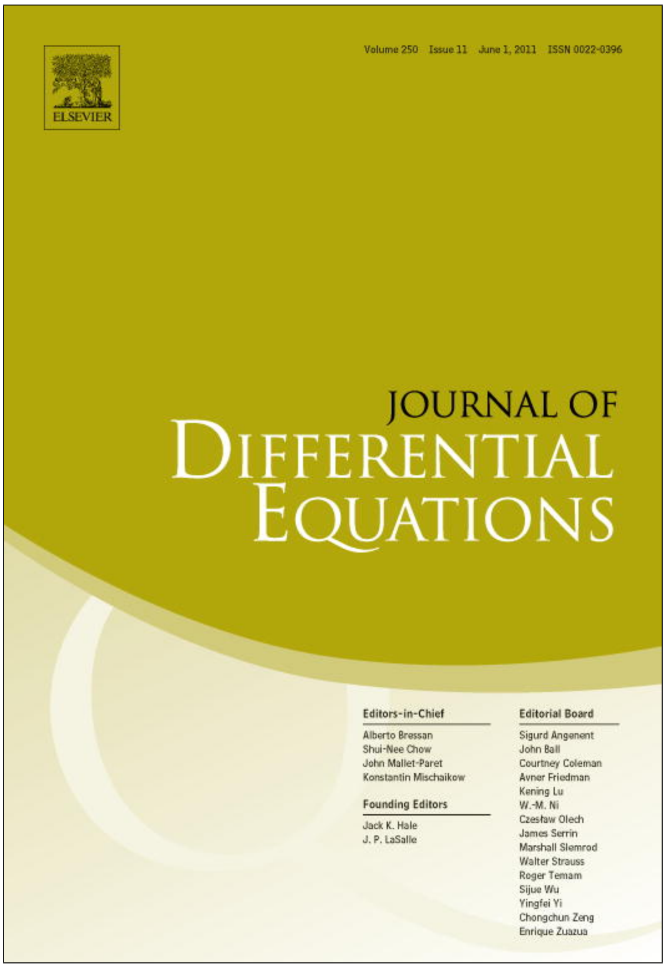
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Stability of periodic solutions of state-dependent delay-differential equations

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ABSTRACT

We consider a class of autonomous delay-differential equations

$$\dot{z}(t) = f(z_t)$$

which includes equations of the form

$$\begin{aligned} &\dot{z}(t) = g\big(z(t), z(t-r_1), \ldots, z(t-r_n)\big), \\ &r_i = r_i\big(z(t)\big) \quad \text{for } 1 \leqslant i \leqslant n, \end{aligned} \tag{$*$}$$

with state-dependent delays $r_i(z(t)) \geqslant 0$. The functions g and r_i satisfy appropriate smoothness conditions.

We assume there exists a periodic solution $z = x(t)$ which is linearly asymptotically stable, namely with all nontrivial characteristic multipliers μ satisfying $|\mu| < 1$. We prove that the appropriate nonlinear stability properties hold for $x(t)$, namely, that this solution is asymptotically orbitally stable with asymptotic phase, and enjoys an exponential rate of attraction given in terms of the leading nontrivial characteristic multiplier.

A principal difficulty which distinguishes the analysis of equations such as $(*)$ from ones with constant delays, is that even with g and r_i smooth, the associated function f is not smooth in function space. Techniques of Hartung, Krisztin, Walther, and Wu are employed to resolve these issues.

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1. Introduction

In this paper we study stability questions for a broad class of autonomous state-dependent delay-differential equations. Specifically, we prove that linearized asymptotic stability of a periodic solution $x(t)$ implies nonlinear (Lyapunov) stability of that solution, in fact, asymptotic orbital stability with asymptotic phase, and exponential attraction at a rate determined by the leading nontrivial characteristic multiplier. This is, of course, the analog of a classic theorem in ordinary differential equations; see, for example, [1]. The corresponding result for retarded equations with constant delay also has been known for many years; see [7].

Among the equations we treat are those with pointwise state-dependent delays such as

$$\dot{z}(t) = g\big(z(t), z(t-r_1), \ldots, z(t-r_n)\big), \qquad r_i = r_i\big(z(t)\big) \quad \text{for } 1 \leqslant i \leqslant n, \tag{1.1}$$

where

$$g : U_g \subseteq \mathbf{R}^{m(n+1)} \to \mathbf{R}^m, \qquad r_i : U_{r_i} \subseteq \mathbf{R}^m \to [0, R] \quad \text{for } 1 \leqslant i \leqslant n,$$

for some (typically open) sets U_g and U_{r_i}. In the case $n = 1$ this equation takes the form

$$\dot{z}(t) = g\big(z(t), z(t-r)\big), \qquad r = r\big(z(t)\big), \tag{1.2}$$

where

$$g : U_g \subseteq \mathbf{R}^{2m} \to \mathbf{R}^m, \qquad r : U_r \subseteq \mathbf{R}^m \to [0, R]. \tag{1.3}$$

The model equation

$$\varepsilon\dot{z}(t) = -z(t) - kz(t-r), \qquad r = r\big(z(t)\big) = 1 + z(t) \tag{1.4}$$

with $\varepsilon > 0$ and $k > 1$, considered in [5] (see also [4]), is a special case.

Generally, we follow the setting of Walther [8] for state-dependent equations (see also Hartung, Krisztin, Walther, and Wu [3]), which we now outline. Consider an autonomous equation

$$\dot{z}(t) = f(z_t) \tag{1.5}$$

where

$$\begin{aligned} &f : U_X \subseteq X \to \mathbf{R}^m \quad \text{is continuous}, \qquad X = C\big([-R, 0], \mathbf{R}^m\big), \\ &z_t \in X \quad \text{is given by } z_t(\theta) = z(t+\theta) \quad \text{for } \theta \in [-R, 0], \end{aligned} \tag{1.6}$$

and where U_X is an open subset of X. This is the classic setting of Hale, as described in the book of Hale and Verduyn Lunel [2]. Local existence of the initial value problem

$$z_0 = \varphi \tag{1.7}$$

for any $\varphi \in U_X$ in forward time is guaranteed, that is, the problem (1.5), (1.7) has a solution $z(t)$ for $0 \leqslant t < \delta$ for some $\delta > 0$. Note that Eq. (1.2) falls into this class by taking

$$f(\varphi) = g\big(\varphi(0), \varphi\big(-r\big(\varphi(0)\big)\big)\big). \tag{1.8}$$

For this equation we assume that $U_g \subseteq \mathbf{R}^{2m}$ and $U_r \subseteq \mathbf{R}^m$ in (1.3) are open sets on which g and r are continuous, and

$$U_X = \{\varphi \in X \mid (\varphi(0), \varphi(-r(\varphi(0)))) \in U_g \text{ and } \varphi(0) \in U_r\}. \tag{1.9}$$

For the model equation (1.4) one has $g(z,\zeta) = -z - k\zeta$ and $r(z) = 1 + z$, with $U_g = \mathbf{R}^2$ and $U_r = (-1, R-1)$, where typically one takes $R \geqslant k+1$.

One has the analogous facts for the more general multiple-delay equation (1.1) as for the single-delay equation (1.2).

In general the solution of the initial value problem (1.5), (1.7) is not unique, although it is unique if f is locally lipschitz. We remark that for the special case of (1.8) and its multiple-delay generalization, the map f is generally not locally lipschitz even if the functions g and r are smooth, unless r is a constant. Thus in general, there is no assurance of a unique solution to the initial value problem (1.2), (1.7). On the other hand, if the initial function φ is lipschitz and g and r are both locally lipschitz, uniqueness does hold for (1.2) with (1.7).

Following Walther, let us consider Eq. (1.5) with a nonlinearity f for which

(H1) $f : U_Y \subseteq Y \to \mathbf{R}^m$ is C^1-smooth, where $Y = C^1([-R,0], \mathbf{R}^m)$ and U_Y is open in Y.

Here we do not necessarily assume that f is defined in a domain $U_X \subseteq X$ as in (1.6), although we do keep the notation $z_t(\theta) = z(t+\theta)$ for $\theta \in [-R, 0]$. For definiteness we shall always take

$$\|\varphi\|_X = \sup_{\theta\in[-R,0]} |\varphi(\theta)|, \qquad \|\varphi\|_Y = \max\{\|\varphi\|_X, \|\dot\varphi\|_X\}, \tag{1.10}$$

for the norm of $\varphi \in X$ or $\varphi \in Y$, respectively. We consider solutions $z(t)$ of (1.5) with initial conditions (1.7) which are C^1-smooth on their initial interval $[-R, 0]$, and which in addition satisfy the compatibility condition $\dot\varphi(0) = f(\varphi)$. This condition ensures that the left- and right-hand derivatives of any such solution $z(t)$ at $t = 0$ are equal. Such solutions are thus C^1 on any maximal interval $[-R, \omega)$ of existence. In this spirit Walther introduces the so-called **solution manifold** $\mathcal{S} \subseteq Y$, defined to be

$$\mathcal{S} = \{\varphi \in U_Y \mid \dot\varphi(0) = f(\varphi)\},$$

which is a relatively closed subset of U_Y. It is easily seen that any solution $z(t)$ with initial condition $z_0 \in \mathcal{S}$ on the solution manifold satisfies $z_t \in \mathcal{S}$ for all $t \geqslant 0$ in its maximal interval of existence.

In general, condition (H1) alone is not enough to guarantee local existence of solutions to (1.5), (1.7), with $\varphi \in \mathcal{S}$. For example, if $f(\psi) = \psi(0) + \dot\psi(0)$ for every $\psi \in U_Y = Y$, then $\mathcal{S} = \{\psi \in Y \mid \psi(0) = 0\}$, and so any solution $z(t)$ to this initial value problem would have to satisfy $z_t \in \mathcal{S}$ for every $t \geqslant 0$ in the interval of existence, namely, $z_t(0) = z(t) = 0$ for such t. However, if the initial condition φ satisfied $\varphi(0) = 0$ but $\dot\varphi(0) \neq 0$, we see that $\dot z(t)$ would be discontinuous at $t = 0$. Thus $z(t)$ could not be considered a solution, as this would mean $z_t \notin Y$ for $t \in (0, R)$ in the interval of existence.

We recall two other conditions on f which play a central role in [3] and in [8], and which with (H1) are sufficient to obtain unique local solutions:

(H2) The Fréchet derivative $Df(\varphi) \in \mathcal{L}(Y, \mathbf{R}^m)$ at any $\varphi \in U_Y$ has a (necessarily unique) continuous linear extension, denoted $D_e f(\varphi) \in \mathcal{L}(X, \mathbf{R}^m)$, to the larger space X.

