C^\circ$ has the fixed point property with respect to d_C, by the Brouwer Fixed Point Theorem. In infinite dimensional spaces, sufficient conditions were obtained by Nussbaum in [41, Theorem 3.10] in terms of "condensing functions and measures of noncompactness".

Let us now formulate the main result of this section.

Theorem 3.2. *Let C be a normal closed cone with nonempty interior in a Banach space $(X, \|\cdot\|)$, and let $f\colon C^\circ \to C^\circ$ be a fixed point free Thompson metric nonexpansive mapping satisfying the fixed point property on C° with respect to d_C. If $\mathcal{O}(x_0; f)$ has compact closure in the norm topology for some $x_0 \in C^\circ$, then there exists a convex set $\Omega \subseteq \partial C$ such that $\omega(x; f) \subseteq \Omega$ for all $x \in C^\circ$.*

This result confirms [41, Conjecture 4.23] by Nussbaum in case the mapping has a pre-compact orbit. Also note that Theorem 3.2 implies Theorem 1.3, as each nonexpansive mapping has the fixed point property on C° with respect to d_C when C is finite dimensional.

The following proposition plays a central role in the proof.

Proposition 3.3. *If $\{x_k\}_k$ is a sequence in the interior of a closed cone C in a Banach space X such that $\{x_k\colon k = 0, 1, 2, \dots\}$ has compact closure in the norm topology on X,*

$$d_C(x_{m+k}, x_{n+k}) \leq d_C(x_m, x_n) \text{ for all } k, m, n \geq 0, \tag{3}$$

and

$$\lim_{k\to\infty} d_C(x_k, x_0) = \infty, \tag{4}$$

then there exist a subsequence $\{x_{k_i}\}_i$ of $\{x_k\}_k$ and $\eta \in \partial C$ such that $\lim_{i\to\infty} x_{k_i} = \eta$ *and*

$$d_C(x_m, x_0) < d_C(x_{k_i}, x_0) \text{ for all } m < k_i. \tag{5}$$

Moreover, there exist $\varphi, \sigma_{jm} \in \Sigma_u^*$ for $j, m \geq 1$ with $\sigma_{jm}(\eta) = 0$ such that

$$\frac{\varphi(x_{k_j+m})}{\varphi(x_0)} \leq \frac{\sigma_{jm}(x_{k_j})}{\sigma_{jm}(x_0)} \tag{6}$$

for all $j, m \geq 1$.

Proof. Take $u \in C^\circ$ fixed, and let $R := 1/d(u, X \setminus C)$, so that Σ_u^* is bounded by R, by Lemma 2.1(3). Let Y be the closed linear span of $\{x_k : k = 0, 1, 2, \ldots\} \cup \{u\}$, and write $K = C \cap Y$. Then Y is separable, and K is a closed cone in Y with u in its interior. Note that

$$M(x/y; C) = M(x/y; K) \text{ for all } x, y \in K^\circ,$$

and hence $d_K(x, y) = d_C(x, y)$ on K°. As $Y \cap C^\circ$ is nonempty, Y is a majorizing subspace of X, meaning that for each $x \in X$ there exists $y \in Y$ such that $x \le_C y$. By Kantorovich's theorem [5, Theorem 1.30], each positive linear functional on Y can be extended as a positive functional to all of X. Thus we may assume from the outset that X is separable.

By Lemma 2.1(3) Σ_u^* is weak* sequentially compact. By (4), we can find a subsequence $\{x_{k_i}\}_i$ of $\{x_k\}_k$ satisfying (5). Furthermore, as $\{x_k : k = 0, 1, 2, \ldots\}$ has compact closure in the norm topology, we can take a further subsequence and assume that $\{x_{k_i}\}_i$ converges to $\eta \in \partial C$. (Note that η cannot lie in C°; as otherwise, $d_C(\eta, x_0) < \infty$, which violates (4).)

By Lemma 2.2, for each $i, j, m \ge 1$ with $m \le k_i$, there exist $\varphi_i, \psi_{im}, \sigma_{ijm} \in \Sigma_u^*$ such that

$$\begin{aligned} M\left(\frac{x_0}{x_{k_i}}\right) &= \frac{\varphi_i(x_0)}{\varphi_i(x_{k_i})}, \\ M\left(\frac{x_0}{x_{k_i-m}}\right) &= \frac{\psi_{im}(x_0)}{\psi_{im}(x_{k_i-m})}, \\ M\left(\frac{x_{k_j}}{x_{k_i-m}}\right) &= \frac{\sigma_{ijm}(x_{k_j})}{\sigma_{ijm}(x_{k_i-m})}. \end{aligned} \tag{7}$$

We claim that for all i sufficiently large, the Thompson metric distance equals the logarithm of the M-functions in (7). We prove this only for $M(x_{k_j}/x_{k_i-m})$; the arguments for the other functions are similar and left to the reader.

First note that

$$d_C(x_{k_i}, x_0) - d_C(x_{k_j+m}, x_0) \le d_C(x_{k_j+m}, x_{k_i}) \le d_C(x_{k_j}, x_{k_i-m}).$$

As the left hand side tends to ∞ as $i \to \infty$, we find that $d_C(x_{k_j}, x_{k_i-m}) \to \infty$ as $i \to \infty$. For $i \ge 1$, let $\hat\sigma_{ijm} \in \Sigma_u^*$ be such that

$$M\left(\frac{x_{k_i-m}}{x_{k_j}}\right) = \frac{\hat\sigma_{ijm}(x_{k_i-m})}{\hat\sigma_{ijm}(x_{k_j})}.$$

By weak* compactness of Σ_u^*, the sequence $\{\hat\sigma_{ijm}(x_{k_j})\}_i$ is bounded from below by a positive real; hence

$$M\left(\frac{x_{k_i-m}}{x_{k_j}}\right) = \frac{\hat\sigma_{ijm}(x_{k_i-m})}{\hat\sigma_{ijm}(x_{k_j})} \le \frac{R\|x_{k_i-m}\|}{\hat\sigma_{ijm}(x_{k_j})}$$

is bounded from above by a positive real, since $(\|x_{k_i-m}\|)_i$ is bounded. Thus, for all i sufficiently large,

$$d_C(x_{k_j}, x_{k_i-m}) = \log \frac{\sigma_{ijm}(x_{k_j})}{\sigma_{ijm}(x_{k_i-m})}.$$

From now on we assume that i is so large that the Thompson metric distance is given by the logarithm of the M-functions in (7).

By (5),

$$\log\left(\frac{\psi_{im}(x_0)}{\psi_{im}(x_{k_i-m})}\right) = d_C(x_0, x_{k_i-m}) \leq d_C(x_0, x_{k_i}) = \log\left(\frac{\varphi_i(x_0)}{\varphi_i(x_{k_i})}\right),$$

so that

$$\frac{\varphi_i(x_{k_i})}{\psi_{im}(x_{k_i-m})} \leq \frac{\varphi_i(x_0)}{\psi_{im}(x_0)}. \tag{8}$$

Note also that by the definition of $\psi_{im} \in \Sigma_u^*$,

$$\frac{\sigma_{ijm}(x_0)}{\sigma_{ijm}(x_{k_i-m})} \leq \frac{\psi_{im}(x_0)}{\psi_{im}(x_{k_i-m})},$$

so that

$$\frac{\psi_{im}(x_{k_i-m})}{\sigma_{ijm}(x_{k_i-m})} \leq \frac{\psi_{im}(x_0)}{\sigma_{ijm}(x_0)}. \tag{9}$$

Now using equations (3), (8), and (9), we get that

$$\begin{aligned}\frac{\varphi_i(x_{k_j+m})}{\varphi_i(x_0)} &= \frac{\varphi_i(x_{k_j+m})}{\varphi_i(x_{k_i})}\frac{\varphi_i(x_{k_i})}{\varphi_i(x_0)} \leq e^{d_C(x_{k_j+m}, x_{k_i})}\frac{\varphi_i(x_{k_i})}{\varphi_i(x_0)} \\ &\leq e^{d_C(x_{k_j}, x_{k_i-m})}\frac{\varphi_i(x_{k_i})}{\varphi_i(x_0)} = \frac{\sigma_{ijm}(x_{k_j})}{\sigma_{ijm}(x_{k_i-m})}\frac{\varphi_i(x_{k_i})}{\varphi_i(x_0)} \\ &= \frac{\sigma_{ijm}(x_{k_j})}{\varphi_i(x_0)}\frac{\varphi_i(x_{k_i})}{\psi_{im}(x_{k_i-m})}\frac{\psi_{im}(x_{k_i-m})}{\sigma_{ijm}(x_{k_i-m})} \\ &\leq \frac{\sigma_{ijm}(x_{k_j})}{\varphi_i(x_0)}\frac{\varphi_i(x_0)}{\psi_{im}(x_0)}\frac{\psi_{im}(x_0)}{\sigma_{ijm}(x_0)} = \frac{\sigma_{ijm}(x_{k_j})}{\sigma_{ijm}(x_0)}.\end{aligned}$$

As Σ_u^* is sequentially weak* compact, we can pass to a subsequence twice and assume that $\varphi_i \to \varphi \in \Sigma_u^*$ and $\sigma_{ijm} \to \sigma_{jm} \in \Sigma_u^*$ in the weak* topology as $i \to \infty$, which proves (6).

It remains to show that $\sigma_{jm}(\eta) = 0$ for all $j, m \geq 1$. As

$$d_C(x_{k_j}, x_{k_i-m}) = \log\left(\sigma_{ijm}(x_{k_j})/\sigma_{ijm}(x_{k_i-m})\right) \to \infty$$

as $i \to \infty$, we know that $\sigma_{ijm}(x_{k_i-m}) \to 0$ as $i \to \infty$. Moreover,

$$\log \frac{\sigma_{ijm}(x_{k_i})}{\sigma_{ijm}(x_{k_i-m})} \le d_C(x_{k_i}, x_{k_i-m}) \le d_C(x_m, x_0),$$

which implies that $\sigma_{ijm}(x_{k_i}) \to 0$ as $i \to \infty$. As

$$\lim_{i\to\infty} |\sigma_{ijm}(x_{k_i} - \eta)| \le \lim_{i\to\infty} R\|x_{k_i} - \eta\| = 0$$

and $\sigma_{ijm}(x_{k_i} - \eta) \to -\sigma_{jm}(\eta)$ as $i \to \infty$, we see that $\sigma_{jm}(\eta) = 0$. □

Before proceeding. we mention a useful result of Całka . Recall that a metric space (M, ρ) is said to be **finitely totally bounded** if for each bounded set $S \subseteq M$ and each $\varepsilon > 0$, the set S can be covered with finitely many balls of radius ε.

Theorem 3.4 (Całka [12, Theorem 5.6])**.** *If $f : M \to M$ is a nonexpansive mapping on a finitely totally bounded metric space (M, ρ) and there exists $x_0 \in M$ such that $\mathcal{O}(x_0; f)$ has a bounded subsequence, then $\mathcal{O}(x; f)$ is bounded for every $x \in M$.*

Using Całka's theorem and Proposition 3.3, we now derive the following consequence.

Corollary 3.5. *Let C be a normal closed cone with nonempty interior in a Banach space $(X, \|\cdot\|)$ and let $f : C^\circ \to C^\circ$ be a fixed point free Thompson metric nonexpansive mapping satisfying the fixed point property on C° with respect to d_C. If $x_0 \in C^\circ$ is such that $\mathcal{O}(x_0; f)$ has compact closure in the norm topology, then there exists $\varphi \in C^* \setminus \{0\}$ such that $\omega(x_0; f) \subseteq \ker(\varphi) \cap C$.*

Proof. Take $u \in C^\circ$ fixed, and let $R := 1/d(u, X \setminus C)$. Recall that Σ_u^* is bounded by R, by Lemma 2.1(3).

We first prove that

$$\lim_{k\to\infty} d_C(f^k(x_0), x_0) = \infty.$$

Suppose, by way of contradiction, that there exist $r > 0$ and a subsequence $\{f^{k_i}(x_0)\}_i$ of $\{f^k(x_0)\}$ such that $d_C(f^{k_i}(x_0), x_0) \le r$ for all i. Define M to be the norm closure of $\mathcal{O}(x_0; f)$, which is compact in the norm topology, by assumption. As the topology of d_C coincides with the norm topology on C°, $M \cap B_\delta(x_0)$ is compact with respect to d_C for each closed ball $B_\delta(x_0) := \{y \in C^\circ : d_C(x_0, y) \le \delta\}$. So, for each $\varepsilon > 0$, the set $\mathcal{O}(x_0; f) \cap B_\delta(x_0)$ can be covered with finitely many balls of radius ε. This shows that $(\mathcal{O}(x_0; f), d_C)$ is finitely totally bounded. Using Całka's theorem, we see that $\mathcal{O}(x_0; f)$ is bounded with respect to d_C. This implies that $\omega(x_0; f) \subseteq C^\circ$ is a nonempty and bounded with respect to d_C. As f has the fixed point property on C° with respect to d_C, we can apply [41, Theorem 3.11] to conclude that the mapping f has a fixed point in C°, which contradicts our assumption.

For $k \geq 1$ let $x_k := f^k(x_0)$, so $\{x_k\}_k$ satisfies the assumptions of Proposition 3.3. We find $\varphi, \sigma_{jm} \in \Sigma_u^*$ such that

$$\frac{\varphi(x_{k_j+m})}{\varphi(x_0)} \leq \frac{\sigma_{jm}(x_{k_j})}{\sigma_{jm}(x_0)} \tag{10}$$

and $\sigma_{jm}(\eta) = 0$, where $x_{k_i} \to \eta \in \partial C$ as $i \to \infty$.

By weak* compactness of Σ_u^*, $\sigma_{jm}(x_0)$ is uniformly bounded from below in j and m, and

$$\lim_{j\to\infty} |\sigma_{jm}(x_{k_j})| = \lim_{j\to\infty} |\sigma_{jm}(x_{k_j} - \eta)| \leq \lim_{j\to\infty} R\|x_{k_j} - \eta\| = 0,$$

by Lemma 2.1(2). This implies that the right hand side of (10) converges to 0 uniformly in m, and hence

$$\lim_{j\to\infty} \varphi(x_{k_j+m}) = 0 \tag{11}$$

uniformly in m.

Now if $\xi \in \omega(x_0; f)$, there exists a subsequence $\{x_{k_{j_n}+m_n}\}_j$ of $\{x_{k_n} + m_n\}$ with $x_{k_{j_n}+m_n} \to \xi$ and $k_{j_n} \to \infty$ as $n \to \infty$. It follows from (11) that $\varphi(\xi) = 0$, and hence ξ is in the kernel $\ker(\varphi)$ of the positive functional φ; so, $\omega(x_0, f) \subseteq \ker(\varphi) \cap C$. ☐

We can now prove Theorem 3.2.

Proof of Theorem 3.2. By Corollary 3.5, $\omega(x_0; f) \subset \partial C$. It remains to show that the convex hull of $\cup_{x\in C^\circ}\omega(x; f)$, denoted Ω, is contained in ∂C. The argument is similar to that given in [41, Theorem 5.3] and relies on the fact that the closure of $\mathcal{O}(x_0; f)$ is compact.

Let $z \in C^\circ$ and $\zeta \in \omega(z; f)$. Then there exists a subsequence $\{f^{k_i}(z)\}_i$ of $\{f^k(z)\}_k$ converging to ζ in the norm topology. As $\mathcal{O}(x_0; f)$ has a compact closure, we may assume, after possibly taking a further subsequence, that $f^{k_i}(x_0)$ converges to some $\xi \in \omega(x_0; f)$. Obviously, $d_C(f^{k_i}(x_0), f^{k_i}(z)) \leq d_C(x_0, z)$ for all i, and hence $\xi \sim_C \zeta$, by [41, Lemma 5.2].

Now let $\eta \in \Omega$. Then there exist $z_1, \dots, z_n \in C^\circ$, $0 < \lambda_1, \dots, \lambda_n < 1$ with $\sum_{i=1}^n \lambda_i = 1$, and $\zeta_i \in \omega(z_i; f)$ for $i = 1, \dots, n$ such that $\eta = \sum_{i=1}^n \lambda_i\zeta_i$. For each $i = 1, \dots, n$, there exists $\xi_i \in \omega(x_0; f)$ with $\xi_i \sim_C \zeta_i$. Clearly, $\nu := \sum_{i=1}^n \lambda_i\xi_i$ is in the convex hull of $\omega(x_0; f)$ and $\nu \sim_C \eta$.

