Asymptotic Fixed Point Theory and the Beer Barrel Theorem

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Abstract

In Sections 2 and 3 of this paper we refine and generalize theorems of Nussbaum (see [42]) concerning the approximate fixed point index and the fixed point index class. In Section 4 we indicate how these results imply a wide variety of asymptotic fixed point theorems. In Section 5 we prove a generalization of the mod p theorem: if p is a prime number, f belongs to the fixed point index class and f satisfies certain natural hypothesis, then the fixed point index of f^p is congruent mod p to the fixed point index of f. In Section 6 we give a counterexample to part of an asymptotic fixed point theorem of A. Tromba [55]. Sections 2, 3, and 4 comprise both new and expository material. Sections 5 and 6 comprise new results.

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1 Introduction

By "asymptotic fixed point theory" we mean results in which the existence of a fixed point of a map f or information about the fixed point index or Lefschetz number of f is obtained with the aid of assumptions on certain iterates f^j of f. A famous example of such a result is the "mod p theorem" (see [31], [47], [51] and [52]): if p is a prime and natural assumptions are satisfied, then the fixed point index of f^p is congruent to the fixed point index of f, mod p. Information about the fixed point index of a map may be very useful in applications (see, for example, [34], [44] and [45], where fixed point index calculations and global bifurcation theorems yield otherwise inaccessible results concerning nonlinear differential-delay equations); and generalizations of the fixed point index will be one focus of this paper.

Some of the earliest asymptotic fixed point theorems were obtained by Felix Browder; see [5]-[9]. See, also, work by J. Leray [33], who contributed to the development of the fixed point index and who introduced a generalized Lefschetz number which has proved very useful in the subject.

The "beer barrel theorem" of our title is a reference to a paper by Anthony Tromba [54], entitled "The Beer Barrel Theorem". As described in [54], H.-O. Peitgen challenged Tromba to give an "easy proof" of a thorem proved by R.D. Nussbaum in [41]. The prize for success was a barrel of beer.

The following result, which is a very special case of Corollary 9 on page 367 of [41], is the focus of Tromba's paper:

Theorem 1.1. *(See Corollary 9, page 367, in [41].) Let G be an open convex set in a Banach space and $f : \overline{G} \to \overline{G}$ a continuous map. Assume that there is an integer N such that $\overline{f^N(G)}$ is compact, $\overline{f^N(G)} \subset G$ and f is continuously Fréchet differentiable on some open neighborhood of $\overline{f^N(G)}$. Then f has a fixed point in G.*

One of the oldest unresolved conjectures in asymptotic fixed point theory (see the remarks on page 363 in [41]) asks whether Theorem 1.1 remains true without the assumption of continuous Fréchet differentiability.

In [54] Tromba introduces the nice idea of using an infinite dimensional variant of a transversality theorem of Ralph Abraham [1] in the context of fixed point theory. With the aid of Abraham's transversality theorem, he claims to prove Theorem 1.1 under the additional assumptions that $N = 2$ and that f is C^1 on G. The claim is made that the given proof actually applies under the less restrictive assumption that f is C^1 on an open neighborhood of $\{x \in G \,|\, f^2(x) = x\}$. It is also claimed (no proof) that "a slightly more technical argument than that presented here should also work" for the case $N > 2$. On this basis Tromba won the beer barrel.

Perhaps because of its catchy title, many mathematicians are aware of Tromba's paper and its basic claims; far fewer have read the paper carefully. There are, in fact, some leaks in the beer barrel. The proof, as given, use Smale's mod 2 degree theory, which requires C^2 Fredholm maps which are proper at 0, while the maps in Theorem 1.1 are only C^1. This potential difficulty is not discussed in [54].

The question of minimal differentiability assumptions is also delicate. If, for example, $N = 3$ in Theorem 1.1 and f is only assumed C^1 on an open neighborhood of $\{x \in G \mid f^3(x) = x\}$, then the direct argument used in [54], depending as it does on mod 2 calculations and Smale's mod 2 degree theory, cannot work.

In a more sedately titled later paper [55], "A General Asymptotic Fixed Point Theorem", Tromba claims the following result:

Claim 1.2. *(See Theorem 2 in [55].) Let G be an open subset of a Banach space E with $T : G \to G$ a C^1 map such that $\overline{T^n(G)}$ is compact in G for some integer $n \geq 1$. Assume that either*

(1) as $m \to \infty$, the Lefschetz number $L(T^m)$ of T^m is bounded, and $L(T^m) \neq 0$ for all m sufficiently large; or

(2) for all m sufficiently large $L(T^m)$ is even and non-zero.

Then T has a fixed point.

Unfortunately, Claim 1.2 under assumption (2) is not true in general, and in Section 6 we present a two dimensional counterexample. Claim 1.2 under assumption (1) is true, and we note that it follows from Corollary 4.7 of the present paper.

As in [54], the heart of the argument given in [55] is a nice application of an infinite dimensional version of a transversality theorem of R. Abraham [1]. The proof also uses a degree theory of Elworthy and Tromba [16] which requires that the maps in question be C^2, while in the above claim T is only assumed C^1. As in [54], this difficulty is never discussed.

However, the main difficulty with [55] is that all of its theorems are straightforward consequences of much more general theorems proved by R.D. Nussbaum in the much earlier paper [42], of which Tromba was apparently unaware.

Since it is not generally understood even now that the results in [42] imply a wide variety of fixed point theorems, we take this opportunity to revisit and generalize theorems in [42]. For example, in Section 5, we prove a version of the mod p theorem which is valid for any map f in the so-called "fixed point index class" of [42].

The reader will note that we allow maps f in our discussion such that f^j may not be compact for any $j \geq 1$. As discussed in Hale's book [20], such maps may arise by considering the operation of translation along trajectories for neutral functional differential equations or in the study of certain hyperbolic equations. Our interest comes from a different direction, the problem of generalizing the classical Krein-Rutman theroem [46]. If K is a closed cone in a Banach space E, one can consider maps $f : K \to K$ which are homogeneous of degree one ($f(tx) = tf(x)$ for $x \in K$ and $t \geq 0$), continuous and order-preserving with respect to the partial ordering induced by K. Natural examples of such maps need not be compact or differentiable; see [36]. Associated to such a map is its "cone spectral radius", $r := r_K(f)$; and if $r > 0$, one can seek nonzero fixed points of $g(x) = r^{-1}f(x)$. It turns out that the definition of the "cone essential spectral radius of f", namely the quantity $\rho_K(f)$ given in [36], [46], has serious drawbacks; but in work in progress the authors have found what seems the appropriate definition of $\rho_K(f)$. Of course, $\rho_K(f) = 0$ if f is compact. A natural conjecture is that $g(x) = r^{-1}f(x)$ has a nonzero fixed point in K whenever $\rho_K(f) < r_K(f)$; and we have essentially proved such a theorem when f is linear on K or when f is compact. In general proving such a theorem is closely related to the problem of defining a generalized fixed point index for maps like $g_\lambda(x) = \lambda^{-1}f(x)$, where $\rho_K(f) < \lambda < r_K(f)$.

This is a long paper, so it may be useful to give an outline. Section 2 is devoted to the "approximate fixed point index". The approximate fixed point index was introduced in [42]. Here we introduce a number of refinements and generalizations of theorems in [42].

In Section 3 we use results about the approximate fixed point index to define the "fixed point index class", a broad class of maps for which a reasonable fixed point index can be defined. The actual definition may seem unnatural, but it is justified by the broad range of examples which it covers. Much of Section 3 can be found in [42], and is included for the reader's convenience. However, some new results (see Theorem 3.1 below) and refinements are also given.

Section 4 presents asymptotic fixed point theorems, with Theorem 4.5 a refinement of Proposition

2.4 in [42]. The hypotheses of Theorem 4.5 are satisfied by various classes of examples described in Section 3, and in each case one obtains a corresponding fixed point theorem. In contrast to [42], we explicitly spell out these applications. We also call the reader's attention to Theorem 4.18, which is motivated by fixed point theorems of "Frum-Ketkov type" (see Corollary 4.19) and which may apply to maps which are not in the fixed point index class.

In Section 5 we prove a version of the mod p theorem which is valid for any map in the fixed point index class. This material has no counterpart in [42].

In Section 6 we describe a counterexample to some of Tromba's assertions in [55].

Sections 2-4 comprise a mixture of expository and new material, while the results in Sections 5 and 6 are new.

2 The Approximate Fixed Point Index

Suppose that $(Y, \|\cdot\|)$ is a normed linear space and that X is a closed subset of Y. We shall say that X is complete if X is a complete metric space in the metric from Y. We shall always consider X as a topological space in the topology inherited from Y. In particular, when we say that $W \subset X$ is open, we mean that $W = V \cap X$, where $V \subset Y$ is open. We do *not* assume that the interior of X in Y is nonempty.

If X is a closed subset of a normed linear space $(Y, \|\cdot\|)$, we shall say that $X \in \mathcal{F}$ (we omit explicit mention of Y in the notation) if there exists a locally finite covering $\{C_j \mid j \in J\}$ of X by closed, convex sets $C_j \subset X$. Thus, $X = \bigcup_{j\in J} C_j$ and for each $x \in X$ there exists an open neighborhood W of x in X such that $W \cap C_j$ is empty except for finitely many j. We shall write $X \in \mathcal{F}_0$ if there exist closed, convex sets $C_j \subset X$, for $1 \le j \le n$, with $n < \infty$, where $X = \bigcup_{j=1}^n C_j$. Eventually we shall have to restrict to complete subsets X of Y, but initially this is unnecessary. The classes $\mathcal{F}$ and $\mathcal{F}_0$ will play an important role in our discussion of generalizations of the fixed point index.

We shall also need some facts about the "Kuratowski measure of noncompactness" or, as we shall abbreviate it, the Kuratowski MNC. If (X, ρ) is a general metric space and $A \subset X$, recall that $\operatorname{diam}(A)$, the diameter of A, is defined by $\operatorname{diam}(A) = \sup\{\rho(x, y) \mid x, y \in A\}$ and A is "bounded" if $\operatorname{diam}(A) < \infty$. If A is a bounded subset of X, Kuratowski [32] defined the Kuratowski MNC $\alpha(A)$ by

$$\alpha(A) = \inf\{\delta > 0 \mid A = \bigcup_{i=1}^{n} A_i \text{ for some } A_i \text{ with } \operatorname{diam}(A_i) \le \delta, \text{ for } 1 \le i \le n < \infty\}.$$

If (X, ρ) is a complete metric space, $\alpha(A) = 0$ if and only if $\overline{A}$ is compact. If A_n, for $n \geq 1$, is a decreasing sequence of closed, bounded nonempty subsets of complete metric space (X, ρ) and if $\lim_{n\to\infty} \alpha(A_n) = 0$, Kuratowski [32] proved that $A_\infty := \bigcap_{n\geq 1} A_n$ is compact and nonempty. Furthermore, if U is an open neighborhood A_∞, there exists an integer $n(U)$ such that $A_k \subset U$ for all $k \geq n(U)$.

If $(Y, \|\cdot\|)$ is a normed linear space, the norm $\|\cdot\|$ gives a metric on Y and one can take the Kuratowski MNC α on Y with respect to this metric. If $A \subset Y$, we shall denote the convex hull of A, namely the smallest convex set containing A, by $\mathrm{co}(A)$; and we shall denote the closure of $\mathrm{co}(A)$ by $\overline{\mathrm{co}}(A)$. If A and B are subsets of Y and λ is a scalar, we shall write $A+B = \{a+b | a \in A \text{ and } b \in B\}$ and $\lambda A = \{\lambda a | a \in A\}$. In [10], G. Darbo observed that $\alpha(\mathrm{co}(A)) = \alpha(A)$, and that $\alpha(A+B) \leq \alpha(A)+\alpha(B)$ and $\alpha(\lambda A) = |\lambda|\alpha(A)$. For general metric spaces one has $\alpha(\overline{A}) = \alpha(A)$.

If D_1 is a subset of a metric space (X_1, ρ_1), and (X_2, ρ_2) is a second metric space and $f : D_1 \to X_2$ is a continuous map, we shall say that f is a "k-set-contraction" if $\alpha_2(f(S)) \leq k\alpha_1(S)$ for all bounded sets $S \subset X_1$, where α_j denotes the Kuratowski MNC on (X_j, ρ_j). If D is a subset of (X, ρ) with Kuratowksi MNC α and $f : D \to X$ is a continuous map, f is a "k-set-contraction" (some authors say "k-α-contraction") if $\alpha(f(S)) \leq k\alpha(S)$ for all bounded $S \subset D$. If $U : D \to X$ is a Lipschitz map with Lipschitz constant k and $C : D \to X$ is a compact map and $f(x) = U(x) + C(x)$, then f is a k-set-contraction.

Using these ideas, Darbo [10] proved an elegant fixed point theorem in 1955. A closely related result was obtained by Sadovskii [48] in 1967.

Proposition 2.1. *(See [10].) Let G be a closed bounded convex set in a Banach space $(X, \|\cdot\|)$ and let $f : G \to G$ be a k-set-contraction with $k < 1$. Then f has a fixed point in G.*

As a motivation for our later definitions, it is useful to state a more general version of Proposition 2.1 which follows by essentially the same proof.

Proposition 2.2. *(See Proposition 10 on page 225, Section A of [38].) Let G be a closed, bounded convex set in a Banach space $(X, \|\cdot\|)$ and $f : G \to G$ a continuous map. Define $G_1 = \overline{\mathrm{co}}(f(G))$ and $G_n = \overline{\mathrm{co}}(f(G_{n-1}))$ for $n > 1$ and assume that $\lim_{n\to\infty} \alpha(G_n) = 0$, where α dentoes the Kuratowski MNC. Then f has a fixed point in G.*

If f in Proposition 2.2 is a k-set-contraction with $k < 1$, then $\lim_{n\to\infty} \alpha(G_n) = 0$; but (see Corollary 3 on page 225, Section A of [38]) the conditions of Proposition 2.2 may be satisfied in cases of interest for which f is not a k-set-contraction with $k < 1$.

For the reader's convenience we recall some geometrical results concerning finite unions of closed, convex sets. These results represent the motivation for considering sets $X \in \mathcal{F}_0$ and play a central role in our work here.

Proposition 2.3. *(See Theorem 1 on page 229, Section B of [38].) Let C be a closed, metrizable subset of a Hausdorff locally convex topological vector space X and assume that $C = \bigcup_{i=1}^m C_i$, where $m < \infty$ and C_i is closed and convex for $1 \le i \le m$. Assume that $D = \bigcup_{i=1}^m D_i$, where $D_i \subset C_i$ and D_i is closed and convex for $1 \le i \le m$. For every subset $J \subset \{1, 2, \ldots, m\}$ assume that $C_J := \bigcap_{j \in J} C_j$ is nonempty if and only if $D_J := \bigcap_{j \in J} D_j$ is nonempty. Then there is a retraction R of C onto D such that $R(x) \in C_i \cap D$ whenever $x \in C_i$, for $1 \le i \le m$.*

We shall actually find it more convenient to use a corollary of Proposition 2.3.

Corollary 2.4. *(Compare Lemma 1.1 in [42].) Let $(Y, \|\cdot\|)$ be a normed linear space and let $C \subset Y$ be complete in the metric from Y. Assume that $C = \bigcup_{i=1}^m C_i$, where $m < \infty$ and C_i is closed and convex for $1 \le i \le m$. Let $A \subset C$ be compact. Then there exists a compact set $D \in \mathcal{F}_0$ with $A \subset D \subset C$ and a retraction R of C onto D such that $R(x) \in C_i \cap D$ whenever $x \in C_i$, for $1 \le i \le m$.*

Proof. For each subset $J \subset \{1, 2, \ldots, m\}$ such that $C_J := \bigcap_{j \in J} C_j$ is nonempty, select $x_J \in C_J$. For $1 \le i \le m$, define

$$D_i := \overline{\text{co}}(\{x_J \mid i \in J, \text{ where } J \subset \{1, 2, \ldots, m\}\} \cup (A \cap C_i)).$$

Since the convex closure of a compact set in C_i is compact, D_i is compact and convex. By our construction, for each $J \subset \{1, 2, \ldots, m\}$ such that C_J is nonempty, $x_J \in \bigcap_{i \in J} D_i := D_J$. It follows from Proposition 2.3 that there exists a continuous retraction R of C onto $D := \bigcup_{i=1}^m D_i$ such that $R(x) \in C_i \cap D$ whenever $x \in C_i$, for $1 \le i \le m$. ■

If C and D are as in Corollary 2.4, then D is a finite union of compact, convex sets and hence a "compact metric ANR"; see [24], [25] or [27] for definitions. It is well known that $H_*(D)$ (singular homology over the rationals) is a finite dimensional graded vector space whenever D is a compact metric ANR. Since D is a deformation retraction of C by the homotopy $H(x,t) = (1-t)x + tR(x)$ for $x \in C$ and $0 \le t \le 1$, it follows that $H_*(C)$ is isomorphic to $H_*(D)$, so $H_i(C)$ is a finite dimensional vector space for all i and $H_i(C) = 0$ except for finitely many i. We shall need this remark later.

The assumption in Corollary 2.4 that C is a complete metric space in the metric inherited from $(Y, \|\cdot\|)$ was only used to insure that the set D_i constructed in the proof of Corollary 2.4 is compact. If A is a subset of a finite dimensional vector subspace of Y, then $K_i := \{x_J \mid i \in J, \text{ where } J \subset \{1, 2 \ldots, m\}\} \cup (A \cap C_i)$ is a subset of a finite dimensional vector subspace Y_i of Y. Since any finite dimensional linear subspace of Y is a Banach space, $D_i := \overline{\text{co}}(K_i)$ is a compact, convex set. Thus, if A is a subset of a finite dimensional linear subspace of Y, the assumption in Corollary 2.4 that C is complete in the metric from Y is unnecessary.

We shall need a second corollary of Proposition 2.3. The following result is closely related to Proposition 1.1 in [42].

Corollary 2.5. *Suppose that $X \in \mathcal{F}_0$, so $X = \bigcup_{i=1}^m C_i$, where $m < \infty$ and C_i are closed, convex subsets of a normed linear space $(Y, \|\cdot\|)$. Assume that U is a bounded, open subset of the metric space X, that B is a compact subset of $\overline{U}$ and $f : \overline{U} \to X$ is continuous. Assume that $\alpha(f(\partial U)) < \delta$ and $\alpha(f(\overline{U})) < \eta$, where $\partial U := \overline{U} \backslash U$ and α is the Kuratowski MNC on Y. Then there exists a compact set $D \in \mathcal{F}_0$ with $D \subset X \cap Y_0$, with Y_0 a finite dimensional linear subspace of Y, and a continuous map $g : \overline{U} \to D$ such that*

(1) $\|f(x) - g(x)\| < \delta$ for all $x \in \partial U$;

(2) $\|f(x) - g(x)\| < \eta$ for all $x \in \overline{U}$; and

(3) if $f(x) \in C_i$ for some $x \in \overline{U}$ and some i, then $g(x) \in C_i$.

