

# Extension of order-preserving maps on a cone

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We examine the problem of extending, in a natural way, order-preserving maps that are defined on the *interior* of a closed cone  $K_1$  (taking values in another closed cone  $K_2$ ) to the whole of  $K_1$ .

We give conditions, in considerable generality (for cones in both finite- and infinite-dimensional spaces), under which a natural extension exists and is continuous. We also give weaker conditions under which the extension is upper semi-continuous.

Maps  $f$  defined on the interior of the non-negative cone  $K$  in  $\mathbb{R}^N$ , which are both homogeneous of degree 1 and order preserving, are non-expanding in the Thompson metric, and hence continuous. As a corollary of our main results, we deduce that all such maps have a homogeneous order-preserving continuous extension to the whole cone. It follows that such an extension must have at least one eigenvector in  $K - \{0\}$ . In the case where the cycle time  $\chi(f)$  of the original map does not exist, such eigenvectors must lie in  $\partial K - \{0\}$ .

We conclude with some discussions and applications to operator-valued means. We also extend our results to an ‘intermediate’ situation, which arises in some important application areas, particularly in the construction of diffusions on certain fractals via maps defined on the interior of cones of Dirichlet forms.

## 1. Introduction

In what follows, we typically let  $K_1, K_2$  denote closed cones in topological vector spaces (TVS)  $X_1, X_2$ , respectively, and let  $f : \overset{\circ}{K}_1 \rightarrow K_2$  be a map that is order preserving with respect to the usual partial orderings induced by  $K_1$  and  $K_2$ . We are interested in extending, in a natural way, the map  $f$  to a map  $F : K_1 \rightarrow K_2$  defined on the whole of  $K_1$ .

In § 2, we recall some basic definitions and show (theorem 2.13) that, under the assumption that images of decreasing sequences converge (which we call condition A), an order-preserving map has a natural order-preserving extension.

In § 3, we impose additional constraints under which a continuous map has a continuous extension. In particular, we consider order-preserving continuous maps

$f : \overset{\circ}{K}_1 \rightarrow K_2$ , where  $K_1$  satisfies a geometrical condition (G), where  $K_2$  satisfies a weak normality condition (WN), and where  $f$  satisfies a weak version of homogeneity of degree 1 (WH). In theorem 3.10, we prove that such maps have a natural extension that is also continuous. We show, in lemma 3.3, that in the finite-dimensional case, condition G is equivalent to  $K_1$  being polyhedral. There are interesting examples in both the finite- and infinite-dimensional cases. We also give examples in which the conditions do not hold and the extension is not everywhere continuous.

In §4, we demonstrate that in certain contexts continuity of the original map  $f : \overset{\circ}{K}_1 \rightarrow K_2$  holds automatically and is therefore not needed as an explicit assumption. As an example, we consider maps that are homogeneous of degree 1 and order preserving from the interior of the standard positive cone in  $\mathbb{R}^N$  to itself. Such maps are automatically continuous on the interior of the cone and our results show that they always have a homogeneous order-preserving continuous extension to the whole cone. As a corollary, we deduce that such maps must have at least one eigenvector in  $K_1 - \{0\}$  and hence, in the case where the cycle time  $\chi(f)$  of the map does not exist, that such eigenvectors must lie in  $\partial K_1 - \{0\}$ .

In §5, we return to the setting of §2, i.e. we consider maps that have a natural extension that need not be continuous. We introduce a natural ‘multi-valued’ extension  $\Phi : K_1 \rightarrow \mathcal{P}(X_2)$  and give conditions under which it is upper semi-continuous. We also examine the structure of the set  $\Phi(x)$  for points  $x \in K_1$ .

In §6, we give some further examples and applications to operator-valued means. In particular, we exhibit a map that is homogeneous of degree 1, order preserving and continuous on the interior of a normal cone, whose extension is discontinuous at certain points on the boundary of the cone. This is an example of a cone for which the geometrical condition G, used in theorem 3.10 of §3, does not hold. Specifically, the map is the harmonic mean of two positive semi-definite symmetric real  $2 \times 2$  matrices.

Finally, in §7, we extend our main results slightly to an ‘intermediate’ situation that arises in some important application areas, specifically where one is interested in certain maps,  $f : D_1 \rightarrow D_2$ , on sub-cones  $D_1 \subset K_1$ ,  $D_2 \subset K_2$ , which are order preserving with respect to the partial orderings induced by the larger cones  $K_1, K_2$ . This situation arises in the theory of construction of diffusions on certain fractals via maps defined on the interior of cones of Dirichlet forms. We show how our theory is applied to this situation. (This application area is dealt with in more detail in a forthcoming paper.)

## 2. Natural extension

The main purpose of this section is to give conditions under which an order-preserving map defined on the interior of a closed cone has a natural extension defined on the whole cone.

We first recall some basic definitions.

**DEFINITION 2.1** (closed cone, cone ordering). A closed cone (with vertex at zero) in a topological vector space (TVS)  $X$  is a closed convex subset  $K \subset X$  such that (1)  $K \cap (-K) = \{0\}$  and (2)  $\lambda K \subseteq K$  for all real  $\lambda \geq 0$ . The cone structure induces a partial ordering. We write  $x \leq_K y$  if  $y - x \in K$  (or simply  $x \leq y$ , if  $K$  is

obvious from the context). If  $K$  has non-empty interior,  $\overset{\circ}{K} \neq \emptyset$ , we write  $x \ll y$  if  $y - x \in \overset{\circ}{K}$ .

In what follows, we will usually let  $K_1$  be a closed cone in a Hausdorff TVS  $X_1$  with  $\overset{\circ}{K}_1 \neq \emptyset$  and let  $K_2$  be a closed cone in a TVS  $X_2$ , and consider maps  $f : \overset{\circ}{K}_1 \rightarrow K_2$  that are both (1) continuous and (2) order preserving in the sense that, for all  $x, y \in \overset{\circ}{K}_1$ ,

$$x \leq_{K_1} y \quad \text{implies} \quad f(x) \leq_{K_2} f(y).$$

REMARK 2.2 ('allowable sequences'). When considering extension of such maps to points  $x \in \partial K_1$ , on the boundary of  $K_1$ , it will be natural to consider sequences  $\langle x_n \in \overset{\circ}{K}_1 : n \geq 1 \rangle$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$  and  $x_n \gg x$  for all  $n$ . We will call such sequences 'allowable'.

In other words, we consider sequences  $x_n \rightarrow x$  in  $\overset{\circ}{K}_1$  that lie in a 'copy' of the interior of the cone translated so that the vertex lies at the point  $x$ . Assuming that  $\overset{\circ}{K}_1 \neq \emptyset$ , if we take any  $u \in \overset{\circ}{K}_1$  and define  $x_n := x + n^{-1}u$ , then  $\lim_{n \rightarrow \infty} x_n = x$  and  $x_n \gg x$  for all  $n \geq 1$  (by convexity). So such a sequence always exists.

LEMMA 2.3. *Suppose that  $K$  is a closed cone with non-empty interior  $\overset{\circ}{K} \neq \emptyset$  in a TVS  $X$ . Let  $x \in K$  and let  $\langle x_n \in K : n \geq 1 \rangle$  be an allowable sequence with  $\lim_{n \rightarrow \infty} x_n = x$  (so that  $x_n \gg x$  for all  $n \geq 1$ ). Then there exists an increasing sequence of positive integers  $n_j \uparrow \infty$  such that  $x_m \ll x_{n_j}$  for all  $m \geq n_{j+1}$ .*

*Proof.* Define  $n_1 := 1$ . Assume, by induction, that we have found

$$n_1 < n_2 < \dots < n_k$$

meeting the above conditions. Let  $U_k \subset \overset{\circ}{K}$  be a set, open in  $X$ , such that  $x_{n_k} - x \in U_k$ . Let  $V_k$  be an open neighbourhood of 0 such that  $V_k = -V_k$  and  $x_{n_k} - x + V_k \subset U_k$ . Select  $n_{k+1} > n_k$  such that  $x_m - x \in V_k$  for  $m \geq n_{k+1}$  (because  $\lim_{n \rightarrow \infty} x_n = x$ ). It follows that

$$x_{n_k} - x - (x_m - x) = x_{n_k} - x_m \in (x_{n_k} - x) - V_k \subset U_k$$

for  $m \geq n_{k+1}$ , so that  $x_{n_k} - x_m \in U_k \subset \overset{\circ}{K}$  for all  $m \geq n_{k+1}$ . □

Under the assumption to be given below, namely that images of decreasing sequences converge, we will be able to prove that a natural extension exists.

DEFINITION 2.4 (condition A). Let  $f : \overset{\circ}{K}_1 \rightarrow K_2$ , where  $K_1, K_2$  are closed cones in Hausdorff TVS  $X_1, X_2$ , respectively. Suppose that if  $x_1 \geq x_2 \geq \dots \geq x_k \geq \dots$  is any decreasing sequence in  $\overset{\circ}{K}_1$ , then the sequence  $\langle f(x_j) : j \geq 1 \rangle$  converges in  $K_2$ . If  $f, K_1$  and  $K_2$  satisfy these conditions, then we shall say that condition A is satisfied.

Condition A holds in some interesting cases.

LEMMA 2.5. *Suppose that  $K_1$  is a closed cone in a Hausdorff TVS  $X_1$  and that  $\overset{\circ}{K}_1 \neq \emptyset$ . Assume that  $K_2$  is a closed cone in a finite-dimensional Hausdorff TVS  $X_2$ . Let  $f : \overset{\circ}{K}_1 \rightarrow K_2$  be an order-preserving map. Then condition A holds.*

In the proof of lemma 2.5, we will make use of normality of the cone  $K_2$ ; recall the definition below.

**DEFINITION 2.6** (normal cone). A cone  $K$  in a normed linear space, not necessarily finite dimensional, is said to be ‘normal’ if there exists a constant  $M$  such that  $\|x\| \leq M\|y\|$  whenever  $0 \leq_K x \leq_K y$ .

*Proof of lemma 2.5.* Recall that the topology on any finite-dimensional Hausdorff TVS  $X_2$  of dimension  $n$  is such that  $X_2$  is linearly homeomorphic to  $\mathbb{R}^n$  with the standard Euclidean metric. Thus we can assume that  $X_2$  is a normed linear space. Recall also that any closed cone  $K$  in a finite-dimensional normed linear space is normal.

Suppose that  $\langle x_j \in K_1 : j \geq 1 \rangle$  is a sequence of points with  $x_{j+1} \leq x_j$  for all  $j \geq 1$ . By the order-preserving property of  $f$ , it follows that  $0 \leq_{K_2} f(x_{j+1}) \leq_{K_2} f(x_j) \leq_{K_2} f(x_1)$  for all  $j \geq 1$ . By the normality of  $K_2$ , we have  $\|f(x_j)\| \leq M\|f(x_1)\|$  for all  $j \geq 1$ , for some constant  $M$ .

Writing  $y_j := f(x_j)$ , the sequence  $\langle y_j : j \geq 1 \rangle$  is a bounded set in a finite-dimensional Banach space and so is compact. It follows that there exists a convergent subsequence  $y_{j_i} \rightarrow y$ , for some  $y \in K_2$ , where  $j_i \uparrow \infty$  as  $i \rightarrow \infty$ . Because  $y_m - y = \lim_{k \rightarrow \infty} y_m - y_{j_k}$ , we have  $y_m - y \geq 0$  for all  $m \geq 1$ . If  $m \geq j_i$ , we have  $0 \leq y_m - y \leq y_{j_i} - y$ , so the normality of  $K_2$  implies that  $\|y_m - y\| \leq M\|y_{j_i} - y\|$  for  $m \geq j_i$ . The latter inequality implies that the limit of the full sequence  $\langle y_m \rangle$  exists and that  $\lim_{m \rightarrow \infty} y_m = y$ , giving condition A.  $\square$

**REMARK 2.7.** Note that there are interesting closed cones  $K$  that are *not* finite dimensional, but which have the property that decreasing sequences converge.