Now suppose that there exists $\chi \in \Omega \cap C^\circ$. By the previous observation, there exists ν in the convex hull of $\omega(x_0; f)$ with $\nu \sim_C \chi$. But this implies that $\nu \in C^\circ$, which contradicts the fact that $\omega(x_0; f)$ is contained in ∂C. □

4 The cone spectral radius

In the remainder of this paper, we discuss Denjoy-Wolff type theorems for Hilbert's metric nonexpansive mappings that come from scaling order-preserving homogeneous mappings $f\colon C^\circ \to C^\circ$. More precisely, we consider mappings $g\colon \Sigma^\circ \to \Sigma^\circ$ of the form

$$g(x) := \frac{f(x)}{q(f(x))} \text{ for } x \in \Sigma_q^\circ,$$

where $f\colon C^\circ \to C^\circ$ is an order-preserving homogeneous mapping on the interior of a normal closed cone in a Banach space $(X, \|\cdot\|)$, $q\colon C^\circ \to (0, \infty)$ is a norm continuous homogeneous function, and

$$\Sigma_q^\circ := \{x \in C^\circ : q(x) = 1\}.$$

Typical examples of functions q include strictly positive functionals φ in X^*, $q(\cdot) = \|\cdot\|$, and $q(\cdot) = \|\cdot\|_u$, where $u \in C^\circ$ is fixed.

To analyze the dynamics of such mappings, we need to introduce a spectral radius for order-preserving homogenous mappings $f\colon C^\circ \to C^\circ$. There exist various definitions for the spectral radius for continuous, order-preserving, homogeneous mappings $f\colon C \to C$ if f is defined on the whole of the closed cone C; see [36]. In general, however, $f\colon C^\circ \to C^\circ$ may fail to have a continuous, order-preserving, homogeneous extension to the whole of C; see [11]. So, some additional analysis is needed.

4.1 Approximate eigenvectors. As the definition of, and the results concerning, the cone spectral radius for mappings $f\colon C^\circ \to C^\circ$ are of some independent interest, we work in a slightly more general setting. We consider homogeneous mappings that are defined on a subset of a normal closed cone $C \subseteq X$ and that are order-preserving with respect to a, possibly different, normal closed cone $K \subseteq X$ with $C \subseteq K$.

Throughout this section, we assume that $C \subseteq K$ are normal closed cones in a Banach space $(X, \|\cdot\|)$. For $u \in C$ with $\|u\| = 1$, we denote the part of u (with respect to K) by

$$K_u := \{x \in K : \alpha x \leq_K u \leq_K \beta x \text{ for some } 0 < \alpha \leq \beta\}.$$

We consider homogeneous mappings $f: C \cap K_u \to C \cap K_u$ that are order-preserving with respect to K, so $f(x) \leq_K f(y)$ whenever $x, y \in C \cap K_u$ and $x \leq_K y$. For the applications in this paper, we eventually assume that $u \in C^\circ$ and $K = C$, in which case $K_u = C^\circ$. The reader may wish to make this simplifying assumption.

Definition 4.1. Let $u \in C$, with $\|u\| = 1$, and $f: C \cap K_u \to C \cap K_u$ be homogeneous and order-preserving with respect to K. We say that f is ***u*-bounded** if there exists $M > 0$ such that

$$f(x) \leq_K M\|x\|u \text{ for all } x \in C \cap K_u.$$

Note that if $u \in C^\circ$, with $\|u\| = 1$, and $K = C$, then any homogeneous order-preserving mapping $f: C^\circ \to C^\circ$ is u-bounded. Indeed, as C is a closed normal cone and $u \in C^\circ$, the order-unit norm $\|\cdot\|_u$ is equivalent to $\|\cdot\|$. So there exists a constant $M_1 > 0$ such that $\|x\|_u \leq M_1$ for all $x \in C$ with $\|x\| \leq 1$. This implies that $x \leq_C M_1 u$ for all $x \in C$ with $\|x\| \leq 1$, and hence $f(x) \leq_C M_1 f(u)$. As $u \in C^\circ$, there exists $M_2 > 0$ such that $f(u) \leq_C M_2 u$, so

$$f(x) \leq_C M_1 M_2 u \text{ for all } x \in C \text{ with } \|x\| \leq 1.$$

Given a homogeneous mapping $f: C\cap K_u \to C\cap K_u$ which is order-preserving with respect to K, we define for $k \geq 1$,

$$\|f^k\|_{C\cap K_u} := \sup\{\|f^k(x)\| : x \in C \cap K_u \text{ and } \|x\| \leq 1\}.$$

Lemma 4.2. *If $f: C \cap K_u \to C \cap K_u$ is a homogeneous mapping which is order-preserving with respect to K and there exists an integer $m \geq 1$ such that f^m is u-bounded, then $\|f^k\|_{C\cap K_u} < \infty$ for all $k \geq m$, and*

$$\lim_{k\to\infty} \|f^k\|_{C\cap K_u}^{1/k} = \inf_{k\geq m} \|f^k\|_{C\cap K_u}^{1/k} = \lim_{k\to\infty} \|f^k(u)\|^{1/k}.$$

Proof. We first show that f^k extends continuously to 0 for all $k \geq m$. Note that if $k \geq m$ and $\|x_n\| \to 0$, then

$$f^k(x_n) = f^{k-m}(f^m(x_n)) \leq_K M\|x_n\| f^{k-m}(u) \leq_K M\|x_n\|\beta_k u$$

for some $\beta_k > 0$, as f^m is u-bounded and $f^{k-m}(u) \in C \cap K_u$. Thus, for each $k \geq m$, we have that $f^k(x_n) \to 0$ if $\|x_n\| \to 0$. So, defining $f^k(0) := 0$, we obtain a continuous extension of f^k to 0.

Using the homogeneity of f^k, it is easy to show that $\|f^k\|_{C\cap K_u} < \infty$ for all $k \geq m$. Using sub-additivity, we now show that

$$\lim_{k\to\infty} \|f^k\|_{C\cap K_u}^{1/k} = \inf_{k\geq m} \|f^k\|_{C\cap K_u}^{1/k}.$$

Let $a_n := \log \|f^n\|_{C\cap K_u}$ for all $n \geq 1$. We know that $a_n < \infty$ for all $n \geq m$; and, clearly, $a_{p+q} \leq a_p + a_q$ for all $p, q \geq m$. Let $L := \inf_{n\geq m} a_n/n < \infty$. Take $\varepsilon > 0$ and choose $k \geq m$ such that $a_k/k < L + \varepsilon$. For each $n \geq 2k$ we have that $n =: p_n k + q_n + m$, where $p_n \geq 1$ and $0 \leq q_n < k$, so $a_n \leq a_{p_n k} + a_{q_n+m} \leq p_n a_k + a_{q_n+m}$. This gives the inequality

$$\frac{a_n}{n} \leq \frac{p_n k}{n}\frac{a_k}{k} + \frac{a_{q_n+m}}{n}.$$

Letting $n \to \infty$ shows that

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_k}{k},$$

since $p_n k/n \to 1$ as $n \to \infty$, and $a_{j+m} < \infty$ for all $0 \leq j < k$. Thus,

$$L \leq \liminf_{n\to\infty} \frac{a_n}{n} \leq \limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_k}{k} \leq L + \varepsilon,$$

which shows that $\lim_{n\to\infty} a_n/n = L$.

Write $r := \lim_{k\to\infty} \|f^k\|_{C\cap K_u}^{1/k}$ and $r_k := \|f^k\|_{C\cap K_u}^{1/k}$. It is an easy exercise in calculus to show that $\lim_{k\to\infty} r_{k+n}^{(k+n)/k} = r$ for all $n \geq 1$. Let $\kappa > 0$ be the normality constant of K. If $x \in C \cap K_u$ and $\|x\| \leq 1$, then $f^m(x) \leq_K Mu$, so $f^{k+m}(x) \leq_K Mf^k(u)$, which gives $\|f^{k+m}(x)\| \leq M\kappa\|f^k(u)\|$ for all $x \in C \cap K_u$ with $\|x\| \leq 1$. Thus

$$r_{m+k}^{(m+k)/k} \leq (M\kappa\|f^k(u)\|)^{1/k} \leq (M\kappa)^{1/k} r_k,$$

and hence $\lim_{k\to\infty} \|f^k(u)\|^{1/k} = r$. □

It is well known; see for example [5, Theorem 2.38] or [29, Theorem 4.4], that as K is normal, $(X, \|\cdot\|)$ admits an equivalent **monotone norm** $|\cdot|$, i.e., $|x| \leq |y|$ whenever $0 \leq_K x \leq_K y$. Given a homogeneous mapping $f\colon C \cap K_u \to C \cap K_u$ which is order-preserving with respect to K and $\varepsilon > 0$, define $f_{\varepsilon,u}\colon C \cap K_u \to C \cap K_u$ by

$$f_{\varepsilon,u}(x) := f(x) + \varepsilon|x|u \quad \text{for } x \in C \cap K_u. \tag{12}$$

Note that $f_{\varepsilon,u}$ is homogeneous and order-preserving with respect to K, as $|\cdot|$ is a monotone norm. Moreover, for each $x \in C \cap K_u$ we have that $f(x) \leq_K f_{\varepsilon,u}(x)$ and

$$\sup_{x\in C\cap K_u\colon |x|\leq 1} |f(x) - f_{\varepsilon,u}(x)| \leq \varepsilon|u|.$$

Theorem 4.3. *Let $f\colon C \cap K_u \to C \cap K_u$ be a homogeneous mapping which is order-preserving with respect to K, and let $f_{\varepsilon,u}\colon C \cap K_u \to C \cap K_u$ be given by (12). If f is u-bounded, then*

(i) *for each $\varepsilon > 0$, the mapping $f_{\varepsilon,u}$ has a unique eigenvector $v_{\varepsilon,u} \in C \cap K_u$ with $|v_{\varepsilon,u}| = 1$, and*

$$f_{\varepsilon,u}(v_{\varepsilon,u}) =: r_{\varepsilon,u} v_{\varepsilon,u};$$

(ii) *the mapping* $\varepsilon \mapsto v_{\varepsilon,u}$ *is continuous in the norm topology for* $\varepsilon > 0$*;*

(iii) *for each* $\varepsilon > 0$*,*

$$r_{\varepsilon,u} = \lim_{k\to\infty} \|f^k_{\varepsilon,u}\|^{1/k}_{C\cap K_u} = \inf_{k\geq 1} \|f^k_{\varepsilon,u}\|^{1/k}_{C\cap K_u} = \lim_{k\to\infty} \|f^k_{\varepsilon,u}(u)\|^{1/k},$$

and

$$\lim_{\varepsilon\to 0^+} r_{\varepsilon,u} = \lim_{k\to\infty} \|f^k\|^{1/k}_{C\cap K_u}.$$

Proof. For notational convenience, we write $f_\varepsilon := f_{\varepsilon,u}$, $r_\varepsilon := r_{\varepsilon,u}$, and $v_\varepsilon := v_{\varepsilon,u}$. Let $\Sigma := \{x \in C \cap K_u : |x| = 1\}$ and define $g_\varepsilon : \Sigma \to \Sigma$ by

$$g_\varepsilon(x) := \frac{f_\varepsilon(x)}{|f_\varepsilon(x)|} \quad \text{for } x \in \Sigma.$$

As $\varepsilon u \leq_K f_\varepsilon(x)$ for all $x \in \Sigma$, $\varepsilon|u| = |\varepsilon u| \leq |f_\varepsilon(x)|$ for all $x \in \Sigma$.

Because of the equivalence of the norms $|\cdot|$ and $\|\cdot\|$ and the fact that f is u-bounded, there exists $M_1 > 0$ such that

$$f(x) \leq_K M_1|x|u \quad \text{for all } x \in C \cap K_u.$$

This implies for $x \in \Sigma$ that $f_\varepsilon(x) \leq_K (M_1 + \varepsilon)u$ and $|f_\varepsilon(x)| \leq (M_1 + \varepsilon)|u|$.

By definition of Hilbert's metric δ_K,

$$\delta_K(g_\varepsilon(x), g_\varepsilon(y)) = \delta_K(f_\varepsilon(x), f_\varepsilon(y))$$

for all $x, y \in \Sigma$, and

$$\delta_K(g_\varepsilon(x), u) = \delta_K(f_\varepsilon(x), u) \leq \log\left(\frac{M_1 + \varepsilon}{\varepsilon}\right) \tag{13}$$

for all $x \in \Sigma$.

Let $\Gamma := \{x \in K_u : |x| = 1\}$. We know (see [29, Theorem 4.8]) that the metric space (Γ, δ_K) is complete, as K is a closed normal cone in $(X, \|\cdot\|)$. We now show that Σ is a closed subset of (Γ, δ_K), from which we conclude that (Σ, δ_K) is a complete metric space.

Suppose that $\{x_k\}_k$ is a sequence in Σ converging to ξ in (Γ, δ_K), Then there exist $0 < \alpha_k \leq \beta_k$ such that $\alpha_k\xi \leq_K x_k \leq_K \beta_k\xi$ for $k = 1, 2, \ldots$ and $\log(\beta_k/\alpha_k) \to 0$ as $k \to \infty$. Since

$$\alpha_k = |\alpha_k\xi| \leq |x_k| = 1 \leq |\beta_k\xi| = \beta_k,$$

$\alpha_k \leq 1 \leq \beta_k$, and hence $\lim_{k\to\infty} \alpha_k = 1 = \lim_{k\to\infty} \beta_k$. This implies that

$$0 \leq_k x_k - \alpha_k\xi \leq_K (\beta_k - \alpha_k)\xi,$$

so $|x_k - \alpha_k\xi| \le \beta_k - \alpha_k$, which shows that $\lim_{k\to\infty} |x_k - \xi| = 0$. Since $\|\cdot\|$ and $|\cdot|$ are equivalent, C is closed in $(X, |\cdot|)$, and therefore $\xi \in C$. But also $|\xi| = 1$, which shows that $\xi \in \Sigma$ and hence Σ is closed in (Γ, δ_K).

To proceed, we fix $\varepsilon_1 > 0$. Define

(14) $$R := \log\left(\frac{M_1+\varepsilon_1}{\varepsilon_1}\right) > 0 \quad \text{and} \quad B_R := \{x \in \Sigma : \delta_K(x,u) \le R\};$$

B_R is a closed subset of (Σ, δ_K), so (B_R, δ_K) is a complete metric space. Note that it follows from (13) that $g_\varepsilon(B_R) \subseteq B_R$ for $\varepsilon_1 < \varepsilon$.

To prove that f_ε has a unique normalised eigenvector $v_\varepsilon \in C \cap K_u$, it suffices to show that g_ε has a unique fixed point in B_R, where we choose $0 < \varepsilon_1 < \varepsilon$. The idea is to prove that g_ε is a contraction mapping on the complete metric space (B_R, δ_K).

To this end, we let $x \ne y$ in B_R and define $\alpha := M(x/y)^{-1}$ and $\beta := M(y/x)$, so $\alpha x \le_K y \le_K \beta x$ and $\delta_K(x,y) = \log(\beta/\alpha) > 0$. As $x, y \in \Sigma$, $\alpha = \alpha|x| \le |y| = 1 \le \beta|x| = \beta$, so $\alpha \le 1 \le \beta$. Now $\alpha \le 1$ and $f(x) \le_K M_1 u$ give the inequality

$$\varepsilon(1-\alpha)f(x) \le_K \varepsilon(1-\alpha)M_1 u.$$

Combining this with the inequalities $\alpha M_1 f(x) \le_K M_1 f(y)$ and $\alpha\varepsilon f(x) \le_K \varepsilon f(y)$ gives

$$(\alpha M_1 + \varepsilon)(f(x) + \varepsilon u) \le_K (M_1+\varepsilon)(f(y) + \varepsilon u),$$

which shows that

$$\alpha' f_\varepsilon(x) \le_K f_\varepsilon(y), \quad \text{where } \alpha' := \frac{\alpha M_1 + \varepsilon}{M_1 + \varepsilon}.$$

In a similar way, it can be shown that

$$f_\varepsilon(y) \le_K \beta' f_\varepsilon(x), \quad \text{where } \beta' := \frac{\beta M_1 + \varepsilon}{M_1 + \varepsilon}.$$

So, letting $\varepsilon' := \varepsilon/M_1$, we get

(15) $$\delta_K(f_\varepsilon(x), f_\varepsilon(y)) \le \log\left(\frac{\beta'}{\alpha'}\right) = \log\left(\frac{\beta+\varepsilon'}{\alpha+\varepsilon'}\right).$$

Note that $\delta_K(x,y) \le 2R$, so $\log(\beta/\alpha) \le 2R$, and hence $\beta/\alpha \le e^{2R} = \left(\frac{M_1+\varepsilon_1}{\varepsilon_1}\right)^2$, as $x, y \in B_R$. Thus, to prove that g_ε is a contraction, it suffices to show that there exists $0 \le c < 1$ such that for all $0 < \alpha \le 1 \le \beta$ with $1 < \beta/\alpha \le e^{2R}$, and for all $\varepsilon > 0$ with $\varepsilon_1 < \varepsilon$,

(16) $$\log\left(\frac{\beta+\varepsilon'}{\alpha+\varepsilon'}\right) \le c\log\left(\frac{\beta}{\alpha}\right).$$

Basic algebra gives

$$\log\left(\frac{\beta+\varepsilon'}{\alpha+\varepsilon'}\right) = \log\left(\frac{\beta}{\alpha}\right) + \log\left(1-\left(\frac{\varepsilon'}{\alpha+\varepsilon'}\right)\left(1-\frac{\alpha}{\beta}\right)\right).$$