If X is complete, namely if X is a complete metric space in the metric inherited from Y, then there exists a compact set $D_1 \in \mathcal{F}_0$ with $D_1 \subset X$ and a continuous map $g_1 : \overline{U} \to D_1$ which satisfies properties (1)-(3) and also satisfies

(4) $g_1(x) = f(x)$ for all $x \in B$.

Proof. Since $\alpha(f(\partial U)) < \delta$ and $\alpha(f(\overline{U})) < \eta$, we can write $f(\partial U) = \bigcup_{j=1}^n S_j$ and $f(\overline{U}) = \bigcup_{k=1}^p T_k$, where $\text{diam}(S_j) < \delta$ for $1 \le j \le n$ and $\text{diam}(T_k) < \eta$ for $1 \le k \le p$. We have the inclusion of

$f(\overline{U}) \subset Z$, where $Z \in \mathcal{F}_0$ and

$$Z := \Big(\bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \overline{\mathrm{co}}(C_i \cap S_j) \Big) \cup \Big(\bigcup_{i=1}^{m} \bigcup_{k=1}^{p} \overline{\mathrm{co}}(C_i \cap T_k) \Big).$$

If we use Corollary 2.4 and the remark following Corollary 2.4, we see that there exists a finite dimensional linear subspace Y_0 of Y, a compact set $D \in \mathcal{F}_0$, with $D \subset Z \cap Y_0$ and a continuous retraction $R : Z \to D$ of Z onto D such that for all i, j and k

(5) $R(y) \in \overline{\mathrm{co}}(C_i \cap S_j)$ for all $y \in \overline{\mathrm{co}}(C_i \cap S_j)$; and

(6) $R(y) \in \overline{\mathrm{co}}(C_i \cap T_k)$ for all $y \in C_i \cap T_k$.

If X is complete, Z is complete, and $A := f(B)$ is a compact subset of Z. By using Corollary 2.4, we see in this case that there exists a compact set $D_1 \in \mathcal{F}_0$ with $A \subset D_1 \subset Z$ and a continuous retraction $R_1 : Z \to D_1$ of Z onto D_1 such that conditions (5) and (6) are satisfied by R_1. If for $x \in \overline{U}$ we define $g(x) = R(f(x))$ and (assuming X is complete) $g_1(x) = R_1(f(x))$, properties (5) and (6) insure that $g(x)$ (respectively, $g_1(x)$) lies in C_i whenever $f(x) \in C_i$, for $1 \le i \le m$. Because $R_1(f(x)) = f(x)$ for all $x \in B$, we have $g_1(x) = f(x)$ for all $x \in B$. If $x \in \partial U$, then $f(x) \in C_i \cap S_j$ for some i and j, so $f(x) \in \overline{\mathrm{co}}(C_i \cap S_j)$ and $g(x) \in \overline{\mathrm{co}}(C_i \cap S_j)$. Since $\mathrm{diam}(C_i \cap S_j) < \delta$, we have $\mathrm{diam}(\overline{\mathrm{co}}(C_i \cap S_j)) < \delta$ and $\|f(x) - g(x)\| < \delta$. Similarly, if $x \in \overline{U}$, then $f(x) \in C_i \cap T_k$ for some i and k, so $f(x) \in \overline{\mathrm{co}}(C_i \cap T_k)$ and $g(x) \in \overline{\mathrm{co}}(C_i \cap T_k)$. Since $\mathrm{diam}(C_i \cap T_k) < \eta$, we have $\mathrm{diam}(\overline{\mathrm{co}}(C_i \cap T_k)) < \eta$ and $\|f(x) - g(x)\| < \eta$. The same argument shows that $\|f(x) - g_1(x)\| < \delta$ for all $x \in \partial U$ and $\|f(x) - g_1(x)\| < \eta$ for all $x \in \overline{U}$. ■

If f is as in Corollary 2.5 but $f(\overline{U})$ is unbounded, a slight variant of the above argument shows the existence of a compact map g which satisfies properties (1) and (3) in Corollary 2.5 and also satisfies property (4) if X is complete.

Suppose, now, that $X \in \mathcal{F}_0$ and that $X = \bigcup_{i=1}^{m} C_i$, where $m < \infty$ and C_i, for $1 \le i \le m$, are closed, convex subsets of a given normed linear space $(Y, \|\cdot\|)$. Let U be a bounded, open subset of the topological space X and $f : \overline{U} \to X$ a continuous map such that

$$\inf\{\|f(x) - x\| \mid x \in \partial U\} = \delta > 0. \tag{2.1}$$

A continuous map $g : \overline{U} \to X$ will be called compact if the closure of $g(\overline{U})$ is compact. Generalizing terminology in [42], a continuous map $g : \overline{U} \to X$ is called an "admissible approximation to f with respect to $\{C_i \mid 1 \le i \le m\}$" if

(1) g is compact;

(2) $\|f(x) - g(x)\| < \delta$ for all $x \in \partial U$; and

(3) if $f(x) \in C_i$ for some $x \in \overline{U}$ and some i, then $g(x) \in C_i$.

Corollary 2.5 implies that if $\alpha(f(\partial U)) < \delta$, then there exists an admissible approximation to f with respect to $\{C_i \mid 1 \le i \le m\}$.

In general, if $X \in \mathcal{F}_0$ and U is a bounded open subset of X and $g : \overline{U} \to X$ is a continuous, compact map such that $g(x) \neq x$ for all $x \in \partial U$, then the fixed point index of $g : U \to X$ is defined and satisfies the usual properties of the fixed point index. If either

(1) X is complete; or

(2) g has range in a finite dimensional linear subspace Y_0 of Y (where X is a subset of the normed linear space Y);

then there exists a compact set $D \subset X$, with $D \in \mathcal{F}_0$ and $g(U) \subset D$, and the classical fixed point index $i_D(g, U \cap D)$ (see [4], [11] or [13]) is defined.

By definition, $i_X(g, U) = i_D(g, U \cap D)$, which is independent of the particular compact $D \in \mathcal{F}_0$ as above. If neither (1) nor (2) holds, $i_X(g, U)$ is defined by using Corollary 2.5 to approximate g by an appropriate map $h : \overline{U} \to X$ which has finite dimensional range.

Let $X \in \mathcal{F}_0$, with U a bounded, open subset and $f : \overline{U} \to X$ is a continuous map which satisfies equation (2.1), and assume that $\alpha(f(\partial U)) < \delta$. (If $U = X$, ths condition is vacuous.) Then we define the "approximate fixed point index" $i_X(f, U)$ by

$$i_X(f, U) = i_X(g, U), \tag{2.2}$$

where g is an admissible approximation to f with respect to $\{C_i \mid 1 \le i \le m\}$ and the right hand side of (2.2) is the standard fixed point index of $g : \overline{U} \to X$. Because $\|f(x) - x\| \ge \delta$ and $\|f(x) - g(x)\| < \delta$ for all $x \in \partial U$, we have $g(x) \neq x$ for $x \in \partial U$ and the fixed point index of $g : \overline{U} \to X$ is defined.

We must show that the definition of $i_X(f, U)$ is independent of the particular g selected. The argument roughly follows the lines of the proof of Proposition 1.1 in [42]. If f is as above and $X = \bigcup_{i=1}^m C_i$, where C_i is closed and convex for $1 \le i \le m$, there exists an admissible approximation g_0 to f with respect to $\{C_i \mid 1 \le i \le m\}$. Similarly, if $X = \bigcup_{j=1}^n D_j$, where D_j is closed and convex for

$1 \leq j \leq n$, there exists an admissible approximation g_1 to f with respect to $\{D_j \mid 1 \leq j \leq n\}$. Since $X = \bigcup_{i=1}^m \bigcup_{j=1}^n (C_i \cap D_j)$, there is an admissible approximation g to f with respect to $\{C_i \cap D_j \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. If $x \in \overline{U}$ and $f(x) \in C_i \cap D_j$ for some i and j, then $g_0(x) \in C_i$ and $g(x) \in C_i \cap D_j$, so $(1-t)g_0(x) + tg(x) \in C_i \subset X$ for $0 \leq t \leq 1$. Also, since $\|f(x) - g_0(x)\| < \delta$ and $\|f(x) - g(x)\| < \delta$ for all $x \in \partial U$,

$$\|f(x) - (1-t)g_0(x) - tg(x)\| < \delta$$

for all $x \in \partial U$. Since $\|f(x) - x\| \geq \delta$ for all $x \in \partial U$, we conclude that

$$(1-t)g_0(x) + tg(x) \neq x$$

for $0 \leq t \leq 1$ and $x \in \partial U$. The homotopy property for the fixed point index for compact maps now implies that

$$i_X(g_0, U) = i_X(g, U).$$

The same argument shows that $i_X(g_1, U) = i_X(g, U)$, so $i_X(g_0, U) = i_X(g_1, U)$, and the approximate fixed point index is well defined.

If $f : \overline{U} \to X$ is as above and if, in addition, $\alpha(f(U)) < \eta$, Corollary 2.5 implies that there is a compact map $g : \overline{U} \to X$ such that $\|f(x) - g(x)\| < \eta$ for all $x \in \overline{U}$, with $\|f(x) - g(x)\| < \delta$ for all $x \in \partial U$ and $i_X(f, U) = i_X(g, U)$. If $i_X(f, U) \neq 0$, the properties of the standard fixed point index imply that there exists $x_* \in U$ with $g(x_*) = x_*$, so $\|f(x_*) - x_*\| < \eta$. Roughly speaking, if the approximate fixed point index is nonzero, we obtain an approximate fixed point.

If $X \in \mathcal{F}_0$ and $f : X \to X$ is continuous, we have seen that $i_X(f, X)$ is defined and, because $H_*(X)$ is finitely generated, $L(f)$, the Lefschetz number of $f : X \to X$, is defined. If $X = \bigcup_{i=1}^m C_i$, where C_i is closed and convex, Corollary 2.5 implies that there is a compact set $D \subset X$, with $D \in \mathcal{F}_0$, and a continuous map $g : X \to D$ which is an admissible approximation to f with respect to $\{C_i | 1 \leq i \leq m\}$. It follows that $i_X(f, X) = i_X(g, X)$; and, because $(1-t)f(x) + tg(x) \in X$ for $0 \leq t \leq 1$ and for all $x \in X$, we have $f_* = g_*$ at the homology level and so $L(f) = L(g)$. Because $g(X) \subset D$, properties of the fixed point index and the Lefschetz number imply that $i_X(g, X) = i_D(g_1, D)$ and $L(g) = L(g_1)$, where $g_1 : D \to D$ denotes the map $x \to g(x)$ for $x \in D$. The normalization property for the standard fixed point index implies that $i_D(g_1, D) = L(g_1)$, so $i_X(f, X) = L(f)$.

If $X \in \mathcal{F}_0$ and $f : X \to X$ is a continuous map with $L(f) \neq 0$, it follows that $i_X(f, X) \neq 0$; and if $\alpha(f(X)) < \eta$, we conclude that there exists $x_* \in X$ with $\|f(x_*) - x_*\| < \eta$.

If $X \in \mathcal{F}_0$, then X is a so-called "metric absolute neighborhood retract" or "metric ANR". An intriguing and apparently difficult question is to what extent the results of this section can be generalized to general metric ANR's or even to spaces $X \in \mathcal{F}$. We shall not pursue this point here.

Even in the case that $X \in \mathcal{F}_0$, the approximate fixed point index can sometimes be defined in greater generality than has been given here. To illustrate this point, we shall consider the case that X is convex.

In general, if (X, ρ) is a metric space and $A \subset X$ is a bounded set, define

$$\gamma_X(A) = \inf\{r > 0 \mid A \subset \bigcup_{i=1}^{n} V_r(x_i) \text{ for some } x_i \in X, \text{ for } 1 \le i \le n < \infty\},$$

where $V_r(y) = \{x \in X \mid \rho(x, y) \le r\}$. If X is a Banach space, $\gamma_X(A)$ is sometimes called the "ball measure of noncompactness (ball MNC)" or "Hausdorff MNC" of A; see [2],[39] or [48]. If α denotes the Kuratowski MNC, then $\gamma_X(A) \le \alpha(A) \le 2\gamma_X(A)$ for all bounded sets $A \subset X$.

Now suppose that X is a closed, convex subset of a normed linear space $(Y, \|\cdot\|)$, that $U := V \cap X$, where V is a bounded, open subset of Y and $f : \overline{U} \to X$ is a continuous map such that $\inf\{\|f(x) - x\| \mid x \in \partial U\} := \delta > 0$. Assume that $\gamma_X(f(\partial U)) < \delta$ and that $\gamma_X(f(\overline{U})) < \eta < \infty$. We claim that there is a compact, continuous map $g : \overline{U} \to X$ with

(1) $\|f(x) - g(x)\| < \delta$ for all $x \in \partial U$; and

(2) $\|f(x) - g(x)\| < \eta$ for all $x \in \overline{U}$.

If f is as above but $f(\overline{U})$ is unbounded, a variant of the argument below shows the existence of a compact, continuous g satisfying property (1).

To show the existence of g, select δ' with $\delta' \le \eta$ and $\gamma_X(f(\partial U)) < \delta' < \delta$. Select points $x_i \in X$, for $1 \le i \le m < \infty$, with $f(\partial U) \subset \bigcup_{i=1}^m B_{\delta'}(x_i)$, where $B_{\delta'}(x_i) := \{x \in X \mid \|x - x_i\| < \delta'\}$. There exists a set $W \subset \overline{U}$, relatively open in the topology on $\overline{U}$, with $\partial U \subset W$ and $\overline{W} \subset \bigcup_{i=1}^m f^{-1}(B_{\delta'}(x_i))$. For $1 \le i \le m$, define $W_i = f^{-1}(B_{\delta'}(x_i))$, a relatively open subset of $\overline{U}$. For $m < i \le m + n < \infty$, select $x_i \in X$ with $f(\overline{U}) \subset \bigcup_{i=m+1}^{m+n} B_\eta(x_i)$; and for $m < i \le m + n$ define $W_i := f^{-1}(B_\eta(x_i)) \cap (\overline{U} \backslash \overline{W})$, a relatively open subset of $\overline{U}$. For $1 \le i \le m + n$, let $\theta_i : \overline{U} \to [0, 1]$ be a partition of unity subordinate to the open covering $\{W_i \mid 1 \le i \le m + n\}$ and define $g : \overline{U} \to X$ by $g(y) = \sum_{i=1}^{m+n} \theta_i(y) x_i$.

If $y \in \overline{W}$, our construction implies that $\theta_i(y) = 0$ for $m < i \le m + n$ and that if $\theta_i(y) > 0$ for

$1 \le i \le m$, then $\|f(y) - x_i\| < \delta'$. It follows that for $y \in \overline{W} \supset \partial U$,

$$\|f(y) - g(y)\| = \left\| \sum_{i=1}^{m} \theta_i(y)(f(y) - x_i) \right\| \le \sum_{i=1}^{m} \theta_i(y)\|f(y) - x_i\| < \sum_{i=1}^{m} \theta_i(y)\delta' = \delta'$$

A similar argument shows that $\|f(y) - g(y)\| < \eta$ for all $y \in \overline{U}$.

Now suppose that X is a closed, convex subset of a normed linear space $(Y, \|\cdot\|)$, that U is a bounded, relatively open subset of X and $f : \overline{U} \to X$ is a continuous map such that $\inf\{\|f(x) - x\| \mid x \in \partial U\} = \delta > 0$. If there exists a compact, continuous map $g : \overline{U} \to X$ such that $\|f(x) - g(x)\| < \|f(x) - x\|$ for all $x \in \partial U$, so $g(x) \neq x$ for $x \in \partial U$, we define the approximate fixed point index $i_X(f, U)$ by $i_X(f, U) := i_X(g, U)$. If $g_0 : \overline{U} \to X$ and $g_1 : \overline{U} \to X$ are two such maps $\|f(x) - (1-t)g_0(x) - tg_1(x)\| < \|f(x) - x\|$ for all $x \in \partial U$ and $0 \le t \le 1$, so $(1-t)g_0(x) + tg_1(x) \neq x$ for all $x \in \partial U$ and $0 \le t \le 1$ and the homotopy property of the usual fixed point index implies that $i_X(g_0, U) = i_X(g_1, U)$. (Note that $(1-t)g_0(x) + tg_1(x) \in X$ for $0 \le t \le 1$, because X is convex, but this part of the argument fails if $X \in \mathcal{F}_0$.) Our previous remarks prove the existence of such a g if $\gamma_X(f(\partial U))) < \delta$.

If X is an infinite dimensional Banach space and $U = \{x \in X \mid \|x\| < 1\}$ and $f : \overline{U} \to X$ is a continuous map such that $\inf\{\|f(x) - x\| \mid x \in \partial U\} > 1$ and $f(\partial U) \subset \overline{U}$, the approximate fixed point index in the sense just described is defined, because $\gamma_X(f(\partial U)) \le \gamma_X(\overline{U}) = 1$. However, if $f(x) = -x$ for all $x \in \partial U$, the general framework of this section would not be applicable, since $\alpha(f(\partial U)) = 2$ and $\inf\{\|f(x) - x\| \mid x \in \partial U\} = 2$.

If $X \in \mathcal{F}_0$ and U is a bounded, relatively open subset of X such that $\inf\{\|f(x) - x\| \mid x \in \partial U\} = \delta > 0$ and $\alpha(f(\partial U)) < \delta$, we have seen that the approximate fixed point index $i_X(f, U)$ is defined. If there exists $X_0 \subset X$, with $X_0 \in \mathcal{F}_0$, such that $f(\overline{U}) \subset X_0$, it follows directly from the corresponding property for the classical fixed point index that

$$i_X(f, U) = i_{X_0}(f, U \cap X_0).$$

We shall need this fact later.

It will also be convenient to have a version of the homotopy property for the approximate fixed point index.

Proposition 2.6. *Suppose that $X \in \mathcal{F}_0$, that U is a bounded, relatively open subset of X and that $f : \overline{U} \times [0,1] \to X$ is a continuous map. Define $f_t : \overline{U} \to X$, for $0 \le t \le 1$, by $f_t(x) := f(x,t)$ and* *assume that $t \to f_t(x)$ is continuous, uniformly in $x \in \partial U$. For $0 \le t \le 1$ assume that*

$$\alpha(f_t(\partial U)) < \delta_t := \inf\{\|f_t(x) - x\| \mid x \in \partial U\}.$$

Then the approximate fixed point index $i_X(f_t, U)$ is defined and constant for $0 \le t \le 1$.

