

EIGENVALUES FOR A CLASS OF HOMOGENEOUS CONE MAPS ARISING FROM MAX-PLUS OPERATORS

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Abstract. We study the nonlinear eigenvalue problem $f(x) = \lambda x$ for a class of maps $f : K \rightarrow K$ which are homogeneous of degree one and order-preserving, where $K \subseteq X$ is a closed convex cone in a Banach space X . Solutions are obtained, in part, using a theory of the “cone spectral radius” which we develop. Principal technical tools are the generalized measure of noncompactness and related degree-theoretic techniques. We apply our results to a class of problems

$$\max_{t \in J(s)} a(s, t)x(t) = \lambda x(s)$$

arising from so-called “max-plus operators,” where we seek a nonnegative eigenfunction $x \in C[0, \mu]$ and eigenvalue λ . Here $J(s) = [\alpha(s), \beta(s)] \subseteq [0, \mu]$ for $s \in [0, \mu]$, with a , α , and β given functions, and the function a nonnegative.

1. Introduction. This paper is concerned with maps $f : K \rightarrow K$, where K is a closed convex cone in a Banach space X . We assume that f is homogeneous of degree one, namely that $f(\theta x) = \theta f(x)$ for every $x \in K$ and every nonnegative real θ , and we seek nontrivial solutions $x \in K \setminus \{0\}$ to the problem

$$f(x) = \lambda x \tag{1.1}$$

for some $\lambda \geq 0$. Often, we assume additionally that f is order-preserving with respect to the order on X induced by the cone K . In the simplest finite-dimensional case we have a linear map $f(x) = Ax$ where A is an $n \times n$ matrix with nonnegative entries, and we seek a solution $x \in \mathbb{R}^n \setminus \{0\}$ to (1.1) with $x \geq 0$. Very broadly, in this paper we wish to generalize the well-known theory of such matrices, and the corresponding theory of positive linear operators in Banach spaces, to a class of nonlinear infinite-dimensional maps. We note that there is an extensive literature concerning eigenvectors and fixed points of linear and nonlinear cone-preserving maps. We refer to the classic paper [24] and the book [23]. See also [4], [32], [33], [40], and [41].

We develop our theory from both an abstract point of view, and also as it applies to a specific class of maps. The abstract theory is contained in Sections 2 and 3

AMS 2000 Subject Classification: 47H07, 47H09, 47H10, 47J10.

Key words and phrases: Cone, max-plus operator, measure of noncompactness, nonlinear eigenvalue problem, spectral radius.

This work was partially supported by NSF Grant DMS-9970319 (JM-P) and by NSF Grant DMS-0070829 (RDN).

of this paper, while in Section 4 we apply these results to a class of maps $f = F : C[0, \mu] \rightarrow C[0, \mu]$ of the form

$$(F(x))(s) = \max_{t \in J(s)} a(s, t)x(t), \quad (1.2)$$

which arise from the so-called “max-plus operators” as described below. The maximization in (1.2) is taken over the compact interval

$$J(s) = [\alpha(s), \beta(s)], \quad (1.3)$$

where $\alpha, \beta : [0, \mu] \rightarrow [0, \mu]$ are given continuous functions satisfying $\alpha(s) \leq \beta(s)$ in $[0, \mu]$. The kernel $a : \mathcal{S} \rightarrow [0, \infty)$ is a given nonnegative continuous function, where we denote

$$\mathcal{S} = \{(s, t) \in [0, \mu] \times [0, \mu] \mid t \in J(s)\}, \quad (1.4)$$

which is a compact set. We have in particular that $F : K \rightarrow K$ where K denotes the cone of nonnegative functions in $X = C[0, \mu]$.

In Section 2 we develop the notion of the cone spectral radius $r = r(f) \geq 0$ for a general homogeneous cone map f , where roughly r^n is the typical growth rate of iterates $f^n(x)$ of a point $x \in K$. Several possible definitions for r are presented, and we provide conditions under which they are equal. In Section 3 we define the cone essential spectral radius $\rho = \rho(f) \geq 0$ for f . The definition of ρ depends not only on the map f , but also on a so-called “generalized measure of noncompactness” which must initially be chosen. We prove in Theorem 3.4, under quite general conditions, that if $\rho < r$ then the problem (1.1) has a nontrivial solution in K with $\lambda = r$. Even if $\rho = r$, we obtain the same conclusion in Corollary 3.11 under dynamical conditions (essentially a compactness condition) on the orbit $\{f^n(e)\}_{n=0}^\infty$ of a particular point $e \in K$.

In Section 4 we apply the theory of Sections 2 and 3 to maps F of the form (1.2). A main result is Theorem 4.1, which asserts the existence of an eigenfunction for F with eigenfunction $\lambda = r(F)$ under certain conditions on α , β , and a . The following theorem gives the flavor of this result.

Theorem 1.1. *Assume that $\alpha, \beta : [0, \mu] \rightarrow [0, \mu]$ are C^1 and monotone increasing, that*

$$\alpha(s) < s < \beta(s) \text{ in } (0, \mu), \quad (1.5)$$

and also that $\alpha'(0) < 1$ and $\alpha(\mu) < \mu$, and that $\beta'(\mu) < 1$ and $\beta(0) > 0$. Also assume the function $a : \mathcal{S} \rightarrow (0, \infty)$ is C^1 and strictly positive. Then there exists $x \in C[0, \mu]$ with $x(s) > 0$ in $[0, \mu]$ such that $F(x) = rx$, where $r = r(F) > 0$ is the cone spectral radius of F .

Let us remark that by a **monotone increasing** function g we mean that $g(s_1) \leq g(s_2)$ whenever $s_1 \leq s_2$, that is, g is nondecreasing. Similarly, the term **monotone decreasing** is used for nonincreasing.

We shall not explicitly prove Theorem 1.1, as this result will be superseded by the more general Theorem 4.1. Key components in the proof of this result are Theorem 4.5 and Corollary 4.9, which provide the value of $\rho = \rho(F)$ and a lower bound for $r = r(F)$ respectively, and from which one concludes that $\rho < r$ under appropriate conditions. Theorem 4.1 also covers cases in which $\rho = r$, where the existence of an eigenfunction follows from Corollary 3.11. On the other hand, Proposition 4.23 provides a class of examples for which no eigenfunction exists,

and this result serves to illustrate to some extent the sharpness of some of the conditions of Theorem 4.1. Propositions 4.12 and 4.13 provide conditions under which eigenfunctions of F are strictly positive or monotone.

A crucial part of the analysis in Section 4 involves sequences s_i of points in $[0, \mu]$ which satisfy the admissibility condition $s_i \in J(s_{i-1})$ over some range of i , say for $1 \leq i \leq n$ for some n . The value of the product

$$a(s_0, s_1)a(s_1, s_2) \cdots a(s_{n-1}, s_n) \tag{1.6}$$

along such sequences, or equivalently of the sum

$$w(s_0, s_1) + w(s_1, s_2) + \cdots + w(s_{n-1}, s_n) \tag{1.7}$$

where $w(s, t) = \log a(s, t)$, plays an important role in many of the proofs. Indeed, the product (1.6) arises when one takes iterates $F^n(x)$ of F , and in particular appears in Theorem 4.3 where a formula for r is given. Of course the sum (1.7) is reminiscent of the type of sums encountered in ergodic theory. More formally, one might consider the set

$$\mathcal{S}_\infty = \{s : \mathbb{Z} \rightarrow [0, \mu] \mid s_i \in J(s_{i-1}) \text{ for every } i \in \mathbb{Z}\}$$

of bi-infinite sequences satisfying the admissibility condition, and define the shift map $\mathcal{J} : \mathcal{S}_\infty \rightarrow \mathcal{S}_\infty$ by

$$\mathcal{J}(s)_i = s_{i+1}, \quad i \in \mathbb{Z}.$$

The map \mathcal{J} is a homeomorphism of \mathcal{S}_∞ endowed with the compact-open topology onto itself, and thus can be viewed as a dynamical system. This provides a natural generalization of a dynamical system generated by an interval map $J : [0, \mu] \rightarrow [0, \mu]$ to one generated by a multi-valued map $J : [0, \mu] \rightarrow 2^{[0, \mu]}$ as we have above. We believe that many of the more subtle properties of our eigenvalue problem, and more generally questions involving the asymptotic behavior of iterates $F^n(x)$ of points $x \in K$, are intimately related to properties of the map \mathcal{J} . This clearly represents an area for further study.

The operator F in (1.2) and eigenvalue problem (1.1) with $f = F$ arise in the study of periodic solutions of a class of differential-delay equations

$$\varepsilon y'(t) = g(y(t), y(t - \tau)), \quad \tau = \tau(y(t)), \tag{1.8}$$

with state-dependent delay. Here $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a given nonlinearity, $\tau : \mathbb{R} \rightarrow [0, \infty)$ is a given delay function which is evaluated at the state $y(t)$, and $\varepsilon > 0$ is a singular perturbation parameter which is taken to be small. As described in [27], following the theory of “limiting profiles” developed in [26], the analysis of periodic solutions of equation (1.8) for small ε leads to the study of the additive eigenvalue problem

$$z(s) + p = \max_{t \in J(s)} (w(s, t) + z(t)). \tag{1.9}$$

In this equation, which is considered in the interval $[0, \mu]$, the quantity $p \in \mathbb{R}$ is unknown, a so-called additive eigenvalue, which along with the unknown function $z : [0, \mu] \rightarrow \mathbb{R}$ is sought. The kernel function $w : \mathcal{S} \rightarrow \mathbb{R}$ is given, along with α and β as above. By letting $(\tilde{F}(z))(s)$ denote the right-hand side of (1.9), we thereby define a nonlinear operator $\tilde{F} : C[0, \mu] \rightarrow C[0, \mu]$ which is sometimes called

a **max-plus operator**. The paper [28], which is a companion to [27], describes very explicitly the general solution to a particular class of problems (1.9).

(We remark that the independent variables t and s in equation (1.9) are not the time t in (1.8), but rather correspond to the vertical axis y . That is, the graph of z depicts the limiting graphs of solutions of (1.8) as $\varepsilon \rightarrow 0$, but with the t and y axes interchanged.)

The problem (1.9) is easily reduced to the framework of our paper by exponentiating. Namely, upon setting $a(s, t) = \exp(w(s, t))$ and $x(s) = \exp(z(s))$ and also $\lambda = e^p$ in (1.9), one arrives at the equation

$$\lambda x(s) = \max_{t \in J(s)} a(s, t)x(t), \quad (1.10)$$

which is simply the eigenvalue problem (1.1) with $f = F$ as in (1.2). In this respect the paper [28], which provides a detailed analysis of some very special systems, complements the present paper, which develops a general theory. We remark that in the system (1.9) as it arises in the delay equation problem, it is sometimes the case that a solution takes on the value $z(s) = -\infty$ at points, which corresponds to the value $x(s) = 0$ in the exponentiated problem. It is of interest to avoid such situations, and indeed we provide conditions under which $x(s) > 0$ holds for every $s \in [0, \mu]$, that is, $x \in \text{int}(K)$. Generally however, $x(s) \geq 0$ in $[0, \mu]$, and so $x \in K$.

Equation (1.9) and thus the operator F in (1.2) have arisen in other contexts for the case where $\alpha(s) = 0$ and $\beta(s) = \mu$ are constant functions over the interval $[0, \mu]$. See [7], [8], [18], and [37]. It is known [7], [37], in this case that F is a continuous compact map on $C[0, \mu]$. However, as we shall see below, compactness fails in general when α and β are not constant; and indeed, this failure of compactness is the motivation for much of the work here.

Discrete finite dimensional versions of (1.9) arise in many applications (see, for example, [2], [9], [11], [12], and [19]) wherein this equation takes the form

$$z_i + p = \max_{j \in J(i)} (w_{ij} + z_j).$$

Here $W = (w_{ij})$ is an $n \times n$ matrix, $z \in \mathbb{R}^n$, and $J(i) \subseteq \{1, 2, \dots, n\}$ is a nonempty subset for every $1 \leq i \leq n$. See also [17] and [21] for algorithms to solve this problem.

Max-plus operators arise quite generally in problems of optimal and stochastic control; see, for example, [13], [14], [15], and [16]. For some general references on max-plus analysis see the book [22].

2. The Cone Spectral Radius. Let X be a Banach space. By a **cone** in X we mean a convex set $K \subseteq X$ such that

$$K \cap (-K) = \{0\}, \quad \lambda K \subseteq K \text{ for every } \lambda \geq 0,$$

both hold. Here $-K = \{-x \mid x \in K\}$ and $\lambda K = \{\lambda x \mid x \in K\}$. By a closed cone we mean a cone which is a closed set. Any cone K induces a partial ordering \leq_K defined to be $x \leq_K y$ if and only if $y - x \in K$. If confusion seems unlikely we shall write \leq instead of \leq_K . If W is a compact Hausdorff space and we set $X = C(W)$, the Banach space of continuous real-valued functions $x : W \rightarrow \mathbb{R}$ with the norm $\|x\| = \max\{|x(s)| \mid s \in W\}$, then the set $K = \{x \in X \mid x(s) \geq 0 \text{ for every } s \in W\}$ of nonnegative functions in X is a closed cone.

Let $K \subseteq X$ be a cone in a Banach space. Then a map $f : D \subseteq X \rightarrow X$ from a subset D of X into X is called **order-preserving** (in the partial ordering \leq induced by K) if $f(x) \leq f(y)$ whenever $x, y \in D$ and $x \leq y$. A map $f : K \rightarrow K$ is called **homogeneous of degree one** if $f(\lambda x) = \lambda f(x)$ for every $x \in K$ and every nonnegative real λ . In this paper we shall be interested in maps $f : K \rightarrow K$ which are continuous and homogeneous of degree one; and we shall usually need to assume that f is order-preserving. We make the following formal definition.

Definition. Let K be a cone in a Banach space and let $f : K \rightarrow K$ be a map. We say that f satisfies **Hypothesis A** if f is continuous and homogeneous of degree one, and also the cone K is closed. We say that f satisfies **Hypothesis B** if f satisfies Hypothesis A, and in addition f is order-preserving in the partial ordering induced by K .

Suppose now that $f : K \rightarrow K$ satisfies Hypothesis A. We want to associate to f a nonnegative real number called the **cone spectral radius** of f , but as we shall see, there is more than one reasonable definition of this quantity. Let f^n denote the composition of f with itself n times. Because f^n is continuous at 0, there exists $\delta = \delta_n > 0$ such that $\|f^n(x)\| \leq 1$ for every $x \in K$ with $\|x\| \leq \delta$. It follows by homogeneity that f^n maps bounded subsets of K to bounded subsets of K , and thus we can define a finite quantity

$$b_n = \sup\{\|f^n(x)\| \mid x \in K \text{ and } \|x\| \leq 1\}. \tag{2.1}$$

The homogeneity of f implies that

$$\|f^n(x)\| \leq b_n \|x\| \text{ for every } x \in K, \tag{2.2}$$

and one sees easily from (2.2) that for all positive integers n and m we have

$$b_{n+m} \leq b_n b_m. \tag{2.3}$$

A well-known calculus lemma asserts that for any sequence of nonnegative real numbers which satisfy the inequalities (2.3) we have that

$$\lim_{n \rightarrow \infty} b_n^{1/n} = \inf_{n \geq 1} b_n^{1/n} < \infty. \tag{2.4}$$

With this, we make the following definition.

Definition. If $f : K \rightarrow K$ satisfies Hypothesis A, then we define **Bonsall’s cone spectral radius** of f to be the quantity $\tilde{r}(f)$ given by

$$\tilde{r}(f) = \lim_{n \rightarrow \infty} b_n^{1/n} = \inf_{n \geq 1} b_n^{1/n}, \tag{2.5}$$

where b_n is as in (2.1).

If X is a Banach space containing a closed cone K , and if $f : X \rightarrow X$ is a bounded linear map such that $f(K) \subseteq K$, then Bonsall [4] introduced what we have called Bonsall’s cone spectral radius under the name “the partial spectral radius of f (as a map of K to K).” Recall that a cone K in a Banach space X is called **total** if X equals the closure of its span $\{ax + by \mid a, b \in \mathbb{R} \text{ and } x, y \in K\}$. One might hope that if K is a closed total cone and $f : X \rightarrow X$ is a bounded linear map with spectral radius r for which $f(K) \subseteq K$, then $\tilde{r}(f) = r$. However, Bonsall [4] has

given a simple example which shows this is false in general. On the other hand, if ρ denotes the essential spectral radius of f (see [29] and [32]) and if $\rho < r$, then it is proved in [32] that $r = \tilde{r}(f)$.

If $f : K \rightarrow K$ satisfies Hypothesis A there is an alternate possible definition of the cone spectral radius. Namely, first denote for any $x \in K$ the quantity

$$\begin{aligned} \mu(x) &= \sup\{\lambda > 0 \mid \sup_{n \geq 1} \lambda^{-n} \|f^n(x)\| = \infty\} \\ &= \inf\{\lambda > 0 \mid \sup_{n \geq 1} \lambda^{-n} \|f^n(x)\| < \infty\}, \end{aligned} \quad (2.6)$$

where the reader easily observes that the above sup and inf are identical, and where we make the convention for the empty set that $\sup \phi = 0$ and $\inf \phi = \infty$. The quantity $\mu(x)$ in a crude sense measures the growth rate of $\|f^n(x)\|$ as $n \rightarrow \infty$. We mention here a third formula

$$\mu(x) = \limsup_{n \rightarrow \infty} \|f^n(x)\|^{1/n} \quad (2.7)$$

for the quantity $\mu(x)$. The equivalence of (2.6) and (2.7) is easily established.

Definition. If $f : K \rightarrow K$ satisfies Hypothesis A, then we define the **cone spectral radius** of f to be the quantity $r(f)$ given by

$$r(f) = \sup_{x \in K} \mu(x), \quad (2.8)$$

where $\mu(x)$ is as in (2.6).

Another way of defining a cone spectral radius is in terms of eigenvalues. This definition is most useful when the map f enjoys an appropriate compactness condition.

Definition. If $f : K \rightarrow K$ satisfies Hypothesis A, then we define the **cone eigenvalue spectral radius** of f to be the quantity $\hat{r}(f)$ given by

$$\hat{r}(f) = \sup\{\lambda \geq 0 \mid \text{there exists } x \in K \setminus \{0\} \text{ with } f(x) = \lambda x\},$$

where $\sup \phi = 0$.