This motivates the following definition.

**DEFINITION 2.8** (monotone convergence property). Let  $K$  be a closed cone in a Hausdorff TVS  $X$ . We say that  $K$  has the ‘monotone convergence property’ if, whenever  $\langle y_j : j \geq 1 \rangle$  is a sequence in  $K$  and  $y_{j+1} \leq y_j$  for all  $j \geq 1$ , there exists  $y \in K$  with  $\lim_{j \rightarrow \infty} y_j = y$ .

**EXAMPLE 2.9.** From the above discussion, we see that any finite-dimensional closed cone has the monotone convergence property.

**EXAMPLE 2.10.** Suppose that  $H$  is a real Hilbert space and that  $X$  is the set of bounded self-adjoint linear maps  $A : H \rightarrow H$ . Equip  $X$  with the strong operator topology: if  $\langle A_j : j \geq 1 \rangle$  is a sequence in  $X$ , then  $A_j \rightarrow A$  in the strong operator topology if and only if  $\|A_j(x) - A(x)\| \rightarrow 0$  as  $j \rightarrow \infty$  for all  $x \in H$ . Let  $K$  be the set of positive semi-definite bounded self-adjoint operators in  $X$ , so that  $\langle Ax, x \rangle \geq 0$  for all  $x \in H$ . Since  $I \in \overset{\circ}{K}$ , where  $I : H \rightarrow H$  is the identity operation, we know that  $\overset{\circ}{K}$  is non-empty. It is a standard result that  $K$  has the monotone convergence property [9] (see also example 6.2).

**EXAMPLE 2.11.** Let  $(S, \mathcal{M}, \mu)$  be a measure space and let  $X = L^1(S, \mathcal{M}, \mu)$  denote the usual Banach space of  $\mu$ -integrable real-valued maps. Let  $K$  denote the closed cone in  $X$  of maps that are greater than or equal to zero  $\mu$ -almost everywhere (two

maps in  $X$  being identified if they agree  $\mu$ -almost everywhere). The monotone convergence theorem from real analysis implies that  $K$  has the monotone convergence property.

We now use the monotone convergence property and the previous remarks to give a generalization of lemma 2.5 to the case where  $X_2$  need not be finite dimensional.

LEMMA 2.12. *Suppose that  $K_1$  is a closed cone in a Hausdorff TVS  $X_1$  and that  $\overset{\circ}{K}_1 \neq \emptyset$ . Assume that  $K_2$  is a closed cone in a Hausdorff TVS  $X_2$  and that  $K_2$  satisfies the monotone convergence property. Let  $f : \overset{\circ}{K}_1 \rightarrow K_2$  be a continuous order-preserving map. Then condition A is satisfied.*

The above lemmas and discussion illustrate that there are interesting examples that satisfy condition A.

We now state our first extension theorem.

THEOREM 2.13 (natural extension using allowable sequences). *Let  $K_1$  be a closed cone with non-empty interior in a Hausdorff TVS  $X_1$ . Let  $K_2$  be a closed cone in a Hausdorff TVS  $X_2$ . Assume that  $f : \overset{\circ}{K}_1 \rightarrow K_2$  is order preserving and continuous and that condition A is satisfied. Suppose that  $x \in K_1$  and that  $\langle x_n \in \overset{\circ}{K}_1 : n \geq 1 \rangle$  is an allowable sequence with  $\lim_{n \rightarrow \infty} x_n = x$ . Then we have the following.*

- (a) *There exists  $z = z_x \in K_2$  such that  $\lim_{n \rightarrow \infty} f(x_n) = z_x$ .*
- (b) *Further, if  $\langle y_n \in \overset{\circ}{K}_1 : n \geq 1 \rangle$  is another allowable sequence such that  $\lim_{n \rightarrow \infty} y_n = x$ , then  $\lim_{n \rightarrow \infty} f(y_n) = \lim_{n \rightarrow \infty} f(x_n) = z_x$ .*
- (c) *Further, if we define  $F(x) := z_x$  for all  $x \in K_1$ , then  $F(x) = f(x)$  for all  $x \in \overset{\circ}{K}_1$  (i.e.  $F$  is an extension of  $f$ ). If  $x \in K_1$  and  $\langle x_n \in K_1 : n \geq 1 \rangle$  is any sequence such that  $\lim_{n \rightarrow \infty} x_n = x$  and  $x_{n+1} \ll x_n$  for all  $n \geq 1$ , then  $\lim_{n \rightarrow \infty} F(x_n) = F(x)$ . (In fact, if  $\langle y_n \in K_1 : n \geq 1 \rangle$  is any sequence such that  $y_n \geq x$  for all  $n \geq 1$  and  $\lim_{n \rightarrow \infty} y_n = x$ , it follows that  $\lim_{n \rightarrow \infty} F(y_n) = F(x)$ ).*

*Proof of part (a).* In what follows, suppose that  $x \in K_1$  and that  $\langle x_n \in \overset{\circ}{K}_1 : n \geq 1 \rangle$  is a sequence such that  $x_n \gg x$  for all  $n \geq 1$  and  $\lim_{n \rightarrow \infty} x_n = x$  (i.e.  $\langle x_n \rangle$  is an allowable sequence converging to  $x$ ).

By lemma 2.3, select an increasing sequence  $n_j \uparrow \infty$  such that  $x_m \ll x_{n_j}$  for all  $m \geq n_{j+1}$ . By condition A, there exists  $z \in K_2$  such that

$$\lim_{j \rightarrow \infty} f(x_{n_j}) = z.$$

The order-preserving property of  $f$  implies that  $z \leq f(x_{n_j})$  for all  $j \geq 1$  and  $z \leq f(x_m) \leq f(x_{n_j})$  for all  $m \geq n_{j+1}$ .

Suppose, by way of contradiction, that the full sequence  $\langle f(x_m) : m \geq 1 \rangle$  does not converge to  $z$ . Then there exists an open neighbourhood  $U$  of  $z$  and a sequence  $\nu_k \uparrow \infty$  such that  $f(x_{\nu_k}) \notin U$ . Lemma 2.3 implies that, by taking a further subsequence, which we label the same, we can assume that  $x_{\nu_k} \gg x_m$  for all  $m \geq \nu_{k+1}$  and  $k \geq 1$ .

Condition A implies that there exists  $\zeta \in K_2$  with  $\lim_{k \rightarrow \infty} f(x_{\nu_k}) = \zeta$ . We claim that  $\zeta = z$ , a contradiction, which proves that  $\lim_{m \rightarrow \infty} f(x_m) = z$ . To see this, fix

$j \geq 1$ . Now take  $k$  large enough so that  $\nu_k \geq n_{j+1}$ . We see that  $x_{\nu_k} \ll x_{n_j}$ , so that  $f(x_{\nu_k}) \leq f(x_{n_j})$ , for all large enough  $k$ . Letting  $k \rightarrow \infty$ , we see that  $\zeta \leq f(x_{n_j})$ . Letting  $j \rightarrow \infty$ , we see that  $\zeta \leq z$ . A symmetrical argument shows that  $z \leq \zeta$ , so that  $z = \zeta$ .  $\square$

*Proof of part (b).* Now suppose that  $\langle y_n \in \overset{\circ}{K}_1 : n \geq 1 \rangle$  is another sequence with  $y_n \gg x$  for all  $n \geq 1$  and  $\lim_{n \rightarrow \infty} y_n = x$ . We need to prove that

$$\lim_{n \rightarrow \infty} f(y_n) = \lim_{n \rightarrow \infty} f(x_n),$$

where the sequence  $\langle x_n : n \geq 1 \rangle$  is as above. Define a sequence  $\langle w_n : n \geq 1 \rangle$  by  $w_{2n} := x_n$  and  $w_{2n+1} := y_n$ . Then  $w_n \gg x$  for all  $n \geq 1$  and  $\lim_{n \rightarrow \infty} w_n = x$ . It follows from the proof of part (a) that  $\lim_{n \rightarrow \infty} f(w_n)$  exists. But this implies that

$$\lim_{n \rightarrow \infty} f(y_n) = \lim_{n \rightarrow \infty} f(w_n) = \lim_{n \rightarrow \infty} f(x_n).$$

$\square$

*Proof of part (c).* Now, for all  $x \in K_1$ , define  $F(x)$  as in the theorem. If  $x \in \overset{\circ}{K}_1$ , then continuity of  $f$  implies that  $f(x) = F(x)$ . Thus  $F$  is an extension of  $f$ . Let  $x \in \partial K_1$  and  $\langle x_n \in K_1 : n \geq 1 \rangle$  be a sequence with  $x_n \geq x$  for all  $n \geq 1$  and  $\lim_{n \rightarrow \infty} x_n = x$ . Select  $u \in \overset{\circ}{K}_1$  and note that

$$F(x_n) = \lim_{k \rightarrow \infty} f(x_n + k^{-1}u).$$

Let  $V$  be an open neighbourhood of 0 in  $X_2$  and  $W$  be an open neighbourhood of 0 such that  $-W = W$  and  $W + W \subset V$ . For each  $n \geq 1$ , select an integer  $k_n$  with  $f(x_n + k^{-1}u) \in F(x_n) + W$  for all  $k \geq k_n$ . Note that we can also assume that  $k_n < k_{n+1}$  for  $n \geq 1$ . By definition of  $F(x)$ , there exists an integer  $n_*$  such that  $f(x_n + k_n^{-1}u) \in F(x) + W$  for all  $n \geq n_*$ . It follows that, for  $n \geq n_*$ ,

$$F(x_n) \in f(x_n + k_n^{-1}u) - W = f(x_n + k_n^{-1}u) + W \subset (F(x) + W) + W \subset F(x) + V.$$

Since  $F(x) + V$  was chosen to be an arbitrary open neighbourhood of  $F(x)$ , we conclude that  $\lim_{n \rightarrow \infty} F(x_n) = F(x)$ .  $\square$

REMARK 2.14 (the extended map is order preserving). The map  $F$  is also necessarily order preserving, since if  $x, y \in K_1$  and  $x \leq y$ , we may take  $u \in \overset{\circ}{K}_1$  and note that

$$F(x) = \lim_{n \rightarrow \infty} f(x + n^{-1}u) \leq \lim_{n \rightarrow \infty} f(y + n^{-1}u) = F(y).$$

Note also that theorem 2.13 implies that  $F$  is continuous at  $0 \in K_1$ , since all sequences  $\langle x_n \rangle$  with  $x_n \rightarrow 0$  satisfy  $0 \leq x_n$ , as demanded by part (c) in the proof of the theorem.

### 3. Continuous extension

As we show later (in example 6.2), the map  $F : K_1 \rightarrow K_2$ , as defined in theorem 2.13, need not be everywhere continuous, even if  $K_1$  and  $K_2$  are finite-dimensional closed cones. The aim of this section is to give some further conditions

on  $f$ ,  $K_1$  and  $K_2$ , in considerable generality, which ensure that  $F : K_1 \rightarrow K_2$  is continuous.

We begin with a geometrical condition on  $K_1$ , which, as we prove below, is a generalization of the polyhedral property of a cone.

**DEFINITION 3.1** (geometrical condition G). Let  $K$  be a closed cone in a Hausdorff TVS  $X$ . If  $x \in K$ , we shall say that ‘ $K$  satisfies condition G at  $x$ ’ if, whenever  $\langle x_n \in K : n \geq 1 \rangle$  is a sequence in  $K$  with  $\lim_{n \rightarrow \infty} x_n = x$  and  $\lambda < 1$ , there exists an integer  $n_*$  such that  $\lambda x \leq x_n$  for all  $n \geq n_*$ . We shall say simply that ‘ $K$  satisfies condition G’ if  $K$  satisfies condition G at  $x$  for all  $x \in K$ .