Writing $\rho := \beta/\alpha$, so $1 < \rho \le e^{2R}$, and using the fact that $0 < \alpha \le 1$, we derive that

$$\log\left(\frac{\beta+\varepsilon'}{\alpha+\varepsilon'}\right) \le \log\rho + \log\left(1-\left(\frac{\varepsilon'}{1+\varepsilon'}\right)\left(1-\frac{1}{\rho}\right)\right). \tag{17}$$

Let $\gamma := \varepsilon'/(1+\varepsilon')$, and for $1 < \rho < e^{2R}$, consider the continuous function $\rho \mapsto \psi(\rho)$, where

$$\psi(\rho) = \frac{\log\rho + \log\left(1-\left(\frac{\varepsilon'}{1+\varepsilon'}\right)\left(1-\frac{1}{\rho}\right)\right)}{\log\rho} = 1 + \frac{\log(1-\gamma(1-1/\rho))}{\log\rho}.$$

Thus, to establish (16) it suffices to find $0 < \delta \le 1$, which is independent of $\gamma = \varepsilon'/(1+\varepsilon') = \varepsilon/(M_1+\varepsilon)$ for $0 < \varepsilon_1 < \varepsilon$, such that

$$\begin{aligned}\sup\left\{\frac{\log(1-\gamma(1-1/\rho))}{\log\rho} : 1 < \rho \le e^{2R}\right\} \\ = \sup\left\{\frac{\log(1-\gamma(1-e^{-\sigma}))}{\sigma} : 0 < \sigma \le 2R\right\} \le -\delta.\end{aligned} \tag{18}$$

As $0 < \gamma(1-e^{-\sigma}) < 1$, we can use Taylor's formula to get

$$\frac{1}{\sigma}\log(1-\gamma(1-e^{-\sigma})) = -\sum_{j=1}^{\infty}\frac{(\gamma(1-e^{-\sigma}))^j}{\sigma j} \le -\frac{\gamma(1-e^{-\sigma})}{\sigma}$$

for $0 < \sigma \le 2R$. Now consider the derivative of $\vartheta\colon \sigma \mapsto (1-e^{-\sigma})/\sigma$:

$$\vartheta'(\sigma) = \frac{e^{-\sigma}(\sigma+1-e^{\sigma})}{\sigma^2}.$$

As $e^{\sigma} > 1+\sigma$ for all $\sigma > 0$, we conclude that $\vartheta'(\sigma) < 0$ on $0 < \sigma \le 2R$, and hence

$$\frac{1}{\sigma}\log(1-\gamma(1-e^{-\sigma})) \le -\gamma\left(\frac{1-e^{-2R}}{2R}\right).$$

So, if we let

$$\delta := \left(\frac{\varepsilon_1}{M_1+\varepsilon_1}\right)\left(\frac{1-e^{-2R}}{2R}\right) < 1,$$

then (18) holds for all $\varepsilon_1 < \varepsilon$, as $\varepsilon_1/(M_1+\varepsilon_1) \le \varepsilon/(M_1+\varepsilon)$ for all $\varepsilon_1 \le \varepsilon$.

It now follows from (17) that g_ε is a contraction mapping on the complete metric space (B_R, δ_K) with contraction constant $1-\delta$ for all $\varepsilon_1 < \varepsilon$. So, by the Contraction Mapping Theorem g_ε, $\varepsilon_1 < \varepsilon$ has a unique fixed point $v_\varepsilon \in \Sigma$. Moreover,

v_ε is the unique normalised eigenvector in Σ of f_ε, and $f_\varepsilon(v_\varepsilon) = |f_\varepsilon(v_\varepsilon)|v_\varepsilon$. Writing $r_\varepsilon := |f_\varepsilon(v_\varepsilon)|$ and recalling that $|f_\varepsilon(x)| \geq \varepsilon|u|$ for all $x \in \Sigma$, we see that $r_\varepsilon > \varepsilon|u|$.

We now prove the second assertion, which is also a consequence of the Contraction Mapping Theorem. Indeed, for $\varepsilon_1 > 0$, let R be as in (14) above. For $\varepsilon > \varepsilon_1$, that the mapping g_ε is a contraction mapping on (B_R, δ_K) with contraction constant $0 \leq c < 1$, where c is independent of ε. This implies that

$$\delta_K(v_\mu, v_\varepsilon) \leq \frac{1}{1-c}\delta_K(g_\mu(v_\varepsilon), v_\varepsilon) = \frac{1}{1-c}\delta_K(f_\mu(v_\varepsilon), v_\varepsilon)$$

for $\mu, \varepsilon > \varepsilon_1$. As $v_\varepsilon \in B_R$, there exist $0 < a \leq 1 \leq b$ such that $av_\varepsilon \leq_K u \leq_K bv_\varepsilon$. So, if $\mu > \varepsilon > \varepsilon_1$, then

$$(1 + a(\mu - \varepsilon))v_\varepsilon \leq_K f_\mu(v_\varepsilon) = f_\varepsilon(v_\varepsilon) + (\mu - \varepsilon)u \leq_K (1 + b(\mu - \varepsilon))v_\varepsilon.$$

This implies that

$$\delta_K(v_\mu, v_\varepsilon) \leq \frac{1}{1-c}\delta_K(f_\mu(v_\varepsilon), v_\varepsilon) \leq \frac{1}{1-c}\log\left(\frac{1 + b(\mu - \varepsilon)}{1 + a(\mu - \varepsilon)}\right)$$

for $\varepsilon_1 < \varepsilon < \mu$. A similar argument shows that

$$\delta_K(v_\mu, v_\varepsilon) \leq \frac{1}{1-c}\log\left(\frac{1 + a(\mu - \varepsilon)}{1 + b(\mu - \varepsilon)}\right)$$

for $\varepsilon_1 < \mu < \varepsilon$. So, if $\varepsilon_k \to \mu$, then $\delta_K(v_{\varepsilon_k}, v_\mu) \to 0$, so $\|v_{\varepsilon_k} - v_\mu\| \to 0$, as the topology of δ_K is the same as the topology of $\|\cdot\|$ on B_R.

To prove the third assertion, we can apply Lemma 4.2 to f_ε and f to get

$$\lim_{k\to\infty} \|f_\varepsilon^k\|_{C\cap K_u}^{1/k} = \inf_{k\geq 1} \|f_\varepsilon^k\|_{C\cap K_u}^{1/k} = \lim_{k\to\infty} \|f_\varepsilon^k(u)\|^{1/k} \tag{19}$$

and

$$\lim_{k\to\infty} \|f^k\|_{C\cap K_u}^{1/k} = \inf_{k\geq 1} \|f^k\|_{C\cap K_u}^{1/k} = \lim_{k\to\infty} \|f^k(u)\|^{1/k}. \tag{20}$$

For $0 \leq \varepsilon \leq \mu$, it is easy to see that $f_\varepsilon^k(x) \leq_K f_\mu^k(x)$ for all $x \in C \cap K_u$ and $k \geq 1$. By the normality of K, there exists a constant $M_2 > 0$ (independent of k) such that $\|f_\varepsilon^k\|_{C\cap K_u} \leq M_2\|f_\mu^k\|_{C\cap K_u}$ for $0 \leq \varepsilon \leq \mu$. It follows that

$$|f_\varepsilon^k\|_{C\cap K_u}^{1/k} \leq \left(M_2\|f_\mu^k\|_{C\cap K_u}\right)^{1/k}$$

for $0 \leq \varepsilon \leq \mu$, and hence $\lim_{\varepsilon\to 0^+}\left(\lim_{k\to\infty} \|f_\varepsilon^k\|_{C\cap K_u}^{1/k}\right)$ exists and satisfies

$$\lim_{\varepsilon\to 0^+}\left(\lim_{k\to\infty} \|f_\varepsilon^k\|_{C\cap K_u}^{1/k}\right) \geq \lim_{k\to\infty} \|f^k\|_{C\cap K_u}^{1/k}. \tag{21}$$

Recall that $f_\varepsilon(v_\varepsilon) = r_\varepsilon v_\varepsilon$ and $v_\varepsilon \in \Sigma$. Thus there exist $0 < \lambda_1 \leq \lambda_2$ (depending on ε) such that $\lambda_1 u \leq_K v_\varepsilon \leq_K \lambda_2 u$. This implies that $\lambda_1 f_\varepsilon^k(u) \leq_K r_\varepsilon^k v_\varepsilon \leq_K \lambda_2 f_\varepsilon^k(u)$ for all $k \geq 1$, and hence

$$r_\varepsilon = \lim_{k\to\infty} \|f_\varepsilon^k(u)\|^{1/k}. \tag{22}$$

It remains to show that $\lim_{\varepsilon\to 0^+} r_\varepsilon = \lim_{k\to\infty} \|f^k(u)\|^{1/k}$. Combining (19)–(22), we see that it suffices to show that

$$\lim_{\varepsilon\to 0^+} r_\varepsilon \leq \lim_{k\to\infty} \|f^k(u)\|^{1/k} =: r_{C\cap K_u}(f).$$

First note that there exists $\gamma > 0$, independent of $\varepsilon \geq 0$, such that $u \leq_K \gamma f_\varepsilon(u)$. We also know that for each $|x| \leq 1$ and $0 < \varepsilon < M_1$,

$$f_\varepsilon(x) \leq_K (M_1 + \varepsilon)u \leq_K 2M_1 u.$$

Thus $f_\varepsilon^k(x) \leq_K 2M_1\gamma f_\varepsilon^k(u)$ for each $k \geq 1$ and $|x| \leq 1$. As the norms $|\cdot|$ and $\|\cdot\|$ are equivalent on X, there exists a constant $M_3 > 0$ such that

$$\|f_\varepsilon^k(x)\| \leq M_3 \|f_\varepsilon^k(u)\| \tag{23}$$

for all $x \in C \cap K_u$ and $\|x\| \leq 1$.

Now fix $\eta > 0$ and choose $N \geq 1$ so large that

$$M_3^{1/N} \|f^N(u)\|^{1/N} < r_{C\cap K_u}(f) + \eta/2. \tag{24}$$

Since $\lim_{\varepsilon\to 0^+} \|f_\varepsilon^N(u)\| = \|f^N(u)\|$, there exists $\varepsilon(\eta) > 0$ such that for $0 < \varepsilon < \varepsilon(\eta)$,

$$M_3^{1/N} \|f_\varepsilon^N(u)\|^{1/N} < r_{C\cap K_u}(f) + \eta.$$

From (23), we now deduce that

$$\begin{aligned} \|f_\varepsilon^N\|_{C\cap K_u}^{1/N} &= \sup\{\|f_\varepsilon^N(x)\|^{1/N} : x \in C\cap K_u \text{ and } \|x\| \leq 1\} \\ &\leq M_3^{1/N} \|f_\varepsilon^N(u)\|^{1/N} < r_{C\cap K_u}(f) + \eta \end{aligned}$$

for $0 < \varepsilon < \varepsilon(\eta)$. It now follows from (19) that

$$\lim_{k\to\infty} \|f_\varepsilon^k\|_{C\cap K_u}^{1/k} \leq \|f_\varepsilon^N\|_{C\cap K_u}^{1/N} < r_{C\cap K_u}(f) + \eta.$$

As $\eta > 0$ was arbitrary, we conclude that $\lim_{\varepsilon\to 0^+} r_\varepsilon = r_{C\cap K_u}(f)$. □

Remark 4.4. The general idea of using perturbations of f like $f_{\varepsilon,u}$ has been exploited before; see, for example, [3], [36, Lemma 2.1], and [41, Lemmas 3.2, 3.9, and 4.1]. In particular, cf. [41, Sections 3 and], where results similar to Theorem 4.3 are established, and [3, Lemma 7.6], which provides a proof of [41, Lemma 3.2].

We also remark that if $w \in K_u$ and f is a u-bounded homogeneous mapping which is order-preserving with respect to K, then there exists a constant $M_w > 0$ such that $f(x) \leq_K M_w\|x\|w$ for all $x \in C \cap K_u$. Now, for $\varepsilon > 0$, we can consider the mapping $f_{\varepsilon,w}\colon C \cap K_u \to C \cap K_u$ given by $f_{\varepsilon,w}(x) = f(x) + \varepsilon|x|w$ for all $x \in C \cap K_u$. Then $f_{\varepsilon,w}$ has a unique eigenvector $v_{\varepsilon,w} \in C \cap K_u$ with $\|v_{\varepsilon,w}\| = 1$. A slight variant of the proof of Theorem 4.3 shows that the mapping $(\varepsilon, u) \mapsto v_{\varepsilon,u}$ is norm continuous; see [41, Lemma 4.1] for related results.

For a u-bounded, homogeneous, and order-preserving with respect to K mapping $f\colon C \cap K_u \to C \cap K_u$, we define the **partial spectral radius** of f by

$$r_{C\cap K_u}(f) := \lim_{k\to\infty} \|f^k\|_{C\cap K_u}^{1/k}.$$

Note that, as $\|f^{k+m}\|_{C\cap K_u} \leq \|f^k\|_{C\cap K_u}\|f^m\|_{C\cap K_u}$ for all $m, k \geq 1$, it follows from Fekete's sub-additive lemma that $r_{C\cap K_u}(f) = \inf_k \|f^k\|_{C\cap K_u}^{1/k} < \infty$. Using the notation from Theorem 4.3, we obtain the following immediate corollary, which we need later.

Corollary 4.5. *If $f\colon C \cap K_u \to C \cap K_u$ is u-bounded, homogeneous and order-preserving with respect to K, then*

$$\lim_{\varepsilon\to 0^+} r_{C\cap K_u}(f_{\varepsilon,u}) = r_{C\cap K_u}(f).$$

Of particular interest to us is the case where $K = C$ and $K_u = C^\circ$. In that case, the partial spectral radius $r_{C^\circ}(f)$ satisfies a Collatz-Wielandt formula, which generalizes [18, Corollary 37]; see also [3] and [31, Section 5.6].

Theorem 4.6 (Collatz-Wielandt formula I)**.** *Let C be a closed normal cone with nonempty interior in a Banach space X and $f\colon C^\circ \to C^\circ$ order-preserving and homogeneous. Then*

$$r_{C^\circ}(f) = \inf_{y\in C^\circ} M(f(y)/y).$$

Proof. Let $y \in C^\circ$ and recall that, as C is normal, the norms $\|\cdot\|$ and $\|\cdot\|_y$ are equivalent. For each $k \geq 1$ and $0 \leq_C x \leq_C y$, $\|f^k(x)\|_y \leq \|f^k(y)\|_y$, as f is order-preserving. This implies that

$$\|f^k\|_{y,C^\circ} := \sup\{\|f^k(x)\|_y\colon x \in C^\circ \text{ with } \|x\|_y \leq 1\} = \|f^k(y)\|_y.$$

It now follows from Lemma 4.2 that

$$r_{C^\circ}(f) = \lim_{k\to\infty} \|f^k(y)\|_y^{1/k} = \inf_{k\geq 1} \|f^k(y)\|_y^{1/k} \leq M(f(y)/y),$$

as $\|f(y)\|_y = M(f(y)/y)$, so $r_{C^\circ}(f) \leq \inf_{y\in C^\circ} M(f(y)/y)$.

Now let $\varepsilon > 0$, $u \in C^\circ$, and $f_{\varepsilon,u}$ be as in (12). Then $f(x) \leq_C f_{\varepsilon,u}(x)$ for all $x \in C^\circ$, and $r_{C^\circ}(f_{\varepsilon,u}) \to r_{C^\circ}(f)$ as $\varepsilon \to 0^+$, by Corollary 4.5. By Theorem 4.3, there exist $v_{\varepsilon,u} \in C^\circ$ such that $f_{\varepsilon,u}(v_{\varepsilon,u}) = r_{C^\circ}(f_{\varepsilon,u})v_{\varepsilon,u}$. Thus, $M(f_{\varepsilon,u}(v_{\varepsilon,u})/v_{\varepsilon,u}) = r_{C^\circ}(f_{\varepsilon,u})$, so

$$r_{C^\circ}(f) = \lim_{\varepsilon\to 0^+} M(f_{\varepsilon,u}(v_{\varepsilon,u})/v_{\varepsilon,u}) \geq \liminf_{\varepsilon\to 0^+} M(f(v_{\varepsilon,u})/v_{\varepsilon,u}) \geq \inf_{y\in C^\circ} M(f(y)/y). \qquad \square$$

The following two basic observations concerning $r_{C^\circ}(f)$ will be useful to us later. The first one is essentially [32, Lemma 2.2]; we include the short proof for the reader's convenience.

