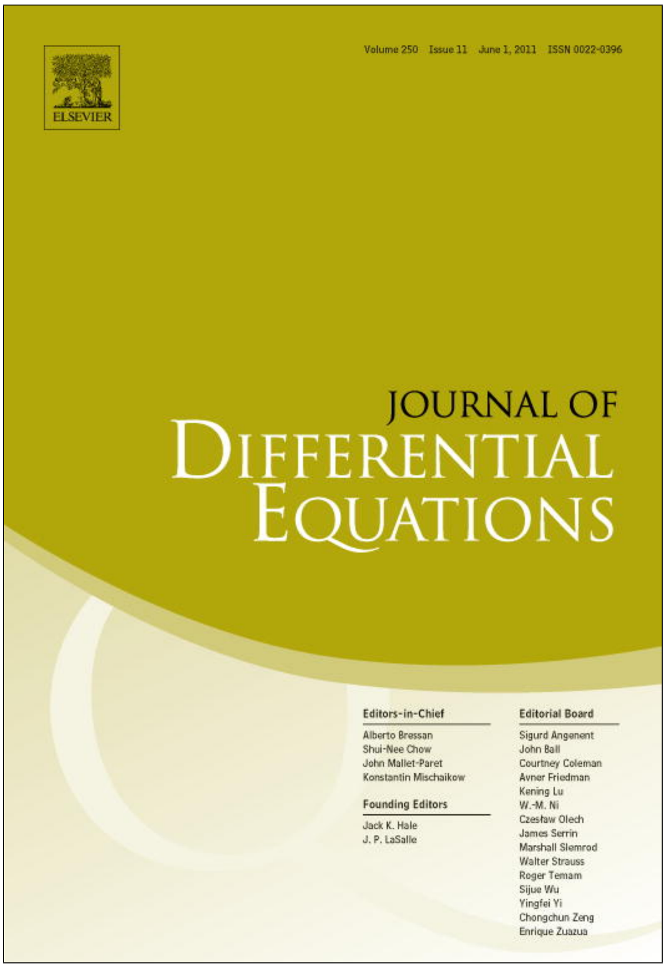
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Superstability and rigorous asymptotics in singularly perturbed state-dependent delay-differential equations

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ABSTRACT

We study the singularly perturbed state-dependent delay-differential equation

$$\varepsilon\dot{x}(t) = -x(t) - kx(t-r), \quad r = r\big(x(t)\big) = 1 + x(t), \qquad (*)$$

which is a special case of the equation

$$\varepsilon\dot{x}(t) = g\big(x(t), x(t-r)\big), \quad r = r\big(x(t)\big).$$

One knows that for every sufficiently small $\varepsilon > 0$, Eq. $(*)$ possesses at least one so-called slowly oscillating periodic solution, and moreover, the graph of every such solution approaches a specific sawtooth-like shape as $\varepsilon \to 0$. In this paper we obtain higher-order asymptotics of the sawtooth, including the location of the minimum and maximum of the solution with the form of the solution near these turning points, and as well an asymptotic formula for the period. Using these and other asymptotic formulas, we further show that the solution enjoys the property of superstability, namely, the nontrivial characteristic multipliers are of size $O(\varepsilon)$ for small ε. This stability property implies that this solution is unique among all slowly oscillating periodic solutions, again for small ε.

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1. Introduction

In this paper we study the state-dependent delay-differential equation

$$\varepsilon\dot{x}(t)=-x(t)-kx(t-r),\quad r=r\big(x(t)\big)=1+x(t),\tag{1.1}$$

where

$$k>1\tag{1.2}$$

is a given constant. In this equation the delay r varies with time as a function of the solution $x(t)$, and we consider only solutions for which $r(x(t))\geqslant 0$, that is, $x(t)\geqslant -1$. We always take $\varepsilon>0$, and generally ε is small, so Eq. (1.1) is in singular perturbation form.

Differential equations with variable delays, and more specifically with state-dependent delays, have been studied for at least 45 years. They often appear in an applied framework, although more recently they have been studied from a theoretical perspective. Among the earliest works are those by Driver [15–18], Driver and Norris [19], and Cooke [11], in the 1960's; and by Winston [58,59], Nussbaum [47], and Alt [2,3], in the 1970's. Numerous models have arisen in the context of biology; see, for example, [1,5,7,22,32–35,38], and [40]. See also [8,10], and [37], for models in economics, and as well [46] for a model of crystal growth.

Although many of the state-dependent problems can be put in the general delay-equation frameworks of [14] and [24], many of the results there (for example, stability results and existence of invariant manifolds) do not directly apply. Much recent work has been aimed at addressing this problem, and establishing a rigorous foundation for the theory of state-dependent delay-differential equations. In this direction we mention the recent foundational work of Walther [51], Hartung, Krisztin, Walther, and Wu [25], and Krisztin [28,29]. Many other theoretical works are to be found, for example, [4,6,9, 10,12,13,23,26,30,31,36,39,49,50,52–56], and [57].

Our analysis in the present paper continues our earlier work [41–43], and [45] on a class of equations of the form

$$\varepsilon\dot{x}(t)=g\big(x(t),x(t-r)\big),\quad r=r\big(x(t)\big).\tag{1.3}$$

In particular, in [43] we obtained general results on the limiting shape of so-called slowly oscillating periodic solutions as $\varepsilon\to 0$. These results are, in a sense, "zeroth-order" results, in that given a sequence $x_i(t)$ of slowly oscillating periodic solutions with corresponding parameter values $\varepsilon_i\to 0$ as $i\to\infty$, the limiting set $\Gamma\subseteq\mathbf{R}^2$ of the graphs of the solutions is determined in a very explicit form. Here Γ is, roughly, the limit $\Gamma_i\to\Gamma$ of the sequence of sets

$$\Gamma_i=\big\{(t,x)\in\mathbf{R}^2\ \big|\ x=x_i(t)\text{ for }t\in\mathbf{R}\big\},\tag{1.4}$$

the limit being taken in a sense similar to Hausdorff convergence. In general the limiting set Γ is not the graph of a function as it can contain vertical line segments. A negative feedback condition was assumed for Eq. (1.3), as well as monotonicity of g in the delay variable $x(t-r)$, and conditions on g yielding instability of the origin and boundedness of solutions. It was also assumed that $r(x(t))\geqslant 0$. A particular class of such equations are those of the form

$$\varepsilon\dot{x}(t)=-x(t)+f\big(x(t-r)\big),\quad r=r\big(x(t)\big),\tag{1.5}$$

where $f(0)=0$, $f'(0)<-1$, and $f'(x)<0$ for all x, along with a boundedness condition on f, and where also $r(0)>0$. Quite generally, under such conditions one has the existence of at least one slowly oscillating periodic solution for every ε in the range $0<\varepsilon<\varepsilon_0$ where ε_0 is the Hopf bifurcation point for slowly oscillating periodic solutions.

The singular parameter ε is very natural in such problems. In particular, taking the singular limit $\varepsilon\to 0$ corresponds, after rescaling time, to a regular problem with a large delay. The advantage of introducing ε is that it provides an "organizing center" at $\varepsilon = 0$ near which precise, quantitative, and global results can be obtained using analytical methods.

In the present paper we refine some of the zeroth-order results by obtaining a "higher-order" description of how the above solutions $x_i(t)$ converge. Our results are expressed as an asymptotic description of these solutions in terms of the parameter ε. The asymptotics are rather subtle, involving logarithmic terms as well as powers of ε. A central result obtained from this analysis is the stability and uniqueness of slowly oscillating periodic solutions of Eq. (1.1) for sufficiently small ε. In fact, we prove a result on so-called **superstability** of these solutions, namely that all nontrivial characteristic (Floquet) multipliers λ satisfy an estimate $|\lambda| = O(\varepsilon)$ as $\varepsilon \to 0$. The stability of all slowly oscillating periodic solutions for small ε in turn implies their uniqueness for small ε, as we show by means of a degree theory argument.

In both the earlier work described above, and in the present paper, only problems with a single delay r are considered. In fact, while the theory of single-delay problems such as (1.3) is now highly developed, there is relatively little known of a global nature about multiple-delay equations such as

$$\varepsilon\dot{x}(t) = g\big(x(t), x(t - r_1), \ldots, x(t - r_m)\big), \quad r_j = r_j\big(x(t)\big), \tag{1.6}$$

and the study of such equations (and also systems) is a significant challenge. We believe that the techniques developed in the present paper are robust enough to allow a much more detailed analysis of problems such as (1.6) than has been possible before. We propose that extending and adapting our techniques to (1.6) and to more general problems is the basis for a challenging and long-term program.

Our purpose in restricting to the very special equation (1.1) is to keep the analysis simple and to provide as much transparency and insight as possible into the underlying dynamical mechanisms. Here we do not consider systems of the form (1.3), or even those of the form (1.5), with general nonlinearities, although this is planned for a future study. Indeed, we believe that the analysis of the model equation (1.1) in the present paper is fundamentally the same and just as intricate as the analysis of a broader class of systems of the form (1.3). Thus the analysis of Eq. (1.1) herein should serve as a template for the study of the more general problem (1.3), and even for the study of multiple-delay problems such as (1.6).

Note finally that one could consider the system

$$\varepsilon\dot{x}(t) = -k_1 x(t) - k_2 x(t - r), \quad r = r\big(x(t)\big) = c_1 + c_2 x(t), \tag{1.7}$$

where $0 < k_1 < k_2$, which is a negative feedback condition along with an instability condition on the origin, and where $c_1, c_2 > 0$. In fact, Eq. (1.7) is easily transformed into Eq. (1.1) by rescaling x, t, and ε. Namely, upon introducing new variables $\bar{x} = c_1^{-1}c_2 x$ and $\bar{t} = c_1^{-1}t$ with $\bar{\varepsilon} = \varepsilon(k_1 c_1)^{-1}$ in (1.7), one obtains (1.1) with $k = k_1^{-1}k_2$.

2. The setting and main results

Consider now Eq. (1.1), where (1.2) is assumed. It is known [42,45] that for every sufficiently small ε, there exists a so-called **slowly oscillating periodic solution (SOPS)** of Eq. (1.1), namely a solution $x(t)$ satisfying the following properties for some discrete set $\{z_n\}_{n=-\infty}^{\infty}$:

$$\begin{aligned} &\big\{t \in \mathbf{R} \bigm| x(t) = 0\big\} = \{z_n\}_{n=-\infty}^{\infty}, \quad \text{where } z_{n+1} - z_n > r(0) = 1 \quad \text{for every } n, \quad \text{and} \\ &x(t + p) = x(t) \quad \text{for all } t, \text{ where } p = z_{n+2} - z_n \quad \text{for every } n. \end{aligned} \tag{2.1}$$

It is easily seen from the differential equation (1.1) that $\dot{x}(z_n) \neq 0$ for every n, so all zeros of $x(t)$ are simple. Thus the solution $x(t)$ changes sign infinitely often, with sign changes separated by a distance greater than the delay $r(0) = 1$ at zero. Also, $x(t)$ repeats periodically after any two consecutive zeros.

It is also known that any SOPS of Eq. (1.1) satisfies several additional properties, in particular

$$\begin{aligned} &-1 < x(t) < k \quad \text{for every } t \in \mathbf{R}, \\ &\bigl\{t \in (z_n, z_{n+1}) \bigm| \dot{x}(t) = 0\bigr\} = \{q_n\} \quad \text{is a singleton for every } n. \end{aligned} \tag{2.2}$$

Thus an SOPS $x(t)$ has exactly one local maximum and one local minimum per period, and these are the global maximum and minimum. In each interval $[q_n, q_{n+1}]$ the solution moves strictly monotonically between these two extrema, and the range of the solution is contained in the open interval $(-1, k)$. Note that this range is contained in the region where $r(x) > 0$. We shall make use of these facts in our analysis.

One easily sees, by repeatedly differentiating (1.1), that the SOPS $x(t)$ is C^∞-smooth. It is an open question, however, as to whether or not $x(t)$ is analytic in t. In this paper we shall use **smooth** to mean C^∞-smooth unless stated otherwise.

We remark that more or less identical results on the existence of SOPS's hold for the more general class of equations (1.3) under appropriate assumptions on the nonlinearity, including negative feedback, monotonicity of g in the delay term, instability at the origin, and a boundedness condition on g, with $r(0) > 0$.

It is not difficult to show directly that the monotonicity property

$$\frac{d}{dt}\bigl(t - r\bigl(x(t)\bigr)\bigr) > 0, \quad \text{equivalently} \quad \dot{x}(t) < 1, \quad \text{for every } t \in \mathbf{R}, \tag{2.3}$$

holds for the so-called **historical time** $t - r(x(t))$ for every SOPS, and indeed, this was shown in [41] for a very general class of equations of the form (1.3). For Eq. (1.1), considered in the present paper, one proves (2.3) by simply observing that whenever $\frac{d}{dt}(t - r(x(t))) = 0$, that is, whenever $\dot{x}(t) = 1$, then (1.1) implies that $\ddot{x}(t) = -\varepsilon^{-1}\dot{x}(t) < 0$ and thus $\frac{d^2}{dt^2}(t - r(x(t))) = -\ddot{x}(t) > 0$. Therefore every critical point of the function $t - r(x(t))$ is a nondegenerate local minimum, and it follows from the fact that $t - r(x(t)) \to -\infty$ as $t \to -\infty$, that no such critical points can exist. (Note that this proof is valid for any solution $x(t)$ of (1.1) which is defined for all $t \in \mathbf{R}$ and bounded as $t \to -\infty$.)

In this paper we shall observe that (2.3) is a consequence of our asymptotic analysis for small ε. Indeed, this follows from Corollaries 5.3 and 6.4, from Proposition 7.5 (which concern Intervals II, III, and IV, respectively), and from Corollary 4.4 which ensures that $\dot{x}(t) < 0$ throughout Interval I. (These four time intervals cover the entire period of $x(t)$, and will be defined below.) We believe these techniques may be useful in proving analogs of (2.3) for other problems, such as Eq. (1.6) with multiple delays, where the proof from [41] is not applicable.

Let us now recall the zeroth-order results for Eq. (1.1). It is known from [43] that as $\varepsilon \to 0$, the asymptotic shape of any SOPS of (1.1) has the form of a sawtooth in the following sense. Define a sawtooth-shaped set $\Gamma \subseteq \mathbf{R}^2$ by

$$\begin{aligned} &\Gamma = \bigcup_{n=-\infty}^{\infty} (V_n \cup S_n), \\ &V_n = \bigl\{(t, x) \in \mathbf{R}^2 \bigm| t = (k+1)n \text{ and } -1 \leqslant x \leqslant k\bigr\}, \\ &S_n = \bigl\{(t, x) \in \mathbf{R}^2 \bigm| x = t - 1 - (k+1)n \text{ and } (k+1)n \leqslant t \leqslant (k+1)(n+1)\bigr\}. \end{aligned}$$

The sets V_n are the vertical parts of the sawtooth, while the S_n are the sloping portions. Now assume we have a sequence $x_i(t)$ of SOPS's of Eq. (1.1) for $\varepsilon = \varepsilon_i$, where $\varepsilon_i \to 0$ is a sequence of positive quantities. Let p_i denote the corresponding periods of these solutions. Also assume, by a time translation, that

$$x_i(0) = 0 \quad \text{and} \quad \dot{x}_i(0) < 0 \quad \text{for every } i. \tag{2.4}$$

Let $\Gamma_i \subseteq \mathbf{R}^2$ as in (1.4) denote the graph of $x_i(t)$ in the plane. Then as proved in [43], for every $R > 0$ there exists a compact set $K_R \subseteq \mathbf{R}^2$ containing the R-ball at the origin, that is

$$\left\{(t, x) \in \mathbf{R}^2 \;\middle|\; t^2 + x^2 \leqslant R^2\right\} \subseteq K_R,$$

such that

$$\lim_{i\to\infty} \operatorname{dist}(\Gamma_i \cap K_R, \Gamma \cap K_R) = 0. \tag{2.5}$$

Here $\operatorname{dist}(\cdot,\cdot)$ denotes the Hausdorff distance between two sets, and so (2.5) says that the graphs of the solutions $x_i(t)$ converge to the sawtooth set Γ. Recall that in any metric space (X, d) the Hausdorff distance is defined as

$$\operatorname{dist}(A, B) = \max\left\{\max_{a\in A} d(a, B), \max_{b\in B} d(b, A)\right\},$$

$$d(a, B) = \min_{b\in B} d(a, b), \qquad d(b, A) = \min_{a\in A} d(b, a),$$

for any nonempty compact sets $A, B \subseteq X$.

The above set Γ is known as the **limiting profile** of the sequence $x_i(t)$ of SOPS's. The vertical portions V_n of Γ arise from transition layers in the solution, namely where $|\dot{x}(t)|$ is of size ε^{-1}, while the sloping portions S_n occur where $|\dot{x}(t)|$ remains bounded as $\varepsilon \to 0$. The vertices or turning points $((k+1)n, -1)$, $((k+1)n, k) \in \Gamma$, where the vertical and sloping portions meet, are of particular interest in our analysis below.

Let us note that the limiting value

$$\lim_{i\to\infty} p_i = k + 1 \tag{2.6}$$

of the period follows directly from the convergence result (2.5). Also observe, for future use, that (2.5) implies the limit

$$\lim_{i\to\infty} x_i(t) = t + k \quad \text{uniformly for } t \text{ in compact subsets of } (-k-1, 0). \tag{2.7}$$

This corresponds to the convergence of a portion of the graph Γ_i to the segment $S_{-1} \subseteq \Gamma$ in the limiting profile.

We regard the above convergence results (2.5), (2.6), and (2.7) as zeroth-order results, in that they give no information on the rate of convergence, or on the detailed shape of Γ_i such as the widths, in terms of ε_i, of the turning points near the maxima and minima of $x_i(t)$. Our object in this paper is to obtain a more precise asymptotic description of the shape of SOPS's in terms of the small parameter ε, thereby refining the result (2.5) on Hausdorff convergence. We shall then use these refined estimates to study the linear variational equation of such SOPS's, and thereby obtain results on the associated characteristic (Floquet) multipliers. In particular, the above-mentioned results on superstability and uniqueness of SOPS's will follow from these estimates.

Let us precisely state our main results. The first theorem concerns uniqueness and stability of SOPS's for small ε. The term superstability refers to the estimate (2.8) below on the nontrivial characteristic multipliers.

Theorem A. *Fix k as in* (1.2)*. Then there exists $\varepsilon_s > 0$ such that if $0 < \varepsilon < \varepsilon_s$ the following properties hold.*

(1) *Eq.* (1.1) *has a unique slowly oscillating periodic solution $x_\varepsilon(t)$.*

(2) *The trivial characteristic multiplier* $\lambda = 1$ *of the periodic solution* $x_\varepsilon(t)$ *has simple algebraic multiplicity, and* $|\lambda| < 1$ *for every nontrivial characteristic multiplier, and so this solution is asymptotically orbitally stable with asymptotic phase. In fact, there exists* $C_1 > 0$ *such that*

$$|\lambda| \leqslant \varepsilon C_1 \tag{2.8}$$

for every nontrivial characteristic multiplier of this solution.

Remark. In the above theorem, ε_s is a sufficiently small quantity. It is an open question as to whether or not the conclusions of this theorem hold for every ε in the range $0 < \varepsilon < \varepsilon_0$, where ε_0 is the location of the Hopf bifurcation of SOPS's from the origin.

A precise description of asymptotic orbital stability with asymptotic phase, in the context of problems with state-dependent delays, is given in [44], which is a companion paper to the present paper, and which can be read independently of the present paper. It is proved there for a general class of state-dependent problems, that the linear stability condition $|\lambda| < 1$ on nontrivial characteristic multipliers implies asymptotic orbital stability with asymptotic phase. In fact, such stability issues are a matter of some delicacy as it is not immediately clear what is the appropriate phase space for many state-dependent problems. However, the recent papers of Walther [51], and Hartung, Krisztin, Walther, and Wu [25], have addressed and clarified many of these issues, and their theory plays a significant role here.

The next result describes some (but by no means all) of the asymptotic results we obtain for SOPS's. Taking ε small, we may assume the conclusions of Theorem A hold, and so the SOPS is unique. We denote this solution by $x_\varepsilon(t)$, and more generally, we use a subscript ε to denote various quantities associated to this solution, such as the zeros $z_n = z_{n,\varepsilon}$ in (2.1) and critical points $q_n = q_{n,\varepsilon}$ in (2.2). Let us also define three quantities Q_+, Q_0, and Q_- by

$$Q_+ = \frac{\log(k+1)}{k-1}, \qquad Q_0 = \frac{\log k}{k-1}, \qquad Q_- = \frac{\log(k-1)}{k-1}. \tag{2.9}$$

These quantities will be regular players in what follows.

Theorem B. *Fix k as in* (1.2) *and let* $x_\varepsilon(t)$ *denote the unique slowly oscillating periodic solution of Eq.* (1.1) *guaranteed by Theorem* A *for small* ε. *Assume that each solution is normalized to satisfy* $x_\varepsilon(0) = 0$ *and* $\dot{x}_\varepsilon(0) < 0$, *and enumerate the zeros so that* $z_0 = z_{0,\varepsilon} = 0$ *in the notation* (2.1)*. Then the location of the first minimum* $q_0 = q_{0,\varepsilon} \in (0, z_{1,\varepsilon})$ *to the right of* $t = 0$ *and the value of the solution there, the location of the first maximum* $q_{-1} = q_{-1,\varepsilon} \in (z_{-1,\varepsilon}, 0)$ *to the left of* $t = 0$ *and the value of the solution there, and the period* $p = p_\varepsilon$ *of the solution, are as given in Table* 1.

The above results follow from precise asymptotic estimates on any SOPS $x(t)$ for small ε. These estimates are summarized in Table 2, as ansätze in various intervals which comprise one period of the solution.