Later, in example 6.2, we exhibit a map  $f : \hat{K}_1 \rightarrow K_2$ , where  $K_1$  is a normal cone for which condition G does not hold. The map is order preserving, homogeneous of degree 1 and continuous on the interior of  $K_1$ , but has an order-preserving homogeneous extension that is *not* continuous at certain points on the boundary  $\partial K_1$ .

However, there are nice cases in which condition G is satisfied, as we illustrate below.

**DEFINITION 3.2** (polyhedral cone). Recall that a closed cone  $K$  in a Hausdorff TVS  $X$  is called ‘polyhedral’ if there exist continuous linear functionals  $\varphi_j : X \rightarrow \mathbb{R}$ ,  $1 \leq j \leq N < \infty$ , such that

$$K = \{x \in X : \varphi_j(x) \geq 0 \text{ for } 1 \leq j \leq N\}.$$

**LEMMA 3.3** (polyhedral is equivalent to finite-dimensional G). *Let  $K$  be a closed cone in a Hausdorff TVS  $X$ . It follows that  $K$  is polyhedral if and only if  $X$  is finite dimensional and  $K$  satisfies condition G.*

We are grateful to Cormac Walsh, who pointed out the converse part of the lemma.

*Proof.* (We first show that a polyhedral cone must be finite dimensional and must satisfy condition G.) If  $K$  is polyhedral, it is not hard to see that  $X$  must be finite dimensional and of dimension less than or equal to  $N$ . (Otherwise, one constructs  $x \in X$ ,  $x \neq 0$ , with  $\varphi_j(x) = 0$  for  $1 \leq j \leq N$ .)

Now we must deduce condition G. Let  $\varphi_j$ ,  $1 \leq j \leq N$ , be continuous linear functionals on  $X$  such that

$$K = \{y \in X : \varphi_j(y) \geq 0 \text{ for } 1 \leq j \leq N\}.$$

Suppose that  $x \in K$  and that  $\langle x_n \in K : n \geq 1 \rangle$  is a sequence such that  $\lim_{n \rightarrow \infty} x_n = x$ . Take  $\lambda < 1$ . We have to show that there exists  $n_*$  such that  $\lambda x \leq x_n$  for all  $n \geq n_*$ . By definition,  $\lambda x \leq x_n$  if and only if  $\varphi_j(x_n - \lambda x) = \varphi_j(x_n) - \lambda \varphi_j(x) \geq 0$  for  $1 \leq j \leq N$ . Because  $x_n \in K$  for  $n \geq 1$ , we know that  $\varphi_j(x_n) \geq 0$  for  $n \geq 1$  and  $1 \leq j \leq N$ . Consider each  $j$ . If  $\varphi_j(x) = 0$ , it follows that  $\varphi_j(x_n) - \lambda \varphi_j(x) \geq 0$  for all  $n \geq 1$ . If  $\varphi_j(x) > 0$ , then, because  $\lim_{n \rightarrow \infty} \varphi_j(x_n) = \varphi_j(x)$ , it follows that  $\varphi_j(x_n) - \lambda \varphi_j(x) \geq 0$  for all sufficiently large  $n$ , say for  $n \geq n_j$ . If we define  $n_* := \max\{n_j : 1 \leq j \leq N\}$ , we see that  $\varphi_j(x_n - \lambda x) \geq 0$  for  $n \geq n_*$  and for  $1 \leq j \leq N$ , so that  $x_n - \lambda x \in K$  for  $n \geq n_*$ . Thus condition G holds.

We now prove the converse. First, note that for any closed cone  $K$  in a finite-dimensional Banach space, there exists a continuous linear functional  $L : X \rightarrow \mathbb{R}$  such that  $L(x) > 0$  for all non-zero  $x \in K$ . Let  $S := \{x \in K : L(x) = 1\}$ . It is straightforward to show that  $S$  is compact and convex. It is known that  $K$  is polyhedral if and only if  $S$  has finitely many extreme points (see, for example, [12]).

Now let  $K$  be a closed cone in a finite-dimensional Banach space  $X$  and suppose that  $K$  satisfies condition G. We will show that  $S$  (as defined above) has finitely many extreme points. The above remark shows that  $K$  must then be polyhedral.

Let  $z \in S$  and  $0 < \alpha < 1$ . By condition G, there exists a neighbourhood  $U$  of  $z$  in  $X$  such that  $\alpha z \leq u$  for all  $u \in U \cap K$ . Now let  $y \in S \cap U$  with  $y \neq z$ , so that  $\alpha z \leq y$ , i.e.  $y - \alpha z \in K$ .

Consider the point  $x := y/(1-\alpha) - \alpha z/(1-\alpha)$ . We have  $x = (y - \alpha z)/(1-\alpha) \in K$  and, further,  $L(x) = 1$ , so that  $x \in S$ . Note that  $y = (1-\alpha)x + \alpha z$ , thus  $y$  lies between  $x$  and  $z$  and cannot be an extreme point of  $S$ . We have shown that for all  $z \in S$  there exists a neighbourhood  $U_z$  in which there are no extreme points of  $S$  except, perhaps, for  $z$  itself.

Note that  $\bigcup_{z \in S} U_z$  is an open cover of  $S$ . Since  $S$  is compact, there exists a finite open sub-cover of  $S$ . By the above, each set in this sub-cover may contain at most one extreme point of  $S$ . Thus  $S$  has only a finite number of extreme points, and we conclude that  $K$  is polyhedral.  $\square$

The next condition that we need is a weak form of *homogeneity* of the map  $f : \mathring{K}_1 \rightarrow K_2$ . Recall the definition of homogeneity.

DEFINITION 3.4 (homogeneity, H). We say that  $f : \mathring{K}_1 \rightarrow K_2$  is ‘homogeneous of degree  $p$ ’ if, for all  $x \in \mathring{K}$  and  $\lambda > 0$ ,

$$f(\lambda x) = \lambda^p f(x).$$

We will need a weakened form of homogeneity of degree 1.

DEFINITION 3.5 (weak homogeneity, WH). Let  $K_1$  be a closed cone with  $\mathring{K}_1 \neq \emptyset$  in a Hausdorff TVS  $X_1$  and let  $K_2$  be a closed cone in a Hausdorff TVS  $X_2$ . Consider a map  $f : \mathring{K}_1 \rightarrow K_2$ . We shall say that ‘ $f$  satisfies condition WH at  $x \in \mathring{K}_1$ ’ if, for every positive real  $\alpha$ ,  $0 < \alpha < 1$ , there exist  $\delta > 0$  and an open neighbourhood  $V$  of  $x$  in  $X_1$  such that, for all  $y \in V \cap \mathring{K}_1$  and for all real  $\lambda \in [1 - \delta, 1]$ ,

$$f(\lambda y) \geq \alpha f(y).$$

If  $f$  satisfies condition WH at  $x$  for every  $x \in \mathring{K}_1$ , then we shall say simply that ‘ $f$  satisfies condition WH’.

EXAMPLE 3.6. For example, if  $f : \mathring{K}_1 \rightarrow K_2$  is continuous, it is easy to prove that  $f$  satisfies condition WH at every  $x \in \mathring{K}_1$  such that  $f(x) \in \mathring{K}_2$ .

EXAMPLE 3.7. If there exists  $\delta_* > 0$  and a map  $\varphi : [1 - \delta_*, 1] \rightarrow (0, \infty)$  with  $\lim_{\lambda \rightarrow 1^-} \varphi(\lambda) = 1$  such that

$$f(\lambda y) \geq \varphi(\lambda) f(y),$$

for all  $y \in \mathring{K}_1$  and  $\lambda \in [1 - \delta_*, 1]$ , then  $f$  satisfies condition WH.



The final condition that we shall need is a variant of the assumption of normality for the cone  $K_2$ , which we call the ‘weak normality’ or ‘sandwich’ condition.

DEFINITION 3.8 (weak normality, WN). Let  $K_2$  be a closed cone in a Hausdorff TVS  $X_2$ . We shall say that ‘ $K_2$  satisfies condition WN’ if, whenever  $\langle x_n : n \geq 1 \rangle$ ,  $\langle y_n : n \geq 1 \rangle$  and  $\langle z_n : n \geq 1 \rangle$  are sequences in  $K_2$  with  $0 \leq x_n \leq y_n \leq z_n$  for all  $n \geq 1$  and  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = y$  for some  $y \in K_2$ , it follows that  $\lim_{n \rightarrow \infty} y_n$  exists and equals  $y$ .

LEMMA 3.9. Let  $K_2$  be a closed cone in a normed linear space. Then  $K_2$  is normal if and only if  $K_2$  satisfies condition WN.

*Proof.* We first prove that normality implies WN. By the definition of normality, there exists a constant  $M$  such that  $\|u\| \leq M\|v\|$  whenever  $0 \leq u \leq v$ . If  $\langle x_n \rangle$ ,  $\langle y_n \rangle$ ,  $\langle z_n \rangle$  are sequences as in definition 3.8 above, we have that  $0 \leq y_n - x_n \leq z_n - x_n$  for all  $n \geq 1$ , so

$$\|y_n - x_n\| \leq M\|z_n - x_n\|.$$

Since  $\lim_{n \rightarrow \infty} (z_n - x_n) = 0$ , it follows that  $\lim_{n \rightarrow \infty} (y_n - x_n) = 0$ , so that we have  $\lim_{n \rightarrow \infty} y_n = y$ , as desired.

Conversely, suppose that  $K_2$  satisfies condition WN. To show that  $K_2$  is normal, we argue by contradiction. Assume that  $K_2$  is not normal. Then there exist sequences  $\langle u_n \rangle$  and  $\langle v_n \rangle$  in  $K_2$  with  $0 \leq u_n \leq v_n$  for all  $n \geq 1$  and  $\|u_n\|/\|v_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ . Let  $x_n := 0$ ,  $y_n := u_n/\|u_n\|$  and  $z_n := v_n/\|u_n\|$ , and note that  $x_n \leq y_n \leq z_n$ ,  $x_n \rightarrow 0$  and  $z_n \rightarrow 0$  as  $n \rightarrow \infty$ . We conclude that  $y_n \rightarrow 0$ , which contradicts the fact that  $\|y_n\| = 1$  for all  $n \geq 1$ .  $\square$

We now state our second main extension theorem; we use the above additional conditions to ensure that the extended map  $F$  is sequentially continuous.

THEOREM 3.10 (continuity of the extended map). Let  $K_1$  be a closed cone with  $\mathring{K}_1 \neq \emptyset$  in a Hausdorff TVS  $X_1$ ,  $K_2$  a closed cone in a Hausdorff TVS  $X_2$  and  $f : K_1 \rightarrow K_2$  a continuous order-preserving map. Assume that condition A is satisfied, that  $K_1$  satisfies condition G at some  $x \in \partial K_1$ , that  $f$  satisfies condition WH at  $x$  and that  $K_2$  satisfies condition WN. Define the extension  $F$  as in theorem 2.13. Then  $F$  is sequentially continuous at  $x$ .