(H3) The map $(\varphi, \psi) \to D_e f(\varphi)\psi$ from $U_Y \times X \subseteq Y \times X$ into $\mathbf{R}^m$ is jointly continuous in φ and ψ.

See Proposition 2.3 below for a detailed statement of the relevant results on existence, uniqueness, and regularity.

Generally, if Z_1 and Z_2 are Banach spaces, we let $\mathcal{L}(Z_1, Z_2)$ denote the Banach space of bounded linear operators from Z_1 to Z_2 endowed with the operator norm. We remark that under the above conditions, the extended map $D_e f(\varphi)$ may not vary continuously with $\varphi \in U_Y$ in its operator norm, that is, as an element of the space $\mathcal{L}(X, \mathbf{R}^m)$.

Let us also remark that conditions (H1), (H2), and (H3) together imply several useful properties of f, in particular that:

(H4) If $\varphi \in U_Y$ then there exists a neighborhood $\varphi \in V \subseteq U_Y$ and a constant $B > 0$ such that $\|D_e f(\widetilde{\varphi})\|_{\mathcal{L}(X,\mathbf{R}^m)} \leqslant B$ for every $\widetilde{\varphi} \in V$.
(H5) If $\varphi \in U_Y$ then there exists a neighborhood $\varphi \in V \subseteq U_Y$ and a constant $B > 0$ such that $|f(\widetilde{\varphi}) - f(\widehat{\varphi})| \leqslant B\|\widetilde{\varphi} - \widehat{\varphi}\|_X$ for every $\widetilde{\varphi}, \widehat{\varphi} \in V$.

Properties (H4) and (H5) are closely related to the property of a map being **almost locally lipschitz**, which was introduced in [6]. Note that while the relevant neighborhoods in these properties are in the space Y, the relevant norms are associated with the space X. The proof of (H4) is an easy exercise using the uniform boundedness principle; indeed, if there does not exist such a neighborhood V for which the required bound B exists, then there would be a sequence $\varphi_n \in U_Y$ with $\varphi_n \to \varphi$ in Y, with $\|D_e f(\varphi_n)\|_{\mathcal{L}(X,\mathbf{R}^m)} \to \infty$. But then for every $\psi \in X$ the sequence $D_e f(\varphi_n)\psi$ would be bounded, with the limit $D_e(\varphi)\psi$, by condition (H3). Thus the uniform boundedness principle would imply that $\|D_e f(\varphi_n)\|_{\mathcal{L}(X,\mathbf{R}^m)}$ is bounded, a contradiction.

Also, (H5) follows from (H4) by a mean-value theorem. In particular, assuming that V in (H4) is convex, one has for any $\widetilde{\varphi}, \widehat{\varphi} \in V$ that

$$f(\widetilde{\varphi}) - f(\widehat{\varphi}) = \int_0^1 Df\big(s\widetilde{\varphi} + (1-s)\widehat{\varphi}\big)(\widetilde{\varphi} - \widehat{\varphi})\, ds, \tag{1.11}$$

as f is C^1 on U_Y. By (H4) we have the bound

$$\big|Df\big(s\widetilde{\varphi} + (1-s)\widehat{\varphi}\big)(\widetilde{\varphi} - \widehat{\varphi})\big| = \big|D_e f\big(s\widetilde{\varphi} + (1-s)\widehat{\varphi}\big)(\widetilde{\varphi} - \widehat{\varphi})\big| \leqslant B\|\widetilde{\varphi} - \widehat{\varphi}\|_X \tag{1.12}$$

on the above integrand, to give (H5).

With the above conditions, we now state the main results of this paper. The following theorem is the basic result on asymptotic stability with asymptotic phase, for the class of equations considered.

Theorem 1.1. *Assume that f satisfies conditions* (H1), (H2), *and* (H3), *and suppose that $z = x(t)$ is a non-constant periodic solution of Eq.* (1.5) *of period $p > 0$, with $x_t \in U_Y$ for every $t \in \mathbf{R}$. Assume that the trivial eigenvalue $\lambda = 1$ of the monodromy operator M of this solution has simple algebraic multiplicity. Also assume that the remaining spectrum of M lies strictly inside the unit circle, specifically that*

$$\sup\big\{|\lambda| \,\big|\, \lambda \in \operatorname{spec}(M) \setminus \{1\}\big\} = \lambda_0 < 1 \tag{1.13}$$

for some λ_0. (*In other words, $z = x(t)$ has linearized asymptotic stability.*)

Then $z = x(t)$ is asymptotically orbitally stable with asymptotic phase in the following sense. There exist $K_1, K_2 > 0$ such that the following holds. Given any

$$\mu < \frac{|\log \lambda_0|}{p}, \tag{1.14}$$

there exists $K_3(\mu) > 0$ such that if $\|\varphi - x_\sigma\|_Y \leqslant K_1$ for some $\varphi \in \mathcal{S}$ and $\sigma \in \mathbf{R}$, then there exists $\theta \in \mathbf{R}$ such that

$$\|z_t - x_{t+\sigma+\theta}\|_Y \leqslant K_3(\mu)e^{-\mu t}\|\varphi - x_\sigma\|_Y, \qquad |\theta| \leqslant K_2\|\varphi - x_\sigma\|_Y, \tag{1.15}$$

for all $t \geqslant 0$, where $z(t)$ is the solution of the initial value problem $z_0 = \varphi$ to Eq. (1.5).

A second result, involving the C^0 norm instead of the C^1 norm, follows more or less directly from the above theorem provided that the following stronger condition holds in place of (H1):

(H1′) Condition (H1) holds. Moreover, there exists a set $U_X \subseteq X$, open in X and with

$$U_Y = U_X \cap Y,$$

to which f has a continuous (in the X-topology) extension. Further, given any compact (in the Y-topology) set $Q \subseteq \mathcal{S}$, there exist quantities $B', B'' > 0$ such that $\widetilde{\varphi} \in U_X$ and $|f(\varphi) - f(\widetilde{\varphi})| \leqslant B' \|\varphi - \widetilde{\varphi}\|_X$ whenever $\|\varphi - \widetilde{\varphi}\|_X \leqslant B''$ with $\varphi \in Q$ and $\widetilde{\varphi} \in X$.

The following result holds.

Corollary 1.2. *Assume all the conditions in the statement of Theorem* 1.1, *except that the stronger condition* (H1′) *is taken in place of condition* (H1). *Then there exist $K_1', K_2' > 0$ such that the following holds. Given any μ as in* (1.14), *there exists $K_3'(\mu) > 0$ such that if $\|\varphi - x_\sigma\|_X \leqslant K_1'$ for some $\varphi \in U_X$ and $\sigma \in \mathbf{R}$, and with $z(t)$ any solution of the initial value problem $z_0 = \varphi$ to Eq.* (1.5), *then*

$$\begin{aligned}
&\|z_t - x_{t+\sigma+\theta}\|_X \leqslant K_3'(\mu) e^{-\mu t} \|\varphi - x_\sigma\|_X \quad \text{for } 0 \leqslant t \leqslant R, \\
&\|z_t - x_{t+\sigma+\theta}\|_Y \leqslant K_3'(\mu) e^{-\mu t} \|\varphi - x_\sigma\|_X \quad \text{for } t \geqslant R, \qquad |\theta| \leqslant K_2' \|\varphi - x_\sigma\|_X,
\end{aligned} \tag{1.16}$$

for some $\theta \in \mathbf{R}$.

Concerning notation, we shall let $z(t)$ denote a general solution of Eq. (1.5), often (although not always) lying on $\mathcal{S} \subseteq Y$, while $x(t)$ is reserved for the periodic solution in the statements of the above results. In the setting of Corollary 1.2, note that $z_t \in \mathcal{S}$ for all $t \geqslant R$. Of course for the periodic solution it is the case that $x_t \in \mathcal{S}$, and also that $\dot{x}_t \neq 0$, for every $t \in \mathbf{R}$.

Let us also remark that in both Theorem 1.1 and Corollary 1.2, it is implicit that the solution $z(t)$ starting near the periodic solution $x(t)$ exists for all $t \geqslant 0$, that is, it is part of the conclusion of these results that the maximal interval of existence of this solution is $[0, \infty)$.

While this paper is a companion to [5], it can also be considered a self-contained work to be read on its own. As such, we do not necessarily follow the notational conventions of [5], and in fact our notation herein adheres more closely to that of [3] and [8].

2. The functional analytic setting

We begin by recalling some basic results from [3] and [8].

Proposition 2.1. *(See [3, Theorem 3.2.1] and [8, Proposition 1].) Assume that f satisfies conditions* (H1), (H2), *and* (H3). *Then for every $\varphi \in \mathcal{S}$ the map $\psi \to \dot{\psi}(0) - Df(\varphi)\psi$ takes Y onto $\mathbf{R}^m$. Thus $\mathcal{S}$ is an embedded C^1 submanifold of Y of codimension m, with tangent space*

$$T_\varphi \mathcal{S} = \{\psi \in Y \mid \dot{\psi}(0) = Df(\varphi)\psi\}$$

at any $\varphi \in \mathcal{S}$.

We remark that the tangent space $T_\varphi \mathcal{S}$ in the above result is dense in the space X. This follows easily from the fact that the operator $Df(\varphi)$ on Y has a continuous extension $D_e f(\varphi)$ to X.