It is worth noting that the asymptotic forms of an SOPS near its minimum and maximum are very different. Roughly speaking, the characteristic widths of the vertices in the graph of $x(t)$ at the minimum and maximum are of orders $O(\varepsilon^2|\log\varepsilon|)$ and $O(\varepsilon|\log\varepsilon|)$, respectively, and so the minimum vertex is "sharper" than the maximum vertex. As is pointed out in Section 5, the solution near the minimum of $x(t)$ has a dynamical interpretation as a turning point near a normally hyperbolic invariant manifold for an ordinary differential equation, in the spirit of Fenichel [20,21], although with a time scaling of ε^2 rather than ε; see the discussion preceding Lemma 5.1. On the other hand, the dynamics near the maximum of $x(t)$ are essentially those of a regular point, but rescaled by a factor of ε; see the ansatz for Interval IV in Table 2 and the limiting formula (2.10) for $\zeta_*(\tau)$ below.

Table 1
The minimum, the maximum, and the period.

$q_{0,\varepsilon} = \varepsilon(Q_+ - Q_0) + o(\varepsilon)$
$\min_{t\in\mathbf{R}} x_\varepsilon(t) = x_\varepsilon(q_{0,\varepsilon}) = -1 + \varepsilon(Q_0 - Q_-) + o(\varepsilon)$
$q_{-1,\varepsilon} = -\left(\frac{\varepsilon\lvert\log\varepsilon\rvert}{k-1}\right) - 2\varepsilon Q_- + o(\varepsilon)$
$\max_{t\in\mathbf{R}} x_\varepsilon(t) = x_\varepsilon(q_{-1,\varepsilon}) = k - \varepsilon\left(\frac{k}{k-1}\right)\bigl(1 + (k-2)Q_0 - (k-1)Q_-\bigr) + o(\varepsilon)$
$p_\varepsilon = k + 1 + \frac{\varepsilon\lvert\log\varepsilon\rvert}{k-1} + \varepsilon\bigl(-1 + Q_+ - kQ_0 + (k+2)Q_-\bigr) + o(\varepsilon)$

To contrast more clearly the form of the solution at its minimum and maximum, we first mention that the minimum and maximum, $t = q_0$ and $t = q_1$, occur in Intervals II and IV, respectively, as outlined in Table 2. In particular, differentiating the ansatz for Interval II gives

$$\dot{x}(t) = 1 - \varepsilon\dot{\sigma}(t) - \varepsilon^{-1}e^{-\eta(\varepsilon^{-2}(t-\varepsilon T_3))}\eta'\bigl(\varepsilon^{-2}(t-\varepsilon T_3)\bigr),$$

where it is the case that $\dot{\sigma}(t) > 0$ is bounded and that $\eta(\theta)$ has linear growth in θ with $\eta(0)$ bounded independent of ε, throughout the range under consideration; see (5.12) in Lemma 5.1, and (5.20) and (5.21) in Proposition 5.2. (Here prime $'$ denotes differentiation with respect to the argument θ of η, and not with respect to t. We use the same convention for ζ below.) Thus

$$\dot{x}(t) = 1 - \varepsilon^{-1}e^{-\eta(\theta)}\eta'(\theta) + O(\varepsilon), \quad t = \varepsilon T_3 + \varepsilon^2\theta,$$

and so $\dot{x}(t)$ is negative and of order $O(\varepsilon^{-1})$ when $\theta = 0$, while $\dot{x}(t)$ is positive and of order $O(1)$ when θ is of order $O(\lvert\log\varepsilon\rvert)$ so that $\varepsilon^{-1}e^{-\eta(\theta)}\eta'(\theta) \ll 1$. In essence, the vertex of the graph of $x(t)$ near its minimum has width $O(\varepsilon^2\lvert\log\varepsilon\rvert)$. By contrast, for the maximum, we differentiate the ansatz for Interval IV to give

$$\dot{x}(t) = -\bigl(e^{(k-1)\tau}\zeta(\tau)\bigr)', \quad t = k + 1 + \varepsilon\tau.$$

As it turns out, the function $\zeta(\tau)$ is C^1 close to the limiting function

$$\zeta_*(\tau) = \frac{e^{(k-1)T_{5,*}}}{(k-1)^2} - \frac{ke^{-(k-1)\tau}}{(k-1)^2}\bigl(1 + (k-1)(\tau - Q_+ + 2Q_0 - Q_-)\bigr) \tag{2.10}$$

as $\varepsilon \to 0$, where $T_{5,*}$ is a certain constant; see Eq. (7.15). Thus $\dot{x}(t)$ is close to

$$-\bigl(e^{(k-1)\tau}\zeta_*(\tau)\bigr)' = \frac{k - e^{(k-1)(\tau+T_{5,*})}}{k-1}.$$

It follows that when $\tau = -T_{5,*}$ then $\dot{x}(t)$ is near $+1$, while if $\tau = \lvert\log\varepsilon\rvert(k-1)^{-1}$ then $\dot{x}(t)$ is negative and of order $O(\varepsilon^{-1})$. One concludes that the vertex of the graph of $x(t)$ near its maximum has width $O(\varepsilon\lvert\log\varepsilon\rvert)$.

Theorem A is proved by first obtaining the asymptotic estimates in Table 2 for any SOPS $x(t)$ for small ε. (In fact, the results of Theorem B are among these estimates, and we shall prove Theorem B before we prove Theorem A.) The estimates on $x(t)$ are then used to obtain further estimates on solutions $y(t)$ of the linear variational equation

$$\varepsilon\dot{y}(t) = a(t)y(t) - ky\bigl(t - r(x(t))\bigr), \quad a(t) = -1 + k\dot{x}\bigl(t - r(x(t))\bigr), \tag{2.11}$$

about $x(t)$. Observe that the variational equation (2.11) for a specific solution $x(t)$ of Eq. (1.1) is obtained by replacing $x(t)$ with $x(t) + \delta y(t)$, and correspondingly replacing $x(t - r(x(t)))$ with $x(t - r(x(t) + \delta y(t))) + \delta y(t - r(x(t) + \delta y(t)))$, then differentiating the resulting equation with respect to δ and setting $\delta = 0$. Generally, the desired estimates on $x(t)$ and $y(t)$ will be obtained straightforwardly by solving the relevant differential equation around one period in a finite number of steps. With this we obtain precise information about the monodromy operator M, which is the solution operator for the linear equation (2.11) around one period.

Note here that property (2.3) implies the estimate

$$a(t) < k - 1 \quad \text{for every } t \in \mathbf{R}, \tag{2.12}$$

for the coefficient in (2.11).

Quite generally, for a given p-periodic solution $x(t)$ of (1.1), we may define the operator M for Eq. (2.11) by prescribing an initial condition

$$y(t) = \psi(t) \quad \text{for } t \in J \tag{2.13}$$

on a suitable compact interval J, and then defining $M\psi$ to be the function

$$(M\psi)(t) = y(t + p) \quad \text{for } t \in J. \tag{2.14}$$

Here $\psi, M\psi \in X$ where $X = C(J)$ is the space of continuous functions on J, and where the interval J must be such that the solution to (2.11), (2.13) is defined for forward time. Specifically, $J = [t_-, t_+]$ must have the property that $t - r(x(t)) \geqslant t_-$ holds whenever $t \geqslant t_+$. If this is so, then the monodromy operator $M : X \to X$ is a bounded linear operator, and additionally it is compact if $t_+ - t_- \leqslant p$. Moreover, M always has the trivial eigenvector

$$M\psi_0 = \psi_0, \quad \psi_0(t) = \dot{x}(t) \quad \text{for } t \in J, \tag{2.15}$$

corresponding to the p-periodic solution $y(t) = \dot{x}(t)$ of Eq. (2.11).

In our analysis leading to the proof of Theorem A, we show that for a suitably chosen interval $J = J_\varepsilon$, the monodromy operator $M = M_\varepsilon$ has the form $M_\varepsilon = M_{0,\varepsilon} + M_{1,\varepsilon}$, where $M_{1,\varepsilon}$ is a rank-one operator with the trivial eigenfunction $M_{1,\varepsilon}\psi_{0,\varepsilon} = \psi_{0,\varepsilon}$, where the operator $M_{0,\varepsilon}$ has small norm, and where the norm of $M_{1,\varepsilon}$ is uniformly bounded in ε. To be precise, we show that $\|M_{0,\varepsilon}\| \leqslant \varepsilon C_1$ and $\|M_{1,\varepsilon}\| \leqslant C_1$ for some C_1 independent of small ε, where $\|\cdot\|$ denotes the operator norm for $C(J_\varepsilon)$, and that additionally $M_{0,\varepsilon}M_{1,\varepsilon} = 0$. The interval J_ε must be chosen carefully, and has the form $J_\varepsilon = [-\varepsilon T_1, \varepsilon T_{2,\varepsilon}]$, for some fixed $T_1 > 0$, and for $T_{2,\varepsilon} > 0$ which approaches a finite positive limit as $\varepsilon \to 0$. Establishing these facts requires delicate and precise estimates on the periodic solution $x(t)$ and on solutions $y(t)$ of the variational equation.

Let us outline the structure of the paper. In Section 3 we describe the four fundamental time intervals, denoted I, II, III, and IV, on which the SOPS $x(t)$ is to be analyzed. These intervals cover one period of $x(t)$. We also describe the form (ansatz) of the solution in each interval. We then treat Intervals I, II, III, and IV in Sections 4, 5, 6, and 7, respectively, obtaining precise and rigorous bounds for the solution. With the aid of these bounds, in Section 8 we obtain asymptotic forms for the maximum, minimum, and period of $x(t)$, thereby proving Theorem B. In Section 9 we study the linear variational equation (2.11) about $x(t)$, and obtain bounds for the solution $y(t)$ of this equation successively in each of the four fundamental intervals. With the aid of these bounds we prove Part 2 of Theorem A, namely superstability, also in Section 9. Finally, in Section 10 we prove Part 1 of Theorem A, namely the uniqueness claims about $x(t)$. This is done with the aid of a degree argument and the stability results of the companion paper [44].

Table 2
Ansätze for the solution in various intervals.

Interval I: $-\varepsilon T_1 \leqslant t \leqslant \varepsilon T_2$,
$x(t) = \varphi(\tau), \qquad t = \varepsilon\tau.$
Interval II: $\varepsilon T_3 \leqslant t \leqslant k + 1 - \varepsilon T_4$,
$x(t) = t - 1 - \varepsilon\sigma(t) + \varepsilon e^{-\eta(\varepsilon^{-2}(t-\varepsilon T_3))}.$
Interval III: $k + 1 - \varepsilon T_4 \leqslant t \leqslant k + 1 - \varepsilon T_5$,
$x(t) = k + \varepsilon(\tau - T_3) - \varepsilon^2\alpha(\tau), \qquad t = k + 1 + \varepsilon\tau.$
Interval IV: $k + 1 - \varepsilon T_5 \leqslant t \leqslant k + 1 + \dfrac{\varepsilon\lvert\log\varepsilon\rvert}{k-1} + \varepsilon T_6$,
$x(t) = k - \varepsilon e^{(k-1)\tau}\zeta(\tau), \qquad t = k + 1 + \varepsilon\tau.$

3. Outline of the proofs

In this section we describe the proofs of our results in more detail. Let us first remark that we do not know *a priori* that for any ε there is a unique SOPS $x_\varepsilon(t)$, and so we may not regard such a solution as a well-defined function of ε. Thus in what follows, we shall instead consider a general sequence $x_i(t)$ of SOPS's with positive $\varepsilon = \varepsilon_i \to 0$, as $i \to \infty$. We shall establish various estimates involving these solutions, and will need to keep track of how the estimates depend on ε_i. We thus adopt the following conventions, which generally will be used without further comment.

Standing Assumptions. For the remainder of this paper, unless stated otherwise, we assume that k is fixed as in (1.2), that we have a sequence $x_i(t)$ of SOPS's of Eq. (1.1) for positive $\varepsilon = \varepsilon_i \to 0$, and that these solutions have been normalized by a time translation so as to satisfy (2.4).

Notation. We let T_m, for $m \geqslant 1$, denote certain times which arise as we estimate the solution $x_i(t)$, although with a possible scaling of the time. Similarly, H_m will denote certain values in the range of the solution, in x-space, again with an appropriate scaling of x. The index m will be such that H_m is associated to T_m.

In some cases the quantities $T_m = T_{m,i}$ and $H_m = H_{m,i}$ will depend on the index i of the sequence $x_i(t)$ of SOPS's. In such a case the sequence $T_{m,i}$ or $H_{m,i}$ will be bounded in i as $i \to \infty$. Often we shall omit the index i, writing $x(t)$, ε, T_m, and H_m, for $x_i(t)$, ε_i, $T_{m,i}$, and $H_{m,i}$, and so forth, to keep the notation simple.

We shall also let C_m denote various constants appearing in our analysis, usually as uniform estimates or bounds on the solution or on related quantities. That is, the constants C_m will be independent of the index i, and typically the associated estimates will hold for large i. The indices m here have no special meaning, being assigned sequentially as new constants are needed.

Finally, we shall let a dot $\dot{}$ denote the derivative with respect to the time variable t. We use a prime $'$ to denote derivatives with respect to all other variables, including scaled time variables such as $\tau = \varepsilon^{-1}t$ as for example in Interval I below.

Generally, our strategy is to obtain rigorous asymptotic expressions for the solution $x(t)$ in a set of intervals which covers one period of the solution. In each interval an ansatz for the solution will be given, along with precise estimates on the solution. As a rule, different scalings of t and x will be used in different intervals. The estimates on the solution in each interval will be obtained from the form of the solution in a previous interval, using the delay structure of the differential equation. The initial estimate, to start the process, is the convergence result (2.7). Table 2 provides a list of the intervals and the associated ansätze for the solution.

We shall see from their definitions that $T_3 < T_2$, and so there is an overlap between Intervals I and II. This overlap is small, and is only needed for technical reasons. Also, the point $t = p$, where $p = p_i$ is the period of $x_i(t)$, is contained in Interval IV. Thus these four intervals cover a full period of the solution.

Let us outline some of the basic properties of the functions appearing in the above ansätze. These properties will be established in the sections that follows. The functions $\varphi(\tau)$, $\sigma(t)$, and $\zeta(\tau)$, which depend on the index i, and the derivatives $\varphi'(\tau)$, $\dot\sigma(t)$, and $\zeta'(\tau)$ with respect to their arguments, are all uniformly bounded in their respective domains, and also are bounded with respect to i. That is, they enjoy bounds which are independent of τ or t, and are also independent of i. The derivative $\eta'(\theta)$ of the function $\eta(\theta)$ with respect to its argument $\theta = \varepsilon^{-2}(t - \varepsilon T_3)$ also enjoys this boundedness property, and in fact it is bounded between two positive constants. Thus $\eta(\theta)$ has uniform linear growth in θ. However, the function $\eta(\theta)$ itself is not uniformly bounded, as its domain $0 \leqslant \theta \leqslant \varepsilon^{-2}(k+1-\varepsilon(T_3+T_4))$ is an interval of size $O(\varepsilon^{-2})$. The initial value $\eta(0) = \eta_i(0)$ is bounded with respect to i, but $\eta(\theta)$ becomes arbitrarily large as θ becomes large. (Note that this growth implies that the exponential term in the formula of $x(t)$ for Interval II becomes extremely small.) Finally, although $\alpha(\tau)$ and its derivative $\alpha'(\tau)$ are unbounded, they are of size at most $O(|\log\varepsilon|)$ and $O(\varepsilon^{-1/2})$, respectively. In fact, $\alpha(\tau)$ has roughly the same character as $\log\tau$, for τ of size $O(\varepsilon^{1/2})$.

As stated above, we obtain new information about the solution in each interval by using information obtained earlier in a previous interval. To describe this approach in more detail, let us write the differential equation (1.1) as

$$\varepsilon\dot x_{\rm c}(t) = -x_{\rm c}(t) - kx_{\rm h}\bigl(t - r\bigl(x_{\rm c}(t)\bigr)\bigr), \tag{3.1}$$

where $x_{\rm h}(\cdot)$ denotes the historical solution from a previous interval, and $x_{\rm c}(\cdot)$ denotes the current solution which is to be analyzed. For example, in Section 6 we study the solution in Interval III by taking the ansatz

$$x_{\rm c}(t) = k + \varepsilon(\tau - T_3) - \varepsilon^2\alpha(\tau) \tag{3.2}$$

from the list above for the current solution. Noting from this that $x_{\rm c}(t) = t - 1 - \varepsilon T_3 - \varepsilon^2\alpha(\varepsilon^{-1}(t - k - 1))$, since $t = k + 1 + \varepsilon\tau$, we see also that

$$\dot x_{\rm c}(t) = 1 - \varepsilon\alpha'(\tau). \tag{3.3}$$

As it will turn out, the historical time $t - r(x_{\rm c}(t))$ belongs to Interval II whenever t is in Interval III. Thus we take the ansatz of Interval II to express the historical solution in (3.1), namely

$$\begin{aligned} x_{\rm h}\bigl(t - r\bigl(x_{\rm c}(t)\bigr)\bigr) &= \bigl(s - 1 - \varepsilon\sigma(s) + \varepsilon e^{-\eta(\varepsilon^{-2}(s-\varepsilon T_3))}\bigr)\big|_{s=t-r(x_{\rm c}(t))} \\ &= \bigl(s - 1 - \varepsilon\sigma(s) + \varepsilon e^{-\eta(\varepsilon^{-2}(s-\varepsilon T_3))}\bigr)\big|_{s=\varepsilon T_3+\varepsilon^2\alpha(\tau)}. \end{aligned} \tag{3.4}$$

We regard the functions σ and η as known, as they occur in the (known) historical solution, while the function α appearing in the current solution is regarded as unknown. Upon substitution of the formulas (3.2), (3.3), and (3.4) into (3.1), one thereby obtains an ordinary differential equation for $\alpha(\tau)$. Analysis of this ordinary differential equation yields an asymptotic description of $\alpha(\tau)$.

We remark that the historical interval is not necessarily the interval immediately preceding the current interval. In particular, in Section 7 we have Interval IV for the current interval, although Interval II is the historical one.

To obtain estimates on solutions, it will be useful to construct sets in the (t,x)-plane to which these solutions are confined. The following technical result, which describes an invariance property of certain sets $U \subseteq \mathbf{R}^2$, will be used to this end. In the notation below, solutions beginning in the set U can only exit U at points where the "exit function" $E(t,x)$ vanishes.

Proposition 3.1. *Suppose that*

$$W_j : O \to \mathbf{R} \quad \textit{for } 1 \leqslant j \leqslant m, \qquad E : O \to \mathbf{R}$$

are C^1 functions defined in an open set $O \subseteq \mathbf{R}^2$ of the (t,x)-plane. Let

$$U = \{(t,x) \in O \mid W_j(t,x) \geqslant 0 \text{ for } 1 \leqslant j \leqslant m \text{ and } E(t,x) \geqslant 0\},$$

and assume that U is a closed subset of $\mathbf{R}^2$ (equivalently, that $\partial U \subseteq O$). Suppose also that

$$f : U \to \mathbf{R}$$

is continuous in (t,x) and locally Lipschitz in x, and also assume that for each j we have that

$$D_t W_j(t,x) > 0 \quad \text{whenever } (t,x) \in U \text{ and } W_j(t,x) = 0,$$

where D_t denotes the total derivative along the vector field

$$D_t W_j(t,x) = \frac{\partial W_j(t,x)}{\partial t} + \frac{\partial W_j(t,x)}{\partial x} f(t,x).$$

Let $(t_0,x_0) \in U$ be such that $E(t_0,x_0) > 0$. Then there exists a solution $x = \psi(t)$ to the initial value problem

$$\dot{x} = f(t,x), \quad x(t_0) = x_0, \tag{3.5}$$

on an interval I, with $(t,\psi(t)) \in U$ for all $t \in I$, and where either

(1) $I = [t_0, \infty)$;
(2) $I = [t_0, \omega)$ *with* $t_0 < \omega < \infty$ *and* $\lim_{t\to\omega} |\psi(t)| = \infty$; *or*
(3) $I = [t_0, \omega]$ *with* $t_0 < \omega < \infty$ *and* $E(\omega, \psi(\omega)) = 0$.

In any case, one has for every j that $W_j(t,\psi(t)) > 0$ for all $t \in I \setminus \{t_0\}$, including the endpoint $t = \omega$ if case (3) holds.

Proof. Extend the function f from its domain U, which is a closed set, to a continuous function $f : \mathbf{R}^2 \to \mathbf{R}$ defined on the plane. Let $\psi(t)$ be a solution to the initial value problem (3.5) on a maximal interval $\widetilde{I}$ to the right of t_0. (As the extended f need not be Lipschitz outside of U, this solution need not be unique. However it is unique up to such time where it may leave U.) The interval $\widetilde{I}$ has the form $\widetilde{I} = [t_0, \widetilde{\omega})$ with $t_0 < \widetilde{\omega} \leqslant \infty$, and with $\lim_{t\to\widetilde{\omega}} |\psi(t)| = \infty$ if $\widetilde{\omega} < \infty$. If there exists a point $t_* \in \widetilde{I}$ for which $(t_*, \psi(t_*)) \notin U$, then let

$$\omega = \sup\{t \in \widetilde{I} \mid (s, \psi(s)) \in U \text{ for every } s \in [t_0, t]\}, \quad I = [t_0, \omega].$$

If no such point t_* exists, let $\omega = \widetilde{\omega}$ and $I = \widetilde{I}$. We see that in any case, $(t,\psi(t)) \in U$ for every $t \in I$.