Lemma 4.7. *Let C be a normal closed cone with nonempty interior in a Banach space X. If $f\colon C^\circ \to C^\circ$ is an order-preserving homogeneous mapping, then $r_{C^\circ}(f) > 0$.*

Proof. Pick $u, x \in C^\circ$. As $f(x) \in C^\circ$, there exists $\alpha > 0$ such that $\alpha x \leq_C f(x)$, so $\alpha^k x \leq_C f^k(x)$ for all $k \geq 1$. We show that $\alpha \leq r_{C^\circ}(f)$. Suppose that $\alpha > r_{C^\circ}(f) + \varepsilon$ for some $\varepsilon > 0$. The definition of $r_{C^\circ}(f)$ implies that $\|f^k(x)\| \leq (r_{C^\circ}(f)+\varepsilon)^k$ for all $k \geq 1$ sufficiently large. Since $\alpha > r_{C^\circ}(f)+\varepsilon$, we conclude that $\alpha^{-k} f^k(x) \to 0$ as $k \to \infty$. However, $\alpha^{-k} f^k(x) - x \in C$ for all $k \geq 1$, so $-x \in C$, which is impossible. $\square$

Lemma 4.8. *Let C be normal closed cone with nonempty interior in a Banach space X. If $f\colon C^\circ \to C^\circ$ is an order-preserving homogeneous mapping with $r_{C^\circ}(f) = 1$, then, for each $x \in C^\circ$, the orbit $\mathcal{O}(x; f)$ does not accumulate at* 0.

Proof. Let $u \in C^\circ$ and $v_{\varepsilon,u} \in C^\circ$ be as in Theorem 4.3. As C is a normal cone, the norms $\|\cdot\|$ and $\|\cdot\|_u$ are equivalent, and hence there exists a constant $M > 0$ such that $\|v_{\varepsilon,u}\|_u \leq M$ for all $0 < \varepsilon \leq 1$. If $x \in C^\circ$, then $u \leq_C \beta x$ for some $\beta > 0$. So letting $z := \beta M x$, we obtain $v_{\varepsilon,u} \leq_C z$ for all $0 < \varepsilon \leq 1$.

We show that $\mathcal{O}(z; f)$ does not accumulate at 0. As $f\colon C^\circ \to C^\circ$, for each $n \geq 1$ there exists $\beta_n > 0$ (depending only on $f^n(z) \in C^\circ$) such that $|f^{n-1}(z)|u \leq_C \beta_n f^n(z)$. Thus

$$f_{\varepsilon,u}(f^{n-1}(z)) \leq_C (1 + \varepsilon\beta_n) f^n(z)$$

for all $n \geq 1$. Now fix $k \geq 1$ and note that, as $v_{\varepsilon,u} \leq_C z$,

$$v_{\varepsilon,u} \leq_C f_{\varepsilon,u}^{k-1}(f_{\varepsilon,u}(z)) \leq_C (1+\varepsilon\beta_1) f_{\varepsilon,u}^{k-1}(f(z)) \leq_C \dots \leq_C \prod_{i=1}^{k}(1+\varepsilon\beta_i) f^k(z).$$

It follows that $1 \leq \prod_{i=1}^{k}(1+\varepsilon\beta_i)|f^k(z)|$. So, letting $\varepsilon \to 0$, we see that $1 \leq |f^k(z)|$. Thus $\mathcal{O}(z; f)$ does not accumulate at 0. As f is homogeneous, no orbit inside C° can accumulate at 0. □

Later, in Theorem 6.1, we need to assume that the set $\{v_{\varepsilon,u} \colon 0 < \varepsilon \leq 1\}$ in Theorem 4.3 contains a convergent subsequence in the norm topology. This is always the case in finite dimensional spaces but not in infinite dimensional spaces. For this reason, we introduce the following terminology.

Definition 4.9. Let C be normal closed cone with nonempty interior in a Banach space X. If $f \colon C^\circ \to C^\circ$ is an order-preserving homogeneous mapping and the set $\{v_{\varepsilon,u} \colon 0 < \varepsilon \leq 1\}$ in Theorem 4.3 contains a convergent subsequence in the norm topology, we say that f has **converging approximate eigenvectors**.

In the next subsection, using so-called generalised measures of noncompactness or simply generalised MNC's, we establish several sufficient conditions for a mapping to have converging approximate eigenvectors.

4.2 Generalised measures of non-compactness. Let X be a (real or complex) Banach space, and denote by $\mathcal{B}(X)$ the collection of all bounded, nonempty, subsets of X. Given $S, T \in \mathcal{B}(X)$, we denote by co(S) the convex hull of S, $S + T := \{s + t \colon s \in S \text{ and } t \in T\}$, and $\lambda S := \{\lambda s \colon s \in S\}$ for all λ in the scalar field. Following the terminology from [36], we call a mapping $\beta \colon \mathcal{B}(X) \to [0, \infty)$ a **generalised homogeneous measure of non-compactness (MNC)** if it satisfies the following conditions:

(A1) for all $S \in \mathcal{B}(X)$, $\beta(S) = 0$ if and only if $\overline{S}$ is compact;
(A2) for all $S \in \mathcal{B}(X)$, with $S \subseteq T$, $\beta(S) \leq \beta(T)$;
(A3) for all $S \in \mathcal{B}(X)$ and $x_0 \in X$, $\beta(S \cup \{x_0\}) = \beta(S)$;
(A4) for all $S \in \mathcal{B}(X)$, $\beta(\overline{S}) = \beta(S)$;
(A5) for all $S \in \mathcal{B}(X)$, $\beta(\text{co}(S)) = \beta(S)$;
(A6) for all $S, T \in \mathcal{B}(X)$, $\beta(S + T) \leq \beta(S) + \beta(T)$;
(A7) for all $S \in \mathcal{B}(X)$ and all scalars λ, $\beta(\lambda S) = |\lambda|\beta(S)$.

Property (A7) is called the **homogeneity property** of β. Some treatments of MNC's assume that β satisfies the so-called **set additive property**:

(A8) for all $S, T \in \mathcal{B}(X)$, $\beta(S \cup T) = \max\{\beta(S), \beta(T)\}$.

However, we do not assume (A8).

A fundamental example is the **Kuratowski measure of non-compactness**

$$\alpha(S) := \inf\left\{\delta > 0 \colon S = \bigcup_{i=1}^{n} S_i \text{ with } \operatorname{diam}(S_i) \leq \delta \text{ for all } 1 \leq i \leq n < \infty\right\}$$

for $S \in \mathcal{B}(X)$. The Kuratowski MNC satisfies properties (A1)–(A8). Notice that (A1) and (A8) imply (A2) and (A3), but there are many interesting examples of generalised homogeneous MNC's that do not satisfy (A8).

Using the generalised homogeneous MNC's, we can formulate a condition under which the set $\{v_{\varepsilon,u} \colon 0 < \varepsilon \leq 1\}$ in Theorem 4.3 has a compact norm closure.

Theorem 4.10. *Let $f\colon C \cap K_u \to C \cap K_u$ be a homogeneous mapping which is order-preserving with respect to K and u-bounded. Let $\{v_{\varepsilon,u} \colon 0 < \varepsilon \leq 1\}$ be as in Theorem 4.3. If $r_{C\cap K_u}(f) > 0$ and there exists a generalised homogeneous MNC β such that for each $A \in \mathcal{B}(X)$ with $A \subseteq C \cap K_u$ and $\beta(A) > 0$*

$$\beta(f(A)) < r_{C\cap K_u}(f)\beta(A),$$

then $\{v_{\varepsilon,u} \colon 0 < \varepsilon \leq 1\}$ has a compact closure in the norm topology.

Proof. For simplicity, we write $S := \{v_{\varepsilon,u} \colon 0 < \varepsilon \leq 1\}$ and $r := r_{C\cap K_u}(f) > 0$. It suffices to show that $\beta(S) = 0$, by (A1). Define $g(x) := f(x)/r$ for all $x \in C \cap K_u$. Then $\beta(g(A)) < \beta(A)$ for all $A \in \mathcal{B}(X)$ with $A \subseteq C \cap K_u$ and $\beta(A) > 0$, by (A7).

As $|v_{\varepsilon,u}| = 1$ and $f_{\varepsilon,u}(v_{\varepsilon,u}) = f(v_{\varepsilon,u}) + \varepsilon u = r_\varepsilon v_{\varepsilon,u}$, where $r_\varepsilon := r_{C\cap K_u}(f_{\varepsilon,u})$, we have

$$g(v_{\varepsilon,u}) + \frac{\varepsilon}{r}u + \left(1 - \frac{r_\varepsilon}{r}\right)v_{\varepsilon,u} = v_{\varepsilon,u}.$$

Define $T := \{\frac{\varepsilon}{r}u + (1 - \frac{r_\varepsilon}{r})v_{\varepsilon,u} \colon 0 < \varepsilon \leq 1\}$. Note that Corollary 4.5 implies that $\lim_{\varepsilon\to 0^+} \frac{r_\varepsilon}{r} = 1$, and hence

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon}{r}u + \left(1 - \frac{r_\varepsilon}{r}\right)v_{\varepsilon,u} = 0.$$

Thus, the mapping $\sigma\colon \varepsilon \mapsto \frac{\varepsilon}{r}u + (1 - \frac{r_\varepsilon}{r})v_{\varepsilon,u}$ is a norm continuous mapping on $[0, 1]$, by Theorem 4.3. This implies that $\overline{T}$ is compact, so $\beta(T) = 0$. Since $S \subseteq g(S) + T$, we conclude from (A2) and (A6) that

$$\beta(S) \leq \beta(g(S) + T) = \beta(g(S)) + \beta(T) = \beta(g(S)),$$

so $\beta(S) = 0$. □

Notice that in the proof of Theorem 4.10 we have used only properties (A1), (A2), (A6) and (A7) of β. Another sufficient condition is given in the following result.

Theorem 4.11. *Let* $f: C \cap K_u \to C \cap K_u$ *be a homogeneous mapping which is order-preserving with respect to* K*, u-bounded, and satisfies* $r_{C\cap K_u}(f) = 1$. *Let* $\{v_{\varepsilon,u}: 0 < \varepsilon \le 1\}$ *be as in Theorem 4.3. If there exists a generalised homogeneous MNC* β *such that* $\liminf_{m\to\infty} \beta(f^m(V)) = 0$*, where* $V := \{x \in C \cap K_u : |x| \le 1\}$*, and* f *is uniformly continuous on* V *in the norm topology, then* $\{v_{\varepsilon,u}: 0 < \varepsilon \le 1\}$ *has a compact closure in the norm topology.*

Proof. Note that by Theorem 4.3(ii), it suffices to prove that $\beta(\{v_{\varepsilon,u}: 0 < \varepsilon \le \varepsilon_0 \le 1\}) = 0$, where $\varepsilon_0 > 0$ can be arbitrary small. Now let $\eta > 0$ be given.

We first show that for each $m \ge 1$ and $\sigma > 0$, there exists $\varepsilon_0 := \varepsilon_0(\sigma, m) > 0$ such that

$$|f^m(v_{\varepsilon,u}) - v_{\varepsilon,u}| \le \sigma \quad \text{for all } 0 < \varepsilon \le \varepsilon_0. \tag{25}$$

For $m = 1$, the assertion follows from the fact that

$$|f(v_{\varepsilon,u}) - v_{\varepsilon,u}| \le |f_{\varepsilon,u}(v_{\varepsilon,u}) - \varepsilon u - v_{\varepsilon,u}| \le |r_{C\cap K_u}(f_{\varepsilon,u})v_{\varepsilon,u} - \varepsilon u - v_{\varepsilon,u}| \to 0,$$

as $\varepsilon \to 0^+$, since $r_{C\cap K_u}(f_{\varepsilon,u}) \to r_{C\cap K_u}(f) = 1$, by Corollary 4.5.

Now suppose the assertion holds for all $1 \le j < m$. As f is uniformly continuous on V, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \le \sigma/4 \text{ for all } x, y \in V \text{ with } |x - y| \le \delta.$$

As f is homogenous, $|f(x) - f(y)| \le \sigma/2$ for all $x, y \in C \cap K_u$ with $|x|, |y| \le 2$ and $|x - y| \le 2\delta$.

As $|v_{\varepsilon,u}| = 1$ for all $0 < \varepsilon \le 1$, we can use the induction hypothesis to find $\varepsilon_0 > 0$ such that $|f^{m-1}(v_{\varepsilon,u}) - v_{\varepsilon,u}| \le 2\delta$ and $|f^{m-1}(v_{\varepsilon,u})| \le 2$ for all $0 < \varepsilon \le \varepsilon_0$. Using uniform continuity of f, we deduce that

$$|f^m(v_{\varepsilon,u}) - f(v_{\varepsilon,u})| = |f(f^{m-1}(v_{\varepsilon,u})) - f(v_{\varepsilon,u})| \le \sigma/2$$

for all $0 < \varepsilon \le \varepsilon_0$. Applying the induction hypothesis again, and possibly decreasing $\varepsilon_0 > 0$, we may also assume that $|f(v_{\varepsilon,u}) - v_{\varepsilon,u}| \le \sigma/2$ for all $0 < \varepsilon \le \varepsilon_0$. Combining these inequalities gives

$$|f^m(v_{\varepsilon,u}) - v_{\varepsilon,u}| \le |f(f^{m-1}(v_{\varepsilon,u})) - f(v_{\varepsilon,u})| + |f(v_{\varepsilon,u}) - v_{\varepsilon,u}| \le \sigma/2 + \sigma/2 \le \sigma$$

for all $0 < \varepsilon \le \varepsilon_0$.

As $\liminf_{m\to\infty} \beta(f^m(V)) = 0$, there exists $m_0 \geq 1$ such that $\beta(f^{m_0}(V)) \leq \eta/2$. Define $\Gamma_{\varepsilon_0} := \{f^{m_0}(v_{\varepsilon,u})\colon 0 < \varepsilon \leq \varepsilon_0\}$. Taking $\sigma = \frac{\eta}{2\beta(B_1(0))}$ in (25), we find an $\varepsilon_0 > 0$ such that

$$\{v_{\varepsilon,u}\colon 0 < \varepsilon \leq \varepsilon_0\} \subseteq \Gamma_{\varepsilon_0} + \left\{x \in C \cap K_u\colon |x| \leq \frac{\eta}{2\beta(B_1(0))}\right\},$$

where $B_1(0) := \{x \in X\colon |x| \leq 1\}$. This implies that

$$\beta(\{v_{\varepsilon,u}\colon 0 < \varepsilon \leq \varepsilon_0\}) \leq \beta(\Gamma_{\varepsilon_0}) + \frac{\eta}{2\beta(B_1(0))}\beta(B_1(0)) \leq \eta/2 + \eta/2 = \eta,$$

as $\Gamma_{\varepsilon_0} \subseteq f^{m_0}(V)$. Thus $\beta(\{v_{\varepsilon,u}\colon 0 < \varepsilon \leq 1\}) = 0$. □

Remark 4.12. For a generalised homogeneous MNC β on a Banach space X and a bounded linear map $g\colon X \to X$, one can define $\beta(g)$ as in Theorem 4.10, i.e.,

$$\beta(g) := \inf\{c > 0\colon \beta(g(A)) \leq c\beta(A) \text{ for all bounded subsets } A \text{ of } X\}.$$

However, as follows from [37, Theorem 8], it may happen that $\beta(g^m) = \infty$ for infinitely many positive integers m.

5 Horofunctions of Hilbert's metric

The horofunction boundary, which goes back to Gromov [21], is known to be a useful tool for proving Denjoy-Wolff type theorems for fixed point free nonexpansive mappings on a variety of metric spaces; see [18, 25, 34, 41]. We also exploit horofunctions here. We sfollow Walsh [50], who made detailed study of the horofunction boundary of finite dimensional Hilbert's metric spaces, and use the so-called Funk and reverse Funk (weak) metrics.

Let C be a closed cone with nonempty interior in a Banach space X. For $x, y \in C^\circ$ the **Funk (weak) metric** is given by

$$\text{Funk}_C(x, y) := \log M(x/y). \tag{26}$$

and the **reverse Funk (weak) metric** is given by

$$\text{RFunk}_C(x, y) := \log M(y/x). \tag{27}$$

Hilbert's (projective) metric satisfies

$$\delta_C(x, y) = \text{Funk}_C(x, y) + \text{RFunk}_C(x, y), \tag{28}$$

and Thompson's metric satisfies

$$d_C(x, y) = \max\{\text{Funk}_C(x, y), \text{Funk}_C(y, x)\} \tag{29}$$

for all $x, y \in C^\circ$.

The reader can check that both the Funk metric and reverse Funk metric satisfy the triangle inequality on $C^\circ \times C^\circ$, but are clearly neither symmetric nor non-negative functions. They are named after P. Funk who studied them in [17] in connection with Hilbert's fourth problem; see [43] for more details.