Proof. Given $t_0 \in J := [0,1]$, it follows from the fact that $t \to f_t(x)$ is continuous uniformly in $x \in \partial U$ that there exists $\eta > 0$ such that

$$\alpha(\{f(x,t) \mid x \in \partial U \text{ with } |t - t_0| \le \eta \text{ and } t \in J\})$$
$$< \sigma < \rho := \inf\{\|f(x,t) - x\| \mid x \in \partial U \text{ with } |t - t_0| \le \eta \text{ and } t \in J\}.$$

Following the argument in Corollary 2.5, there exist sets S_j, for $1 \le j \le n < \infty$, such that $\operatorname{diam}(S_j) < \sigma$ and

$$\{f(x,t) \mid x \in \partial U \text{ with } |t - t_0| \le \eta \text{ and } t \in J\} = \bigcup_{j=1}^{n} S_j.$$

Because $X \in \mathcal{F}_0$, we can write $X = \bigcup_{i=1}^{m} C_i$, where C_i, for $1 \le i \le m$, is a closed, convex subset of a normed linear space $(Y, \|\cdot\|)$. Define $T := \{f(x,t) \mid x \in \overline{U} \text{ with } |t - t_0| \le \eta \text{ and } t \in J\}$ and

$$Z := \left(\bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \overline{\mathrm{co}}(C_i \cap S_j)\right) \cup \left(\bigcup_{i=1}^{n} \overline{\mathrm{co}}(C_i \cap T)\right).$$

By Corollary 2.4, there exists a finite dimensional linear subspace $Y_0 \subset Y$, a compact set $D \in \mathcal{F}_0$ with $D \subset Z \cap Y_0$ and a continuous retraction R of Z onto D such that

(1) $R(y) \in \overline{\mathrm{co}}(C_i \cap S_j) \cap D$ for all $y \in \overline{\mathrm{co}}(C_i \cap S_j)$; and

(2) $R(y) \in \overline{\mathrm{co}}(C_i \cap T)$ for all $y \in C_i \cap T$.

It follows that for $x \in \overline{U}$ and $|t - t_0| \le \eta$, with $t \in J$, that $g_t(x) = R(f_t(x)) \in C_i$ whenever $f_t(x) \in C_i$ and $g_t : \overline{U} \to D$ is a compact, continuous map. If $x \in \partial U$ and $|t - t_0| \le \eta$, with $t \in J$, then $f_t(x) \in C_i \cap S_j$ for some i and j, so $g_t(x) \in C_i \cap S_j$ and $\|f_t(x) - g_t(x)\| < \sigma < \delta_t$. It follows that $i_X(f_t, U) = i_X(g_t, U)$. By the homotopy property for the standard fixed point index, $i_X(g_t, U)$ is constant for $|t - t_0| \le \eta$, with $t \in J$.

For each integer N, let $J_N = \{t \in [0,1] \mid i_X(f_t, U) = N\}$. The above argument shows that J_N is open in the relative topology on J. On the other hand, J_N is the complement of $\bigcup_{M \ne N} J_M$, so J_N

is closed. By the connectedness of J, for every integer N the set J_N is either empty or $J_N = J$. It follows that if N is chosen so J_N is not empty, then $J_N = J$. ■

Corollary 2.7. *Suppose that $X \in \mathcal{F}_0$, that U is a bounded, relatively open subset of X and $f : \overline{U} \to X$ is a continuous map such that $\delta := \inf\{\|f(x) - x\| \mid x \in \partial U\} > 0$ and $\alpha(f(\partial U)) := \eta < \delta$. Then $i_X(f, U)$, the approximate fixed point index, is defined. If $g : \overline{U} \to X$ is a continuous, compact map such that $\sup\{\|f(x) - g(x)\| \mid x \in \partial U\} := \sigma < \delta$ and $(1 - t)f(x) + tg(x) \in X$ for all $x \in U$ and $t \in [0, 1]$, then $i_X(f, U) = i_X(g, U)$*

Proof. For $(x, t) \in \overline{U} \times [0, 1]$, define $f_t(x) = (1 - t)f(x) + tg(x) \in X$. Notice that $(x, t) \to f_t(x)$ is continuous, uniformly in $x \in \partial U$. For $0 \le t \le 1$, we have $\alpha(f_t(\partial U)) = (1 - t)\eta$. If $x \in \partial U$ and $0 \le t \le 1$, then $\|f_t(x) - x\| = \|f(x) - x - t(f(x) - g(x))\| \ge \delta - t\sigma > (1 - t)\eta$, where we have used that $\sigma < \delta$ and $\eta < \delta$. This shows that the hypothesis of Proposition 2.6 are satisfied, so $i_X(f_t, U)$ is constant for $0 \le t \le 1$. ■

If $X = \bigcup_{i=1}^m C_i$, where C_i, for $1 \le i \le m$, is convex and g is an admissible approximation to f with respect to $\{C_i \mid 1 \le i \le m\}$, then g satisfies the conditions in Corollary 2.7. However, it may easily happen that the conditions on g in Corollary 2.7 are satisfied even if g is not such an admissible approximation.

3 Generalizing the Fixed Point Index

Our primary interest in the approximate fixed point index is as a convenient tool to generalize the fixed point index to various classes of possibly noncompact maps. This approach has been used in [42]. Here we shall recall for the reader's convenience some results from [42]. We shall also present refinements of theorems in [42] and we shall explicitly spell out some corollaries which were previously left implicit.

To begin, suppose that U is an open subset of a complete metric space $X \in \mathcal{F}$ and that $f : U \to X$ is a continuous map such that $S := \{x \in U \mid f(x) = x\}$ is compact (possibly empty). Note that we do not assume that U is bounded or that f is defined and continuous on $\overline{U}$. We wish to describe a large class of examples for which one can define a reasonable fixed point index for $f : U \to X$.

For f, U, S and X as above, suppose that there exists a bounded open neighborhood W of S in

X with $\overline{W} \subset U$ and a decreasing sequence of spaces $K_n \in \mathcal{F}_0$ with $K_n \subset X$ for all n such that

(1) $W \subset K_1$;

(2) $f(W \cap K_n) \subset K_{n+1}$ for all $n \geq 1$; and

(3) $\lim_{n\to\infty} \alpha(K_n) = 0$, where α denotes the Kuratowski MNC on X.

We allow the possibility that W is empty if S is empty or that K_n is empty for some n. If the above conditions are satisfied for some W and some decreasing sequence of spaces K_n, for $n \geq 1$, we say that "f belongs to the fixed point index class". It is proved in Proposition 2.1 of [42] that the approximate fixed point index $i_{K_n}(f, W \cap K_n)$ is defined and constant for all sufficiently large n, so we define

$$i_X(f, U) := \lim_{n\to\infty} i_{K_n}(f, W \cap K_n). \tag{3.1}$$

If $i_X(f, U) \neq 0$, it is proved in [42] that f has a fixed point in U. For the above definition to be reasonable, we have to show that $i_X(f, U)$ is independent of the particular open neighborhood W of S and the particular decreasing sequence of sets $K_n \in \mathcal{F}_0$. Thus suppose that V is another bounded open neighborhood of S with $\overline{V} \subset U$ and that we are given a decreasing sequence of spaces $H_n \in \mathcal{F}_0$ with $H_n \subset X$ for all $n \geq 1$ and

(4) $V \subset H_1$;

(5) $f(V \cap H_n) \subset H_{n+1}$ for all $n \geq 1$; and

(6) $\lim_{n\to\infty} \alpha(H_n) = 0$.

Then it is proved in Proposition 2.1 of [42] that

$$\lim_{n\to\infty} i_{K_n}(f, W \cap K_n) = \lim_{n\to\infty} i_{H_n}(f, V \cap H_n).$$

It is proved in Section 2 of [42] that with the above definition we obtain a fixed point index which satisfies the usual properties: additivity, homotopy invariance, commutativity and connections to the Lefschetz fixed point theorem. Moreover, although the definition may seem artificial, it is actually very much in the spirit of Proposition 2.2 and Darbo's old argument in [10]. Furthermore, as we shall show below, the approach in [42] provides a unified way of treating a wide variety of seemingly disparate examples.

There are, however, some subtleties here. If f is in the fixed point index class, one might expect that $F(x) := x - f(x)$ is "proper at 0" when F is restricted to $\overline{H}$ and H is a suitable open neighborhood of S. Specifically, one might expect that there is an open neighborhood H of S in X such that if x_n, for $n \geq 1$, is any sequence in $\overline{H}$ with $x_n - f(x_n) \to 0$, then x_n has a convergent subsequence. This would imply that $\inf\{\|x - f(x)\| \mid x \in \partial H := \overline{H} \backslash H\} > 0$. However, in this generality there seems no reason why $F|\overline{H}$ has to be proper at 0 for some open neighborhood H of S.

If X is itself a Banach space or a closed cone in a Banach space and U and S are as above and f belongs to the fixed point index class, one might hope that small perturbations of f necessarily belong to the fixed point index class. Specifically, for $y \in X$ and $\varepsilon > 0$, define $f_\varepsilon(x) = f(x) + \varepsilon y$. Is it necessarily true that f_ε belongs to the fixed point index class for $\varepsilon > 0$ small, say for $0 < \varepsilon < \varepsilon(y)$, and if so, that $i_X(f_\varepsilon, U) = i_X(f, U)$ for such ε? In the absence of more restrictive assumptions on f, it is unclear whether f_ε belongs to the fixed point index class for small $\varepsilon > 0$.

In the situation described above, it is frequently the case that we have much more information about the sets K_n, and this added information leads to a refinement of equation (3.1).

Theorem 3.1. *Suppose that U is an open subset of a complete space $X \in \mathcal{F}$ and $f : U \to X$ is a continuous map such that $S := \{x \in U \mid f(x) = x\}$ is compact (possibly empty). Assume that there exists an open neighborhood W of S in X and a sequence of sets $K_n \in \mathcal{F}_0$ with $K_n \subset X$ such that*

(1) $W \subset K_1$;

(2) $f(W \cap K_n) \subset K_{n+1}$ for all $n \geq 1$; and

(3) $\lim_{n\to\infty} \alpha(K_n) = 0$, where α denotes the Kuratowski MNC on X.

Assume also that there exist integers m and N such that $K_n = \bigcup_{j=1}^N C_{j,n}$ for all $n \geq m$, where $C_{j,n}$ is closed and convex with $C_{j,n} \supset C_{j,n+1}$ for all $n \geq m$ and $1 \leq j \leq N$. Then $K_\infty := \bigcap_{n\geq 1} K_n = \bigcup_{j=1}^N C_{j,\infty}$, where $C_{j,\infty} := \bigcap_{n\geq m} C_{j,n}$, so K_∞ is a compact metric ANR. Furthermore, f belongs to the fixed point index class and

$$i_X(f, U) = i_{K_\infty}(f, W \cap K_\infty).$$

Proof. Because $\alpha(K_n) \to 0$, we have $\alpha(K_\infty) = 0$ and K_∞ is compact (possibly empty). As is noted in [38], page 226, any space $X \in \mathcal{F}$ is a metric ANR, so K_∞ is a compact metric ANR.

For any subset $J \subset \{1, 2, \ldots, N\}$ and any $n \geq m$, define

$$C_{J,n} := \bigcap_{j \in J} C_{j,n}.$$

Because $\alpha(C_{J,n}) \leq \alpha(K_n)$, we have $\lim_{n\to\infty} \alpha(C_{J,n}) = 0$. If $C_{J,n}$ is nonempty for all $n \geq m$, it follows from Kuratowski's theorem that $C_{J,\infty} := \bigcap_{n \geq m} C_{J,n}$ is nonempty. Using this fact, we see that there exists an integer ν such that for $n \geq \nu$ and any subset $J \subset \{1, 2 \ldots, N\}$, the set $C_{J,n}$ is nonempty if and only if $C_{J,\infty}$ is nonempty. By relabeling the index set $1 \leq j \leq N$, and possibly decreasing N, we can assume that $C_{j,n}$ is nonempty for all $n \geq \nu$ and for $1 \leq j \leq N$. Select $\delta_n > \alpha(K_n)$ such that $\lim_{n\to\infty} \delta_n = 0$. It follows from Theorem 1 in Section B of [38] that for $n \geq \nu$ there exists a retraction R_n of K_n onto K_∞ such that $\|R_n(x) - x\| < \delta_n$ for all $x \in K_n$ and $R_n(x) \in C_{j,n} \cap K_\infty$ whenever $x \in C_{j,n}$. We know that there exist $\delta > 0$ and an integer $\nu_1 \geq \nu$ such that if $n \geq \nu_1$, then $\|f(x) - x\| \geq \delta$ for all $x \in \partial W \cap K_n$. If we choose ν_2 so that $\delta_n \leq \delta$ for all $n \geq \nu_2$, it follows that $g_n : \overline{W} \cap K_n \to K_n$ defined by $g_n(x) := R_n(f(x))$ is an admissible approximation to $f : \overline{W} \cap K_n \to K_n$ with respect to $\{C_{j,n} \,|\, 1 \leq j \leq N\}$. By our definition of the generalized fixed point index, $i_X(f, U) = i_{K_n}(g_n, W \cap K_n)$ for $n \geq \nu_2$. However, $g_n(W \cap K_n) \subset K_\infty$ and $g_n(x) = f(x)$ for all $x \in W \cap K_\infty$, so the commutativity property of the fixed point index implies that

$$i_{K_n}(g_n, W \cap K_n) = i_{K_\infty}(g_n, W \cap K_\infty) = i_{K_\infty}(f, W \cap K_\infty),$$

which completes the proof. ■

The point of the definition embodied in equation (3.1) is that it covers many examples, so before proceeding further we recall for the reader's convenience some classes of maps f which fall into the fixed point index class.

Example 1. (See Proposition 3.1 in [42].) Suppose that U is an open subset of a complete space $X \in \mathcal{F}$ and $f : U \to X$ is a continuous map. Assume that $\Gamma \subset U$ is a compact set with $f(\Gamma) \subset \Gamma$ and that there exists a bounded open neighborhood V of Γ in X with $V \subset U$ such that $f|V$ is a k-set-contraction, $k < 1$. Then there exists a bounded, open neighborhood W of Γ with $\overline{W} \subset V$ and a decreasing sequence of sets $K_n \in \mathcal{F}_0$, with $K_n \subset X$, such that conditions (1)-(3) of Theorem 3.1 hold. Furthermore, there exists an integer N such that $K_n = \bigcup_{j=1}^{N} C_{j,n}$ for all $n \geq 1$, where $C_{j,n}$ is closed and convex and $C_{j,n} \supset C_{j,n+1}$ for $1 \leq j \leq N$ and $n \geq 1$. In particular, if $S := \{x \in U \mid f(x) = x\}$ is compact and $S \subset \Gamma$ for Γ as above, then f belongs to the fixed point index class and

$$i_X(f, U) = i_{K_\infty}(f, W \cap K_\infty), \tag{3.2}$$

where $K_\infty := \bigcap_{n \geq 1} K_n$.

The argument to prove the assertion above is straightforward. For a sufficiently small neighborhood W of Γ, define $K_1 = (\overline{\text{co}}(W) \cap X) \cup (\overline{\text{co}}(f(W)) \cap X)$, and $K_2 = \overline{\text{co}}(f(W)) \cap X)$ and then $K_n = \overline{\text{co}}(f(W \cap K_{n-1})) \cap X$ for $n \geq 3$. Theorem 3.1 implies that $i_X(f, U) = i_{K_\infty}(f, W \cap K_\infty)$. In this context, recall that X is a complete subset of a normed linear sapce $(Y, \|\cdot\|)$ and the Kuratowski MNC α is that inherited from Y. Note that if $f|\overline{W}$ is a compact map, then f is a 0-set-contraction, a special case of our set-up.

In the spirit of Proposition 2.2, note that instead of assuming that $f|V$ is a k-set-contraction with $k < 1$, it suffices to assume that there exists a bounded open neighborhood W of Γ in X, with $W \subset U$, such that if $G_1 := \overline{\text{co}}(f(W)) \cap X$ and $G_n := \overline{\text{co}}(f(W \cap G_{n-1})) \cap X$ for $n \geq 2$, then $\lim_{n \to \infty} \alpha(G_n) = 0$.

Example 2. (See Lemmas 3.1 and 3.2 and Proposition 3.3, all in [42], and Lemma 14 in [41].) Let U be an open subset of a Banach space X and $f : U \to X$ a continuous map. Assume $\Gamma \subset U$ is compact and $f(\Gamma) \subset \Gamma$ and f is C^1 (continuously Fréchet differentiable) on an open neighborhood V of Γ. Assume also that there exists an integer $N \geq 1$ such that f^N is defined on V and and $f^N|V$ is a c-set-contraction with $c < 1$. Then there exists a bounded open neighborhood W of Γ, with $\overline{W} \subset V$, and a decreasing sequence of sets $K_n \in \mathcal{F}_0$ such that conditions (1)-(3) of Theorem 3.1 hold. Furthermore, W and K_n, for $n \geq 1$, can be chosen so that $K_n = \bigcup_{j=1}^p C_{j,n}$ for all n, where p is independent of n, and $C_{j,n}$, for $1 \leq j \leq p$, is closed and convex, and $C_{j,n} \supset C_{j,n+1}$ for $1 \leq j \leq p$ and $n \geq 1$.

If $S := \{x \in U \mid f(x) = x\}$ is compact and $S \subset \Gamma$ for some Γ meeting the above conditions, then f belongs to the fixed point index class and (by Theorem 3.1) equation (3.2) holds, where $K_\infty := \bigcap_{n \geq 1} K_n$.

Example 2 has a number of useful variants.

Example 3. Suppose that $X \in \mathcal{F}_0$ and X is a closed subset of a Banach space $(Y, \|\cdot\|)$. Let U be an open subset of X and $f : U \to X$ a continuous map. Assume that $\Gamma \subset U$ is compact and $f(\Gamma) \subset \Gamma$. Assume that there exists an open neighborhood $\widehat{V}$ of Γ in Y and an extension $\widehat{f} : \widehat{V} \to Y$ of $f|\widehat{V} \cap X$ such that

(1) $\widehat{f}$ is C^1 on $\widehat{V}$; and

(2) for some integer $N \geq 1$, the iterate $\widehat{f}^N$ is defined on an open neighborhood of Γ in Y and $\widehat{f}^N$ is a c-set-contraction with $c < 1$.

By Example 2, there exists a bounded open neighborhood $\widehat{W}$ of Γ in Y and a decreasing sequence of sets $\widehat{K}_n \in \mathcal{F}_0$, with $\widehat{K}_n \subset Y$, which meet conditions (1)-(3) of Theorem 3.1. By assumption, $X = \bigcup_{j \in J} C_j$, where C_j, for $j \in J$, is closed and convex and J is a finite set. It follows that $K_n := \widehat{K}_n \cap X \in \mathcal{F}_0$, and $W := \widehat{W} \cap X$ is an open neighborhood of Γ in X with $\overline{W} \subset V$ and that conditions (1)-(3) of Theorem 3.1 hold.

If $S = \{x \in U \mid f(x) = x\}$ is compact and $S \subset \Gamma$ for some Γ meeting the above conditions, it follows as in Example 2 that f belongs to the fixed point index class and that equation (3.2) holds, where $K_\infty := \bigcap_{n \geq 1} K_n$.