The spectral radii $\tilde{r}(f)$, $r(f)$, and $\hat{r}(f)$ defined above of course depend on the choice of cone K as well as on the map f . However, as we generally work with a fixed cone, in our notation we shall usually suppress the dependence of these quantities on K . We shall write $\tilde{r}_K(f)$, $r_K(f)$, and $\hat{r}_K(f)$ when we need to indicate which cone is taken.

Let us note that the inequalities

$$\hat{r}(f) \leq r(f), \quad \hat{r}(f) \leq \tilde{r}(f), \quad \hat{r}(f) \leq \hat{r}(f^m)^{1/m}, \quad (2.9)$$

always hold. The cone spectral radius will be more convenient for us than Bonsall's cone spectral radius, but if f satisfies Hypothesis B and if K is a so-called normal cone, then we shall see below that $r(f) = \tilde{r}(f)$. Under appropriate compactness conditions, including the finite dimensional case (see [34, Definition 3.2, page 89]), we will show that

$$r(f) = \tilde{r}(f) = \hat{r}(f).$$

Our next proposition lists some elementary properties of the cone spectral radius and Bonsall's cone spectral radius.

Proposition 2.1. *If $f : K \rightarrow K$ satisfies Hypothesis A, then*

$$r(f) \leq \tilde{r}(f) < \infty. \tag{2.10}$$

If m is a positive integer then

$$r(f^m) = r(f)^m, \quad \tilde{r}(f^m) = \tilde{r}(f)^m. \tag{2.11}$$

If $\lambda > r(f)$ and $x \in K$, then $\lim_{n \rightarrow \infty} \lambda^{-n} \|f^n(x)\| = 0$.

Proof. Assuming that $r(f) > 0$, fix $0 < \lambda < r(f)$. Then there exists $x \in K$ with $\mu(x) > \lambda$ by the definition (2.8) of $r(f)$. Thus $\sup_{n \geq 1} \lambda^{-n} \|f^n(x)\| = \infty$ by (2.6), so in particular there exists a subsequence $n_i \rightarrow \infty$ for which

$$\|f^{n_i}(x)\| \geq \lambda^{n_i}, \tag{2.12}$$

and thus from (2.2) and (2.12) we have that

$$b_{n_i} \geq \lambda^{n_i} \|x\|^{-1}. \tag{2.13}$$

Upon taking the n_i^{th} root in (2.13) and passing to the limit we conclude that $\tilde{r}(f) \geq \lambda$. As λ is arbitrary, we conclude that $\tilde{r}(f) \geq r(f)$, as desired. The finiteness of $\tilde{r}(f)$ was noted earlier (2.4), and so this establishes (2.10).

We shall show that the first equation in (2.11) follows from the formula

$$\mu(x) = \max_{0 \leq i \leq m-1} \mu_m(f^i(x))^{1/m}, \tag{2.14}$$

where $\mu_m(x)$ denotes the quantity as in (2.6) or (2.7), but with f^m replacing f , that is, with f^{mn} replacing f^n in (2.6) and (2.7) and so

$$\mu_m(x) = \limsup_{n \rightarrow \infty} \|f^{mn}(x)\|^{1/n}. \tag{2.15}$$

To prove (2.14) we first note that by (2.15)

$$\begin{aligned} \mu_m(f^i(x))^{1/m} &= \limsup_{n \rightarrow \infty} \|f^{mn+i}(x)\|^{1/mn} \\ &= \limsup_{n \rightarrow \infty} \left(\|f^{mn+i}(x)\|^{1/(mn+i)} \right)^{1+i/mn} \\ &= \limsup_{n \rightarrow \infty} \|f^{mn+i}(x)\|^{1/(mn+i)}. \end{aligned} \tag{2.16}$$

Taking the maximum of the quantities $\mu_m(f^i(x))^{1/m}$ over the range $0 \leq i \leq m - 1$ yields the lim sup of the sequence which is the union of the sequences in the right-hand side of (2.16), namely $\mu(x) = \limsup_{k \rightarrow \infty} \|f^k(x)\|^{1/k}$, as one easily sees. This establishes (2.14).

To prove now that (2.14) implies the first equation in (2.11), we first observe the upper bound $\mu_m(f^k(x))^{1/m} \leq r(f^m)^{1/m}$ by (2.8). Applying this bound to the right-hand side of (2.14) and then taking the supremum of the left-hand side over all

$x \in K$ yields $r(f) \leq r(f^m)^{1/m}$. To obtain the opposite inequality we first note that $\mu_m(x)^{1/m} \leq \mu(x) \leq r(f)$ by (2.14) and (2.8). Taking the supremum of $\mu_m(x)^{1/m}$ thus gives $r(f^m)^{1/m} \leq r(f)$, as desired.

The second equation in (2.11) follows directly from (2.1) and (2.5) applied to f^m , which gives

$$\tilde{r}(f^m) = \lim_{n \rightarrow \infty} b_{mn}^{1/n} = \left(\lim_{n \rightarrow \infty} b_{mn}^{1/mn} \right)^m = \tilde{r}(f)^m.$$

The proof of the final sentence in the statement of the proposition is straightforward, and is omitted. \square

A closed cone K in a Banach space X is called **normal** if there exists a constant C such that $\|x\| \leq C\|y\|$ whenever $0 \leq x \leq y$. If K is normal, it is known [41] that there exists an equivalent norm $\|\cdot\|$ on X such that $\|x\| \leq \|y\|$ whenever $0 \leq x \leq y$. It is known that K is normal if X is finite dimensional. Also, the set K of nonnegative functions in $C(W)$ is normal, where W is a compact Hausdorff space.

Theorem 2.2. *If $f : K \rightarrow K$ satisfies Hypothesis B and the cone K is normal, then*

$$r(f) = \tilde{r}(f). \quad (2.17)$$

If in addition $y \in \text{int}(K)$ where $\text{int}(K)$ denotes the interior of K , then

$$r(f) = \tilde{r}(f) = \mu(y). \quad (2.18)$$

Also, if $e \in K$ is such that $e \geq x$ for every $x \in K$ with $\|x\| \leq 1$, then

$$r(f) = \tilde{r}(f) = \mu(e), \quad (2.19)$$

and in fact $\lim_{n \rightarrow \infty} \|f^n(e)\|^{1/n}$ exists in this case.

Remark. It may be that $\text{int}(K) = \phi$ in which case the statement (2.18) is vacuous, and likewise with (2.19) if the element e does not exist. If K is the cone of nonnegative functions in $C(W)$, then we can take e in Theorem 2.2 to be the function identically equal to 1.

Remark. The assumption that f is order-preserving (which is part of Hypothesis B) is essential in Theorem 2.2, as the following example shows. Let X be the space of all bounded bi-infinite sequences $\{x_i\}_{i=-\infty}^{\infty}$ of real numbers for which $\lim_{i \rightarrow \pm\infty} x_i = 0$, endowed with the norm $\|x\| = \sup_{i \in \mathbb{Z}} |x_i|$, and let $K \subseteq X$ be the set of all $x \in X$ for which $x_i \geq 0$ for every $i \in \mathbb{Z}$. Certainly, the cone K is closed and normal. Define $f : K \rightarrow K$ by

$$f(x)_i = |x_0| \|x\|^{-1} x_{i+1}, \quad f(0) = 0,$$

that is, f is a shift followed by a rescaling by a factor $|x_0| \|x\|^{-1}$. Clearly f is continuous and homogeneous of degree one. One can check that

$$f^n(x)_i = |x_0 x_1 \cdots x_{n-1}| \|x\|^{-n} x_{i+n}, \quad \|f^n(x)\| = |x_0 x_1 \cdots x_{n-1}| \|x\|^{-n+1},$$

for $x \neq 0$ and every $n \geq 1$, and it follows from this and the fact that $\lim_{i \rightarrow \infty} x_i = 0$ that we have $\lim_{n \rightarrow \infty} \lambda^{-n} \|f^n(x)\| = 0$ for every $\lambda > 0$. Thus $\mu(x) = 0$ and so

$r(f) = 0$. On the other hand, for any $m \geq 1$ consider the particular element $x^m \in K$ given by

$$x_i^m = \begin{cases} 1, & \text{for } 0 \leq i \leq m - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|x^m\| = 1$ and $\|f^m(x^m)\| = 1$, which implies that $b_m \geq 1$ for the quantity (2.1). In fact, $b_m = 1$ as $\|f^m(x)\| \leq \|x\|$ holds for every x , and so $\tilde{r}(f) = 1$.

Remark. One might also ask whether the normality of K is essential to Theorem 2.2, that is, whether there exists an example of a continuous map $f : K \rightarrow K$, which is homogeneous of degree one and order-preserving, and for which $r(f) < \tilde{r}(f)$. For such an example K cannot be normal. Moreover, f cannot be linear or compact by remarks later in this section. At this point, we do not know whether such an example exists.

Proof of Theorem 2.2. We know from Proposition 2.1 that $r(f) \leq \tilde{r}(f)$, so it suffices to prove that $\tilde{r}(f) \leq r(f)$. Fix $\lambda > r(f)$ and define $p, p_n : K \rightarrow [0, \infty)$ by

$$p(x) = \sup_{n \geq 1} p_n(x), \quad p_n(x) = \lambda^{-n} \|f^n(x)\|,$$

where the choice of λ implies the finiteness of $p(x)$. Then for each $m \geq 1$ the set $W_m \subseteq K$ defined to be

$$W_m = \{x \in K \mid p(x) \leq m\} = \bigcap_{n=1}^{\infty} \{x \in K \mid p_n(x) \leq m\}$$

is closed as each function p_n is continuous. Also, as $p(x) < \infty$ for each $x \in K$ we have that

$$K = \bigcup_{m=1}^{\infty} W_m.$$

As K is a complete metric space, the Baire category theorem implies that for some $m_0 \geq 1$ the set W_{m_0} has nonempty interior in the relative topology on K . That is, there exists $x_0 \in W_{m_0}$ and $\varepsilon > 0$ such that $B_\varepsilon(x_0) \cap K \subseteq W_{m_0}$, where $B_\varepsilon(x_0) = \{x \in X \mid \|x - x_0\| < \varepsilon\}$. This implies that $x_0 + z \in W_{m_0}$ for every $z \in B_\varepsilon(0) \cap K$ with $\|z\| < \varepsilon$, so for every $n \geq 1$ and all such z we obtain

$$\lambda^{-n} \|f^n(x_0 + z)\| \leq m_0. \tag{2.20}$$

Because f is order-preserving, $0 \leq f^n(z) \leq f^n(x_0 + z)$ for $z \in K$, so the inequality (2.20) and the normality of K imply that

$$\lambda^{-n} \|f^n(z)\| \leq Cm_0$$

for every $n \geq 1$ and $z \in B_\varepsilon(0) \cap K$. With b_n defined as in (2.1), it follows that

$$b_n \leq Cm_0 \varepsilon^{-1} \lambda^n,$$

implying that $\tilde{r}(f) \leq \lambda$. Since $\lambda > r(f)$ was arbitrary, we conclude that $\tilde{r}(f) \leq r(f)$, as desired.

Suppose next that $y \in \text{int}(K)$. If $x \in K$ then there exists a constant $\delta > 0$ (depending on x) such that $y - \delta x \in K$, and so $x \leq \delta^{-1}y$. Because f is order-preserving and K is normal we have that

$$\|f^n(x)\| \leq C\delta^{-1}\|f^n(y)\|$$

for every $n \geq 1$, and thus $\mu(x) \leq \mu(y)$. From (2.8) it follows that $r(f) = \mu(y)$, which gives (2.18).

If there exists e as in the statement of the theorem, then $x \leq \|x\|e$ and hence $f^n(x) \leq \|x\|f^n(e)$ for every $x \in K$ and $n \geq 1$. By normality we have $\|f^n(x)\| \leq C\|x\|\|f^n(e)\|$ and we conclude that $\mu(x) \leq \mu(e)$. As before, $r(f) = \mu(e)$, which gives (2.19). Taking in particular $x = f^m(e)$ gives $\|f^{n+m}(e)\| \leq C\|f^n(e)\|\|f^m(e)\|$, and hence $a_{n+m} \leq a_n a_m$ where $a_k = C\|f^k(e)\|$ for every $k \geq 1$. With this the existence of the limit $\lim_{n \rightarrow \infty} \|f^n(e)\|^{1/n} = \lim_{n \rightarrow \infty} a_n^{1/n}$ is established. \square

Remark. If X is a Banach space and $K \subseteq X$, then K is called a **closed wedge** if K is closed and convex, and if $\lambda K \subseteq K$ for every $\lambda \geq 0$. Closed cones in X and closed linear subspaces of X are all closed wedges. If K is a closed wedge in X and if $f : K \rightarrow K$ is continuous and homogeneous of degree one, then one can still define $\tilde{r}(f)$ by equation (2.5) and $r(f)$ by equation (2.8). An examination of the proof of Proposition 2.1 shows that this result remains true when K is a closed wedge.

Remark. Suppose that $K \subseteq X$ is a closed wedge and $f : X \rightarrow X$ is a bounded linear map such that $f(K) \subseteq K$. We claim that $r(f) = \tilde{r}(f)$. Since $r(f) \leq \tilde{r}(f)$, it suffices to take $\lambda > r(f)$ and prove that $\lambda \geq \tilde{r}(f)$. The same argument as in the proof of Theorem 2.2 shows that there exist $x_0 \in K$ and $\varepsilon > 0$, and an integer m_0 , such that (2.20) holds for every $n \geq 1$ and every $z \in B_\varepsilon(0) \cap K$. In particular, taking $z = 0$ in (2.20) gives

$$\lambda^{-n}\|f^n(x_0)\| \leq m_0. \quad (2.21)$$

Using the triangle inequality and the linearity of f^n , we conclude from (2.20) and (2.21) that

$$\lambda^{-n}\|f^n(z)\| \leq 2m_0$$

for every $n \geq 1$ and every $z \in B_\varepsilon(0) \cap K$, and so $\tilde{r}(f) \leq \lambda$ by arguing as in the proof of Theorem 2.2. Thus $\tilde{r}(f) = r(f)$. In particular, this shows that for a bounded linear map f Theorem 2.2 is true without any assumption of normality for K .

Remark. A slight generalization of Theorem 2.2, in which we have two cones $K_1 \subseteq K$, can be given and is sometimes useful. Namely, we assume that f satisfies the conditions of Theorem 2.2 with respect to the cone K as stated. Additionally, we assume that $f(K_1) \subseteq K_1$, where the cone $K_1 \subseteq K$ is closed. Then one concludes (2.17), (2.18), and (2.19), but with $r(f) = r_K(f)$ and $\tilde{r}(f) = \tilde{r}_K(f)$ replaced with $r_{K_1}(f)$ and $\tilde{r}_{K_1}(f)$, with $y \in \text{int}(K_1)$ assumed in (2.18), and where $e \in K_1$ is assumed to satisfy $e \geq_K x$ for every $x \in K_1$ with $\|x\| \leq 1$. The proof of these facts is essentially the same as the proof of Theorem 2.2.

Note that all the assumptions involving order, in particular Hypothesis B, are taken with respect to the order \leq_K induced by the larger cone. The conclusions about growth rates, on the other hand, are made with respect to the smaller cone K_1 . Note also that there is no assumption that f is order-preserving with respect to \leq_{K_1} , and indeed, this provides the motivation for this generalization. Namely, it may happen that while a nonlinear map f is not order-preserving with respect to a

cone K_1 , it is order-preserving with respect to a larger cone K . It may also happen that verifying the order-preserving property with respect to \leq_K is easy, but that checking whether it is order-preserving with respect to \leq_{K_1} is harder.

It is possible to remove both the assumptions in Theorem 2.2 that f is order-preserving and that K is normal, at the expense of assuming a compactness condition, and conclude that $r(f) = \tilde{r}(f)$. Generally, we say that $f : K \rightarrow K$ is a **compact map** if the set $\overline{f(B)}$ is compact whenever the set $B \subseteq K$ is bounded. The approach in the proof below is typically associated with asymptotic fixed point results [5], [6] and point dissipative maps [20].

Theorem 2.3. *If $f : K \rightarrow K$ satisfies Hypothesis A and if for some $m \geq 1$ the map f^m is compact, then (2.17) holds.*

Proof. By (2.11) of Proposition 2.1 it is enough to consider the case $m = 1$, and by (2.10) of that result it is enough to prove that $\tilde{r}(f) \leq r(f)$. Fix any $\lambda > r(f)$. We must prove that $\lambda \geq \tilde{r}(f)$. Letting $g(x) = \lambda^{-1}f(x)$, we see that $r(g) < 1$ hence $\lim_{n \rightarrow \infty} g^n(x) = 0$ for every $x \in K$. Let $B = \{x \in K \mid \|x\| < 1\}$ and denote $Q = \overline{g(B)}$. Then $Q \subseteq K$ is a compact set, and for every $x \in Q$ there exists an integer $n = n(x) \geq 1$ such that $g^{n-1}(x) \in B$ and hence $g^n(x) \in Q$. By continuity, there exists an open neighborhood $U_x \subseteq K$ of x (open in the relative topology on K) such that $g^{n-1}(y) \in B$ and hence $g^n(y) \in Q$ for every $y \in U_x$. By compactness, there exists a finite collection of points $x_i \in Q$, for $1 \leq i \leq k$, such that

$$Q \subseteq \bigcup_{i=1}^k U_{x_i}.$$

Let us define $n_0 = \max_{1 \leq i \leq k} n_i$ where for ease of notation we write $n_i = n(x_i)$. Also set

$$Q_0 = \bigcup_{i=0}^{n_0-1} g^i(Q),$$

which is a compact subset of K . We claim that $g(Q_0) \subseteq Q_0$. Clearly, it is sufficient to prove that $g^{n_0}(Q) \subseteq Q_0$ to establish this fact. Taking any $x \in Q$, denote $y = g^{n_0}(x)$, which is a typical point in $g^{n_0}(Q)$. We have that $x \in U_{x_i}$ for some $1 \leq i \leq k$ and so $g^{n_i}(x) \in Q$. Denoting $z = g^{n_i}(x)$, we have that

$$y = g^{n_0-n_i}(z) \in g^{n_0-n_i}(Q),$$

and as $0 \leq n_0 - n_i \leq n_0 - 1$ we conclude that $y \in Q_0$. This establishes the claim.