Let us observe now, that if $t \in I$ is such that $W_j(t,\psi(t)) = 0$ for some j, then $\frac{d}{dt} W_j(t,\psi(t)) = D_t W_j(t,\psi(t)) > 0$ at this point. If $t > t_0$ for such a point, then $W_j(t-\delta, \psi(t-\delta)) < 0$ for all small $\delta > 0$, implying from the definition of U that $(t-\delta, \psi(t-\delta)) \notin U$. But this is false, as $t - \delta \in I$. We thus conclude that $W_j(t,\psi(t)) > 0$ must hold for all $t \in I \setminus \{t_0\}$, for every j.

We next prove the strict inequality $\omega > t_0$. If $W_{j_0}(t_0,\psi(t_0)) = 0$ holds for some j_0, then $D_t W_{j_0}(t_0,\psi(t_0)) > 0$ and thus $W_{j_0}(t,\psi(t)) > 0$ for all $t > t_0$ near t_0. It follows that there exists $\delta > 0$ such that $W_j(t,\psi(t)) > 0$ for all $t \in (t_0, t_0+\delta]$ and every j. By decreasing δ if necessary, we additionally have that $E(t,\psi(t)) > 0$ for all such t, because $E(t_0,\psi(t_0)) = E(t_0,x_0) > 0$ is assumed. It follows that $(t,\psi(t)) \in U$ for every $t \in [t_0, t_0+\delta]$, and thus $\omega \geqslant t_0 + \delta$, as desired.

At this point observe that if our solution satisfies $(t,\psi(t)) \in U$ for all $t \in \widetilde{I}$, then either case (1) or case (2) holds and we are done. Assume therefore that $(t_*, \psi(t_*)) \notin U$ for some $t_* \in \widetilde{I}$. Then $(\omega, \psi(\omega)) \in \partial U$. As $W_j(\omega,\psi(\omega)) > 0$ for every j, necessarily $E(\omega,\psi(\omega)) = 0$, and it follows that case (3) holds as claimed. With this the proposition is proved. □

Remark. Perhaps the simplest example of a set U as in Proposition 3.1 is one of the form

$$U = \{(t,x) \mid t_- \leqslant t \leqslant t_+ \text{ and } \beta(t) \leqslant x \leqslant \gamma(t)\},$$

where $t_- < t_+$ and $\beta, \gamma : [t_-, t_+] \to \mathbf{R}$ are C^1 and satisfy $\beta(t) < \gamma(t)$ throughout their domain. Taking

$$W_1(t,x) = x - \beta(t), \qquad W_2(t,x) = -x + \gamma(t), \qquad W_3(t,x) = t - t_-, \qquad E(t,x) = -t + t_+,$$

we see that Proposition 3.1 applies provided that

$$\left.\begin{aligned} D_t W_1\big(t,\beta(t)\big) &= f\big(t,\beta(t)\big) - \dot\beta(t) > 0,\\ D_t W_2\big(t,\gamma(t)\big) &= -f\big(t,\gamma(t)\big) + \dot\gamma(t) > 0, \end{aligned}\right\} \quad \text{for every } t \in [t_-, t_+]. \tag{3.6}$$

(Technically, one must make C^1-smooth extensions of the functions $\beta(t)$ and $\gamma(t)$ to some open interval containing $[t_-, t_+]$ in order that W_1 and W_2 are defined on an open set O containing U.)

The conditions (3.6) simply say that the vector field points inward to U along the upper and lower boundaries of U. The condition $D_t W_3(t,x) = 1 > 0$ of course holds everywhere. Proposition 3.1 implies that for any initial condition $(t_0, x_0) \in U$, with $t_0 < t_+$, the solution $(t, \psi(t))$ stays in the interior of U for $t_0 < t < t_+$ until it exits at some point $(t_+, \psi(t_+))$ with $\beta(t_+) < \psi(t_+) < \gamma(t_+)$.

4. Interval I

Recall the sequence $x_i(t)$ of SOPS's, with $\varepsilon_i \to 0$, and with the normalization (2.4), as per our Standing Assumptions. In this section we establish some basic properties of the transition layers, namely the portions of the solutions that converge to the vertical segments V_n of the limiting profile as $i \to \infty$. The main result here is Corollary 4.4, which provides a description of the solution in an interval of size $O(\varepsilon)$ about $t = 0$, namely in Interval I. To begin, we scale time to define the function $\varphi(\tau) = \varphi_i(\tau)$ by

$$\varphi(\tau) = x(\varepsilon\tau). \tag{4.1}$$

(Recall that we sometimes suppress the index i for notational simplicity.) Our interest in this function is for τ in bounded intervals, which corresponds to neighborhoods of size $O(\varepsilon)$ in the original time t, but of course the formula (4.1) defines $\varphi(\tau)$ for any real τ. We note the uniform bound

$$\big|\varphi'(\tau)\big| = \varepsilon\big|\dot x(t)\big| < k(k+1) \tag{4.2}$$

for all τ or all t, following from the differential equation (1.1) and the general bounds (2.2) on the solution $x(t)$.

Proposition 4.1. *Fix quantities τ_1, τ_2 (of either sign) satisfying*

$$\tau_1 < \tau_2 < Q_+ - Q_0, \tag{4.3}$$

where Q_+ and Q_0 are as in (2.9). Then

$$\lim_{i\to\infty} \varphi_i(\tau) = \varphi_*(\tau) \quad \text{where } \varphi_*(\tau) = k\big(1 - e^{(k-1)\tau}\big), \tag{4.4}$$

with the convergence in (4.4) in the space $C^1[\tau_1, \tau_2]$.

Remark. Note that $\varphi_*(\tau)$ is strictly decreasing in τ, with a nonzero derivative. Moreover,

$$\varphi_*(-\infty) = k, \qquad \varphi_*(Q_+ - Q_0) = -1 \tag{4.5}$$

both hold, which is consistent with the inequalities in the hypothesis (4.3) of the result.

Proof of Proposition 4.1. We make the replacement $x_c(t) = \varphi(\tau)$ in the differential equation (3.1), with $\tau = \varepsilon^{-1}t$. For the historical term we write

$$x_h\big(t - r\big(x_c(t)\big)\big) = x_h\big(t - 1 - x_c(t)\big) = x\big(\varepsilon\tau - 1 - \varphi(\tau)\big),$$

dropping the subscript "h" at the end for simplicity of notation. Using a dot $\dot{}$ to denote the derivative with respect to t and a prime $'$ to denote the derivative with respect to τ, we have that $\varepsilon\dot{x}_c(t) = \varphi'(\tau)$. Thus $y = \varphi(\tau)$ is seen to satisfy the ordinary differential equation

$$\begin{aligned} &y' = f_1(\tau, y), \quad y(0) = 0, \\ &f_1(\tau, y) = f_{1,i}(\tau, y) = -y - kx(\varepsilon\tau - 1 - y), \end{aligned}$$

where here we regard $x(\cdot)$ as a known function. The convergence result (2.7) implies that

$$\begin{aligned} &\lim_{i\to\infty} f_{1,i}(\tau, y) = f_{1,*}(y) \quad \text{uniformly for } (\tau, y) \text{ in compact subsets of } U_1, \text{ where} \\ &f_{1,*}(y) = -y - k(k - 1 - y) = (k - 1)(y - k), \\ &U_1 = \big\{(\tau, y) \in \mathbf{R}^2 \;\big|\; -1 < y < k\big\}. \end{aligned}$$

(The subscript 1 will distinguish the function $f_1(\tau, y)$ and the region U_1 from analogous functions and regions introduced later.) Observe now that the solution of the limiting equation $y' = f_{1,*}(y)$ with the initial condition $y(0) = 0$ is the function $y = \varphi_*(\tau)$ in the statement of the result. Also,

$$\big(\tau, \varphi_*(\tau)\big) \in U_1 \quad \text{for every } \tau \in [\tau_1, \tau_2]$$

holds with τ_1 and τ_2 as given, from the monotonicity of the function $\varphi_*(\tau)$ and from (4.3) and (4.5). Thus by a standard result on continuous dependence of solutions for ordinary differential equations, the convergence of $\varphi_i(\tau)$ to $\varphi_*(\tau)$ in $C^1[\tau_1, \tau_2]$ holds. □

A variant of the above result, stated in terms of the range of the solution, can also be given.

Corollary 4.2. *Fix quantities ξ_1, ξ_2 satisfying*

$$-1 < \xi_2 < \xi_1 < k.$$

Then there exists $C > 0$ such that for all sufficiently large i, there exist quantities $t_{1,i} < t_{2,i}$ such that

$$\begin{aligned} &x(t_{1,i}) = \xi_1, \qquad x(t_{2,i}) = \xi_2, \qquad |t_{1,i}| \leqslant \frac{\varepsilon|\xi^1|}{C}, \qquad |t_{2,i}| \leqslant \frac{\varepsilon|\xi^2|}{C}, \\ &\dot{x}(t) < -\frac{C}{\varepsilon} \quad \text{for every } t \in [t_{1,i}, t_{2,i}]. \end{aligned} \tag{4.6}$$

Proof. From (4.5) we may fix $\tau_1 < 0 < \tau_2 < Q_+ - Q_0$ to satisfy

$$-1 < \varphi_*(\tau_2) < \xi_2 < \xi_1 < \varphi_*(\tau_1) < k.$$

Then the convergence (4.4) ensured by Proposition 4.1 implies that $\varphi_i(\theta_{1,i}) = \xi_1$ and $\varphi_i(\theta_{2,i}) = \xi_2$ for some $\theta_{1,i}, \theta_{2,i} \in [\tau_1, \tau_2]$, for sufficiently large i, and so $t_{1,i} = \varepsilon\theta_{1,i}$ and $t_{2,i} = \varepsilon\theta_{2,i}$ satisfy the two equalities in (4.6). We have directly from the formula (4.4) for $\varphi_*(\tau)$ that $\varphi_*'(\tau) < -C$ for every $\tau \in [\tau_1, \tau_2]$, for some $C > 0$, and thus $\varphi_i'(\tau) < -C$ also holds for such τ, for every large i, due to the convergence of $\varphi_i(\tau)$ to $\varphi_*(\tau)$ in the space $C^1[\tau_1, \tau_2]$. Now with the formula $\dot{x}(t) = \varepsilon^{-1}\varphi_i'(\varepsilon^{-1}t)$ we conclude the final inequality in (4.6), in fact for t in the larger interval $[\varepsilon\tau_1, \varepsilon\tau_2]$ which contains the origin. The first two inequalities in (4.6) now follow from this, as $x(0) = 0$. □

The next result extends Corollary 4.2, to show that the quantity ξ_2 in that result can be taken within $O(\varepsilon)$ of the value -1.

Proposition 4.3. *Fix ξ satisfying*

$$-1 < \xi < k.$$

Then there exist $H > 0$ and $C > 0$ such that for all large i, there exist $t_i < \widetilde{t}_i$ such that

$$x(t_i) = \xi, \qquad -1 < x(\widetilde{t}_i) < -1 + \varepsilon H, \qquad |t_i| \leqslant \frac{\varepsilon|\xi|}{C}, \qquad |\widetilde{t}_i| < \frac{\varepsilon}{C},$$
$$\dot{x}(t) < -\frac{C}{\varepsilon} \quad \textit{for every } t \in [t_i, \widetilde{t}_i]. \tag{4.7}$$

Proof. Without loss $\xi > k^{-1}$. Denote $\xi_1 = \xi$. Fix ξ_2 to satisfy all the inequalities

$$\xi_2 < \xi_1, \qquad -1 < \xi_2 < k - 1 - \xi_1, \tag{4.8}$$

and let $t_{1,i} < t_{2,i}$ be as in the statement of Corollary 4.2. Then

$$\lim_{i\to\infty}\bigl(t_{2,i} - r\bigl(x(t_{2,i})\bigr)\bigr) = \lim_{i\to\infty}\bigl(t_{2,i} - r(\xi_2)\bigr) = -1 - \xi_2 \in (-k, 0) \subseteq (-k-1, 0) \tag{4.9}$$

and so

$$\lim_{i\to\infty} x\bigl(t_{2,i} - r\bigl(x(t_{2,i})\bigr)\bigr) = k - 1 - \xi_2 > \xi_1 \tag{4.10}$$

by (2.7), using (4.8). Assuming i is large enough so that $x(t_{2,i} - r(x(t_{2,i}))) > \xi_1$, define

$$t_{3,i} = \inf\bigl\{t > t_{2,i} \bigm| x\bigl(t - r\bigl(x(t)\bigr)\bigr) = \xi_1\bigr\} \tag{4.11}$$

and observe that

$$t_{2,i} < t_{3,i}, \qquad x\bigl(t - r\bigl(x(t)\bigr)\bigr) \geqslant \xi_1 \quad \text{for every } t \in [t_{2,i}, t_{3,i}]. \tag{4.12}$$

Now define $t_i = t_{1,i}$ and $\widetilde{t}_i = t_{3,i}$. We shall show that the statement of the proposition holds with these values. Already, we have the first equality in (4.7) by definition. Also, the upper bound for $\dot{x}(t)$ in (4.7) holds in $[t_i, t_{2,i}]$ by Corollary 4.2, so we need only establish this bound (with possibly a different constant C) in the interval $[t_{2,i}, \widetilde{t}_i]$. We claim that this follows easily from the lower bound -1 on the solution in (2.2) and from (4.12). Indeed, throughout the interval $[t_{2,i}, \widetilde{t}_i]$ we have

$$\varepsilon\dot{x}(t) = -x(t) - kx\big(t - r\big(x(t)\big)\big) < 1 - k\xi_1 < 0,$$

and so the desired bound (4.7) on $\dot{x}(t)$ holds there provided that $C \leqslant k\xi_1 - 1$.

We next establish the bound

$$x(\widetilde{t}_i) < -1 + \varepsilon H \tag{4.13}$$

for some $H > 0$ independent of i. (The lower bound $x(t) > -1$ of course holds for every t, by (2.2).) We claim that (4.13) follows from the relation

$$\widetilde{t}_i - r\big(x(\widetilde{t}_i)\big) = t_i, \tag{4.14}$$

which is to be shown. Indeed, if (4.14) holds, then

$$1 + x(\widetilde{t}_i) = r\big(x(\widetilde{t}_i)\big) = \widetilde{t}_i - t_i < \frac{\varepsilon|x(\widetilde{t}_i) - x(t_i)|}{C} < \frac{\varepsilon(k+1)}{C}, \tag{4.15}$$

with the first inequality in (4.15) following from the upper bound (4.7) on $\dot{x}(t)$ and the second inequality in (4.15) following from the general bounds (2.2) on $x(t)$. Clearly, (4.15) implies (4.13) with $H = (k+1)C^{-1}$.

To prove (4.14), observe first that

$$x\big(\widetilde{t}_i - r\big(x(\widetilde{t}_i)\big)\big) = \xi = x(t_i)$$

from the definition (4.11) of $\widetilde{t}_i = t_{3,i}$. Now let $\{z_n\}_{n=-\infty}^{\infty} = \{z_{n,i}\}_{n=-\infty}^{\infty}$ denote the zeros of $x(t) = x_i(t)$ as in (2.1), normalized so that $z_{0,i} = 0$. Thus

$$z_{-1,i} = \sup\big\{t < 0 \bigm| x(t) = 0\big\}$$

denotes the zero of $x(t)$ immediately to the left of $t = 0$. The slowly oscillating property of $x(t)$ implies that $z_{-1,i} < -1$, and from (2.7) we see further that

$$\lim_{i\to\infty} z_{-1,i} = -k. \tag{4.16}$$

We have that $x(t) > 0$ throughout $(z_{-1,i}, 0)$, with $\dot{x}(t) = 0$ at exactly one point, where $x(t)$ achieves its maximum (recall (2.2)). From (2.7) we see that the value of that maximum approaches k as $i \to \infty$, and so $x(t) = \xi$ at exactly two points $t_{-,i} < t_{+,i}$ in $(z_{-1,i}, 0)$, for large i. The limiting values of these points are clear from (2.7), namely

$$\big\{t \in (z_{-1,i}, 0) \bigm| x(t) = \xi\big\} = \{t_{-,i}, t_{+,i}\}, \qquad \lim_{i\to\infty} t_{-,i} = -k + \xi, \qquad \lim_{i\to\infty} t_{+,i} = 0. \tag{4.17}$$

Now (4.9) and (4.16) imply that $t_{2,i} - r(x(t_{2,i})) \in (z_{-1,i}, 0)$ for large i, and with (4.10) and the above we conclude that

$$t_{-,i} < t_{2,i} - r\big(x(t_{2,i})\big) < t_{+,i} \tag{4.18}$$

for large i. Also,

$$\frac{d}{dt}\big(t - r\big(x(t)\big)\big) = 1 - \dot{x}(t) > 1 + \frac{C}{\varepsilon} > 0 \quad \text{for every } t \in [t_i, \widetilde{t}_i]$$

and so

$$t_{2,i} - r\big(x(t_{2,i})\big) < \widetilde{t}_i - r\big(x(\widetilde{t}_i)\big). \tag{4.19}$$

We conclude now from (4.11), (4.18), and (4.19) that

$$\widetilde{t}_i - r\big(x(\widetilde{t}_i)\big) = t_{+,i}.$$

We must show that $t_i = t_{+,i}$ for large i in order to establish (4.14). If $t_i < 0$ then this conclusion follows from (4.17), the fact that $x(t_i) = \xi$, and the fact that $t_i \to 0$ as $i \to \infty$, by Corollary 4.2. On the other hand $t_i \geqslant 0$ is impossible for large i, as $x(t) < 0$ throughout at least the interval $(0, 1]$ due the fact that this solution is slowly oscillating, which would force $t_i > 1$ and preclude the limit $t_i \to 0$. Thus (4.14) is established, completing the proof of (4.13).

Finally, the estimates on $|t_i|$ and $|\widetilde{t}_i|$ in (4.7) follow from the estimate on $\dot{x}(t)$ in (4.7), upon noting that $t_i < 0 < \widetilde{t}_i$. □

A refinement of the above result allows us to fix an interval $-T_1 \leqslant \tau \leqslant T_2$ on which we shall work. More precisely, we have the following result.

Corollary 4.4. *Let $\varphi(\tau) = \varphi_i(\tau)$ be as in* (4.1)*. Then there exist positive quantities T_1, H_2, and C_2 independent of i, and positive quantities $T_2 = T_{2,i}$, such that for all large i*

$$\varphi'(\tau) < -C_2 \quad \textit{for every } \tau \in [-T_1, T_2], \tag{4.20}$$

and also

$$\varphi(T_2) = -1 + \varepsilon H_2. \tag{4.21}$$

Moreover T_1 and H_2 can be chosen to satisfy

$$Q_0 - Q_- < H_2 < Q_+ - Q_0 + T_1, \tag{4.22}$$

where $Q_\pm$ and Q_0 are as in (2.9)*. Also,*

$$\lim_{i\to\infty} T_{2,i} = Q_+ - Q_0 \tag{4.23}$$

holds. (*More precisely, there exist H_2 and $T_2 = T_{2,i}$ such that for all sufficiently large T_1, there exists C_2 such that the above holds.*)

Remark. The inequalities in (4.22) will be needed in our later analysis.

Proof of Corollary 4.4. Take $\xi = 0$ in Proposition 4.3. Observe that H in the statement of that result can be increased if desired, so that without loss we may assume that it satisfies $H > Q_0 - Q_-$. Defining $H_2 = H$ for this H, we have that the first inequality of (4.22) holds. Also define $T_2 = T_{2,i}$ such that εT_2 is the unique point in the interval $[t_i, \widetilde{t}_i] = [0, \widetilde{t}_i]$ for which $x(\varepsilon T_2) = -1 + \varepsilon H_2$. This gives (4.21). Finally, let C_0 denote the quantity C which appears in the statement of Proposition 4.3.

Now fix $T_1 > 0$ large enough that the second inequality in (4.22) holds, and take $\tau_1 = -T_1$ and $\tau_2 = 0$ in the statement of Proposition 4.1. The convergence (4.4) in that result and the fact that $\varphi_*'(\tau) < 0$, ensure that there is a constant $C_{00} > 0$ such that $\varphi'(\tau) < -C_{00}$ for every $\tau \in [-T_1, 0]$, for all large i. One now easily sees that upon taking $C_2 = \min\{C_0, C_{00}\}$, the inequality (4.20) holds for all large i.

There remains to prove the limit (4.23). Fix any quantity τ_0 satisfying $0 < \tau_0 < Q_+ - Q_0$. Then the convergence (4.4) in the interval $[0, \tau_0]$, along with the fact that

$$\varphi_*(\tau_0) > \varphi_*(Q_+ - Q_0) = -1,$$

from (4.5), implies that $T_{2,i} > \tau_0$ for all large i and hence

$$\liminf_{i\to\infty} T_{2,i} \geqslant Q_+ - Q_0.$$

Thus for large i we have that $T_{2,i} > \tau_0$, and so from (4.20)

$$\varphi_*(Q_+ - Q_0) = -1 < \varphi(T_{2,i}) < \varphi(\tau_0) - C_2(T_{2,i} - \tau_0),$$

and one concludes from this that

$$\limsup_{i\to\infty} T_{2,i} \leqslant \tau_0 + \frac{\varphi_*(\tau_0) - \varphi_*(Q_+ - Q_0)}{C_2}.$$

This in turn implies that

$$\limsup_{i\to\infty} T_{2,i} \leqslant Q_+ - Q_0,$$

as we see by taking τ_0 arbitrarily close to $Q_+ - Q_0$, and this completes the proof. □

At this point, let us fix the quantities T_1, $T_{2,i}$, H_2, and C_2 as in the statement of Corollary 4.4, keeping them fixed for the remainder of the paper. We see that Corollary 4.4 provides a precise description of the solution $x(t)$ for t in Interval I, namely for $t \in [-\varepsilon T_1, \varepsilon T_{2,i}]$.