Proposition 2.2. *Assume that $g : U_g \subseteq \mathbf{R}^{2m} \to \mathbf{R}^m$ and $r : U_r \subseteq \mathbf{R}^m \to [0, R]$ are C^1 functions, where U_g and U_r are open subsets of $\mathbf{R}^{2m}$ and $\mathbf{R}^m$. Let U_X be as in* (1.9) *and $U_Y = U_X \cap Y$, and let $f(\varphi)$ be defined by Eq.* (1.8) *for* $\varphi \in U_X$*. Then* $f : U_X \to \mathbf{R}$ *satisfies condition* (H1′) (*and thus condition* (H1)), *as well as conditions* (H2) *and* (H3)*. The Fréchet derivative of* f *is given by*

$$Df(\varphi)\psi = \big(D_1 g\big(\varphi(0), \varphi\big(-r\big(\varphi(0)\big)\big)\big) - D_2 g\big(\varphi(0), \varphi\big(-r\big(\varphi(0)\big)\big)\big)\dot{\varphi}\big(-r\big(\varphi(0)\big)\big)r'\big(\varphi(0)\big)\big)\psi(0) + \big(D_2 g\big(\varphi(0), \varphi\big(-r\big(\varphi(0)\big)\big)\big)\big)\psi\big(-r\big(\varphi(0)\big)\big) \tag{2.1}$$

for any $\varphi \in U_Y$ *and* $\psi \in Y$*, and with the same formula for the extension* $D_e f(\varphi)$ *with* $\varphi \in U_Y$ *and* $\psi \in X$*.*

The analogous results hold for the function f *associated to the multiple-delay problem* (1.1)*.*

Proof. For simplicity we consider only the single-delay problem, with f as in (1.8). Following [8], we write

$$f(\varphi) = g\big(\mathrm{Ev}(\varphi, 0), \mathrm{Ev}\big(\varphi, -r\big(\mathrm{Ev}(\varphi, 0)\big)\big)\big)$$

for $\varphi \in U_Y$, where $\mathrm{Ev} : Y \times [-R, 0] \to \mathbf{R}^m$ is the evaluation map given by

$$\mathrm{Ev}(\varphi, t) = \varphi(t).$$

As noted in [8], the function Ev is C^1-smooth with derivative

$$D\,\mathrm{Ev}(\varphi, t)(\psi, s) = \psi(t) + \dot{\varphi}(t)s, \tag{2.2}$$

so in particular the function f is C^1 on U_Y. This establishes (H1). Further, using (2.2) one sees that the derivative of f is

$$\begin{aligned} Df(\varphi)\psi = {} & D_1 g\big(\mathrm{Ev}(\varphi, 0), \mathrm{Ev}\big(\varphi, -r\big(\mathrm{Ev}(\varphi, 0)\big)\big)\big)\psi(0) \\ & + D_2 g\big(\mathrm{Ev}(\varphi, 0), \mathrm{Ev}\big(\varphi, -r\big(\mathrm{Ev}(\varphi, 0)\big)\big)\big)\big(\psi\big(-r\big(\mathrm{Ev}(\varphi, 0)\big)\big) \\ & - \dot{\varphi}\big(-r\big(\mathrm{Ev}(\varphi, 0)\big)\big)r'\big(\mathrm{Ev}(\varphi, 0)\big)\psi(0)\big), \end{aligned}$$

which is identical to Eq. (2.1). From this formula one sees easily that both (H2) and (H3) hold.

There remains to prove condition (H1′). Certainly f is continuous on the set U_X given by (1.9). Now with $Q \subseteq \mathcal{S}$ as in the statement of condition (H1′), note that $\{\varphi(0) | \varphi \in Q\}$ is a compact subset of the open set $U_r \subseteq \mathbf{R}^m$. Thus there exists a quantity $B'' > 0$ such that $\varphi(0) + \beta \in U_r$ whenever $\varphi \in Q$ and $|\beta| \leqslant B''$. Further, there exists $B_1 > 0$ such that $|r'(\varphi(0) + \beta)| \leqslant B_1$ whenever $\varphi \in Q$ and $|\beta| \leqslant B''$.

Next let $B_2 > 0$ be such that $\|\varphi\|_Y \leqslant B_2$ for every $\varphi \in Q$, where we are using the fact that Q is compact in Y. Noting that the set $\{(\varphi(0), \varphi(-r(\varphi(0)))) \mid \varphi \in Q\}$ is a compact subset of the open set $U_g \subseteq \mathbf{R}^{2m}$, we see that by decreasing B'' if necessary, we may assume that $(\varphi(0) + \beta, \varphi(-r(\varphi(0))) + \gamma) \in U_g$ whenever $\varphi \in Q$ with $|\beta| \leqslant B''$ and $|\gamma| \leqslant (B_1 B_2 + 1)B''$. Further, there exist $B_3, B_4 > 0$ such that

$$\big|D_1 g\big(\varphi(0) + \beta, \varphi\big(-r\big(\varphi(0)\big)\big) + \gamma\big)\big| \leqslant B_3, \qquad \big|D_2 g\big(\varphi(0) + \beta, \varphi\big(-r\big(\varphi(0)\big)\big) + \gamma\big)\big| \leqslant B_4,$$

for any such φ, β, and γ.

With B'' as above, assume that $\varphi \in Q$ and $\widetilde{\varphi} \in X$ with $\|\varphi - \widetilde{\varphi}\|_X \leqslant B''$. Then $|\varphi(0) - \widetilde{\varphi}(0)| \leqslant B''$ and so $\widetilde{\varphi}(0) \in U_r$. Also,

$$\begin{aligned}
&\bigl|\varphi\bigl(-r\bigl(\varphi(0)\bigr)\bigr)-\widetilde{\varphi}\bigl(-r\bigl(\widetilde{\varphi}(0)\bigr)\bigr)\bigr| \\
&\quad\leqslant \bigl|\varphi\bigl(-r\bigl(\varphi(0)\bigr)\bigr)-\varphi\bigl(-r\bigl(\widetilde{\varphi}(0)\bigr)\bigr)\bigr|+\bigl|\varphi\bigl(-r\bigl(\widetilde{\varphi}(0)\bigr)\bigr)-\widetilde{\varphi}\bigl(-r\bigl(\widetilde{\varphi}(0)\bigr)\bigr)\bigr| \\
&\quad\leqslant B_2\bigl|r\bigl(\varphi(0)\bigr)-r\bigl(\widetilde{\varphi}(0)\bigr)\bigr|+\|\varphi-\widetilde{\varphi}\|_X \\
&\quad\leqslant (B_1B_2+1)\|\varphi-\widetilde{\varphi}\|_X \\
&\quad\leqslant (B_1B_2+1)B'',
\end{aligned}$$

and therefore $(\widetilde{\varphi}(0),\widetilde{\varphi}(-r(\widetilde{\varphi}(0))))\in U_g$. Thus $\widetilde{\varphi}\in U_X$. Finally,

$$\begin{aligned}
\bigl|f(\varphi)-f(\widetilde{\varphi})\bigr| &= \bigl|g\bigl(\varphi(0),\varphi\bigl(-r\bigl(\varphi(0)\bigr)\bigr)\bigr)-g\bigl(\widetilde{\varphi}(0),\widetilde{\varphi}\bigl(-r\bigl(\widetilde{\varphi}(0)\bigr)\bigr)\bigr)\bigr| \\
&\leqslant B_3\bigl|\varphi(0)-\widetilde{\varphi}(0)\bigr|+B_4\bigl|\varphi\bigl(-r\bigl(\varphi(0)\bigr)\bigr)-\widetilde{\varphi}\bigl(-r\bigl(\widetilde{\varphi}(0)\bigr)\bigr)\bigr| \\
&\leqslant \bigl(B_3+B_4(B_1B_2+1)\bigr)\|\varphi-\widetilde{\varphi}\|_X.
\end{aligned}$$

Upon setting $B'=B_3+B_4(B_1B_2+1)$, we obtain the desired result. □

Proposition 2.3. *(See [3, Theorem 3.2.1] and [8, Theorem 1].) Assume that f satisfies conditions* (H1), (H2), *and* (H3). *Then for every $\varphi\in\mathcal{S}$ the initial value problem* (1.5), (1.7) *has a unique solution $z(t)=z(t,\varphi)$ on a maximal interval $0\leqslant t<\omega(\varphi)$, where $0<\omega(\varphi)\leqslant\infty$. Moreover, $z_t(\varphi)\in\mathcal{S}$ for every such t. Denote*

$$\begin{aligned}
&\mathcal{D}=\bigl\{(t,\varphi)\in[0,\infty)\times\mathcal{S}\bigm| t<\omega(\varphi)\bigr\}, \\
&\mathcal{D}_t=\bigl\{\varphi\in\mathcal{S}\bigm|(t,\varphi)\in\mathcal{D}\bigr\},\qquad \mathcal{D}_*=\bigl\{(t,\varphi)\in\mathcal{D}\bigm| t>R\bigr\},
\end{aligned}$$

where $t\geqslant 0$ in the definition of $\mathcal{D}_t$, and denote

$$Z(t,\varphi)=z_t(\varphi)\quad\text{for } (t,\varphi)\in\mathcal{D},$$

and so $Z:\mathcal{D}\to\mathcal{S}$ is the semiflow map. Then $\mathcal{D}$ is a relatively open subset of $[0,\infty)\times\mathcal{S}$, and Z is continuous, jointly in t and φ, on $\mathcal{D}$. Also, for every fixed $t\geqslant 0$ the map $\varphi\to Z(t,\varphi)$ is C^1-smooth on $\mathcal{D}_t$. Further, the map Z is C^1-smooth, again jointly in t and φ, on $\mathcal{D}_*$. Finally, the derivative $D_2Z(t,\varphi)$ of the semiflow map with respect to its initial condition is given by

$$D_2Z(t,\varphi)\psi=y_t \tag{2.3}$$

where $y(t)$ is the unique solution of the linear variational equation

$$\dot y(t)=Df(z_t)y_t, \tag{2.4}$$

with initial condition $y_0=\psi\in T_{z_0}\mathcal{S}$, with $z_t=Z(t,\varphi)$ on its maximal interval, and where $y_t\in T_{z_t}\mathcal{S}$ holds.

Let us remark that the proof of local existence in the C^1 setting of Proposition 2.3 above is not a direct application of the usual Picard iteration. Rather it is a variant of the Picard method which involves both C^1 and C^0 estimates in the spirit of (1.11) and (1.12), where both conditions (H4) and (H5) come into play.

As in [8], given a solution $z(t)$ of Eq. (1.5) with z_t lying on the solution manifold $\mathcal{S}$, one may generally consider the linear variational equation (2.4). Proposition 2.3 asserts the existence and uniqueness of a solution to (2.4) with the initial condition y_0 taken in the tangent space $T_{z_0}\mathcal{S}$, as variations are taken only with respect to the C^1 norm in $\mathcal{S}$. Thus one has $y_t\in T_{z_t}\mathcal{S}$ for every $t\geqslant 0$ in the interval of existence, and so $D_2Z(t,\varphi) \in \mathcal{L}(T_{z_0}\mathcal{S}, T_{z_t}\mathcal{S})$. In particular, $T_{z_0}\mathcal{S}$ and $T_{z_t}\mathcal{S}$ are subspaces of Y and are thus endowed with the C^1 norm.