*Proof.* Suppose that  $\langle v_n \in K_1 : n \geq 1 \rangle$  is any sequence with  $\lim_{n \rightarrow \infty} v_n = x$ . (Note that we do not insist that the sequence be ‘allowable’.) We have to prove that  $\lim_{n \rightarrow \infty} F(v_n) = F(x)$ . Given  $\alpha < 1$ , definition 3.5 (condition WH) implies that there exist  $\delta > 0$  and an open neighbourhood  $V$  of  $x$  such that  $f(\lambda y) \geq \alpha f(y)$  for all  $y \in V \cap \mathring{K}_1$  and all  $\lambda \in [1 - \delta, 1]$ . Select  $u \in \mathring{K}_1$ . If  $y \in V \cap K_1$ , then  $y + k^{-1}u \in V$  for all large integers  $k$ . It follows that

$$f(\lambda(y + k^{-1}u)) \geq \alpha f(y + k^{-1}u)$$

for  $k \in \mathbb{N}$ ,  $k$  large and  $\lambda \in [1 - \delta, 1]$ . Letting  $k \rightarrow \infty$ , we find that

$$F(\lambda y) \geq \alpha F(y).$$

For each  $n$ , define

$$\alpha_n := \sup\{\alpha \leq 1 : \alpha F(x) \leq F(v_k) \text{ for all } k \geq n\}$$

and select

$$\delta_n := \inf\{\delta \geq 1/n : x + \delta u \geq v_k \text{ for all } k \geq n\}.$$

Clearly,  $\delta_n$  is a decreasing sequence and  $\alpha_n$  is an increasing sequence. We leave to the reader the exercise of proving that  $\lim_{n \rightarrow \infty} \delta_n = 0$ .

We claim that  $\lim_{n \rightarrow \infty} \alpha_n = 1$ . To see this, take  $\alpha$  with  $0 < \alpha < 1$ . By our previous remarks, there exist  $\delta > 0$  and an open neighbourhood  $V$  of  $x$  such that  $F(\lambda y) \geq \alpha F(y)$  for all  $y \in V \cap K_1$  and  $\lambda \in [1 - \delta, 1]$ . In particular, we have that  $F(\lambda x) \geq \alpha F(x)$  for  $\lambda \in [1 - \delta, 1]$ . Put  $\lambda := 1 - \delta$ . Because  $K_1$  satisfies condition G at  $x$ , there exists  $n_*$  such that  $\lambda x \leq v_n$  for all  $n \geq n_*$ . It follows that

$$\alpha F(x) \leq F(\lambda x) \leq F(v_n) \quad \text{for all } n \geq n_*.$$

It follows from this equation that  $\alpha_{n_*} \geq \alpha$ . Since  $\alpha < 1$  was arbitrary, we see that  $\lim_{n \rightarrow \infty} \alpha_n = 1$ . We have

$$x_n := \alpha_n F(x) \leq F(v_n) =: y_n \leq F(x + \delta_n u) =: z_n.$$

Since  $\lim_n \alpha_n = 1$  and  $\lim_n \delta_n = 0$ , then  $\lim_n x_n = F(x) = \lim_n z_n$ . Because  $K_2$  satisfies condition WN, we finally deduce that  $\lim_n y_n = \lim_n F(v_n) = F(x)$ , proving that  $F$  is sequentially continuous at  $x$ .  $\square$

**COROLLARY 3.11.** *Let  $K_1$  be a closed polyhedral cone with  $\overset{\circ}{K}_1 \neq \emptyset$  in a Hausdorff TVS  $X_1$ . Let  $K_2$  be a closed cone in a Hausdorff TVS  $X_2$  and assume either that (i)  $K_2$  is finite dimensional or, more generally, that (ii)  $K_2$  has the monotone convergence property and satisfies condition WN. Suppose that  $f : \overset{\circ}{K}_1 \rightarrow K_2$  is continuous and order preserving and satisfies condition WH on  $K_1$ . Then  $f$  has a sequentially continuous extension  $F : K_1 \rightarrow K_2$ . Further,  $F$  is order preserving and clearly satisfies condition WH on  $K_1$ .*

*Proof.* Firstly, we make some observations for case (i). Note that any finite-dimensional closed cone  $K_2$  in a Hausdorff TVS has the monotone convergence property. Also, any finite-dimensional Hausdorff TVS  $X_2$  of dimension  $n$  is linearly homeomorphic to  $\mathbb{R}^n$  with the topology induced by the standard Euclidean metric. So we can assume that  $X_2$  is a normed linear space. Since any closed cone in a finite-dimensional normed space is normal, we deduce that in case (i)  $K_2$  is certainly weakly normal, i.e. condition WN holds. Thus both monotone convergence and weak normality of  $K_2$  hold in each case.

Lemma 2.12 implies that  $f$  satisfies condition A. It follows that  $f$  has an extension  $F : K_1 \rightarrow K_2$  as defined in theorem 2.13. Lemma 3.3 implies that  $K_1$  satisfies condition G. Theorem 3.10 now implies that  $F$  is sequentially continuous at  $x$  for all  $x \in K_1$ . If  $0 \leq x \leq y$  (in the partial ordering from  $\overset{\circ}{K}_1$ ), select  $u \in \overset{\circ}{K}_1$  and note that

$$0 \leq x + k^{-1}u \leq y + k^{-1}u$$

for every positive integer  $k$ . Because  $f$  is order preserving,

$$0 \leq f(x + k^{-1}u) \leq f(y + k^{-1}u),$$

in the partial ordering from  $K_2$ . Taking limits as  $k \rightarrow \infty$ , we see that

$$F(x) = \lim_{k \rightarrow \infty} f(x + k^{-1}u) \leq \lim_{k \rightarrow \infty} f(y + k^{-1}u) = F(y),$$

and we conclude that  $F$  is order preserving. Finally, since  $F = f$  on  $\overset{\circ}{K}_1$  and  $f$  satisfies condition WH, then  $F$  also satisfies condition WH.  $\square$

REMARK 3.12. Theorem 3.10 enables one to remove some extraneous assumptions in the literature. See, for example, the statement of theorem 2.1 on p. 28 in [7].

We now give an alternative to conditions G and WH in theorem 3.10.

REMARK 3.13 (alternative condition to G and WH). Suppose that  $K_1$  is a closed cone with  $\overset{\circ}{K}_1 \neq \emptyset$  in a Hausdorff TVS  $X_1$  and that  $K_2$  is a closed cone in a Hausdorff TVS  $X_2$  and that  $K_2$  satisfies condition WN. Assume that  $f : \overset{\circ}{K}_1 \rightarrow K_2$  is continuous, order preserving and satisfies condition A. Thus, by theorem 2.13,  $f$  has an order-preserving extension  $F : K_1 \rightarrow K_2$ . Now suppose that  $x \in \partial K_1$  and that for every  $\alpha < 1$  and every sequence  $\langle v_n : n \geq 1 \rangle$  in  $K_1$  with  $\lim_{n \rightarrow \infty} v_n = x$ , there exists  $n_*$  with  $\alpha F(x) \leq F(v_n)$  for all  $n \geq n_*$ . An examination of the proof of theorem 3.10 shows that  $F$  is sequentially continuous at  $x$ , even if  $K_1$  does not satisfy condition G at  $x$  or  $f$  does not satisfy condition WH at  $x$ . In particular, if  $F(x) = 0$ , then  $F$  is sequentially continuous at  $x$ .

The following example shows that the extended map  $F$  need not be continuous at every  $x \in \partial K_1$  (although it is continuous at every  $x$  for which  $F(x) = 0$ ).

EXAMPLE 3.14. Let  $K_1$  be the cone of non-negative continuous maps  $x \in C[0, 1]$  and let  $K_2 = [0, \infty)$ , the non-negative reals. Obviously,  $K_2$  has the monotone convergence property and satisfies condition WN. Define  $f : \overset{\circ}{K}_1 \rightarrow K_2$  by

$$f(x) := \left( \int_0^1 x(t)^{-1} dt \right)^{-1}.$$

Then  $f$  is continuous and order preserving and homogeneous of degree 1 (so it certainly satisfies condition WH). It follows that  $f$  satisfies condition A and, by theorem 2.13, that  $f$  has an extension  $F : K_1 \rightarrow K_2$  that is order preserving. The definition of  $F$  shows that for  $x \in \partial K_1$ ,

$$F(x) = \lim_{k \rightarrow \infty} \left( \int_0^1 \left( x(t) + \frac{1}{k} \right)^{-1} dt \right)^{-1}.$$

By using the monotone convergence theorem, we see that

$$F(x) = \left( \int_0^1 x(t)^{-1} dt \right)^{-1},$$

where  $I(x) := \int_0^1 x(t)^{-1} dt$  is interpreted as a Lebesgue integral and  $F(x) = 0$  if and only if  $I(x) = +\infty$ , i.e. if and only if  $x(t)^{-1}$  is not Lebesgue integrable on  $[0, 1]$ . By remark 3.13,  $F$  is continuous at  $x \in \partial K_1$  whenever  $I(x) = +\infty$  (in particular,  $F$  is continuous at  $x \in \partial K_1$  when  $x(t) = 0$  on a set of positive Lebesgue measure).

Conversely, if  $x \in \partial K_1$  and  $I(x) < +\infty$ , then one can prove that  $F$  is *not* continuous at  $x$ . The argument goes as follows. By using the monotone convergence theorem, one shows that  $F(x) = I(x)^{-1}$ , where  $I(x)$  is positive. On the other hand, since  $x \in \partial K_1$ ,  $x$  vanishes at some point  $\tau$ , we can find a sequence  $\langle x_n \rangle$  in  $K_1$  that converges to  $x$  in the sup norm and which has the property that each  $x_n$  vanishes on

an interval containing  $\tau$ . But, by our previous remarks, this implies that  $F(x_n) = 0$ , so  $F(x_n)$  does not approach  $F(x)$  as  $n \rightarrow \infty$ . Thus  $F$  is not continuous at  $x$ .

In the above, we have demonstrated that the conditions we have given for continuity of the extended map are quite weak, holding in many contexts. We have also given examples in which the conditions fail and the extended map is not continuous. We have also given an alternative to some of our conditions.

#### 4. Automatic continuity and the standard positive cone

In this section, we show that continuity of the original map  $f$  on  $\overset{\circ}{K}_1$  holds implicitly in some contexts, and so is not required as an explicit assumption. In particular, this is true for certain classes of maps, defined on the interior of the standard positive cone in  $\mathbb{R}^N$ , that arise in the study of discrete event systems.

First, we work in a more general setting. Recall the definition of comparability.

**DEFINITION 4.1** (comparability). Let  $v, y$  denote elements of a cone  $K$ . We say that ‘ $v$  and  $y$  are comparable’ if there exist reals  $\alpha > 0$  and  $\beta > 0$  with  $\alpha y \leq_K v \leq_K \beta y$ . The notion of comparability divides  $K$  into disjoint equivalence classes called the ‘components’ or ‘parts’ of  $K$ ; if  $v \in K - \{0\}$ , we let  $K(v)$  denote the set of points  $u \in K$  that are comparable with  $v$ .

Recall also the definition of the (Thompson) part metric.

**DEFINITION 4.2** (Thompson’s part metric). Let  $v, y$  denote comparable elements of a cone  $K$ . We define the positive real quantity,

$$M(v/y) := \inf\{\beta > 0 : v \leq \beta y\}.$$

For  $v, y$  comparable, we then define

$$\bar{d}(v, y) := \max\{\log M(v/y), \log M(y/v)\}.$$

It is straightforward to prove that if  $K(v)$  is a part of  $K$ , then the restriction of  $\bar{d}$  to  $K(v)$  is a metric, called ‘Thompson’s (part) metric on  $K(v)$ ’. By defining  $\bar{d}(v, y) := +\infty$  if  $v, y$  are non-comparable elements of  $K$ , we extend  $\bar{d}$  to the whole of  $K$  (note that only the restrictions of  $\bar{d}$  to each part are actually metrics).