Therefore, if $x \in B$ then $g(x) \in Q \subseteq Q_0$ and so $g^i(x) \in Q_0$ for every $i \geq 1$. As Q_0 is compact and hence bounded, we have that $\tilde{r}(g) \leq 1$, or equivalently, that $\tilde{r}(f) \leq \lambda$, as desired. \square

Remark. As is the case with Theorem 2.2, the above result remains true when K is merely a closed wedge instead of a closed cone.

3. Eigenvectors for the Cone Spectral Radius. Suppose that K is a closed cone in a Banach space X and that $f : K \rightarrow K$ is a map which satisfies Hypothesis B. Letting $r = r(f)$, one may ask whether there exists $x \in K \setminus \{0\}$ with $f(x) = rx$. Without some sort of compactness condition on f the answer is negative, even for bounded linear maps. For example, consider the cone K of nonnegative functions in

$C[0, 1]$ and define $f : K \rightarrow K$ by $(f(x))(s) = sx(s)$ for each $x \in K$, where $0 \leq s \leq 1$. One easily checks that $r(f) = 1$ but that $f(x) \neq x$ for every $x \in K \setminus \{0\}$.

The kinds of compactness conditions we shall need are best described in terms of “generalized measures of noncompactness.” Recall (see [33, page 28]) that if ν is a map which assigns to each bounded subset A of a Banach space X a nonnegative, finite number $\nu(A)$, then ν is called a **generalized measure of noncompactness** if ν satisfies the following four conditions:

$$\nu(A) = 0 \text{ if and only if } \overline{A} \text{ is compact;} \quad (3.1)$$

$$\nu(A + B) \leq \nu(A) + \nu(B); \quad (3.2)$$

$$\nu(\overline{\text{co}(A)}) = \nu(A); \text{ and} \quad (3.3)$$

$$\nu(A \cup B) = \max\{\nu(A), \nu(B)\}. \quad (3.4)$$

Here we denote $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$, and $\overline{\text{co}(A)}$ denotes the smallest closed convex set containing A . We mention the books [1] and [3] as references for generalized measures of noncompactness.

Example. If (X, d) is a complete metric space and A is a bounded subset of X , then C. Kuratowski [25] has defined a quantity $\nu(A)$ by

$$\nu(A) = \inf\{\delta > 0 \mid \text{there exist } S_i \subseteq X \text{ for } 1 \leq i \leq k, \\ \text{for some } k \geq 1, \text{ such that } A = \bigcup_{i=1}^k S_i \text{ and } \text{diam}(S_i) \leq \delta\},$$

where $\text{diam}(\cdot)$ denotes the diameter of a set. He proved that if A_n , for $n \geq 1$, is a monotone decreasing sequence of closed bounded nonempty sets, and if $\nu(A_n) \rightarrow 0$ as $n \rightarrow \infty$, then the intersection

$$A_\infty = \bigcap_{n=1}^{\infty} A_n$$

is compact and nonempty. Moreover, for any open set $U \supseteq A_\infty$, there exists an integer $m = m(U)$ such that $A_n \subseteq U$ for every $n \geq m$.

One easily verifies (3.1) and (3.4) for the above example, and G. Darbo [10] has observed that if X is a Banach space then (3.2) and (3.3) are also satisfied. One can also see that if X is a Banach space then for any nonnegative real number λ and any bounded set $A \subseteq X$ one has that

$$\nu(\lambda A) = \lambda \nu(A). \quad (3.5)$$

In general, if a generalized measure of noncompactness ν on a Banach space X satisfies equation (3.5) for all bounded sets A and every $\lambda \geq 0$, we say that ν is a **homogeneous generalized measure of noncompactness**. It is not difficult to show that if ν is a homogeneous generalized measure of noncompactness then there exists a constant $C > 0$ such that

$$\nu(A) \leq C\alpha(A)$$

for every bounded set $A \subseteq X$, where α is the Kuratowski-Darbo generalized measure of noncompactness of the example above. Indeed, for any $\varepsilon > 0$ one can cover A

with a finite number of balls of radius $\alpha(A) + \varepsilon$, and so $\nu(A) \leq C(\alpha(A) + \varepsilon)$ using (3.4), where $C = \nu(B)$ with B the unit ball. We are unfortunately not aware of any result which provides the opposite inequality $\alpha(A) \leq \tilde{C}\nu(A)$ for general ν . If such a result were available then one could conclude that the definition below of the cone essential spectral radius would be independent of the choice of ν .

Example. Let (W, d) be a compact metric space and let $X = C(W)$. If $A \subseteq X$ is bounded and if $\delta > 0$, we define $\omega_\delta(A)$ by

$$\omega_\delta(A) = \sup\{|x(t) - x(s)| \mid x \in A, \text{ with } t, s \in W \text{ satisfying } d(t, s) \leq \delta\}, \quad (3.6)$$

and we define $\omega(A)$ by

$$\omega(A) = \inf_{\delta > 0} \omega_\delta(A) = \lim_{\delta \rightarrow 0^+} \omega_\delta(A). \quad (3.7)$$

If $\alpha(A)$ denotes the Kuratowski-Darbo generalized measure of noncompactness defined in the previous example, then it is a special case of Theorem 1 in [30] that

$$\alpha(A) \leq \omega(A) \leq 2\alpha(A) \quad (3.8)$$

for every bounded set $A \subseteq X$. Equation (3.8) implies that $\omega(A) = 0$ if and only if \bar{A} is compact, which is the Ascoli-Arzelà theorem. We leave to the reader the routine verifications that ω satisfies (3.2), (3.3), (3.4), and (3.5), and so ω is a homogeneous generalized measure of noncompactness. We shall always write ω for this generalized measure of noncompactness.

Quite generally, suppose that $f : K \rightarrow K$ satisfies Hypothesis A and that ν is a homogeneous generalized measure of noncompactness on the underlying Banach space X . We may define a quantity

$$\nu(f) = \inf\{\lambda > 0 \mid \nu(f(A)) \leq \lambda\nu(A) \text{ for every bounded set } A \subseteq K\}, \quad (3.9)$$

where we set $\inf \phi = \infty$. By analogy with (2.1) we may now define quantities

$$c_n = \nu(f^n) \quad (3.10)$$

for $n \geq 1$. One easily checks that $c_{n+m} \leq c_n c_m$ for every $n, m \geq 1$ for which both c_n and c_m are finite. (In contrast to the quantities b_n which are easily seen to be finite, it is not evident that $c_n < \infty$, although this will always be the case for the examples we study.) With this, and with a slight extension of the calculus lemma mentioned in the previous section to deal with the case where some c_n are infinite, we may make the following definition.

Definition. Let $f : K \rightarrow K$ satisfy Hypothesis A and let ν be a homogeneous generalized measure of noncompactness on X . We define the **cone essential spectral radius** of f to be the quantity $\rho(f)$ given by

$$\rho(f) = \lim_{n \rightarrow \infty} c_n^{1/n} = \inf_{n \geq 1} c_n^{1/n}, \quad (3.11)$$

provided $c_n < \infty$ except for finitely many n , where c_n is as in (3.9), (3.10). (If $c_n = \infty$ for infinitely many n then one might define $\rho(f) = \infty$, although as noted this case will not arise in our work.)

As noted above, the quantity $\rho(f)$ would seem to depend on the choice of the generalized measure of noncompactness ν . We shall refrain from explicitly indicating this dependence as typically ν will be fixed throughout our analysis.

We may now state our first main result of this section.

Theorem 3.1. *Let $f : K \rightarrow K$ satisfy Hypothesis B. Denoting $r = r(f)$, suppose there exists a homogeneous generalized measure of noncompactness ν such that $c_m^{1/m} < r$ for some $m \geq 1$, with c_m as in (3.10). Then there exists $x_m \in K$ with $\|x_m\| = 1$ satisfying*

$$f^m(x_m) = r^m x_m. \quad (3.12)$$

In particular, if we have the inequality

$$\rho(f) < r(f) \quad (3.13)$$

between the cone essential spectral radius and the cone spectral radius, then there exists $m_0 \geq 1$ such that for every $m \geq m_0$ there exists x_m as above.

The following lemma is given as Theorem 2.1 of [32]. The reader should also compare Proposition 6 on page 252 of [31], which, if the cone is normal, provides a more general result.

Lemma 3.2 ([32, Theorem 2.1]). *Let $f : K \rightarrow K$ satisfy Hypothesis B. Suppose there exists a homogeneous generalized measure of noncompactness ν and a constant $0 \leq \lambda < 1$ such that $\nu(f(A)) \leq \lambda\nu(A)$ for every bounded set $A \subseteq K$. Assume also that there exists $y \in K$ such that the set $\{\|f^n(y)\| \mid n \geq 1\}$ is unbounded. Then there exists $z \in K$ with $\|z\| = 1$, and $\theta \geq 1$, such that $f(z) = \theta z$.*

Further, if $f(x) \neq x$ for every $x \in K$ with $\|x\| = 1$, and if we denote $B = \{x \in K \mid \|x\| < 1\}$, then $i_K(f, B) = 0$ where $i_K(f, B)$ denotes the fixed point index of $f : B \rightarrow K$.

Remark. The proof of the above lemma is an exercise in the fixed point index for maps in cones. In fact, the final statement of this result, that $i_K(f, B) = 0$, is the crucial step in the proof of the rest of the result. An exposition of the basic properties of the fixed point index can be found in Section 1 of [33].

Proof of Theorem 3.1. First observe that it is sufficient to prove the existence of x_m satisfying (3.12) in the case $m = 1$. Indeed, from the first equation in (2.11) one has that $c_m^{1/m} < r(f)$ if and only if $c_m < r(f^m)$, and so one may argue by replacing f with f^m . We therefore take $m = 1$ below.

Assuming that $c_1 < r$, fix a quantity λ satisfying $c_1 < \lambda < r$ and choose a sequence of numbers λ_k such that $\lambda < \lambda_k < r$ and $\lim_{k \rightarrow \infty} \lambda_k = r$. Consider the map f_k defined as $f_k(x) = \lambda_k^{-1} f(x)$. Then $\nu(f_k(A)) \leq \lambda_k^{-1} c_1 \nu(A)$ for every bounded set $A \subseteq K$ by the definition (3.10) of c_1 . Also, by the definition (2.8) of $r(f)$ there exists an element $y_k \in K$ for which $\lambda_k < \mu(y_k) \leq r$, with μ as in (2.6) for the map f . It follows that for every $k \geq 1$ the sequence

$$\lambda_k^{-n} \|f^n(y_k)\| = \|f_k^n(y_k)\|$$

is unbounded as $n \rightarrow \infty$. Applying Lemma 3.2 to the map f_k , we see that there exist $z_k \in K$ and $\theta_k \geq 1$, with $\|z_k\| = 1$, such that $f_k(z_k) = \theta_k z_k$, that is,

$$f(z_k) = \lambda_k \theta_k z_k. \quad (3.14)$$

We claim that the set $A = \{z_k \mid k \geq 1\}$ has compact closure. To prove this it suffices to show that $\nu(A) = 0$. Since $\lambda_k \theta_k \geq \lambda_k > \lambda$, we have that

$$A = \{(\lambda_k \theta_k)^{-1} f(z_k) \mid k \geq 1\} \subseteq \overline{\text{co}(\lambda^{-1} f(A) \cup \{0\})}$$

and hence

$$\nu(A) \leq \nu(\overline{\text{co}(\lambda^{-1} f(A) \cup \{0\})}) = \lambda^{-1} \nu(f(A)) \leq \lambda^{-1} c_1 \nu(A).$$

As $c_1 < \lambda$, it follows that $\nu(A) = 0$ and that A has compact closure.

We may now take convergent subsequences $z_{k_i} \rightarrow x_1$ and $\theta_{k_i} \rightarrow \theta \geq 1$, where $\|x_1\| = 1$. Recalling that $\lambda_k \rightarrow r$, and passing to the limit in (3.14), we obtain $f(x_1) = \sigma x_1$ for some $\sigma \geq r$. Clearly $\mu(x_1) = \sigma$, and from the definition of r we have $\sigma \leq r$. Thus $\sigma = r$, and we have (3.12), as desired.

Let us finally note from the definition (3.11) of the cone essential spectral radius that (3.13) implies that $c_m^{1/m} < r$ for all large m , and hence the existence of x_m . \square

Corollary 3.3. *Assume that $f : K \rightarrow K$ satisfies the conditions of Theorem 3.1, in particular, that $c_m^{1/m} < r(f)$ for some $m \geq 1$. Then*

$$r(f) = \hat{r}(f^m)^{1/m} = \sup_{n \geq 1} \hat{r}(f^n)^{1/n}$$

holds.

Proof. By (2.9) and (2.11) one has that

$$\sup_{n \geq 1} \hat{r}(f^n)^{1/n} \leq \sup_{n \geq 1} r(f^n)^{1/n} = r(f),$$

and so it is sufficient to prove that $r(f) \leq \hat{r}(f^m)^{1/m}$. But this is immediate from Theorem 3.1, which asserts the existence of an eigenvector for f^m in K with corresponding eigenvalue $r(f)^m$. \square

We say that a Banach space X is a **vector lattice** with respect to the ordering induced by a cone K (or we say that a cone K induces a vector lattice on X) if for every $x, y \in X$ there exists a least upper bound $z \in X$ for x and y , that is, we have $x \leq z$ and $y \leq z$, and also $z \leq u$ for every $u \in X$ for which $x \leq u$ and $y \leq u$. The element z , which one thinks of as the maximum of x and y , is clearly unique, and we denote it by $z = x \vee y$. We also define $x \wedge y = -((-x) \vee (-y))$. Thus $w = x \wedge y$ satisfies $w \leq x$ and $w \leq y$, and w is the maximal such element. We shall say that X is a **topological vector lattice** if it is a vector lattice for which the mapping $(x, y) \rightarrow x \vee y$ from $X \times X \rightarrow X$ is continuous. One easily checks that if X is a topological vector lattice then the associated cone K is closed. The cone K of nonnegative functions in $X = C(W)$, where W is a compact Hausdorff space, is a topological vector lattice. On the other hand, the cone of nonnegative functions in $C^1[0, 1]$ is not a vector lattice as the maximum of two C^1 functions need not be C^1 .

Now suppose that a cone K induces a vector lattice on X . If $f : K \rightarrow K$ then it may happen that

$$f(x \vee y) = f(x) \vee f(y) \text{ for every } x, y \in K. \tag{3.15}$$

Indeed, if equation (3.15) holds (as will be the case for the class of examples studied in the next section), then Theorem 3.1 takes a stronger form. Here we clearly see the importance of verifying the inequality (3.13).

Theorem 3.4. *Let $f : K \rightarrow K$ satisfy Hypothesis B. Suppose also that K induces a vector lattice on X , and that (3.15) holds. Further, assume that ν is a homogeneous generalized measure of noncompactness on X for which the inequality (3.13) holds. Then there exists $y \in K \setminus \{0\}$ for which*

$$f(y) = ry, \tag{3.16}$$

where we denote $r = r(f)$.

Proof. Defining $g(x) = r^{-1}f(x)$, we see by Theorem 3.1 that there exists $x_0 \in K \setminus \{0\}$ and $m \geq 1$ such that $g^m(x_0) = x_0$. Define $y \in K \setminus \{0\}$ by $y = x_0 \vee g(x_0) \vee g^2(x_0) \vee \dots \vee g^{m-1}(x_0)$. One now sees from equation (3.15) that $g(y) = y$, and thus (3.16) holds as desired. \square

Remark. As with Theorem 2.2, one may generalize Theorems 3.1 and 3.4 and Corollary 3.3 to the case of two cones $K_1 \subseteq K$. We assume that $f : K \rightarrow K$ satisfies Hypothesis B as stated, and in Theorem 3.4 that K induces a vector lattice on X . Additionally we assume that $f(K_1) \subseteq K_1$ where $K_1 \subseteq K$ is a closed cone. In the statement of these results the quantities $r(f)$, $\hat{r}(f)$, and $\rho(f)$ are replaced with $r_{K_1}(f)$, $\hat{r}_{K_1}(f)$, and $\rho_{K_1}(f)$, and $c_n = \nu(f^n)$ replaced with the corresponding quantity $\nu(f^n|_{K_1})$ for the restriction of f^n to K_1 . Then the conclusions of these results hold, except that $x_m \in K_1$ in Theorem 3.1. In Theorem 3.4 we have only $y \in K$ and not $y \in K_1$ in the absence of further information, as K_1 is not assumed to generate a lattice.

In making the generalization of Theorem 3.1 we require an appropriate extension of Lemma 3.2, which is easily given following the proof in [32].

If the cone K in Theorem 3.1 or Theorem 3.4 has nonempty interior, it is frequently important to know whether the map f has an eigenvector in the interior of this cone. If such an eigenvector exists then the corresponding eigenvalue is necessarily $r(f)$, as follows from Proposition 3.8 below. We shall be interested in this question for the class of examples in Section 4. However, it is known that, in general, the question of existence of eigenvectors in $\text{int}(K)$ may be quite difficult even for finite dimensional cones. We refer the reader to [34], [35], [36], and [38] for a discussion of this issue and some instructive examples.

The proofs of Theorems 3.1 and 3.4 involve the use of the fixed point index and provide no hints as to how to construct the eigenvector. At the cost of somewhat more restrictive hypotheses, we now present a variant of Theorem 3.4 which can be proved without the use of the fixed point index, and which provides a construction of the eigenvector. We begin by recalling a lemma from [39].

Lemma 3.5 ([39, page 954]). *Let K be a closed normal cone which induces a vector lattice on X , and suppose that $\text{int}(K) \neq \emptyset$. Let $A \subseteq X$ be compact and let $B \subseteq X$ denote the smallest closed set such that $A \subseteq B$ and $x \vee y, x \wedge y \in B$ whenever $x, y \in B$. Then B is compact.*

An easy consequence of the above is the following result.

Lemma 3.6. *Let K be a closed normal cone which induces a vector lattice on X , and suppose that $\text{int}(K) \neq \emptyset$. Let $A \subseteq X$ be compact and let $Q \subseteq X$ denote the*

smallest closed set such that $A \subseteq Q$ and $x \vee y \in Q$ whenever $x, y \in Q$. Then Q is compact.