Lemma 5.1. *Let C be a closed cone with nonempty interior in a Banach space X. For each $y \in C^\circ$, the functions $x \mapsto \mathrm{Funk}_C(x, y)$ and $x \mapsto \mathrm{RFunk}_C(x, y)$ are Lipschitz with Lipschitz constant 1 with respect to d_C on C°.*

Proof. For $x_1, x_2 \in C^\circ$,

$$x_1 \leq_C M(x_1/x_2)x_2 \leq_C M(x_1/x_2)M(x_2/y)y,$$

so $M(x_1/y) \leq M(x_1/x_2)M(x_2/y)$. This implies that

$$\mathrm{Funk}_C(x_1, y) \leq \mathrm{Funk}_C(x_1, x_2) + \mathrm{Funk}_C(x_2, y).$$

Interchanging the roles of x_1 and x_2 gives

$$\mathrm{Funk}_C(x_2, y) \leq \mathrm{Funk}_C(x_2, x_1) + \mathrm{Funk}_C(x_1, y),$$

so

$$|\mathrm{Funk}_C(x_1, y) - \mathrm{Funk}_C(x_2, y)| \leq d_C(x_1, x_2).$$

The argument for RFunk_C goes in a similar fashion. □

It follows from Lemma 5.1 and (28) that for each $y \in C^\circ$, the function $x \mapsto \delta_C(x, y)$ is Lipschitz with Lipschitz constant 2 with respect to d_C on C°.

The following lemma lists some basic properties of Funk_C that are immediate from the definition.

Lemma 5.2. *Let C be a closed cone with nonempty interior in a Banach space X. Then*

1. *for $x, y \in C^\circ$, and $\alpha, \beta > 0$,*

$$\mathrm{Funk}_C(\alpha x, \beta y) = \mathrm{Funk}_C(x, y) + \log \alpha - \log \beta;$$

2. *if $x_1, x_2 \in C^\circ$ with $x_1 \leq_C x_2$, and $y \in C^\circ$, then*

$$\mathrm{Funk}_C(x_1, y) \leq_C \mathrm{Funk}_C(x_2, y);$$

3. *if $y_1, y_2 \in C^\circ$ with $y_1 \leq_C y_2$, and $x \in C^\circ$, then*

$$\mathrm{Funk}_C(x, y_2) \leq_C \mathrm{Funk}_C(x, y_1);$$

4. *if* $f: C^\circ \to C^\circ$ *is an order-preserving homogeneous mapping, then*

$$\mathrm{Funk}_C(f(x), f(y)) \le \mathrm{Funk}_C(x, y) \quad \text{for all } x, y \in C^\circ.$$

Following Walsh [50], we now define the horofunction boundaries for the Funk metric, RFunk metric, and δ_C. Fix a base point $b \in C^\circ$ and let ρ be either Funk_C, RFunk_C, or, δ_C. Denote by $\mathcal{C}(C^\circ)$ the space of continuous functions from (C°, d_C) into $\mathbb{R}$, equipped with the topology of compact convergence (also called the topology of uniform convergence on compact sets); see [39, Section 46]. Define $i_\rho: C^\circ \to \mathcal{C}(C^\circ)$ by $i_\rho(y)(x) := \rho(x, y) - \rho(b, y)$. Note that for each $x, x' \in C^\circ$,

$$|i_\rho(y)(x) - i_\rho(y)(x')| = |\rho(x, y) - \rho(x', y)| \le 2d_C(x, x')$$

for all $y \in C^\circ$, by Lemma 5.1, and hence $i_\rho(C^\circ) := \{i_\rho(y): y \in C^\circ\}$ is an equicontinuous family in $\mathcal{C}(C^\circ)$. Furthermore, if ρ is Funk_C or RFunk_C, then for each $x \in C^\circ$, $|i_\rho(y)(x)| \le d_C(x, b)$ for all $y \in C^\circ$, by Lemma 5.1. Also, if $\rho = \delta_C$, then for each $x \in C^\circ$, $|i_\rho(y)(x)| \le 2d_C(x, b)$ for all $y \in C^\circ$. Thus, for each fixed $x \in C^\circ$, the set $\{i_\rho(y)(x): y \in C^\circ\}$ has compact closure in $\mathbb{R}$. It now follows from Ascoli's Theorem [39, Theorem 47.1] that $i_\rho(C^\circ)$ has compact closure in $\mathcal{C}(C^\circ)$ with respect to the topology of compact convergence.

The boundary, $\overline{i_\rho(C^\circ)} \setminus i_\rho(C^\circ)$, is called the **horofunction boundary** and its elements are called **horofunctions**. Note that $i_\rho(\alpha y) = i_\rho(y)$ for all $\alpha > 0$ and $y \in C^\circ$. Thus, letting $S := \{y \in C^\circ: \|y\| = 1\}$, we see that $\mathcal{H}_\rho = \overline{i_\rho(S)} \setminus i_\rho(S)$. For simplicity, we write $i_F := i_\rho$ and $\mathcal{H}_F := \overline{i_F(C^\circ)} \setminus i_F(C^\circ)$ if $\rho = \mathrm{Funk}_C$. Likewise, we write i_R and $\mathcal{H}_R$ for $\rho = \mathrm{RFunk}_C$, and i_H, respectively, and $\mathcal{H}_H$ for $\rho = \delta_C$.

On $\overline{i_\rho(C^\circ)}$, the topology of compact convergence agrees with the topology of pointwise convergence. It also coincides with the compact open topology; see [39, Section 46]. If C is a finite dimensional cone, the metric space (C°, d_C) is σ-compact, viz, the union of countably many compact sets. In that case, the topology of compact convergence on $\mathcal{C}(C^\circ)$ is metrizable, and hence each horofunction h in $\mathcal{H}_\rho$ is the limit of a sequence $\{i_\rho(y_n)\}_n$ where $(y_n)_n$ is in C°. However, if C is infinite dimensional, (C°, d_C) is no longer σ-compact, and the topology of compact convergence is not metrizable. Therefore we work with nets instead of sequences. For each $h \in \mathcal{H}_\rho$, there exists a net $(i_\rho(y_\alpha))_\alpha$ such that $i_\rho(y_\alpha) \to h$, where $y_\alpha \in C^\circ$ for all α. Moreover, every net $(i_\rho(y_\alpha))_\alpha$ in $\overline{i_\rho(C^\circ)}$ has a convergent subnet, as $\overline{i_\rho(C^\circ)}$ is compact.

The next lemma is an infinite dimensional version of [50, Lemma 2.4].

Lemma 5.3. *Let C be a closed cone with nonempty interior in a Banach space X, and let $(i_R(y_\alpha))_\alpha$ be a net converging to $h \in \mathcal{H}_R$. If $(y_\alpha)_\alpha$ has a subnet* *converging to* $y \in C \setminus \{0\}$ *in the norm topology, then* $y \in \partial C$ *and*

$$h(x) = \mathrm{RFunk}_C(x, y) - \mathrm{RFunk}_C(b, y) \tag{30}$$

for all $x \in C^\circ$.

Proof. Let $(y_\beta)_\beta$ be a subnet of $(y_\alpha)_\alpha$ converging to $y \in C \setminus \{0\}$ in the norm topology. By Lemma 2.2, $\mathrm{RFunk}_C(x, y_\beta) \to \mathrm{RFunk}_C(x, y)$ for all $x \in C^\circ$, and hence $i_R(y_\beta)$ converges to $x \mapsto \mathrm{RFunk}_C(x, y) - \mathrm{RFunk}_C(b, y)$. This proves (30). Note also that, as $h \in \mathcal{H}_R$, the point $y \in \partial C$; as otherwise, $h \in i_R(C^\circ)$. □

In general, it appears to be difficult to completely characterize $\mathcal{H}_F$. Instead, we observe that all Funk horofunctions have a kind of sub-gradient, which will prove useful later.

Lemma 5.4. *Let* C *be a closed cone with nonempty interior in a Banach space* X*. If* $h \in \mathcal{H}_F$*, there exists* $\varphi \in C^* \setminus \{0\}$ *such that* $\log \varphi(x) \leq h(x)$ *for all* $x \in C^\circ$.

Proof. Let $(i(y_\alpha))_\alpha$ be a net converging to $h \in \mathcal{H}_F$. For each α there exists $\varphi_\alpha \in \Sigma_b^*$ such that $\mathrm{Funk}_C(b, y_\alpha) = \log \frac{\varphi_\alpha(b)}{\varphi_\alpha(y_\alpha)}$, by Lemma 2.2. So, for each α and each $x \in C^\circ$,

$$\begin{aligned}\mathrm{Funk}_C(x, y_\alpha) - \mathrm{Funk}_C(b, y_\alpha) &\geq \log \frac{\varphi_\alpha(x)}{\varphi_\alpha(y_\alpha)} - \log \frac{\varphi_\alpha(b)}{\varphi_\alpha(y_\alpha)} \\ &= \log \varphi_\alpha(x) - \log \varphi_\alpha(b) = \log \varphi_\alpha(x).\end{aligned}$$

As Σ_b^* is weak* compact, there exists a subnet on which φ_α converges to a point $\varphi \in \Sigma_b^*$ in the weak* topology. Thus, $h(x) \geq \log \varphi(x)$ for all $x \in C^\circ$. □

We also need the following fact.

Proposition 5.5. *Let* $(y_\alpha)_\alpha$ *be a net in* C° *such that* $y_\alpha \to y \in \partial C \setminus \{0\}$*. Then* $i_R(y_\alpha) \to h_R \in \mathcal{C}(C^\circ)$*, where* $h_R(x) = \mathrm{RFunk}_C(x, y) - \mathrm{RFunk}_C(b, y)$ *for all* $x \in C^\circ$ *and* $h_R \in \mathcal{H}_R$*. If* $(y_\beta)_\beta$ *is a subnet of* $(y_\alpha)_\alpha$*, then* $i_F(y_\beta)$ *converges in* $\mathcal{C}(C^\circ)$ *if and only if* $i_H(y_\beta)$ *converges in* $\mathcal{C}(C^\circ)$*. Moreover, if* $i_F(y_\beta)$ *converges to* $h_F \in \mathcal{C}(C^\circ)$ *and* $i_H(y_\beta)$ *converges to* $h_H \in \mathcal{C}(C^\circ)$*, then* $h_F \in \mathcal{H}_F$ *and* $h_H \in \mathcal{H}_H$.

Proof. It follows from Lemma 2.2 that $i_R(y_\alpha)$ converges to $h_R \in \mathcal{C}(C^\circ)$, where $h_R(x) = \mathrm{RFunk}_C(x, y) - \mathrm{RFunk}_C(b, y)$ for all $x \in C^\circ$. To show that $h_R \in \mathcal{H}_R$, we need to prove that there does not exist $v \in C^\circ$ such that

$$h_R(x) = \mathrm{RFunk}_C(x, v) - \mathrm{RFunk}_C(b, v) \tag{31}$$

for all $x \in C^\circ$. We argue by contradiction. Suppose there exists $v \in C^\circ$ satisfying (31). Let $\{\varepsilon_k\}_k$ be a sequence of reals with $0 < \varepsilon_k < 1$ and $\lim_{k\to\infty} \varepsilon_k = 0$. Define $x_k := \varepsilon_k b + (1-\varepsilon_k)y$ for all $k \geq 1$. Because $y \leq_C \left(\frac{1}{1-\varepsilon_k}\right)x_k$, we see that $\log M(y/x_k) \leq -\log(1-\varepsilon_k)$. As $y \neq 0$, $-\infty < \log M(y/b) < \infty$, so

$$\limsup_{k\to\infty} h_R(x_k) = \limsup_{k\to\infty} \mathrm{RFunk}_C(x_k, y) - \mathrm{RFunk}_C(b, y) < \infty. \tag{32}$$

On the other hand,

$$\mathrm{RFunk}_C(x_k, v) = \log M(v/\varepsilon_k b + (1-\varepsilon_k)y) \to \infty$$

as $k \to \infty$, because $v \in C^\circ$ and $\varepsilon_k b+(1-\varepsilon_k)y \to y \in \partial C$. Moreover, $\mathrm{RFunk}_C(b, v)$ is finite, as $b \in C^\circ$. So, if there exists $v \in C^\circ$ satisfying (31), then

$$\lim_{k\to\infty} h_R(x_k) = \infty, \tag{33}$$

which contradicts (32).

Now suppose that $(y_\beta)_\beta$ is a subnet of $(y_\alpha)_\alpha$. Then $i_R(y_\beta)$ still converges in $\mathcal{C}(C^\circ)$; and because $i_R(y_\beta) + i_F(y_\beta) = i_H(y_\beta)$, the convergence of $i_F(y_\beta)$ in $\mathcal{C}(C^\circ)$ is equivalent to the convergence of $i_H(y_\beta)$ in $\mathcal{C}(C^\circ)$. Suppose that $i_F(y_\beta)$ converges to $h_F \in \mathcal{C}(C^\circ)$ and $i_H(y_\beta)$ converges to $h_H \in \mathcal{C}(C^\circ)$. It remains to show that $h_F \in \mathcal{H}_F$ and $h_H \in \mathcal{H}_H$.

To prove that $h_F \in \mathcal{H}_F$, we need to show that there does not exist $v \in C^\circ$ such that

$$h_F(x) = \mathrm{Funk}_C(x, v) - \mathrm{Funk}_C(b, v) \tag{34}$$

for all $x \in C^\circ$. Let x_k be as above. Note that for each β,

$$\begin{aligned} i_F(y_\beta)(x_k) &= \mathrm{Funk}_C(x_k, y_\beta) - \mathrm{Funk}_C(b, y_\beta) \\ &= \log M(\varepsilon_k b + (1-\varepsilon_k)y/y_\beta) - \log M(b/y_\beta) \\ &\leq \log M(\varepsilon_k b + (1-\varepsilon_k)y/\varepsilon_k b + (1-\varepsilon_k)y_\beta) \\ &\qquad + \log M(\varepsilon_k b + (1-\varepsilon_k)y_\beta/y_\beta) - \log M(b/y_\beta). \end{aligned}$$

We know that y_β converges to y, so Lemma 2.2 implies that for each fixed $k \geq 1$,

$$M(\varepsilon_k b + (1-\varepsilon_k)y/\varepsilon_k b + (1-\varepsilon_k)y_\beta) \to 0.$$

Also,

$$M(\varepsilon_k b + (1-\varepsilon_k)y_\beta/y_\beta) \leq \varepsilon_k M(b/y_\beta) + (1-\varepsilon_k)$$

and $M(b/y_\beta) \to \infty$ as $y_\beta \to y \in \partial C$. Thus

$$i_F(y_\beta)(x_k) \leq M\left(\varepsilon_k b + \frac{(1-\varepsilon_k)y}{\varepsilon_k b} + (1-\varepsilon_k)y_\beta\right) + \log\left(\frac{\varepsilon_k M(b/y_\beta) + (1-\varepsilon_k)}{M(b/y_\beta)}\right). \tag{35}$$

The right hand side of (35) converges to $\log(\varepsilon_k)$ as $y_\beta \to y$, and hence $h_F(x_k) \leq \log(\varepsilon_k)$ for $k \geq 1$. Thus

$$\lim_{k\to\infty} h_F(x_k) = -\infty. \tag{36}$$

On the other hand, if there exists a $v \in C^\circ$ satisfying (34), then it follows from Lemma 2.2 that

$$\lim_{k\to\infty} h_F(x_k) = \log M(y/v) - \log M(b/v) > -\infty, \tag{37}$$

which contradicts (36).

If there exists $v \in C^\circ$ such that $h_H = i_R(v) + i_F(v)$, the estimates in (33) and (37) show that $\lim_{k\to\infty} i_R(x_k) = \infty$ and $\lim_{k\to\infty} i_F(x_k) > -\infty$, which implies that

$$\lim_{k\to\infty} h_H(x_k) = \infty. \tag{38}$$

On the other hand, $h_H(x) = h_R(x) + h_F(x)$ for all $x \in C^\circ$. Equations (32) and (36) show that $\limsup_{k\to\infty} h_R(x_k) < \infty$ and $\lim_{k\to\infty} h_F(x_k) = -\infty$, so $\lim_{k\to\infty} h_H(x_k) = -\infty$, which contradicts (38) and shows that $h_H \in \mathcal{H}_H$. □

Note that if ρ is Funk_C, RFunk_C or δ_C, and $y \in C^\circ$, then

$$i_\rho(y)(x) = \rho(x, y) - \rho(b, y) = \rho(x, y/\|y\|) - \rho(b, y/\|y\|)$$

for all $x \in C^\circ$. Thus, any horofunction is the limit of a net $(i_\rho(y_\alpha))_\alpha$ where $\|y_\alpha\| = 1$ for all α. If C is a finite dimensional cone, any sequence $(y_n)_n$ with $\|y_n\| = 1$ for all n has a limit point $y \in C$ with $\|y\| = 1$. In that case, it follows from Lemma 5.3 and Proposition 5.5 that

$$\mathcal{H}_R = \{x \mapsto \mathrm{RFunk}_C(x, y) - \mathrm{RFunk}_C(b, y) \colon y \in \partial C \text{ and } \|y\| = 1\};$$

cf. [50, Proposition 2.5].