By a slight variant of the argument in this example we could allow $X \in \mathcal{F}$. We have assumed that $X \in \mathcal{F}_0$ only for simplicity.

Before describing our next example, we shall need some results about Banach manifolds. We refer to Elworthy's paper [15] for further details. Recall that a Banach space $(X, \|\cdot\|)$ is called a C^1 Banach space if there exists a C^1 map $f : X \to \mathbb{R}$ which has bounded support but is not identically zero. Elworthy [15] has proved that if M is a C^1 Banach manifold modelled on a C^1 Banach space, then there is a C^1 map j of M into a Banach space Z such that $j(M)$ is closed, an open neighborhood H of $j(M)$ in Z and a C^1 map $r : H \to M$ such that rj is the identity on M. It may easily happen that M is a C^1 Banach manifold which is not modelled on a C^1 Banach space but for which C^1 maps j and r as above exist: if M is any closed, complemented linear subspace of a Banach space Z, take j to be the inclusion map and r to be a bounded linear projection of Z onto M. For this reason we make the following *ad hoc* definition, motivated by Elworthy's theorem:

Definition 3.2. A C^1 Banach manifold M can be "nicely embedded in a Banach space Z" if there exists a C^1 map $j : M \to Z$ such that $j(M)$ is a closed subset of Z, an open neighborhood H of $j(M)$ in Z and a C^1 map $r : H \to M$ such that rj is the identity map on M.

Example 4. (See Section 3 of [42].) Let M be a C^1 Banach manifold which can be nicely embedded in a Banach space Z and let j, r and H be as in Definition 3.2. Assume that M is a subset of a Banach space Y and that the inclusion of M into Y is C^1. Let U be an open subset of M and $f : U \to M$ a continuous map. Assume that $\Gamma \subset U$ is a compact set such that $f(\Gamma) \subset \Gamma$ and suppose that there exists an open neighborhood V of Γ in M and an integer N such that f is C^1 on V and $f^N|V$ is a c-set-contraction, with $c < 1$. (The Kuratowski MNC α from Y is used.) Define $g = jfr : r^{-1}(U) \to Z$. Then g satisfies the conditions described in Example 2, so if $S := \{x \in U \mid f(x) = x\}$ and $S \subset \Gamma$ for Γ as above, we find that g is in the fixed point index class. To see this, notice that $\Gamma_1 := j(\Gamma)$ is compact, $g(\Gamma_1) \subset \Gamma_1$ and g is C^1 on an open neighborhood of Γ_1 in Z. If V_1 is an open neighborhood of Γ_1 on which g^m is defined, note that $g^m = jf^m r$. Because j and r are C^1, there is a constant C and there exist open neighborhoods V_2 of Γ in M and V_3 of Γ_1 in Z such that $j : V_2 \to Z$ and $r : V_3 \to M$ are Lipschitz maps with Lipschitz constant C and therefore are C-set-contractions. It follows that on a small neighborhood V_4 of Γ_1, that $g^{Np} = j(f^N)^p r$ is a $c^p C^2$-set-contraction. By choosing p so large that $c^p C^2 < 1$, we find that there is an integer $N_1 = Np$ and an open neighborhood V_4 of Γ_1 such that $g^{N_1}|V_4$ is a c_1-set-contraction, where $c_1 = c^p C^2 < 1$.

If $f^N|V$ is a compact map, we have a special case of this example for which the above argument takes a simpler form.

In Section 3 of [42] it is assumed that M is a closed subset of Y, but this hypothesis is not actually used in [42].

It may, of course, happen that a C^1 Banach manifold can be nicely embedded in a Banach space Z_1 and in a Banach space Z_2. Thus, for $k = 1, 2$, there are C^1 maps $j_k : M \to j_k(M) \subset Z_k$, an open neighborhood H_k of the closed set $j_k(M)$ in Z_k and C^1 maps $r_k : H_k \to M$ with $r_k(j_k(x)) = x$ for all $x \in M$. If $S := \Gamma := \{x \in U \mid f(x) = x\}$ is compact and $f : U \to M$ satisfies the assumptions in Example 4, remarks in Example 4 imply that $g_k := j_k f r_k : r_k^{-1}(U) \to Z_k$ belongs to the fixed point index class and $i_{Z_k}(g_k, r_k^{-1}(U))$ is defined for $k = 1, 2$. If one uses the fact that $g_1 = h_1 h_2$, where $h_1 = j_1 f r_2$ and $h_2 = j_2 r_1$ and $g_2 = h_2 h_1$, then the commutativity property for the generalized fixed point index (see Propositions 2.5 and 3.5 in [42]) implies that $i_{Z_1}(g_1, r_1^{-1}(U)) = i_{Z_2}(g_2, r_2^{-1}(U))$. We omit the details of the argument. Using this observation, we shall define $i_M(f, U) := i_Z(jfr, r^{-1}(U))$, since our definition is independent of the particular maps j and r as in Definition 3.2.

Our next example follows directly from Proposition 3.6 in [42]. Note that the absence of a continuous differentiability assumption on f forces us to assume that f, rather than f^N for some $N \geq 1$, is a k-set-contraction with $k < 1$.

Example 5. (See Proposition 3.6 in [42].) Let M be a C^1 Banach manifold which can be nicely embedded in a Banach space Z and let j, r and H be as in Definition 3.2. Assume that M is a subset of a Banach space Y and that the inclusion of M into Y is C^1. Let U be an open subset of M and $f : U \to M$ be a continuous map. Assume that $\Gamma \subset U$ is a compact set with $f(\Gamma) \subset \Gamma$ and that there exists an open neighborhood V of Γ in M such that $f|V$ is a k-set-contraction with $k < 1$. Define $g = jfr : r^{-1}(U) \to Z$. Then there exists a bounded open neighborhood W_1 of $\Gamma_1 := j(\Gamma)$ with $\overline{W}_1 \subset r^{-1}(V)$ and a decreasing sequence of sets $K_n \in \mathcal{F}_0, K_n \subset r^{-1}(V)$, such that

(1) $W_1 \subset K_1$;

(2) $g(W_1 \cap K_n) \subset K_{n+1}$ for all $n \geq 1$; and

(3) $\lim_{n\to\infty} \alpha_Z(K_n) = 0$, where α_Z denotes the Kuratowski MNC on Z.

Furthermore, there exists an integer N such that for all $n \geq 1, K_n = \bigcup_{j=1}^{N} C_{j,n}$, where $C_{j,n}$ is closed and convex and $C_{j,n} \supset C_{j,n+1}$ for $1 \leq j \leq N$ and $n \geq 1$. In particular, if $S := \{x \in U \mid f(x) = x\}$ and $S \subset \Gamma$ for Γ as above, then g belongs to the fixed point index class and (by Theorem 3.1)

$$i_Z(g, r^{-1}(U)) = i_{K_\infty}(g, W_1 \cap K_\infty),$$

where $K_\infty := \bigcap_{n\geq 1} K_n$.

As in Example 4, M may be nicely embedded in a Banach space Z_1 and in a Banach space Z_2. In the notation of Example 4, one can use the commutativity property of the fixed point index to prove that $i_{Z_1}(g_1, r_1^{-1}(U)) = i_{Z_2}(g_2, r_2^{-1}(U))$, where $g_k = j_k f r_k$. Thus, in the context of Example 5, we shall define $i_M(f, U) := i_Z(g, r^{-1}(U))$.

If U is an open subset of $X \in \mathcal{F}, f : U \to X$ belongs to the fixed point index class and $\theta : U \to V \subset Y$ is a homeomorphism onto an open subset V of $Y \in \mathcal{F}$, one might hope that $\theta f \theta^{-1} : V \to Y$ belongs to the fixed point index class and that $i_X(f, U) = i_Y(\theta f \theta^{-1}, V)$. We have no results in this generality; but if θ and θ^{-1} are C^1, such a result is frequently true. As an example, we restate for the reader's convenience Proposition 3.8 in [42].

Example 6. (Compare Proposition 3.8 in [42].) Let U be an open subset of a Banach space X and $f : U \to X$ a continuous map such that $S := \{x \in U \mid f(x) = x\}$ is compact. Assume that there exists an open neighborhood V of S such that $f|V$ is a k-set-contraction with $k < 1$. Assume also that $\theta : U \to G$ is a C^1 diffeomorphism onto an open subset G of a Banach space Y and that θ^{-1} is C^1. Then $\theta f \theta^{-1} : G \to Y$ belongs to the fixed point index class and

$$i_X(f, U) = i_Y(\theta f \theta^{-1}, G).$$

Furthermore, $\theta f \theta^{-1}$ satisfies the assumptions of Theorem 3.1, so if K_n, for $n \geq 1$, are sets as in Theorem 3.1 and W is an appropriate open neighborhood of $\theta(S)$, the fixed point set of $\theta f \theta^{-1}$ in G, then

$$i_Y(\theta f \theta^{-1}, G) = i_{K_\infty}(\theta f \theta^{-1}, W \cap K_\infty).$$

The proof of Proposition 3.8 in [42] actually shows that if $\Gamma \subset U$ is a compact set in U such that $f(\Gamma) \subset \Gamma$ and if there is an open neighborhood V of Γ such that $f|V$ is a k-set contraction with $k < 1$, then we obtain an open neighborhood W of $\theta(\Gamma)$ and sets $K_n \in \mathcal{F}_0$ as in Example 1 for the map $g := \theta f \theta^{-1}$, in other words

(1) $W \subset K_1$;

(2) $g(W \cap K_n) \subset K_{n+1}$ for all $n \geq 1$; and

(3) $\lim_{n \to \infty} \alpha_Y(K_n) = 0$, where α_Y denotes the Kuratowski MNC on Y.

In Examples 1-6 above, the maps f satisfy the hypotheses of Theorem 3.1. For the following class of maps, treated in Proposition 3.2 of [42], Theorem 3.1 may not be applicable.

Example 7. (Compare [18], Theorem 16.3 in [9], Corollary 1 in [41] and Proposition 3.2 in [42].) Suppose that $X \in \mathcal{F}$, where X is complete, and that U is an open subset of X with $f : U \to X$ a continuous map. Assume that $\Gamma \subset U$ is a compact set such that $f(\Gamma) \subset \Gamma$. Assume also that there exists a compact set M with $\Gamma \subset M \subset X$, a constant c with $0 \leq c < 1$ and a number $r_0 > 0$ such that whenever $x \in U$ and $d(x, M) \leq r_0$, then $d(f(x), M) \leq c d(x, M)$. (Here, $d(x, M)$ denotes the distance of x to M.) Then there exist a decreasing sequence of sets $K_n \in \mathcal{F}_0$ with $K_n \subset X$ and an open neighborhood W of Γ such that conditions (1)-(3) of Theorem 3.1 hold. In particular, if $S := \{x \in U \mid f(x) = x\}$ and $S \subset \Gamma$ for Γ as above, then f belongs to the fixed point index class.

The proof in [42] is stated only for the case that $\Gamma := S$ and S is compact, but the argument is the same for general Γ.

4 Asymptotic Fixed Point Theorems

In this section we shall use the results described in Sections 2 and 3 to prove "asymptotic fixed point theorems". Roughly speaking, by an asymptotic fixed point theorem we mean a result in which assumptions about iterates of a map f are used to prove that f has a fixed point or to obtain information about the fixed point index or Lefschetz number of f. We mention two old unresolved conjectures of this type.

Conjecture 4.1. *Let G be a closed, bounded convex set in a Banach space X and $f : G \to G$ a continuous map. Assume that there exists an integer $N \geq 2$ such that $\overline{f^N(G)}$ is compact. Then f has a fixed point.*

Conjecture 4.2. *Let G be a closed, bounded convex set in a Banach space X and $f : G \to G$ a continuous map. Assume that there is an integer $N \geq 2$ such that f^N is a k-set-contraction with $k < 1$. Then f has a fixed point.*

Since the map f^N in Conjecture 4.1 is a k-set-contraction with $k = 0$, Conjecture 4.2 is more general. As we shall see, both conjectures are true if one has slightly more information about the map f. However, this fact has a negative side: if the conjectures are false, finding counterexamples is likely to be difficult.

We begin by recalling some definitions. If Z is a Hausdorff topological space and $f : Z \to Z$ is a continuous map, we shall call a compact, nonempty set $\Gamma \subset Z$ a "compact attractor for f" if

(1) $f(\Gamma) \subset \Gamma$; and

(2) given any open neighborhood U of Γ and any compact set $A \subset Z$, there exists an integer $n = n(U, A)$ with $f^m(A) \subset U$ for all $m \geq n$.

Terminology here is far from uniform. We refer the reader, for example, to a literature which refers to "dissipative processes": see [3], [20], [22] and [23]. Some authors would demand that a set Γ which we call here a compact attractor for f should also satisfy $f(\Gamma) = \Gamma$ instead of just $f(\Gamma) \subset \Gamma$. If Γ is any nonempty, compact, Hausdorff space and $f : \Gamma \to \Gamma$ is a continuous map and if we define $\Gamma_\infty := \bigcap_{n \geq 1} f^n(\Gamma)$, it is well-known and easily verified that $f(\Gamma_\infty) = \Gamma_\infty$ and that given any open neighborhood U of Γ_∞, there exists an integer $N = N(U)$ such that $f^n(\Gamma) \subset U$ for all $n \geq N$. Using this fact, one can show that if $f : Z \to Z$ is a continuous map and $\Gamma \subset Z$ is a compact attractor for Γ (in our sense) then $\Gamma_\infty := \bigcap_{n\geq 1} f^n(\Gamma)$ is a compact attractor for f and $f(\Gamma_\infty) = \Gamma_\infty$. Thus, with no loss of generality, we could also demand in our definition of a compact attractor Γ for f that $f(\Gamma) = \Gamma$.

In general, if X is a Hausdorff topological space and $f : X \to X$ is a continuous map, the question of whether f has a compact attractor may be nontrivial and must be handled on a case by case basis. Indeed, this is the case even if $X = \mathbb{R}^n$, finite dimensional Euclidean space. If f^{n_0} is a compact map for some $n_0 \geq 1$, results of Billotti and LaSalle [3] simplify the problem of proving existence of a compact attractor for f.

Proposition 4.3. *(See [3].) Let (X, ρ) be a complete metric space and $f : X \to X$ a continuous map which takes bounded sets to bounded sets. Assume that there exists an integer $n_0 \geq 1$ such that f^{n_0} is compact, so $\overline{f^{n_0}(A)}$ is compact for every bounded set $A \subset X$. Assume that there exists a bounded set $B \subset X$ with the following property: for every $x \in X$, there exists an integer $m(x) \geq 1$ with $f^{m(x)}(x) \in B$. Then there exists a compact set Γ which is a compact attractor for f.*

For the reader's convenience we mention one other set of conditions which insures the existence of a compact attractor.

Proposition 4.4. *Suppose that (X, ρ) is a complete metric space with Kuratowski MNC α. Let $U \subset X$ be an open set and $f : U \to U$ a continuous map. Assume that there exists a closed, bounded set $B \subset U$ such that*

(1) $f(B) \subset B$; and

(2) for every compact set $A \subset U$, there exists an integer $n = n(A)$ with $f^n(A) \subset B$.

Assume either

(3) $\lim_{j\to\infty} \alpha(f^j(B)) = 0$; or (less generally)

(4) there exists an integer m such that $f^m|B$ is a c-set-contraction with $c < 1$.

Then it follows that $\Gamma := \bigcap_{j\geq 1} \overline{f^j(B)}$ is a compact attractor for $f : U \to U$ and $f(\Gamma) = \Gamma$.

Proof. The sets $\overline{f^j(B)}$, for $j \geq 1$, form a decreasing sequence of closed, bounded, nonempty subsets of B. Under assumption (3)

$$\lim_{j\to\infty}\alpha(\overline{f^j(B)}) = \lim_{j\to\infty}\alpha(f^j(B)) = 0.$$

Under assumption (4),

$$\alpha(\overline{f^{jm}(B)}) = \alpha(f^{jm}(B)) \leq c^j\alpha(B),$$

so $\lim_{j\to\infty}\alpha(\overline{f^{jm}(B)}) = 0$ and hence $\lim_{j\to\infty}\alpha(\overline{f^j(B)}) = 0$.

By Kuratowski's theorem, $\Gamma := \bigcap_{j\geq 1}\overline{f^j(B)}$ is compact and nonempty; and for any open neighborhood V of Γ, there is an integer $j(V)$ such that $\overline{f^j(B)} \subset V$ for all $j \geq j(V)$. Because $f(\overline{f^j(B)}) \subset \overline{f^{j+1}(B)}$, we see that $f(\Gamma) \subset \bigcap_{j\geq 1}\overline{f^{j+1}(B)} = \Gamma$. If $f(\Gamma) \neq \Gamma$, there is an open neighborhood W of $f(\Gamma)$ such that $\Gamma \not\subset W$ and an open neighborhood $V = f^{-1}(W)$ of Γ such that $f(V) \subset W$. We use here that $f(\Gamma)$ is compact. By the remarks above, there is an integer $j(V)$ with $\overline{f^j(B)} \subset V$ for $j \geq j(V)$. If $j \geq j(V)$, we see that $\Gamma \subset \overline{f^{j+1}(B)} \subset f(V) \subset W$, a contradiction.

If A is a compact subset of U, then by our assumptions there is an integer n_1 with $f^{n_1}(A) \subset B$. If V is an open neighborhood of Γ, we have proved that there exists n_2 such that $\overline{f^j(B)} \subset V$ for all $j \geq n_2$. It follows that $f^j(A) \subset V$ for all $j \geq n_1 + n_2$, so Γ is a compact attractor for f. ■

We also need to recall the generalized Lefschetz number introduced by Leray [33]. If V is a vector space over the rationals $\mathbb{Q}$ and $A : V \to V$ is a linear map, define

$$N := \{v \in V \mid A^k v = 0 \text{ for some } k \geq 1\}.$$

It may happen that the quotient space V/N is finite dimensional. If V/N is finite dimensional and if $\overline{A} : V/N \to V/N$ denotes the linear map induced by A, Leray defines $\mathrm{tr}_{\mathrm{gen}}(A) := \mathrm{tr}(\overline{A})$, where $\mathrm{tr}_{\mathrm{gen}}(A)$ is the generalized trace of A and $\mathrm{tr}(\overline{A})$ is the trace of $\overline{A}$. If V is finite dimensional, it is not hard to show that $\mathrm{tr}_{\mathrm{gen}}(A) = \mathrm{tr}(A)$. If V and W are vector spaces over $\mathbb{Q}$ and $A : V \to W$ and $B : W \to V$ are linear maps, Leray [33] has proved that $\mathrm{tr}_{\mathrm{gen}}(AB)$ is defined if and only if $\mathrm{tr}_{\mathrm{gen}}(BA)$ is defined and if both are defined, $\mathrm{tr}_{\mathrm{gen}}(AB) = \mathrm{tr}_{\mathrm{gen}}(BA)$. If Z is a topological space and $f : Z \to Z$ is a continuous map, $H_i(Z)$ (singular homology over $\mathbb{Q}$) is a vector space over $\mathbb{Q}$ and $f_{*,i} : H_i(Z) \to H_i(Z)$ is a linear map. If $\mathrm{tr}_{\mathrm{gen}}(f_{*,i})$ is defined for all $i \geq 0$ and equals 0 except for finitely many i, Leray defines the generalized Lefschetz number $L_{\mathrm{gen}}(f)$ of f by

$$L_{\mathrm{gen}}(f) = \sum_{i\geq 0}(-1)^i \operatorname*{tr}_{\mathrm{gen}}(f_{*,i}).$$

If Z_1 and Z_2 are topological spaces and $f : Z_1 \to Z_2$ and $g : Z_2 \to Z_1$ are continuous maps, $L_{\text{gen}}(gf)$ is defined if and only if $L_{\text{gen}}(fg)$ is defined and we have

$$L_{\text{gen}}(gf) = L_{\text{gen}}(fg)$$

We refer to [8], [17], [33] and [38], pages 248-250, for further information about the generalized Lefschetz number.