5. Interval II

We next consider the solution for

$$\varepsilon T_3 \leqslant t \leqslant k + 1 - \varepsilon T_4, \tag{5.1}$$

which is Interval II. Here we define

$$T_3 = T_2 - 2\varepsilon^2 C_3, \qquad T_4 = kH_2 + 1 - T_2, \qquad C_3 = \frac{1}{(kC_2)^2}. \tag{5.2}$$

Of course $T_3 = T_{3,i}$ and $T_4 = T_{4,i}$ depend on i, as do $T_2 = T_{2,i}$ and $\varepsilon = \varepsilon_i$, but one sees that they are bounded sequences. (As before, we sometimes suppress the index i for ease of notation. Also, our statements generally hold for sufficiently large i, that is, for sufficiently small $\varepsilon = \varepsilon_i$.) For large i the left-hand endpoint εT_3 of this interval is seen to lie in Interval I, which is the interval $[-\varepsilon T_1, \varepsilon T_2]$ considered in Corollary 4.4, and thus $\dot{x}(\varepsilon T_3) < 0$. We see also that Interval II contains the diagonally sloping part of the sawtooth, and as well the point near $t = 0$ where $x(t)$ achieves its minimum. We shall provide a precise asymptotic description of how the solution passes through its minimum, changing from the vertical downward transition layer to the upward sloping part of the sawtooth.

For future use let us define $H_3 = H_{3,i}$ by

$$\varphi(T_3) = -1 + \varepsilon H_3. \tag{5.3}$$

We note the bounds

$$0 < H_3 - H_2 < 2\varepsilon k(k+1)C_3 \tag{5.4}$$

for large i, which follow from the formula (4.21), from the monotonicity of $\varphi(\tau)$ in Corollary 4.4, and from the bound (4.2) on $\varphi'(\tau)$.

We prescribe an ansatz for the solution in Interval II as

$$x(t) = t - 1 - \varepsilon\sigma(t) + \varepsilon e^{-\eta(\varepsilon^{-2}(t-\varepsilon T_3))}, \quad \varepsilon T_3 \leqslant t \leqslant k + 1 - \varepsilon T_4. \tag{5.5}$$

The function $\sigma(t) = \sigma_i(t)$ will be defined to be the solution of the initial value problem

$$\begin{aligned} &\varepsilon^2\dot{\sigma} = f_2(t,\sigma), \qquad \sigma(-T_0) = \widetilde{\sigma}(-T_0), \\ &f_2(t,\sigma) = f_{2,i}(t,\sigma) = t - 1 + \varepsilon - \varepsilon\sigma + k\varphi(\sigma), \end{aligned} \tag{5.6}$$

for an appropriately chosen $T_0 > 0$. Here $\varphi(\tau) = x(\varepsilon\tau)$ as before. The related function $\widetilde{\sigma}(t) = \widetilde{\sigma}_i(t)$, which determines the initial condition in (5.6), will be defined to be a particular solution of the functional equation

$$f_{2,i}(t,\widetilde{\sigma}) = t - 1 + \varepsilon - \varepsilon\widetilde{\sigma} + k\varphi(\widetilde{\sigma}) = 0. \tag{5.7}$$

By way of motivation, the differential equation in (5.6) can be rewritten as the two-dimensional autonomous system

$$\begin{aligned} &\sigma' = f_2(t,\sigma), \\ &t' = \varepsilon^2, \end{aligned} \tag{5.8}$$

after a rescaling of the time. One has a limiting graph $\sigma = \sigma_*(t)$, defined to be the solution of the limit $t - 1 + k\varphi_*(\sigma_*) = 0$ of Eq. (5.7), and which is a normally hyperbolic invariant manifold for (5.8) when $\varepsilon = 0$, in the spirit of Fenichel [20,21]. (See also the comprehensive article [27] of Jones.) The graph of $\sigma(t)$ is thus a smooth perturbation of this invariant manifold for small $\varepsilon > 0$, and the term $\varepsilon e^{-\eta(\varepsilon^{-2}(t-\varepsilon T_3))}$ in (5.5) describes the rapid approach to this manifold as one passes through the turning point at the minimum of $x(t)$.

In the next result we define $\sigma(t)$ and $\widetilde{\sigma}(t)$ precisely for $t \in [-T_0, k+1-\varepsilon T_4]$, and hence for t in Interval II. We also provide bounds for $\sigma(t)$ in terms of $\widetilde{\sigma}(t)$.

Lemma 5.1. *Fix T_0 satisfying*

$$0 < T_0 < k\varphi_*(-T_1) - 1, \tag{5.9}$$

where we observe that

$$k\varphi_*(-T_1) - 1 > k\varphi_*(Q_+ - 2Q_0 + Q_-) - 1 = 0$$

holds in view of (2.9), (4.4), *and* (4.22). *Then for large i*

$$\begin{aligned} &\sigma : [-T_0, k+1-\varepsilon T_4] \to [-T_1, T_2], \\ &\widetilde{\sigma} : [-T_0, k+1-\varepsilon T_4] \to [-T_1 + \varepsilon, T_2] \end{aligned} \tag{5.10}$$

are uniquely defined as the solutions of (5.6) *and* (5.7) *with the domains and ranges indicated. Both functions are smooth in* t*. In addition*

$$\widetilde{\sigma}(k+1-\varepsilon T_4) = T_2, \tag{5.11}$$

and

$$\begin{aligned} &0 < \dot{\sigma}(t) < 2k^2(k+1)C_3 \quad \textit{and} \quad \widetilde{\sigma}(t) - \varepsilon^2 C_3 < \sigma(t) < \widetilde{\sigma}(t) \\ &\textit{for every } t \in (-T_0, k+1-\varepsilon T_4], \end{aligned} \tag{5.12}$$

with C_3 *as in* (5.2)*. Finally,*

$$\lim_{i\to\infty} \sigma_i(t_i) = Q_+ - 2Q_0 + Q_- \tag{5.13}$$

for any sequence t_i *satisfying* $\lim_{i\to\infty} t_i = 0$*.*

Proof. We first consider $\widetilde{\sigma}(t)$. Denote $\widetilde{\varphi}(\tau) = \varphi(\tau) + \varepsilon k^{-1}(1-\tau)$ and observe from (4.20) that $\widetilde{\varphi}'(\tau) < -C_2$ for $\tau \in [-T_1, T_2]$. Next observe that (5.7) can be rewritten as $-k\widetilde{\varphi}(\widetilde{\sigma}) = t-1$. We see that $\widetilde{\sigma}(t)$ is uniquely defined as a solution of this equation with the domain and range in (5.10) provided that

$$-k\widetilde{\varphi}(-T_1+\varepsilon) \leqslant -T_0 - 1 < k - \varepsilon T_4 \leqslant -k\widetilde{\varphi}(T_2). \tag{5.14}$$

For large i the first inequality of (5.14) follows from the fact that

$$\begin{aligned} -k\widetilde{\varphi}(-T_1+\varepsilon) &= -k\varphi(-T_1+\varepsilon) - \varepsilon(T_1+1) + \varepsilon^2 \\ &\to -k\varphi_*(-T_1) < -T_0 - 1 \quad \text{as } i \to \infty, \end{aligned}$$

using Proposition 4.1 and (5.9). It is clear that the middle inequality in (5.14) holds for large i. The final inequality in (5.14) is in fact an equality, as is easily seen from the definition (5.2) of T_4 and from the formula (4.21) involving H_2. This equality in turn implies that Eq. (5.11) holds. Let us also observe here that $\widetilde{\sigma}(t)$ is smooth in t by the implicit function theorem. Indeed,

$$\dot{\widetilde{\sigma}}(t) = \frac{-1}{k\varphi'(\widetilde{\sigma}(t)) - \varepsilon} > 0 \tag{5.15}$$

by (5.7).

Now consider $\sigma(t)$. To show this function is well-defined as in (5.10) and satisfies the bounds in (5.12), for large i we define the set $U_2 = U_{2,i}$ by

$$\begin{aligned} U_2 &= \bigl\{(t,\sigma) \in \mathbf{R}^2 \bigm| -T_0 \leqslant t \leqslant k+1-\varepsilon T_4 \text{ and } \widetilde{\sigma}(t) - \varepsilon^2 C_3 \leqslant \sigma \leqslant \widetilde{\sigma}(t)\bigr\} \\ &\subseteq [-T_0, k+1-\varepsilon T_4] \times [-T_1, T_2], \end{aligned}$$

and show that it satisfies the conditions of Proposition 3.1 for the differential equation (5.6). More precisely, we show that U_2 satisfies the conditions of the remark immediately following the proof of Proposition 3.1, and so with the vector field pointing strictly inward along the upper and lower boundaries of U_2. Thus U_2 enjoys the invariance property of Proposition 3.1, wherein solutions of the differential equation in U_2 can only leave that set at the right-hand boundary $t = k+1-\varepsilon T_4$. As the initial condition $\sigma(-T_0) = \widetilde{\sigma}(-T_0)$ of $\sigma(t)$ lies in U_2, the desired bounds on $\sigma(t)$ in (5.12) follow. (The bounds on $\dot{\sigma}(t)$ will be proved later.)

To establish these facts, and specifically the inequalities (3.6) in this setting, let $\sigma = \widetilde{\sigma}(t)$ for some $t \in [-T_0, k+1-\varepsilon T_4]$, and so (t,σ) is a point on the upper boundary of U_2. Then $f_2(t,\widetilde{\sigma}(t)) = 0$ from (5.7). Thus $\varepsilon^{-2} f_2(t,\widetilde{\sigma}(t)) - \dot{\widetilde{\sigma}}(t) < 0$ by (5.15), and so the vector field in (5.6) points strictly inward into U_2 along its upper boundary. Next let $\sigma = \widetilde{\sigma}(t) - \varepsilon^2 C_3$ for some t in the same interval, so (t,σ) lies on the lower boundary of U_2. From the bound $\varphi'(\tau) < -C_2$ and by the mean value theorem, we have that

$$f_2\big(t,\widetilde{\sigma}(t) - \varepsilon^2 C_3\big) = f_2\big(t,\widetilde{\sigma}(t) - \varepsilon^2 C_3\big) - f_2\big(t,\widetilde{\sigma}(t)\big) > (\varepsilon + kC_2)\varepsilon^2 C_3 > \varepsilon^2 kC_2C_3 \quad (5.16)$$

for this point. However, from (5.15) we have that $\dot{\widetilde{\sigma}}(t) < (kC_2+\varepsilon)^{-1} < (kC_2)^{-1} = kC_2C_3$. Combining this inequality with (5.16) yields $\varepsilon^{-2} f_2(t,\widetilde{\sigma}(t) - \varepsilon^2 C_3) - \dot{\widetilde{\sigma}}(t) > 0$, and thus the vector field points strictly inward into U_2 along its lower boundary. This verifies the desired conditions on f_2 and U_2.

We now must establish the bounds on $\dot{\sigma}(t)$ in (5.12). We note that $f_2(t,\sigma)$ is a decreasing function of σ in U_2, as $\varphi'(\tau) < 0$ for $\tau \in [-T_1, T_2]$. Thus $f_2(t,\sigma) > f_2(t,\widetilde{\sigma}(t)) = 0$ for $(t,\sigma) \in U_2$ provided $\sigma < \widetilde{\sigma}(t)$, and hence $\dot{\sigma}(t) > 0$ for t as in (5.12), as claimed. Also, $\varphi'(\tau) > -k(k+1)$ from (4.2) and so for large i

$$\frac{\partial f_2(t,\sigma)}{\partial \sigma} > -\varepsilon - k^2(k+1) > -2k^2(k+1).$$

Thus

$$f_2(t,\sigma) < f_2\big(t,\widetilde{\sigma}(t) - \varepsilon^2 C_3\big) < f_2\big(t,\widetilde{\sigma}(t)\big) + 2\varepsilon^2 k^2(k+1)C_3 = 2\varepsilon^2 k^2(k+1)C_3$$

in U_2, provided that $\sigma > \widetilde{\sigma}(t) - \varepsilon^2 C_3$. This now gives the upper bound on $\dot{\sigma}(t)$ in (5.12).

Finally, to prove (5.13) it is enough to take the limit of $\widetilde{\sigma}_i(0)$ as $i \to \infty$, in light of the bounds (5.12). From the definition (5.7) of $\widetilde{\sigma}(t)$ we have that

$$\lim_{i\to\infty} \widetilde{\sigma}_i(0) = \varphi_*^{-1}\big(k^{-1}\big) = Q_+ - 2Q_0 + Q_-,$$

where the formula (4.4) for $\varphi_*(\tau)$ is used. □

Let T_0 and $\sigma(t) = \sigma_i(t)$ be as in the statement of Lemma 5.1. We keep these fixed for the remainder of the paper.

We now wish to obtain an ordinary differential equation for the function $\eta(\theta)$ which occurs in the ansatz (5.5). Here $\theta = \varepsilon^{-2}(t - \varepsilon T_3)$ is the independent variable, which by (5.1) lies in the interval

$$0 \leqslant \theta \leqslant \frac{k+1-\varepsilon(T_3+T_4)}{\varepsilon^2}. \quad (5.17)$$

Let us first use the ansatz (4.1) of Interval I to substitute for the historical term in Eq. (3.1), that is, we let $x_{\mathrm{h}}(t - r(x_{\mathrm{c}}(t))) = \varphi(\varepsilon^{-1}(t - r(x_{\mathrm{c}}(t))))$. We shall justify this later, by showing that $t - r(x_{\mathrm{c}}(t))$ lies in Interval I whenever t lies in Interval II. This substitution gives

$$\varepsilon \dot{x}_{\mathrm{c}}(t) = -x_{\mathrm{c}}(t) - k\varphi\big(\varepsilon^{-1}\big(t - r\big(x_{\mathrm{c}}(t)\big)\big)\big) = -x_{\mathrm{c}}(t) - k\varphi\big(\varepsilon^{-1}\big(t - 1 - x_{\mathrm{c}}(t)\big)\big). \quad (5.18)$$

Now substitute the ansatz (5.5) for $x_{\mathrm{c}}(t)$ in (5.18). This yields, after a calculation in which the differential equation (5.6) for $\sigma(t)$ is used, the ordinary differential equation

$$\eta' = f_3(\theta,\eta), \qquad \eta(0) = -\log\big(H_3 - T_3 + \sigma(\varepsilon T_3)\big),$$
$$f_3(\theta,\eta) = f_{3,i}(\theta,\eta) = \varepsilon + ke^{\eta}\big(\varphi\big(\sigma\big(\varepsilon T_3 + \varepsilon^2\theta\big) - e^{-\eta}\big) - \varphi\big(\sigma\big(\varepsilon T_3 + \varepsilon^2\theta\big)\big)\big), \quad (5.19)$$

where prime $'$ denotes differentiation with respect to θ. Note also that formula (5.3) relating $x(\varepsilon T_3) = \varphi(T_3)$ and H_3 has been used in obtaining the initial condition.

We remark that the ansatz (5.5) presumes that the final term (the exponential term) in the expression for $x(t)$ is positive. This assumption will be justified in our analysis below. In particular, we must ensure that the solution $\eta(\theta)$ to (5.19) exists throughout the interval (5.17). In doing so we must check that the initial condition $\eta(0)$ is well-defined by the above formula, insofar that it is the logarithm of a positive quantity. The following result establishes these facts, and more.

Proposition 5.2. *There exist constants C_4 and C_5, independent of i, such that*

$$e^{-C_5} < H_3 - T_3 + \sigma(\varepsilon T_3) < e^{-C_4}$$

for large i, and so $\eta(0)$ is well-defined and bounded for large i, satisfying

$$C_4 < \eta(0) < C_5. \tag{5.20}$$

Additionally, the solution $\eta(\theta)$ to (5.19) *exists and satisfies*

$$\varepsilon + kC_2 < \eta'(\theta) < \varepsilon + k^2(k+1) \tag{5.21}$$

for θ in the interval (5.17)*, so in particular*

$$C_4 + kC_2\theta < \eta(\theta) < C_5 + 2k^2(k+1)\theta \tag{5.22}$$

holds throughout this interval.

Proof. We have that $\lim_{i\to\infty} H_{3,i} = H_2$ by (5.4), where H_2 defined in Corollary 4.4 is independent of i. Also $\lim_{i\to\infty} T_{3,i} = \lim_{i\to\infty} T_{2,i} = Q_+ - Q_0$ from the definition (5.2) of T_3 and by (4.23) in Corollary 4.4. Thus

$$\lim_{i\to\infty}\big(H_{3,i} - T_{3,i} + \sigma_i(\varepsilon_i T_{3,i})\big) = H_2 - (Q_+ - Q_0) + (Q_+ - 2Q_0 + Q_-) = H_2 - Q_0 + Q_- > 0$$

by (5.13) and by the first inequality in (4.22). The bounds on $\eta(0)$ follow directly.

Now for large i define the set $U_3 = U_{3,i}$ by

$$U_3 = \big\{(\theta,\eta) \in \mathbf{R}^2 \;\big|\; 0 \leqslant \theta \leqslant \varepsilon^{-2}\big(k+1-\varepsilon(T_3+T_4)\big) \text{ and } \eta \geqslant \eta(0)\big\},$$

and observe that $f_3(\theta,\eta)$ is well-defined in U_3 with the quantity $\varepsilon T_3 + \varepsilon^2\theta$ lying in the domain $[-T_0, k+1-\varepsilon T_4]$ of $\sigma(t)$ for θ in the interval (5.17) of U_3. Thus

$$\sigma\big(\varepsilon T_3 + \varepsilon^2\theta\big) \leqslant T_2,$$

by (5.10) of Lemma 5.1. Also, from the fact that $\dot\sigma(t) > 0$, we have for $(\theta,\eta) \in U_3$ that

$$\sigma\big(\varepsilon T_3 + \varepsilon^2\theta\big) - e^{-\eta} \geqslant \sigma(\varepsilon T_3) - e^{-\eta(0)} = T_3 - H_3 > -T_1, \tag{5.23}$$

where the final inequality in (5.23) holds for large i, and follows from the limit $T_{3,i} \to Q_+ - Q_0$ noted above, and from the second inequality in (4.22), where again (5.4) is used. Thus the arguments appearing in the function $\varphi(\tau)$ in the formula (5.19) for $f_3(\theta,\eta)$ lie in the interval $[-T_1, T_2]$ in which

$\varphi'(\tau) < -C_2$. Recalling the general bound (4.2) on $\varphi'(\tau)$, we conclude from the mean value theorem that

$$\varepsilon + kC_2 < f_3(\theta, \eta) < \varepsilon + k^2(k+1) \quad \text{for every } (\theta, \eta) \in U_3. \tag{5.24}$$

It is now clear from (5.24) and from the form of U_3, that (5.21) and (5.22) hold for θ in (5.17). (One may appeal to Proposition 3.1, although the conclusion here is straightforward.) □

In the next result, as in Corollary 6.4 and Proposition 7.5, we obtain a partial confirmation of the monotonicity property (2.3). The second conclusion of this result, namely the mapping property of the intervals, will be needed later.

Corollary 5.3. *We have for large i that*

$$\varepsilon T_3 \leqslant t \leqslant k+1-\varepsilon T_4 \quad \Longrightarrow \quad \begin{cases} \dfrac{d}{dt}\big(t - r\big(x(t)\big)\big) > 0 \quad \text{and} \\ -\varepsilon T_1 < t - r\big(x(t)\big) < \varepsilon T_2. \end{cases} \tag{5.25}$$

In particular, if t belongs to Interval II then $t - r(x(t))$ belongs to the interior of Interval I.

Proof. From the ansatz (5.5) for Interval II we have that

$$t - r\big(x(t)\big) = t - 1 - x(t) = \varepsilon\big(\sigma(t) - e^{-\eta(\varepsilon^{-2}(t-\varepsilon T_3))}\big). \tag{5.26}$$

The inequalities $\dot\sigma(t) > 0$ in (5.12) and $\eta'(\theta) > 0$ in (5.21) imply the first conclusion in (5.25). Taking the left endpoint $t = \varepsilon T_3$ of Interval II in (5.26) gives

$$\big(t - r\big(x(t)\big)\big)\big|_{t=\varepsilon T_3} = \varepsilon\big(\sigma(\varepsilon T_3) - e^{-\eta(0)}\big) > -\varepsilon T_1$$

as in (5.23). And taking the right endpoint $t = k+1-\varepsilon T_4$ in (5.26) gives

$$\big(t - r\big(x(t)\big)\big)\big|_{t=k+1-\varepsilon T_4} < \varepsilon\sigma(k+1-\varepsilon T_4) \leqslant \varepsilon T_2$$

by (5.10). The bounds on $t - r(x(t))$ in (5.25) follow directly. □

6. Interval III

We next consider the solution in Interval III, namely for

$$k+1-\varepsilon T_4 \leqslant t \leqslant k+1-\varepsilon T_5, \tag{6.1}$$

where we define $T_5 = T_{5,i}$ to be

$$T_5 = 1 + (k-1)T_3 - k\sigma(\varepsilon T_3) + 2\varepsilon^{1/2}k. \tag{6.2}$$

As always, we take i sufficiently large. We note that

$$\begin{aligned} T_{4,*} &= \lim_{i\to\infty} T_{4,i} = kH_2 + 1 - Q_+ + Q_0, \\ T_{5,*} &= \lim_{i\to\infty} T_{5,i} = 1 - Q_+ + (k+1)Q_0 - kQ_-, \end{aligned} \tag{6.3}$$

by (4.23), (5.2), and (5.13), and thus $T_{4,*} > T_{5,*}$ by the first inequality in (4.22), and where the above limits serve as the definitions of $T_{4,*}$ and $T_{5,*}$. In particular, $T_{4,i} > T_{5,i}$ for large i and so the inequalities in (6.1) make sense.