**REMARK 4.3** (automatic continuity). Suppose that  $K_1$  is a normal cone with non-empty interior in a Banach space  $X_1$  and that  $K_2$  is a normal cone in a Banach space  $X_2$ . Let  $\bar{d}_1$  denote Thompson’s metric on  $\overset{\circ}{K}_1$ . It is known that  $\bar{d}_1$  gives the same topology on  $\overset{\circ}{K}_1$  as the topology induced by the norm on  $X_1$ . (See proposition 1.1 in [8] for a proof and references to the literature. See also [11].) If  $f : \overset{\circ}{K}_1 \rightarrow K_2$  is homogeneous of degree 1 and order preserving, then there exists  $v \in K_2$  such that  $f(x)$  is comparable to  $v$  in  $K_2$  for all  $x \in \overset{\circ}{K}_1$  (i.e. the interior maps to a single part). If  $K_2(v)$  is the part corresponding to  $v$ , i.e. the set of elements in  $K_2$  comparable to  $v$ , and  $\bar{d}_2$  is the (Thompson) part metric on  $K_2(v)$ , then

$$\bar{d}_2(f(x_1), f(x_2)) \leq \bar{d}_1(x_1, x_2)$$

for all  $x_1, x_2 \in \overset{\circ}{K}_1$ , i.e.  $f$  is Thompson non-expanding on  $\overset{\circ}{K}_1$ . It follows that  $f$  is automatically continuous as a map from  $\overset{\circ}{K}_1$  (with the norm topology) to  $K_2$

(with the norm topology). Thus sometimes we do not need to assume explicitly that  $f : \mathring{K}_1 \rightarrow K_2$  is continuous.

EXAMPLE 4.4 (monotone homogeneous maps on the positive cone). Let

$$K_1 = K_2 = K := \mathbb{R}_+^N,$$

the standard positive cone in  $\mathbb{R}^N$ . Maps  $f : \mathring{K} \rightarrow \mathring{K}$  that are order preserving and homogeneous of degree 1 are non-expanding with respect to the Thompson metric on  $\mathring{K}$ . (In fact, if homogeneity holds, then being order preserving is equivalent to being non-expanding. See [3].) Hence such maps are continuous on  $\mathring{K}$ .

REMARK 4.5 (topical maps and additive homogeneity). The maps of example 4.4 are sometimes called ‘topical’ in the literature and are of interest for certain classes of discrete event systems. They may be viewed as the image under the bijection (component-wise exponentiation)  $\exp : \mathbb{R}^N \rightarrow (\mathring{\mathbb{R}}_+^N) = \mathring{K}$  of maps  $g : \mathbb{R}^N \rightarrow \mathbb{R}^N$  that are *additively* homogeneous and order preserving and, consequently, non-expanding in the supremum ( $\ell_\infty$ ) norm. To each such map  $g$  there corresponds a map  $f : \mathring{K} \rightarrow \mathring{K}$  with  $f(x) = (\exp \circ g \circ \log)(x)$ .

With suitable modifications of the proofs to take advantage of full, rather than weak, homogeneity, our results imply the following corollary. (This result was proved previously, by more direct means, in [1].)

COROLLARY 4.6 (continuous extension of maps on the positive cone). *All homogeneous order-preserving (and hence Thompson non-expanding and continuous) maps  $f : \mathring{K} \rightarrow \mathring{K}$  with  $K = \mathbb{R}_+^N$  have an extension  $F : K \rightarrow K$  that is order preserving, homogeneous and continuous on the whole of  $K$ .*

REMARK 4.7. We can also show (see [1]) that the extended map  $F$  is Thompson non-expanding, taking all parts of  $K$  to parts.

COROLLARY 4.8 (existence of eigenvectors). *Let  $f : \mathring{K} \rightarrow \mathring{K}$ , with  $K = \mathbb{R}_+^N$ , be an order-preserving homogeneous map with the order-preserving homogeneous (and continuous) extension  $F : K \rightarrow K$ . Then  $F$  has at least one eigenvector in  $K - \{0\}$ .*

*Proof.* Let  $\Pi \subset K$  denote the intersection of the positive cone with the surface of the ( $\ell_2$ ) unit hyper-sphere,

$$\Pi := \{x \in K : \|x\|_2 = 1\},$$

and let  $\pi$  denote the projection (normalization),

$$\pi : K - \{0\} \rightarrow \Pi, \quad x \mapsto \frac{x}{\|x\|_2}.$$

We have seen that  $f$  has a continuous extension  $F$  to the whole cone  $K$ . If  $F(x) = 0$  for some  $x \in \Pi$  then, by definition,  $F$  has an eigenvector with eigenvalue 0 and we are done. (This happens, for example, in the case  $f : \mathring{\mathbb{R}}_+^2 \rightarrow \mathring{\mathbb{R}}_+^2$  with  $f(x_1, x_2) := \sqrt{x_1 x_2}(1, 1)$ , where all  $x \in \partial\Pi$  are mapped to the vertex 0.) Suppose, on the other hand, that  $F(x) \neq 0$  for all  $x \in \Pi$ , then the projected map  $\pi \circ F : \Pi \rightarrow \Pi$  is well defined and continuous on  $\Pi$ . Further,  $\Pi$  is homeomorphic

to a compact convex set. Hence, by Brouwer's fixed-point theorem,  $\pi \circ F$  has at least one fixed point in  $\Pi$ . By homogeneity, it follows that  $F$  itself must have at least one eigenvector in  $K - \{0\}$ .  $\square$

In fact, using a more sophisticated argument given below, one can prove the following stronger result.

LEMMA 4.9 (eigenvector with non-zero eigenvalue). *Let  $C$  be a closed cone with non-empty interior in a finite-dimensional Banach space  $X$ . Let  $F : C \rightarrow C$  be a continuous map that is order preserving, homogeneous of degree one and maps the interior of  $C$  into itself. Then  $F$  has an eigenvector in  $C - \{0\}$  with non-zero eigenvalue.*

REMARK 4.10. The result follows from theorem 2.1 in [6]. Let  $C$  be a closed cone in a Banach space  $X$  and let  $g : C \rightarrow C$  be a continuous compact map that is order preserving and homogeneous of degree one. Assume that there exists  $u \in C$  such that  $\{g^m(u) : m \geq 1\}$  is unbounded in norm. Then there exists  $x \in C$  with  $\|x\| = 1$  and  $t \geq 1$  such that  $g(x) = tx$ . The assumption of compactness can be weakened.

*Proof of lemma 4.9.* If  $u \in \overset{\circ}{C}$ , there exists  $c > 0$  such that  $F(u) \geq cu$  (since  $F(u) \in \overset{\circ}{C}$ ). Take  $a > 1$  and define  $g(x) := (a/c)F(x)$ , so that  $g(u) \geq au$  and  $\{g^m(u) : m \geq 1\}$  is unbounded in norm. It follows from the theorem mentioned above (in remark 4.10) that there exists  $x \in C$  with  $\|x\| = 1$  and  $t \geq 1$  such that  $g(x) = tx$ , so that  $F(x) = (tc/a)x$ . A simple limiting and compactness argument, letting  $a \rightarrow 1$ , shows that  $F$  has an eigenvector with eigenvalue not less than  $c$ . The above argument actually shows that if we define

$$c := \sup\{k : F(x) \geq kx \text{ for some } x \in C \text{ with } \|x\| = 1\},$$

then there exists an eigenvector of  $F$  with eigenvalue  $c$ .  $\square$

DEFINITION 4.11 (cycle-time vector). From the viewpoint of applications, a natural question is whether the map  $f : \overset{\circ}{K} \rightarrow \overset{\circ}{K}$  has a *cycle-time vector*, defined formally by

$$\chi(f) := \lim_{k \rightarrow \infty} (f^k(x))^{1/k}$$

(where  $x^{1/k}$  indicates the component-wise  $k$ th root of  $x$ ). If this limit exists for some  $x \in \overset{\circ}{K}$ , then it follows, from the fact that  $f$  is non-expanding, that it exists for all  $y \in \overset{\circ}{K}$  (for  $K = \mathbb{R}_+^N$ ) and takes the same value everywhere. Thus the cycle time is naturally regarded as a property of the map itself.

Existence of an eigenvector  $x \in \overset{\circ}{K}$  in the interior with, say,  $f(x) = \lambda x$  for some  $\lambda > 0$ , implies directly the existence of the cycle time (with  $\chi(f) = \lambda \mathbf{1}$ ). Thus our above result establishes the following corollary.

COROLLARY 4.12 (no cycle-time implies an eigenvector on the boundary). *If  $\chi(f)$  does not exist, for  $f : \overset{\circ}{K} \rightarrow \overset{\circ}{K}$  homogeneous and order preserving on the positive cone  $K$ , then the extended map  $F$  has at least one eigenvector in  $\partial K - \{0\}$  with non-zero eigenvalue (and there are no eigenvectors in  $\overset{\circ}{K}$ ).*

The cycle-time vector is known to exist for certain classes of maps in general dimension  $N$ . Specifically, a nonlinear hierarchy of such maps may be built from simple maps by closure under a finite set of operations (see [4]). The cycle-time also exists for all order-preserving homogeneous maps with  $N = 1, 2$ .

However,  $\chi(f)$  need not exist in general for  $N \geq 3$ , as illustrated by a family of maps introduced in [5]. Consider a sequence of reals  $\langle a^k \in [0, 1] : k \geq 1 \rangle$  and let

$$\sigma^k := \sum_{j=1}^k a^j$$

with  $\sigma^0 := 0$ . Then there exists a homogeneous order-preserving map  $f : \overset{\circ}{K} \rightarrow \overset{\circ}{K}$  with  $K := \mathbb{R}_+^3$ , such that

$$f^k(1, 1, 1) = (1, \exp(\sigma^k), \exp(k)) \quad \text{for all } k \geq 0.$$

For suitable choices of the sequence  $\langle a^k \rangle$ , we can arrange that the sequence  $\langle \sigma^k/k \rangle$  does not converge and, hence, that  $\chi(f)$  does not exist.

The construction of a particular family of such maps, for which there is no cycle time, reveals that  $\pi \circ F$  fixes a continuum of points on one edge of  $\overset{\circ}{K}$ . It would be interesting to have a characterization of the possible fixed-point sets for a general projected map.

### 5. Upper semi-continuity

In this section, we return to the setting of theorem 2.13, in which the map  $f : \overset{\circ}{K}_1 \rightarrow K_2$  has a natural extension  $F : K_1 \rightarrow K_2$  that is not necessarily continuous. In this situation, we define a natural multi-valued extension  $\Phi$  and give conditions under which this extension is upper semi-continuous. We then examine the structure of the image set  $\Phi(x)$  of a single point  $x \in K_1$  under this multi-valued map.

Let  $K_1$  be a closed cone with non-empty interior  $\overset{\circ}{K}_1$  in a Hausdorff TVS  $X_1$  and let  $K_2$  be a closed cone in a TVS  $X_2$ . Let  $f : \overset{\circ}{K}_1 \rightarrow K_2$  be continuous and order preserving, and assume condition A: if  $\langle x_k \in \overset{\circ}{K}_1 : k \geq 1 \rangle$  is any sequence such that  $x_{k+1} \leq x_k$  for all  $k \geq 1$ , then the sequence  $\langle f(x_j) : j \geq 1 \rangle$  converges to a point  $y \in K_2$ .

If condition A holds, we have seen that we can define a map  $F : K_1 \rightarrow K_2$  in a natural way, by  $F(x) := y$ , where  $x \in K_1$ ,  $\langle x_n : n \geq 1 \rangle$  is any ‘allowable’ sequence in  $\overset{\circ}{K}_1$  (i.e.  $x_n \gg x$  for all  $n$ ) such that  $\lim_{n \rightarrow \infty} x_n = x$ , and where  $y := \lim_{n \rightarrow \infty} f(x_n)$ . (We saw that this limit necessarily exists and is independent of the particular sequence  $\langle x_n \rangle$ .) The map  $F$ , thus defined, is order preserving and extends  $f$ . Further, we saw that if  $x \in K_1$  and  $\langle x_n \rangle$  is any sequence in  $K_1$  such that  $x_n \gg x$  for all  $n$  and  $\lim_{n \rightarrow \infty} x_n = x$ , then  $\lim_{n \rightarrow \infty} F(x_n) = F(x)$ . However, for a general sequence  $x_n \rightarrow x$ , lack of sequential continuity means that  $\lim_{n \rightarrow \infty} F(x_n)$  need not take the value  $F(x)$ .