On the other hand, one can instead consider the related equation

$$\dot{y}(t) = D_e f(z_t) y_t, \tag{2.5}$$

and with arbitrary continuous (not necessarily smooth) initial conditions $y_0 = \psi \in X$. This amounts to taking variations in the larger space X and with respect to the C^0 norm. (We still, however, have $z_t \in \mathcal{S}$ for the solution of the underlying nonlinear equation.) As noted earlier, the coefficient operator $D_e f(z_t)$ need not vary continuously with t in the space $\mathcal{L}(X, \mathbf{R}^m)$. However, it follows easily from property (H4) that for every $t > 0$ in the maximal interval of the solution $z(t)$, there is a uniform bound

$$\left\| D_e f(z_s) \right\|_{\mathcal{L}(X,\mathbf{R}^m)} \leqslant B_0 = B_0(t) \quad \text{for } 0 \leqslant s \leqslant t \tag{2.6}$$

for the norm of $D_e f(z_s)$. With this, and again using the continuity condition (H3), one easily shows the existence of a unique solution to Eq. (2.5) for any $y_0 \in X$, along with a standard bound

$$\|y_s\|_X \leqslant e^{B_0 s} \|y_0\|_X \quad \text{for } 0 \leqslant s \leqslant t. \tag{2.7}$$

In the special case that $y_0 \in T_{z_0}\mathcal{S}$, then we have the bound $|\dot{y}(s)| \leqslant \|\dot{y}_0\|_X$ for $-R \leqslant s \leqslant 0$, and $|\dot{y}(s)| \leqslant B_0 \|y_s\|_X \leqslant B_0 e^{B_0 s} \|y_0\|_X$ for $0 \leqslant s \leqslant t$, using (2.5) and (2.7). With (2.7) this gives the bound

$$\|y_s\|_Y \leqslant B_{00} e^{B_0 s} \|y_0\|_Y \quad \text{for } 0 \leqslant s \leqslant t, \qquad B_{00} = \max\{B_0, 1\}, \tag{2.8}$$

where the definition (1.10) of the norm on Y is used.

3. The monodromy operator

The monodromy operator of our periodic solution can be defined using either Eq. (2.4) or Eq. (2.5). As we shall see, for the class of equations considered here the two approaches are essentially equivalent.

Throughout this section we assume all the conditions in the statement of Theorem 1.1. Let us note that we do not require that $p > 0$ be the minimal period; unless noted otherwise, p can be any integer multiple of the minimal period. Of course the monodromy operator will depend on which such period is chosen.

In defining the monodromy operator, we may, on the one hand, consider the operator $\widetilde{M}$ defined by

$$\widetilde{M} = D_2 Z(p, x_0)$$

as in (2.3), and so $\widetilde{M} \in \mathcal{L}(T_{x_0}\mathcal{S}, T_{x_0}\mathcal{S})$. By Proposition 2.3 the operator $\widetilde{M}$ is given by the formula

$$\widetilde{M} y_0 = y_p, \quad y_0 \in T_{x_0}\mathcal{S},$$

where $y(t)$ is the solution of the linear equation (2.4) with the periodic solution x_t in place of z_t. On the other hand, a formally different operator M is obtained by solving equation (2.5), again with x_t in place of z_t, and defining

$$M y_0 = y_p, \quad y_0 \in X.$$

Then $M \in \mathcal{L}(X, X)$ for this operator, and M agrees with $\widetilde{M}$ on $T_{x_0}\mathcal{S}$, which is densely contained in X. We observe that the Banach spaces X and $T_{x_0}\mathcal{S}$ on which M and $\widetilde{M}$ act are endowed with different norms, namely the C^0 and C^1 norms, respectively.

Differentiating the identity $\dot{x}(t) = f(x_t)$ with respect to t gives $\dot{y}(t) = Df(x_t)y_t$ where $y(t) = \dot{x}(t)$. Then $\widetilde{M}y_0 = y_p$, and as $y_p = y_0 \neq 0$, it follows that

$$M\dot{x}_0 = \widetilde{M}\dot{x}_0 = \dot{x}_0,$$

which is the trivial eigenvector of the monodromy operator with trivial eigenvalue $\lambda = 1$.

In a standard fashion, M^n and $\widetilde{M}^n$ are the monodromy operators for $x(t)$ considered with period np, where M and $\widetilde{M}$ as above are the monodromy operators corresponding to period p. In the case that $np \geqslant R$, we have the following compactness result.

Proposition 3.1. *Assume all the conditions in the statement of Theorem* 1.1 *hold, and let M and $\widetilde{M}$ denote the monodromy operators of $x(t)$ with period p as above. Also assume that $n \geqslant 1$ is such that $np \geqslant R$. Then the monodromy operators $M^n \in \mathcal{L}(X, X)$ and $\widetilde{M}^n \in \mathcal{L}(T_{x_0}\mathcal{S}, T_{x_0}\mathcal{S})$ of $x(t)$ with period np are compact.*

Proof. Let B_0 be as in (2.6) with $t = np$, with z_s replaced by x_s, and let B_{00} be as in the equality in (2.8). Considering first the operator M^n, let $y(t)$ satisfy Eq. (2.5), with z_t replaced by x_t. Then $M^n y_0 = y_{np}$, and we have

$$\|y_{np}\|_X \leqslant e^{B_0 np}\|y_0\|_X \tag{3.1}$$

by (2.7). Further, as $np \geqslant R$ we have that $[np - R, np] \subseteq [0, np]$, and so $|\dot{y}(s)| \leqslant B_0 e^{B_0 s}\|y_0\|_X$ for $s \in [np - R, np]$, from the differential equation (2.5) and (2.7). Thus

$$\|\dot{y}_{np}\|_X \leqslant B_0 e^{B_0 np}\|y_0\|_X. \tag{3.2}$$

Thus the elements y_{np} are uniformly bounded and equicontinuous for y_0 in bounded set of X, and so M^n is compact.

Now consider $\widetilde{M}^n$. Here we take $y_0 \in T_{x_0}\mathcal{S} \subseteq Y$, with $y(t)$ the solution of (2.4), again with z_t replaced by x_t, and $\widetilde{M}^n y_0 = y_{np}$. The bounds (3.1) and (3.2) still hold, and as $\|y_0\|_X \leqslant \|y_0\|_Y$, we again obtain uniform bounds on $\|y_{np}\|_X$ and $\|\dot{y}_{np}\|_X$ for y_0 in bounded subsets of $T_{x_0}\mathcal{S}$. We must establish equicontinuity of the elements $\dot{y}_{np}$ for such y_0 in order to conclude compactness of $\widetilde{M}^n$. To this end, take $s_1, s_2 \in [np - R, np] \subseteq [0, np]$, and using (2.4) along with (2.8), we have the estimate

$$\begin{aligned}
\bigl|\dot{y}(s_1) - \dot{y}(s_2)\bigr| &\leqslant \bigl|Df(x_{s_1})y_{s_1} - Df(x_{s_1})y_{s_2}\bigr| + \bigl|Df(x_{s_1})y_{s_2} - Df(x_{s_2})y_{s_2}\bigr| \\
&\leqslant B_0\|y_{s_1} - y_{s_2}\|_X + \bigl\|Df(x_{s_1}) - Df(x_{s_2})\bigr\|_{\mathcal{L}(Y,\mathbf{R}^m)}\|y_{s_2}\|_Y \\
&\leqslant B_{00}e^{B_0 np}\bigl(B_0|s_1 - s_2| + \bigl\|Df(x_{s_1}) - Df(x_{s_2})\bigr\|_{\mathcal{L}(Y,\mathbf{R}^m)}\bigr)\|y_0\|_Y.
\end{aligned}$$

The continuity of $Df(x_s)$ in s, in the operator norm, with the above estimate, immediately gives the desired equicontinuity property. □

The following result clarifies the relation between the spectra of the operators M and $\widetilde{M}$.

Proposition 3.2. *Assume all the conditions in the statement of Theorem* 1.1 *hold, and let M and $\widetilde{M}$ denote the monodromy operators of $x(t)$ with period p as above. Then $M \in \mathcal{L}(X, X)$ and $\widetilde{M} \in \mathcal{L}(T_{x_0}\mathcal{S}, T_{x_0}\mathcal{S})$ have the same spectrum*

$$\operatorname{spec}(M) = \operatorname{spec}(\widetilde{M}). \tag{3.3}$$

Further, if $\lambda \neq 0$ then for every $k \geqslant 1$ we have that

$$\ker(\lambda I - M)^k = \ker(\lambda I - \widetilde{M})^k, \tag{3.4}$$

so in particular, the algebraic multiplicities of λ as an eigenvalue of M and of $\widetilde{M}$ are equal.

Proof. Fix $n \geqslant 1$ so that $np \geqslant R$. Then both M^n and $\widetilde{M}^n$ are compact operators by Proposition 3.1, and so for both these operators the essential spectrum is just the point $\{0\}$, and every nonzero point of the spectrum is an isolated point of finite algebraic multiplicity. From this we see that (3.3) follows directly from (3.4).

To prove (3.4), fix any $\lambda \neq 0$ and $k \geqslant 1$. Clearly $\ker(\lambda I - \widetilde{M})^k \subseteq \ker(\lambda I - M)^k$ as M is an extension of $\widetilde{M}$. To prove the opposite inclusion, let us first observe that

$$\operatorname{range}\bigl(M^n\bigr) \subseteq T_{x_0}\mathcal{S}. \tag{3.5}$$

The proof of (3.5) is simply the observation that if $y(t)$ satisfies (2.5) (with x_t in place of z_t), then $y_{np} \in Y$ as $np \geqslant R$, and also $\dot{y}(np) = D_e f(x_{np}) y_{np} = D_e f(x_0) y_{np} = Df(x_0) y_{np}$. This implies that $y_{np} \in T_{x_0}\mathcal{S}$, as desired.

Next we write $(\lambda I - M)^k = \lambda^k I - M\Psi(M)$ where Ψ is a polynomial depending on λ and k. Similarly $(\lambda I - \widetilde{M})^k = \lambda^k I - \widetilde{M}\Psi(\widetilde{M})$. Take any $\psi \in \ker(\lambda I - M)^k$. Then $M\Psi(M)\psi = \lambda^k \psi$ and thus $M^n \Psi(M)^n \psi = \lambda^{kn}\psi$. This implies, with (3.5), that $\psi \in T_{x_0}\mathcal{S}$, which is the domain of $\widetilde{M}$. It follows that $\widetilde{M}\Psi(\widetilde{M})\psi = \lambda^k \psi$, equivalently, $\psi \in \ker(\lambda I - \widetilde{M})^k$. This completes the proof. □

4. The Poincaré map

Here we construct a Poincaré map in the solution manifold $\mathcal{S}$ for our periodic solution $x(t)$. We continue to assume that the conditions in the statement of Theorem 1.1 hold. In addition, we assume the (not necessarily minimal) period p satisfies

$$p > R. \tag{4.1}$$

Let us note that from the point of view of Theorem 1.1 and Corollary 1.2, condition (4.1) is in fact no restriction at all. Indeed, replacing the period p with any multiple np does not affect the inequality (1.14) in the statement of Theorem 1.1, as this would replace λ_0 with λ_0^n, leaving the right-hand side of that inequality unchanged. Thus, in proving Theorem 1.1 and Corollary 1.2, we may assume without loss of generality that p is any multiple of the minimal period for which (4.1) holds.