If Z is a Hausdorff topological space and $f : Z \to Z$ is a continuous map, suppose that there exists a subspace Y of Z such that

(1) $f(Y) \subset Y$; and

(2) for every compact set $A \subset Z$, there exists an integer $n = n(A)$ with $f^n(A) \subset Y$.

If f_Y denotes $f : Y \to Y$, then G. Fournier [17] has proved that $L_{\text{gen}}(f)$ is defined if and only if $L_{\text{gen}}(f_Y)$ is defined and $L_{\text{gen}}(f) = L_{\text{gen}}(f_Y)$. We shall use Fournier's result later.

We now present a refinement of Proposition 2.4 in [42]. Theorem 4.5 below, in combination with remarks in Examples 1-7 in Section 3, will yield a variety of fixed point theorems.

Recall that $X \in \mathcal{F}$ means that X is a complete subset of a normed linear space $(Y, \|\cdot\|)$ and X is a locally finite union of closed, convex sets. As usual, α will denote the Kuratowski MNC on Y.

Theorem 4.5. *(Compare Proposition 2.4 in [42].) Suppose that $X \in \mathcal{F}$, that U is an open subset of X and that $f : U \to U$ is a continuous map which has a compact, nonempty attractor $\Gamma \subset U$. Assume that there exists a bounded open neighborhood V of Γ in X with $\overline{V} \subset U$ and a decreasing sequence of sets $K_n \in \mathcal{F}_0$ with $K_n \subset X$ such that*

(1) $V \subset K_1$;

(2) $f(V \cap K_n) \subset K_{n+1}$ for all $n \geq 1$; and

(3) $\lim_{n\to\infty} \alpha(K_n) = 0$, where α denotes the Kuratowski MNC on X.

Then, for every integer $p \geq 1$, the map f^p belongs to the fixed point index class, $L_{\text{gen}}(f^p)$, the generalized Lefschetz number of f^p, is defined and

$$i_X(f^p, U) = L_{\text{gen}}(f^p) \tag{4.1}$$

If p is a prime, we have that

$$L_{\text{gen}}(f^p) \equiv L_{\text{gen}}(f) \mod p \tag{4.2}$$

If $L_{\text{gen}}(f) \neq 0$, then f has a fixed point in U. If there exists a strictly increasing sequence of prime numbers p_j, for $j \geq 1$, such that $L_{\text{gen}}(f^{p_j}) \neq 0$ for all $j \geq 1$ and $\{L_{\text{gen}}(f^{p_j}) \mid j \geq 1\}$ is bounded, then f has a fixed point in U and $L_{\text{gen}}(f) = L_{\text{gen}}(f^{p_j})$ for all large j.

Proof. The same argument used in the first paragraph of the proof of Proposition 2.4 in [42] shows that there exists an open neighborhood W of Γ in X, with $\overline{W} \subset V$, such that $\overline{f(W)} \subset W$. It follows that $\overline{f(W \cap K_n)} \subset W \cap K_{n+1}$ for all $n \geq 1$, so $\overline{f^p(W \cap K_n)} \subset W \cap K_{n+p} \subset W \cap K_{n+1}$ for all integers $p \geq 1$, so f^p belongs to the fixed point index class.

The proof of Proposition 2.4 in [42] shows that there exists a decreasing sequence of sets $A_n \in \mathcal{F}_0$, with $A_n \subset X$, and an integer $m \geq 1$, such that

(4) $W \subset A_1$;

(5) $f(W \cap A_n) \subset A_{n+1}$ for all $n \geq 1$;

(6) $\lim_{n\to\infty} \alpha(A_n) = 0$, where α denotes the Kuratowski MNC on X; and

(7) $A_n \subset W$ for all $n \geq m+1$.

Because $\overline{f(W)} \subset W$, we see as before that $f^p(W \cap A_n) \subset A_{n+p} \subset A_{n+1}$ for all integers $p \geq 1$. By our definition of the generalized fixed point index, we have that

$$i_X(f^p, U) = \lim_{n\to\infty} i_{A_n}(f^p, U \cap A_n).$$

If $n \geq m+1$, then $U \cap A_n = A_n$, so

$$i_{A_n}(f^p, U \cap A_n) = i_{A_n}(f^p, A_n). \tag{4.3}$$

As was proved in Section 2, the approximate fixed point index $i_{A_n}(f^p, A_n)$ is defined for $n \geq m+1$, the Lefschetz number $L(f^p : A_n \to A_n)$ of $f^p : A_n \to A_n$ is defined for $n \geq m+1$, and

$$i_{A_n}(f^p, A_n) = L(f^p : A_n \to A_n). \tag{4.4}$$

Equations (4.3) and (4.4) imply that

$$i_X(f^p, U) = \lim_{n\to\infty} L(f^p : A_n \to A_n). \tag{4.5}$$

We now apply Fournier's theorem. If K is a compact subset of U, then because Γ is a compact attractor for f (and hence for f^p for all integers $p \geq 1$), there exists an integer q so $g^q(K) \subset W$, where $g := f^p$. Since $g(W) \subset W$, Fournier's theorem implies that $L_{\text{gen}}(g)$ is defined if and only if $L_{\text{gen}}(g : W \to W)$ is defined and $L_{\text{gen}}(g) = L_{\text{gen}}(g : W \to W)$. Since $g(W) \subset W \cap A_2$ and $g(W \cap A_2) \subset W \cap A_2$, a second application of Fournier's theorem shows that $L_{\text{gen}}(g)$ is defined if and only if $L_{\text{gen}}(g : W \cap A_2 \to W \cap A_2)$ is defined and $L_{\text{gen}}(g) = L_{\text{gen}}(g : W \cap A_2 \to W \cap A_2)$.

Repeating this argument and using the fact that $g(W \cap A_n) \subset W \cap A_{n+1}$ and $g(W \cap A_{n+1}) \subset W \cap A_{n+1}$, we see that for all $n \geq 1$, the quantity $L_{\text{gen}}(g)$ is defined if and only if $L_{\text{gen}}(g : W \cap A_n \to W \cap A_n)$ is defined and that $L_{\text{gen}}(g) = L_{\text{gen}}(g : W \cap A_n \to W \cap A_n)$. However, if $n \geq m+1$, then $W \cap A_n = A_n$, and we already know that $L_{\text{gen}}(g : A_n \to A_n)$ is defined and equals $L(g : A_n \to A_n)$ for $n \geq m+1$. It follows that for $n \geq m+1$

$$L_{\text{gen}}(f^p) = L(f^p : A_n \to A_n). \tag{4.6}$$

Combining equation (4.5) and equation (4.6) we obtain equation (4.1).

If p is a prime, to prove equation (4.2) it suffices by virtue of equation (4.6), to prove that if $A \in \mathcal{F}_0$ with $\alpha(A) < \infty$ and $h : A \to A$ is a continuous map, then

$$L(h^p) \equiv L(h) \mod p.$$

By Corollary 2.5 in Section 2, there exists a compact, continuous map $\gamma : A \to A$ such that γ is homotopic to h. There exists a compact set $B \subset A$, with $B \in \mathcal{F}_0$ and $\gamma(A) \subset B$; namely, if $A = \bigcup_{j=1}^{\nu} C_j$, where C_j is closed and convex, define B, a finite union of compact convex sets by

$$B = \bigcup_{j=1}^{\nu} \overline{\text{co}}(\gamma(A) \cap C_j).$$

By the invariance of the Lefschetz number under homotopy, $L(h^p) = L(\gamma^p)$ and $L(h) = L(\gamma)$. Since $\gamma(A) \subset B$, we have $L(\gamma : A \to A) = L(\gamma : B \to B)$ and $L(\gamma^p : A \to A) = L(\gamma^p : B \to B)$. Thus it suffices to prove that

$$L(\gamma^p : B \to B) \equiv L(\gamma : B \to B) \mod p. \tag{4.7}$$

However, equation (4.7) is a special case of the so-called mod p theorem for the Lefschetz number (see [47]), which is, in turn, considerably easier to prove than the general mod p theorem for the Brouwer degree and the fixed point index (see [31], [51] and [52]).

We have proved that f^p lies in the fixed point index class for every integer $p \geq 1$ and that equations (4.1) and (4.2) hold. Thus, if $L_{\text{gen}}(f^p) \neq 0$, then $i_X(f^p, U) \neq 0$ and f^p has a fixed point. In particular, if $L_{\text{gen}}(f) \neq 0$, then $i_X(f, U) \neq 0$ and f has a fixed point in U.

To complete the proof, suppose that there exists a sequence of primes $p_j \to \infty$ such that $L_{\text{gen}}(f^{p_j}) \neq 0$ and $\{L_{\text{gen}}(f^{p_j}) \mid j \geq 1\}$ is bounded. Define M by

$$M = \sup\{|L_{\text{gen}}(f^{p_j})| \mid j \geq 1\} + |L_{\text{gen}}(f)|.$$

If $p_j > M$, we have that $|L_{\text{gen}}(f) - L_{\text{gen}}(f^{p_j})| = \nu_j p_j$, where ν_j is an integer. Since $|L_{\text{gen}}(f) - L_{\text{gen}}(f^{p_j})| \leq M$ and $p_j > M$, we must have that $\nu_j = 0$. ■

Remark 4.6. Theorem 4.5 shows that if there exists a sequence of prime numbers $p_j \to \infty$ such that $L_{\text{gen}}(f^{p_j}) \neq 0$ for all j and $\{L_{\text{gen}}(f^{p_j}) \mid j \geq 1\}$ is bounded, then $L_{\text{gen}}(f) \neq 0$ and f has a fixed point. However, simple finite dimensional examples show that often no such sequence of prime numbers exists, even though $L_{\text{gen}}(f) \neq 0$. To see this, consider S^1, the unit circle in $\mathbb{C} = \mathbb{R}^2$. For d an integer, define $f : S^1 \to S^1$ by $z \to z^d$. One has $L(f) = 1 - d$, so $L(f) \neq 0$ for $d \neq 1$. However, if $p_j \to \infty$ is a sequence of primes, $L(f^{p_j}) = 1 - d^{p_j}$, so $\{L(f^{p_j}) \mid j \geq 1\}$ is bounded if and only if $d = \pm 1$. Of course, $S^1 \notin \mathcal{F}_0$, but S^1 is homeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| = 1\} \in \mathcal{F}_0$. Alternately, there is an open neighborhood U of S^1 and a retraction r of U onto S^1 and one can work with the composition $g = fr : U \to S^1$, since $L(g^p) = L(f^p)$.

If Theorem 4.5 is applied to each of Examples 1-7 in Section 3, we obtain corresponding fixed point theorems.

Corollary 4.7. *(Compare Example 1 in Section 3.) Suppose $X \in \mathcal{F}$, with $f : U \to U$ a continuous map and U an open subset of X, and assume $\Gamma \subset U$ is a compact, nonempty set which is a compact attractor for f. Assume that there exists an open neighborhood G of Γ such that $f|G$ is a k-set-contraction with $k < 1$. Then for every integer $p \geq 1$, the map f^p belongs to the fixed point index class, $L_{\text{gen}}(f^p)$ is defined and $i_X(f^p, U) = L_{\text{gen}}(f^p)$. If $L_{\text{gen}}(f) \neq 0$, then f has a fixed point in U. If there exists a sequence of primes $p_j \to \infty$ such that $L_{\text{gen}}(f^{p_j}) \neq 0$ and $\{L_{\text{gen}}(f^{p_j}) \mid j \geq 1\}$ is bounded, then $L_{\text{gen}}(f) \neq 0$ and f has a fixed point.*

Proof. As noted in Example 1 in Section 3, there exists a sequence $K_n \in \mathcal{F}_0$, for $n \geq 1$, satisfying the conditions of Theorem 4.5, so Corollary 4.7 is an immediate consequence of Theorem 4.5. ■

Remark 4.8. It often happens that $L_{\rm gen}(f)$ can be directly computed. Suppose, for example, that U is a topological space, $f : U \to U$ is continuous and there exists an integer N such that $f^N(U) \subset Z \subset U$, where Z is contractible in itself to a point. If p is a prime number and $p \geq N$, it follows that $f^p(U) \subset Z$, so $L_{\rm gen}(f^p) = L_{\rm gen}(f^p : Z \to Z) = 1$. If we are in the framework of Theorem 4.5, it follows that $L_{\rm gen}(f) = 1$. In particular, if there exists an integer $N \geq 0$ such that $Z := f^N(U)$ is contractible in itself to a point, we find that $L_{\rm gen}(f) = 1$.

Corollary 4.9. *(See Example 2 in Section 3.) Let U be an open subset of a Banach space X and let $f : U \to U$ be a continuous map. Assume that $\Gamma \subset U$ is a compact, nonempty set which is a compact attractor for f and suppose that there is an open neighborhood V of Γ and an integer $N \geq 1$ such that $f|V$ is C^1 and $f^N|V$ is a c-set-contraction with $c < 1$. Then for every integer $p \geq 1$, the map f^p belongs to the fixed point index class, $L_{\rm gen}(f^p)$ is defined and $i_X(f^p, U) = L_{\rm gen}(f^p)$. If $L_{\rm gen}(f) \neq 0$, then f has a fixed point in U. If there exists a sequence of prime numbers $p_j \to \infty$ such that $L_{\rm gen}(f^{p_j}) \neq 0$ and $\{L_{\rm gen}(f^{p_j}) \mid j \geq 1\}$ is bounded, then $L_{\rm gen}(f) \neq 0$ and $L_{\rm gen}(f) = L_{\rm gen}(f^{p_j})$ for all large j. If there exists an integer m and a set $Z \subset U$ such that $f^m(U) \subset Z$ and Z is contractible in itself to a point, then $L_{\rm gen}(f) = 1$.*

Proof. The discussion in Example 2 in Section 3 shows that the hypotheses of Theorem 4.5 are satisfied. The final statement of Corollary 4.9 follows from Remark 4.8. ■

The following simple variant of Corollary 4.9 follows immediately from Corollary 4.9 and Proposition 4.4.

Corollary 4.10. *Let U be an open subset of a Banach space X and $f : U \to U$ a continuous map. Assume that there exists an integer $m \geq 1$ such that $B := \overline{f^m(U)}$ is a bounded subset of U and $\lim_{j\to\infty} \alpha(f^j(B)) = 0$, so it follows from Proposition 4.4 that $\Gamma := \bigcap_{j\geq 1} \overline{f^j(B)}$ is a compact, nonempty attractor for f. Assume that there exists an open neighborhood V of Γ and an integer $N \geq 1$ such that $f|V$ is C^1 and $f^N|V$ is a k-set-contraction with $k < 1$. Then for every $p \geq 1$, the map f^p belongs to the fixed point index class, $L_{\rm gen}(f^p)$ is defined and $i_X(f^p, U) = L_{\rm gen}(f^p)$. If $L_{\rm gen}(f) \neq 0$, then f has a fixed point in U. If there exists a sequence of prime numbers $p_j \to \infty$ such that $L_{\rm gen}(f^{p_j}) \neq 0$*

and $\{L_{\text{gen}}(f^{p_j}) \mid j \geq 1\}$ is bounded, then $L_{\text{gen}}(f) \neq 0$ and $L_{\text{gen}}(f) = L_{\text{gen}}(f^{p_j})$ for all large j. If there exists a set $Z \subset U$ which is contractible in itself to a point and if $f^j(U) \subset Z$ for some $j \geq 1$, then $L_{\text{gen}}(f) = 1$.

Remark 4.11. The reader should note that most of the assertions in Corollaries 4.9 and 4.10 above follow easily from Proposition 2.4, Lemma 3.1 and Proposition 3.3, all in [42].

Remark 4.12. In Theorem 2 of [55], Tromba assumes that U is an open subset of a Banach space X, that $f : U \to U$ is a C^1 map and that there exists an integer $n \geq 1$ with $\overline{f^n(U)}$ a compact subset of U. If $\{L_{\text{gen}}(f^j) \mid j \geq n\}$ is bounded and if there exists $\nu \geq 1$ with $L_{\text{gen}}(f^j) \neq 0$ for all $j \geq \nu$, Tromba's theorem asserts that f has a fixed point. If we take $B = \overline{f^n(U)}$, this result is a very special case of Corollary 4.10. Tromba also claims that if $L_{\text{gen}}(f^m)$ is even and nonzero for all sufficiently large m, then f has a fixed point. We shall prove in Section 6 that this assertion is false even when X is two dimensional.

Tromba's argument in [55] requires the use of perturbations. Starting with a map f as in Tromba's theorem, one obtains a map g which is close to f. However, although there exists an open neighborhood V of $\Gamma := \bigcap_{j\geq 1} \overline{f^j(B)}$ such that $f^n|V$ is a compact map, it need not be true that $g^n|V$ is compact; and this point leads to technical complications in [55]. Nevertheless, one can always arrange that $g^n|V$ is a k-set-contraction, where k can be chosen independent of g and $k < 1$. This would seem an argument for working in the generality of Corollary 4.9 or 4.10, even when $\overline{f^n(U)}$ is compact and $\overline{f^n(U)} \subset U$.