If further K induces a topological vector lattice on X , and we let

$$D = \{x_1 \vee x_2 \vee \dots \vee x_n \mid x_i \in A \text{ for } 1 \leq i \leq n, \text{ for some } n \geq 1\},$$

then $\overline{D} = Q$. \square

Proof. Clearly $Q \subseteq B$ with B as in Lemma 3.5. By that result B is compact, hence so is Q . With D defined as in the statement of the lemma, we easily see that $\overline{D} \subseteq Q$. Assuming that K induces a topological vector lattice, we see also that $x \vee y \in \overline{D}$ whenever $x, y \in \overline{D}$, and so the definition of Q implies that $\overline{D} = Q$. \square

Remark. The conclusions of Lemmas 3.5 and 3.6 can be false if K is not a normal cone. For example, let $X = W^{1,\infty}(0, 1)$, the space of lipschitz functions $x : [0, 1] \rightarrow \mathbb{R}$ with the norm $\|x\| = \max_{s \in [0,1]} |x(s)| + \text{ess sup}_{s \in [0,1]} |x'(s)|$, and let $K \subseteq X$ be the set of nonnegative functions. As the maximum of two lipschitz functions is lipschitz, one sees that K is a vector lattice. Taking $x_a(s) = s - a$ in $[0, 1]$ and letting 0 denote the zero function in X , one has that $y_a = 0 \vee x_a$, namely $y_a(s) = \max\{0, s - a\}$, does not vary continuously in X with $a \in [0, 1]$ even though x_a does. Indeed, if $0 \leq a_1 < a_2 \leq 1$, then $\|y_{a_1} - y_{a_2}\| \geq \text{ess sup}_{s \in [0,1]} |y'_{a_1}(s) - y'_{a_2}(s)| = 1$. Thus if $A \subseteq X$ denotes the compact set consisting of all x_a for $a \in [0, 1]$ together with the zero function, then the sets B and Q in Lemmas 3.5 and 3.6 contain uncountably many points y_a which are pairwise separated by a distance at least 1. Thus neither B nor Q is compact.

Another interesting point about Lemma 3.5 is that its conclusion can be false if $\text{int}(K) = \phi$. An example is given in [39, page 955] for the case that $X = L^p[0, 1]$ for $1 \leq p < \infty$, and where K is the cone of nonnegative functions in X .

It is useful here to recall some basic notions of dynamical systems. Generally, if $g : X \rightarrow X$ is continuous where X is any metric space, one may consider the **forward orbit**

$$\gamma^+(x) = \{g^n(x) \mid n \geq 0\}$$

of any point $x \in X$. The **omega limit set** of a point x is defined to be the set

$$\omega(x) = \{y \in X \mid \text{there exists } n_i \rightarrow \infty \text{ for which } \lim_{i \rightarrow \infty} g^{n_i}(x) = y\}.$$

It is a well-known and easily proved result that $\omega(x) \subseteq \overline{\gamma^+(x)} \subseteq X$ is a closed set which satisfies $g(\omega(x)) = \omega(x)$. In addition, if the closure $\overline{\gamma^+(x)}$ of the forward orbit of x is compact, then $\omega(x)$ is compact and $\omega(x) \neq \phi$.

In order to make clear the map g in question, we shall write $\gamma_g^+(x)$ and $\omega_g(x)$ for the forward orbit and omega limit set of a point x .

Lemma 3.7. *Let $f : K \rightarrow K$ satisfy Hypothesis A, and let ν be a homogeneous generalized measure of noncompactness on X . Assume the inequality (3.13) holds, and let $g(x) = r^{-1}f(x)$ where $r = r(f)$. Suppose that $A \subseteq K$ is a bounded set for which $g(A) \subseteq A$ and such that \overline{Z} is compact, where $Z = A \setminus g(A)$. Then \overline{A} is compact.*

In particular, if the forward orbit $\gamma_g^+(y)$ of some point $y \in K$ under g is bounded, then $\overline{\gamma_g^+(y)}$ is compact.

Proof. One easily shows by induction that the set $g^{n-1}(A) \setminus g^n(A)$ has compact closure for every $n \geq 1$, and thus $A \setminus g^n(A)$ has compact closure. We may therefore

write $A = g^n(A) \cup Z_n$ where $\nu(Z_n) = 0$. Now taking c_n as in (3.10), one has from (3.11) that $c_m^{1/m} < r$ for some $m \geq 1$. Fix this m and observe from the definition of c_m that $\nu(f^m(A)) \leq c_m \nu(A)$. Then

$$\nu(A) = \nu(g^m(A) \cup Z_m) = \nu(g^m(A)) = r^{-m} \nu(f^m(A)) \leq r^{-m} c_m \nu(A),$$

and as $c_m < r^m$ we conclude that $\nu(A) = 0$. Thus \bar{A} is compact.

The final sentence in the statement of the lemma is proved by taking $A = \gamma_g^+(y)$, as $g(A) \cup \{y\} = A$ for this set. \square

Our next proposition gives a necessary condition for a map to have an eigenvector in the interior of a cone.

Proposition 3.8. *Let $f : K \rightarrow K$ satisfy Hypothesis B, and suppose the cone K is normal. Assuming that $\text{int}(K) \neq \emptyset$, suppose there exists $y \in \text{int}(K)$ and $\lambda \geq 0$ for which $f(y) = \lambda y$. Then $\lambda = r(f)$, and for every $x \in \text{int}(K)$ there exist positive constants a_1, a_2 such that*

$$a_1 r(f)^n \leq \|f^n(x)\| \leq a_2 r(f)^n \tag{3.17}$$

for every $n \geq 1$. If only $x \in K$ then there exists a_2 such that the right-hand inequality in (3.17) holds for every $n \geq 1$.

Proof. Assuming first that both $x, y \in \text{int}(K)$, we see that there exists $\varepsilon > 0$ such that $x - \varepsilon y, y - \varepsilon x \in K$, and so $\varepsilon y \leq x \leq \varepsilon^{-1} y$. Because f^n is order-preserving, we have that

$$\varepsilon \lambda^n y = \varepsilon f^n(y) \leq f^n(x) \leq \varepsilon^{-1} f^n(y) = \varepsilon^{-1} \lambda^n y, \tag{3.18}$$

and it follows that

$$C^{-1} \varepsilon \lambda^n \|y\| \leq \|f^n(x)\| \leq C \varepsilon^{-1} \lambda^n \|y\| \tag{3.19}$$

where C is the constant in the definition of cone normality. Noting that $\|f^n(y)\| = \lambda^n \|y\|$ and so $\mu(y) = \lambda$, we have from (2.18) of Theorem 2.2 that $\lambda = r(f)$. This completes the result, with $a_1 = C^{-1} \varepsilon \|y\|$ and $a_2 = C \varepsilon^{-1} \|y\|$.

If $x \in K$ then one has only that $y - \varepsilon x \in K$ for some $\varepsilon > 0$. This yields the right-hand inequalities in (3.18) and (3.19), and hence the right-hand inequality in (3.17). \square

Lemma 3.9. *Let K be a closed cone which induces a vector lattice on X . Let $Q \subseteq X$ be a compact set such that $x \vee y \in Q$ whenever $x, y \in Q$. Then there exists $z \in Q$ such that $z \geq x$ for every $x \in Q$.*

Proof. Because Q is compact, there exists a countable subset $D \subseteq Q$ which is dense in Q . Denote $D = \{x_n\}_{n=1}^\infty$. For each $n \geq 1$ let $z_n = x_1 \vee x_2 \vee \dots \vee x_n$, so that $z_n \in Q$ and $z_n \geq x_k$ for $1 \leq k \leq n$. Take a subsequence z_{n_i} converging to a point $\lim_{i \rightarrow \infty} z_{n_i} = z \in Q$, and observe that $z \geq x_k$ for every $k \geq 1$ follows from the fact that K is closed. It follows further from the closedness of K that $z \geq x$ for every $x \in \bar{D} = Q$, as desired. \square

Theorem 3.10. *Let $f : K \rightarrow K$ satisfy both Hypothesis B and (3.15), and suppose the cone K is normal and induces a topological vector lattice on X . Assume that $\text{int}(K) \neq \emptyset$ and that there exist $y \in K$ and positive constants a_1 and a_2 such that*

$$a_1 r^n \leq \|f^n(y)\| \leq a_2 r^n \tag{3.20}$$

for every $n \geq 1$, where we denote $r = r(f)$. Finally, assume either the inequality (3.13) holds, or more generally that $r > 0$ and that the closure $\overline{\gamma_g^+(y)}$ of the forward orbit of y under the map $g(x) = r^{-1}f(x)$ is compact. Then there exists $z \in K \setminus \{0\}$ such that $f(z) = rz$ and $z \geq x$ for every $x \in \omega_g(y)$.

Proof. The inequalities (3.20) imply that $a_1 \leq \|x\| \leq a_2$ for every $x \in \gamma_g^+(y)$, so if (3.13) holds then Lemma 3.7 implies that $\overline{\gamma_g^+(y)}$ is compact. Thus $\omega_g(y)$ is compact and nonempty. Now let

$$D = \{x_1 \vee x_2 \vee \dots \vee x_n \mid x_i \in \omega_g(y) \text{ for } 1 \leq i \leq n \text{ for some } n \geq 1\},$$

and set $Q = \overline{D}$. As $g(\omega_g(y)) = \omega_g(y)$, one sees by (3.15) that $g(D) = D$. By Lemma 3.6 the set \overline{D} is compact, and so $g(\overline{D}) = \overline{g(D)}$. Thus $g(Q) = Q$.

The set Q is closed under the operation \vee , and so Lemma 3.9 now implies that there exists an element $z \in Q$ such that $z \geq x$ for every $x \in Q$. Clearly such z is unique. The order-preserving property of g implies that $g(z) \geq g(x)$ for every $x \in Q$, and as $g(Q) = Q$ it follows that $g(z) \in Q$ and that $g(z) \geq x$ for every $x \in Q$. The uniqueness of the maximal element z of Q now implies that $g(z) = z$, that is, $f(z) = rz$. \square

Remark. In most respects Theorem 3.10 is less general than Theorem 3.4. However, the elementary proof, which avoids the use of the fixed point index, may be of interest. Moreover, if one can obtain information about the omega limit set $\omega_g(y)$ in Theorem 3.10, then this result may provide more information than Theorem 3.4.

Corollary 3.11. *Let $X = C(W)$ where W is a compact Hausdorff space, and let $K \subseteq X$ denote the cone of nonnegative functions in X . Assume that $f : K \rightarrow K$ satisfies both Hypothesis B and (3.15). Let $e \in X$ denote the function identically equal to 1 and assume that there exists a positive constant a such that*

$$\|f^n(e)\| \leq ar^n \tag{3.21}$$

for every $n \geq 1$, where we denote $r = r(f)$. Finally, assume either the inequality (3.13) holds, or more generally that $r > 0$ and that $\overline{\gamma_g^+(e)}$ is compact where $g(x) = r^{-1}f(x)$. Then there exists $z \in K \setminus \{0\}$ such that $f(z) = rz$ and $z \geq x$ for every $x \in \omega_g(e)$.

Proof. The result follows directly from Theorem 3.10 once one establishes the existence of a constant $a_1 > 0$ in (3.20), with $y = e$. One has $x \leq e$ for every $x \in K$ with $\|x\| = 1$, and hence $f^n(x) \leq f^n(e)$, which implies that $\|f^n(x)\| \leq \|f^n(e)\|$. Therefore $b_n = \|f^n(e)\|$ for the quantities (2.1). By (2.5) and also by (2.17) of Theorem 2.2 one has that

$$r^n = \tilde{r}(f)^n \leq b_n = \|f^n(e)\|,$$

which provides $a_1 = 1$. \square

Under the hypotheses of Theorem 3.4 or Corollary 3.11 the map f may possess a “dominant eigenvector.” To make this idea precise, we introduce a definition.

Definition. If K is a closed cone, $A \subseteq K$, and $y \in K$, we say that **y dominates A** if there exists $\lambda > 0$ such that $x \leq \lambda y$ for every $x \in A$.

Proposition 3.12. *Let $f : K \rightarrow K$ satisfy both Hypothesis B and (3.15), and suppose the cone K is normal and induces a vector lattice on X . Assume also that $\text{int}(K) \neq \phi$. Let*

$$A = \{x \in K \mid \|x\| = 1 \text{ and } f(x) = rx\}$$

where we denote $r = r(f)$, and assume either the inequality (3.13) holds, or more generally that the set A is compact and nonempty. Then there exists $y \in A$ such that y dominates A .

Proof. We first note that (3.13) implies that A is compact and nonempty. Indeed, $A \neq \phi$ by Theorem 3.4, and A is clearly closed and bounded. As $f(A) = rA$, we have by Lemma 3.7 that A is compact.

Now let $Q \subseteq K$ be the smallest closed set containing A and which is closed under the operation \vee . Then Q is compact by Lemma 3.6. Thus by Lemma 3.9 there exists $z \in Q$ such that $z \geq x$ for every $x \in Q$. We now claim that $f(x) = rx$ for every $x \in Q$, which we see implies that every element of Q is a multiple of some element of A . Indeed, let $R = \{x \in Q \mid f(x) = rx\}$, and observe that $A \subseteq R$, that R is closed, and that $x \vee y \in R$ whenever $x, y \in R$ by (3.15). Thus $R = Q$ from the definition of Q . One easily sees now that the normalized element $y = z\|z\|^{-1}$ belongs to A and dominates A . \square

4. Positive Eigenfunctions for Some Max-Type Operators. As noted in the Introduction, we are interested in finding solutions to equation (1.10) with $x(s) \geq 0$ and with $\lambda \geq 0$. To this end we shall formulate equation (1.10) as a nonlinear eigenvalue problem in a cone, and then apply the theory of the previous two sections.

Standing Hypotheses and Notation. For the remainder of this section we shall denote $X = C[0, \mu]$ where μ is a fixed positive number, and we let $K \subseteq X$ denote the cone of nonnegative functions in X . We let $\alpha, \beta : [0, \mu] \rightarrow [0, \mu]$ denote given continuous functions which satisfy $\alpha(s) \leq \beta(s)$ in $[0, \mu]$. We let $a : \mathcal{S} \rightarrow [0, \infty)$ denote a given nonnegative continuous function where the set $\mathcal{S} \subseteq [0, \mu] \times [0, \mu]$ is defined by (1.4), and we denote

$$A_- = \min_{(s,t) \in \mathcal{S}} a(s,t), \quad A_+ = \max_{(s,t) \in \mathcal{S}} a(s,t).$$

The interval $J(s)$ is as in (1.3), and we denote

$$\mathcal{S}_\delta^L = ([0, \delta] \times [0, \delta]) \cap \mathcal{S}, \quad \mathcal{S}_\delta^R = ([\mu - \delta, \mu] \times [\mu - \delta, \mu]) \cap \mathcal{S}$$

for any $0 < \delta \leq \mu$.

It will also be useful to state several additional hypotheses, to be assumed as needed. In contrast to the above hypotheses which hold throughout this section, the ones below are discretionary in that we do not assume them unless explicitly indicated. The reason for the somewhat curious labeling of these hypotheses will be apparent from the statement of Lemma 4.26.

Definition. We introduce several hypotheses defined as follows.

Hypothesis X. $\alpha(\mu) < \mu$ and $\beta(0) > 0$.

Hypothesis Y. $\alpha(\mu) < \mu$ and $a(0, 0) < r = r(F)$.

Hypothesis Y'. α is lipschitz in $[0, \delta]$ with lipschitz constant c , and a is lipschitz in \mathcal{S}_δ^L .

Hypothesis Z. $\beta(0) > 0$ and $a(\mu, \mu) < r = r(F)$.

Hypothesis Z'. β is lipschitz in $[\mu - \delta, \mu]$ with lipschitz constant c , and a is lipschitz in \mathcal{S}_δ^R .

The quantities δ and c in Hypotheses Y' and Z' will always be indicated when these hypotheses are made.

With the above, let the function $F(x)$ be given by (1.2) for every $x \in X$, and so $F : X \rightarrow X$. We shall shortly show that $F : K \rightarrow K$ is continuous and homogeneous of degree one, and with this there is defined the cone spectral radius $r(F)$ which appears in Hypotheses Y and Z. However, let us first state the following theorem on the map F , which is a principal result of this section.

Theorem 4.1. *Assume that α and β are monotone increasing in $[0, \mu]$ and that (1.5) holds. Also assume the function a is strictly positive in \mathcal{S} and denote*

$$a_+ = \max_{s \in [0, \mu]} a(s, s). \tag{4.1}$$

Finally, if $a(0, 0) = a_+$ then assume both Hypothesis X holds and Hypothesis Y' holds for some $\delta > 0$ and $c < 1$, and if $a(\mu, \mu) = a_+$ then assume both Hypothesis X holds and Hypothesis Z' holds for some $\delta > 0$ and $c < 1$. Then there exists $x \in K$ which is strictly positive in $(0, \mu)$ such that $F(x) = rx$, where $r = r(F)$ satisfies $r > 0$. If also $\alpha(\mu) < \mu$ then $x(\mu) > 0$, and if $\beta(0) > 0$ then $x(0) > 0$.

Another result, similar to the one above, is close to the examples considered in [27] and [28].

Theorem 4.2. *Assume for some $\beta_0 \in (0, \mu]$ that α is monotone increasing in $[0, \beta_0]$ with $\alpha(s) < s$ in $(0, \beta_0]$, that $\beta(s) = \beta_0$ in $[0, \beta_0]$ with $\beta(s) \leq \beta_0$ in $[\beta_0, \mu]$, and that the function a is strictly positive in \mathcal{S} . Denote*

$$a_0 = \max_{s \in [0, \beta_0]} a(s, s),$$

and if $a(0, 0) = a_0$ then assume Hypothesis Y' holds for some $\delta > 0$ and $c < 1$, and if $a(\beta_0, \beta_0) = a_0$ then assume a is lipschitz in $([\beta_0 - \delta, \beta_0] \times [\beta_0 - \delta, \beta_0]) \cap \mathcal{S}$ for some $\delta > 0$. Then there exists $x \in K$ which is strictly positive in $[0, \mu]$ such that $F(x) = rx$, where $r = r(F)$ satisfies $r > 0$.

Let us remark that below we shall give other conditions not covered by the above theorems which also imply the existence of an eigenfunction $x \in K \setminus \{0\}$ of F with eigenvalue $r = r(F)$. In particular, Corollaries 4.21 and 4.22 provide such conditions.