Let $U \subseteq \mathcal{S}$ be an open set (in the relative topology of $\mathcal{S}$) which contains the initial point x_0 of the periodic orbit, and let $H : U \to \mathbf{R}$ be a C^1-smooth function with

$$H(x_0) = 0, \qquad DH(x_0)\dot{x}_0 \neq 0. \tag{4.2}$$

Define the set

$$\mathcal{P} = \bigl\{\varphi \in U \bigm| H(\varphi) = 0\bigr\}. \tag{4.3}$$

Then in a sufficiently small neighborhood of x_0, the set $\mathcal{P}$ is a C^1 manifold containing x_0 and which is transverse to the periodic orbit in $\mathcal{S}$ at x_0. The tangent space of $\mathcal{P}$ at x_0 is the subspace

$$T_{x_0}\mathcal{P} = \ker\bigl(DH(x_0)\bigr) = \bigl\{\psi \in T_{x_0}\mathcal{S} \bigm| DH(x_0)\psi = 0\bigr\}, \tag{4.4}$$

which is a subspace of $T_{x_0}\mathcal{S}$ of codimension one. (We shall only be interested in points of $\mathcal{P}$ in a sufficiently small neighborhood of x_0; outside a neighborhood of this point the set $\mathcal{P}$ need not be a manifold.) The set $\mathcal{P}$, in a sufficiently small neighborhood of x_0 in $\mathcal{S}$, is a Poincaré section to the periodic orbit. As an example of such a function H, one could take $H(\varphi) = \Lambda(\varphi - x_0)$ where Λ is any bounded linear functional on Y for which $\Lambda\dot{x}_0 \neq 0$.

The following result defines a return map to the Poincaré section $\mathcal{P}$ in a sufficiently small neighborhood of x_0.

Proposition 4.1. *Assume all the conditions in the statement of Theorem* 1.1 *hold. Assume further that the period p of $x(t)$ satisfies* (4.1), *and fix a function $H : U \to \mathbf{R}$ with Poincaré section $\mathcal{P}$ as above in* (4.2), (4.3). *Then there exists a neighborhood $V \subseteq U$ containing x_0, and a unique C^1 function $\delta : V \to \mathbf{R}$ with $\delta(x_0) = 0$, such that*

$$Z\big(p + \delta(\varphi), \varphi\big) \in \mathcal{P}$$

for every $\varphi \in V$.

Proof. Consider the function $\overline{H}(\delta, \varphi) = H(Z(p + \delta, \varphi))$ defined in a sufficiently small neighborhood of $(\delta, \varphi) = (0, x_0)$ in $\mathbf{R} \times \mathcal{S}$. Then due to condition (4.1), the function $\overline{H}$ is C^1 with $\overline{H}(0, x_0) = H(Z(p, x_0)) = H(x_0) = 0$. Moreover,

$$\frac{d}{d\delta}\overline{H}(\delta, x_0) = \frac{d}{d\delta}H\big(Z(p + \delta, x_0)\big) = \frac{d}{d\delta}H(x_{p+\delta}) = \frac{d}{d\delta}H(x_\delta) = DH(x_\delta)\dot{x}_\delta \tag{4.5}$$

for $|\delta|$ sufficiently small, and so the quantity (4.5) is nonzero at $\delta = 0$ by (4.2). The result now follows by applying the implicit function theorem to the function $\overline{H}$. □

Let us note that the neighborhood V of x_0 in the above result is a neighborhood in the solution manifold $\mathcal{S}$, and not just a neighborhood in the Poincaré section $\mathcal{P}$ which is locally a submanifold of $\mathcal{S}$. Thus the above result allows for initial conditions φ which do not necessarily lie on the Poincaré section. In this context we define the return map Γ by

$$\Gamma(\varphi) = Z\big(p + \delta(\varphi), \varphi\big) \in \mathcal{P} \tag{4.6}$$

for $\varphi \in V$, and so $\Gamma : V \to \mathcal{P}$. Certainly $\Gamma(x_0) = x_0$, and in fact $\Gamma(x_t) = x_0$ if $|t|$ is sufficiently small. Because the range of Γ lies in $\mathcal{P}$, we have that

$$\operatorname{range}\big(D\Gamma(x_0)\big) \subseteq T_{x_0}\mathcal{P}. \tag{4.7}$$

Thus $D\Gamma(x_0) \in \mathcal{L}(T_{x_0}\mathcal{S}, T_{x_0}\mathcal{P})$, although we may also regard $D\Gamma(x_0) \in \mathcal{L}(T_{x_0}\mathcal{S}, T_{x_0}\mathcal{S})$ due to the inclusion $T_{x_0}\mathcal{P} \subseteq T_{x_0}\mathcal{S}$ as a subspace.

The next result relates the spectrum of $D\Gamma(x_0)$ to the spectrum of the monodromy operators. In preparation for this, observe the direct sum decomposition

$$T_{x_0}\mathcal{S} = \langle\dot{x}_0\rangle \oplus T_{x_0}\mathcal{P}, \tag{4.8}$$

which holds by (4.4) because $DH(x_0)\dot{x}_0 \neq 0$. Here $\langle\dot{x}_0\rangle$ denotes the one-dimensional span of $\dot{x}_0$.

Proposition 4.2. *Assume all the conditions in the statement of Theorem* 1.1 *hold. Assume further that the period p of $x(t)$ satisfies* (4.1), *and fix a function H with Poincaré section $\mathcal{P}$ as above in* (4.2), (4.3). *Let the neighborhood V and map δ be as in the statement of Proposition* 4.1, *with return map Γ as in* (4.6). *Then relative to the decomposition* (4.8), *the operators $\widetilde{M}, D\Gamma(x_0) \in \mathcal{L}(T_{x_0}\mathcal{S}, T_{x_0}\mathcal{S})$ have the matrix representations*

$$\widetilde{M} = \begin{pmatrix} 1 & A \\ 0 & \widehat{M} \end{pmatrix}, \qquad D\Gamma(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{M} \end{pmatrix},$$
$$A = -D\delta(x_0)|_{T_{x_0}\mathcal{P}}, \qquad \widehat{M} = D\Gamma(x_0)|_{T_{x_0}\mathcal{P}}. \tag{4.9}$$

Here $A \in \mathcal{L}(T_{x_0}\mathcal{P}, \mathbf{R})$ and $\widehat{M} \in \mathcal{L}(T_{x_0}\mathcal{P}, T_{x_0}\mathcal{P})$, and further, $\widehat{M}$ and $D\Gamma(x_0)$ are compact operators, as are $\widetilde{M}$ and M. If $k \geqslant 1$ denotes the algebraic multiplicity of $\lambda = 1$ as an eigenvalue of the operator $\widetilde{M}$, or equivalently of M, then

$$\operatorname{spec}(\widehat{M}) = \operatorname{spec}\bigl(D\Gamma(x_0)\bigr) = \begin{cases} \operatorname{spec}(\widetilde{M}) \setminus \{1\} = \operatorname{spec}(M) \setminus \{1\} & \text{if } k = 1, \\ \operatorname{spec}(\widetilde{M}) = \operatorname{spec}(M) & \text{if } k > 1, \end{cases} \tag{4.10}$$

and if $k > 1$ then $\lambda = 1$ has algebraic multiplicity $k - 1$ as an eigenvalue of $\widehat{M}$ and of $D\Gamma(x_0)$. Further, the multiplicity of any given $\lambda \neq 0, 1$ in the spectrum of $\widehat{M}$, $D\Gamma(x_0)$, $\widetilde{M}$, or M is the same for these four operators.

Proof. First, we have the trivial eigenvector $\widetilde{M}\dot{x}_0 = \dot{x}_0$, to give the first column of the matrix for $\widetilde{M}$. Next let $A \in \mathcal{L}(T_{x_0}\mathcal{P}, \mathbf{R})$ and $\widehat{M} \in \mathcal{L}(T_{x_0}\mathcal{P}, T_{x_0}\mathcal{P})$ be defined to be the matrix entries in the formula for $\widetilde{M}$ in (4.9), that is, the first equation in (4.9). We must verify the remaining formulas in (4.9). Differentiating the formula (4.6) for Γ at x_0 gives

$$D\Gamma(x_0) = D_1 Z(p, x_0) D\delta(x_0) + D_2 Z(p, x_0) = \dot{x}_p D\delta(x_0) + \widetilde{M} = \dot{x}_0 D\delta(x_0) + \widetilde{M}$$

which we may write as

$$\widetilde{M} = -\dot{x}_0 D\delta(x_0) + D\Gamma(x_0). \tag{4.11}$$

The two terms in the right-hand side of (4.11) correspond to the decomposition (4.8), in particular because of (4.7). Thus letting $P \in \mathcal{L}(T_{x_0}\mathcal{S}, T_{x_0}\mathcal{S})$ denote the projection from $T_{x_0}\mathcal{S}$ onto $T_{x_0}\mathcal{P}$ with kernel $\langle \dot{x}_0 \rangle$, we have that $P = \operatorname{diag}(0, I)$ in matrix form and it follows that

$$-\dot{x}_0 D\delta(x_0) = (I - P)\widetilde{M} = \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix}, \qquad D\Gamma(x_0) = P\widetilde{M} = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{M} \end{pmatrix}. \tag{4.12}$$

This gives the matrix representation of $D\Gamma(x_0)$ in (4.9). One also immediately reads off from (4.12) the formulas for A and $\widehat{M}$ as claimed in (4.9).

The operators M and $\widetilde{M}$ are compact by Proposition 3.1 and the inequality (4.1), and from the matrix representations (4.9) it follows that $\widehat{M}$ and $D\Gamma(x_0)$ are also compact. The remaining claims about the spectra of the various operators follows simply and directly from the matrix representations (4.9), using in particular the upper triangular structure, and from Proposition 3.2. □

5. Proofs of the main results

In this section we prove both Theorem 1.1 and Corollary 1.2. Before doing this, we need the following result.