DEFINITION 5.1 (multi-valued extension  $\Phi$ ). Assuming condition A and that  $f$  is continuous and order preserving, we now define a natural multi-valued version  $\Phi$  of

the extended map  $F$  by

$$\Phi(x) := \bigcap_{V \in \mathcal{N}(x)} \overline{f(V \cap \overset{\circ}{K}_1)}, \quad (5.1)$$

where  $\mathcal{N}(x)$  denotes the collection of open neighbourhoods  $V \subset X_1$  that contain  $x$ .

We now show that at every point  $x \in K_1$ , our single-valued extension  $F$  takes a maximal value in the set  $\Phi(x)$ .

LEMMA 5.2 (the single-valued extension is maximal). *Let  $f$  be continuous and order preserving and assume condition A. Let  $\Phi(x)$  be defined as in 5.1. Then  $F(x) \in \Phi(x)$  for all  $x \in K_1$ . Further, if  $y \in \Phi(x)$ , then  $y \leq F(x)$ .*

*Proof.* Given  $x \in K_1$ , let  $\langle x_n : n \geq 1 \rangle$  be a sequence in  $\overset{\circ}{K}_1$  with  $x_n \gg x$  for all  $n$  and  $\lim_{n \rightarrow \infty} x_n = x$ . Given any open neighbourhood  $V$  of  $x$ , we have  $x_n \in V$  for all  $n \geq n_V$ , say.

It follows that

$$F(x) = \lim_{n \rightarrow \infty} f(x_n) \in \overline{f(V \cap \overset{\circ}{K}_1)}.$$

Since  $V$  was arbitrary, we have

$$F(x) \in \bigcap_{V \in \mathcal{N}(x)} \overline{f(V \cap \overset{\circ}{K}_1)}.$$

Suppose now that

$$y \in \bigcap_{V \in \mathcal{N}(x)} \overline{f(V \cap \overset{\circ}{K}_1)}.$$

For a given  $n$ , there exists an open neighbourhood  $V$  of  $x$  such that  $x_n - V \in \overset{\circ}{K}_1$  (because  $x_n - x \in \overset{\circ}{K}_1$ ). It follows that  $f(x_n) \geq f(z)$  for all  $z \in V \cap \overset{\circ}{K}_1$ . So, since  $K_2$  is closed,

$$f(x_n) \geq \zeta \quad \text{for all } \zeta \in \overline{f(V \cap \overset{\circ}{K}_1)}.$$

This implies that  $f(x_n) \geq y$ . Since  $n \geq 1$  was arbitrary and  $K_2$  is closed, we conclude that  $F(x) = \lim_{n \rightarrow \infty} f(x_n) \geq y$ .  $\square$

DEFINITION 5.3 (upper semi-continuity). Given a map  $\Phi$  from a topological space  $X$  to  $\mathcal{P}(Y)$ , the power set of a topological space  $Y$ , we say that ‘ $\Phi$  is upper semi-continuous at  $x \in X$ ’ if, for every open neighbourhood  $W$  of  $\Phi(x)$ , there exists an open neighbourhood  $V$  of  $x$  with  $\Phi(V) \subset W$ . We say that ‘ $\Phi$  is upper semi-continuous’ if it is upper semi-continuous at  $x$  for all  $x \in X$ . (Here,  $\Phi(V) := \bigcup_{x \in V} \Phi(x)$ .)

In other words, the ‘multi-valued’ map  $\Phi$  is ‘upper semi-continuous’ if and only if it is continuous when viewed as a map  $\Phi : X \rightarrow \mathcal{P}(Y)$ .

Under an additional compactness assumption, we deduce upper semi-continuity of  $\Phi$ .



**THEOREM 5.4** (compactness and upper semi-continuity). *Assume that  $f$  is continuous and order preserving and that condition A holds. For each  $x \in \partial K_1$ , assume that there exists an open neighbourhood  $V$  of  $x$  such that*

$$\overline{f(V \cap \overset{\circ}{K}_1)}$$

*is compact. Let  $\Phi(x)$  be defined as in 5.1. Then  $\Phi$  is upper semi-continuous.*

*Proof.* By virtue of lemma 5.2, it suffices to prove upper semi-continuity at  $x_*$  for  $x_* \in \partial K_1$ . Let  $V_* \subset X_1$  be an open neighbourhood of  $x_*$  such that

$$M := \overline{f(V_* \cap \overset{\circ}{K}_1)}$$

is compact. If  $\mathcal{O} \subset X_2$  is an open neighbourhood of  $\Phi(x_*)$ , we need to prove that there exists  $V \in \mathcal{N}(x_*)$ ,  $V \subset V_*$ , such that  $\Phi(x) \subset \mathcal{O}$  for all  $x \in V \cap K_1$ . Since

$$\Phi(x) \subset \overline{f(V \cap \overset{\circ}{K}_1)}$$

whenever  $x \in K_1 \cap V$  and  $V$  is open, it suffices to show that there exists  $V \in \mathcal{N}(x_*)$ ,  $V \subset V_*$ , with

$$\overline{f(V \cap \overset{\circ}{K}_1)} \subset \mathcal{O}.$$

If not, then for every  $V \in \mathcal{N}(x_*)$ ,  $V \subset V_*$ , we have

$$M_V := \overline{f(V \cap \overset{\circ}{K}_1)} \cap (\mathcal{O}' \cap M) \neq \emptyset,$$

where  $\mathcal{O}'$  denotes the complement,  $(X_2 - \mathcal{O})$ , of  $\mathcal{O}$ . Now,  $M_V$  is a compact non-empty subset of the compact Hausdorff space  $\mathcal{O}' \cap M$  for  $V \in \mathcal{N}(x_*)$ ,  $V \subset V_*$ . If  $n$  is an integer and  $V_j \in \mathcal{N}(x_*)$ ,  $V_j \subset V_*$ , for  $1 \leq j \leq n$ , then

$$\bigcap_{j=1}^n M_{V_j} \neq \emptyset,$$

because  $\bigcap_{j=1}^n V_j =: W \in \mathcal{N}(x_*)$ ,  $W \subset V_*$  and  $M_W \subset \bigcap_{j=1}^n M_{V_j} \neq \emptyset$ .

It follows that

$$\bigcap_{V \in \mathcal{N}(x_*), V \subset V_*} M_V \neq \emptyset.$$

Since

$$\bigcap_{V \in \mathcal{N}(x_*), V \subset V_*} M_V \subset \bigcap_{V \in \mathcal{N}(x_*)} \overline{f(V \cap \overset{\circ}{K}_1)} =: \Phi(x) \subset \mathcal{O}$$

and

$$\bigcap_{V \in \mathcal{N}(x_*), V \subset V_*} M_V \subset (\mathcal{O}' \cap M),$$

we have a contradiction and thus deduce upper semi-continuity at  $x_*$ . Since  $x_* \in \partial K_1$  was arbitrary, this completes the proof.  $\square$

**LEMMA 5.5.** *Assume the hypotheses in the statement of theorem 5.4 and assume, in addition, that  $X_1$  is a locally convex topological vector space. Then  $\Phi(x)$  is compact and connected (when viewed as a subset of  $Y$ ) for all  $x \in K_1$ .*

*Proof.* We have already noted that  $\Phi(x)$  is compact and non-empty. For a given  $x \in \partial K_1$ , select  $V_* \in \mathcal{N}(x)$  such that

$$\overline{f(V_* \cap \overset{\circ}{K}_1)}$$

is compact. Let  $\mathcal{H} = \mathcal{H}(x)$  denote the collection of open convex sets  $V \subset X_1$  such that  $V \subset V_*$  and  $x \in V$ . The assumption that  $X_1$  is a locally convex TVS implies that for every  $W \in \mathcal{N}(x)$ , there exists  $V \in \mathcal{H}$  with  $V \subset W$ . Thus we have

$$\Phi(x) = \bigcap_{V \in \mathcal{N}(x)} \overline{f(V \cap \overset{\circ}{K}_1)} = \bigcap_{V \in \mathcal{H}} \overline{f(V \cap \overset{\circ}{K}_1)}.$$

Note that  $V \cap \overset{\circ}{K}_1$  is convex, and hence connected, for  $V \in \mathcal{H}(x)$ . Thus  $f(V \cap \overset{\circ}{K}_1)$ , the continuous image of a connected set, is also connected. Finally,

$$\overline{f(V \cap \overset{\circ}{K}_1)},$$

the closure of a connected set, is connected.

Let

$$M := \overline{f(V_* \cap \overset{\circ}{K}_1)},$$

a compact set. Assume, by way of contradiction, that  $\Phi(x)$  is not connected. Then there exist disjoint relatively open subsets  $\mathcal{O}_j \subset M$ ,  $j = 1, 2$ , with  $\mathcal{O}_j \cap \Phi(x) \neq \emptyset$  and  $\Phi(x) \subset (\mathcal{O}_1 \cup \mathcal{O}_2)$ . If  $V_1, V_2, \dots, V_n$  are in  $\mathcal{H}$ , note that  $\bigcap_{j=1}^n V_j \in \mathcal{H}$ . Thus the same argument used in the proof of theorem 5.4 shows that there exists  $V \in \mathcal{H}$  with

$$\overline{f(V \cap \overset{\circ}{K}_1)} \subset (\mathcal{O}_1 \cup \mathcal{O}_2).$$

Because  $\overline{f(V \cap \overset{\circ}{K}_1)}$  is connected, we must have that

$$\overline{f(V \cap \overset{\circ}{K}_1)} \subset \mathcal{O}_j \quad \text{for } j = 1, 2.$$

Assume, without loss of generality, that

$$\overline{f(V \cap \overset{\circ}{K}_1)} \subset \mathcal{O}_1.$$

But then  $\overline{f(V \cap \overset{\circ}{K}_1)} \cap \mathcal{O}_2 = \emptyset$ , which is a contradiction. Thus  $\Phi(x)$  is connected.  $\square$

## 6. Applications to operator-valued means

REMARK 6.1 (operator-valued means). Theorem 2.13 can be applied to the study of ‘operator-valued means’ (see, for example, [2] and references to the literature in that paper). In the finite-dimensional case, theorem 5.4 is also applicable to operator-valued means, but in the infinite-dimensional case the assumptions are too restrictive. It would be interesting to have a version of theorem 5.4 that could be applied to the study of operator-valued means for linear operators on separable Hilbert spaces.

We now give a map on a normal cone that is continuous, order preserving and homogeneous of order 1 and that has an order-preserving homogeneous extension that is *not* continuous (the cone does not satisfy condition G of theorem 3.10). This

map arises naturally in the study of operator-valued means; the extension problem is a natural one for operator-valued means and there are many other related questions.

In what follows, let  $H$  be a real Hilbert space, with inner product  $\langle x, y \rangle$ . Let  $\mathcal{L}(H)$  denote the bounded linear maps  $L : H \rightarrow H$  and let  $\mathcal{A}(H) \subset \mathcal{L}(H)$  denote the *self-adjoint* bounded linear maps  $A : H \rightarrow H$ . Note that  $\mathcal{A}(H)$  forms a Banach space when equipped with the usual operator norm  $\|A\| := \sup\{\|Ax\| : \|x\| = 1\}$ .