With $t = k + 1 + \varepsilon\tau$ in the interval (6.1), we prescribe the ansatz

$$\begin{aligned} x_c(t) &= x_c(k+1+\varepsilon\tau) = t - 1 - \varepsilon T_3 - \varepsilon^2\alpha\bigl(\varepsilon^{-1}(t-k-1)\bigr) \\ &= k + \varepsilon(\tau - T_3) - \varepsilon^2\alpha(\tau), \quad -T_4 \leqslant \tau \leqslant -T_5, \end{aligned} \tag{6.4}$$

and we will derive an ordinary differential equation for $\alpha(\tau)$. We first observe that

$$t - r\bigl(x_c(t)\bigr) = t - 1 - x_c(t) = k + \varepsilon\tau - x_c(t) = \varepsilon T_3 + \varepsilon^2\alpha(\tau). \tag{6.5}$$

Differentiating (6.4) and substituting this and (6.5) into the differential equation (3.1) now gives

$$\begin{aligned} \varepsilon\dot{x}_c(t) &= \varepsilon - \varepsilon^2\alpha'(\tau) = -x_c(t) - kx_h\bigl(t - r\bigl(x_c(t)\bigr)\bigr) \\ &= -k - \varepsilon(\tau - T_3) + \varepsilon^2\alpha(\tau) - kx_h\bigl(\varepsilon T_3 + \varepsilon^2\alpha(\tau)\bigr), \end{aligned} \tag{6.6}$$

where prime $'$ denotes the derivative with respect to τ. We use the ansatz (5.5) of Interval II for the historical term in (6.6), to give

$$x_h\bigl(\varepsilon T_3 + \varepsilon^2\alpha(\tau)\bigr) = \varepsilon T_3 + \varepsilon^2\alpha(\tau) - 1 - \varepsilon\sigma\bigl(\varepsilon T_3 + \varepsilon^2\alpha(\tau)\bigr) + \varepsilon e^{-\eta(\alpha(\tau))}. \tag{6.7}$$

The use of this ansatz is justified as long as α is such that $\varepsilon T_3 + \varepsilon^2\alpha$ belongs to Interval II, equivalently, that

$$0 \leqslant \alpha \leqslant \frac{k+1-\varepsilon(T_3+T_4)}{\varepsilon^2}. \tag{6.8}$$

Combining (6.6) and (6.7) now gives, with some calculation, the ordinary differential equation

$$\begin{aligned} &\varepsilon\alpha' = \tau + f_4(\alpha), \\ &f_4(\alpha) = f_{4,i}(\alpha) = 1 + (k-1)(T_3 + \varepsilon\alpha) - k\sigma\bigl(\varepsilon T_3 + \varepsilon^2\alpha\bigr) + ke^{-\eta(\alpha)}. \end{aligned} \tag{6.9}$$

We note that the function f_4 is defined for α satisfying (6.8).

The initial condition for (6.9) is taken at $\tau = -T_4$, corresponding to the point $t = k + 1 - \varepsilon T_4$ where Intervals II and III meet. For ease of notation let us denote

$$v = k + 1 - \varepsilon(T_3 + T_4).$$

Then we have

$$x(k+1-\varepsilon T_4) = k - \varepsilon T_4 - \varepsilon\sigma(k+1-\varepsilon T_4) + \varepsilon e^{-\eta(\varepsilon^{-2}v)}$$

from the ansatz (5.5) for Interval II, and also

$$x(k+1-\varepsilon T_4) = k - \varepsilon(T_3+T_4) - \varepsilon^2\alpha(-T_4)$$

from the ansatz (6.4) for Interval III. Equating these gives

$$\alpha(-T_4) = -\left(\frac{T_3 - \sigma(k+1-\varepsilon T_4) + e^{-\eta(\varepsilon^{-2}v)}}{\varepsilon}\right). \tag{6.10}$$

The quantity $\alpha(-T_4)$, despite the term ε in the denominator of the formula (6.10), is actually small, as the following result shows.

Lemma 6.1. *The bounds*

$$0 < \alpha(-T_4) < 2\varepsilon C_3, \qquad 0 < -T_4 + f_4\big(\alpha(-T_4)\big) < 4\varepsilon k^2(k+1)C_3 \tag{6.11}$$

hold for large i.

Proof. We have first that

$$T_2 - \varepsilon^2 C_3 < \sigma(k+1-\varepsilon T_4) < T_2 \tag{6.12}$$

from (5.11) and (5.12). Also, the lower bound in (5.22) ensures that

$$e^{-\eta(\varepsilon^{-2}v)} < \varepsilon^2 C_3 \tag{6.13}$$

for large i, and in fact the left-hand side of (6.13) decays faster than any power of ε. With the formula (6.10) for $\alpha(-T_4)$ one easily checks, using (6.12) and (6.13) and also using the formula (5.2) relating T_2 and T_3, that the bounds on $\alpha(-T_4)$ in (6.11) hold.

Let us now estimate the quantity $-T_4 + f_4(\alpha(-T_4))$. We do this by calculating $\dot{x}(k+1-\varepsilon T_4)$ using the two ansätze (5.5) and (6.4). Differentiating (5.5), we have that

$$\dot{x}(k+1-\varepsilon T_4) = 1 - \varepsilon\dot{\sigma}(k+1-\varepsilon T_4) - \varepsilon^{-1}e^{-\eta(\varepsilon^{-2}v)}\eta'\big(\varepsilon^{-2}v\big),$$

while from (6.4), and more specifically from the first equality in (6.6), we have that

$$\dot{x}(k+1-\varepsilon T_4) = 1 - \varepsilon\alpha'(-T_4).$$

Equating these and taking note of the differential equation (6.9) gives

$$\varepsilon\alpha'(-T_4) = -T_4 + f_4\big(\alpha(-T_4)\big) = \varepsilon\dot{\sigma}(k+1-\varepsilon T_4) + \varepsilon^{-1}e^{-\eta(\varepsilon^{-2}v)}\eta'\big(\varepsilon^{-2}v\big).$$

The bounds (5.21) and (6.13) imply that

$$0 < \varepsilon^{-1}e^{-\eta(\varepsilon^{-2}v)}\eta'\big(\varepsilon^{-2}v\big) < \varepsilon\big(\varepsilon + k^2(k+1)\big)C_3 < 2\varepsilon k^2(k+1)C_3, \tag{6.14}$$

with the final inequality in (6.14) holding for small ε, that is, for large i. With this, and with the bounds (5.12) on $\dot{\sigma}(t)$ in Lemma 5.1, we obtain the bounds (6.11) on $-T_4 + f_4(\alpha(-T_4))$, as desired. □

Now for large i define the set $U_{4a} = U_{4a,i}$ by

$$U_{4a} = \left\{(\tau,\alpha) \in \mathbf{R}^2 \;\middle|\; -T_4 \leqslant \tau \leqslant -T_5 \text{ and } \alpha(-T_4) \leqslant \alpha \leqslant \eta^{-1}\left(\frac{1}{2}|\log\varepsilon|\right),\right.$$
$$\left.\text{where also } 0 \leqslant \tau + f_4(\alpha) \leqslant \varepsilon^{1/2}C_6\right\},$$
$$C_6 = \frac{2}{k^2 C_2}. \tag{6.15}$$

The notation U_{4a} rather than U_4 reflects the fact that in the next section we shall consider a similarly defined set U_{4b}, with the same ansatz (6.4).

In the definition of U_{4a} we observe that the quantity $\eta^{-1}(\frac{1}{2}|\log\varepsilon|)$ is well-defined, at least for large i, in light of the uniform positive lower bound (5.21) on $\eta'(\theta)$, the uniform bounds (5.20) on $\eta(0)$, and the interval (5.17) of size $O(\varepsilon^{-2})$ on which $\eta(\theta)$ is defined. Still, in order for U_{4a} to be well-defined, we must ensure that the interval $\alpha(-T_4) \leqslant \alpha \leqslant \eta^{-1}(\frac{1}{2}|\log\varepsilon|)$ lies within the domain of the function f_4. This and more is established by the following result.

Lemma 6.2. *We have for large i that*

$$0 < \alpha(-T_4) < \eta^{-1}\left(\frac{1}{2}|\log\varepsilon|\right) < \frac{k+1-\varepsilon(T_3+T_4)}{\varepsilon^2} \tag{6.16}$$

and thus the set U_{4a} and the differential equation (6.9) *in U_{4a} are well-defined. Also,*

$$\alpha < \frac{|\log\varepsilon|}{2kC_2} \tag{6.17}$$

holds for every $(\tau,\alpha) \in U_{4a}$. Finally,

$$\alpha > \frac{|\log\varepsilon|}{5k^2(k+1)} \tag{6.18}$$

holds on the right-hand boundary of U_{4a}, that is, whenever $(-T_5,\alpha) \in U_{4a}$.

Proof. As always, we assume that i is sufficiently large. The bounds (5.22) on $\eta(\theta)$ imply that

$$\frac{|\log\varepsilon| - 2C_5}{4k^2(k+1)} < \eta^{-1}\left(\frac{1}{2}|\log\varepsilon|\right) < \frac{|\log\varepsilon| - 2C_4}{2kC_2},$$

and with the bounds (6.11) on $\alpha(-T_4)$ we have (6.16). Further, as $\alpha \leqslant \eta^{-1}(\frac{1}{2}|\log\varepsilon|)$ whenever $(\tau,\alpha) \in U_{4a}$, we immediately obtain (6.17).

Now assume that $(-T_5,\alpha) \in U_{4a}$. From the definition (6.2) of T_5 and the formula (6.9) for $f_4(\alpha)$, we see that

$$\begin{aligned} T_5 - f_4(\alpha) + ke^{-\eta(\alpha)} &= -\varepsilon(k-1)\alpha + k\big(\sigma(\varepsilon T_3 + \varepsilon^2\alpha) - \sigma(\varepsilon T_3)\big) + 2\varepsilon^{1/2}k \\ &< \varepsilon^2 k\alpha\big(2k^2(k+1)C_3\big) + 2\varepsilon^{1/2}k \\ &< \varepsilon^2|\log\varepsilon|\left(\frac{k^2(k+1)C_3}{C_2}\right) + 2\varepsilon^{1/2}k < 3\varepsilon^{1/2}k, \end{aligned}$$

where the nonnegativity of α (because $\alpha \geqslant \alpha(-T_4) > 0$) and the bound (5.12) on $\dot{\sigma}(t)$ are used in obtaining the first inequality, and the upper bound (6.17) on α is used in obtaining the second inequality. From this we have that

$$e^{-\eta(\alpha)} < 3\varepsilon^{1/2} + \frac{-T_5 + f_4(\alpha)}{k} \leqslant \varepsilon^{1/2}\left(3 + \frac{C_6}{k}\right),$$

where the second inequality holds because $(-T_5, \alpha) \in U_{4a}$. Taking logarithms and using (5.22) gives

$$C_5 + 2k^2(k+1)\alpha > \eta(\alpha) > \frac{|\log \varepsilon|}{2} - \log\left(3 + \frac{C_6}{k}\right),$$

which implies (6.18), as desired. □

We see from (6.11) that

$$\big(-T_4, \alpha(-T_4)\big) \in U_{4a}$$

for the initial condition of the differential equation (6.9). With the next result, which establishes the invariance property for U_{4a} as in Proposition 3.1, we conclude from Lemma 6.2 that

$$\frac{|\log \varepsilon|}{5k^2(k+1)} < \alpha(-T_5) < \frac{|\log \varepsilon|}{2kC_2} \tag{6.19}$$

at the point where this solution leaves U_{4a}. The following result also justifies the ansatz (5.5) used above in (6.7). This is stated more explicitly in Corollary 6.4.

Proposition 6.3. *For large i the set U_{4a} with the differential equation* (6.9) *satisfies conclusion* (3) *of Proposition* 3.1 *with*

$$W_1(\tau, \alpha) = \alpha - \alpha(-T_4), \qquad W_2(\tau, \alpha) = -\alpha + \eta^{-1}\left(\frac{1}{2}|\log \varepsilon|\right),$$
$$W_3(\tau, \alpha) = \tau + f_4(\alpha), \qquad W_4(\tau, \alpha) = -\big(\tau + f_4(\alpha)\big) + \varepsilon^{1/2}C_6,$$
$$W_5(\tau, \alpha) = \tau + T_4, \qquad E(\tau, \alpha) = -\tau - T_5.$$

Thus $(\tau, \alpha(\tau)) \in U_{4a}$ for $-T_4 \leqslant \tau \leqslant -T_5$. Moreover, $\varepsilon\alpha'(\tau) = W_3(\tau, \alpha(\tau)) > 0$ throughout this range.

Proof. The six functions W_j, for $1 \leqslant j \leqslant 5$, and E, define the set U_{4a} as in the statement of Proposition 3.1. Taking i sufficiently large, let us verify the inward-pointing conditions $D_\tau W_j(\tau, \alpha) > 0$, where D_τ denotes the total derivative, as in Proposition 3.1. In the calculations below we always take $(\tau, \alpha) \in U_{4a}$ satisfying $W_j(\tau, \alpha) = 0$, as in the statement of that result. It is enough to consider the cases $1 \leqslant j \leqslant 4$, the case $j = 5$ being trivial.

When $j = 1$ we have, with $\alpha = \alpha(-T_4)$ because $W_1(\tau, \alpha) = 0$, that

$$\varepsilon D_\tau W_1(\tau, \alpha) = \varepsilon\alpha' = \tau + f_4\big(\alpha(-T_4)\big) \geqslant -T_4 + f_4\big(\alpha(-T_4)\big) > 0$$

by (6.11), as desired.

When $j = 3$ we have that

$$D_\tau W_3(\tau, \alpha) = 1 + f_4'(\alpha)\alpha' = 1 > 0,$$

as desired, since $\varepsilon\alpha' = \tau + f_4(\alpha) = 0$ from the differential equation (6.9) and because $W_3(\tau,\alpha) = 0$ is assumed.

When $j = 4$ we have again from (6.9) and the fact that $W_4(\tau,\alpha) = 0$, that $\varepsilon\alpha' = \varepsilon^{1/2}C_6$, and so

$$D_\tau W_4(\tau,\alpha) = -\bigl(1 + f_4'(\alpha)\alpha'\bigr) = -\left(1 + \frac{f_4'(\alpha)C_6}{\varepsilon^{1/2}}\right). \tag{6.20}$$

To verify the inward-pointing condition, we must show that the quantity in (6.20) is positive. Differentiating the formula (6.9) for $f_4(\alpha)$, we have

$$\begin{aligned} 1 + \frac{f_4'(\alpha)C_6}{\varepsilon^{1/2}} &= 1 + \varepsilon^{1/2}(k-1)C_6 - \varepsilon^{3/2}kC_6\dot{\sigma}\bigl(\varepsilon T_3 + \varepsilon^2\alpha\bigr) - \left(\frac{kC_6e^{-\eta(\alpha)}\eta'(\alpha)}{\varepsilon^{1/2}}\right) \\ &< 1 + \varepsilon^{1/2}(k-1)C_6 - kC_6\eta'(\alpha) \\ &< 1 + \varepsilon^{1/2}(k-1)C_6 - k^2C_2C_6 = -1 + \varepsilon^{1/2}(k-1)C_6 < 0, \end{aligned} \tag{6.21}$$

where we have used the fact that $\dot{\sigma}(t) > 0$ from (5.12), that $e^{-\eta(\alpha)} \geqslant \varepsilon^{1/2}$ following from $\alpha \leqslant \eta^{-1}(\frac{1}{2}|\log\varepsilon|)$ in (6.15), and as well the lower bound on $\eta'(\theta)$ in (5.21) and the definition (6.15) of C_6. As the final quantity in (6.21) is negative, the inward-pointing condition holds.

We lastly consider the case $j = 2$. We shall in fact show this case is vacuous, namely, that there are no points of U_{4a} satisfying $W_2(\tau,\alpha) = 0$. Assuming to the contrary that (τ,α) is such a point, we have from Lemma 6.2 that (6.17) holds. Also, $e^{-\eta(\alpha)} = \varepsilon^{1/2}$ holds. Using (6.17) along with the formula (6.9) for $f_4(\alpha)$ and the inequality $\dot{\sigma}(t) > 0$, we have for this point that

$$\begin{aligned} f_4(\alpha) &= 1 + (k-1)(T_3 + \varepsilon\alpha) - k\sigma\bigl(\varepsilon T_3 + \varepsilon^2\alpha\bigr) + \varepsilon^{1/2}k \\ &< 1 + (k-1)\left(T_3 + \frac{\varepsilon|\log\varepsilon|}{2kC_2}\right) - k\sigma(\varepsilon T_3) + \varepsilon^{1/2}k < T_5, \end{aligned}$$

where the definition (6.2) of T_5 is used in the final inequality. Thus

$$W_3(\tau,\alpha) = \tau + f_4(\alpha) \leqslant -T_5 + f_4(\alpha) < 0,$$

contradicting the assumption that $(\tau,\alpha) \in U_{4a}$.

The fact that the set U_{4a} is bounded implies that it is case (3) in the statement of Proposition 3.1 that holds, as claimed.

The equation in the final sentence of the proposition is just the differential equation (6.9), and this quantity is strictly positive for $-T_4 < \tau \leqslant -T_5$ by Proposition 3.1. Positivity of this quantity at $\tau = -T_4$ holds by Lemma 6.1. □

Corollary 6.4. *We have for large i that*

$$k + 1 - \varepsilon T_4 \leqslant t \leqslant k + 1 - \varepsilon T_5 \quad \Longrightarrow \quad \begin{cases} \dfrac{d}{dt}\bigl(t - r\bigl(x(t)\bigr)\bigr) > 0 & \text{and} \\ \varepsilon T_3 < t - r\bigl(x(t)\bigr) < k + 1 - \varepsilon T_4. \end{cases}$$

In particular, if t belongs to Interval III then $t - r(x(t))$ belongs to the interior of Interval II.

Proof. This follows from (6.5) and because $\alpha'(\tau) > 0$ for $-T_4 \leqslant \tau \leqslant -T_5$ by the final sentence of Proposition 6.3, and from the bounds $\alpha(-T_4) > 0$ in (6.11) and $\alpha(-T_5) < (2kC_2)^{-1}|\log\varepsilon|$ in (6.17). □

7. Interval IV

We finally consider the solution in Interval IV, namely for

$$k+1-\varepsilon T_5 \leqslant t \leqslant k+1+\frac{\varepsilon|\log\varepsilon|}{k-1}+\varepsilon T_6, \tag{7.1}$$

where $T_6 = T_{6,i}$ will be suitably chosen. This interval will contain the point at which $x(t)$ achieves its maximum, and as well will contain much of the vertical transition layer of the sawtooth near $t = k+1$. With this we will be able to compare the form of the solution near $t = 0$ and near $t = k+1$, on successive transition layers, and thereby obtain an asymptotic expression for the period of $x(t)$.

Our analysis for Interval IV will proceed in two steps. First, we use the same ansatz as for Interval III, that is, $t = k+1+\varepsilon\tau$ and the formula (6.4) for $x_c(t)$. However, we will use these formulas in a different region U_{4b} in the plane and with the interval

$$-T_5 \leqslant \tau \leqslant \frac{|\log\varepsilon|}{k-1}+T_6, \tag{7.2}$$

which corresponds to the interval (7.1). This will provide crude bounds on the solution. Following this, we will introduce a new ansatz with which we shall refine our estimates of the solution in the same interval (7.2).

Let us therefore define the set $U_{4b} = U_{4b,i}$ by

$$\begin{aligned} U_{4b} = \big\{(\tau,\alpha)\in\mathbf{R}^2 \;\big|\; &\tau \geqslant -T_5 \text{ and } \alpha(-T_5) \leqslant \alpha \leqslant \varepsilon^{-2}\big(k+1-\varepsilon(T_3+T_4)\big), \\ &\text{where also } \tau + f_4(\alpha) \geqslant 0\big\} \end{aligned} \tag{7.3}$$

for large i. The quantity $\alpha(-T_5)$ in the above definition is the same one as in the previous section. As $\alpha(-T_5) > \alpha(-T_4) > 0$, one sees that α in the above definition lies in the interval (6.8) where $f_4(\alpha)$ is defined, and thus U_{4b} is well-defined. And using the ansatz for Interval II exactly as in the previous section, one again obtains the differential equation (6.9) which is valid throughout the set U_{4b}.

Observe that U_{4b}, unlike U_{4a}, is unbounded in the positive τ-direction. We have not yet chosen the quantity T_6 which will define the right-hand endpoint of Interval IV. In fact this will be done via the following result, by taking the exit time from U_{4b}, with the exit occurring on the upper boundary of that set.

Proposition 7.1. *For large i the set U_{4b} with the differential equation* (6.9) *satisfies conclusion* (3) *of Proposition* 3.1 *with*

$$W_1(\tau,\alpha) = \alpha - \alpha(-T_5), \qquad W_2(\tau,\alpha) = \tau + f_4(\alpha),$$
$$W_3(\tau,\alpha) = \tau + T_5, \qquad E(\tau,\alpha) = -\alpha + \left(\frac{k+1-\varepsilon(T_3+T_4)}{\varepsilon^2}\right).$$

Thus $(\tau,\alpha(\tau)) \in U_{4b}$ for $-T_5 \leqslant \tau \leqslant \omega$ for some finite $\omega > -T_5$, with

$$\alpha(\omega) = \frac{k+1-\varepsilon(T_3+T_4)}{\varepsilon^2} \tag{7.4}$$

at the right-hand endpoint of this interval. Moreover, $\varepsilon\alpha'(\tau) = W_2(\tau,\alpha(\tau)) > 0$ holds throughout this interval.