In order to apply our results of the previous sections to the map F , we must first verify that $F(x)$ indeed belongs to K for $x \in K$, and that F satisfies the appropriate properties, in particular, Hypothesis B and condition (3.15). This will be done in Proposition 4.7. We wish to obtain lower bounds for the cone spectral radius $r(F)$ and upper bounds for the cone essential spectral radius $\rho(F)$ in hopes of finding conditions under which $\rho(F) < r(F)$, so that Theorem 3.4 can be used (which is basically reformulated for max-plus operators as Theorem 4.4). Theorem 4.3 below is a crucial result by which such estimates can be obtained, and in particular it

provides upper bounds for $\rho(F)$ and a formula for the quantities b_n in (2.1) which enter into the definition of $r(F)$. Corollary 4.9 provides lower bounds for $r(F)$ in a form which is easy to use. In Lemma 4.11 the quantities $c_n = \omega(F^n)$ in (3.10), which enter into the definition of $\rho(F)$, are bounded $c_n \leq \lambda_n$ by the quantities appearing in Theorem 4.3. If the functions α and β are monotone increasing then Lemma 4.16 gives an equality $c_n = \lambda_n$, and in Theorem 4.5 we are able to give the exact value of $\rho(F)$ in a very explicit form.

We note that Theorem 4.1 above applies even in some cases where $\rho(F) = r(F)$ holds, as shown by an example below. On the other hand, Proposition 4.23 below provides a class of examples for which $\rho(F) = r(F)$ and for which no eigenfunction in K exists. We shall in fact use Corollary 3.11 to prove some of the cases of Theorem 4.1, wherein the compactness of $\overline{\gamma_g^+(e)}$ must be verified rather than having to check (3.13).

Results on the positivity and monotonicity of the eigenfunction x are given in Propositions 4.12 and 4.13, respectively.

Except for Theorems 4.1 through 4.5, we shall generally prove results when they are stated. The proofs of Theorems 4.3 and 4.4 are given after the proof of Lemma 4.11, and the proof of Theorem 4.5 is given after that of Lemma 4.20, as the necessary theory must first be developed. Similarly, Theorems 4.1 and 4.2 are proved at the end of this section.

The definition of the cone essential spectral radius depends upon the choice of a homogeneous generalized measure of noncompactness ν on X . In what follows below, we shall always take $\nu = \omega$ with ω as in (3.7), and with ω_δ for each $\delta > 0$ as in (3.6). The notation $\omega(F)$ denotes the quantity (3.9) with ω and F in place of ν and f .

The estimates on the spectral radii involve iterates F^n of the operator F , and as such, it will be convenient to introduce some additional notation and terminology before stating further results. Consider for each $n \geq 1$ the set \mathcal{S}_n of $(n+1)$ -tuples $\sigma = (s_0, s_1, s_2, \dots, s_n)$ defined as

$$\mathcal{S}_n = \{(s_0, s_1, s_2, \dots, s_n) \mid s_0 \in [0, \mu] \text{ and } s_i \in J(s_{i-1}) \text{ for } 1 \leq i \leq n\}.$$

Thus $\mathcal{S}_1 = \mathcal{S}$. An element $\sigma \in \mathcal{S}_n$ will be called an **admissible n -sequence**. If some $\sigma \in \mathcal{S}_n$ satisfies $s_0 = s_n$, then we shall say that σ is an **n -cycle**. Let us define a function $a_n : \mathcal{S}_n \rightarrow [0, \infty)$ by setting

$$a_n(\sigma) = a(s_0, s_1)a(s_1, s_2) \cdots a(s_{n-1}, s_n). \quad (4.2)$$

Note that the set $\mathcal{S}_n \subseteq [0, \mu]^{n+1}$ is compact and is the maximal set on which the formula (4.2) for $a_n(\sigma)$ is defined, and that the function a_n is continuous on \mathcal{S}_n . For later use let us also denote

$$J(L) = \bigcup_{s \in L} J(s)$$

for any set $L \subseteq [0, \mu]$, and define inductively

$$J^{n+1}(L) = J(J^n(L)) \text{ for } n \geq 0,$$

with $J^0(L) = L$. Of course $J^n(s)$ means $J^n(L)$ with $L = \{s\}$. It is easy to check that if L is a compact interval then so is $J(L)$, so by iterating we see that $J^n(L)$ is a compact interval for every $n \geq 1$.

The following definition will play a key role in our obtaining estimates of $\rho(F)$.

Definition. Let $\sigma = (s_0, s_1, s_2, \dots, s_n) \in \mathcal{S}_n$ be given, and let $p_0 \in [0, \mu]$ be given. We say an element $\pi \in \mathcal{S}_n$ given as $\pi = (p_0, p_1, p_2, \dots, p_n)$ is the **best-approximating n -sequence to σ through p_0** if for every $1 \leq i \leq n$ the point p_i is the point in the interval $J(p_{i-1})$ nearest to s_i .

Let us note that given any $\sigma \in \mathcal{S}_n$, then π above is uniquely determined by the point p_0 . Also note that if $p_k = s_k$ for some k , then $p_i = s_i$ for every $i \geq k$, where $0 \leq k \leq i \leq n$. One therefore has that if $p_n \neq s_n$, then $p_i \neq s_i$ for every $0 \leq i \leq n$. In this spirit it is convenient to introduce the following notation.

Notation. If $\sigma, \pi \in \mathcal{S}_n$ then we shall always let s_i and p_i denote the coordinates of these $(n + 1)$ -tuples, as above. We shall also write

$$\begin{aligned} \pi|\sigma &\iff \pi \text{ is the best-approximating } n\text{-sequence to } \sigma \text{ through } p_0, \\ \pi||\sigma &\iff \pi|\sigma \text{ and } p_i \neq s_i \text{ for every } 0 \leq i \leq n, \\ \pi|_\delta\sigma &\iff \pi|\sigma \text{ and } |p_0 - s_0| \leq \delta, \text{ and} \\ \pi||_\delta\sigma &\iff \pi||\sigma \text{ and } |p_0 - s_0| \leq \delta, \end{aligned}$$

where $\delta > 0$.

Now observe that if $\pi|\sigma$ and $p_i \neq s_i$ for some $1 \leq i \leq n$, then either $p_i = \alpha(p_{i-1}) > s_i$ or else $p_i = \beta(p_{i-1}) < s_i$. Thus if $\pi||\sigma$ then $\pi \in \mathcal{E}_n$ where $\mathcal{E}_n \subseteq \mathcal{S}_n$ is defined to be

$$\mathcal{E}_n = \{\pi \in \mathcal{S}_n \mid p_i \in \{\alpha(p_{i-1}), \beta(p_{i-1})\} \text{ for every } 1 \leq i \leq n\}.$$

That is, \mathcal{E}_n is the set of admissible n -sequences whose coordinates p_i , for $1 \leq i \leq n$, are each endpoints of the allowable interval $J(p_{i-1})$. One sees that if the strict inequality $\alpha(s) < \beta(s)$ holds everywhere, then there exist exactly 2^n elements of \mathcal{E}_n for every given $p_0 \in [0, \mu]$, and that in any case the set \mathcal{E}_n has at most 2^n elements for every p_0 .

Let us also introduce the set

$$\mathcal{A}_{n,\delta} = \{\pi \in \mathcal{S}_n \mid \pi||_\delta\sigma \text{ for some } \sigma \in \mathcal{S}_n\}$$

for every $\delta > 0$, which will play an important role below. We have that $\mathcal{A}_{n,\delta} \subseteq \mathcal{E}_n$ and that $\mathcal{A}_{n,\delta_1} \subseteq \mathcal{A}_{n,\delta_2}$ for $\delta_1 < \delta_2$. With this, we are now able to state two main theorems concerning spectral radii and eigenfunctions of F .

Theorem 4.3. *We have that*

$$r(F) = \lim_{n \rightarrow \infty} b_n^{1/n} = \inf_{n \geq 1} b_n^{1/n}, \tag{4.3}$$

where

$$b_n = \max_{\sigma \in \mathcal{S}_n} a_n(\sigma), \tag{4.4}$$

and that

$$\rho(F) \leq \inf_{n \geq 1} \lambda_n^{1/n}, \tag{4.5}$$

where

$$\lambda_n = \lim_{\delta \rightarrow 0+} \lambda_n(\delta) = \inf_{\delta > 0} \lambda_n(\delta), \quad \lambda_n(\delta) = \sup_{\pi \in \mathcal{A}_{n,\delta}} a_n(\pi), \tag{4.6}$$

for the cone spectral radius $r(F)$ and the cone essential spectral radius $\rho(F)$ of the map $F : K \rightarrow K$ in (1.2). (Recall that if $\mathcal{A}_{n,\delta} = \phi$ then $\lambda_n(\delta) = 0$.)

Theorem 4.4. *Suppose that $\rho(F) < r(F)$ for the map $F : K \rightarrow K$ in (1.2), or more generally that $\lambda_n < r^n$ for some $n \geq 1$, where λ_n is as in (4.6) and $r = r(F)$. Then there exists $x \in K \setminus \{0\}$ with $F(x) = rx$.*

In the special case that the functions α and β are monotone increasing it is possible to give the exact value of $\rho(F)$ in an explicit form. Generally, if $g : [c, d] \rightarrow [c, d]$ is a continuous monotone increasing function, then the iterates $s_i = g^i(s_0)$ of any point tend to a fixed point $s_i \rightarrow s_* = g(s_*)$ of g as $i \rightarrow \infty$. Because of this, if α and β are both monotone increasing in $[0, \mu]$ we might expect the fixed points of these functions to play a special role in estimating the quantities λ_n and $\rho(F)$ in Theorem 4.3, and indeed this is the case. Let us say that $s \in [c, d]$ is a **point of constancy** of g if there exists $\varepsilon > 0$ such that $g(t) = g(s)$ for every $t \in [s - \varepsilon, s + \varepsilon] \cap [c, d]$. Let us also define sets

$$\begin{aligned} Q_g &= \{s \in [c, d] \mid s \text{ is a point of constancy of } g\}, \\ S_g &= \{s \in [c, d] \mid g(s) = s\}, \quad C_g = S_g \cap Q_g, \quad D_g = S_g \setminus Q_g, \end{aligned} \tag{4.7}$$

for such a function. With this we may state another main result of this section.

Theorem 4.5. *Suppose that both functions α and β are monotone increasing in $[0, \mu]$. Then the inequality in (4.5) is an equality and in fact*

$$\rho(F) = \lim_{n \rightarrow \infty} \lambda_n^{1/n} = \inf_{n \geq 1} \lambda_n^{1/n}. \tag{4.8}$$

Additionally,

$$\rho(F) = \sup_{s \in D_\alpha \cup D_\beta} a(s, s) \tag{4.9}$$

where $D_\alpha, D_\beta \subseteq [0, \mu]$ are the sets (4.7) for the functions α and β .

Remark. Clearly $\lambda_n \leq b_n$ and so $\rho(F) \leq r(F)$ by Theorem 4.3 for the map F . However, let us remark that in the abstract setting of the previous section it is not clear that $\rho(f) \leq r(f)$ always holds for general maps f and for general measures of noncompactness ν .

Remark. Although $\mathcal{A}_{n,\delta} \subseteq \mathcal{E}_n$, not every element of \mathcal{E}_n need belong to $\mathcal{A}_{n,\delta}$ for some δ . Indeed, one difficulty in applying Theorem 4.3 is that the sets $\mathcal{A}_{n,\delta}$ figure in the definition of $\lambda_n(\delta)$ and λ_n , and in general determining $\mathcal{A}_{n,\delta}$ precisely poses difficulties. However, if α and β are monotone increasing as in Theorem 4.5 then it is possible to give explicit descriptions of $\mathcal{A}_{n,\delta}$ and of the quantities $\lambda_n(\delta)$ and λ_n . In this case Lemma 4.15 below shows that every $\pi \in \mathcal{A}_{n,\delta}$ necessarily is of the form either $p_i = \alpha^i(p_0)$ for every $1 \leq i \leq n$, or else $p_i = \beta^i(p_0)$ for every $1 \leq i \leq n$, and in fact we provide a necessary and sufficient condition for such a sequence to belong to $\mathcal{A}_{n,\delta}$. Note that here at most two of the 2^n elements $\pi \in \mathcal{E}_n$ with a given $p_0 \in [0, \mu]$ belong to $\mathcal{A}_{n,\delta}$ for any δ .

Remark. If both functions α and β are constant, say $\alpha(s) = \alpha_0$ and $\beta(s) = \beta_0$ identically in $[0, \mu]$, it is known that F is a compact map. In fact, this is a special case of Theorem 4.3. One has for such α and β that whenever $\sigma, \pi \in \mathcal{S}_n$ satisfy $\pi|\sigma$ then necessarily $p_i = s_i$ for $1 \leq i \leq n$. This implies that $\pi|\sigma$ can never hold and hence that $\mathcal{A}_{n,\delta} = \phi$ for $n \geq 1$ and $\delta > 0$. Thus $\lambda_n = \lambda_n(\delta) = 0$ in (4.6) and

so $\rho(F) = 0$. Also, Lemma 4.11 below implies that $\omega(F(A)) = 0$ hence $\overline{F(A)}$ is compact whenever $A \subseteq K$ is bounded.

Example. We give an example to which Theorem 4.1 applies, yet for which $\rho(F) = r(F)$. With $[0, \mu] = [0, 1]$, let $\alpha(s) = s/2$ and $\beta(s) = 1$ in that interval, and let $a(s, t) = 1$ identically in \mathcal{S} . It is clear that the hypotheses of Theorem 4.1 hold. Also, we have that each $b_n = 1$ and so $r(F) = 1$, by Theorem 4.3, and that $D_\alpha = \{0\}$ and $D_\beta = \phi$ and so $\rho(F) = a(0, 0) = 1$, by Theorem 4.5. One sees easily that

$$(F(x))(s) = \max_{t \in [s/2, 1]} x(t)$$

for any $x \in K$, and that the constant function which equals 1 identically is the eigenfunction guaranteed by Theorem 4.1.

For notational convenience let us define a function $q : [0, \mu] \times [0, \mu] \rightarrow [0, \mu]$ by letting $q(s, t)$ denote the point in the interval $J(s)$ which is closest to t . That is,

$$q(s, t) = \min\{\max\{t, \alpha(s)\}, \beta(s)\}, \tag{4.10}$$

which equals t if $t \in J(s)$, and which equals $\alpha(s)$ or $\beta(s)$ if $t \leq \alpha(s)$ or $t \geq \beta(s)$, respectively. Let us also set

$$\begin{aligned} \psi(\delta) &= \max\{\psi_\alpha(\delta), \psi_\beta(\delta)\}, \\ \psi_\tau(\delta) &= \max\{|\tau(s) - \tau(\tilde{s})| \mid s, \tilde{s} \in [0, \mu] \text{ and } |s - \tilde{s}| \leq \delta\}, \end{aligned} \tag{4.11}$$

for every $\delta > 0$, where $\tau = \alpha$ or β . The function ψ , which measures the modulus of continuity of α and β , is monotone increasing and satisfies $\lim_{\delta \rightarrow 0^+} \psi(\delta) = 0$.

We present a technical lemma, followed by a result which places the map F within our theory.

Lemma 4.6. *Let $s, \tilde{s} \in [0, \mu]$ and $t \in J(s)$, and set $\tilde{t} = q(\tilde{s}, t)$. Then $|t - \tilde{t}| \leq \psi(|s - \tilde{s}|)$ with q and ψ as in (4.10) and (4.11).*

Proof. It is sufficient to show that

$$|t - \tilde{t}| \leq \max\{|\alpha(s) - \alpha(\tilde{s})|, |\beta(s) - \beta(\tilde{s})|\}, \tag{4.12}$$

by (4.11). Indeed, if $t \in J(\tilde{s})$ then $t = \tilde{t}$. If $t < \alpha(\tilde{s})$ then $\tilde{t} = \alpha(\tilde{s})$, and as $t \in J(s)$ we have $t \geq \alpha(s)$, to give (4.12). Similarly, if $t > \beta(\tilde{s})$ then (4.12) holds. This proves the result. \square

Proposition 4.7. *The map $F : K \rightarrow K$ in (1.2) satisfies Hypothesis B and condition (3.15).*

Proof. We leave the verification that F is homogeneous of degree one, order-preserving and satisfies (3.15) to the reader. We have to prove that if $x \in K$ then $F(x) \in K$, and also that $F : K \rightarrow K$ is continuous.

First fix $x \in K$ and let $y = F(x)$ be given by (1.2). We must prove that $y(s)$ is continuous in s (it is obviously nonnegative). Letting $\varepsilon > 0$, we have that there exists $\delta > 0$ such that if $(s, t), (\tilde{s}, \tilde{t}) \in \mathcal{S}$ satisfy $|s - \tilde{s}| \leq \delta$ and $|t - \tilde{t}| \leq \psi(\delta)$, then

$$|a(s, t)x(t) - a(\tilde{s}, \tilde{t})x(\tilde{t})| \leq \varepsilon.$$

In particular, take arbitrary points $s, \tilde{s} \in [0, \mu]$ with $|s - \tilde{s}| \leq \delta$, let $t \in J(s)$ be such that $y(s) = a(s, t)x(t)$, and choose $\tilde{t} = q(\tilde{s}, t)$, that is, \tilde{t} is the nearest point in $J(\tilde{s})$ to t . Then $|t - \tilde{t}| \leq \psi(\delta)$ by Lemma 4.6 and so

$$y(s) - y(\tilde{s}) = a(s, t)x(t) - y(\tilde{s}) \leq a(s, t)x(t) - a(\tilde{s}, \tilde{t})x(\tilde{t}) \leq \varepsilon.$$

Our argument is symmetric in the roles of s and \tilde{s} , so we also find that $y(\tilde{s}) - y(s) \leq \varepsilon$, and we conclude that $|y(s) - y(\tilde{s})| \leq \varepsilon$ whenever $|s - \tilde{s}| \leq \delta$. As $\varepsilon > 0$ was arbitrary, we conclude that y is continuous.

It remains to prove that F is continuous on K . Let $x_1, x_2 \in K$ be given, denote $\varepsilon = \|x_1 - x_2\|$, and let $y_i = F(x_i)$ for $i = 1, 2$. Then $x_1(t) \leq x_2(t) + \varepsilon$ for every $t \in [0, \mu]$, and so

$$\begin{aligned} y_1(s) &= \max_{t \in J(s)} a(s, t)x_1(t) \leq \max_{t \in J(s)} a(s, t)(x_2(t) + \varepsilon) \\ &\leq \max_{t \in J(s)} a(s, t)x_2(t) + A_+\varepsilon = y_2(s) + A_+\varepsilon, \end{aligned}$$

for every $s \in [0, \mu]$, where we recall that A_+ is the maximum of a in \mathcal{S} . Similarly $y_2(s) \leq y_1(s) + A_+\varepsilon$ for every s . We conclude from this that $\|F(x_1) - F(x_2)\| \leq A_+\|x_1 - x_2\|$ for every $x_1, x_2 \in K$, thereby establishing the continuity (in fact, lipschitz continuity) of F . \square

The following results allow for estimates of $r(F)$.