Corollary 4.13. *(See Example 4 in Section 3.) Let M be a C^1 Banach manifold which can be nicely embedded in a Banach space Z and let j, r and H be as in Definition 3.2. Assume that M is a subset of a Banach space Y and that the inclusion of M into Y is C^1. Let U be an open subset of M and $f : U \to U$ a continuous map which has a compact, nonempty attractor $\Gamma \subset U$. Assume that there exists an open neighborhood V of Γ in M and an integer N such that $f|V$ is C^1 and $f^N|V$ is a c-set-contraction with $c < 1$ (with respect to the Kuratowski MNC on Y). Define $g = jfr : r^{-1}(U) \to r^{-1}(U)$. Then for every integer $p \geq 1$, the map g^p belongs to the fixed point index class, $L_{\text{gen}}(g^p)$ is defined and $i_Z(g^p, r^{-1}(U)) = L_{\text{gen}}(g^p)$. Furthermore, we have that $L_{\text{gen}}(f^p) = L_{\text{gen}}(g^p)$. If $L_{\text{gen}}(f) \neq 0$, then f has a fixed point in U and g has a fixed point in $r^{-1}(U)$. If there exists a sequence of prime numbers $p_j \to \infty$ such that $L_{\text{gen}}(f^{p_j}) \neq 0$ and $\{L_{\text{gen}}(f^{p_j}) \mid j \geq 1\}$ is bounded, then $L_{\text{gen}}(f) \neq 0$ and $L_{\text{gen}}(f) = L_{\text{gen}}(f^{p_j})$ for all large j.*

Proof. Notice that $g^p = jf^pr$, so it is easy to see that $j(\Gamma)$ is a nonempty compact attractor for $g : r^{-1}(U) \to r^{-1}(U)$. Arguing as in Example 4 of Section 3, we see that there is an open neighborhood V_1 of $j(\Gamma)$ in H and an integer $N_1 \geq 1$ such that $g|V_1$ is C^1 and $g^{N_1}|V_1$ is a k-set-contraction for some $k < 1$. We are now in the framework of Corollary 4.9. Because $g^p = jf^pr$, the properties of the generalized Lefschetz number imply that

$$L_{\text{gen}}(g^p) = L_{\text{gen}}(jf^pr) = L_{\text{gen}}(rjf^p) = L_{\text{gen}}(f^p). \tag{4.8}$$

Using equation (4.8), Corollary 4.13 follows from Corollary 4.9. ■

Remark 4.14. If M and U are as in Corollary 4.13 and $f : U \to U$ is a continuous map, assume that there is an integer m such that $\overline{f^m(U)} := B$ is a closed, bounded subset of U and $\lim_{j\to\infty} \alpha(f^j(B)) = 0$. It then follows from Proposition 4.4 that $\Gamma := \bigcap_{j\geq 1} \overline{f^j(B)}$ is a nonempty, compact set which is a compact attractor for f. If there exists an open neighborhood V of Γ and an integer N such that $f|V$ is C^1 and $f^N|V$ is a c-set-contraction with $c < 1$, we are in the framework of Corollary 4.13. In particular, $L_{\text{gen}}(f)$ is defined; and if $L_{\text{gen}}(f) \neq 0$, then f has a fixed point in U. If there exists a sequence of primes $p_j \to \infty$ such that $L_{\text{gen}}(f^{p_j}) \neq 0$ for all j and $\{L_{\text{gen}}(f^{p_j}) \mid j \geq 1\}$ is bounded, then $L_{\text{gen}}(f) \neq 0$ and f has a fixed point.

Remark 4.15. Tromba [55] considers a special case of Corollary 4.13 and of Remark 4.14 in Theorem 3 of [55]. Suppose that $M = U$ is a C^1 Banach manifold as in Corollary 4.13, that $f : M \to M$ is a C^1 map and that there exists an integer $n \geq 1$ such that $\overline{f^n(M)}$ is compact. Assume that there exists an integer $\nu \geq 1$ such that $L_{\text{gen}}(f^j) \neq 0$ for all $j \geq \nu$ and $\{L_{\text{gen}}(f^j) \mid j \geq \nu\}$ is bounded. Then Theorem 3 of [55] asserts that f has a fixed point in M. Theorem 3 in [55] also makes the incorrect assertion that if $L_{\text{gen}}(f^j)$ is nonzero and even for all large j, then f has a fixed point.

We next consider a generalization of Proposition 3.7 in [42]. The proof will follow easily from Theorem 4.5 and the discussion in Example 5 in Section 3.

Corollary 4.16. *(Compare Proposition 3.7 in [42].) Let notation and assumptions be as in the first two sentences of Example 5 in Section 3. Let U be an open subset of M and $f : U \to U$ a continuous map and assume that there exists a compact, nonempty set $\Gamma \subset U$ which is a compact attractor for f. Assume also that there exists an open neighborhood V of Γ in M such that $f|V$ is* *a k-set-contraction with $k < 1$ (with respect to the Kuratowski MNC on Y). Then $j(\Gamma)$ is a compact attractor for $g : r^{-1}(U) \to r^{-1}(U)$, where $g = jfr$. For every integer $p \geq 1$, the map g^p belongs to the fixed point index class, $L_{\text{gen}}(g^p)$ is defined, $L_{\text{gen}}(f^p)$ is defined and equation (4.2) is satisfied. If $L_{\text{gen}}(f) \neq 0$, then f has a fixed point in U. If there exists a sequence of primes $p_j \to \infty$ such that $L_{\text{gen}}(f^{p_j}) \neq 0$ for all j and $\{L_{\text{gen}}(f^{p_j}) \mid j \geq 1\}$ is bounded, then $L_{\text{gen}}(f) \neq 0$ and $L_{\text{gen}}(f) = L_{\text{gen}}(f^{p_j})$ for all large j.*

Proof. Clearly we have $g^p = jf^pr$. Using this fact, the reader can verify that $j(\Gamma)$ is a compact attractor for g. By the discussion in Example 5 in Section 3 we see that g satisfies the hypotheses of Theorem 4.5. It follows that g^p belongs to the fixed point index class for all $p \geq 1$, that $L_{\text{gen}}(g^p)$ is defined for all $p \geq 1$ and $i_Z(g^p, r^{-1}(U)) = L_{\text{gen}}(g^p)$. We also have that

$$L_{\text{gen}}(g^p) = L_{\text{gen}}(jf^pr) = L_{\text{gen}}(rjf^p) = L_{\text{gen}}(f^p),$$

so all the claims of Corollary 4.16 follow directly from Theorem 4.5. ■

If we use Example 3 in Section 3, we obtain a variant of Corollary 4.9.

Corollary 4.17. *(Compare Example 3 in Section 3.) Suppose that X is a closed subset of a Banach space Y and $X = \bigcup_{j=1}^{n} C_j$, where C_j is closed and convex, for $1 \leq j \leq n$. Let $U \subset X$ be open in the relative topology on X (possibly $U = X$) and let $f : U \to U$ be a continuous map. Assume that there exists a compact, nonempty set $\Gamma \subset U$ which is a compact attractor for f. Assume that there exists an open neighborhood $\widehat{V}$ of Γ in Y and an extension $\widehat{f} : \widehat{V} \to Y$ of $f|\widehat{V} \cap X$ such that $\widehat{f}$ is C^1 on $\widehat{V}$ and for some integer N, the map $\widehat{f}^N$ is defined on an open neighborhood $\widehat{V}_1$ of Γ in Y and $\widehat{f}^N|\widehat{V}_1$ is a c-set-contraction with $c < 1$. Then all the conclusions of Theorem 4.5 are satisfied. In particular, for each $p \geq 1$, the map f^p belongs to the fixed point index class, $L_{\text{gen}}(f^p)$ is defined and $i_X(f^p, U) = L_{\text{gen}}(f^p)$. If $L_{\text{gen}}(f) \neq 0$, then f has a fixed point. If p is a prime, then $L_{\text{gen}}(f^p) \equiv L_{\text{gen}}(f) \pmod{p}$.*

Proof. By using the results of Example 3 in Section 3, we obtain an open neighborhood V of Γ in X and a sequence of sets $K_n \in \mathcal{F}_0$, with $K_n \subset X$, which satisfy the conditions of Theorem 4.5. ■

If one does not demand that f belong to the fixed point index class, a more general form of Theorem 4.5 is available. The following theorem should be compared to Theorem 1 in [41], where it is assumed that X is a closed, convex set and $U = X$f.

Theorem 4.18. *(Compare Theorem 1 in [41].) Suppose that $X \in \mathcal{F}$, with U an open subset of X and $f : U \to U$ a continuous map. Assume that there exists a sequence K_j, for $j \geq 1$, of subsets of X such that*

(1) $K_j \subset U$ and $K_j \in \mathcal{F}_0$ for all $j \geq 1$;

(2) $f(K_j) \subset K_j$ for all $j \geq 1$;

(3) for every compact set $A \subset U$ and every $j \geq 1$ there exists an integer $\nu = \nu(A, j)$ with $f^\nu(A) \subset K_j$; and

(4) $\lim_{j\to\infty} \alpha(K_j) = 0$, where α denotes the Kuratowski MNC on X.

Then $\Gamma := \bigcap_{j\geq 1} K_j$ is compact and nonempty and Γ is a compact attractor for f. For every integer $p \geq 1$, the quantity $L_{\text{gen}}(f^p)$ is defined and $L_{\text{gen}}(f^p) = L_{\text{gen}}(f^p : K_j \to K_j)$ for every $j \geq 1$. If $L_{\text{gen}}(f) \neq 0$, then f has a fixed point. If p is a prime number, then

$$L_{\text{gen}}(f^p) \equiv L_{\text{gen}}(f) \mod p. \tag{4.9}$$

If there exists a sequence of prime numbers $p_j \to \infty$ such that $L_{\text{gen}}(f^{p_j}) \neq 0$ for all j and such that $\{L_{\text{gen}}(f^{p_j}) \mid j \geq 1\}$ is bounded, then $L_{\text{gen}}(f) \neq 0$ and $L_{\text{gen}}(f) = L_{\text{gen}}(f^{p_j})$ for all large j. If there is an open neighborhood W of Γ, with $W \subset U$, such that $f(W) \subset W$, and a set $Z \subset W$ such that Z is contractible in itself to a point and $f^j(W) \subset Z$ for some j, then $L_{\text{gen}}(f) = 1$.

Proof. We first prove that $\Gamma := \bigcap_{j\geq 1} K_j \subset U$ is a compact, nonempty attractor for f. Here we can replace assumption (1) in the theorem by the weaker assumption that K_j is closed and $K_j \subset U$ for all $j \geq 1$. If $A \subset U$ is a compact set, there is an integer $m_j \geq 1$ with $f^{m_j}(A) \subset K_j$; and since $f(K_j) \subset K_j$, we have that $f^\nu(A) \subset K_j$ for all $\nu \geq m_j$. We can assume that $m_j \leq m_{j+1}$ for all $j \geq 1$, and we define

$$B_j = \overline{\bigcup_{\nu \geq m_j} f^\nu(A)} \subset K_j.$$

It follows that B_j, for $j \geq 1$, is a decreasing sequence of closed, bounded nonempty sets in K_1. Since $\alpha(B_j) \leq \alpha(K_j)$, we have $\lim_{j\to\infty} \alpha(B_j) = 0$. It follows from Kuratowski's theorem that $B_\infty := \bigcap_{j\geq 1} B_j$ is a compact, nonempty set and, by our construction, $B_\infty \subset \bigcap_{j\geq 1} K_j := \Gamma$. Since $\alpha(\Gamma) \leq \alpha(K_j)$ for all $j \geq 1$, we have $\alpha(\Gamma) = 0$, so Γ is compact and Γ is nonempty because B_∞ is nonempty. Because $f(K_j) \subset K_j$ for all j, we have $f(\Gamma) \subset \Gamma$. Kuratowski's theorem implies that if $W \subset U$ is any open neighborhood of Γ, then $B_j \subset W$ for all large j. It follows that if W_1 is any open neighborhood of Γ, with $W_1 \subset U$, then $f^j(A) \subset W_1$ for all large j, so Γ is a compact attractor for f.

For the remainder of the proof we assume that $K_j \subset U$ and $K_j \in \mathcal{F}_0$ for $j \geq 1$. Since $f^p : K_j \to K_j$ for $p \geq 1$, the argument in the proof of Theorem 4.5 shows that $L_{\text{gen}}(f^p : K_j \to K_j)$ and, if p is a prime,

$$L_{\text{gen}}(f^p : K_j \to K_j) \equiv L_{\text{gen}}(f : K_j \to K_j) \mod p.$$

If we use assumption (3) in Theorem 4.18 and Fournier's theorem [17], we see that for every $p \geq 1$ and $j \geq 1$, we have $L_{\text{gen}}(f^p) = L_{\text{gen}}(f^p : K_j \to K_j)$. These remarks also imply equation (4.9).

For $k \geq 1$ we define $C_k := \bigcap_{j=1}^k K_j$, so $C_k \supset \Gamma$ for all $k \geq 1$. Then C_k, for $k \geq 1$, is a decreasing sequence of closed, bounded subsets of U; and because $\alpha(C_k) \leq \alpha(K_k)$, we have $\lim_{k\to\infty} \alpha(C_k) = 0$. Thus Kuratowski's theorem implies that if x_k, for $k \geq 1$, is a sequence of points with $x_k \in C_k$, there is a convergent subsequence $x_{k_i} \to x \in \Gamma$. The reader can verify that $C_k \in \mathcal{F}_0$ and $f(C_k) \subset C_k$ for all $k \geq 1$. Also, if A is a compact subset of U and $k \geq 1$, there exists an integer $n = n(A, k)$ so $f^n(A) \subset C_k$. It follows that our arguments above apply and for all $k \geq 1$,

$$L_{\text{gen}}(f) = L_{\text{gen}}(f : C_k \to C_k).$$

Using results of Section 2, we see that the approximate fixed point index $i_{C_k}(f, C_k)$ is defined for all $k \geq 1$ and

$$i_{C_k}(f, C_k) = L_{\text{gen}}(f : C_k \to C_k) = L_{\text{gen}}(f).$$

If $L_{\text{gen}}(f) \neq 0$ and if $\delta_k > \alpha(C_k)$ and $\lim_{k\to\infty} \delta_k = 0$, we also obtain from Section 2 that there is a point $x_k \in C_k$ with $\|f(x_k) - x_k\| < \delta_k$. By taking a subsequence, we can assume that $x_k \to x \in \Gamma$ and so $f(x) = x$ by the continuity of f.

If p_j is a sequence of prime numbers as in Theorem 4.18, the conclusions in Theorem 4.18 follow by the same argument as in Theorem 4.5.

If W is an open neighborhood of Γ and $f(W) \subset W$, Fournier's theorem [17] implies that $L_{\text{gen}}(f : U \to U) = L_{\text{gen}}(f : W \to W)$. Since $C_k \subset W$ for all large k, the same argument used for $f : U \to U$ shows that for all $p \geq 1$ and for k large,

$$L_{\text{gen}}(f^p : W \to W) = L_{\text{gen}}(f^p : C_k \to C_k).$$

Since $C_k \in \mathcal{F}_0$, we know that for any prime p

$$L_{\text{gen}}(f^p : C_k \to C_k) \equiv L_{\text{gen}}(f : C_k \to C_k) \mod p,$$

and we conclude that for any prime p,

$$L_{\text{gen}}(f^p : W \to W) \equiv L_{\text{gen}}(f : W \to W) \mod p.$$

If $f^j(W) \subset Z$ for all $j \geq N$, we see that for any prime $p \geq N$,

$$L_{\text{gen}}(f^p : W \to W) = L_{\text{gen}}(f^p : Z \to Z) = 1,$$

and the latter equation implies, as in Remark 4.8, that $L_{\text{gen}}(f) = 1$. ■

With the aid of Theorem 4.18, we can prove a theorem of "Frum-Ketkov type". We refer to [41], pages 349-350, for a discussion of the background behind such results.

Corollary 4.19. *(Compare Example 7 in Section 3.) Suppose that $X \in \mathcal{F}$, that U is an open subset of X and $f : U \to U$ is a continuous map. Assume that there exists a compact, nonempty set $\Gamma \subset U$ which is a compact attractor for f. If $d(x, \Gamma)$ denotes the distance of $x \in X$ to Γ and $N_r(\Gamma) = \{x \in X \,|\, d(x, \Gamma) \leq r\}$, assume that there exist sequences r_k and s_k, for $k \geq 1$, with $0 \leq s_k < r_k$ for all $k \geq 1$, with $\lim_{k\to\infty} r_k = 0$, and $f(N_{r_k}(\Gamma)) \subset N_{s_k}(\Gamma)$ and $N_{r_k}(\Gamma) \subset U$ for all $k \geq 1$. Then there exists a sequence K_j, for $j \geq 1$, of subspaces of X which meet conditions (1)-(4) of Theorem 4.18 and also satisfy $N_{s_j}(\Gamma) \subset K_j \subset N_{r_j}(\Gamma)$ for $j \geq 1$. It follows that f satisfies all the conclusions of Theorem 4.18. In particular, $L_{\text{gen}}(f)$ is defined and if $L_{\text{gen}}(f) \neq 0$, then f has a fixed point. If U is contractible in itself to a point or if U is convex, then $L_{\text{gen}}(f) = 1$ and f has a fixed point.*

Proof. It suffices to prove that there exist sets $K_j \in \mathcal{F}_0$ such that $N_{s_j}(\Gamma) \subset K_j \subset N_{r_j}(\Gamma)$ since then K_j will contain an open neighborhood of Γ, with $f(K_j) \subset K_j$, and one can easily verify that conditions (1)-(4) are satisfied. Define $\rho_j = \frac{s_j + r_j}{2}$ and $\varepsilon_j = \frac{r_j - s_j}{8}$ and let $z_1, \ldots, z_m$ be an ε_j-net for Γ, so $z_i \in \Gamma$ for $1 \leq i \leq m = m(j)$, and if $z \in \Gamma$ then $\|z - z_i\| < \varepsilon_j$ for some i. If $B_\rho(z)$ denotes a closed ball of radius ρ and center z, define K_j by $K_j = \bigcup_{i=1}^m B_{\rho_j}(z_i)$. Clearly, we have $K_j \in \mathcal{F}_0$ and $K_j \subset N_{r_j}(\Gamma)$. Since $N_{s_j}(\Gamma)$ is the closure of $\{x \in X \mid d(x, \Gamma) < s_j\}$, to prove that $N_{s_j}(\Gamma) \subset K_j$, it suffices to prove that if $d(x, \Gamma) < s_j$ then $x \in K_j$. However, if $d(x, \Gamma) < s_j$, there exists $z \in \Gamma$ with $\|x - z\| < s_j$ and there exists $z_i \in \Gamma$ with $\|z - z_i\| < \frac{r_j - s_j}{8}$, so $\|x - z_i\| < s_j + \frac{r_j - s_j}{8} < \rho_j$ and $x \in K_j$. ■

The condition that sets lie in $\mathcal{F}_0$, namely that they be finite unions of closed, convex sets, has played a crucial role in our arguments in this section. If $K \in \mathcal{F}_0$, with $f : K \to K$ a continuous map with $L(f) \neq 0$ and $\alpha(K) < \delta$, we know that there exists $x_* \in K$ with $\|f(x_*) - x_*\| < \delta$. If we assume, instead of $K \in \mathcal{F}_0$, that K is a complete metric ANR with $\alpha(K) < \delta$, we have no directly corresponding result. Nevertheless, some of our arguments can be generalized to more general classes of spaces than $\mathcal{F}_0$.

We close this section by returning to Conjecture 4.2 and applying the results of this section.