5.1 The horofunction boundary of a symmetric cone. If C° is a symmetric cone, there exists a particularly simple description of $\mathcal{H}_F$. Recall that a **symmetric cone** is the interior of the cone of squares in a euclidean Jordan algebra. A detailed exposition of the theory of symmetric cones can be found in [16] by Faraut and Korányi. We follow their notation and terminology. A **euclidean Jordan algebra** $(X, \bullet)$ is a finite dimensional real inner product space $(X, \langle\cdot,\cdot\rangle)$ equipped with a bilinear product $x \bullet y$ such that for each $x, y \in X$

(1) $x \bullet y = y \bullet x$,
(2) $x \bullet (x^2 \bullet y) = x^2 \bullet (x \bullet y)$,

(3) the linear map $L(x)\colon X \to X$ given by $L(x)w := x \bullet w$ satisfies

$$\langle L(x)w, z\rangle = \langle w, L(x)z\rangle \text{ for all } w, z \in X.$$

The collection of squares in $(X, \bullet)$ forms a cone C, and its interior is called a symmetric cone. We denote the unit element in $(X, \bullet)$ by e, which is an element of C°. It is a basic consequence of the Spectral Decomposition Theorem [16, Theorem III.1.2] that $\|x\|_e := \inf\{\lambda > 0\colon -\lambda e \leq_C x \leq_C \lambda e\} = \max\{|\lambda|\colon \lambda \in \sigma(x)\}$. For $x \in X$, the linear mapping $P(x)\colon X \to X$ given by $P(x) := 2L(x)^2 - L(x^2)$ is called the **quadratic representation** of x. Note that $P(x^{-1/2})x = e$ for all $x \in C^\circ$. The mapping $P(x)$ maps the symmetric cone C onto itself if $x \in X$ is invertible; see [16, Proposition III.2.2]; hence it preserves Funk_C, by Lemma 5.2. So, for $x, y \in C^\circ$,

$$M(x/y) = M(P(y^{-1/2})x/e) = \max\{\lambda\colon \lambda \in \sigma(P(y^{-1/2})x)\},$$

where the second equality follows from the Spectral Decomposition Theorem.

Theorem 5.6. *If C° is a symmetric cone in a euclidean Jordan algebra $(X, \bullet)$ and the unit $e \in C^\circ$ is a base point, then*

(i) *$\mathcal{H}_F$ consists of those $f \in \mathcal{C}(C^\circ)$ for which there exists $z \in \partial C$ with $\|z\|_e = 1$ such that $f(x) = \mathrm{RFunk}_C(x^{-1}, z)$ for all $x \in C^\circ$,*

(ii) *$\mathcal{H}_R$ consists of those $g \in \mathcal{C}(C^\circ)$ for which there exists $y \in \partial C$ with $\|y\|_e = 1$ such that $g(x) = \mathrm{RFunk}_C(x, y)$ for all $x \in C^\circ$,*

(iii) *$\mathcal{H}_H$ consists of those $h \in \mathcal{C}(C^\circ)$ for which there exist $y, z \in \partial C$ with $\|y\|_e = \|z\|_e = 1$ and $y \bullet z = 0$ such that*

$$h(x) = \mathrm{RFunk}_C(x^{-1}, z) + \mathrm{RFunk}_C(x, y)$$

for all $x \in C^\circ$.

Proof. Let $g \in \mathcal{H}_R$ and let $\{y_n\}_n$ be a sequence in C°, with $\|y_n\|_e = 1$ for all n, such that $i_R(y_n) \to g$. Taking subsequences, if necessary, we may assume that $y_n \to y \in \partial C \setminus \{0\}$. It follows from Lemma 5.3 that

$$g(x) = \lim_{n\to\infty} i_R(y_n)(x) = \mathrm{RFunk}_C(x, y) - \mathrm{RFunk}_C(e, y)$$

for all $x \in C^\circ$. But $\mathrm{RFunk}_C(e, y) = \log M(y/e) = \log \|y\|_e = 0$, so $g(x) = \mathrm{RFunk}_C(x, y)$ for all $x \in C^\circ$. On the other hand, if $y \in \partial C$ with $\|y\|_e = 1$, there exists a sequence $\{y_n\}_n$ in C° with $\|y_n\|_e = 1$ for all n such that $y_n \to y$. Taking a subsequence, if necessary, we can also ensure that $\{i_R(y_n)\}_n$ converges to an element in $\mathcal{H}_R$. So, by Lemma 5.3,

$$\lim_{n\to\infty} i_R(y_n)(x) = \mathrm{RFunk}_C(x, y) - \mathrm{RFunk}_C(e, y) = \mathrm{RFunk}_C(x, y)$$

for all $x \in C^\circ$, and hence $x \mapsto \mathrm{RFunk}_C(x, y) \in \mathcal{H}_R$. This completes the proof of part (ii).

Let $f \in \mathcal{H}_F$ and $\{y_n\}_n$ be a sequence in C°, with $\|y_n\|_e = 1$ for all n, such that $i_F(y_n) \to f$. Taking subsequences, if necessary, we may assume that $y_n \to y \in \partial C \setminus \{0\}$ and $y_n^{-1}/\|y_n^{-1}\|_e \to z \in C$. Note that as $y \in \partial C \setminus \{0\}$, it follows from the Spectral Decomposition Theorem that $\|y_n^{-1}\|_e \to \infty$. This implies that

$$y \bullet z = \lim_{n\to\infty} y_n \bullet \left(\frac{y_n^{-1}}{\|y_n^{-1}\|_e} \right) = \lim_{n\to\infty} \frac{e}{\|y_n^{-1}\|_e} = 0.$$

It follows from [16, Exercise 3.3] that $\langle y, z\rangle = 0$, and hence $z \in \partial C$, as $\langle v, w\rangle > 0$ for all $w \in C^\circ$ and $v \in C \setminus \{0\}$.

The inverse operation $w \mapsto w^{-1}$ on C° is known to be an order-reversing homogeneous of degree -1 involution; see [23, Proposition 3.2]. This implies that $\mathrm{Funk}_C(u, v) = \mathrm{RFunk}_C(u^{-1}, v^{-1})$ for all $u, v \in C^\circ$. Using Lemma 5.3 again, we see that

$$\begin{aligned} f(x) &= \lim_{n\to\infty} \mathrm{Funk}_C(x, y_n) - \mathrm{Funk}_C(e, y_n) \\ &= \lim_{n\to\infty} \mathrm{RFunk}_C(x^{-1}, y_n^{-1}/\|y_n^{-1}\|_e) - \mathrm{RFunk}_C(e, y_n^{-1}/\|y_n^{-1}\|_e) \\ &= \mathrm{RFunk}_C(x^{-1}, z) - \mathrm{RFunk}_C(e, z) \\ &= \mathrm{RFunk}_C(x^{-1}, z) \end{aligned} \tag{39}$$

for all $x \in C^\circ$, as $\mathrm{RFunk}_C(e, z) = \log \|z\|_e = 0$.

On the other hand, if $y, z \in \partial C$ with $\|y\|_e = \|z\|_e = 1$ and $y \bullet z = 0$, there exists a Jordan frame $\{c_1, \dots, c_k\}$ such that $y = \sum_{i=1}^p \lambda_i c_i$ and $z = \sum_{i=p+1}^q \mu_i c_i$ with $1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p > 0$, $1 = \mu_1 \geq \mu_2 \geq \dots \geq \mu_q > 0$, and $p < q \leq k$. For $n \geq 1$, define

$$y_n := \sum_{i=1}^p \lambda_i c_i + \sum_{i=p+1}^q \frac{1}{n^2 \mu_i} c_i + \sum_{i=q+1}^k \frac{1}{n} c_i \in C^\circ. \tag{40}$$

For sufficiently large n, $\|y_n\|_e = 1$ and

$$y_n^{-1} = \sum_{i=1}^p \frac{1}{\lambda_i} c_i + \sum_{i=p+1}^q n^2 \mu_i c_i + \sum_{i=q+1}^k n c_i \in C^\circ.$$

Note that $\|y_n^{-1}\|_e = n^2 \mu_1 = n^2$ for all large n, so

$$\frac{y_n^{-1}}{\|y_n^{-1}\|_e} = \sum_{i=1}^p \frac{1}{n^2 \lambda_i} c_i + \sum_{i=p+1}^q \mu_i c_i + \sum_{i=q+1}^k \frac{1}{n} c_i \in C^\circ$$

for all large n, which converges to z as $n \to \infty$. Taking a further subsequence, if necessary, we may assume that $\{i_F(y_n)\}_n$ converges to a point in $\mathcal{H}_F$. Using the same equations as in (39), we see that $i_F(y_n)(x) \to \mathrm{RFunk}_C(x^{-1}, z)$ for all $x \in C^\circ$, which completes the proof of part (i).

If $h \in \mathcal{H}_H$, there exists a sequence $\{y_n\}_n$ in C° with $\|y_n\|_e = 1$ for all n such that $i_H(y_n) \to h$. Taking a subsequence, if necessary, we may assume that $y_n \to y \in \partial C$ and $y_n^{-1}/\|y_n^{-1}\|_e \to z \in C$. By the same argument as before, we see that $y \bullet z = 0$ and $z \in \partial C$. Taking a further subsequence, if necessary, we may also assume that $i_F(y_n) \to f \in \mathcal{H}_F$ and $i_R(y_n) \to g \in \mathcal{H}_R$, where $f(x) = \mathrm{RFunk}_C(x^{-1}, z)$ and $g(x) = \mathrm{RFunk}_C(x, y)$ for all $x \in C^\circ$. This shows that $h(x) = \mathrm{RFunk}_C(x^{-1}, z) + \mathrm{RFunk}_C(x, y)$ for all $x \in C^\circ$.

To prove the other inclusion, suppose that $y, z \in \partial C$ with $\|y\|_e = \|z\|_e = 1$ and $y \bullet z = 0$. Then we can define y_n as in (40) for all $n \geq 1$. Taking a subsequence, if necessary, we may assume that $i_F(y_n) \to f \in \mathcal{H}_F$, $i_R(y_n) \to g \in \mathcal{H}_R$, and $i_H(y_n) \to h \in \mathcal{H}_H$. So $h(x) = f(x) + g(x)$ for all $x \in C^\circ$. By the previous arguments, $f(x) = \mathrm{RFunk}_C(x^{-1}, z)$ and $g(x) = \mathrm{RFunk}_C(x, y)$ for all $x \in C^\circ$. □

Remark 5.7. If C° is the symmetric cone of self-adjoint positive definite matrices over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, then for each $x, y \in C^\circ$,

$$M(x/y) = \max \sigma(P(y^{-1/2})x) = \max \sigma(y^{-1/2}xy^{-1/2}) = \max\{\lambda : \lambda \in \sigma(y^{-1}x)\}.$$

So, in that case, the horofunctions are given by

1. $h_F(x) = \log \max \sigma(xz)$,
2. $h_R(x) = \log \max \sigma(x^{-1}y)$,
3. $h_H(x) = \log \max \sigma(xz) + \log \max \sigma(x^{-1}y)$,

where $y, z \in \partial C$ are such that $\|y\|_e = \|z\|_e = 1$ and $y \bullet z = 0$.

We also find a description of the horofunctions of Hilbert's metric on the interior of the standard positive cone $(\mathbb{R}^n_+)^\circ = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } i\}$ alternative to the one given in [27]. Indeed, in that case, Theorem 5.6 gives

1. $h_F(x) = \log \max_i x_i z_i$,
2. $h_R(x) = \log \max_i x_i^{-1} y_i$,
3. $h_H(x) = \log \max_i x_i z_i + \log \max_i x_i^{-1} y_i$,

where $y, z \in \partial \mathbb{R}^n_+$ are such that $\|y\|_\infty = \|z\|_\infty = 1$ and $y_i z_i = 0$ for all i.

6 A Wolff type theorem for cones

Let $f : \Omega \to \Omega$ be a fixed point free nonexpansive mapping on a finite dimensional Hilbert's metric space. Then there exists a horofunction in $h \in \mathcal{H}_H$ such that

$h(f(x)) \leq h(x)$ for all $x \in \Omega$; see [18, Theorem 16] and [25, Theorem 3.4]. The next theorem gives an analogous result for order-preserving homogenous mappings $f: C^\circ \to C^\circ$ that do not have an eigenvector in C°, where the cone can be infinite dimensional.

Theorem 6.1. *Let C be closed normal cone with nonempty interior in a Banach space X. If $f: C^\circ \to C^\circ$ is an order-preserving homogeneous mapping with no eigenvector in C° and f has converging approximate eigenvectors, then there exists a net (v_α) in C° with $v_\alpha \to v \in \partial C$ and $|v| = 1$ such that $i_F(v_\alpha) \to h_F \in \mathcal{H}_F$, $i_R(v_\alpha) \to h_R \in \mathcal{H}_R$, and $i_H(v_\alpha) \to h_H \in \mathcal{H}_H$ with $h_H(x) = h_F(x) + h_R(x)$ for all $x \in C^\circ$ such that*

1. $h_F(f(x)) \leq h_F(x) + \log r_{C^\circ}(f)$,
2. $h_R(f(x)) \leq h_R(x) - \log r_{C^\circ}(f)$,
3. $h_H(f(x)) \leq h_H(x)$.

Moreover, there exists $y \in \partial C \setminus \{0\}$ such that $h_R(x) = \mathrm{RFunk}_C(x, y)$ for all $x \in C^\circ$.

Proof. Let $u \in C^\circ$ be the base point to construct the horofunction boundaries. By Theorem 4.3, for each $\varepsilon > 0$, there exists $v_{\varepsilon,u} \in C^\circ$ such that $|v_{\varepsilon,u}| = 1$ and $f_{\varepsilon,u}(v_{\varepsilon,u}) = r_{\varepsilon,u}v_{\varepsilon,u}$. For simplicity, we write $v_\varepsilon := v_{\varepsilon,u}$, $r_\varepsilon := r_{\varepsilon,u}$, and $f_\varepsilon := f_{\varepsilon,u}$. It also follows from Theorem 4.3 that $r_\varepsilon \to r_{C^\circ}(f)$ as $\varepsilon \to 0$.

Since f has converging approximate eigenvectors, $\{v_\varepsilon : 0 < \varepsilon < 1\}$ contains a convergent subsequence $\{v_{\varepsilon_n}\}_n$ with limit, say, $v \in C$. Note that $|v| = 1$, and $v \in \partial C$; as otherwise, v is an eigenvector of f in C°. By Proposition 5.5, there exists a subnet (v_{ε_α}) such that $i_F(v_{\varepsilon_\alpha}) \to h_F \in \mathcal{H}_F$, $i_R(v_{\varepsilon_\alpha}) \to h_R \in \mathcal{H}_R$, and $i_H(v_{\varepsilon_\alpha}) \to h_H \in \mathcal{H}_H$. By construction, $h_H(x) = h_F(x) + h_R(x)$ for all $x \in C^\circ$. Thus, to prove the third inequality it suffices to show the first two. From Lemma 5.2, we see that for each α and $x \in C^\circ$,

$$\begin{aligned}\mathrm{Funk}_C(f(x), v_{\varepsilon_\alpha}) - \mathrm{Funk}_C(u, v_{\varepsilon_\alpha}) &\leq \mathrm{Funk}_C(f_{\varepsilon_\alpha}(x), v_{\varepsilon_\alpha}) - \mathrm{Funk}_C(u, v_{\varepsilon_\alpha}) \\ &= \mathrm{Funk}_C(f_{\varepsilon_\alpha}(x), f_{\varepsilon_\alpha}(v_{\varepsilon_\alpha})) \\ &\qquad + \log r_{\varepsilon_\alpha} - \mathrm{Funk}_C(u, v_{\varepsilon_\alpha}) \\ &\leq \mathrm{Funk}_C(x, v_{\varepsilon_\alpha}) + \log r_{\varepsilon_\alpha} - \mathrm{Funk}_C(u, v_{\varepsilon_\alpha}).\end{aligned}$$

Thus, $h_F(f(x)) \leq h_F(x) + \log r_{C^\circ}(f)$ for all $x \in C^\circ$.

To prove the second inequality, fix $x \in C^\circ$, Note that since $f(x) \in C^\circ$, there exists a constant $\beta > 0$, depending on x, such that $|x|u \leq_C \beta f(x)$, and hence

$(1+\beta\varepsilon)^{-1}f_\varepsilon(x) \leq_C f(x)$. From Lemma 5.2, we see that for each α,

$$\begin{aligned}\mathrm{RFunk}_C(f(x), v_{\varepsilon_\alpha}) - \mathrm{RFunk}_C(u, v_{\varepsilon_\alpha}) &\leq \mathrm{RFunk}_C((1+\beta\varepsilon_\alpha)^{-1}f_{\varepsilon_\alpha}(x), v_{\varepsilon_\alpha}) \\ &\qquad - \mathrm{RFunk}_C(u, v_{\varepsilon_\alpha}) \\ &= \mathrm{RFunk}_C(f_{\varepsilon_\alpha}(x), f_{\varepsilon_\alpha}(v_{\varepsilon_\alpha})) \\ &\qquad - \mathrm{RFunk}_C(u, v_{\varepsilon_\alpha}) + \log(1+\beta\varepsilon_\alpha) \\ &\qquad - \log r_{\varepsilon_\alpha} \\ &\leq \mathrm{RFunk}_C(x, v_{\varepsilon_\alpha}) - \mathrm{RFunk}_C(u, v_{\varepsilon_\alpha}) \\ &\qquad + \log(1+\beta\varepsilon_\alpha) - \log r_{\varepsilon_\alpha}.\end{aligned}$$

Thus, $h_R(f(x)) \leq h_R(x) - \log r_{C^\circ}(f)$ for all $x \in C^\circ$.