Proposition 4.8. *If $x \in K$ and $n \geq 1$, then*

$$(F^n(x))(s) = \max\{a_n(\sigma)x(s_n) \mid \sigma \in \mathcal{S}_n \text{ and } s_0 = s\}. \tag{4.13}$$

The quantities b_n as in (2.1), but for the map F in place of f , are given by (4.4) and we have that (4.3) holds.

Proof. If $n = 1$, then equation (4.13) is the definition of $F(x)$. The formula for general $n \geq 1$ follows by an induction argument, which we leave to the reader.

If e denotes the function identically equal to 1, then it is immediate from (4.13) that $\|F^n(x)\| \leq \|F^n(e)\|$ for every $x \in K$ with $\|x\| \leq 1$, and thus $b_n = \|F^n(e)\|$. With this, one now sees that (4.4) follows directly from (4.13). Also, we have the equality $\tilde{r}(F) = r(F)$ from Theorem 2.2, which with (2.5) gives us (4.3). \square

Corollary 4.9. *We have that*

$$r(F) \geq a_n(\sigma)^{1/n}$$

if $\sigma \in \mathcal{S}_n$ is an n -cycle.

Proof. We shall use the formulas (4.3) and (4.4), as established by Proposition 4.8. Let σ be an n -cycle and extend it periodically to an infinite sequence (s_0, s_1, s_2, \dots) which satisfies $s_i = s_{i+n}$ for every $i \geq 0$. Then every truncated $(m + 1)$ -tuple $\sigma_m = (s_0, s_1, s_2, \dots, s_m)$ satisfies $\sigma_m \in \mathcal{S}_m$ and so $b_m \geq a_m(\sigma_m)$. In particular, take $m = kn$ and note from the formula (4.2) that $a_{kn}(\sigma_{kn}) = a_n(\sigma_n)^k$, and so $b_{kn} \geq a_n(\sigma_n)^k$. Taking the $(kn)^{\text{th}}$ root of this inequality and letting $k \rightarrow \infty$ gives $r(F) \geq a_n(\sigma_n)^{1/n}$, which is as desired. \square

We give a technical lemma, followed by a result which allows us to estimate $\omega(F^n(A))$ in terms of $\omega(A)$ for any bounded set $A \subseteq K$. The proofs of Theorems 4.3 and 4.4 follow directly.

Lemma 4.10. *There exist nonnegative monotone increasing functions $\psi_i, \theta_n : (0, \infty) \rightarrow (0, \infty)$ for $i, n \geq 1$, satisfying $\lim_{\delta \rightarrow 0^+} \psi_i(\delta) = 0$ and $\lim_{\delta \rightarrow 0^+} \theta_n(\delta) = 0$, and with the following properties: If $\sigma, \pi \in \mathcal{S}_n$ satisfy $\pi|_\delta \sigma$ for some $\delta > 0$, with $n \geq 1$, then $|s_i - p_i| \leq \psi_i(\delta)$ for $1 \leq i \leq n$, and $|a_n(\sigma) - a_n(\pi)| \leq \theta_n(\delta)$.*

Proof. By Lemma 4.6 we have that $|s_i - p_i| \leq \psi(|s_{i-1} - p_{i-1}|)$ for $1 \leq i \leq n$. Thus by defining recursively $\psi_i(\delta) = \psi(\psi_{i-1}(\delta))$, with $\psi_1(\delta) = \psi(\delta)$, we obtain $|s_i - p_i| \leq \psi_i(\delta)$. One easily checks that the function ψ_i satisfies the desired properties. Now set

$$\theta_n(\delta) = \max\{|a_n(\sigma) - a_n(\pi)| \mid \sigma, \pi \in \mathcal{S}_n \text{ and } |s_i - p_i| \leq \psi_i(\delta) \text{ for } 0 \leq i \leq n\},$$

where $\psi_0(\delta) = \delta$. Again, one checks that θ_n is as desired. \square

Lemma 4.11. *For every $n \geq 0$ we have for every bounded set $A \subseteq K$ that $\omega(F^n(A)) \leq \lambda_n \omega(A)$, and thus*

$$\omega(F^n) \leq \lambda_n \tag{4.14}$$

with λ_n as in (4.6).

Proof. Let $A \subseteq K$ be a bounded set with $\|x\| \leq M$ for every $x \in A$, and fix $n \geq 1$. Take any $x \in A$ and let $y = F^n(x)$. In order to estimate the value of $\omega(F^n(A))$, we need to estimate the difference $|y(s_0) - y(p_0)|$ for nearby points s_0 and p_0 . Fix $\delta > 0$ and let $s_0, p_0 \in [0, \mu]$ satisfy $|s_0 - p_0| \leq \delta$. By possibly interchanging the roles of s_0 and p_0 , we can assume without loss that $y(s_0) \geq y(p_0)$. By (4.13) of Proposition 4.8 there exists $\sigma \in \mathcal{S}_n$, with initial coordinate s_0 , such that $y(s_0) = a_n(\sigma)x(s_n)$. Define $\pi \in \mathcal{S}_n$ to be the best-approximating n -sequence to σ through p_0 , and so $\pi|_\delta \sigma$. Then $y(p_0) \geq a_n(\pi)x(p_n)$ again by (4.13), and so

$$\begin{aligned} 0 \leq y(s_0) - y(p_0) &\leq a_n(\sigma)x(s_n) - a_n(\pi)x(p_n) \\ &\leq |a_n(\sigma) - a_n(\pi)|x(s_n) + a_n(\pi)|x(s_n) - x(p_n)| \\ &\leq \theta_n(\delta)M + a_n(\pi)|x(s_n) - x(p_n)| \\ &\leq \theta_n(\delta)M + \lambda_n(\delta)|x(s_n) - x(p_n)| \\ &\leq \theta_n(\delta)M + \lambda_n(\delta)\omega_\eta(A), \end{aligned} \tag{4.15}$$

where $\eta = \psi_n(\delta)$ in the last line of (4.15). We have used Lemma 4.10 and the fact that $\pi|_\delta \sigma$ to obtain the fourth inequality in (4.15). To obtain the fifth inequality in (4.15) we may assume that $x(s_n) \neq x(p_n)$, in which case $\pi|_\delta \sigma$ holds, and so $\pi \in \mathcal{A}_{n,\delta}$ and $a_n(\pi) \leq \lambda_n(\delta)$ by (4.6). The final inequality in (4.15) follows from the fact, by Lemma 4.10, that $|s_n - p_n| \leq \psi_n(\delta) = \eta$.

Let us now take the supremum in the first line of (4.15), over all $y \in F^n(A)$ and over all points s_0 and p_0 separated by a distance at most δ . This gives

$$\omega_\delta(F^n(A)) \leq \theta_n(\delta)M + \lambda_n(\delta)\omega_\eta(A). \tag{4.16}$$

Upon letting $\delta \rightarrow 0$ in (4.16) we obtain $\omega(F^n(A)) \leq \lambda_n \omega(A)$ as claimed. This immediately gives the inequality (4.14) using the definition (3.9). \square

Proof of Theorem 4.3. Proposition 4.8 gives the formulas (4.3), (4.4) for $r(F)$. The inequality (4.5) for $\rho(F)$ follows from the inequality (4.14) in Lemma 4.11 together with the definition (3.11) with (3.10), where $\nu = \omega$. \square

Proof of Theorem 4.4. If $\lambda_n < r^n$ for some $n \geq 1$ then $\rho(F) < r(F)$. The existence of the eigenfunction x with eigenvalue r now follows from Theorem 3.4, using Proposition 4.7. \square

Before proceeding toward the proof of Theorem 4.5, we give two propositions which provide criteria under which eigenfunctions of F must be either strictly positive, or else monotone.

Proposition 4.12. *Suppose that $x \in K \setminus \{0\}$ satisfies $F(x) = \lambda x$ for some $\lambda \geq 0$, and that the function a is strictly positive throughout \mathcal{S} . Let*

$$Z = \{s \in [0, \mu] \mid x(s) = 0\}.$$

Then we have

$$J(Z) \subseteq Z, \tag{4.17}$$

that is, $J(s) \subseteq Z$ for every $s \in Z$.

If furthermore (1.5) holds then $Z \cap (0, \mu) = \emptyset$, and so $x(s) > 0$ for every $s \in (0, \mu)$. Also, we have that $\lambda > 0$.

If in addition to (1.5) we have $\alpha(\mu) < \mu$ then $x(\mu) > 0$, and if $\beta(0) > 0$ then $x(0) > 0$.

Proof. If $s \in Z$ then by (1.10), using the strict positivity of a , we have $x(t) = 0$ for every $t \in J(s)$. This gives $J(s) \subseteq Z$, and so (4.17) holds.

Assuming (1.5), take any $s \in Z \cap (0, \mu)$ and let $I = [s_1, s_2]$ be the maximal interval contained in Z which satisfies $s \in I \subseteq [0, \mu]$, with possibly $s_1 = s_2$. As $J(s_1) \subseteq Z$ we have $[\alpha(s_1), s_1] \subseteq J(s_1) \subseteq Z$, and thus

$$[\alpha(s_1), s_2] = [\alpha(s_1), s_1] \cup I \subseteq Z. \tag{4.18}$$

If $s_1 > 0$ then $\alpha(s_1) < s_1$, and so (4.18) contradicts the maximality of the interval I . Thus $s_1 = 0$. Similarly $s_2 = \mu$, and this implies that x is the zero function, a contradiction. Thus $Z \cap (0, \mu) = \emptyset$ as claimed. From this one sees the left-hand side of (1.10) is strictly positive in $(0, \mu)$, and so $\lambda > 0$.

Assuming (1.5), if $\alpha(\mu) < \mu$ then $\mu \in Z$ would imply the existence of some $s \in J(\mu) \subseteq Z$ with $s \in (0, \mu)$, a contradiction. Thus $\mu \notin Z$ and so $x(\mu) > 0$. Similarly $\beta(0) > 0$ implies that $x(0) > 0$. \square

Remark. Suppose that $s_1 \leq \alpha(s) < s < \beta(s) \leq s_2$ for every $s \in (s_1, s_2)$, for some $s_1 < s_2$ with $s_1, s_2 \in [0, \mu]$. Then if $F(x) = \lambda x$ for some $\lambda \geq 0$ and $x \in K \setminus \{0\}$, and if a is strictly positive in \mathcal{S} , it follows from Proposition 4.12 that either $x(s) > 0$ for every $s \in (s_1, s_2)$ or else that $x(s) = 0$ for every $s \in [s_1, s_2]$.

Proposition 4.13. *Suppose that $x \in K \setminus \{0\}$ satisfies $F(x) = \lambda x$ for some $\lambda > 0$. If $J(s_1) \subseteq J(s_2)$ and $a(s_1, t) \leq a(s_2, t)$ whenever $0 \leq s_1 < s_2 \leq \mu$ and $t \in J(s_1)$, then x is monotone increasing in $[0, \mu]$. If $J(s_1) \supseteq J(s_2)$ and $a(s_1, t) \geq a(s_2, t)$ whenever $0 \leq s_1 < s_2 \leq \mu$ and $t \in J(s_2)$, then x is monotone decreasing in $[0, \mu]$.*

Proof. Under the first set of hypotheses, if $0 \leq s_1 < s_2 \leq \mu$ we have that

$$\lambda x(s_1) = \max_{t \in J(s_1)} a(s_1, t)x(t) \leq \max_{t \in J(s_1)} a(s_2, t)x(t) \leq \max_{t \in J(s_2)} a(s_2, t)x(t) = \lambda x(s_2).$$

This implies that $x(s_1) \leq x(s_2)$, as desired. A similar argument gives the second part of the proposition. \square

As a practical matter, we need an efficient method for estimating the quantities λ_n appearing in Theorem 4.3 in order to use that result, and also to prove Theorem 4.5. In this direction Lemma 4.15 below characterizes the sets $\mathcal{A}_{n,\delta}$ when α and β are monotone increasing. Indeed, in most of the following results we shall assume that both α and β are monotone increasing in $[0, \mu]$. The same methods yield theorems for the case that both α and β are monotone decreasing.

We need to introduce some additional notation before proceeding. For any $s \in [0, \mu]$ define admissible n -sequences $\zeta_n^\pm(s) \in \mathcal{E}_n$ by

$$\zeta_n^-(s) = (s, \alpha(s), \alpha^2(s), \dots, \alpha^n(s)), \quad \zeta_n^+(s) = (s, \beta(s), \beta^2(s), \dots, \beta^n(s)).$$

Also define functions $u_n^\pm : [0, \mu] \rightarrow [0, \infty)$ by

$$u_n^\pm(s) = a_n(\zeta_n^\pm(s)), \tag{4.19}$$

with a_n as in (4.2).

We first prove a technical lemma.

Lemma 4.14. *Assume that both functions α and β are monotone increasing in $[0, \mu]$. For any $n \geq 1$ take any $\sigma, \pi \in \mathcal{S}_n$ for which $\pi|\sigma$ and with $p_0 > s_0$. Then there exists $0 \leq k \leq n$ such that*

$$\alpha^i(s_0) \leq s_i < p_i = \alpha^i(p_0) \text{ for } 0 \leq i \leq k, \quad p_i = s_i \text{ for } k < i \leq n. \tag{4.20}$$

If $\pi|\sigma$ with $p_0 < s_0$ then the corresponding result holds.

Proof. If $p_i = s_i$ for some $1 \leq i \leq n-1$ then $p_{i+1} = s_{i+1}$ as $\pi|\sigma$. If $p_i < s_i$ for some $1 \leq i \leq n$ then necessarily $p_i = \beta(p_{i-1}) < s_i$, and as $s_i \leq \beta(s_{i-1})$ must hold and β is monotone increasing, we conclude that $p_{i-1} < s_{i-1}$. From this it follows that $p_0 < s_0$, a contradiction. Thus $p_i < s_i$ is impossible, and so we see that there exists $0 \leq k \leq n$ such that $p_i > s_i$ for $0 \leq i \leq k$ while $p_i = s_i$ for $k < i \leq n$. Necessarily $p_i = \alpha(p_{i-1})$ for $1 \leq i \leq k$ as $\pi|\sigma$, and so $p_i = \alpha^i(p_0)$ for such i . As $\alpha(s_{i-1}) \leq s_i$ must hold for every i , we have that $\alpha^i(s_0) \leq s_i$ as α is monotone increasing. With this we have (4.20). \square

Lemma 4.15. *Assume that both functions α and β are monotone increasing in $[0, \mu]$. Then for every $\delta > 0$ we have that*

$$\mathcal{A}_{n,\delta} = \mathcal{A}_{n,\delta}^- \cup \mathcal{A}_{n,\delta}^+, \tag{4.21}$$

where

$$\begin{aligned} \mathcal{A}_{n,\delta}^- &= \{ \pi \in \mathcal{S}_n \mid \pi = \zeta_n^-(p_0) \text{ for some } p_0 \in (0, \mu], \\ &\quad \text{with } \alpha^n(s_*) < \alpha^n(p_0) \text{ where } s_* = (p_0 - \delta) \vee 0 \}, \\ \mathcal{A}_{n,\delta}^+ &= \{ \pi \in \mathcal{S}_n \mid \pi = \zeta_n^+(p_0) \text{ for some } p_0 \in [0, \mu], \\ &\quad \text{with } \beta^n(p_0) < \beta^n(s_*) \text{ where } s_* = (p_0 + \delta) \wedge \mu \}. \end{aligned} \tag{4.22}$$

If α and β are strictly increasing in $[0, \mu]$, then $\mathcal{A}_{n,\delta}$ is independent of δ with the conditions on s_* in (4.22) automatically satisfied. If α is constant in $[0, \mu]$ with β monotone increasing there then $\mathcal{A}_{n,\delta} = \mathcal{A}_{n,\delta}^+$, while if α is monotone increasing and β is constant then $\mathcal{A}_{n,\delta} = \mathcal{A}_{n,\delta}^-$.

Proof. All assertions of the lemma follow from equations (4.21), (4.22), so it suffices to prove these. We first show that $\mathcal{A}_{n,\delta}$ is contained in the right-hand side of (4.21).

Take any $\pi \in \mathcal{A}_{n,\delta}$, and let $\sigma \in \mathcal{S}_n$ be such that $\pi \parallel_\delta \sigma$. Let us first observe that as $\alpha(s_{i-1}) \leq s_i \leq \beta(s_{i-1})$ for every $1 \leq i \leq n$, one has from the monotonicity of α and β that $\alpha^i(s_0) \leq s_i \leq \beta^i(s_0)$ for such i .

Assuming for definiteness that $p_0 > s_0$, we have by Lemma 4.14 that (4.20) holds, and moreover $k = n$. We have further that $\alpha^n(s_*) \leq \alpha^n(s_0)$ where $s_* = (p_0 - \delta) \vee 0 \leq s_0$, and thus $\alpha^n(s_*) < \alpha^n(p_0)$. This shows that π belongs to the right-hand side of (4.21), in fact $\pi \in \mathcal{A}_{n,\delta}^-$. One similarly concludes that $\pi \in \mathcal{A}_{n,\delta}^+$ in the case that $p_0 < s_0$.

To show the opposite inclusion, let π belong to the right-hand side of (4.21), say $\pi = \zeta_n^-(p_0) \in \mathcal{A}_{n,\delta}^-$ for definiteness. Letting $s_0 = (p_0 - \delta) \vee 0$ and $s_i = \alpha^i(s_0)$ for $1 \leq i \leq n$, one sees that $s_i < p_i$ for these points (the strict inequality holding because $s_n < p_n$), and so $\pi \parallel_\delta \sigma$. Thus $\pi \in \mathcal{A}_{n,\delta}$. A similar argument applies if $\pi \in \mathcal{A}_{n,\delta}^+$, and so we obtain equality in (4.21). \square

With α and β monotone increasing we have for either choice of sign \pm that $\mathcal{A}_{n,\delta_1}^\pm \subseteq \mathcal{A}_{n,\delta_2}^\pm$ when $\delta_1 < \delta_2$, for the sets given by (4.22). Thus we may define quantities

$$\lambda_n^\pm = \lim_{\delta \rightarrow 0^+} \lambda_n^\pm(\delta) = \inf_{\delta > 0} \lambda_n^\pm(\delta), \quad \lambda_n^\pm(\delta) = \sup_{\pi \in \mathcal{A}_{n,\delta}^\pm} a_n(\pi), \quad (4.23)$$

where we have from Lemma 4.15 that $\lambda_n = \max\{\lambda_n^-, \lambda_n^+\}$ and that $\lambda_n(\delta) = \max\{\lambda_n^-(\delta), \lambda_n^+(\delta)\}$ for the quantities in (4.6). With this we are able to show the inequality (4.14) is an equality when α and β are monotone increasing.