Corollary 4.20. *Let G be a closed, bounded convex set in a Banach space X and $f : G \to G$ a continuous map. Assume that there exists an integer $N \geq 1$ such that f^N is a k-set-contraction with $k < 1$. Then if $\Gamma := \bigcap_{j\geq 1} \overline{f^j(G)}, \Gamma$ is a compact, nonempty set which is a compact attractor for f and $f(\Gamma) = \Gamma$. Assume that f satisfies one of the following hypotheses:*

(1) there is an open neighborhood V of Γ in X and an extension $F : V \to X$ of $f|V \cap G \to G$ such that

(i) F is C^1 on V; and

(ii) for some integer $m \geq 1$ and some open neighborhood V_1 of Γ in X, the map F^m is defined on V_1 and $F^m|V_1$ is a c-set-contraction for some $c < 1$;

(2) there is an open neighborhood U of Γ in X such that $f|U \cap G$ is a c-set-contraction for some $c < 1$; or

(3) there is a compact set M with $\Gamma \subset M \subset G$ and sequences of reals s_n and r_n, for $n \geq 1$, such that

(i) $0 \leq s_n < r_n$ for all $n \geq 1$, and $\lim_{n\to\infty} r_n = 0$; and

(ii) $f(N_{r_n}(M)) \subset N_{s_n}(M)$ for all $n \geq 1$, where $N_r(M) := \{x \in G \,|\, d(x, M) \leq r\}$ and $d(x, M) = \inf\{\|x - y\| \mid y \in M\}$.

Then f has a fixed point in G.

Proof. Proposition 4.4 implies that Γ is nonempty, compact attractor for f and $f(\Gamma) = \Gamma$. Because G is contractible in itself to a point, $L(f) = 1$. If hypothesis (1) holds, Corollary 4.17 implies that f has a fixed point. If hypothesis (2) holds, Corollary 4.7 implies that f has a fixed point. If hypothesis (3) holds, Corollary 4.19 implies that f has a fixed point. ∎

Remark 4.21. We could replace hypothesis (2) in Corollary 4.20 by a weaker assumption:

($2'$) there exists an open neighborhood V of Γ in X such that if we define $G_1 = \overline{\text{co}}(f(V \cap G))$ and $G_n = \overline{\text{co}}(f(V \cap G_{n-1}))$ for $n > 1$, then $\lim_{n \to \infty} \alpha(G_n) = 0$.

See the paragraph immediately preceding Example 2 in Section 3.

Remark 4.22. If $\text{int}(G)$, the interior of G, is nonempty and $\Gamma \subset \text{int}(G)$, hypothesis (1) reduces to the assumption that f is C^1 on some open neighborhood of Γ. However, if $\text{int}(G)$ is empty or if Γ intersects ∂G, hypothesis (1) becomes more problematic. We conjecture that hypothesis (1) can be replaced by a weaker assumption:

($1'$) there exists an open neighborhood V of Γ in X and an extension $F : V \to X$ of $f|V \cap G$ such that F is C^1.

5 The mod p Theorem

Suppose that U is an open subset of a Hausdorff topological space X and that $f : U \to X$ is a continuous map. For a prime number p, let $V \subset U$ be an open set such that $f^j(V) \subset U$ for $1 \le j \le p$ and define $\Sigma = \{x \in V \mid f^p(x) = x\}$. Assume that Σ is compact (possibly empty) and $f(\Sigma) = \Sigma$.

Under the above assumptions, we could (assuming X is a metric space) take a bounded open neighborhood W of Σ with $\overline{W} \subset V$. We could then have that

(1) f^j is defined and continuous on $\overline{W}$ for $1 \le j \le p$;

(2) $f^p(x) \ne x$ for all $x \in \partial W := \overline{W} \backslash W$;

(3) $\Sigma := \{x \in \overline{W} \mid f^p(x) = x\}$ is compact; and

(4) $f(\Sigma) = \Sigma$;

and we could take this as our starting point.

Under additional assumptions, it may happen that generalized fixed point indices $i_X(f^p, V)$ and $i_X(f, V)$ can be defined. Under these circumstances, simple but unrigorous arguments suggest that it "should be" true that

$$i_X(f^p, V) \equiv i_X(f, V) \mod p. \tag{5.1}$$

If $X = \mathbb{R}^n$, equation (5.1) was proved independently by Krasnoselskii and Zabreiko [32] and Steinlein [51]; and an expository treatment of this case is given in Chapter 3 of [46]. Note that when $X = \mathbb{R}^n$ and I denotes the identity map, equation (5.1) is equivalent to a statement about Brouwer degree:

$$\deg(I - f^p, V, 0) \equiv \deg(I - f, V, 0) \mod p. \tag{5.2}$$

Steinlein [52] has generalized equation (5.2) and proved equation (5.1) when $X \in \mathcal{F}$ and $f : U \to X$ is a k-set-contraction with $k < 1$. Dold [12] has given some interesting extensions of equation (5.1) for Euclidean neighborhood retracts or ENR's. It seems very likely that the approximation methods of this section can be used to extend Dold's theorems to the context of maps which lie in the fixed point index class, but for simplicity we shall restrict ourselves here to the mod p theorem. For our purposes here it will suffice to know that equation (5.1) is valid when $X \in \mathcal{F}_0$ and f is a compact map, a case which can be obtained fairly easily from equation (5.2) or directly from Steinlein's theorem in [52]. Our goal is to prove the following theorem, which, in conjunction with Examples 1-7 in Section 3, covers a variety of interesting classes of maps.

Theorem 5.1. *Suppose that $X \in \mathcal{F}$, that U is an open subset of X and $f : U \to X$ is a continuous map. For a prime number p, let $V \subset U$ be an open set such that $f^j(V) \subset U$ for $1 \leq j \leq p$ and let $\Sigma := \{x \in V \mid f^p(x) = x\}$. Assume that Σ is compact (possibly empty) and that $f(\Sigma) = \Sigma$. Assume also that there exist a bounded open neighborhood W of Σ, with $\overline{W} \subset V$, and a decreasing sequence of sets $K_n \in \mathcal{F}_0$ with $K_n \subset X$ such that*

(1) $W \subset K_1$;

(2) $f(W \cap K_n) \subset K_{n+1}$ *for all* $n \geq 1$; *and*

(3) $\lim_{n\to\infty} \alpha(K_n) = 0$, *where α denotes the Kuratowski MNC on X.*

Then it follows that f and f^p belong to the fixed point index class and equation (5.1) *is satisfied.*

Remark 5.2. It frequently happens (see Examples 1-6 in Section 3) that there exist sets K_n as above which also satisfy the conditions of Theorem 3.1. In that case we know that $K_\infty := \bigcap_{n\geq 1} K_n \in \mathcal{F}_0$ and K_∞ is compact and, by virtue of Theorem 3.1,

$$i_X(f^p, V) = i_{K_\infty}(f^p, W_1 \cap K_\infty) \quad \text{and} \quad i_X(f, V) = i_{K_\infty}(f, W_1 \cap K_\infty).$$

Here W_1 is chosen to be an open neighborhood of Σ in X such that $\overline{f^j(W_1)} \subset W$ for $0 \le j \le p$. Since $\Sigma \subset W_1 \cap K_\infty$ and $f(\Sigma) = \Sigma$, the known case of the mod p theorem implies that

$$i_{K_\infty}(f^p, W_1 \cap K_\infty) \equiv i_{K_\infty}(f, W_1 \cap K_\infty) \mod p$$

and proves Theorem 5.1. However, Example 7 in Section 3 shows that we cannot always assume that the sets K_n satisfy the conditions of Theorem 3.1.

We now begin the proof of Theorem 5.1 in the general case. The proof will be divided into a series of technical steps.

Proof of Theorem 5.1.

Step 1: Let W_1 be a bounded open neighborhood of Σ such that $\overline{W}_1 \subset W$ and $\overline{f^j(W_1)} \subset W$ for $1 \le j \le p$. Such a neighborhood of Σ exists because $f(\Sigma) = \Sigma$ and f is continuous. It follows easily that $\overline{W}_1 \subset K_1$ and $\overline{f^j(W_1 \cap K_n)} \subset W \cap K_{n+j}$ for $n \ge 1$ and $1 \le j \le p$. Since $K_{n+j} \subset K_{n+1}$ for $j \ge 1$, we see that $\overline{f^j(W_1 \cap K_n)} \subset W \cap K_{n+1}$ for $1 \le j \le p$ and for $n \ge 1$. It follows that f and f^p belong to the fixed point index class and, by our definition of $i_X(f, V)$ and $i_X(f^p, V)$,

$$i_X(f, V) = \lim_{n\to\infty} i_{K_n}(f, W_1 \cap K_n) \tag{5.3}$$

and

$$i_X(f^p, V) = \lim_{n\to\infty} i_{K_n}(f^p, W_1 \cap K_n). \tag{5.4}$$

Recall that the right hand side in equations (5.3) and (5.4) denotes an approximate fixed point index, which is defined and constant for all large n.

Henceforth, we shall assume that Σ is nonempty, since the properties of the fixed point index imply that $i_X(f, V) = 0 = i_X(f^p, V)$ if Σ is empty. For notational convenience, we define a sequence δ_n, for $n \ge 1$, such that

$$\alpha(K_n) < \delta_n < \alpha(K_n) + \frac{1}{n}.$$

We define K_∞ by

$$K_\infty := \bigcap_{n \ge 1} K_n$$

and we note that K_∞ is nonempty because $\Sigma \subset K_\infty$ and compact because $\alpha(K_\infty) \le \lim_{n\to\infty} \alpha(K_n) = 0$ and K_∞ is closed.

Step 2: Because $K_n \in \mathcal{F}_0$, with $K_n \subset X$ and $\alpha(K_n) < \delta_n$, we can write

$$K_n = \bigcup_{j=1}^{\nu(n)} C_{j,n},$$

where $C_{j,n} \subset X$ is closed and convex for $1 \le j \le \nu(n) < \infty$ and $\operatorname{diam}(C_{j,n}) < \delta_n$ for $1 \le j \le \nu(n)$. By Corollary 2.4, there exists a compact, continuous map $R_n : K_n \to K_n$ such that

(1) $R_n(x) \in C_{j,n}$ for all $x \in C_{j,n}$, for $1 \le j \le \nu(n)$;

(2) $\|R_n(x) - x\| < \delta_n$ for all $x \in K_n$; and

(3) $R_n(x) = x$ for all $x \in K_\infty$.

For $x \in W \cap K_n$, we define $g_n(x) \in K_n$ by

$$g_n(x) := R_n(f(x)).$$

If $0 \le t \le 1$ and $x \in W \cap K_n$, we define

$$h_{n,t}(x) := h_n(x;t) = (1-t)f(x) + tR_n(f(x)).$$

Note that $h_n(x;t) \in K_n$ and $\|f(x) - h_n(x,t)\| < \delta_n$ for $x \in W \cap K_n$ and $0 \le t \le 1$, that $g_n(\overline{W})$ is compact and $h_n(x,t) = f(x)$ for $x \in W \cap K_\infty$ and $0 \le t \le 1$. If $B \subset X$ and $c > 0$, it is convenient to define $N_c(B) = \inf\{y \in X \mid d(y,B) \le c\}$, where $d(y,B) = \inf\{\|y - x\| \mid x \in B\}$.

With these notational preliminaries, we claim that there exists $c > 0$ such that for $0 \le j \le p$ and for all sufficiently large n,

(1) $N_c(f^j(W_1 \cap K_n)) \subset W$; and

(2) $f^j(N_c(W_1 \cap K_n)) \subset W$.

To prove inclusion (1), recall that W_1 was selected so $f^j(\overline{W}_1) \subset \overline{f^j(W_1)} \subset W$ for $0 \le j \le p$, so certainly $f^j(\overline{W}_1 \cap K_\infty) \subset W$ for $0 \le j \le p$. Since $f^j(\overline{W}_1 \cap K_\infty)$ is compact for $0 \le j \le p$, there exists $c > 0$ with $N_{2c}(f^j(\overline{W}_1 \cap K_\infty)) \subset W$ for $0 \le j \le p$. By the continuity of f^j for $0 \le j \le p$, there exists an open neighborhood H of $\overline{W}_1 \cap K_\infty$ in X so $f^j(H) \subset N_c(f^j(\overline{W}_1 \cap K_\infty))$ for $0 \le j \le p$ and, consequently, $N_c(f^j(H)) \subset N_{2c}(f^j(\overline{W}_1 \cap K_\infty))$ for $0 \le j \le p$. Now $\alpha(\overline{W}_1 \cap K_n) \to 0$ as $n \to \infty$, so by

Kuratowski's theorem, $\overline{W}_1 \cap K_n \subset H$ for all large n, so $N_c(f^j(\overline{W}_1 \cap K_n)) \subset W$ for $0 \le j \le p$ and all sufficiently large n.

The proof of inclusion (2) is similar. We know that $f^j(\overline{W}_1 \cap K_\infty)$ is a compact subset of W for $0 \le j \le p$. By the continuity of f, there exists an open neighborhood H_1 of $\overline{W}_1 \cap K_\infty$ in X such that $f^j(H_1) \subset W$ for $0 \le j \le p$. By decreasing c if necessary, we can arrange that $N_{2c}(\overline{W}_1 \cap K_\infty) \subset H_1$. Since $\alpha(\overline{W}_1 \cap K_n) \to 0$ as $n \to \infty$, Kuratowski's theorem implies that $\overline{W}_1 \cap K_n \subset N_c(\overline{W}_1 \cap K_\infty)$ for all large n, so $N_c(\overline{W}_1 \cap K_n) \subset H_1$ and $f^j(N_c(\overline{W}_1 \cap K_n)) \subset W$ for $0 \le j \le p$ and for all large n.

For the remainder of the proof, $c > 0$ will be a constant as above.

Step 3: We next claim that for every $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and an integer $n(\varepsilon)$ such that for all $x, y \in \overline{W} \cap K_n$ with $n \ge n(\varepsilon)$ and $\|x - y\| < \delta(\varepsilon)$, we have $\|f^j(x) - f^j(y)\| < \varepsilon$ for $0 \le j \le p$. By the compactness of $\overline{W} \cap K_\infty$ and the continuity of f^j on $\overline{W} \cap K_\infty$ for $0 \le j \le p$, there exists $\delta > 0$ such that $\|f^j(x) - f^j(y)\| < \varepsilon$ if $x, y \in \overline{W} \cap K_\infty$, $\|x - y\| < 2\delta$ and $0 \le j \le p$. We claim that there exists $\eta > 0$ such that if $x, y \in N_\eta(\overline{W} \cap K_\infty)$ and $\|x - y\| < \delta$, then $\|f^j(x) - f^j(y)\| < \varepsilon$ for $0 \le j \le p$. We argue by contradiction. If not, there exists a sequence $\eta_k \to 0^+$ and points $x_k, y_k \in N_{\eta_k}(\overline{W} \cap K_\infty)$ with $\|x_k - y_k\| < \delta$ and $\|f^j(x_k) - f^j(y_k)\| \ge \varepsilon$ for some fixed j with $0 \le j \le p$. Since $\overline{W} \cap K_\infty$ is compact, we can assume, by taking a subsequence, that $x_k \to x \in \overline{W} \cap K_\infty$ and $y_k \to y \in \overline{W} \cap K_\infty$. But then we see that $\|f^j(x) - f^j(y)\| \ge \varepsilon$ and $\|x - y\| \le \delta$, which contradicts our choice of δ. It follows that such an η exists. Since $\alpha(\overline{W} \cap K_n) \to 0$ as $n \to \infty$, Kuratowski's theorem implies $\overline{W} \cap K_n \subset N_\eta(\overline{W} \cap K_\infty)$ for all sufficiently large n, say for $n \ge n(\varepsilon)$. If $x, y \in \overline{W} \cap K_n$ and $n \ge n(\varepsilon)$ and $\|x - y\| < \delta$, our construction shows that $\|f^j(x) - f^j(y)\| < \varepsilon$ for $0 \le j \le p$.

Step 4: For all $\varepsilon > 0$, we claim that there exists an integer $n(\varepsilon)$ such that for all $n \ge n(\varepsilon)$, for all $x \in \overline{W}_1 \cap K_n$, for all real numbers t_j with $1 \le j \le p$ and with $0 \le t_j \le 1$ for $1 \le j \le p$, and for all k with $1 \le k \le p$, we have

$$\|f^k(x) - h_{n,t_k} h_{n,t_{k-1}} \cdots h_{n,t_1}(x)\| < \varepsilon. \tag{5.5}$$

If $k = 1$, notice that for all $x \in \overline{W}_1 \cap K_n$

$$\|f(x) - h_{n,t_1}(x)\| = t_1 \|f(x) - R_n f(x)\| < \delta_n.$$

Since $\lim_{n\to\infty} \delta_n = 0$, we can assume that $\delta_n < \varepsilon$ for all large n.

We now argue by finite induction on k, for $1 \le k \le p$. (If $k > 1$, part of our inductive hypothesis is that, for all large n, we have $h_{n,t_{k-1}} h_{n,t_{k-2}} \cdots h_{n,t_1}(x) \in W$ for all $x \in \overline{W}_1 \cap K_n$ and for all real numbers $t_1, t_2, \ldots, t_{k-1}$ with $0 \le t_j \le 1$ for $1 \le j \le k-1$.) Assume that, for some k with $1 \le k < p$, we have proved the inductive hypothesis for all j with $1 \leq j \leq k$. For $c > 0$ as in Step 2, the inductive hypothesis implies that for all sufficiently large n and for $1 \leq j \leq k$ we have, for all $x \in \overline{W}_1 \cap K_n$,

$$\|f^j(x) - h_{n,t_j} h_{n,t_{j-1}} \cdots h_{n,t_1}(x)\| < c.$$

Using Step 2, this implies that for $1 \leq j \leq k$ and n sufficiently large

$$h_{n,t_j} h_{n,t_{j-1}} \cdots h_{n,t_1}(x) \in W \cap K_n$$

for all $x \in \overline{W}_1 \cap K_n$. Of course, we also know that $f^j(x) \in W \cap K_n$ for all $x \in \overline{W}_1 \cap K_n$.