Now, let  $C := \{A \in \mathcal{A}(H) : \langle Ax, x \rangle \geq 0 \text{ for all } x \in H\}$ , the *positive semi-definite* operators in  $\mathcal{A}(H)$ . Since  $\|A\| = \sup\{\langle Ax, x \rangle : \|x\| = 1\}$  for  $A \in C$ , it follows that  $C$  is a normal cone and that  $\overset{\circ}{C} \neq \emptyset$  (the identity operator,  $I \in \overset{\circ}{C}$ ).

EXAMPLE 6.2 (map without continuous extension). With  $C$  defined as above, let  $K_1 := C \times C$ , a cone in the Banach space  $\mathcal{A}(H) \times \mathcal{A}(H)$ , and let  $K_2 := C$ , a cone in the Banach space  $\mathcal{A}(H)$ . Define the map  $f : \overset{\circ}{C} \times \overset{\circ}{C} = \overset{\circ}{K}_1 \rightarrow \overset{\circ}{K}_2 = \overset{\circ}{C}$  to be the ‘harmonic mean’,

$$f(A, B) := (A^{-1} + B^{-1})^{-1}. \tag{6.1}$$

We claim that the map  $f$  is order preserving, homogeneous of degree 1 and continuous on  $\overset{\circ}{K}_1$ , but that  $f$  need *not* have a continuous extension to  $\partial K_1$ . In particular, for the case where  $C$  is the cone of positive semi-definite symmetric real  $n \times n$  matrices, we claim that  $f$  may not be extended continuously even in the case  $n = 2$ .

It is known that  $A \rightarrow A^{-1}$  is order reversing on  $\overset{\circ}{C}$ . Thus we have that  $A \leq_C B$  implies  $B^{-1} \leq_C A^{-1}$ , where  $\leq_C$  is the standard ordering induced by  $C$ . This means that  $\langle Ax, x \rangle \leq \langle Bx, x \rangle$  for all  $x \in H$ . It follows that  $f$  is order preserving and homogeneous of degree 1.

We have, for  $A, B \in \overset{\circ}{C}$ ,

$$A^{-1} + B^{-1} = A^{-1}(A + B)B^{-1} = B^{-1}(A + B)A^{-1},$$

and thus

$$f(A, B) = A(A + B)^{-1}B = B(A + B)^{-1}A. \tag{6.2}$$

If  $A, B \in C$  and  $A + B \in \overset{\circ}{C}$ , then the above formula may be used to extend  $f$  continuously to a relatively open neighbourhood of  $(A, B)$  in  $C \times C$ .

LEMMA 6.3 (discontinuity of the extension). *Now consider  $H = \mathbb{R}^N$  with inner product  $\langle x, y \rangle := \sum_{i=1}^N x_i y_i$ . We may then identify  $\mathcal{A}(H)$  with the  $n \times n$  symmetric real matrices, and  $C$  is the set of positive semi-definite  $n \times n$  symmetric real matrices. If  $A, B \in C$ , then we may define*

$$f(A, B) = \lim_{t \rightarrow 0+} ((A + tI)^{-1} + (B + tI)^{-1})^{-1}. \tag{6.3}$$

If  $A + B$  is invertible, then we have, using 6.2,

$$f(A, B) = A(A + B)^{-1}B = B(A + B)^{-1}A, \tag{6.4}$$

and  $f$  is continuous at  $(A, B) \in K_1 := C \times C$ .

However, if  $A + B$  is not invertible, we claim that  $f$  may not be continuous at  $(A, B)$ , even in the case  $n = 2$ .

*Proof.* Take  $H = \mathbb{R}^2$  with the usual inner product and take

$$A = B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in C.$$

We have

$$\begin{aligned} f(A, B) &= \lim_{t \rightarrow 0^+} (A + tI)(A + B + 2tI)^{-1}(B + tI) \\ &= \lim_{t \rightarrow 0^+} \begin{pmatrix} 1+t & 0 \\ 0 & t \end{pmatrix} \frac{1}{2t(2+2t)} \begin{pmatrix} 2t & 0 \\ 0 & 2+2t \end{pmatrix} \begin{pmatrix} 1+t & 0 \\ 0 & t \end{pmatrix} \\ &= \lim_{t \rightarrow 0^+} \frac{1}{2t(2+2t)} \begin{pmatrix} 2t(1+t)^2 & 0 \\ 0 & t^2(2+2t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Recall that  $f$  is order preserving. Suppose that  $(A_m, B_m) \in \mathring{K}_1$  and  $(A_m, B_m) \rightarrow (A, B)$  and  $f(A_m, B_m) \rightarrow L$ . Since  $(A_m, B_m) \leq (A, B) + \delta(I, I)$  for  $\delta > 0$  and  $m$  sufficiently large, then, for  $m$  large, we have

$$f(A_m, B_m) \leq f(A + \delta I, B + \delta I),$$

and  $L \leq f(A + \delta I, B + \delta I)$  for every  $\delta > 0$ . This implies that

$$0 \leq_C L \leq_C \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows from properties of the ordering  $\leq_C$ , induced by  $C$ , that the only possible form for  $L$  is a matrix

$$L(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } 0 \leq \alpha \leq \frac{1}{2}.$$

We claim that, for every  $\alpha \in [0, \frac{1}{2}]$ , there is a sequence  $(A_m, B_m) \in \mathring{K}_1$  with  $(A_m, B_m) \rightarrow (A, B)$  and  $\lim_{m \rightarrow \infty} f(A_m, B_m) = L(\alpha)$ .

In the above calculation, we exhibited an example with  $\alpha = \frac{1}{2}$ . Therefore, it will suffice to take  $0 \leq \alpha < \frac{1}{2}$ . If we let

$$\tilde{A} = \begin{pmatrix} 1 & \delta_1 \\ \delta_1 & \varepsilon_1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & \delta_2 \\ \delta_2 & \varepsilon_2 \end{pmatrix},$$

then  $\tilde{A} \in \mathring{C}$  if and only if  $\varepsilon_1 > \delta_1^2$ . Similarly,  $\tilde{B} \in \mathring{C}$  if and only if  $\varepsilon_2 > \delta_2^2$ . If  $\varepsilon_j \geq \delta_j^2 > 0$  for  $j = 1, 2$  and  $\delta_1 \neq \delta_2$  (so that  $\delta_1^2 + \delta_2^2 > 2\delta_1\delta_2$ ), then we calculate, using 6.4, that

$$\begin{aligned} f(\tilde{A}, \tilde{B}) &= \frac{1}{2(\varepsilon_1 + \varepsilon_2) - (\delta_1 + \delta_2)^2} \\ &\quad \times \begin{pmatrix} (\varepsilon_1 - \delta_1^2) + (\varepsilon_2 - \delta_2^2) & \delta_1(\varepsilon_2 - \delta_2^2) + \delta_2(\varepsilon_1 - \delta_1^2) \\ \delta_1(\varepsilon_2 - \delta_2^2) + \delta_2(\varepsilon_1 - \delta_1^2) & \varepsilon_1(\varepsilon_2 - \delta_2^2) + \varepsilon_2(\varepsilon_1 - \delta_1^2) \end{pmatrix}. \end{aligned}$$

Notice that

$$\begin{aligned} 2(\varepsilon_1 + \varepsilon_2) - (\delta_1 + \delta_2)^2 &= (\varepsilon_1 - \delta_1^2) - (\varepsilon_2 - \delta_2^2) + (\varepsilon_1 + \varepsilon_2 - 2\delta_1\delta_2) \\ &\geq (\varepsilon_1 - \delta_1^2) + (\varepsilon_2 - \delta_2^2) + (\varepsilon_1 + \varepsilon_2 - \delta_1^2 - \delta_2^2) \\ &= 2(\varepsilon_1 - \delta_1^2) + 2(\varepsilon_2 - \delta_2^2). \end{aligned}$$

It follows that

$$\begin{aligned} \left| \frac{\delta_1(\varepsilon_2 - \delta_2^2) + \delta_2(\varepsilon_1 - \delta_1^2)}{2(\varepsilon_1 + \varepsilon_2) - (\delta_1 + \delta_2)^2} \right| &\leq \frac{\max(|\delta_1|, |\delta_2|)[(\varepsilon_2 - \delta_2^2) + (\varepsilon_1 - \delta_1^2)]}{2[(\varepsilon_1 - \delta_1^2) + (\varepsilon_2 - \delta_2^2)]} \\ &\leq \frac{1}{2} \max(|\delta_1|, |\delta_2|) \end{aligned} \quad (6.5)$$

and

$$\begin{aligned} 0 &\leq \frac{\varepsilon_1(\varepsilon_2 - \delta_2^2) + \varepsilon_2(\varepsilon_1 - \delta_1^2)}{2(\varepsilon_1 + \varepsilon_2) - (\delta_1 + \delta_2)^2} \\ &\leq \frac{\max(\varepsilon_1, \varepsilon_2)[(\varepsilon_2 - \delta_2^2) + (\varepsilon_1 - \delta_1^2)]}{2[(\varepsilon_1 - \delta_1^2) + (\varepsilon_2 - \delta_2^2)]} \\ &\leq \frac{1}{2} \max(\varepsilon_1, \varepsilon_2). \end{aligned} \quad (6.6)$$

Now select constants  $k \geq 1$  and  $M > 1$  and define

$$\delta_1 = \frac{1}{m}, \quad \delta_2 = \frac{M}{m}, \quad \varepsilon_1 = k\delta_1^2 = \frac{k}{m^2} \quad \text{and} \quad \varepsilon_2 = k\delta_2^2 = \frac{kM^2}{m^2},$$

so that

$$\tilde{A} := A_m = \begin{pmatrix} 1 & 1/m \\ 1/m & k/m^2 \end{pmatrix} \rightarrow A$$

and

$$\tilde{B} := B_m = \begin{pmatrix} 1 & M/m \\ M/m & kM^2/m^2 \end{pmatrix} \rightarrow B.$$

If we define

$$f(A_m, B_m) =: \begin{pmatrix} a_m & b_m \\ b_m & c_m \end{pmatrix},$$

then (6.5) and (6.6) imply that  $\lim_{m \rightarrow \infty} b_m = \lim_{m \rightarrow \infty} c_m = 0$ , and our formula for  $f(\tilde{A}, \tilde{B})$  implies that

$$a_m = \frac{(k-1) + (k-1)M^2}{(k-1) + (k-1)M^2 + k + kM^2 - 2M},$$

which is independent of  $m$ . (Note that  $\delta_2 \neq \delta_1$ , because  $M > 1$  and thus

$$k + kM^2 - 2M \geq (M-1)^2 > 0$$

and the denominator of  $a_m$  is always positive for  $k \geq 1$ .) When  $k = 1$ ,  $a_m = 0$ . When  $k \rightarrow \infty$ ,

$$\lim_{k \rightarrow \infty} a_m = \lim_{k \rightarrow \infty} \frac{k + kM^2}{k + kM^2 + k + kM^2} = \frac{1}{2}.$$

Since  $a_m$  is a continuous map of  $k \geq 1$ , we conclude that every value  $\alpha$ ,  $0 \leq \alpha < \frac{1}{2}$ , is taken for some  $k \geq 1$  and, as a consequence, that the extension  $F$  cannot be continuous at  $(A, B) \in \partial K_1$ .  $\square$

REMARK 6.4 (maps that are not order preserving). In the study of operator-valued means, one may also come across the problem of extending naturally maps that are *not* order preserving. For example, let  $C$  denote the cone of  $n \times n$  real self-adjoint positive semi-definite matrices in the vector space of real  $n \times n$  self-adjoint matrices. Let  $K_1 = C \times C$ ,  $K_2 = C$ , and define  $f : \overset{\circ}{K}_1 \rightarrow \overset{\circ}{K}_2$  by

$$f(A, B) := \exp\left(\frac{1}{2} \log(A) + \frac{1}{2} \log(B)\right).$$

Note that  $(A, B) \mapsto \frac{1}{2} \log A + \frac{1}{2} \log(B)$  is order preserving, but  $f$  is not. R. D. Nussbaum (unpublished notes) has shown that  $f$  has a ‘natural’ extension  $F : K_1 \rightarrow K_2$  given by

$$F(A, B) = \lim_{k \rightarrow \infty} \exp\left(\frac{1}{2} \log(A + k^{-1}I) + \frac{1}{2} \log(B + k^{-1}I)\right).$$

(The problem is to show that the above limit exists for all  $(A, B) \in K_1$ .)