Lemma 4.16. *Assume that both the functions α and β are monotone increasing in $[0, \mu]$. Then*

$$\omega(F^n) = \lambda_n$$

for every $n \geq 1$.

Proof. As $\omega(F^n) \leq \lambda_n$ holds by Lemma 4.11, it is enough to prove that $\lambda_n^\pm \leq \omega(F^n)$ for both choices of sign \pm in order to prove the result. We shall in fact prove only that

$$\lambda_n^- \leq \omega(F^n), \quad (4.24)$$

the proof for λ_n^+ being similar.

Fix any $\delta > 0$. Then for any $\varepsilon > 0$ there exists $\pi = \zeta_n^-(p_0) \in \mathcal{A}_{n,\delta}^-$ such that

$$a_n(\pi) \geq \lambda_n^-(\delta) - \varepsilon, \quad (4.25)$$

from the definition (4.23). Keeping ε and π fixed, let $\sigma = \zeta_n^-(s_*) \in \mathcal{E}_n$ where $s_* = (p_0 - \delta) \vee 0$ is as in (4.22). Then $s_n < p_n$, and denoting $B = \{x \in K \mid \|x\| \leq 1\}$ we see that there exists an element $x \in B$ with $x(s) = 1$ in $[0, s_n]$ and $x(s) = 0$ in $[p_n, \mu]$. Let this x be fixed and denote $y = F^n(x)$.

Now notice that if $\tau \in \mathcal{S}_n$ is such that $t_0 = p_0$, then $t_i \geq p_i = \alpha^i(p_0)$ for $1 \leq i \leq n$, where we denote $\tau = (t_0, t_1, t_2, \dots, t_n)$. This observation follows easily from the monotonicity of α . Thus $x(t_n) = 0$ as $t_n \geq p_n$ and so $a_n(\tau)x(t_n) = 0$. In particular $y(p_0) = a_n(\tau)x(t_n)$ for some such τ by (4.13), and thus $y(p_0) = 0$. On the other hand we have that $y(s_0) \geq a_n(\sigma)x(s_n) = a_n(\sigma)$, again by (4.13), and therefore

$$\omega_\delta(F^n(B)) \geq y(s_0) - y(p_0) = a_n(\sigma) \quad (4.26)$$

as $y \in F^n(B)$. Noting that $\pi|_\delta\sigma$, we have further that

$$a_n(\sigma) \geq a_n(\pi) - \theta_n(\delta) \geq \lambda_n^-(\delta) - \varepsilon - \theta_n(\delta) \tag{4.27}$$

by Lemma 4.10 and (4.25). Combining (4.26) and (4.27) and letting both $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$, where we have that $\theta_n(\delta) \rightarrow 0$, gives

$$\omega(F^n(B)) \geq \lambda_n^-.$$

Noting finally that $\omega(B) = 1$, we see that (4.24) holds, as desired. \square

Although Lemma 4.16 provides an exact formula for $\omega(F^n)$, it is still desirable to give a better, more easily computed formula for λ_n . We can do this in terms of points of constancy of α^n and β^n . Let us first recast the definitions (4.23). For any $\delta > 0$ define the sets

$$\begin{aligned} G_n^-(\delta) &= \{s \in (0, \mu] \mid \alpha^n((s - \delta) \vee 0) < \alpha^n(s)\}, & \Gamma_n^-(\delta) &= \overline{G_n^-(\delta)}, \\ G_n^+(\delta) &= \{s \in [0, \mu) \mid \beta^n(s) < \beta^n((s + \delta) \wedge 0)\}, & \Gamma_n^+(\delta) &= \overline{G_n^+(\delta)}. \end{aligned}$$

Then if the functions α and β are monotone increasing, we have from (4.23), using (4.19), (4.21), and (4.22), that

$$\lambda_n^\pm(\delta) = \sup_{s \in G_n^\pm(\delta)} u_n^\pm(s) = \max_{s \in \Gamma_n^\pm(\delta)} u_n^\pm(s).$$

In addition $\Gamma_n^\pm(\delta_1) \subseteq \Gamma_n^\pm(\delta_2)$ for $\delta_1 < \delta_2$, and so it is natural to define the sets

$$\Gamma_n^\pm = \bigcap_{\delta > 0} \Gamma_n^\pm(\delta),$$

which are compact. (We also have that $\Gamma_n^- \neq \emptyset$ provided $\Gamma_n^-(\delta) \neq \emptyset$ for every positive δ , and similarly with Γ_n^+ .) It is now easy to see that

$$\lambda_n^\pm = \max_{s \in \Gamma_n^\pm} u_n^\pm(s). \tag{4.28}$$

Clearly, it is important to characterize the sets Γ_n^\pm , which we do in the following result.

Lemma 4.17. *Assume that the function α is monotone increasing in $[0, \mu]$. Then $\Gamma_n^- = [0, \mu] \setminus Q_{\alpha^n}$ with Q_{α^n} as in (4.7). The corresponding result for Γ_n^+ holds if β is monotone increasing.*

Proof. We shall only prove the result for Γ_n^- . We first prove that $Q_{\alpha^n} \subseteq [0, \mu] \setminus \Gamma_n^-$. Take any $s_0 \in Q_{\alpha^n}$. Then $\alpha^n(s) = \alpha^n(s_0)$ for every $s \in [0, \mu]$ near s_0 , say for every $s \in (s_0 - \delta, s_0 + \delta) \cap [0, \mu]$ for some $\delta > 0$. It follows that if $s \in (s_0 - \delta/2, s_0 + \delta) \cap [0, \mu]$ then $\alpha^n((s - \delta/2) \vee 0) = \alpha^n(s)$ and so $s \notin G_n^-(\delta/2)$. This immediately implies that $s_0 \notin \Gamma_n^-(\delta/2)$, and therefore $s_0 \notin \Gamma_n^-$, as desired.

Now we show that $[0, \mu] \setminus Q_{\alpha^n} \subseteq \Gamma_n^-$. Fix any $s_0 \in [0, \mu] \setminus Q_{\alpha^n}$. We consider two cases: the first, in which $\alpha^n(s) < \alpha^n(s_0)$ for every $s \in [0, s_0)$ and where $s_0 \neq 0$; and the second, in which $\alpha^n(s) > \alpha^n(s_0)$ for every $s \in (s_0, \mu]$ and where $s_0 \neq \mu$. In the first case it is clear that $s_0 \in G_n^-(\delta)$ for every $\delta > 0$, and thus $s_0 \in \Gamma_n^-$, as desired. In the second case we have for every δ , that $s > s_0 \geq (s - \delta) \vee 0$ for every $s > s_0$

sufficiently near s_0 , and thus $\alpha^n(s) > \alpha^n(s_0) \geq \alpha^n((s - \delta) \vee 0)$. Thus $s \in G_n^-(\delta)$ for such s , which implies that $s_0 \in \overline{G_n^-(\delta)} = \Gamma_n^-(\delta)$. As δ is arbitrary, we conclude that $s_0 \in \Gamma_n^-$, again as desired. \square

We present three technical lemmas, followed by the proof of Theorem 4.5. It will be convenient here to denote the set

$$N_\varepsilon(A) = \{s \in W \mid d(s, t) < \varepsilon \text{ for some } t \in A\}$$

where generally (W, d) is a metric space with $A \subseteq W$ and $\varepsilon > 0$.

Lemma 4.18. *Let $g : [c, d] \rightarrow [c, d]$ be continuous and monotone increasing and suppose that $g(s_0) = s_0$ for some $s_0 \in [c, d]$. Let $n \geq 1$. Then s_0 is a point of constancy of g if and only if it is a point of constancy of g^n .*

Proof. It is easily seen that if s_0 is a point of constancy of g then it is a point of constancy of g^n , so we only prove the converse. Suppose therefore that s_0 is not a point of constancy of g . Then either $g(s) < g(s_0) = s_0$ for every $s \in [c, s_0]$ where $s_0 \neq c$, or else $g(s) > g(s_0) = s_0$ for every $s \in (s_0, d]$ where $s_0 \neq d$. In the former case an easy induction shows that $g^n(s) < g^n(s_0) = s_0$ for every $s \in [c, s_0]$ and every n , and so s_0 is not a point of constancy of g^n . A similar argument treats the second case. \square

Lemma 4.19. *Let $g : [c, d] \rightarrow \mathbb{R}$ be continuous. Then for every $\varepsilon > 0$ the set $\Psi_{g, \varepsilon} = S_g \setminus N_\varepsilon(D_g)$ is a finite subset of C_g .*

Proof. Obviously $\Psi_{g, \varepsilon} \subseteq C_g$. We next observe that between any two points of C_g there must lie a point of D_g . Indeed, suppose $s_1, s_2 \in C_g$ with $s_1 < s_2$. Then $g(s) < s$ immediately to the right of s_1 while $g(s) > s$ immediately to the left of s_2 , so necessarily $g(s) = s$ for some $s \in (s_1, s_2)$. Let s_3 denote the smallest such s . Thus $g(s_3) = s_3$ and so $s_3 \in S_g$, while $g(s) < s$ for every $s \in (s_1, s_3)$, which implies that $s_3 \notin C_g$. Thus $s_3 \in D_g$ as desired.

Now take any two points $t_1, t_2 \in \Psi_{g, \varepsilon}$, assuming without loss that $t_1 < t_2$. Then $t_1, t_2 \in C_g$ and so there exists a point $t_3 \in (t_1, t_2) \cap D_g$. But $t_1 \notin N_\varepsilon(D_g)$ and so $|t_1 - t_3| \geq \varepsilon$, which implies that $|t_1 - t_2| \geq \varepsilon$. Thus the points of $\Psi_{g, \varepsilon}$ are separated by a minimum distance ε , so it follows that $\Psi_{g, \varepsilon}$ is a finite set. \square

Lemma 4.20. *Let $g : [c, d] \rightarrow [c, d]$ be continuous and monotone increasing and let $U \subseteq [c, d]$ be a (relatively) open neighborhood of the set S_g of fixed points of g . Then there exists an integer $m \geq 1$ such that for every $s \in [c, d]$ we have that $g^i(s) \in U$ except for at most m indices $i \geq 0$.*

Proof. Without loss we may assume that $U = N_\varepsilon(S_g) \cap [c, d]$ for some $\varepsilon > 0$. Define a quantity

$$\gamma = \min_{s \in [c, d] \setminus U} |g(s) - s|,$$

note that $\gamma > 0$, and let m be an integer such that $m\gamma \geq d - c$. We shall prove that m satisfies the conditions in the statement of the lemma.

Take any $s \in [c, d]$ and consider the points $s_i = g^i(s)$ for $i \geq 1$, with $s_0 = s$. If $g(s) = s$ there is nothing to prove as each $s_i \in S_g$, so assume that $g(s) \neq s$. Without loss we assume that $s_1 = g(s) > s = s_0$. Then using the monotonicity of g one easily proves by induction that $s_{i+1} \geq s_i$ for every i , and denoting $s_* = \lim_{i \rightarrow \infty} s_i$ one checks that $g(s_*) = s_*$. Moreover, $[s_0, s_*) \cap S_g = \emptyset$, that is, the interval $[s_0, s_*)$ does not contain any fixed points of g . This last fact follows because if

$t \in (s_0, s_*)$ were fixed by g , then applying g^i to the inequality $s_0 < t$ would yield $s_i = g^i(s_0) \leq g^i(t) = t$, which is false for large i .

Let $m_+ \geq 0$ denote the unique integer such that

$$|s_i - s_*| < \varepsilon \text{ for } i \geq m_+, \quad |s_i - s_*| \geq \varepsilon \text{ for } 0 \leq i < m_+. \tag{4.29}$$

Now suppose that $0 \leq i < m_+$ is such that $s_i \in U$. Then there exists a fixed point $t \in S_g$ of g with $|s_i - t| < \varepsilon$, and by (4.29) we must have $t \neq s_*$. Indeed, $t > s_*$ is impossible as $s_i \leq s_* - \varepsilon$, and so $t < s_0$ in light of the fact that $[s_0, s_*)$ contains no fixed points. It follows that for $0 \leq j \leq i$ we have $|s_j - t| \leq |s_i - t| < \varepsilon$ and so $s_j \in U$ for such j . Letting $m_- = \min\{i \mid 0 \leq i < m_+ \text{ and } s_i \notin U\}$, with $m_- = m_+$ if this set is empty, one concludes that $s_i \notin U$ if and only if $m_- \leq i < m_+$.

Consider now i in the range $m_- \leq i < m_+$. For such i we have $|s_{i+1} - s_i| = |g(s_i) - s_i| \geq \gamma$, and so $s_{i+1} \geq s_i + \gamma$. Thus $s_{m_+} \geq s_{m_-} + m_0\gamma$ where $m_0 = m_+ - m_-$ is exactly the number of indices $i \geq 0$ for which $s_i \notin U$. Necessarily $m_0\gamma \leq s_{m_+} - s_{m_-} \leq d - c$, and so $m_0 \leq m$ with m as above. This proves the result. \square

Proof of Theorem 4.5. The formula (4.8) follows from Lemma 4.16, together with the definition (3.11) with (3.10), where $\nu = \omega$.

To establish the formula (4.9) we first prove that

$$\rho(F) \leq \sup_{s \in D_\alpha \cup D_\beta} a(s, s). \tag{4.30}$$

Fix $M > \sup_{(s,t) \in \mathcal{S}} a(s, t)$. Let $n \geq 1$ be fixed and take any $\gamma > \sup_{s \in D_\alpha \cup D_\beta} a(s, s)$ which also satisfies $\gamma \leq M$. We shall show that there exists an integer $m = m(\gamma)$ which does not depend on n (but may depend on γ) and a quantity $\delta_0 = \delta_0(n, \gamma)$ such that

$$\lambda_n(\delta) \leq M^m \gamma^{n-m} \tag{4.31}$$

for every $\delta \leq \delta_0$. If this is shown, then by letting $\delta \rightarrow 0+$ in (4.31) one concludes that $\lambda_n \leq M^m \gamma^{n-m}$. By further taking the n^{th} root of this inequality and letting $n \rightarrow \infty$ one has $\rho(F) \leq \gamma$. From the arbitrary choice of γ one concludes (4.30).

With n and γ fixed as above, let $\varepsilon > 0$ be such that

$$\sup_{s \in N_{2\varepsilon}(D_\alpha)} a(s, \alpha(s)) \leq \gamma, \quad \sup_{s \in N_{2\varepsilon}(D_\beta)} a(s, \beta(s)) \leq \gamma, \tag{4.32}$$

both hold. By Lemma 4.19 the sets $\Psi_{\alpha, \varepsilon}$ and $\Psi_{\beta, \varepsilon}$ are finite, and moreover are subsets of Q_α and Q_β respectively, and thus there exists a quantity $\tilde{\varepsilon} > 0$ such that

$$\begin{aligned} \alpha(s) &= \alpha(t) \text{ for every } s \in [t - 2\tilde{\varepsilon}, t + 2\tilde{\varepsilon}] \cap [0, \mu], \text{ if } t \in \Psi_{\alpha, \varepsilon}, \\ \beta(s) &= \beta(t) \text{ for every } s \in [t - 2\tilde{\varepsilon}, t + 2\tilde{\varepsilon}] \cap [0, \mu], \text{ if } t \in \Psi_{\beta, \varepsilon}, \end{aligned} \tag{4.33}$$

where we may also assume without loss that $\tilde{\varepsilon} \leq \varepsilon$. Now let $U_\alpha = N_{\tilde{\varepsilon}}(S_\alpha)$ and $U_\beta = N_{\tilde{\varepsilon}}(S_\beta)$, let m_α and m_β be the integers associated to the sets U_α and U_β with maps α and β , respectively, by Lemma 4.20, and let $m = \max\{m_\alpha, m_\beta\}$. Observe that m does not depend on n . Finally, let $\delta_0 > 0$ be small enough so that

$$|\alpha^i((s - \delta) \vee 0) - \alpha^i(s)| \leq \tilde{\varepsilon}, \quad |\beta^i((s + \delta) \wedge \mu) - \beta^i(s)| \leq \tilde{\varepsilon} \tag{4.34}$$

both hold for every $s \in [0, \mu]$ and $\delta \in [0, \delta_0]$, and $1 \leq i \leq n$.

Now let $\pi \in \mathcal{A}_{n,\delta}$ where $\delta \leq \delta_0$. Let us suppose that $\pi \in \mathcal{A}_{n,\delta}^-$ as in Lemma 4.15, the case $\pi \in \mathcal{A}_{n,\delta}^+$ being handled similarly. We have that $p_i = \alpha^i(p_0)$ for $1 \leq i \leq n$ and we denote $s_* = (p_0 - \delta) \vee 0$ as in (4.22). Let us first show that $\alpha^i(p_0) \notin N_{\tilde{\varepsilon}}(\Psi_{\alpha,\varepsilon})$ for every such i . Indeed, suppose to the contrary that there exists $t \in \Psi_{\alpha,\varepsilon}$ with $|\alpha^i(p_0) - t| < \tilde{\varepsilon}$. Then one has by (4.34) that $|\alpha^i(s_*) - \alpha^i(p_0)| \leq \tilde{\varepsilon}$ and hence that $\alpha^i(s_*), \alpha^i(p_0) \in (t - 2\tilde{\varepsilon}, t + 2\tilde{\varepsilon})$. It now follows from (4.33) that $\alpha^i(p_0) = \alpha^i(s_*)$, and therefore $\alpha^n(p_0) = \alpha^n(s_*)$, a contradiction to (4.22).