To prove the final assertion, note that $h_R(x) = \mathrm{RFunk}_C(x, v) - \mathrm{RFunk}_C(u, v)$ for all $x \in C^\circ$, by Lemma 5.3. Letting $y := M(v/u)^{-1}v$, we get that $h_R(x) = \mathrm{RFunk}_C(x, y)$. □

The following example shows that equality can hold simultaneously in the three inequalities in Theorem 6.1.

Example 6.2. Consider the linear mapping $f(X) := MXM^*$, where

$$M := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

on the cone $\Pi_2(\mathbb{R})$ consisting of positive semi-definite 2×2 real matrices in the Jordan algebra of 2×2 symmetric matrices. An elementary computation shows that for $k \geq 1$,

$$f^k(X) = \begin{pmatrix} a + 2kb + k^2c & b + kc \\ b + kc & c \end{pmatrix} \text{ for } X = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \Pi_2(\mathbb{R}), \tag{41}$$

and hence $r_{\Pi_2(\mathbb{R})^\circ}(f) = 1$.

Define the mapping g on the set Σ° consisting of invertible trace 1 matrices in $\Pi_2(\mathbb{R})$, by $g(X) := f(X)/\mathrm{tr}(f(X))$. As f is an invertible linear mapping from $\Pi_2(\mathbb{R})$ onto itself, the mapping g is an Hilbert metric isometry on Σ°. In fact, g corresponds to a parabolic isometry of the hyperbolic plane. To see this, let

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

By the above computation, $g^k(X) \to Y$ for all $X \in \Sigma^\circ$, and hence f has no eigenvector in $\Pi_2(\mathbb{R})^\circ$. It follows from Theorem 5.6 that there exist horofuntions $h_F(X) = \mathrm{RFunk}_C(X^{-1}, Z) \in \mathcal{H}_F$, $h_R(X) = \mathrm{RFunk}_C(X, Y) \in \mathcal{H}_R$, and $h_H = h_F + h_R \in \mathcal{H}_H$, where we have taken the identity matrix I as the base point. Note that

$$h_F(X) = \mathrm{RFunk}_C(X^{-1}, Z) = \log \max \sigma(XZ) = \log c$$

and

$$h_R(X) = \mathrm{RFunk}_C(X, Y) = \log \max \sigma(X^{-1}Y) = \log(c/\det(X)).$$

for all $X \in \Pi_2(\mathbb{R})^\circ$. Since $\det(f(X)) = \det(X)$, we deduce from (41) that

$$h_F(f(X)) = \log c = h_F(X) \quad \text{and} \quad h_R(f(X)) = \log(c/\det(X)) = h_R(X)$$

for all $X \in \Pi_2(\mathbb{R})^\circ$. Thus, for each $X \in \Pi_2(\mathbb{R})^\circ$,

$$h_H(f(X)) = \log c + \log(c/\det(X)) = h_H(X).$$

The level sets of h_F and h_R are depicted in Figure 1.

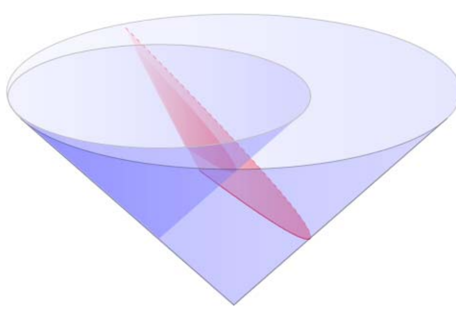

Figure 1. Funk and reverse-Funk horofunction level sets in $\Pi_2(\mathbb{R})^\circ$.

The next corollary generalizes results from [13] and [18] and is an immediate consequence of Lemma 5.4 and Theorem 6.1.

Corollary 6.3. *Let C, X, f, y and h_F be as in Theorem 6.1. Then*
(i) *there exists $\varphi \in C^* \setminus \{0\}$ such that* $\log \varphi(f^k(x)) \leq h_F(x) + k \log r_{C^\circ}(f)$ *for all $x \in C^\circ$ and $k \geq 1$;*
(ii) *for all $x \in C^\circ$ such that $y \leq_C x$, $r_{C^\circ}(f) y \leq_C f(x)$.*

Another consequence of Theorem 6.1 concerns the linear escape rate studied in [18]. Recall that for an order-preserving homogeneous mapping $f : C^\circ \to C^\circ$, the **linear escape rate** is defined by

$$\rho(f) := \lim_{k\to\infty} \frac{\mathrm{RFunk}_C(x, f^k(x))}{k}.$$

Note that

$$\frac{\mathrm{RFunk}_C(x, f^k(x))}{k} = \log M(f^k(x)/x)^{1/k} = \log \|f^k(x)\|_x^{1/k} \to \log r_{C^\circ}(f),$$

as $k \to \infty$, so $\rho(f) = \log r_{C^\circ}(f)$. The following characterization of $\rho(f)$ extends [18, Theorem 1].

Corollary 6.4. *Let C be closed normal cone with nonempty interior in a Banach space X. If $f : C^\circ \to C^\circ$ is an order-preserving homogeneous mapping and f has converging approximate eigenvectors, then*

$$\rho(f) = \max_{h\in\mathcal{A}_R} \inf_{x\in C^\circ} h(x) - h(f(x)), \tag{42}$$

where $\mathcal{A}_R$ consists of those $h \in \overline{i_R(C^\circ)}$ for which there exists a net (y_α) in C°, with $y_\alpha \to y \in C$ and $\|y\|_b = 1$, such that $i_R(y_\alpha) \to h$ in $\mathcal{C}(C^\circ)$. If f has no eigenvector in C°, the maximum is attained at some $h \in \mathcal{A}_R \cap \mathcal{H}_R$.

Proof. Denote by $b \in C^\circ$ the base point for the horofunctions. By Proposition 5.5, for each element $h \in \mathcal{A}_R$ there exists $y \in C$ with $\|y\|_b = 1$ such that

$$h(x) = \mathrm{RFunk}_C(x, y) - \mathrm{RFunk}_C(b, y) = \mathrm{RFunk}_C(x, y) \quad \text{for all } x \in C^\circ.$$

So

$$h(x) - h(f(x)) = \mathrm{RFunk}_C(x, y) - \mathrm{RFunk}_C(f(x), y) \le \mathrm{RFunk}_C(x, f(x))$$

for all $x \in C^\circ$, and hence

$$\sup_{h\in\mathcal{A}_R} \inf_{x\in C^\circ} h(x) - h(f(x)) \le \inf_{x\in C^\circ} \mathrm{RFunk}_C(x, f(x)) = \log r_{C^\circ}(f),$$

by Theorem 4.6.

If f has an eigenvector $v \in C^\circ$, then $f(v) = r_{C^\circ}(f)v$ and $h(v) - h(f(v)) = \log r_{C^\circ}(f)$ for all $h \in \overline{i_R(C^\circ)}$. Since $\rho(f) = \log r_{C^\circ}(f)$, the identity holds if f has an eigenvector in C°.

If f has no eigenvector in C°, then by Theorem 6.1, there exists $h_R \in \mathcal{A}_R \cap \mathcal{H}_R$ such that $\log r_{C^\circ}(f) \le h_R(x) - h_R(f(x))$ for all $x \in C^\circ$. Thus,

$$\rho(f) = \max_{h\in\mathcal{A}_R} \inf_{x\in C^\circ} h(x) - h(f(x)).$$

□

Note that if in Corollary 6.4 the cone C is finite dimensional, then $\mathcal{A}_R = \overline{i_R(C^\circ)}$.

Having established (42), we can now use identical arguments as those used by Gaubert and Vigeral in [18, Lemma 36 and Corollary 37] to obtain a second Collatz-Wielandt formula for $r_{C^\circ}(f)$, which generalises that given in [18, Corollary 37]. The details are left to the reader. To formulate it, we need to recall the following concept. The **radial extension** of an order-preserving homogenous mapping $f\colon C^\circ \to C^\circ$ on the interior of a closed cone in a finite dimensional vector space X is given by

$$\hat{f}(x) := \lim_{\varepsilon \to 0^+} f(x + \varepsilon u) \quad \text{for all } x \in C,$$

where $u \in C^\circ$ is fixed. (It is easy to verify that $\hat{f}$ is an order-preserving homogeneous mapping, and the limit exists and is independent of $u \in C^\circ$, as f is order-preserving and C is finite dimensional.)

Theorem 6.5 (Collatz-Wielandt formula II)**.** *Let C be closed cone with non-empty interior in a finite dimensional vector space X, and $f\colon C^\circ \to C^\circ$ be an order-preserving homogeneous mapping. Then*

$$r_{C^\circ}(f) = \max_{y \in C \setminus \{0\}} m(\hat{f}(y)/y),$$

where $m(\hat{f}(y)/y) := \sup\{\alpha \geq 0\colon \alpha y \leq_C \hat{f}(y)\}$ *for* $y \in C \setminus \{0\}$.

Theorem 6.5 should be compared with [31, Corollary 5.4.2], which implies that if $f\colon C \to C$ is a continuous, order-preserving, homogeneous mapping on a closed cone in a finite dimensional vector space X, then

$$r_C(f) = \max\{\alpha \geq 0\colon f(y) = \alpha y \text{ for some } y \in C \setminus \{0\}\}.$$

The main difference is that in our case, the mapping is defined only on C° and need not have a continuous extension to the boundary; see [11].

7 Denjoy-Wolff theorems for Hilbert's metric

In this section, we prove Denjoy-Wolff type theorems for Hilbert's metric nonexpansive mappings on possibly infinite dimensional domains. We consider mappings $g\colon \Sigma^\circ \to \Sigma^\circ$ of the form

$$g(x) = \frac{f(x)}{q(f(x))} \quad \text{for } x \in \Sigma^\circ := \{x \in C^\circ\colon q(x) = 1\}, \tag{43}$$

where $f\colon C^\circ \to C^\circ$ is an order-preserving homogeneous mapping on the interior of a normal closed C in a Banach space X with $r_{C^\circ}(f) = 1$ and $q\colon C^\circ \to (0, \infty)$

is a continuous homogenous mapping. Typical examples of q are the norm (or an equivalent norm to the norm) of X and strictly positive functionals $q \in C^\circ$. Mappings g of this form are nonexpansive under Hilbert's metric; see, e.g., [31, Section 2.1], Note also that by Lemma 4.7, we can always renormalize f so that $r_{C^\circ}(f) = 1$ without changing g.

Theorem 7.1. *Let C be normal closed cone with nonempty interior in a Banach space X, and let $f\colon C^\circ \to C^\circ$ be a fixed point free order-preserving homogeneous mapping, with $r_{C^\circ}(f) = 1$, satisfying the fixed point property on C° with respect to d_C. Suppose that the mapping $g\colon \Sigma^\circ \to \Sigma^\circ$ is given by* (43). *If there exists $x_0 \in C^\circ$ such that $\mathcal{O}(x_0; f)$ and $\mathcal{O}(x_0/q(x_0); g)$ have compact closures in the norm topology, then there exists a convex set $\mathbf{\Omega} \subseteq \partial C$ such that $\omega(z; g) \subseteq \mathbf{\Omega}$ for all $z \in \Sigma^\circ$.*

Proof. Denote by $\mathbf{\Omega}_0$ the convex hull of $\omega(x_0; f)$. The mapping $f\colon C^\circ \to C^\circ$ is nonexpansive under Thompson's metric, as it is order-preserving and homogeneous; see e.g. [31, Section 2.1]. So we obtain from Corollary 3.5 that $\mathbf{\Omega}_0 \subseteq \partial C$. Using the Hahn-Banach separation theorem, we find $\varphi \in X^*$ such that $\mathbf{\Omega}_0 \subseteq \ker(\varphi)$ and $\varphi(z) > 0$ for all $z \in C^\circ$. Now let $y_0 := x_0/q(x_0)$ and $\eta \in \omega(y_0; g)$. Then there exists a subsequence $\{g^{k_i}(y_0)\}_i$ which converges to η. As $\mathcal{O}(x_0; f)$ has compact closure in the norm topology, we may assume, after taking a further subsequence, if necessary, that $f^{k_i}(x_0)$ converges to say, ξ. It follows from Lemma 4.8 that $\xi \neq 0$, and hence $q(\xi) > 0$. So

$$\varphi(\eta) = \lim_{i\to\infty} \varphi\left(\frac{f^{k_i}(x_0)}{q(f^{k_i}(x_0))}\right) = \lim_{i\to\infty} \frac{\varphi(f^{k_i}(x_0))}{q(f^{k_i}(x_0))} = \frac{\varphi(\xi)}{q(\xi)} = 0,$$

which shows that $\omega(y_0; g) \subseteq \ker(\varphi) \cap C$. As $\mathcal{O}(x_0/q(x_0); g)$ has a compact closure in the norm topology, we can apply [41, Theorem 5.3] to conclude that

$$\bigcup_{z\in\Sigma^\circ} \omega(z; g) \subset \partial C.$$

□

Remark 7.2. It is interesting to note that the assumption that $f\colon C^\circ \to C^\circ$ is a continuous order-preserving mapping such that for each $x \in C^\circ$ the orbit $\mathcal{O}(x; f)$ has a compact closure in the norm topology and all accumulation points of $\mathcal{O}(x; f)$ lie inside ∂C is sufficient for one to prove that $\omega(x; f)$ is contained in a convex subset of ∂C for each $x \in C^\circ$. The argument goes as follows.

Let $x \in C^\circ$, and note that $\omega(x; f)$ is a closed subset of X. As $\omega(x; f)$ is contained in the closure of $\mathcal{O}(x; f)$, which is compact, $\omega(x; f)$ is compact. Hence there exists

$y \in C^\circ$ with $z \leq_C y$ for all $z \in \mathrm{cl}(\mathcal{O}(x; f))$. Indeed, there exists $R > 0$ such that $\omega(x; f) \subseteq B_R(0) := \{u \in X : \|u\| \leq R\}$. Now let $y_0 \in C^\circ$. Then there exists $\delta > 0$ such that $B_\delta(y_0) := \{u \in X : \|y_0 - u\| \leq \delta\} \subseteq C$. Letting $y = \frac{R}{\delta} y_0$, we see that for each $z \in X$, with $\|z\| \leq R$,

$$y - z = \frac{R}{\delta}(y_0 - \frac{\delta}{R} z) =: \frac{R}{\delta}(y_0 - z_0) \in C,$$

as $z_0 = \frac{\delta}{R} z \in X$ with $\|z_0\| \leq \delta$.

By assumption, $\omega(y; f)$ is a nonempty compact subset of ∂C. Let $w \in \omega(y; f)$. As $w \in \partial C$, there exists $\varphi \in C^* \setminus \{0\}$ such that $\varphi(w) = 0$.

We now show that $z \leq_C w$ for all $z \in \omega(x; f)$. If $\{m_i\}_i$ is a sequence such that $f^{m_i}(y) \to w$, and $\{k_j\}_j$ is a sequence such that $f^{k_j}(x) \to z$, then $f^{k_j}(x) \leq_C f^{m_i}(y)$ for all $k_j \geq m_i$, as $f^k(x) \leq_C y$ for all $k \geq 0$. Taking the limit for $j \to \infty$, we get $z \leq_C f^{m_i}(y)$ for all m_i. Now letting $i \to \infty$, we find that $z \leq_C w$. As $\varphi(w) = 0$, we conclude that $\varphi(z) = 0$ and hence $\omega(x; f) \subseteq \ker(\varphi) \cap \partial C$.