Proposition 5.1. *Assume all the conditions in the statement of Theorem* 1.1 *hold. Assume further that the period p of $x(t)$ satisfies* (4.1), *and fix a function H with Poincaré section $\mathcal{P}$ as above in* (4.2), (4.3). *Let the neighborhood V and map δ be as in the statement of Proposition* 4.1, *with return map Γ as in* (4.6). *Then given any λ satisfying $\lambda_0 < \lambda < 1$, where λ_0 is as in* (1.13), *there exists a neighborhood $V^\lambda \subseteq V$ containing x_0, a C^1 function $\delta^\lambda : V^\lambda \to \mathbf{R}$ with $\delta^\lambda(x_0) = 0$, and an integer $n \geqslant 1$, such that*

$$\Gamma^n(\varphi) = Z\bigl(np + \delta^\lambda(\varphi), \varphi\bigr) \in \mathcal{P}, \qquad \bigl\|\Gamma^n(\varphi) - x_0\bigr\|_Y \leqslant \lambda^n \|\varphi - x_0\|_Y \tag{5.1}$$

for every $\varphi \in V^\lambda$.

Proof. We have $\operatorname{spec}(D\Gamma(x_0)) = \operatorname{spec}(M) \setminus \{1\}$ by (4.10) of Proposition 4.2, as the eigenvalue $\lambda = 1$ of M is assumed to have simple algebraic multiplicity. Thus the spectral radius of $D\Gamma(x_0)$ equals λ_0, and so there exists $n \geqslant 1$ such that

$$\left\| D\Gamma(x_0)^n \right\|_{\mathcal{L}(T_{x_0}\mathcal{S}, T_{x_0}\mathcal{S})} \leqslant \frac{\lambda^n}{2}. \tag{5.2}$$

Fix any such n. Then with V as in Proposition 4.1, there exists a neighborhood $\widetilde{V} \subseteq V$ containing x_0 such that if $\varphi \in \widetilde{V}$ then we have the iterates $\Gamma^k(\varphi) \in V$ for every $1 \leqslant k \leqslant n-1$ and so $\Gamma^n(\varphi)$ is well-defined. We claim that for such φ we have

$$\Gamma^k(\varphi) = Z\bigl(kp + \delta(\varphi,k), \varphi\bigr), \qquad \delta(\varphi,k) = \sum_{j=0}^{k-1} \delta\bigl(\Gamma^j(\varphi)\bigr), \tag{5.3}$$

holding for $1 \leqslant k \leqslant n$. Indeed, one easily proves (5.3) by finite induction on k. The claim holds by definition if $k = 1$. If (5.3) holds for some k with $1 \leqslant k \leqslant n-1$ then

$$\begin{aligned} \Gamma^{k+1}(\varphi) &= \Gamma\bigl(\Gamma^k(\varphi)\bigr) = Z\bigl(p + \delta\bigl(\Gamma^k(\varphi)\bigr), \Gamma^k(\varphi)\bigr) = Z\bigl(p + \delta\bigl(\Gamma^k(\varphi)\bigr), Z\bigl(kp + \delta(\varphi,k), \varphi\bigr)\bigr) \\ &= Z\bigl((k+1)p + \delta(\varphi,k) + \delta\bigl(\Gamma^k(\varphi)\bigr), \varphi\bigr) = Z\bigl((k+1)p + \delta(\varphi,k+1), \varphi\bigr), \end{aligned}$$

to give it for $k+1$. Note that the semiflow property of Z is used in the penultimate equality above.

As x_0 is a fixed point of Γ, we have that $D\Gamma^n(x_0) = D\Gamma(x_0)^n$ for the derivative of the nth iterate Γ^n. Thus by (5.2) there exists a neighborhood $V^\lambda \subseteq \widetilde{V}$ containing x_0 such that

$$\|\Gamma^n(\varphi) - x_0\|_Y \leqslant 2\|D\Gamma(x_0)^n\|_{\mathcal{L}(T_{x_0}\mathcal{S}, T_{x_0}\mathcal{S})} \|\varphi - x_0\|_Y \leqslant \lambda^n \|\varphi - x_0\|_Y$$

for every $\varphi \in V^\lambda$. With (5.3) and upon setting $\delta^\lambda(\varphi) = \delta(\varphi, n)$, we have (5.1). □

Proof of Theorem 1.1. Throughout this proof $z(t)$, for $t \geqslant 0$, is the solution of (1.5) with initial condition $z_0 = \varphi$, where $\varphi \in \mathcal{S}$ is as in the statement of the theorem. In particular, $z_t = Z(t, \varphi)$. Also, as noted earlier, we may assume without loss that condition (4.1) on the period p of the periodic solution $x(t)$ holds. Additionally, all norms in this proof are either the norm of Y or else C^1-type norms of associated spaces of operators, unless noted otherwise. Finally, we shall introduce various constants K_j, for $1 \leqslant j \leqslant 18$. Some of these constants will depend on the choice of μ, in which case we write $K_j(\mu)$. We write simply K_j if the constant is independent of μ.

Fix $K_4 > 0$ such that we have the bounds

$$\bigl|\dot{x}(t)\bigr|, \bigl|\ddot{x}(t)\bigr| \leqslant K_4 \quad \text{for all } t \in \mathbf{R}, \tag{5.4}$$

for the periodic solution $x(t)$, and note that for any $t_1, t_2 \in \mathbf{R}$ we have

$$\|x_{t_1} - x_{t_2}\| \leqslant K_4 |t_1 - t_2|, \tag{5.5}$$

here using the Y-norm. In particular, the bounds $\|x_{t_1} - x_{t_2}\|_X \leqslant K_4|t_1 - t_2|$ and $\|\dot{x}_{t_1} - \dot{x}_{t_2}\|_X \leqslant K_4|t_1 - t_2|$ in the X-norm follow from (5.4), and together these give (5.5).

We shall first prove that for every $\mu > 0$ satisfying the inequality (1.14) in the statement of the theorem, there exist quantities $K_5(\mu), K_6(\mu), K_7(\mu) > 0$ such that if $\|\varphi - x_0\| \leqslant K_5(\mu)$ and $\varphi \in \mathcal{S}$, then there exists $\theta \in \mathbf{R}$ such that

$$\|z_t - x_{t+\theta}\| \leqslant K_7(\mu)e^{-\mu t}\|\varphi - x_0\|, \qquad |\theta| \leqslant K_6(\mu)\|\varphi - x_0\|, \tag{5.6}$$

for all $t \geqslant 0$. Note that the conclusion (5.6) is weaker than the desired conclusion (1.15), as the constants $K_5(\mu)$ and $K_6(\mu)$ depend on μ, unlike K_1 and K_2 in the statement of the theorem. Also, the initial condition φ is compared only to x_0, and not to x_σ as in the statement of the theorem.

To prove this claim, first take any μ as above and let $\lambda = e^{-\mu p}$. Then $\lambda_0 < \lambda < 1$ holds and so Proposition 5.1 applies. Fix n as in Proposition 5.1, let V^λ and δ^λ be as in that result, and let $K_5(\mu), K_8(\mu) > 0$ and $W(\mu) \subseteq \mathcal{S}$ be such that

$$\begin{aligned} &W(\mu) = \bigl\{\varphi \in \mathcal{S} \bigm| \|\varphi - x_0\| \leqslant K_5(\mu)\bigr\} \subseteq V^\lambda, \\ &\bigl|\delta^\lambda(\varphi)\bigr| \leqslant K_8(\mu)\|\varphi - x_0\| \quad \text{for every } \varphi \in W(\mu). \end{aligned} \tag{5.7}$$

In particular, we take $K_5(\mu)$ sufficiently small and we use the smoothness of $\delta^\lambda(\varphi)$ in φ to obtain $K_8(\mu)$. By possibly decreasing $K_5(\mu)$ further, we may also assume that there exists $K_9(\mu) > 0$ such that

$$\bigl\|Z(t,\varphi) - x_t\bigr\| \leqslant K_9(\mu)\|\varphi - x_0\| \quad \text{if } \varphi \in W(\mu) \text{ and } 0 \leqslant t \leqslant np + K_5(\mu)K_8(\mu). \tag{5.8}$$

Indeed, the fact that $Z(t,\varphi)$ is C^1 on the set $\mathcal{D}_*$ of Proposition 2.3 ensures that the first inequality in (5.8) holds at least for $\varphi \in W(\mu)$ and $2R \leqslant t \leqslant \max\{2R, np + K_5(\mu)K_8(\mu)\}$, provided that the radius $K_5(\mu)$ of $W(\mu)$ is small enough. Further, this inequality holds also at $t = R$ for small enough $K_5(\mu)$, as $Z(t,\varphi)$ is C^1 in φ for fixed $t = R$. But now with the first inequality in (5.8) holding at both $t = R$ and $t = 2R$, and assuming without loss that $K_9(\mu) \geqslant 1$ so that it also holds at $t = 0$, it is elementary to see that (5.8) holds throughout the interval $0 \leqslant t \leqslant 2R$ since (roughly speaking) the segment $z_t = Z(t,\varphi) \in \mathcal{S} \subseteq C^1[-R,0]$ is composed of pieces taken from the segments $z_0 = \varphi$, z_R, and z_{2R}. Thus (5.8) holds for all φ and t as stated.

Now suppose that $\varphi \in W(\mu)$ for the initial condition of $z(t)$. Then $\varphi \in V^\lambda$, and so by Proposition 5.1 and the fact that $W(\mu)$ is a ball, we have for every $k \geqslant 1$ that $\Gamma^{kn}(\varphi) \in W(\mu)$ with

$$\bigl\|\Gamma^{kn}(\varphi) - x_0\bigr\| \leqslant \lambda^{kn}\|\varphi - x_0\| = e^{-\mu knp}\|\varphi - x_0\|. \tag{5.9}$$

In particular, the solution $z(t)$ exists for all $t \geqslant 0$. Further, we have from Proposition 5.1 that

$$\Gamma^{kn}(\varphi) = Z(t_k,\varphi) = z_{t_k}, \qquad t_k = knp + \delta^\lambda(\varphi,k), \qquad \delta^\lambda(\varphi,k) = \sum_{j=0}^{k-1} \delta^\lambda\bigl(\Gamma^{jn}(\varphi)\bigr), \tag{5.10}$$

for every $k \geqslant 1$, as is easily proved by induction on k, and where the above formulas define t_k and $\delta^\lambda(\varphi,k)$. Let us also define $t_0 = 0$, and note that $t_k < t_{k+1}$ for every $k \geqslant 0$. Now set

$$\theta = -\lim_{k\to\infty} \delta^\lambda(\varphi,k), \tag{5.11}$$

where the estimate

$$\bigl|\delta^\lambda\bigl(\Gamma^{jn}(\varphi)\bigr)\bigr| \leqslant K_8(\mu)\bigl\|\Gamma^{jn}(\varphi) - x_0\bigr\| \leqslant K_8(\mu)e^{-\mu jnp}\|\varphi - x_0\| \tag{5.12}$$

follows from (5.7) and (5.9) and ensures the convergence of the limit (5.11). In fact, upon summing the right-hand side of (5.12) we obtain the estimates

$$\left|\delta^\lambda(\varphi,k)\right|, |\theta| \leqslant \sum_{j=0}^{\infty}\left|\delta^\lambda\left(\Gamma^{jn}(\varphi)\right)\right| \leqslant K_6(\mu)\|\varphi - x_0\|,$$

$$\left|\theta + \delta^\lambda(\varphi,k)\right| \leqslant \sum_{j=k}^{\infty}\left|\delta^\lambda\left(\Gamma^{jn}(\varphi)\right)\right| \leqslant K_6(\mu)e^{-\mu knp}\|\varphi - x_0\|, \qquad K_6(\mu) = \frac{K_8(\mu)}{1-e^{-\mu np}}, \tag{5.13}$$

with the above equation defining $K_6(\mu)$. At this point notice that the second inequality in (5.6) holds.