Proof. Taking $(\tau,\alpha)\in U_{4b}$ with $W_j(\tau,\alpha)=0$ for some j, we must show that $D_\tau W_j(\tau,\alpha)>0$. The case $j=2$ is identical to the case $j=3$ of Proposition 6.3, and the case $j=3$ in the present proposition is trivial. For $j=1$ we have $\alpha=\alpha(-T_5)$ and thus

$$\varepsilon D_\tau W_1(\tau,\alpha)=\varepsilon\alpha'=\tau+f_4\bigl(\alpha(-T_5)\bigr)\geqslant -T_5+f_4\bigl(\alpha(-T_5)\bigr)=\varepsilon\alpha'(-T_5)>0,$$

which holds by the final sentence of Proposition 6.3.

There remains to show that conclusion (3) of Proposition 3.1 holds. Conclusion (2) is impossible due to the boundedness of the set U_{4b} in the α coordinate. Suppose then that conclusion (1) holds, namely, that $(\tau,\alpha(\tau))\in U_{4b}$ for all $\tau\geqslant -T_5$. Consider in particular $\tau\geqslant\tau_0$ where

$$\tau_0=1-\min_{\alpha\in J}f_4(\alpha),\qquad J=\bigl[\alpha(-T_5),\varepsilon^{-2}\bigl(k+1-\varepsilon(T_3+T_4)\bigr)\bigr].$$

Then $\tau+f_4(\alpha)\geqslant 1$ for all $\tau\geqslant\tau_0$ for which $(\tau,\alpha)\in U_{4b}$, and so $\alpha'(\tau)\geqslant\varepsilon^{-1}$ for all such τ. But then $(\tau,\alpha(\tau))$ would have to leave U_{4b} in finite time, due to the boundedness in α of this set, a contradiction. Thus conclusion (3) holds.

As in the proof of Proposition 6.3, the equation in the final sentence of the proposition is the differential equation (6.9), and the quantity there is strictly positive for $-T_5<\tau\leqslant\omega$ by Proposition 3.1. Positivity at $\tau=-T_5$ holds by Proposition 6.3. □

Now define the quantity

$$T_6=\omega-\frac{|\log\varepsilon|}{k-1},\tag{7.5}$$

where ω is as in Proposition 7.1. That is, ω is the right-hand endpoint of the interval (7.2) and is the exit time from U_{4b}. We have that $\omega=\omega_i$ and $T_6=T_{6,i}$ depend on the index i, and later in this section we shall show that $T_{6,i}$ is a bounded sequence in i, although this is not evident here. In this direction let us now introduce a new ansatz

$$x_c(k+1+\varepsilon\tau)=k-\varepsilon e^{(k-1)\tau}\zeta(\tau)\tag{7.6}$$

for the solution in Interval IV, that is, for τ in the same interval (7.2) as above. We are thus making a τ-dependent change of the independent variable, replacing α in (6.4) with

$$\zeta=e^{-(k-1)\tau}(\varepsilon\alpha+T_3-\tau),\tag{7.7}$$

and as such we may rewrite the differential equation (6.9) in terms of the new variable ζ. Indeed, after a straightforward but tedious calculation, we obtain

$$\begin{aligned}
&\zeta'=f_5(\tau,\zeta),\\
&f_5(\tau,\zeta)=f_{5,i}(\tau,\zeta)\\
&\qquad=ke^{-(k-1)\tau}\bigl(\tau-\sigma\bigl(\varepsilon\bigl(\tau+e^{(k-1)\tau}\zeta\bigr)\bigr)+\exp\bigl(-\eta\bigl(\varepsilon^{-1}\bigl(\tau+e^{(k-1)\tau}\zeta-T_3\bigr)\bigr)\bigr)\bigr).
\end{aligned}\tag{7.8}$$

The initial condition for Eq. (7.8) is taken at $\tau=-T_5$, and is given as

$$\zeta(-T_5)=H_5=e^{(k-1)T_5}\bigl(\varepsilon\alpha(-T_5)+T_3+T_5\bigr),\tag{7.9}$$

where the formula above is the definition of $H_5=H_{5,i}$. Under the above transformation of α to ζ, the set U_{4b} is mapped into (but not onto) the set $U_5=U_{5,i}$ given by

$$\begin{aligned} U_5 &= \{(\tau,\zeta)\in \mathbf{R}^2 \mid \tau \geqslant -T_5 \text{ and } T_3+\varepsilon\alpha(-T_5)\leqslant \tau + e^{(k-1)\tau}\zeta \leqslant \varepsilon^{-1}(k+1)-T_4\} \\ &= \{(\tau,\zeta)\in \mathbf{R}^2 \mid \tau \geqslant -T_5 \text{ and } -T_5+e^{-(k-1)T_5}H_5\leqslant \tau + e^{(k-1)\tau}\zeta \leqslant \varepsilon^{-1}(k+1)-T_4\}, \end{aligned} \tag{7.10}$$

where we have used (7.9). Note in particular that the constraint $\tau + f_4(\alpha) \geqslant 0$ in the definition (7.3) of the set U_{4b} has been omitted in the definition (7.10) of U_5, and so the image of U_{4b} under the transformation is strictly contained in U_5. Also note that the transformed differential equation (7.8) is nevertheless well-defined throughout the set U_5, with the arguments $\varepsilon(\tau + e^{(k-1)\tau}\zeta)$ and $\varepsilon^{-1}(\tau + e^{(k-1)\tau}\zeta - T_3)$ of σ and ζ always within the domains (5.10) and (5.17) of these functions.

The advantage of using the transformed differential equation (7.8) is that the nonlinearity f_5 enjoys an integrable bound, and this will imply that $\zeta(\tau)$ approaches a finite limit for large τ. We observe that

$$\begin{aligned} &\left|f_5(\tau,\zeta)\right| \leqslant ke^{-(k-1)\tau}\bigl(|\tau| + \max\{T_1,T_2\} + \exp\bigl(-\eta\bigl(\varepsilon^{-1}\bigl(\tau+e^{(k-1)\tau}\zeta - T_3\bigr)\bigr)\bigr)\bigr) \\ &\text{for every } (\tau,\zeta)\in U_5, \end{aligned} \tag{7.11}$$

which follows from the formula (7.8) for $f_5(\tau,\zeta)$ using the range (5.10) of $\sigma(t)$. We also note the bound

$$\begin{aligned} &\exp\bigl(-\eta\bigl(\varepsilon^{-1}\bigl(\tau+e^{(k-1)\tau}\zeta - T_3\bigr)\bigr)\bigr) \\ &\quad \leqslant e^{-\eta(\alpha(-T_5))} < e^{-C_4-kC_2\alpha(-T_5)} < \exp\left(-C_4 - \frac{C_2|\log\varepsilon|}{5k(k+1)}\right) < \varepsilon^{\rho}, \quad \rho = \frac{C_2}{6k(k+1)}, \end{aligned} \tag{7.12}$$

which is valid for large i and is obtained using the lower bound on $\tau + e^{(k-1)\tau}\zeta$ in (7.10), the monotonicity (5.21) of $\eta(\theta)$, and the estimates (5.22) and (6.19). Combining the bounds (7.11) and (7.12) and recalling the limit (4.23), we have that

$$\left|f_5(\tau,\zeta)\right| < g(\tau) = C_7 e^{-(k-1)\tau}\bigl(|\tau|+1\bigr) \tag{7.13}$$

throughout U_5, with $C_7 = k(\max\{T_1, Q_+ - Q_0\} + 1)$, and where the above formula serves as the definition of $g(\tau)$. Let us observe that $g(\tau)$ is both uniformly bounded and integrable as $\tau \to \infty$. Also note that as T_1 is independent of i (recall Corollary 4.4), then so are C_7 and thus $g(\tau)$.

We now wish to determine the limit of the solution sequence $\zeta_i(\tau)$ as $i \to \infty$. We do this by taking the appropriate limit in the differential equation (7.8) and the initial condition (7.9), making use of the bound (7.13). In doing this we shall maintain uniform estimates in τ, with the result that the limiting behavior of the solution for large τ will also be determined. Of course some care must be taken as the intervals on which the solutions are defined depend on i. The following result is the first step to this end.

Proposition 7.2. *We have the limits*

$$\begin{aligned} &\lim_{i\to\infty} H_{5,i} = H_{5,*} = e^{(k-1)T_{5,*}}(1+kQ_0-kQ_-), \\ &\lim_{i\to\infty} f_{5,i}(\tau,\zeta) = f_{5,*}(\tau) = ke^{-(k-1)\tau}(\tau - Q_+ + 2Q_0 - Q_-), \end{aligned} \tag{7.14}$$

where the above formulas serve as the definitions of $H_{5,*}$ and $f_{5,*}(\tau)$, and where $T_{5,*}$ is as in (6.3)*. The convergence of $f_{5,i}(\tau,\zeta)$ is uniform on compact subsets of $U_5 = U_{5,i}$ in the sense that*

$$\lim_{i\to\infty}\Bigl(\sup_{B\cap U_{5,i}}\bigl|f_{5,i}(\tau,\zeta)-f_{5,*}(\tau)\bigr|\Bigr)=0$$

for every compact set $B\subseteq\mathbf{R}^2$.

Proof. For the first limit in (7.14) we have from (7.9), using the bounds (6.19), and from (4.23), (5.2), and (6.3), that

$$\lim_{i\to\infty}H_{5,i}=e^{(k-1)T_{5,*}}(Q_+-Q_0+T_{5,*})=e^{(k-1)T_{5,*}}\bigl(Q_+-Q_0+1-Q_++(k+1)Q_0-kQ_-\bigr).$$

The second limit in (7.14) follows from the formula (7.8) for $f_{5,i}(\tau,\zeta)$, with the limit (5.13) and the bound (7.12). □

Motivated by the above results, let us define the function

$$\begin{aligned}\zeta_*(\tau)&=H_{5,*}+\int_{-T_{5,*}}^{\tau}f_{5,*}(s)\,ds\\&=\frac{e^{(k-1)T_{5,*}}}{(k-1)^2}-\frac{ke^{-(k-1)\tau}}{(k-1)^2}\bigl(1+(k-1)(\tau-Q_++2Q_0-Q_-)\bigr),\end{aligned}\tag{7.15}$$

and as well denoting

$$\zeta_*(\infty)=\frac{e^{(k-1)T_{5,*}}}{(k-1)^2}.\tag{7.16}$$

As the following result shows, this function is the limit of the sequence of solutions $\zeta_i(\tau)$.

Proposition 7.3. *There exists a uniform bound*

$$\bigl|\zeta_i(\tau)\bigr|<C_8\quad\textit{for every }\tau\in[-T_{5,i},\omega_i],$$

with C_8 independent of large i. Moreover, let $\tau_i\in[-T_{5,i},\omega_i]$ be any sequence for which the (possibly infinite) limit $\lim_{i\to\infty}\tau_i=\tau_*$ exists. Then

$$\lim_{i\to\infty}\zeta_i(\tau_i)=\zeta_*(\tau_*),\qquad\lim_{i\to\infty}\zeta_i'(\tau_i)=\zeta_*'(\tau_*)\tag{7.17}$$

both hold, where we interpret $\zeta_*'(\infty)=0$.

Proof. Writing

$$\zeta_i(\tau)=H_{5,i}+\int_{-T_{5,i}}^{\tau}f_{5,i}\bigl(s,\zeta_i(s)\bigr)\,ds\tag{7.18}$$

for any $\tau\in[-T_{5,i},\omega_i]$, we obtain the desired bound

$$\bigl|\zeta_i(\tau)\bigr|<|H_{5,i}|+\int_{-T_{5,i}}^{\infty}g(s)\,ds\leqslant C_8\tag{7.19}$$

for some C_8 independent of i and τ, in light of the bound (7.13), and because $H_{5,i}$ and $T_{5,i}$ are bounded sequences.

Now taking a sequence τ_i as in the statement of the proposition, suppose first that $\tau_* < \infty$. Then with $\tau = \tau_i$ in (7.18), we may take the limit there by virtue of Proposition 7.2 and the bound (7.19), to obtain the first limit in (7.17). The second limit in (7.17) follows by writing $\zeta_i'(\tau_i) = f_{5,i}(\tau_i, \zeta_i(\tau_i))$ and taking the limit there.

Now suppose that $\tau_* = \infty$. Fixing any $T > -T_{5,*}$, we have $\tau_i \in [T, \omega_i]$ for large i. Integrating (7.8) from T to τ_i, we have that

$$\left|\zeta_i(\tau_i) - \zeta_*(\tau_i)\right| < \left|\zeta_i(T) - \zeta_*(T)\right| + 2\int_T^\infty g(s)\,ds$$

from the bound (7.13). Thus

$$\limsup_{i\to\infty}\left|\zeta_i(\tau_i) - \zeta_*(\tau_i)\right| \leqslant \limsup_{i\to\infty}\left|\zeta_i(T) - \zeta_*(T)\right| + 2\int_T^\infty g(s) = 2\int_T^\infty g(s)\,ds,$$

in particular because $\lim_{i\to\infty}\zeta_i(T) = \zeta_*(T)$ from the first part of this proof. As T is arbitrary, we obtain the first limit in (7.17). For the second limit in (7.17) we have that

$$\left|\zeta_i'(\tau_i)\right| = \left|f_{5,i}\big(\tau_i, \zeta_i(\tau_i)\big)\right| < g(\tau_i) \to 0,$$

as desired. □

Let us now show that the sequence $T_{6,i}$ is bounded, and in fact approaches a specific limit.

Proposition 7.4. *The limit*

$$T_{6,*} = \lim_{i\to\infty} T_{6,i} = -1 + 2Q_+ - (k+1)Q_0 + (k+2)Q_- \tag{7.20}$$

holds, where the above formula serves as the definition of $T_{6,*}$.

Proof. Inserting Eq. (7.4) into the change of variables formula (7.7), we obtain

$$\zeta(\omega) = e^{-(k-1)\omega}\left(\frac{k+1}{\varepsilon} - T_4 - \omega\right).$$

Then making the replacement (7.5) for $\omega = \omega_i$ in the right-hand side of the above equation, we obtain

$$\zeta(\omega) = e^{-(k-1)T_6}\left(k + 1 - \varepsilon\left(T_4 + T_6 + \frac{|\log\varepsilon|}{k-1}\right)\right). \tag{7.21}$$

Also, by passing to a subsequence, we may assume without loss that the (possibly infinite) limit $T_6 = T_{6,i} \to T_{6,*}$ exists. We shall show that in fact only one possible such limit $T_{6,*}$ can occur, namely the one given in (7.20).

Suppose first that $T_{6,*} = -\infty$. Then we have from (7.21) that

$$\liminf_{i\to\infty} e^{(k-1)T_{6,i}}\zeta_i(\omega_i) \geqslant k+1,$$

and thus $\zeta_i(\omega_i) \to \infty$. But this contradicts Proposition 7.3.

Next suppose that $T_{6,*} = \infty$. Then $\omega_i \to \infty$ by (7.5), and so the left-hand side of (7.21) approaches $\zeta_*(\infty)$ by Proposition 7.3. But the right-hand side of (7.21) approaches zero as $i \to \infty$, contradicting (7.16).

Thus $T_{6,*}$ is a finite quantity. Again from (7.5) we have that $\omega_i \to \infty$, and so taking the limit in Eq. (7.21) gives $\zeta_*(\infty) = e^{-(k-1)T_{6,*}}(k+1)$, and so

$$e^{(k-1)T_{6,*}} = (k+1)(k-1)^2 e^{-(k-1)T_{5,*}}$$

by (7.16). From this, using (2.9) and (6.3), we obtain (7.20), as desired. □

Proposition 7.5. *We have for large i that*

$$k+1-\varepsilon T_5 \leqslant t \leqslant k+1+\frac{\varepsilon|\log\varepsilon|}{k-1}+\varepsilon T_6 \quad\Longrightarrow\quad \begin{cases} \dfrac{d}{dt}\big(t-r\big(x(t)\big)\big) > 0 \quad \text{and} \\ \varepsilon T_3 < t-r\big(x(t)\big) \leqslant k+1-\varepsilon T_4. \end{cases}$$

In particular, if t belongs to Interval IV then $t-r(x(t))$ belongs to Interval II.

Proof. As in the proof of Corollary 6.4 we use (6.5), and as well the fact that $\alpha'(\tau) > 0$ for $-T_5 \leqslant \tau \leqslant \omega$ from the final sentence in Proposition 7.1. We also note that $t-r(x(t))$ belongs to the interior of Interval II for $t = k+1-\varepsilon T_5$, by Corollary 6.4. For $t = k+1+\varepsilon|\log\varepsilon|(k-1)^{-1}+\varepsilon T_6$ one checks that $t-r(x(t)) = k+1-\varepsilon T_4$ using (7.4) and (7.5). □

8. The minimum, the maximum, and the period

This section is devoted to the proof of Theorem B. Prior to giving this proof, we need the following technical result.

Lemma 8.1. *Fix any $\beta > 0$. Then the limit*

$$\lim_{i\to\infty} \frac{\eta_i(\beta|\log\varepsilon_i|)}{|\log\varepsilon_i|} = \beta\big(k^2-1\big)(k-1) \tag{8.1}$$

holds.

Proof. We have from the differential equation (5.19) for $\eta(\theta)$ that

$$\frac{\eta_i(\beta|\log\varepsilon_i|)-\eta_i(0)}{|\log\varepsilon_i|} = \int_0^\beta \eta_i'\big(s|\log\varepsilon_i|\big)\,ds = \int_0^\beta f_{3,i}\big(s|\log\varepsilon_i|, \eta_i\big(s|\log\varepsilon_i|\big)\big)\,ds. \tag{8.2}$$

The integrand is uniformly bounded in i, in light of (5.21). If we show also that it enjoys the pointwise convergence property

$$\lim_{i\to\infty} f_{3,i}\big(s|\log\varepsilon_i|, \eta_i\big(s|\log\varepsilon_i|\big)\big) = \big(k^2-1\big)(k-1) \tag{8.3}$$

for every $0 < s \leqslant \beta$, then (8.1) follows from (8.2) and from the boundedness of the sequence $\eta_i(0)$, using the Lebesgue dominated convergence theorem.

To establish (8.3), we note the limit

$$\lim_{i\to\infty} \sigma_i\big(\varepsilon_i T_{3,i}+\varepsilon_i^2\theta\big)\big|_{\theta=s|\log\varepsilon_i|} = Q_+ - 2Q_0 + Q_- \tag{8.4}$$

of one of the terms appearing in the formula (5.19) for $f_{3,i}(\theta,\eta)$ arising when we attempt to take the limit (8.3). In particular, (8.4) holds by (5.13) of Lemma 5.1. Also noting that $e^{-\eta_i(s|\log\varepsilon_i|)} \to 0$ as $i\to\infty$ from the growth estimate (5.22) for $\eta_i(\theta)$ and because $s>0$, we conclude with the help of Proposition 4.1 and the formulas (2.9) and (5.19) that

$$\lim_{i\to\infty} f_{3,i}\big(s|\log\varepsilon_i|, \eta_i\big(s|\log\varepsilon_i|\big)\big) = -k\varphi_*'(Q_+ - 2Q_0 + Q_-) = \big(k^2-1\big)(k-1),$$

as desired. □

Proof of Theorem B. As we have not yet proved there is a unique SOPS $x_\varepsilon(t)$ as per Theorem A, we continue to work with a sequence $x_i(t)$ of SOPS's with $\varepsilon_i \to 0$.

To determine the location of the minimum of $x(t) = x_i(t)$, we take $t = \varepsilon T_3 + \beta\varepsilon^2|\log\varepsilon|$ with $\beta>0$ fixed independent of i. Then t belongs to Interval II for sufficiently large i. For such t we have, from the ansatz (5.5), that

$$\dot x\big(\varepsilon T_3 + \beta\varepsilon^2|\log\varepsilon|\big) = 1 - \varepsilon\dot\sigma\big(\varepsilon T_3 + \beta\varepsilon^2|\log\varepsilon|\big) - \left(\frac{e^{-\eta(\beta|\log\varepsilon|)}\eta'(\beta|\log\varepsilon|)}{\varepsilon}\right). \tag{8.5}$$

Now using Lemma 8.1, which implies that $e^{-\eta(\beta|\log\varepsilon|)}$ decays roughly like ε^ρ for $\rho = \beta(k^2-1)(k-1)$, and as well noting the uniform positive bounds (5.21) on $\eta'(\theta)$ and the bounds (5.12) on $\dot\sigma(t)$, we have from (8.5) that

$$\lim_{i\to\infty} \dot x_i\big(\varepsilon_i T_{3,i} + \beta\varepsilon_i^2|\log\varepsilon_i|\big) = \begin{cases} -\infty, & \text{if } \beta < \frac{1}{(k^2-1)(k-1)},\\ 1, & \text{if } \beta > \frac{1}{(k^2-1)(k-1)}.\end{cases}$$

It follows that the location $t = q_0 = q_{0,i}$ of the minimum of $x(t)$ in Interval II satisfies

$$q_0 = \varepsilon T_3 + \frac{\varepsilon^2|\log\varepsilon|}{(k^2-1)(k-1)} + o\big(\varepsilon^2|\log\varepsilon|\big),$$

and with the formulas (4.23) and (5.2) we obtain the first line of Table 1. As for the second line of Table 1, that is, the value of the solution at this point, first let β_i be defined by $q_{0,i} = \varepsilon_i T_{3,i} + \beta_i\varepsilon_i^2|\log\varepsilon_i|$. Thus $\beta_i \to (k^2-1)^{-1}(k-1)^{-1}$ as $i\to\infty$, and we have again from the ansatz (5.5) that

$$\begin{aligned}\min_{t\in\mathbf{R}} x(t) = x(q_0) &= -1 + \varepsilon T_3 - \varepsilon\sigma(q_0) + \varepsilon e^{-\eta(\beta_i|\log\varepsilon|)} + o(\varepsilon)\\ &= -1 + \varepsilon(Q_+ - Q_0) - \varepsilon(Q_+ - 2Q_0 + Q_-) + o(\varepsilon) = -1 + \varepsilon(Q_0 - Q_-) + o(\varepsilon),\end{aligned}$$

where (4.23), (5.2), and (5.13) are used.