## 7. Extension of our results to an intermediate situation

In this section, we extend our results to an ‘intermediate’ situation that arises in some important applications and in which essentially the same arguments that we have used previously will give continuity.

EXAMPLE 7.1 (construction of diffusions on finitely ramified fractals). The paper of Sabot [10], for example, concerns the construction of diffusions on certain types of finitely ramified fractals. The construction is achieved with the aid of so-called ‘reproduction-decimation operators’. One considers the real Banach space  $X$  of  $n \times n$  real symmetric matrices  $A$  such that  $A\mathbf{1} = 0$ , where  $\mathbf{1}$  denotes the column vector all of whose entries are identically 1. In  $X$ , one considers the cone  $K$  of the positive semi-definite elements in  $X$  and a closed cone  $D$ , contained in  $K$ , of so-called ‘Dirichlet forms’. On  $\overset{\circ}{K}$ , the interior of  $K$ , one defines a reproduction-decimation operator  $g : \overset{\circ}{K} \rightarrow \overset{\circ}{K}$ .

One can prove that  $g(\overset{\circ}{K} \cap D) \subset D$ , so that it makes sense to consider  $f := g|_{(\overset{\circ}{K} \cap D)}$  viewed as a map  $f : \overset{\circ}{K} \cap D \rightarrow D$ . The map  $g$  is homogeneous of degree 1 and order preserving with respect to the ordering induced by  $K$ . Thus  $f$  is homogeneous of degree 1 and order preserving. However,  $f$  is not order preserving with respect to the natural ordering induced by  $D$ , but rather with respect to the ordering induced by the larger cone  $K$ .

The important point, from the viewpoint of applications, is to establish the existence and uniqueness of suitably non-degenerate eigenvectors of the reproduction-decimation operator  $f$ , and for this purpose it is useful to know that  $f$  can always be extended to a continuous map  $F : D \rightarrow D$ . In what follows, we present a straightforward generalization of our results to establish that this is indeed the case.

For simplicity, we restrict ourselves to the following situation. Let  $K_1$  and  $K_2$  be closed cones in finite-dimensional Banach spaces  $X_1$  and  $X_2$ , respectively. Suppose

also that the interior,  $\overset{\circ}{K}_1$ , of  $K_1$  is non-empty. Finally, assume that  $f : \overset{\circ}{K}_1 \rightarrow K_2$  is homogeneous of degree one and order preserving.

In what follows, the term ‘partial ordering’ will always refer to the partial orderings  $\leq_1$  and  $\leq_2$ . The particular partial ordering in use will always be evident from the context, so we shall sometimes abuse notation and write  $\leq$  instead of the more explicit  $\leq_i$ .

Given the above assumptions, the results that we present here are very special cases of our more general theorems presented in this paper.

We have already shown that there exists an order-preserving, homogeneous of degree one, map  $F : K_1 \rightarrow K_2$  such that  $F|_{\overset{\circ}{K}_1} = f$ , i.e. the map  $F$  extends  $f$ .

Suppose that the map  $f$  is continuous. If  $x \in K_1$  and  $\langle x_n : n \geq 1 \rangle$  is any sequence such that  $x_n - x \in \overset{\circ}{K}_1$  for all  $n \geq 1$ , then  $F(x) = \lim_{n \rightarrow \infty} f(x_n)$ . If  $\langle u_n : n \geq 1 \rangle$  is a sequence in  $K_1$  such that  $u_n \geq u_{n+1}$  for all  $n$  and if  $\lim_{n \rightarrow \infty} u_n = x$ , then we also have  $\lim_{n \rightarrow \infty} F(u_n) = F(x)$ .

Recall that  $F$  need not be continuous, as we demonstrated above. Recall, however, that if  $K_1$  is polyhedral, then it follows that  $F$  is continuous.

### 7.1. Continuous extension to sub-cones

**THEOREM 7.2** (continuous extension to sub-cones). *Let our assumptions be as above. In addition, assume that there exist closed cones  $D_i \subset K_i$  ( $i = 1, 2$ ) such that  $\overset{\circ}{D}_1 \neq \emptyset$  and  $f(\overset{\circ}{K}_1 \cap D_1) \subset D_2$ . Assume also that  $D_1$  is polyhedral. Then  $F(D_1) \subset D_2$  and  $F|_{D_1}$  is continuous.*

*Proof.* Take  $u \in \overset{\circ}{D}_1 \subset \overset{\circ}{K}_1$ . If  $x \in D_1$ , the remarks above show that

$$F(x) = \lim_{n \rightarrow \infty} f(x + \varepsilon_n u) \in D_2$$

whenever  $\langle \varepsilon_n \rangle$  is a sequence of positive reals converging to zero.

Since we assume that  $D_1$  is polyhedral, there exist continuous linear functionals  $\psi_1, \psi_2, \dots, \psi_N$  on  $X_1$  such that

$$D_1 = \{y \in X : \psi_i(y) \geq 0 \text{ for } 1 \leq i \leq N\}.$$

Take  $\lambda$ ,  $0 < \lambda < 1$ , and let  $\langle x_n : n \geq 1 \rangle$  be any sequence of points in  $D_1$  with  $\lim_{n \rightarrow \infty} x_n = x \in D_1$ . We claim that there exists an integer  $n(\lambda)$  such that  $x_n - \lambda x \in D_1$  for all  $n \geq n(\lambda)$ . The proof is the same as one given earlier in this paper.

If  $\psi_i(x) = 0$ , then  $\psi_i(x_n - \lambda x) = \psi_i(x_n) \geq 0$  for all  $n$ . If  $\psi_i(x) > 0$ , then  $\psi_i(x_n - \lambda x) \rightarrow (1 - \lambda)\psi_i(x) > 0$ , so that  $\psi_i(x_n) > 0$  for all  $n$  sufficiently large. Thus, for all  $n$  sufficiently large ( $n \geq n(\lambda)$ ),  $\psi_i(x_n - \lambda x) \geq 0$  for  $1 \leq i \leq N$ . It follows that  $x_n - \lambda x \in D_1 \subset K_1$  for  $n \geq n(\lambda)$ . Because  $F$  is order preserving (with respect to the partial orders from  $K_1$  and  $K_2$ ), it follows that  $F(x_n) \geq F(\lambda x) = \lambda F(x)$  for  $n \geq n(\lambda)$ .

Now we change the notation slightly. Given a sequence  $\langle x_n : n \geq 1 \rangle$  in  $D_1$  with  $\lim_{n \rightarrow \infty} x_n = x$ , for  $n \geq 1$  we define

$$\lambda_n := \sup\{\lambda \leq 1 : x_k - \lambda x \in D_1 \text{ for all } k \geq n\}.$$

The argument above shows that  $\lim_{n \rightarrow \infty} \lambda_n = 1$  and, of course,  $\lambda_n \leq \lambda_{n+1}$  for all  $n$ .

Similarly, for  $u \in \overset{\circ}{D}_1$ , define

$$\varepsilon_n := \inf\{\varepsilon > 0 : x + \varepsilon u - x_k \in D_1 \text{ for all } k \geq n\}.$$

One checks readily that  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $F$  is order preserving on  $K_1$  and  $\lambda_n x \leq x_n \leq x + \varepsilon_n u$ , we have

$$F(\lambda_n x) = \lambda_n F(x) \leq F(x_n) \leq F(x + \varepsilon_n u).$$

We have that  $\lim_{n \rightarrow \infty} F(\lambda_n x) = F(x) = \lim_{n \rightarrow \infty} F(x + \varepsilon_n u)$ . Because  $K_2$  is normal (recall that any closed cone a finite-dimensional normed space is normal), we conclude that

$$\lim_{n \rightarrow \infty} F(x_n) = F(x) = \lim_{n \rightarrow \infty} F(\lambda_n x) = \lim_{n \rightarrow \infty} F(x + \varepsilon_n u).$$

□

## 7.2. Application to maps on cones of Dirichlet forms

We apply this result to the following situation. Consider the real Hilbert space  $\mathbb{R}^m$  with the usual inner product. Let  $\mathbf{1}$  denote the column vector in  $\mathbb{R}^m$ , all of whose components are identically equal to 1. If  $f, g \in \mathbb{R}^m$ , then let  $f \wedge g$  denote the pointwise minimum of  $f$  and  $g$ , i.e.  $(f \wedge g)_i = f_i \wedge g_i := \min(f_i, g_i)$ . Let  $\mathcal{A}_m$  denote the set of symmetric  $m \times m$  real matrices  $A$  such that  $A\mathbf{1} = 0$  and let  $K_1$  denote the set of positive semi-definite matrices in  $\mathcal{A}_m$ . Of course,  $\mathcal{A}_m$  is a Banach space and  $K_1$  is a closed cone with non-empty interior. We define  $D_1 \subset K_1$  by saying that  $A \in D_1$  if and only if  $A \in K_1$  and

$$\langle A(f \wedge \mathbf{1}), f \wedge \mathbf{1} \rangle \leq \langle Af, f \rangle \quad \text{for all } f \in \mathbb{R}^m.$$

The bilinear forms induced by  $A \in D_1$  are called Dirichlet forms. One can prove that  $A \in D_1$  if and only if there exists  $c_{ij} \geq 0$  for  $1 \leq i < j \leq m$  such that

$$\langle Af, f \rangle = \sum_{i < j} c_{ij} (f_i - f_j)^2.$$

This implies that if  $A = (a_{ij})$ , then  $a_{ij} = -c_{ij}$  for  $1 \leq i < j \leq m$ ,  $a_{ij} = a_{ji}$  and  $a_{ii} = -\sum_{j \neq i} a_{ij}$  for  $1 \leq i \leq m$ . It follows that  $D_1$  is linearly isomorphic to the standard cone in  $\mathbb{R}^{m(m-1)/2}$ , so that  $D_1$  is polyhedral.

We can carry out this construction in  $\mathbb{R}^n$  also, obtaining  $\mathcal{A}_n$  (the set of  $n \times n$  symmetric matrices  $B$  such that  $B\mathbf{1} = 0$ ; a Banach space),  $K_2$  (the set of positive semi-definite matrices in  $\mathcal{A}_n$ ) and  $D_2$  (the corresponding elements in  $K_2$  that give Dirichlet forms). If  $f : \overset{\circ}{K}_1 \rightarrow K_2$  is homogeneous of degree one and order preserving and if  $f(D_1 \cap \overset{\circ}{K}_1) \subset D_2$ , then it follows that  $f$  extends to a continuous, homogeneous of degree one, order-preserving map  $F : D_1 \rightarrow D_2$ .

If  $m \geq n$ , then there are natural ‘decimation’ maps from  $\overset{\circ}{K}_1 \rightarrow \overset{\circ}{K}_2$  that satisfy these conditions, and which thus have a continuous extension  $F : D_1 \rightarrow D_2$ .

In a forthcoming paper, we provide full details of the application to the ‘reproduction-decimation’ operators arising in the construction of diffusions on certain fractals. The above remarks are intended to supplement the main results of the present paper.



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