Now let $I_k = [t_k, t_{k+1}]$. Certainly $t_k \to \infty$ as $k \to \infty$, and so the intervals I_k for $k \geqslant 0$ cover $[0, \infty)$. It is thus enough that we establish the first inequality of (5.6) in each I_k, of course with the constant $K_7(\mu)$ independent of k. Consider then the solution $z(t)$ for $t \in I_k$. Note first from (5.10) and (5.12) and because $\varphi \in W(\mu)$ that

$$t_{k+1} - t_k = np + \delta^\lambda\left(\Gamma^{kn}(\varphi)\right) \leqslant np + K_8(\mu)e^{-\mu knp}\|\varphi - x_0\| \leqslant np + K_5(\mu)K_8(\mu). \tag{5.14}$$

Then (5.8) applied with the initial point $z_{t_k} = \Gamma^{kn}(\varphi)$, with the aid of the inequalities (5.9) and (5.14), implies that

$$\begin{aligned}\|z_t - x_{t-t_k}\| &= \left\|Z\left(t-t_k, \Gamma^{kn}(\varphi)\right) - x_{t-t_k}\right\| \\ &\leqslant K_9(\mu)\left\|\Gamma^{kn}(\varphi) - x_0\right\| \leqslant K_9(\mu)e^{-\mu knp}\|\varphi - x_0\|\end{aligned} \tag{5.15}$$

for every $t \in I_k$. (Note that t has been shifted by t_k in the above formula vis-à-vis the formula (5.8).) Also, from (5.5), (5.10), and (5.13) we have that

$$\begin{aligned}\|x_{t-t_k} - x_{t+\theta}\| &= \|x_{t-t_k} - x_{t+\theta-knp}\| \\ &\leqslant K_4|\theta - knp + t_k| = K_4\left|\theta + \delta^\lambda(\varphi,k)\right| \leqslant K_4K_6(\mu)e^{-\mu knp}\|\varphi - x_0\|,\end{aligned}$$

and combining this with (5.15) gives

$$\|z_t - x_{t+\theta}\| \leqslant K_{10}(\mu)e^{-\mu knp}\|\varphi - x_0\|, \qquad K_{10}(\mu) = K_9(\mu) + K_4K_6(\mu), \tag{5.16}$$

for every $t \in I_k$, with the above equation defining $K_{10}(\mu)$. Observing further from (5.13) that

$$t_{k+1} - knp = np + \delta^\lambda(\varphi, k+1) \leqslant np + K_6(\mu)\|\varphi - x_0\| \leqslant np + K_5(\mu)K_6(\mu),$$

we have from (5.16) that for $t \in I_k$

$$\|z_t - x_{t+\theta}\| \leqslant K_{10}(\mu)e^{\mu(-knp+t_{k+1}-t)}\|\varphi - x_0\| \leqslant K_{10}(\mu)e^{\mu(np+K_5(\mu)K_6(\mu))}e^{-\mu t}\|\varphi - x_0\|,$$

to give the desired result (5.6) with $K_7(\mu) = K_{10}(\mu)e^{\mu(np+K_5(\mu)K_6(\mu))}$.

We next refine the bounds (5.6) by showing that there exist $K_{11}, K_{12} > 0$ such that the following holds. Given $\mu > 0$ as in (1.14), there exists $K_{13}(\mu) > 0$ such that if $\|\varphi - x_0\| \leqslant K_{11}$ and $\varphi \in \mathcal{S}$, then there exists $\theta \in \mathbf{R}$ such that

$$\|z_t - x_{t+\theta}\| \leqslant K_{13}(\mu)e^{-\mu t}\|\varphi - x_0\|, \qquad |\theta| \leqslant K_{12}\|\varphi - x_0\|, \tag{5.17}$$

for all $t \geqslant 0$. Here K_{11} and K_{12} are independent of μ, as are K_1 and K_2 in the statement of the theorem. However, the initial condition φ is still compared only to x_0 rather than to the more general x_σ as in the statement of the theorem.

To prove this claim, let us first fix some $\overline{\mu} > 0$ satisfying (1.14). The quantity $\overline{\mu}$ will stay fixed for the remainder of the proof, while the quantity μ as before is allowed to take on any positive value satisfying (1.14). Let us also assume, without loss, that

$$K_5(\mu)K_6(\mu) < p \tag{5.18}$$

holds for every such μ. This is accomplished for each μ by decreasing $K_5(\mu)$ sufficiently. With $\overline{\mu}$ now fixed, set

$$K_{11} = K_5(\overline{\mu}), \qquad K_{12} = K_6(\overline{\mu}),$$

and assume that $\varphi \in \mathcal{S}$ satisfies $\|\varphi - x_0\| \leqslant K_{11}$ as in the claim. Then from the earlier and now established claim (5.6) in this proof, we have that

$$\|z_t - x_{t+\theta}\| \leqslant K_7(\overline{\mu})e^{-\overline{\mu}t}\|\varphi - x_0\| \tag{5.19}$$

for all $t \geqslant 0$, for some $\theta \in \mathbf{R}$ satisfying the second inequality in (5.17). Now given any $\mu > 0$ satisfying (1.14), set

$$K_{14}(\mu) = \max\left\{0, \frac{1}{\overline{\mu}}\log\left(\frac{K_7(\overline{\mu})K_{11}}{K_5(\mu)}\right)\right\}.$$

We see from (5.19) that if $t \geqslant K_{14}(\mu)$ then

$$\|z_t - x_{t+\theta}\| \leqslant K_7(\overline{\mu})e^{-\overline{\mu}K_{14}(\mu)}\|\varphi - x_0\| \leqslant \frac{K_5(\mu)\|\varphi - x_0\|}{K_{11}} \leqslant K_5(\mu). \tag{5.20}$$

Further, let

$$K_{15}(\mu) = 1 + \left[\frac{K_{11}K_{12} + K_{14}(\mu)}{p}\right], \qquad t_* = K_{15}(\mu)p - \theta. \tag{5.21}$$

Here $[c]$, for any $c \in \mathbf{R}$, denotes the unique integer m satisfying $m \leqslant c < m+1$. Then from the second inequality in (5.17) we have that $t_* \geqslant K_{15}(\mu)p - K_{11}K_{12} \geqslant K_{14}(\mu)$, and so from (5.20) we have

$$\|z_{t_*} - x_0\| = \|z_{t_*} - x_{t_*+\theta}\| \leqslant K_5(\mu), \tag{5.22}$$

in particular because $t_* + \theta$ is an integer multiple of the period p. Upon letting $\widetilde{z}(t) = z(t + t_*)$, and so $\|\widetilde{z}_0 - x_0\| \leqslant K_5(\mu)$, we have from the earlier claim (5.6) but for $\widetilde{z}(t)$, and then from (5.19), that

$$\begin{aligned}\|z_{t+t_*} - x_{t+\widetilde{\theta}}\| = \|\widetilde{z}_t - x_{t+\widetilde{\theta}}\| &\leqslant K_7(\mu)e^{-\mu t}\|z_{t_*} - x_0\| \\ &= K_7(\mu)e^{-\mu t}\|z_{t_*} - x_{t_*+\theta}\| \leqslant K_7(\mu)K_7(\overline{\mu})e^{-\mu t - \overline{\mu}t_*}\|\varphi - x_0\|\end{aligned} \tag{5.23}$$

for all $t \geqslant 0$, for some $\widetilde{\theta} \in \mathbf{R}$ satisfying

$$|\widetilde{\theta}| \leqslant K_6(\mu)\|z_{t_*} - x_0\| \leqslant K_5(\mu)K_6(\mu) < p, \tag{5.24}$$

by (5.6) and (5.22), and by (5.18). We have next that

$$\|x_t - x_{t+\widetilde{\theta}}\| = \|x_{t+t_*+\theta} - x_{t+\widetilde{\theta}}\| \leqslant \|x_{t+t_*+\theta} - z_{t+t_*}\| + \|z_{t+t_*} - x_{t+\widetilde{\theta}}\| \to 0 \tag{5.25}$$

as $t \to \infty$, by (5.19) with t replaced with $t + t_*$ and by (5.23), and by again noting that $t_* + \theta$ is an integer multiple of p. It follows from (5.25) that $\widetilde{\theta}$ must be an integer multiple of p, and thus $\widetilde{\theta} = 0$ by (5.24). Let us also note the upper bound $t_* \leqslant (K_{15}(\mu) + 1)p$, which holds by (5.21) and because $|\theta| \leqslant K_{11}K_{12} = K_5(\overline{\mu})K_6(\overline{\mu}) < p$. We see from these facts and from (5.23), wherein we replace t with $t - t_*$, that

$$\begin{aligned}\|z_t - x_{t+\theta}\| = \|z_t - x_{t-t_*}\| &\leqslant K_7(\mu)K_7(\overline{\mu})e^{-\mu(t-t_*)-\overline{\mu}t_*}\|\varphi - x_0\| \\ &\leqslant K_7(\mu)K_7(\overline{\mu})e^{\mu(K_{15}(\mu)+1)p-\mu t}\|\varphi - x_0\| = K_{16}(\mu)e^{-\mu t}\|\varphi - x_0\|\end{aligned} \tag{5.26}$$

for $t \geqslant t_*$, where the final equality above serves as the definition of $K_{16}(\mu)$. Next, for $0 \leqslant t \leqslant t_*$ we have from (5.19) and again from the upper bound on t_*, that

$$\begin{aligned}\|z_t - x_{t+\theta}\| &\leqslant K_7(\overline{\mu})e^{(\mu-\overline{\mu})t-\mu t}\|\varphi - x_0\| \leqslant K_7(\overline{\mu})e^{\mu t_*-\mu t}\|\varphi - x_0\| \\ &\leqslant K_7(\overline{\mu})e^{\mu(K_{15}(\mu)+1)p-\mu t}\|\varphi - x_0\| = K_{17}(\mu)e^{-\mu t}\|\varphi - x_0\|,\end{aligned} \tag{5.27}$$

with the final equality above serving to define $K_{17}(\mu)$. Combining (5.26) and (5.27) and letting

$$K_{13}(\mu) = \max\bigl\{K_{16}(\mu), K_{17}(\mu)\bigr\},$$

we have that the first inequality in (5.17) holds for all $t \geqslant 0$, as claimed.