To determine the location of the maximum of $x(t)$, we take $t = k+1+\varepsilon\tau$ where $\tau > -T_{5,*}$ is fixed. Thus t belongs to Interval IV if ε is small enough. Note that here we are determining the location of the point q_1, to the right of zero. The point q_{-1} as in the statement of Theorem B lies to the left of zero and is given by $q_{-1} = q_1 - p$ where p is the period. From the ansatz (7.6) and from (7.15) with Proposition 7.3 we have that

$$\lim_{i\to\infty} \dot x_i(k+1+\varepsilon_i\tau) = -\lim_{i\to\infty}\big(e^{(k-1)\tau}\zeta_i(\tau)\big)' = -\big(e^{(k-1)\tau}\zeta_*(\tau)\big)' = \frac{k - e^{(k-1)(T_{5,*}+\tau)}}{k-1}.$$

The value of τ at which this limiting value vanishes is

$$\tau_0 = -T_{5,*} + \frac{\log k}{k-1} = -T_{5,*} + Q_0,$$

and the limiting value is positive for $-T_{5,*} < \tau < \tau_0$, and negative for $\tau > \tau_0$. It follows that the location $t = q_1 = q_{1,i}$ of the maximum of $x(t)$ in Interval IV satisfies

$$\begin{aligned} q_1 &= k + 1 + \varepsilon(-T_{5,*} + Q_0) + o(\varepsilon) \\ &= k + 1 + \varepsilon(-1 + Q_+ - kQ_0 + kQ_-) + o(\varepsilon), \end{aligned} \tag{8.6}$$

where we have used the value (6.3) of $T_{5,*}$. The value of the solution at this point is

$$\begin{aligned} \max_{t\in\mathbf{R}} x(t) = x(q_1) &= k - \varepsilon e^{(k-1)\tau_0}\zeta_*(\tau_0) + o(\varepsilon) \\ &= k - \frac{\varepsilon e^{(k-1)(T_{5,*}+\tau_0)}}{(k-1)^2} + \frac{\varepsilon k}{(k-1)^2}\big(1 + (k-1)(\tau_0 - Q_+ + 2Q_0 - Q_-)\big) + o(\varepsilon) \\ &= k + \frac{\varepsilon k}{k-1}(\tau_0 - Q_+ + 2Q_0 - Q_-) + o(\varepsilon) \\ &= k - \frac{\varepsilon k}{k-1}\big(1 + (k-2)Q_0 - (k-1)Q_-\big) + o(\varepsilon), \end{aligned}$$

where again the ansatz (7.6), with (7.15) and Proposition 7.3, are used. This gives the fourth line of Table 1.

Consider now the period $p = p_i$ of $x(t)$. Fix $\beta < T_{6,*}$ and take

$$t = k + 1 + \frac{\varepsilon|\log\varepsilon|}{k-1} + \varepsilon\beta,$$

noting that this point lies in Interval IV for large i. Then

$$\begin{aligned} &\lim_{i\to\infty} x_i\big(k + 1 + \varepsilon_i|\log\varepsilon_i|(k-1)^{-1} + \varepsilon_i\beta\big) \\ &\quad = k - \lim_{i\to\infty}\big(\varepsilon_i e^{|\log\varepsilon_i| + (k-1)\beta}\zeta_i\big(|\log\varepsilon_i|(k-1)^{-1} + \beta\big)\big) \\ &\quad = k - \lim_{i\to\infty}\big(e^{(k-1)\beta}\zeta_i\big(|\log\varepsilon_i|(k-1)^{-1} + \beta\big)\big) \\ &\quad = k - e^{(k-1)\beta}\zeta_*(\infty) = k - \left(\frac{e^{(k-1)(T_{5,*}+\beta)}}{(k-1)^2}\right). \end{aligned}$$

The value of β at which this limiting value vanishes is

$$\beta_0 = -T_{5,*} + \frac{\log(k(k-1)^2)}{k-1} = -T_{5,*} + Q_0 + 2Q_- = -1 + Q_+ - kQ_0 + (k+2)Q_-,$$

and arguing as before gives the period of $x(t)$ as

$$\begin{aligned} p &= k + 1 + \frac{\varepsilon|\log\varepsilon|}{k-1} + \varepsilon\beta_0 + o(\varepsilon) \\ &= k + 1 + \frac{\varepsilon|\log\varepsilon|}{k-1} + \varepsilon\big(-1 + Q_+ - kQ_0 + (k+2)Q_-\big) + o(\varepsilon). \end{aligned} \tag{8.7}$$

This gives the fifth line of Table 1. We note in particular that the condition $\beta_0 < T_{6,*}$, as required above, holds by (7.20).

Finally, from (8.6) and (8.7) we calculate $q_{-1} = q_1 - p$ to obtain the third line of Table 1. This completes the proof of the theorem. □

9. Superstability

This section is devoted to the proof of Part 2 of Theorem A. Again, we work with a sequence $x_i(t)$ of SOPS's, with $\varepsilon = \varepsilon_i \to 0$, and we shall obtain the relevant conclusions, including the estimate (2.8), for large i. Here $y(t)$ will denote a general solution of the linear variational equation (2.11), with initial condition $\psi \in X = C(J)$ with $J = [-\varepsilon T_1, \varepsilon T_2]$ as in (2.13). Note that the interval $J = J_i$ and thus the space $X = X_i$ depend on the index i of our sequence. The proof of Part 2 of Theorem A will be given following several lemmas which provide bounds for $y(t)$ in various intervals. Prior to proving these lemmas, we establish the following basic existence result for the linear variational equation which is needed to ensure that the monodromy operator M is well-defined.

Proposition 9.1. *We have for large i that*

$$t - r\big(x(t)\big) > -\varepsilon T_1 \quad \text{whenever } t \geqslant \varepsilon T_2. \tag{9.1}$$

Thus for any given $\psi \in X$, Eq. (2.11) *with the initial condition* (2.13) *has a unique solution $y(t)$ for $t \geqslant \varepsilon T_2$.*

Proof. It is enough to prove (9.1). In fact, by (2.3) it is enough to prove that $t - r(x(t)) > -\varepsilon T_1$ for $t = \varepsilon T_2$. But this follows immediately from Corollary 5.3 using the fact that $T_2 > T_3$, by (5.2). □

Our estimates on $y(t)$ will be derived using the variation of constants formula

$$y(t) = A(t, t_0)y(t_0) - \frac{k}{\varepsilon}\int_{t_0}^{t} A(t, s)y\big(s - r\big(x(s)\big)\big)\, ds \tag{9.2}$$

for the variational equation (2.11), where $t, t_0 \geqslant \varepsilon T_2$ and where

$$A(t, s) = \exp\left(\int_{s}^{t} \frac{a(u)}{\varepsilon}\, du\right), \tag{9.3}$$

where $a(t)$ is as in (2.11). It is useful to note that

$$\dot{x}(t) = A(t, t_0)\dot{x}(t_0) - \frac{k}{\varepsilon}\int_{t_0}^{t} A(t, s)\dot{x}\big(s - r\big(x(s)\big)\big)\, ds, \tag{9.4}$$

as $\dot{x}(t)$ satisfies the variational equation. We define the function $\psi_0 \in X$ to be the initial condition of the solution $\dot{x}(t)$, as in (2.15), and note that its norm in X satisfies

$$\|\psi_0\| < \frac{k(k+1)}{\varepsilon} \tag{9.5}$$

from the general bound (4.2).

We now estimate $y(t)$ in various intervals, beginning with Interval II.

Lemma 9.2. *There exists a constant C_9 such that for large i, the bound*

$$\left|y(t)\right| \leqslant \varepsilon C_9\left(\left|\dot{x}(t)\right| + 1\right)\|\psi\| \quad \text{for every } t \in [\varepsilon T_3, k+1-\varepsilon T_4] \tag{9.6}$$

holds for t in Interval II, for any solution of (2.11), (2.13)*.*

Proof. Without loss $\|\psi\| = 1$. With $t_0 = \varepsilon T_3$, which lies in Interval II, and taking t in Interval II, we have from Corollary 5.3 that the argument $s - r(x(s))$ in the integrand of (9.2) lies in Interval I. The point εT_3 also lies in Interval I, and thus

$$\left|y(\varepsilon T_3)\right| = \left|\psi(\varepsilon T_3)\right| \leqslant 1, \qquad \left|y\left(s - r\left(x(s)\right)\right)\right| = \left|\psi\left(s - r\left(x(s)\right)\right)\right| \leqslant 1,$$

in (9.2). Also note the bounds

$$\varepsilon\dot{x}(\varepsilon T_3) < -C_2, \qquad -k(k+1) < \varepsilon\dot{x}\left(t - r\left(x(t)\right)\right) < -C_2,$$

again for t in Interval II and thus with $t - r(x(t))$ in Interval I, and which follow from (4.2) and (4.20). We thus have from the above estimates and from (9.2), and also from (9.4) and (9.5), that

$$\begin{aligned}
\left|y(t)\right| &\leqslant A(t, \varepsilon T_3) + \frac{k}{\varepsilon}\int_{\varepsilon T_3}^{t} A(t,s)\,ds < -\left(\frac{\varepsilon A(t,\varepsilon T_3)\dot{x}(\varepsilon T_3)}{C_2}\right) + \frac{k}{\varepsilon}\int_{\varepsilon T_3}^{t} A(t,s)\,ds \\
&= -\left(\frac{\varepsilon\dot{x}(t)}{C_2}\right) - \frac{k}{C_2}\int_{\varepsilon T_3}^{t} A(t,s)\dot{x}\left(s - r\left(x(s)\right)\right)ds + \frac{k}{\varepsilon}\int_{\varepsilon T_3}^{t} A(t,s)\,ds \\
&\leqslant -\left(\frac{\varepsilon\dot{x}(t)}{C_2}\right) + \frac{k}{\varepsilon}\left(\frac{k(k+1)}{C_2} + 1\right)\int_{\varepsilon T_3}^{t} A(t,s)\,ds
\end{aligned} \tag{9.7}$$

for t in Interval II. We have the bound $a(t) < k\dot{x}(t - r(x(t))) < -\varepsilon^{-1}kC_2$ for t in Interval II, and thus $A(t,s) < e^{-\varepsilon^{-2}kC_2(t-s)}$ for $t > s$ both in this interval. Inserting this bound into the final integral in (9.7) gives

$$\left|y(t)\right| < \varepsilon\left(\frac{|\dot{x}(t)|}{C_2} + \frac{k(k+1) + C_2}{C_2^2}\right),$$

which establishes the result with $C_9 = \max\{C_2^{-1}, C_2^{-2}(k(k+1) + C_2)\}$. □

The next two lemmas provide bounds for $y(t)$ in Intervals III and IV, respectively.

Lemma 9.3. *For large i, the bound*

$$\left|y(t)\right| \leqslant 2\varepsilon C_9\|\psi\| \quad \text{for every } t \in [k+1-\varepsilon T_4, k+1-\varepsilon T_5] \tag{9.8}$$

holds for t in Interval III, for any solution of (2.11), (2.13)*. Here C_9 is as in Lemma* 9.2*.*

Proof. Without loss $\|\psi\| = 1$. We first estimate the coefficient $a(t)$ in Eq. (2.11). Writing $t = k+1+\varepsilon\tau$, where t is in Interval III, and using the ansatz (6.4) for this interval, we have the formula (6.5) for the historical time, namely $t - r(x(t)) = \varepsilon T_3 + \varepsilon^2\alpha(\tau)$. Now $t - r(x(t))$ belongs to Interval II, by Corollary 6.4, and thus from the ansatz (5.5) for Interval II we have that

$$\dot{x}\big(t - r\big(x(t)\big)\big) = \dot{x}\big(\varepsilon T_3 + \varepsilon^2 \alpha(\tau)\big) = 1 - \varepsilon \dot{\sigma}\left(\varepsilon T_3 + \varepsilon^2 \alpha(\tau)\right) - \left(\frac{e^{-\eta(\alpha(\tau))}\eta'(\alpha(\tau))}{\varepsilon}\right)$$
$$< 1 - \left(\frac{e^{-\eta(\alpha(\tau))}\eta'(\alpha(\tau))}{\varepsilon}\right) < 1 - \left(\frac{kC_2}{\varepsilon^{1/2}}\right). \tag{9.9}$$

We have used (5.12) to obtain the first inequality above. The second inequality follows from (5.21) and also from the fact that $(\tau, \alpha(\tau)) \in U_{4a}$ as t is in Interval III, implying $\alpha(\tau) \leqslant \eta^{-1}(\frac{1}{2}|\log \varepsilon|)$ as per the definition (6.15) of U_{4a}. We thus have for large i that

$$a(t) < k\dot{x}\big(t - r\big(x(t)\big)\big) < k - \left(\frac{k^2 C_2}{\varepsilon^{1/2}}\right) < -\left(\frac{k^2 C_2}{2\varepsilon^{1/2}}\right) \tag{9.10}$$

from (9.9) and from the definition (2.11) of $a(t)$.

Now let us write the differential equation (2.11) as

$$\varepsilon \dot{y} = a(t)y + b(t), \quad b(t) = -ky\big(t - r\big(x(t)\big)\big), \tag{9.11}$$

where we regard $b(t)$ as a known forcing term in a linear equation for y. Again noting that $t - r(x(t))$ belongs to Interval II, we see from (9.6) of Lemma 9.2 that

$$\big|b(t)\big| \leqslant \varepsilon k C_9\big(\big|\dot{x}\big(t - r\big(x(t)\big)\big)\big| + 1\big)$$
$$= \varepsilon k C_9\left(\frac{|a(t) + 1|}{k} + 1\right) < 2\varepsilon C_9\big|a(t)\big|, \tag{9.12}$$

with the final inequality holding as $a(t)$ is negative with large norm in view of (9.10). It follows immediately from (9.12) and from the fact that $a(t) < 0$ that Proposition 3.1 can be applied to the set

$$U_6 = \big\{(t, y) \in \mathbf{R}^2 \;\big|\; k + 1 - \varepsilon T_4 \leqslant t \leqslant k + 1 - \varepsilon T_5 \text{ and } |y| \leqslant 2\varepsilon C_9\big\}$$

for Eq. (9.11), and that one concludes that any solution of (9.11) in the set U_6 can only exit that set on the right-hand boundary $t = k + 1 - \varepsilon T_5$. Thus in order to establish the desired bound (9.8) for the solution, it is enough to show that $|y(k + 1 - \varepsilon T_4)| \leqslant 2\varepsilon C_9$. We have that

$$\big|\dot{x}(k + 1 - \varepsilon T_4)\big| = \big|1 - \varepsilon\alpha'(-T_4)\big| = \big|1 + T_4 - f_4\big(\alpha(-T_4)\big)\big| < 1 - 4\varepsilon k^2(k + 1)C_3 < 1$$

by (6.4), (6.9), and (6.11). With (9.6) this now gives

$$\big|y(k + 1 - \varepsilon T_4)\big| \leqslant \varepsilon C_9\big(\big|\dot{x}(k + 1 - \varepsilon T_4)\big| + 1\big) < 2\varepsilon C_9,$$

as desired. □

Lemma 9.4. *There exists a constant C_{10} such that for large i, the bound*

$$\big|y(t)\big| \leqslant C_{10}\|\psi\| \quad \text{for every } t \in \big[k + 1 - \varepsilon T_5, k + 1 + \varepsilon|\log \varepsilon|(k - 1)^{-1} + \varepsilon T_6\big], \tag{9.13}$$

holds for t in Interval IV, and also such that

$$\int_{t_4}^{t_6} \big|y(s)\big|\, ds \leqslant \varepsilon C_{10}\|\psi\|. \tag{9.14}$$

Here $t_4 = t_{4,i}$ and $t_6 = t_{6,i}$ are defined as

$$t_4 = k + 1 - \varepsilon T_4, \qquad t_6 = k + 1 + \frac{\varepsilon |\log \varepsilon|}{k-1} + \varepsilon T_6, \tag{9.15}$$

namely the left-hand endpoint of Interval III and the right-hand endpoint of Interval IV, respectively.

Proof. Again without loss $\|\psi\| = 1$. In this proof we consider t in both Intervals III and IV, that is, $t_4 \leqslant t \leqslant t_6$. Note that in any case, the quantity $t - r(x(t))$ belongs to Interval II, by either Corollary 6.4 or by Proposition 7.5.

By (2.12) we have that $a(t) < k - 1$. Recalling the definition (9.3) of $A(t,s)$, we see that

$$\begin{aligned} A(t,t_4) &\leqslant e^{\varepsilon^{-1}(k-1)(t-t_4)} \leqslant \frac{e^{(k-1)(T_4+T_6)}}{\varepsilon}, \\ \int_{t_4}^{t} A(t,s)\,ds &< \frac{\varepsilon e^{\varepsilon^{-1}(k-1)(t-t_4)}}{k-1} \leqslant \frac{e^{(k-1)(T_4+T_6)}}{k-1}, \\ \int_{t_4}^{t}\int_{t_4}^{s} A(t,u)\,du\,ds &< \frac{\varepsilon^2 e^{\varepsilon^{-1}(k-1)(t-t_4)}}{(k-1)^2} \leqslant \frac{\varepsilon e^{(k-1)(T_4+T_6)}}{(k-1)^2}. \end{aligned} \tag{9.16}$$

Note that some of the inequalities in (9.16) are obtained by setting $t = t_6$, which is the right-hand endpoint of Interval IV.

We again estimate $y(t)$ using the variation of constants formula (9.2). We take $t_0 = t_4$ there, and so the argument $s - r(x(s))$ in the integrand of (9.2) lies in Interval II. Thus by Lemma 9.2 we have for such s that

$$\begin{aligned} \bigl|y\bigl(s - r\bigl(x(s)\bigr)\bigr)\bigr| &\leqslant \varepsilon C_9\bigl(\bigl|\dot{x}\bigl(s - r\bigl(x(s)\bigr)\bigr)\bigr| + 1\bigr) \\ &\leqslant \varepsilon C_9\bigl(\bigl|\dot{x}\bigl(s - r\bigl(x(s)\bigr)\bigr) - 1\bigr| + 2\bigr) = \varepsilon C_9\bigl(-\dot{x}\bigl(s - r\bigl(x(s)\bigr)\bigr) + 3\bigr), \end{aligned} \tag{9.17}$$

where (2.3) has been used to obtain the final equality in (9.17). Let us also note the bounds

$$\bigl|y(t_4)\bigr| \leqslant 2\varepsilon C_9, \qquad \bigl|\dot{x}(t_4)\bigr| < 2k(k+1)C_9, \tag{9.18}$$

which hold by (9.8) of Lemma 9.3, and because $\dot{x}(t)$ is a solution of (2.11) with initial condition $\dot{x}(t) = \psi_0(t)$ satisfying the bound (9.5). We thus obtain from the variation of constants formula (9.2), using first the bounds (9.17) and (9.18), and then the formula (9.4), that

$$\begin{aligned} \bigl|y(t)\bigr| &\leqslant 2\varepsilon C_9 A(t,t_4) + kC_9 \int_{t_4}^{t} A(t,s)\bigl(-\dot{x}\bigl(s - r\bigl(x(s)\bigr)\bigr) + 3\bigr)\,ds \\ &= \varepsilon C_9\bigl(2 - \dot{x}(t_4)\bigr)A(t,t_4) + \varepsilon C_9 \dot{x}(t) + 3kC_9 \int_{t_4}^{t} A(t,s)\,ds. \end{aligned} \tag{9.19}$$

The desired pointwise bound (9.13) now follows immediately from (9.19) using the bounds (9.16), the bound (9.18) on $\dot{x}(t_4)$, and the general bound (4.2) on $\dot{x}(t)$.

For the integral bound (9.14), we integrate the inequality (9.19) from t_4 to t_6, and again use the bounds in (9.16). We also note, in performing this integration, that the term $\varepsilon C_9\dot{x}(t)$ in (9.19) does not contain an absolute value sign, and so integrating it contributes a term $\varepsilon C_9(x(t_6) - x(t_4))$ of size $O(\varepsilon)$. We have that

$$\int_{t_4}^{t_6} \bigl|y(s)\bigr|\,ds < \varepsilon C_9\left(\frac{2+|\dot{x}(t_4)|}{k-1}\right)e^{(k-1)(T_4+T_6)} + \varepsilon C_9\bigl(x(t_6) - x(t_4)\bigr) + \varepsilon\left(\frac{3kC_9}{(k-1)^2}\right)e^{(k-1)(T_4+T_6)},$$

after a straightforward calculation. This, with the bound (9.18) on $\dot{x}(t_4)$, gives the desired result (9.14). □

Lemma 9.5. *Let $T_7 = T_{7,i}$ and $T_8 = T_{8,i}$ be given by*

$$T_7 = -\varepsilon T_1 + p - r\bigl(x(-\varepsilon T_1 + p)\bigr), \qquad T_8 = \varepsilon T_2 + p - r\bigl(x(\varepsilon T_2 + p)\bigr), \tag{9.20}$$

where $p = p_i$ is the period of $x(t)$. Then

$$\varepsilon T_3 < T_7 < T_8 < k + 1 + \frac{\varepsilon|\log\varepsilon|}{k-1} + \varepsilon T_6 \tag{9.21}$$

holds for large i. Thus the interval $[T_7, T_8]$ is contained in the union of Intervals II, III, and IV.