We now use Step 3. Given $\varepsilon > 0$, select $\delta > 0$ such that for all $u, v \in W \cap K_n$ with $\|u - v\| < \delta$ and all large n, we have $\|f(u) - f(v)\| < \frac{\varepsilon}{2}$. We can assume that $\delta < \varepsilon$ and $\delta < c$. By our inductive hypothesis, for all n sufficiently large, for all $x \in \overline{W}_1 \cap K_n$, for all reals t_j, with $1 \leq j \leq k$, where $0 \leq t_j \leq 1$, we have, for $1 \leq j \leq k$,

$$\|f^j(x) - h_{n,t_j} h_{n,t_{j-1}} \cdots h_{n,t_1}(x)\| < \delta.$$

If we write $v = h_{n,t_k} h_{n,t_{k-1}} \cdots h_{n,t_1}(x)$ and $u = f^k(x)$, it follows, for all $x \in \overline{W}_1 \cap K_n$ and all sufficiently large n that $u, v \in W \cap K_n$ and $\|u - v\| < \delta$, so

$$\|f(u) - f(v)\| = \|f^{k+1}(x) - f^k(v)\| < \frac{\varepsilon}{2}.$$

If $0 \leq t_{k+1} \leq 1$, we also have, for $v \in W \cap K_n$, that

$$\|f(v) - (1 - t_{k+1})f(v) - t_{k+1} R_n f(v)\| \leq \|f(v) - R_n(f(v))\| < \delta_n.$$

It follows that for all sufficiently large n, all $x \in \overline{W}_1 \cap K_n$ and all real numbers t_j, with $1 \leq j \leq k+1$, where $0 \leq t_j \leq 1$ for $1 \leq j \leq k+1$, we have

$$\|f^{k+1}(x) - h_{n,t_{k+1}} h_{n,t_k} \cdots h_{n,t_1}(x)\| < \frac{\varepsilon}{2} + \delta_n < \varepsilon.$$

This completes the inductive step, so for all $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ so that equation (5.5) is satisfied for all $n \geq N(\varepsilon)$, for all $x \in \overline{W}_1 \cap K_n$ and for $1 \leq j \leq p$.

Step 5: We first note that there exists $b > 0$ such that for all sufficiently large n, we have $\|f^p(x) - x\| \geq b > 0$ and $\|f(x) - x\| \geq b$ for all $x \in \partial W_1 \cap K_n$. The proof of this assertion is similar to the argument in Step 2, and is left to the reader. If n is also so large that $\delta_n < b$, the definition of the approximate fixed point index implies that

$$i_{K_n}(f^p, W_1 \cap K_n) = i_{K_n}(g_n f^{p-1}, W_1 \cap K_n) \tag{5.6}$$

and

$$i_{K_n}(f, W_1 \cap K_n) = i_{K_n}(g_n, W_1 \cap K_n). \tag{5.7}$$

Using Step 4, we see that for all large n, all $x \in \overline{W}_1 \cap K_n$ and all t with $0 \le t \le 1$, we have $\|f^p(x) - g_n h_{n,t}^{p-1}(x)\| < b$. Since $\|f^p(x) - x\| \ge b$ for all $x \in \partial W_1 \cap K_n$ and all large n, it follows that for all sufficiently large n and all $x \in \partial W_1 \cap K_n$ and $0 \le t \le 1$, we have $g_n(h_{n,t}^{p-1}(x)) \ne x$. Since $\{g_n(h_{n,t}^{p-1}(x)) \mid x \in \overline{W}_1 \cap K_n, 0 \le t \le 1\}$ has compact closure, the homotopy property of the classical fixed point index implies that for all large n,

$$i_{K_n}(g_n f^{p-1}, W_1 \cap K_n) = i_{K_n}(g_n^p, W_1 \cap K_n). \tag{5.8}$$

Combining equation (5.6) and (5.8) we see that for all large n,

$$i_{K_n}(f^p, W_1 \cap K_n) = i_{K_n}(g_n^p, W_1 \cap K_n). \tag{5.9}$$

Step 6: By virtue of equations (5.7) and (5.9), to complete the proof of Theorem 5.1 it suffices to prove that for all large n

$$i_{K_n}(g_n^p, W_1 \cap K_n) \equiv i_{K_n}(g_n, W_1 \cap K_n) \mod p. \tag{5.10}$$

Because $\Sigma \subset W_1$ and Σ is compact, there exists $\varepsilon_1 > 0$ so that $N_{2\varepsilon_1}(\Sigma) \subset W_1$. Because $f(\Sigma) = \Sigma$ and f is continuous, there exists $\varepsilon_2 > 0$ such that $f(N_{\varepsilon_2}(\Sigma)) \subset N_{\varepsilon_1}(\Sigma)$.

We define $G := N_{\varepsilon_2}(\Sigma)$, so $N_{\varepsilon_1}(f(G)) \subset N_{2\varepsilon_1}(\Sigma) \subset W_1$. The usual sort of argument, which we leave to the reader, shows that there exists a constant $d > 0$ such that for all sufficiently large n and all $x \in (\overline{W}_1 \backslash G) \cap K_n$,

$$\|f^p(x) - x\| \ge d.$$

Select $\varepsilon_3 > 0$ with $\varepsilon_3 < \min\{\varepsilon_1, d\}$. By Step 4, for all sufficiently large n, all $x \in \overline{W}_1 \cap K_n$ and all integers j with $1 \le j \le p$,

$$\|f^j(x) - g_n^j(x)\| < \varepsilon_3.$$

It follows that for all $x \in (\overline{W}_1 \backslash G) \cap K_n$ and all n sufficiently large, $g_n^p(x) \ne x$. Also, since (for n large) $g_n(\overline{W}_1 \cap K_n) \subset K_n$ and $g_n(G) \subset N_{\varepsilon_1}(f(G)) \subset W_1$, we have $g_n(G \cap K_n) \subset W_1 \cap K_n$ for all large n.

Let $S_n = \{x \in \overline{W}_1 \cap K_n \mid g_n^p(x) = x\}$. Because $g_n(x) = f(x)$ for all $x \in \Sigma$, we have $\Sigma \subset S_n$. The remarks above show that $S_n \subset G$ for all large n. It follows that if $x \in S_n$, then $g_n(x) \in g_n(G) \cap K_n \subset W_1 \cap K_n$. Thus $g_n^p(g_n(x))$ is defined and $g_n^p(g_n(x)) = g_n(g_n^p(x)) = g_n(x)$, so $g_n(x) \in S_n$ for all $x \in S_n$.

It follows easily that $g_n(S_n) = S_n$, so by the mod p theorem for compact maps on spaces $K \in \mathcal{F}_0$ (see [52]), equation (5.10) is valid and the theorem is proved. ■

Theorem 5.1 implies in particular that the mod p theorem is valid for all the cases considered in Section 3. For definiteness, we mention two particular examples.

Corollary 5.3. *Suppose that M is a C^1 Banach manifold which can be nicely embedded in a Banach space Z (in the sense of Definition 3.2 of Section 3). (In particular, M can be any Banach space or any C^1 Banach manifold modelled on a C^1 Banach space.) Assume that M is a subset of a Banach space $(Y, \|\cdot\|)$ from which it inherits its topology and that the inclusion map of M into Y is C^1. Let $U \subset M$ be an open subset of M and $f : U \to M$ a continuous map. For a prime number p, let $V \subset U$ be an open set in M such that $f^j(V) \subset U$ for $1 \leq j \leq p$, let $\Sigma = \{x \in V \mid f^p(x) = x\}$ and assume that Σ is compact (possibly empty) and $f(\Sigma) = \Sigma$. Assume either*

(1) there exists an open neighborhood W of Σ in M such that $f|W$ is a k-set-contraction for some $k < 1$, with respect to the Kuratowski MNC on $(Y, \|\cdot\|)$; or

(2) there exists an open neighborhood W of Σ in M and an integer $n \geq 1$ such that f is C^1 and f^n is defined on W and f^n is a k-set-contraction for some $k < 1$.

Then it follows that

$$i_M(f^p, V) \equiv i_M(f, V) \mod p.$$

Remark 5.4. Under the hypothesis of Corollary 5.3,

$$i_M(f^p, V) := i_Z(g^p, r^{-1}(V))$$

and

$$i_M(f, V) := i_Z(g, r^{-1}(V)),$$

where $g := jfr$ and j and r are as in Definition 3.2. As noted in Section 3, this definition is independent of the particular maps j and r as in Definition 3.2.

Corollary 5.5. *Suppose that $X \in \mathcal{F}$, that U is an open subset of X and $f : U \to X$ is a continuous map. For a prime number p, let $V \subset U$ be an open set such that $f^j(V) \subset U$ for $1 \leq j \leq p$. Define*

$\Sigma = \{x \in V \mid f^p(x) = x\}$ and assume that Σ is compact and $f(\Sigma) = \Sigma$. Assume that there exists a compact set $\Gamma \supset \Sigma$, a number $r_0 > 0$ and a number c with $0 \leq c < 1$ such that

$$d(f(x), \Gamma) \leq cd(x, \Gamma)$$

for all $x \in U$ with $d(x, \Gamma) < r_0$. (Here $d(y, \Gamma) := \inf\{\|y - x\| \mid x \in \Gamma\}$.) Then it follows that

$$i_X(f^p, V) \equiv i_X(f, V) \mod p.$$

Theorems which rely on the mod p theorem to prove existence of a fixed point of a map f may be conceptually different from results which assume that f has a compact attractor. This is reflected in the fact that different hypotheses, some stronger and some weaker, are required. Our next corollary illustrates this point.

Corollary 5.6. *Let U be an open subset of a Banach space X and $f : U \to X$ a continuous map. Let V be a bounded, convex, open subset of U with $\overline{V} \subset U$ and assume, for some prime number p, that $f^j(\overline{V}) \subset U$ for $1 \leq j \leq p$, that $f^p(\partial V) \subset V$ and $f^p|\overline{V}$ is a k-set-contraction for some $k < 1$. Let $\Sigma = \{x \in \overline{V} \mid f^p(x) = x\}$ and assume that $f(\Sigma) = \Sigma$. Assume that there exists an open neighborhood W of Σ and a decreasing sequence of sets K_n, for $n \geq 1$, which satisfy the conditions in Theorem 5.1. (This will be true if, for example, there exists an open neighborhood W_1 of Σ such that $f|W_1$ is C^1 or if $f|W_1$ is a c-set-contraction for some $c < 1$.) Then $i_X(f, V)$ is defined, $i_X(f, V) \equiv 1 \pmod p$ and f has a fixed point in V.*

Proof. Select $x_0 \in V$ and consider the homotopy $g_t(x) = (1 - t)x_0 + tf^p(x)$ for $x \in \overline{V}$ and $0 \leq t \leq 1$. Since we assume that V is convex and bounded and $f^p(\partial V) \subset V$, we have $g_t(x) \neq x$ for $0 \leq t \leq 1$ and $x \in \partial V$. Because $f^p|\overline{V}$ is a k-set-contraction with $k < 1$, the homotopy property of the fixed point index for k-set-contractions [38] implies that $i_X(f^p, V) = i_X(g_0, V) = 1$. Because $f^p(\Sigma) = \Sigma$ and f^p is a k-set-contraction with $k < 1$, we have that Σ is compact; and $\Sigma \subset V$ because $f^p(x) \neq x$ for $x \in \partial V$. Theorem 5.1 now implies that $i_X(f, V) \equiv i_X(f^p, V) \equiv 1 \pmod p$, so f has a fixed point in V. If there exists an open neighborhood W_1 of Σ such that $f(W_1)$ is C^1 or if $f|W_1$ is a c-set-contraction for some $c < 1$, then it follows from results in Section 3 that there exist an open neighborhood W of Σ and sets K_n as in Theorem 5.1. ■

If we assume that $\overline{f(V)} \subset V$, we could try to apply theorems in Section 4 to obtain Corollary 5.6. The assumption that f^p is a k-set-contraction with $k < 1$ implies $\Gamma := \bigcap_{j \geq 1} \overline{f^j(V)}$ is a compact attractor for $f : \overline{V} \subset \overline{V}$ and $\Gamma \subset V$ because $\overline{f(V)} \subset V$. If we assumed that f is C^1 on an open neighborhood of Γ, then results in Section 4 would imply that $i_X(f, V) = 1$ and f has a fixed point in V. However, assuming that f is C^1 on a neighborhood of Γ is more restrictive than assuming that f is C^1 on a neighborhood of Σ, and it is unclear how to prove Corollary 5.6 without the mod p theorem.

6 A Counterexample

In this section we wish to describe a counterexample to part of the main theorem in [55]. In Theorem 2 of [55], A. Tromba claims the following result:

Claim 6.1. *(See Theorem 2 in [55].) Let G be an open subset of a Banach space E with $T : G \to G$ a C^1 map such that $\overline{T^n(G)}$ is compact in G for some integer $n \geq 1$. Assume that for all m sufficiently large, $L(T^m)$, the Lefschetz number of T^m, is even and nonzero. Then T has a fixed point.*

We shall give a counterexample to Claim 6.1 when G is a bounded, open subset of $\mathbb{R}^2$. The key observation is contained in the following lemma.

Lemma 6.2. *Let $C_a = \{(x, y) \in \mathbb{R}^2 \mid (x+1)^2 + y^2 = 1\}$ and $C_b = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\}$, and let $X = C_a \cup C_b$. There exists a continuous map $f : X \to X$ such that f has no fixed points in X, but where $L(f^m)$, the Lefschetz number of f^m, is even and nonzero for all $m \geq 2$.*

Proof. We shall identify $\mathbb{R}^2$ with $\mathbb{C}$ so $X = \{z - 1 \mid |z| = 1\} \cup \{z + 1 \mid |z| = 1\}$. Roughly speaking, f will map C_a around itself one time and around C_b two times (counterclockwise); it will map C_b around C_a one time (counterclockwise). To guarantee that f has no fixed points, we shall take a small perturbation.

For $0 \leq \varepsilon < \frac{2\pi}{3}$, define $f_\varepsilon : X \to X$ as follows. In the left-hand circle C_a set

$$f_\varepsilon(z-1) = \begin{cases} (e^{i\varepsilon} z)^3 - 1, & \text{for} \quad 0 \leq \theta \leq \frac{2\pi}{3} - \varepsilon, \\ -(e^{i\varepsilon} z)^3 + 1, & \text{for} \quad \frac{2\pi}{3} - \varepsilon \leq \theta \leq 2\pi - \varepsilon, \\ (e^{i\varepsilon} z)^3 - 1, & \text{for} \quad 2\pi - \varepsilon \leq \theta \leq 2\pi, \end{cases}$$

while in the right-hand circle C_b set

$$f_\varepsilon(z+1) = -e^{3i\varepsilon}z - 1, \text{ for } 0 \le \theta \le 2\pi,$$

where in both cases $z = e^{i\theta}$. For $\varepsilon \neq 0$ and in the above range, the reader can directly verify that f_ε has no fixed points in X. Obviously, the maps f_ε are homotopic, so f_ε and f_0 induce the same maps in homology, and thus $L(f_\varepsilon^m) = L(f_0^m)$ for $m \ge 1$ and $0 \le \varepsilon < \frac{2\pi}{3}$.

It is well-known that if we take singular homology with coefficients in $\mathbb{Q}$, then $H_0(X) = \mathbb{Q}$ and $H_1(X) = \mathbb{Q} \oplus \mathbb{Q}$. For convenience, we define $g = f_0$. Identifying elements of $H_1(X)$ with column vectors, we see that our construction insures that $g_{*,1}$ acts on $H_1(X)$ by left multiplication by the matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. It follows that

$$L(g^m) = 1 - \operatorname{tr}(A^m)$$

where tr denotes trace. A calculation shows that the eigenvalues of A are 2 and -1, so

$$L(g^m) = 1 - (2^m + (-1)^m).$$

It is immediate from this formula that $L(g^m)$ is even and nonzero for $m \ge 2$, so we can define $f = f_\varepsilon$ for fixed ε satisfying $0 < \varepsilon < \frac{2\pi}{3}$. ■

With the aid of Lemma 6.2, our main theorem follows easily.

Theorem 6.3. *There exists a bounded, connected open set G in $\mathbb{R}^2$ and a C^∞ map $T : G \to G$ such that*

(1) $\overline{T(G)} \subset G$;

(2) T has no fixed points in G; and

(3) $L(T^m)$ is even and nonzero for all $m \ge 2$, where $L(T^m)$ denotes the Lefschetz number of T^m.

Proof. If X is as in Lemma 6.2 and $\delta > 0$, define $N_\delta(X)$ to be the δ neighborhood $N_\delta(X) := \{p \in \mathbb{R}^2 \mid d(p, X) < \delta\}$ of X, where $d(p, X)$ denotes the distance of p to X. If distance is measured in the usual Euclidean metric and $\delta < 1$, one can easily see that there is a continuous retraction r of $\overline{N_\delta(X)}$ onto X. We fix a number δ_0 with $0 < \delta_0 < 1$, define $G = N_{\delta_0}(X)$ and let r be a continuous retraction of $\overline{G}$ onto X. If f is as in Lemma 6.2, we define $F : \overline{G} \to X \subset G$ by $F(x, y) = f(r(x, y))$. It follows that $L(F^m) = L(f^m)$ for all $m \geq 1$, and so $L(f^m)$ is nonzero and even for $m \geq 2$. The map F can be extended to a continuous map of $\mathbb{R}^2$ to $\mathbb{R}^2$, and we shall use the same letter to denote the extended map . Since $F(p) = f(r(p))$ for all $p \in \overline{G}$, any fixed point of F in $\overline{G}$ must lie in X and be a fixed point of f, which has no fixed points. Thus, there exists $\eta > 0$ such that $\|F(p) - p\| \geq \eta$ for all $p \in \overline{G}$.

We could define $T = F$ in Theorem 6.3 if F were C^∞. To handle this problem, we use the standard device of mollifiers to approximate F by a nonnegative C^∞ function. Let $\theta : \mathbb{R}^2 \to [0, \infty)$ be a C^∞ function with compact support in the unit ball centered at the origin. We can also arrange that $\int_{\mathbb{R}^2} \theta(p)\, dp = 1$. As usual, for $\varepsilon > 0$ define $\theta_\varepsilon(p) = \varepsilon^{-2}\theta(\varepsilon^{-1}p)$ for $p \in \mathbb{R}^2$, so θ_ε is a nonnegative C^∞ function, $\int_{\mathbb{R}^2} \theta_\varepsilon(p)\, dp = 1$ and θ_ε has support in the ball of radius ε. Define $F_\varepsilon(p)$ by

$$F_\varepsilon(p) := (F * \theta_\varepsilon)(p) := \int_{\mathbb{R}^2} F(q)\theta_\varepsilon(p - q)\, dq.$$

Standard arguments show that F_ε is a C^∞ function. Furthermore, we can assume by taking $\varepsilon > 0$ sufficiently small that for all $p \in \overline{G}$,

$$\|F_\varepsilon(p) - F(p)\| < \min\{\eta, \delta_0\}. \tag{6.1}$$

Since $F(p) \in X$ for all $p \in \overline{G}$, it follows that $\overline{F_\varepsilon(G)} \subset G$ and $(1 - t)F_\varepsilon(p) + tF(p) \in G$ for $0 \leq t \leq 1$ and $p \in G$. Thus, if we consider F_ε as a map of G to G and F as a map of G to G, then F_ε and F are homotopic and $L(F_\varepsilon^m) = L(F^m)$ for all $m \geq 1$. Since $\|F(p) - p\| \geq \eta$ for all $p \in \overline{G}$ and equation (6.1) is satisfied, F_ε has no fixed points in G. Thus, for $\varepsilon > 0$ sufficiently small, we can define $T : G \to G$ by $T(p) = F_\varepsilon(p)$. ■

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