To complete the proof of the theorem, let $K_1, K_{18} > 0$ be such that

$$\begin{aligned}&\bigl\|Z(t,\varphi) - x_{t+\sigma}\bigr\| \leqslant K_{18}\|\varphi - x_\sigma\| \quad \text{if } 0 \leqslant t \leqslant p, \text{ and} \\ &\varphi \in \mathcal{S} \text{ is such that } \|\varphi - x_\sigma\| \leqslant K_1 \text{ for some } \sigma \in \mathbf{R},\end{aligned} \tag{5.28}$$

and also

$$K_1 K_{18} \leqslant K_{11}. \tag{5.29}$$

The proof of the existence of such K_1 and K_{18} follows the same lines as the proof of (5.8). Suppose now that $\varphi \in \mathcal{S}$ satisfies $\|\varphi - x_\sigma\| \leqslant K_1$ for some $\sigma \in \mathbf{R}$, and again let $z(t)$ be the solution with initial condition φ. Also, fix any $\mu > 0$ satisfying (1.14), as in the statement of the theorem. Assuming without loss that $0 \leqslant \sigma < p$, we have from (5.28) that

$$\|z_t - x_{t+\sigma}\| \leqslant K_{18}\|\varphi - x_\sigma\| \leqslant K_{18}e^{\mu(p-t)}\|\varphi - x_\sigma\| \tag{5.30}$$

for $0 \leqslant t \leqslant p - \sigma$. In particular, at $t = p - \sigma$ we have, using (5.29), that

$$\|z_{p-\sigma} - x_0\| \leqslant K_{18}\|\varphi - x_\sigma\| \leqslant K_1K_{18} \leqslant K_{11}.$$

Thus from (5.17), but with initial condition $z_{p-\sigma}$ there, there exists $\theta \in \mathbf{R}$ such that for $t \geqslant 0$ we have

$$\begin{aligned}&\|z_{t+p-\sigma} - x_{t+\theta}\| \leqslant K_{13}(\mu)e^{-\mu t}\|z_{p-\sigma} - x_0\| \leqslant K_{13}(\mu)K_{18}e^{-\mu t}\|\varphi - x_\sigma\|, \\ &|\theta| \leqslant K_{12}\|z_{p-\sigma} - x_0\| \leqslant K_{12}K_{18}\|\varphi - x_\sigma\|.\end{aligned} \tag{5.31}$$

Upon setting $K_2 = K_{12}K_{18}$, we see from this that the second formula in (1.15) holds. Also, for $t \geqslant p-\sigma$ we have that

$$\|z_t - x_{t+\sigma+\theta}\| \leqslant K_{13}(\mu)K_{18}e^{\mu(p-\sigma-t)}\|\varphi - x_\sigma\| \leqslant K_{13}(\mu)K_{18}e^{\mu(p-t)}\|\varphi - x_\sigma\|. \tag{5.32}$$

Finally, from (5.30) and (5.31), and using (5.5), we have for $0 \leqslant t \leqslant p-\sigma$ that

$$\begin{aligned}\|z_t - x_{t+\sigma+\theta}\| &\leqslant \|z_t - x_{t+\sigma}\| + \|x_{t+\sigma} - x_{t+\sigma+\theta}\| \\ &\leqslant K_{18}e^{\mu(p-t)}\|\varphi - x_\sigma\| + K_4|\theta| \leqslant (1+K_4K_{12})K_{18}e^{\mu(p-t)}\|\varphi - x_\sigma\|. \end{aligned} \tag{5.33}$$

Upon setting $K_3(\mu) = K_{18}e^{\mu p}\max\{K_{13}(\mu), 1+K_4K_{12}\}$, the desired conclusion (1.15) of the theorem follows directly from (5.32) and (5.33). □

Proof of Corollary 1.2. Let $Q = \{x_t \mid t \in \mathbf{R}\} \subseteq \mathcal{S}$ denote the periodic orbit, which is a compact subset of $\mathcal{S} \subseteq Y$, and let B' and B'' be as in condition (H1′). Take any $\varphi \in X$ satisfying

$$\|\varphi - x_\sigma\|_X \leqslant K_1', \qquad K_1' = \min\{B'', K_1, K_1(B')^{-1}\}e^{-B'R}$$

for some $\sigma \in \mathbf{R}$, where K_1 is as in the statement of Theorem 1.1 and where the above formula defines K_1'. Then $\|\varphi - x_\sigma\|_X < B''$ hence $\varphi \in U_X$ by (H1′). Let $z(t)$ be any solution of (1.5) with $z_0 = \varphi$ on its maximal interval, which we denote by $[0,\omega)$, where $0 < \omega \leqslant \infty$. (We recall that there is no assurance that such a solution is unique.) Also let $\omega_R = \min\{\omega, R\}$. We claim that

$$\|z_t - x_{t+\sigma}\|_X \leqslant e^{B't}\|\varphi - x_\sigma\|_X < e^{B'R}K_1' \tag{5.34}$$

whenever $0 \leqslant t < \omega_R$. To prove this, let

$$t_0 = \sup\{t \in [0,\omega_R) \mid \|z_{t_1} - x_{t_1+\sigma}\|_X < e^{B'R}K_1' \quad \text{for every } t_1 \in [0,t]\},$$

noting that $\|z_0 - x_\sigma\|_X = \|\varphi - x_\sigma\|_X < e^{B'R}K_1'$. Then if $0 \leqslant t < t_0$ we have by (H1′) that

$$\begin{aligned}|z(t_1) - x(t_1+\sigma)| &\leqslant \|\varphi - x_\sigma\|_X + \int_0^{t_1} |f(z_s) - f(x_{s+\sigma})|\,ds \\ &\leqslant \|\varphi - x_\sigma\|_X + B'\int_0^{t_1} \|z_s - x_{s+\sigma}\|_X\,ds \\ &\leqslant \|\varphi - x_\sigma\|_X + B'\int_0^{t} \|z_s - x_{s+\sigma}\|_X\,ds, \end{aligned} \tag{5.35}$$

provided that $0 \leqslant t_1 \leqslant t$. Letting

$$\eta(t) = \sup_{s\in[-R,t]} |z(s) - x(s+\sigma)| = \sup_{s\in[0,t]} \|z_s - x_{s+\sigma}\|_X$$

for such t, we have upon taking the supremum of the left-hand side of (5.35) for $-R \leqslant t_1 \leqslant t$ that

$$\eta(t) \leqslant \|\varphi - x_\sigma\|_X + B' \int_0^t \eta(s)\,ds, \quad \text{hence } \eta(t) \leqslant e^{B't}\|\varphi - x_\sigma\|_X < e^{B'R} K_1'$$

by Gronwall's inequality. If $t_0 < \omega_R$ the above inequality is valid for $0 \leqslant t \leqslant t_0$, however, this contradicts the choice of t_0. Thus $t_0 = \omega_R$ and our claim, that (5.34) holds whenever $0 \leqslant t < \omega_R$, follows directly. From this we see further that if $0 \leqslant t < \omega_R$ then

$$\left|\dot z(t) - \dot x(t+\sigma)\right| = \left|f(z_t) - f(x_{t+\sigma})\right| \leqslant B'\|z_t - x_{t+\sigma}\|_X \leqslant B' e^{B't}\|\varphi - x_\sigma\|_X < B' e^{B'R} K_1' \tag{5.36}$$

and thus the limit $\lim_{t\to\omega_R} z(t) = z(\omega_R)$ exists. It follows that $\omega_R < \omega$, that is, $R < \omega$. Moreover, from (5.34) and (5.36) we conclude that

$$\|z_R - x_{R+\sigma}\|_Y \leqslant K_4' \|\varphi - x_\sigma\|_X \leqslant K_1' K_4' \leqslant K_1, \qquad K_4' = \max\{1, B'\} e^{B'R},$$

where the above equality serves as the definition of K_4'. Thus from Theorem 1.1, and with any μ as in the statement of that result, we have

$$\begin{aligned} &\|z_t - x_{t+\sigma+\theta}\|_Y \leqslant K_3(\mu) e^{-\mu(t-R)} \|z_R - x_{R+\sigma}\|_Y \leqslant K_3(\mu) K_4' e^{-\mu(t-R)} \|\varphi - x_\sigma\|_X, \\ &|\theta| \leqslant K_2 \|z_R - x_{R+\sigma}\|_Y \leqslant K_2 K_4' \|\varphi - x_\sigma\|_X, \end{aligned} \tag{5.37}$$

for $t \geqslant R$. Also, for $0 \leqslant t \leqslant R$ we have that

$$\begin{aligned} \|z_t - x_{t+\sigma+\theta}\|_X &\leqslant \|z_t - x_{t+\sigma}\|_X + \|x_{t+\sigma} - x_{t+\sigma+\theta}\|_X \leqslant e^{B't}\|\varphi - x_\sigma\|_X + K_4|\theta| \\ &\leqslant \left(e^{B't} + K_2 K_4 K_4'\right)\|\varphi - x_\sigma\|_X \leqslant \left(e^{B'R} + K_2 K_4 K_4'\right) e^{-\mu(t-R)} \|\varphi - x_\sigma\|_X \end{aligned} \tag{5.38}$$

from (5.34) and (5.37), and where K_4 is as in (5.4). Thus from (5.37) and (5.38), we see that the desired conclusions (1.16) of the corollary hold with $K_2' = K_2 K_4'$ and $K_3'(\mu) = \max\{K_3(\mu) K_4' e^{\mu R}, (e^{B'R} + K_2 K_4 K_4') e^{\mu R}\}$. □

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