Theorem 7.3. *Let C be normal closed cone with nonempty interior in a Banach space X and let $f : C^\circ \to C^\circ$ be a fixed point free order-preserving homogeneous mapping, with $r_{C^\circ}(f) = 1$, having converging approximate eigenvectors. Let $g : \Sigma^\circ \to \Sigma^\circ$ be given by (43), where $\Sigma^\circ = \{x \in C^\circ : q(x) = 1\}$ is bounded in the norm topology. If there exists $x_0 \in C^\circ$ such that $\lim_{k\to\infty} \|f^k(x_0)\| = \infty$ and the orbit $\mathcal{O}(x_0/q(x_0); g)$ has compact closure in the norm topology of X, then there exists a convex set $\Omega \subseteq \partial C$ such that $\omega(z; g) \subseteq \Omega$ for all $z \in \Sigma^\circ$.*

Proof. As $r_{C^\circ}(f) = 1$, Corollary 6.3 assures the existence of $\psi \in C^* \setminus \{0\}$ and $h_F \in \mathcal{H}_F$ such that

$$\log \psi(f^k(x_0)) \leq h_F(x_0) \tag{44}$$

for all $k \geq 1$. As Σ° is bounded in the norm topology, there exists $\delta > 0$ such that $q(x) \geq \delta$ for all $x \in C^\circ$ with $\|x\| = 1$. Indeed, if there exists a sequence $\{u_k\}_k$ in C° such that $\|u_k\| = 1$ and $q(u_k) \leq 1/k$ for all k, then $u_k/q(u_k) \in \Sigma^\circ$, but $\|u_k/q(u_k)\| = 1/q(u_k) \to \infty$ as $k \to \infty$, which contradicts the fact that Σ° is bounded. Combining this with the assumption, $\|f^k(x_0)\| \to \infty$ as $k \to \infty$, we find that

$$q(f^k(x_0)) = \|f^k(x_0)\| q\left(\frac{f^k(x_0)}{\|f^k(x_0)\|}\right) \geq \delta \|f^k(x_0)\| \to \infty \quad \text{as } k \to \infty.$$

So,letting $y_0 := x_0/q(x_0)$, we see from (44) that

$$\psi(g^k(y_0)) = \frac{\psi(f^k(x_0))}{q(f^k(x_0))} \to 0 \quad \text{as } k \to \infty.$$

Thus $\omega(y_0; g) \subseteq \ker(\psi) \cap C$. It now follows from [41, Theorem 5.3] that there exists $\Omega \subseteq \partial C$ convex such that $\omega(z; g) \subseteq \Omega$ for all $z \in \Sigma^\circ$. $\square$

Theorems 7.1 and 7.3 confirm a conjecture by Karlsson and Nussbaum; see [26, 41], for an interesting class of Hilbert's metric nonexpansive mappings. The main point is that the arguments do not rely on the geometry of the domain. They also imply Theorem 1.2, as an order-preserving homogeneous mapping $f: C^\circ \to C^\circ$ always satisfies the fixed point property on C° with respect to d_C, and each orbit of $g: \Sigma^\circ \to \Sigma^\circ$ has a compact closure in the norm topology if the cone is finite dimensional. However, we do not know whether there exists an order-preserving homogenous mapping $f: C^\circ \to C^\circ$, where C is a finite dimensional closed cone, with a point $x \in C^\circ$ such that $\mathcal{O}(x; f)$ has an accumulation point in ∂C and $\mathcal{O}(x; f)$ is unbounded in the norm topology. We conjecture that such a mapping cannot exist, but at present can only prove this for polyhedral cones.

Theorem 7.4. *Let $f: C^\circ \to C^\circ$ be an order-preserving homogeneous mapping on the interior of a polyhedral cone. Then there does not exist a point $x \in C^\circ$ such that $\mathcal{O}(x; f)$ has an accumulation point in ∂C and $\mathcal{O}(x; f)$ is unbounded in the norm topology.*

Theorem 7.4 is a simple consequence of the following proposition.

Proposition 7.5. *Let $f: C^\circ \to C^\circ$ be an order-preserving homogenous mapping on the interior of a polyhedral cone C in a finite dimensional vector space V with $r_{C^\circ}(f) = 1$, and let $x \in C^\circ$ be such that $\mathcal{O}(x; f)$ is unbounded in the norm topology. Then there exists $h_R \in \mathcal{H}_R$ such that*

$$\lim_{k \to \infty} h_R(f^k(x)) = -\infty.$$

Proof. For simplicity we write $x_k := f^k(x)$ and $z_k := x_k / \|x_k\|$ for $k \geq 0$. As $\mathcal{O}(x; f)$ is unbounded in the norm topology, there exists a subsequence $\{x_{k_j}\}_j$ of $\{x_k\}_k$ such that $\|x_m\| < \|x_{k_j}\|$ for all $m < k_j$. Note that we can take a further subsequence such that as $j \to \infty$, $i_R(x_{k_j})$ converges to, say, $h_R \in \mathcal{H}_R$ and $z_{k_j} \to z \in C \setminus \{0\}$. We claim that $z \in \partial C$. Indeed, suppose, for the sake of contradiction, that $z \in C^\circ$. The mapping $g: y \mapsto \frac{f(y)}{\|f(y)\|}$ on $\Sigma^\circ := \{y \in C^\circ : \|y\| = 1\}$ is nonexpansive on (Σ°, δ_C). Moreover,

$$g^{k_j}(z_0) = \frac{f^{k_j}(x_0)}{\|f^{k_j}(x_0)\|} = z_{k_j} \to z \in C^\circ \quad \text{as } j \to \infty.$$

Thus, $\omega(z_0; g) \cap \Sigma^\circ$ is nonempty. It now follows from [31, Corollary 3.2.5] that g has a fixed point, say, $u \in \Sigma^\circ$. The equality $u = g(u) = \frac{f(u)}{\|f(u)\|}$ implies that

$\|f(u)\| = r_{C^\circ}(f) = 1$. Thus f has a fixed point in C°. As f is nonexpansive under d_C on C°, all orbits of f are bounded under d_C and hence also bounded in the norm topology, as the topologies coincide. This contradicts our assumption; so, $z \in \partial C$.

Let E be the extreme points of $\Sigma^*_{z_0} := \{\varphi \in C^*\colon \varphi(z_0) = 1\}$. Note that E is a finite set, as C is polyhedral. Let $E_0 := \{\varphi \in E\colon \varphi(z) = 0\}$ and $E_+ := E \setminus E_0$, which are both nonempty sets.

Observe that for $m \geq 0$ fixed and $\varphi \in E_0$,

$$\log \frac{\varphi(z_{k_j-m})}{\varphi(z_{k_j})} \leq d_H(g^{k_j}(z_0), g^{k_j-m}(z_0)) \leq d_H(g^m(z_0), g(z_0)) < \infty.$$

Thus, for $\varphi \in E_0$, $\varphi(z_{k_j-m}) \to 0$ as $j \to \infty$. As $z_{k_j} \to z$ and $\varphi(z_{k_j}) > 0$ for all j and $\varphi \in E_+$, there exists a constant $\gamma > 0$ such that $\varphi(z_{k_j}) > \gamma$ for all j and $\varphi \in E_+$. Combining these two observations gives

$$\limsup_{j\to\infty} \mathrm{RFunk}_C(z_{k_i}, z_{k_j-m}) = \limsup_{j\to\infty} \log\left(\frac{\varphi_j(z_{k_j-m})}{\varphi_j(z_{k_i})}\right) \leq -\log\gamma$$

for some $\varphi_j \in E_+$, as $\varphi_j(z_{k_j-m}) \leq 1$ and $\varphi(z_{k_i}) > \gamma$. Similarly, for all j sufficiently large,

$$\mathrm{RFunk}_C(z_0, z_{k_j}) = \log\left(\frac{\varphi_j(z_{k_j})}{\varphi_j(z_0)}\right) \geq \log\gamma,$$

where $\varphi_j \in E_+$.

Now fix integers $m, i > 0$ and consider

$$h_R(x_{k_i+m}) = \lim_{j\to\infty} \mathrm{RFunk}_C(x_{k_i+m}, x_{k_j}) - \mathrm{RFunk}_C(x_0, x_{k_j}).$$

As f is order-preserving and homogeneous, it is nonexpansive with respect to RFunk_C; see Lemma 5.2(4). Therefore,

$$\begin{aligned} h_R(x_{k_i+m}) &\leq \limsup_{j\to\infty} \mathrm{RFunk}_C(x_{k_i}, x_{k_j-m}) - \mathrm{RFunk}_C(x_0, x_{k_j}) \\ &\leq \limsup_{j\to\infty} \mathrm{RFunk}_C(z_{k_i}, z_{k_j-m}) - \mathrm{RFunk}_C(z_0, z_{k_j}) + \log\left(\frac{\|x_{k_j-m}\|\,\|x_0\|}{\|x_{k_i}\|\,\|x_{k_j}\|}\right) \\ &\leq \limsup_{j\to\infty} \mathrm{RFunk}_C(z_{k_i}, z_{k_j-m}) - \mathrm{RFunk}_C(z_0, z_{k_j}) + \log\left(\frac{\|x_0\|}{\|x_{k_i}\|}\right) \\ &\leq -2\log\gamma + \log\|x_0\| - \log\|x_{k_i}\|. \end{aligned}$$

As $\|x_{k_i}\| \to \infty$, we see that $\lim_{k\to\infty} h_R(x_k) = -\infty$. ☐

Note that Example 6.2 shows that Proposition 7.5 does not hold for general cones.

Let us now prove Theorem 7.4.

Proof of Theorem 7.4. We argue by contradiction. Suppose that $(f^{m_i}(x))_i$ is a norm bounded subsequence and $\mathcal{O}(x, f)$ is unbounded in the norm topology. Then there exists $\beta > 0$ such that $f^{m_i}(x) \leq \beta x$ for all i. Before we can apply Proposition 7.5, we need to show that $r_{C^\circ}(f) = 1$. Note that $\mathcal{O}(x; f)$ has a convergent subsequence $\{f^{s_j}(x)\}_j$ with limit, say, $\eta \in C$. By Lemma 4.8, $\eta \neq 0$, so $r_{C^\circ}(f) = \lim_{j\to\infty} \|f^{s_j}(x)\|^{1/s_j} = 1$.

By Proposition 7.5 there also exists a subsequence $\{f^{k_j}(x)\}_j$ of $\mathcal{O}(x; f)$ with $\|f^{k_j}(x)\| \to \infty$ such that $i_R(f^{k_j}(x)) \to h_R \in \mathcal{H}_R$, where $h_R(f^m(x)) \to -\infty$ as $m \to \infty$. Note, however, that

$$\begin{aligned} i_R(f^{k_j}(x))(f^{m_i}(x)) &= \mathrm{RFunk}_C(f^{m_i}(x), f^{k_j}(x)) - \mathrm{RFunk}_C(x, f^{k_j}(x)) \\ &\geq \mathrm{RFunk}_C(\beta x, f^{k_j}(x)) - \mathrm{RFunk}_C(x, f^{k_j}(x)) \\ &= -\log \beta \end{aligned}$$

for all i and j, which is absurd. □

Remark 7.6. There exists an alternative proof of Theorem 7.4 which does not rely on horofunctions. We sketch the argument.

As C is a polyhedral cone, the order-preserving homogeneous mapping has a continuous order-preserving homogeneous extension to the whole of C; see [11]. Moreover, it follows from [31, Theorem 5.3.1 and Proposition 5.3.6] that $1 = r_{C^\circ}(f) = \hat{r}_C(f)$, where $\hat{r}_C(f)$ is the Bonsall spectral radius, which is given by $\hat{r}_C(f) := \lim_{k\to\infty} \|f^k\|_C^{1/k}$. Now suppose that $x \in C^\circ$ and $\mathcal{O}(x; f)$ is unbounded in the norm topology. Then there exists a subsequence $\{k_i\}_i$ such that $\lim_{i\to\infty} \|f^{k_i}(x)\| = \infty$ and $\|f^j(x)\| < \|f^{k_i}(x)\|$ for all $j < k_i$ and $i \geq 1$. Assume that we have selected a subsequence of $\{k_i\}_i$, which we also label by k_i, such that

$$\lim_{i\to\infty} \frac{f^{k_i-\nu}(x)}{\|f^{k_i-\nu}(x)\|} =: \eta_\nu \in C$$

for all $\nu = 0, \ldots, m$. Leave the subsequence unchanged for $i \leq m$, and for $i \geq m + 1$, modify the subsequence so that

$$\lim_{i\to\infty} \frac{f^{k_i-(m+1)}(x)}{\|f^{k_i-(m+1)}(x)\|} = \eta_{m+1}$$

for some $\eta_{m+1} \in C$. Continuing in this way, we obtain a subsequence $\{k_i\}_i$ such that

$$\lim_{i\to\infty} \frac{f^{k_i-\nu}(x)}{\|f^{k_i-\nu}(x)\|} =: \eta_\nu$$

for all $\nu \geq 0$. Now note that, as $f^{k_m}(x)$ and $f^{k_m-m}(x)$ in C°, there exist $0 < a_m \leq b_m$ such that

$$a_m f^{k_m-m}(x) \leq_C f^{k_m}(x) \leq_C b_m f^{k_m-m}(x),$$

and hence $a_m f^{k_j-m}(x) \leq_C f^{k_j}(x) \leq_C b_m f^{k_j-m}(x)$ for all $j \geq m$. This gives

$$a_m \frac{f^{k_j-m}(x)}{\|f^{k_j}(x)\|} \leq_C \frac{f^{k_j}(x)}{\|f^{k_j}(x)\|} \leq_C b_m \frac{f^{k_j-m}(x)}{\|f^{k_j}(x)\|}.$$

As

$$\frac{\|f^{k_j-m}(x)\|}{\|f^{k_j}(x)\|} \leq 1 \quad \text{and} \quad \frac{\|f^{k_j-m}(x)\|}{\|f^{k_j}(x)\|} \geq \frac{1}{\|f^m\|_C},$$

we have

$$\frac{a_m}{\|f^m\|_C} \frac{f^{k_j-m}(x)}{\|f^{k_j-m}(x)\|} \leq_C \frac{f^{k_j}(x)}{\|f^{k_j}(x)\|} \leq_C b_m \frac{f^{k_j-m}(x)}{\|f^{k_j-m}(x)\|}.$$

Letting $j \to \infty$ gives

$$\frac{a_m}{\|f^m\|_C} \eta_m \leq_C \eta_0 \leq_C b_m \eta_m$$

for all $m \geq 1$. Thus, $\eta_m \sim_C \eta_0$ for all $m \geq 1$, and hence $\eta_m \sim_C \eta_n$ for all $m, n \geq 1$. In a similar way, it can be shown that $f(\eta_1) \sim_C \eta_0$. As $\eta_1 \sim_C \eta_0$, it follows that for each $x \sim_C \eta_0$, $f(x) \sim_C f(\eta_0) \sim_C \eta_0$, and hence $f(C_0) \subseteq C_0$, where $C_0 := \{x \in C : x \sim_C \eta_0\}$ is the part of η_0. By continuity of $f : C \to C$, we find that $f(\overline{C}_0) \subseteq \overline{C}_0$.

It is known that C_0 is the relative interior of the closed cone $\overline{C}_0$; see [31, Lemma 1.2.2]. We claim that $\hat{r}_{\overline{C}_0}(f_{|\overline{C}_0}) = 1$. Obviously $\hat{r}_{\overline{C}_0}(f_{|\overline{C}_0}) \leq 1$, as $\hat{r}_C(f) = 1$. Note that for all $m \geq 1$, we have $\|\eta_m\| = 1$, $\eta_m \in C_0$, and

$$\|f^m(\eta_m)\| = \lim_{i\to\infty} \frac{\|f^m(f^{k_i-m}(x))\|}{\|f^{k_i-m}(x)\|} = \lim_{i\to\infty} \frac{\|f^{k_i}(x)\|}{\|f^{k_i-m}(x)\|} \geq 1,$$

so $\hat{r}_{\overline{C}_0}(f_{|\overline{C}_0}) = \lim_{k\to\infty} \sup\{\|f^k(x)\|^{1/k} : x \in \overline{C}_0 \text{ with } \|x\| \leq 1\} \geq 1$.

It follows from [31, Corollary 5.4.2] that there exists $v \in \overline{C}_0$ such that $f(v) = v$ and $\|v\| = 1$. As $\eta_0 \in C_0$, there exists $\beta > 0$ such that $v \leq_C \beta\eta_0$. As $f^{k_i}(x)/\|f^{k_i}(x)\| \to \eta_0$ and C is polyhedral, we know; see [31, Lemma 5.1.4], that for each $0 < \lambda < 1$ there exists $i_0 \geq 1$ such that $\lambda\eta_0 \leq_C f^{k_i}(x)/\|f^{k_i}(x)\|$ for all $i \geq i_0$. So, if we fix $0 < \lambda < 1$, and let $b := \beta^{-1}$, we get

$$b\lambda\|f^{k_i}(x)\|v = b\lambda\|f^{k_i}(x)\|f^m(v) \leq_C f^{k_i+m}(x)$$

for all $i \geq i_0$ It follows that $\liminf_{m\to\infty} \|f^{k_i+m}(x)\| \geq b\lambda\kappa^{-1}\|f^{k_i}(x)\|$, where $\kappa > 0$ is the normality constant of C, so $\liminf_{n\to\infty} \|f^n(x)\| = \infty$. Thus $\mathcal{O}(x; f)$ cannot have any accumulation points in C.

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Bas Lemmens
University of Kent
Canterbury, Kent CT2 7NX, UK
email: B.Lemmens@kent.ac.uk

Brian Lins
Hampden-Sydney College
Hampden-Sydney, VA 23943, USA
email: blins@hsc.edu

Roger Nussbaum
Rutgers, the State University of New Jersey
Piscataway, NJ 08854-8019, USA
email: nussbaum@math.rutgers.edu

Marten Wortel
North-West University
Private Bag X6001-209, Potchefstroom 2520, South Africa
email: marten.wortel@gmail.com

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