Also, we have that

$$k + 1 - \varepsilon T_5 < -\varepsilon T_1 + p < k + 1 + \frac{\varepsilon|\log\varepsilon|}{k-1} + \varepsilon T_6, \tag{9.22}$$

and so $-\varepsilon T_1 + p$ lies in Interval IV.

Proof. We have from (2.6) and (4.4) that

$$\lim_{i\to\infty} T_{7,i} = k + 1 - r\bigl(\varphi_*(-T_1)\bigr) = k - \varphi_*(-T_1) > 0,$$

so the first inequality of (9.21) follows directly, for large i. The second inequality of (9.21) holds by (2.3). For the third inequality, we see from (4.21) and from the formula for p in Table 1 that

$$\begin{aligned} &k + 1 + \frac{\varepsilon|\log\varepsilon|}{k-1} + \varepsilon T_6 - T_8 \\ &\quad = k + 1 + \frac{\varepsilon|\log\varepsilon|}{k-1} + \varepsilon(T_6 - T_2 + H_2) - p \\ &\quad = \varepsilon\bigl(T_6 - T_2 + H_2 + 1 - Q_+ + kQ_0 - (k+2)Q_-\bigr) + o(\varepsilon). \end{aligned}$$

From (4.23) and (7.20) we have that

$$\begin{aligned} &\lim_{i\to\infty}(T_{6,i} - T_{2,i}) + H_2 + 1 - Q_+ + kQ_0 - (k+2)Q_- \\ &\quad = -\lim_{i\to\infty} T_{2,i} + H_2 + Q_+ - Q_0 = H_2 > 0, \end{aligned}$$

as desired.

For the first inequality in (9.22) we have, again using the formula for p, that

$$-\varepsilon T_1 + p - (k + 1 - \varepsilon T_5) = \frac{\varepsilon|\log\varepsilon|}{k-1} + O(\varepsilon),$$

which is positive for large i. For the second inequality in (9.22) we have that

$$-\varepsilon T_1 + p < \varepsilon T_2 + p - r\big(x(\varepsilon T_2)\big) = T_8 < k + 1 + \frac{\varepsilon|\log\varepsilon|}{k-1} + \varepsilon T_6, \tag{9.23}$$

where the first inequality in (9.23) follows from Proposition 9.1, and where the second inequality in (9.23) is from (9.21) in the first part of this lemma. □

Lemma 9.6. *There exists a constant $C_{11} \geqslant 1$ such that for all large i*

$$\int_{-\varepsilon T_1+p}^{\varepsilon T_2+p} \big|y\big(s - r\big(x(s)\big)\big)\big|\,ds \leqslant \varepsilon^2 C_{11}\|\psi\|, \tag{9.24}$$

where $p = p_i$ is the period of $x(t)$.

Proof. Again without loss $\|\psi\| = 1$. From (4.20) and the periodicity of $x(t)$ we have the bound

$$\frac{d}{dt}\big(t - r\big(x(t)\big)\big) = 1 - \dot{x}(t) > \frac{C_2}{\varepsilon} \quad \text{for every } t \in [-\varepsilon T_1 + p, \varepsilon T_2 + p],$$

and thus we may make the change of variables $\theta = s - r(x(s))$ in the integral (9.24). Letting $t = \gamma(\theta)$ denote the inverse of the function $\theta = t - r(x(t))$, and with T_7 and T_8 as in (9.20), we have that

$$\begin{aligned}\int_{-\varepsilon T_1+p}^{\varepsilon T_2+p} \big|y\big(s - r\big(x(s)\big)\big)\big|\,ds &= \int_{T_7}^{T_8} \frac{|y(\theta)|}{1 - \dot{x}(\gamma(\theta))}\,d\theta \\ &\leqslant \frac{\varepsilon}{C_2}\int_{T_7}^{T_8} \big|y(\theta)\big|\,d\theta \leqslant \frac{\varepsilon}{C_2}\int_{\varepsilon T_3}^{t_6} \big|y(\theta)\big|\,d\theta,\end{aligned} \tag{9.25}$$

with the final inequality in (9.25) following from Lemma 9.5, where we recall the formula (9.15) for t_6. Also recalling t_4 in (9.15), we have further that

$$\begin{aligned}\int_{\varepsilon T_3}^{t_6} \big|y(\theta)\big|\,d\theta &= \int_{\varepsilon T_3}^{t_4} \big|y(\theta)\big|\,d\theta + \int_{t_4}^{t_6} \big|y(\theta)\big|\,d\theta \\ &\leqslant \varepsilon C_9 \int_{\varepsilon T_3}^{t_4} \big|\dot{x}(t)\big| + 1\,dt + \varepsilon C_{10} \\ &< \varepsilon C_9 \int_{\varepsilon T_3}^{t_4} \big|\dot{x}(t)\big|\,dt + \varepsilon\big((k+1)C_9 + C_{10}\big)\end{aligned} \tag{9.26}$$

by Lemma 9.2 and (9.14) of Lemma 9.4. The fact that $t_4 - \varepsilon T_3 < k + 1$, which follows from (5.2) and (9.15), is also used. Due to the monotonicity properties of $x(t)$, and in particular because it has only one critical point (a minimum) in the interval $[\varepsilon T_3, t_4]$, we have that

$$\int_{\varepsilon T_3}^{t_4} \left|\dot{x}(t)\right| dt \leqslant 2\Bigl(\max_{t\in\mathbf{R}} x(t) - \min_{t\in\mathbf{R}} x(t)\Bigr) < 2(k+1). \tag{9.27}$$

The desired result (9.24) with

$$C_{11} = \max\bigl\{C_2^{-1}\bigl(3(k+1)C_9 + C_{10}\bigr), 1\bigr\}$$

is now obtained directly from (9.25), (9.26), and (9.27). □

Proof of Part 2 of Theorem A. It is enough to prove the claims about the characteristic multipliers in the statement of the theorem. The claims of asymptotic stability with asymptotic phase for the nonlinear equation follow from the results of [44], specifically, from Theorem 1.1 and Corollary 1.2 of that paper.

We recall the monodromy operator $M : X \to X$ for the linear variational equation (2.11), given by (2.13), (2.14), where $X = C(J)$ and $J = [-\varepsilon T_2, \varepsilon T_1]$. The characteristic multipliers λ in the statement of the theorem are the nonzero points in the spectrum of M. We recall that M is a compact operator, and so such points are isolated elements of the point spectrum and have finite algebraic multiplicity. We also have that $M\psi_0 = \psi_0$ where ψ_0 is as in (2.15).

Taking any solution $y(t)$ of (2.11) with initial condition $\psi \in X$ as in (2.13), rewrite Eq. (9.4) as

$$A(t, t_0) = \frac{\dot{x}(t)}{\dot{x}(t_0)} + \frac{k}{\varepsilon\dot{x}(t_0)} \int_{t_0}^{t} A(t, s)\dot{x}\bigl(s - r\bigl(x(s)\bigr)\bigr)\, ds, \tag{9.28}$$

assuming that $\dot{x}(t_0) \neq 0$, and then substitute (9.28) for the first occurrence of $A(t, t_0)$ in (9.2). This gives

$$y(t) = \frac{\dot{x}(t)y(t_0)}{\dot{x}(t_0)} + \frac{ky(t_0)}{\varepsilon\dot{x}(t_0)} \int_{t_0}^{t} A(t, s)\dot{x}\bigl(s - r\bigl(x(s)\bigr)\bigr)\, ds - \frac{k}{\varepsilon} \int_{t_0}^{t} A(t, s)y\bigl(s - r\bigl(x(s)\bigr)\bigr)\, ds.$$

With the particular choice of $t_0 = -\varepsilon T_1 + p$, and also replacing t by $t + p$, and using the periodicity of $x(t)$, we have that

$$\begin{aligned} y(t+p) = {}& \frac{\dot{x}(t)y(-\varepsilon T_1 + p)}{\dot{x}(-\varepsilon T_1)} + \frac{ky(-\varepsilon T_1 + p)}{\varepsilon\dot{x}(-\varepsilon T_1)} \int_{-\varepsilon T_1 + p}^{t+p} A(t+p, s)\dot{x}\bigl(s - r\bigl(x(s)\bigr)\bigr)\, ds \\ & - \frac{k}{\varepsilon} \int_{-\varepsilon T_1 + p}^{t+p} A(t+p, s)y\bigl(s - r\bigl(x(s)\bigr)\bigr)\, ds. \end{aligned}$$

Let us now write the monodromy operator as a sum $M\psi = M_0\psi + M_1\psi$ where

$$(M_0\psi)(t) = \frac{ky(-\varepsilon T_1 + p)}{\varepsilon\dot{x}(-\varepsilon T_1)} \int_{-\varepsilon T_1+p}^{t+p} A(t+p,s)\dot{x}\bigl(s - r\bigl(x(s)\bigr)\bigr)\,ds - \frac{k}{\varepsilon} \int_{-\varepsilon T_1+p}^{t+p} A(t+p,s)y\bigl(s - r\bigl(x(s)\bigr)\bigr)\,ds,$$

$$(M_1\psi)(t) = \frac{\dot{x}(t)y(-\varepsilon T_1 + p)}{\dot{x}(-\varepsilon T_1)} = \frac{y(-\varepsilon T_1 + p)}{\dot{x}(-\varepsilon T_1)}\psi_0(t). \tag{9.29}$$

Here $t \in J$ and ψ_0 is as in (2.15). It is clear that $M_0, M_1 : X \to X$ depend linearly on ψ, since $y(t)$ depends linearly on ψ. It is also clear that M_1 is an operator of rank one. Additionally, in the case that $\psi = \psi_0$ and so $y(t) = \dot{x}(t)$, one has that $M_1\psi_0 = \psi_0$ hence $M_0\psi_0 = 0$, and so

$$M_0 M_1 = 0, \qquad M_1^2 = M_1. \tag{9.30}$$

Let us next note the bounds

$$A(t,s) \leqslant e^{\varepsilon^{-1}(k-1)(t-s)} \leqslant e^{(k-1)(T_1+T_2)} \leqslant C_{12}$$

for some C_{12}, valid for large i and with $t \geqslant s$ both in Interval I, and where without loss we can take $C_{12} \geqslant 1$. This follows directly from the formula (9.3) for $A(t,s)$ and the bound (2.12) on $a(u)$. Also, $A(t+p, s+p) = A(t,s)$ for any t and s, and thus one has the estimate

$$A(t+p,s) \leqslant C_{12}$$

for the kernel as it appears in the two integrals in (9.29), with $t \in J$. It now follows from this, from Lemma 9.6, and from the bound (9.5) on $\|\psi_0\|$, that the integrals in (9.29) enjoy the bounds

$$\left| \int_{-\varepsilon T_1+p}^{t+p} A(t+p,s)\dot{x}\bigl(s - r\bigl(x(s)\bigr)\bigr)\,ds \right| \leqslant \varepsilon^2 C_{11} C_{12} \|\psi_0\| < \varepsilon k(k+1) C_{11} C_{12},$$

$$\left| \int_{-\varepsilon T_1+p}^{t+p} A(t+p,s)y\bigl(s - r\bigl(x(s)\bigr)\bigr)\,ds \right| \leqslant \varepsilon^2 C_{11} C_{12} \|\psi\|. \tag{9.31}$$

We also have that

$$\bigl|y(-\varepsilon T_1 + p)\bigr| \leqslant C_{10}\|\psi\|, \qquad \bigl|\dot{x}(-\varepsilon T_1)\bigr| > \frac{C_2}{\varepsilon}, \tag{9.32}$$

with the first inequality in (9.32) following from (9.13) in Lemma 9.4 and from the final sentence in the statement of Lemma 9.5, and with the second inequality in (9.32) holding by (4.20). Combining the bounds in (9.31) and (9.32) with (9.5), we obtain the desired estimates

$$\|M_0\psi\| \leqslant \varepsilon k C_{11} C_{12} \left(\frac{k(k+1)C_{10}}{C_2} + 1 \right) \|\psi\| = \varepsilon C_1 \|\psi\|,$$

$$\|M_1\psi\| \leqslant \left(\frac{k(k+1)C_{10}}{C_2} \right) \|\psi\| \leqslant C_1 \|\psi\|,$$

where the equality above serves as the definition of the constant C_1. (Note that we use here the fact that $C_{11}, C_{12} \geqslant 1$.)

Let us now establish the claims about the characteristic multipliers, that is, about the spectrum of M. Fix any $\lambda \in \mathbf{C}$ satisfying $|\lambda| > \varepsilon C_1$ and $\lambda \neq 1$. We must show that λ does not belong to the spectrum of M. We have first that $|\lambda| > \|M_0\|$ and so $\lambda I - M_0$ is invertible. We claim that the operator

$$L(\lambda) = \big((\lambda - 1)^{-1} M_1 + I\big)(\lambda I - M_0)^{-1}$$

is the inverse of $\lambda I - M$. To see this, we note from (9.30) that $(\lambda I - M_0)M_1 = \lambda M_1$, hence $(\lambda I - M_0)^{-1} M_1 = \lambda^{-1} M_1$. Thus again using (9.30), we have

$$\begin{aligned} L(\lambda)M &= L(\lambda)(M_0 + M_1) \\ &= \big((\lambda - 1)^{-1} M_1 + I\big)(\lambda I - M_0)^{-1} M_0 + \lambda^{-1}\big((\lambda - 1)^{-1} M_1 + I\big) M_1 \\ &= \big((\lambda - 1)^{-1} M_1 + I\big)(\lambda I - M_0)^{-1} M_0 + (\lambda - 1)^{-1} M_1 \\ &= -(\lambda - 1)^{-1} M_1 - I + \lambda\big((\lambda - 1)^{-1} M_1 + I\big)(\lambda I - M_0)^{-1} + (\lambda - 1)^{-1} M_1 \\ &= -I + \lambda L(\lambda), \end{aligned}$$

and so $L(\lambda)$ is a left inverse of $\lambda I - M$. Also, again using (9.30),

$$\begin{aligned} ML(\lambda) &= (M_0 + M_1)L(\lambda) \\ &= M_0\big((\lambda - 1)^{-1} M_1 + I\big)(\lambda I - M_0)^{-1} + \big((\lambda - 1)^{-1} + 1\big) M_1 (\lambda I - M_0)^{-1} \\ &= M_0(\lambda I - M_0)^{-1} + \lambda(\lambda - 1)^{-1} M_1 (\lambda I - M_0)^{-1} \\ &= -I + \lambda(\lambda I - M_0)^{-1} + \lambda(\lambda - 1)^{-1} M_1 (\lambda I - M_0)^{-1} \\ &= -I + \lambda L(\lambda), \end{aligned}$$

and so $L(\lambda)$ is a right inverse of $\lambda I - M$. Thus $\lambda \notin \operatorname{spec}(M)$, as claimed.

There remains to show that the point $\lambda = 1 \in \operatorname{spec}(M)$ has simple algebraic multiplicity. We establish this by calculating the canonical spectral projection P onto the generalized eigenspace, given by

$$P = \frac{1}{2\pi i} \int\limits_{|\lambda - 1| = \delta} L(\lambda)\, d\lambda = M_1 (I - M_0)^{-1}$$

for sufficiently small δ. Then P has a one-dimensional range, namely the span of ψ_0, as desired. □

10. Uniqueness

In this section we complete the proof of Theorem A by showing that the SOPS of Eq. (1.1) is unique for every sufficiently small ε.

Proof of Part 1 of Theorem A. First recall that any SOPS of Eq. (1.1) enjoys the bounds (2.2), along with the bound (4.2) for its derivative. Note also the bound $r(x(t)) < k + 1$ on the delay, which follows from (2.2). Following [45], define

$$\begin{aligned} K = \big\{ &\psi \in C[-k-1, 0] \,\big|\, -1 \leqslant \psi(\theta) \leqslant k \text{ for every } \theta \in [-k-1, 0], \\ &\psi(\theta) \geqslant 0 \text{ for every } \theta \in [-1, 0], \text{ with } \operatorname{Lip}(\psi) \leqslant \varepsilon^{-1} k(k+1), \text{ and } \psi(0) = 0 \big\}, \end{aligned}$$

which is a compact convex set. Then if $x(t)$ is any SOPS, there exists a unique $\tau \in \mathbf{R}$, modulo the period p, such that $x_\tau \in K$. Here, following [24], we define $z_\tau \in C[-k-1,0]$ by

$$z_\tau(\theta) = z(\tau+\theta), \quad \theta \in [-k-1,0],$$

for any function $z(t)$ which is continuous at least on $[\tau-k-1,\tau]$. Also define the set

$$G = \{\psi \in K \mid \psi(\theta) > 0 \text{ for some } \theta \in [-1,0]\}.$$

Then in fact $x_\tau \in G$ for any such solution, with τ as above, for if $x_\tau \in K \setminus G$ then the unique solution from this Lipschitz initial condition would be identically zero for $t \geqslant 0$.

As in [45], we define a Poincaré map $T : G \to K$ as follows. Let $\psi \in G$ and let $z(t)$ for $t \geqslant 0$ denote the unique solution of (1.1) with $z_0 = \psi$. It is not difficult to check that $z(t)$ enjoys the same bounds (2.2) and (4.2) for $t \geqslant 0$. It is the case that either $z(t) \leqslant 0$ for every $t \geqslant 0$, or else there exists some $t_1 > 0$ such that $z(t) \leqslant 0$ for every $t \in [0,t_1]$, and $z(t_1) = 0$ with $\dot{z}(t_1) > 0$. Clearly such t_1, if it exists, is unique. Moreover $t_1 > 1$ also holds. If t_1 exists, then either $z(t) \geqslant 0$ for every $t \geqslant t_1$, or else there exists some $t_2 > t_1$ such that $z(t) \geqslant 0$ for every $t \in [t_1,t_2]$, and $z(t_2) = 0$ with $\dot{z}(t_2) < 0$. Again, t_2 is unique if it exists, and $t_2 > t_1 + 1$ holds. Now define

$$T\psi = \begin{cases} z_{t_2} & \text{if both } t_1 \text{ and } t_2 \text{ exist}, \\ 0 & \text{otherwise}. \end{cases}$$

One easily sees that $T\psi \in K$, and it is shown in [45] that $T : G \to K$ is continuous. Moreover, there is a one-to-one correspondence between fixed points of T in G and SOPS's, namely, $z(t)$ is an SOPS time-translated so that $z(0) = 0$ and $\dot{z}(0) < 0$, if and only if $T\psi = \psi$ where $\psi = z_0$. Also, the fixed-point set

$$S = \{\psi \in G \mid T\psi = \psi\}$$

is a compact subset of G provided that $\varepsilon \neq \varepsilon_0$ where ε_0 is the point of Hopf bifurcation of SOPS's.

We remark that in general, there is no assurance that T can be extended continuously to all of K. In particular, as $0 \notin G$, we are only considering nonzero fixed points of T, unlike for treatments of related problems in which T is defined and continuous at zero.

If $O \subseteq K$ is a relatively open set such that the set $S \cap O$ is compact, then there is defined the fixed-point index $\iota_K(T,O)$ of T with respect to O. Note that if $O \subseteq K$ is any open set whose closure $\overline{O}$ is contained in G, then the compactness of $S \cap O$ is equivalent to the familiar condition that $S \cap \partial O = \emptyset$, namely that there are no fixed points of T on the boundary of O. However, if $\overline{O}$ is not a subset of G, then it can happen that $S \cap \partial O = \emptyset$ and yet $S \cap O$ is not compact, as O might contain fixed points clustering on $K \setminus G$. More generally, one can define the mth iterate $T^m : G_m \to K$ of T in the obvious fashion, where in particular $G_m \subseteq G$ is the set of $\psi \in G$ such that the points $T^i\psi$ for $1 \leqslant i \leqslant m-1$ are (inductively) well-defined and belong to G. Then $\iota_K(T^m,O)$ is defined as long as $S_m \cap O$ is compact, where $S_m = \{\psi \in G_m \mid T^m\psi = \psi\}$ is the set of m-periodic points of T. See, for example, [48] for an exposition of the basic properties of the fixed-point index in this framework.

It is shown in [45] that

$$\iota_K(T,G) = \begin{cases} 1 & \text{if } 0 < \varepsilon < \varepsilon_0, \\ 0 & \text{if } \varepsilon > \varepsilon_0. \end{cases} \tag{10.1}$$

Also, if O is a ball, say

$$O = O_\delta(\psi_*) = \{\psi \in K \mid \|\psi - \psi_*\| < \delta\}$$

for some $\psi_* \in K$ and $\delta > 0$, and if $T^mO \subseteq O$ and $S_m \cap O$ is compact for some $m \geqslant 1$, then $\iota_K(T^m,O) = 1$. Further, if this condition holds for all sufficiently large m then in fact $\iota_K(T,O) = 1$.

Such is the case when ψ_* is an asymptotically stable fixed point, as would arise from an asymptotically stable SOPS of the differential equation (1.1), with δ sufficiently small. In particular, by Part 2 of Theorem A there exists $\varepsilon_s > 0$ such that if $0 < \varepsilon < \varepsilon_s$ then every fixed point ψ_* of T is such a point, and hence is an isolated point of the fixed-point set S. As S is a compact subset of G, it follows that for every such ε the set $S = \{\psi_1, \psi_2, \ldots, \psi_m\}$ is finite, with $m = m(\varepsilon)$ and the points ψ_i of course depending on ε. Thus for such ε, we have from (10.1) and the remarks following that formula, and from the basic properties of the degree, that for sufficiently small δ

$$1 = \iota_K(T, G) = \sum_{i=1}^{m} \iota_K\big(T, O_\delta(\psi_i)\big) = \sum_{i=1}^{m} 1 = m.$$

With $m = 1$, we conclude that T has a unique fixed point in G if $0 < \varepsilon < \varepsilon_s$, and thus Eq. (1.1) has a unique SOPS. This completes the proof. □

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