Let us next note, by Lemma 4.20, that $\alpha^i(p_0) \in N_{\tilde{\varepsilon}}(S_\alpha)$ for all but at most m indices i in the range $0 \leq i \leq n-1$. For any i in this range for which $\alpha^i(p_0) \in N_{\tilde{\varepsilon}}(S_\alpha)$ there exists a point $t \in S_\alpha$ such that $|\alpha^i(p_0) - t| < \tilde{\varepsilon}$. From the above paragraph we know that $t \notin \Psi_{\alpha,\varepsilon} = S_\alpha \setminus N_\varepsilon(D_\alpha)$, and so $t \in N_\varepsilon(D_\alpha)$. Thus there exists $\tilde{t} \in D_\alpha$ with $|t - \tilde{t}| < \varepsilon$ and hence with $|\alpha^i(p_0) - \tilde{t}| < \tilde{\varepsilon} + \varepsilon \leq 2\varepsilon$. Thus $\alpha^i(p_0) \in N_{2\varepsilon}(D_\alpha)$, which implies by (4.32) that $a(\alpha^i(p_0), \alpha^{i+1}(p_0)) \leq \gamma$. For the other indices i , of which there at most m , one has that $a(\alpha^i(p_0), \alpha^{i+1}(p_0)) \leq M$ and it therefore follows that $a_n(\pi) \leq M^m \gamma^{n-m}$. As $\pi \in \mathcal{A}_{n,\delta}$ was arbitrary, we conclude that (4.31) holds as desired. Thus (4.30) is established.

We now establish the opposite inequality to (4.30). Taking $s_0 \in D_\alpha \cup D_\beta$, say $s_0 \in D_\alpha$ for definiteness, it is enough to prove that

$$\rho(F) \geq a(s_0, s_0). \tag{4.35}$$

By Lemma 4.18 the point s_0 , which is not a point of constancy of α , is also not a point of constancy of α^n , that is, $s_0 \notin Q_{\alpha^n}$. But then Lemma 4.17 implies that $s_0 \in \Gamma_n^-$, and so

$$\omega(F^n) = \lambda_n \geq \lambda_n^- \geq u_n^-(s_0) = a(s_0, s_0)^n \tag{4.36}$$

by Lemma 4.16 and (4.28). Taking the n^{th} root in (4.36), letting $n \rightarrow \infty$, and using (4.8), now gives the desired inequality (4.35). \square

Two corollaries follow directly from Theorem 4.5.

Corollary 4.21. *Assume that both α and β are monotone increasing in $[0, \mu]$. Also assume that there exists $\sigma \in \mathcal{S}_n$ which is an n -cycle such that*

$$a_n(\sigma)^{1/n} > \sup_{s \in D_\alpha \cup D_\beta} a(s, s),$$

a particular case of which (a 1-cycle) is that

$$a(s_0, s_0) > \sup_{s \in D_\alpha \cup D_\beta} a(s, s) \tag{4.37}$$

for some $s_0 \in [0, \mu]$ with $s_0 \in J(s_0)$. Then there exists a solution $x \in K \setminus \{0\}$ to $F(x) = rx$ with $r = r(F)$.

Proof. Theorem 4.5 and Corollary 4.9 ensure that $\rho(F) < r(F)$. The result now follows from Theorem 4.4. \square

Corollary 4.22. *Assume that both α and β are monotone increasing in $[0, \mu]$ and also that (1.5) holds. In addition assume that*

$$\max\{a(0, 0), a(\mu, \mu)\} < a_+ \tag{4.38}$$

where a_+ is as in (4.1). Then there exists a solution $x \in K \setminus \{0\}$ to $F(x) = rx$ with $r = r(F)$.

Proof. We have that $D_\alpha \cup D_\beta \subseteq \{0, \mu\}$ and thus (4.38) implies (4.37) for some $s_0 \in [0, \mu]$ where $s_0 \in J(s_0)$. The result now follows from Corollary 4.21. \square

The next result provides a class of examples for which an eigenfunction fails to exist for the quantity $r = r(F)$. In this example all the conditions of our abstract Theorem 3.4 hold except the requirement that $\rho(F) < r(F)$. This example also shows the necessity of the lipschitz condition $c < 1$ on α and β in Theorem 4.1. Indeed, if α in the following result is lipschitz near 0, then necessarily $c \geq 1$ for its lipschitz constant in a neighborhood.

Proposition 4.23. *Let $\alpha : [0, \mu] \rightarrow [0, \mu]$ be continuous and monotone increasing, and let $\beta : [0, \mu] \rightarrow [0, \mu]$ be continuous. Assume that $\alpha(s) \leq \beta(s)$ for $0 \leq s \leq \mu$ and that $\alpha(s) < s$ for $0 < s < \mu$. Also assume that*

$$\alpha(s) \geq s - ks^2 \tag{4.39}$$

for $0 \leq s \leq \delta$, for some $k > 0$ and $\delta > 0$. Let $a : [0, \mu] \times [0, \mu] \rightarrow [0, \infty)$ be a C^1 function for which $a(0, 0) = 1$ and which satisfies $\frac{\partial a}{\partial s}(s, t) \leq 0$ and $\frac{\partial a}{\partial t}(s, t) \leq 0$ in $[0, \mu] \times [0, \mu]$, with $\frac{\partial a}{\partial s}(0, 0) < 0$ and $\frac{\partial a}{\partial t}(0, 0) < 0$. Then $r(F) = 1$, but there does not exist $x \in K \setminus \{0\}$ with $F(x) = x$.

Proof. As $\sigma = (0, 0) \in \mathcal{S}_1$ is a 1-cycle, we have by Corollary 4.9 that $r(F) \geq a(0, 0) = 1$. On the other hand, our assumptions imply that $a(s, t) \leq 1$ throughout $[0, \mu] \times [0, \mu]$, and so $r(F) \leq 1$ by Theorem 4.3. Thus $r(F) = 1$.

Now suppose there exists $x \in K$, with $\|x\| = 1$, such that $F(x) = x$. We seek a contradiction. Assume without loss that δ is small enough that $2k\delta \leq 1$. Given any $s_0 \in (0, \mu)$ and $n \geq 1$, let $\sigma \in \mathcal{S}_n$ be such that $(F^n(x))(s_0) = a_n(\sigma)x(s_n)$, that is, σ is the element of \mathcal{S}_n at which the maximum (4.13) is achieved. Also define $\pi \in \mathcal{S}_n$ by setting $p_i = \alpha^i(s_0)$ for $0 \leq i \leq n$. By induction one sees from the monotonicity of α that $p_i \leq s_i$ for every such i , indeed, if $p_i \leq s_i$ for some $0 \leq i \leq n - 1$ then $p_{i+1} = \alpha(p_i) \leq \alpha(s_i) \leq s_{i+1}$. Thus $a(s_i, s_{i+1}) \leq a(p_i, p_{i+1})$ by the monotonicity of a , and so $a_n(\sigma) \leq a_n(\pi)$, and hence

$$\begin{aligned} x(s_0) &= (F^n(x))(s_0) = a_n(\sigma)x(s_n) \\ &\leq a_n(\sigma) \leq a_n(\pi) = \prod_{i=0}^{n-1} a(\alpha^i(s_0), \alpha^{i+1}(s_0)). \end{aligned} \tag{4.40}$$

Letting $z_n(s_0)$ denote the final term (the n -fold product) in (4.40), we shall prove that

$$\lim_{n \rightarrow \infty} z_n(s_0) = 0 \tag{4.41}$$

for every $s_0 \in (0, \mu)$. Noting that $z_n(s_0) = z_k(s_0)z_{n-k}(\alpha^k(s_0))$ holds identically, and also because $\lim_{k \rightarrow \infty} \alpha^k(s_0) = 0$, one sees that it suffices to prove that (4.41) holds only for $s_0 \in (0, \delta]$ in order to conclude that it holds for every $s_0 \in (0, \mu)$.

Fixing $s_0 \in (0, \delta]$, we note that $\alpha^i(s_0) \in [0, \delta]$ for every i . Letting $d > 0$ be such that $-d$ is an upper bound for the partial derivatives of a in the square $[0, \delta] \times [0, \delta]$ and that $2d\delta < 1$, we have from (4.40) that

$$x(s_0) \leq \prod_{i=0}^{n-1} \left(1 - d(\alpha^i(s_0) + \alpha^{i+1}(s_0)) \right), \tag{4.42}$$

where we note the terms in this product are all positive. Taking logarithms and making a standard estimate, we see that it is sufficient to prove that

$$\sum_{i=0}^{\infty} \alpha^i(s_0) = \infty \tag{4.43}$$

in order to conclude that the right-hand side of (4.42) approaches zero as $n \rightarrow \infty$. To this end, we claim that

$$\alpha^i(s_0) \geq \frac{s_0}{i+1} \tag{4.44}$$

for every $i \geq 0$, and indeed, (4.43) follows directly from (4.44). Noting that (4.44) holds for $i = 0$, we proceed by induction. If (4.44) holds for a particular i , then from (4.39)

$$\begin{aligned} \alpha^{i+1}(s_0) &\geq \alpha(s_0(i+1)^{-1}) \geq \frac{s_0}{i+1} - k \left(\frac{s_0}{i+1} \right)^2 \\ &\geq \frac{s_0}{i+1} - \frac{s_0}{2(i+1)^2} = \frac{(2i+1)s_0}{2(i+1)^2} \geq \frac{s_0}{i+2}, \end{aligned}$$

where $2ks_0 \leq 2k\delta \leq 1$ was used in the penultimate inequality, and where the final inequality is a simple calculus lemma. Thus (4.44) holds also for $i + 1$, and the proof of the proposition is complete. \square

Proceeding toward the proofs of Theorems 4.1 and 4.2, we need several technical lemmas.

Lemma 4.24. *Let $s, t \in [0, \mu]$ and let $n \geq 1$. Then there exists $\sigma = (s_0, s_1, s_2, \dots, s_n) \in \mathcal{S}_n$ with $s_0 = s$ and $s_n = t$ if and only if $t \in J^n(s)$.*

Proof. This is a simple induction on n , the details of which we omit. \square

For the next lemma we define three quantities which are related to the quantity b_n whose value is given in (4.4). For any sufficiently small $\delta > 0$ we set

$$\begin{aligned} b_n(\delta) &= \max_{\substack{\sigma \in \mathcal{S}_n \\ s_0 \in [\delta, \mu - \delta]}} a_n(\sigma), \\ b_n^R(\delta) &= \max_{\substack{\sigma \in \mathcal{S}_n \\ s_0 \in [0, \mu - \delta]}} a_n(\sigma), & b_n^L(\delta) &= \max_{\substack{\sigma \in \mathcal{S}_n \\ s_0 \in [\delta, \mu]}} a_n(\sigma). \end{aligned} \tag{4.45}$$

The indicated maxima certainly exist, as the function a_n is restricted to a compact subset of \mathcal{S}_n . Note also that

$$b_n(\delta) \leq \min\{b_n^R(\delta), b_n^L(\delta)\} \leq \max\{b_n^R(\delta), b_n^L(\delta)\} = b_n, \tag{4.46}$$

where the final equation in (4.46) requires that $\delta \leq \mu - \delta$. The following lemma describes the growth of the quantities (4.45) with n .

Lemma 4.25. *Assume that α and β are monotone increasing in $[0, \mu]$, that (1.5) holds, and that the function a is strictly positive in \mathcal{S} . Denote $r = r(F)$. Then*

(i) *if Hypothesis X holds then for every $\delta > 0$ there exists $M_\delta > 0$ such that*

$$b_n(\delta) \leq M_\delta r^n \tag{4.47}$$

for every $n \geq 1$;
 (ii) if Hypothesis Y holds then for every $\delta > 0$ there exists $M_\delta^R > 0$ such that

$$b_n^R(\delta) \leq M_\delta^R r^n \tag{4.48}$$

for every $n \geq 1$; and
 (iii) if Hypothesis Z holds then for every $\delta > 0$ there exists $M_\delta^L > 0$ such that

$$b_n^L(\delta) \leq M_\delta^L r^n \tag{4.49}$$

for every $n \geq 1$.

Proof. We begin with some preliminary observations. Suppose that $\sigma \in \mathcal{S}_n$ is such that

$$s_{i_1} \in J^m(s_{i_2}) \tag{4.50}$$

for some $0 \leq i_1 \leq i_2 \leq n$, and some $m \geq 1$. Then by Lemma 4.24 there exists $\pi \in \mathcal{S}_m$ such that $p_0 = s_{i_2}$ and $p_m = s_{i_1}$. It follows that $\tau \in \mathcal{S}_{i_2-i_1+m}$ given by concatenating this subinterval of σ with π , namely

$$\tau = (s_{i_1}, s_{i_1+1}, \dots, s_{i_2-1}, s_{i_2} = p_0, p_1, \dots, p_{m-1}, p_m = s_{i_1}),$$

is a cycle. Thus by Corollary 4.9,

$$\begin{aligned} r^{i_2-i_1+m} &\geq a_{i_2-i_1+m}(\tau) = a_{i_2-i_1}(s_{i_1}, s_{i_1+1}, \dots, s_{i_2-1}, s_{i_2}) a_m(\pi) \\ &\geq a_{i_2-i_1}(s_{i_1}, s_{i_1+1}, \dots, s_{i_2-1}, s_{i_2}) A_-^m, \end{aligned} \tag{4.51}$$

with $A_- > 0$ the minimum of a as defined at the beginning of the section. Let us rewrite (4.51) as

$$a_{i_2-i_1}(s_{i_1}, s_{i_1+1}, \dots, s_{i_2-1}, s_{i_2}) \leq (A_-^{-1}r)^m r^{i_2-i_1}, \tag{4.52}$$

in the form of an upper bound.

We now prove (i). Given $\delta > 0$ let $m = m(\delta)$ be such that $\alpha^m(\mu) \leq \delta$ and $\beta^m(0) \geq \mu - \delta$, the existence of such m following from (1.5) and Hypothesis X. Then

$$J^m(s) = [\alpha^m(s), \beta^m(s)] \supseteq [\alpha^m(\mu), \beta^m(0)] \supseteq [\delta, \mu - \delta] \tag{4.53}$$

for every $s \in [0, \mu]$. Now for any $n \geq 1$ take any $\sigma \in \mathcal{S}_n$ for which $s_0 \in [\delta, \mu - \delta]$, as in the definition of $b_n(\delta)$. Then $s_0 \in J^m(s_n)$ by (4.53), so we may take $i_1 = 0$ and $i_2 = n$ as above, whence (4.52) gives

$$a_n(\sigma) \leq (A_-^{-1}r)^m r^n.$$

Thus (4.47) holds with $M_\delta = (A_-^{-1}r)^m$.

The proof of (ii) is somewhat more technical than the proof of (i), but follows similar lines. Here $\beta(0) = 0$ may occur, although $\beta(\delta) > \delta$ holds. We assume δ is small enough that $a(s, t) \leq r$ for $(s, t) \in \mathcal{S}_\delta^L$, by virtue of the fact that $a(0, 0) < r$. Now with δ fixed we have $\beta : [\delta, \mu] \rightarrow [\delta, \mu]$, and there exists $m = m(\delta)$ such that $\alpha^m(\mu) \leq \delta$ and $\beta^m(\delta) \geq \mu - \delta$. Much as before we have

$$J^m(s) = [\alpha^m(s), \beta^m(s)] \supseteq [\alpha^m(\mu), \beta^m(\delta)] \supseteq [\delta, \mu - \delta], \tag{4.54}$$

but now only for $s \in [\delta, \mu]$. Now for any $n \geq 1$ take any $\sigma \in \mathcal{S}_n$ for which $s_0 \in [0, \mu - \delta]$, as in the definition of $b_n^R(\delta)$. If it is the case that $s_i \leq \delta$ for every $0 \leq i \leq n$, then $a(s_{i-1}, s_i) \leq r$ for $1 \leq i \leq n$, and so $a_n(\sigma) \leq r^n$. This gives (4.48) with $M_\delta^R = 1$, and we are done. Assume therefore that $s_i > \delta$ for some i , and let $i_1 \leq i_2$ denote the first, respectively last, indices in the range $0 \leq i \leq n$ for which $s_i > \delta$. We claim that (4.50) and thus (4.52) hold. Indeed, by (4.54) it is enough to prove that $s_{i_1} \leq \beta^m(\delta)$ in order to conclude (4.50). If $i_1 > 0$, then as $s_{i_1} \in J(s_{i_1-1})$ where $s_{i_1-1} \leq \delta$ we have $s_{i_1} \leq \beta(s_{i_1-1}) \leq \beta(\delta) \leq \beta^m(\delta)$, where we have used the definition of i_1 . And if $i_1 = 0$ then $s_{i_1} = s_0 \leq \mu - \delta \leq \beta^m(\delta)$ by assumption and from (4.54).

Having established (4.52), we may now make the estimate

$$\begin{aligned} a_n(\sigma) &= a_{i_1-1}(s_0, s_1, \dots, s_{i_1-2}, s_{i_1-1}) \\ &\quad \times a(s_{i_1-1}, s_{i_1}) a_{i_2-i_1}(s_{i_1}, s_{i_1+1}, \dots, s_{i_2-1}, s_{i_2}) \\ &\quad \times a(s_{i_2}, s_{i_2+1}) a_{n-i_2-1}(s_{i_2+1}, s_{i_2+2}, \dots, s_{n-1}, s_n) \\ &\leq r^{i_1-1} A_+ (A_-^{-1} r)^m r^{i_2-i_1} A_+ r^{n-i_2-1} = M_\delta^R r^n, \end{aligned} \tag{4.55}$$

with $M_\delta^R = (r^{-1} A_+)^2 (A_-^{-1} r)^m$, with A_+ the maximum of a in \mathcal{S} . In making the estimate (4.55) we have used the fact that $a(s_{i-1}, s_i) \leq r$ for $1 \leq i \leq i_1 - 1$ and for $i_2 + 2 \leq i \leq n$, which follows from the definitions of i_1 and i_2 , and we have also used the estimate (4.52). Note also that (4.55) must be interpreted appropriately when $i_1 = 0$ or 1 , when $i_2 = n - 1$ or n , and when $i_1 = i_2$.

The proof of (iii) is similar to that of (ii), and is omitted. \square

Lemma 4.26. *Assume that α and β are monotone increasing in $[0, \mu]$, that (1.5) holds, and that the function a is strictly positive in \mathcal{S} . Also assume that at least one of Hypotheses X, Y, or Z holds. Additionally, if Hypothesis Y is false then assume there exist $\delta > 0$ and $c < 1$ such that Hypothesis Y' holds; and if Hypothesis Z is false then assume there exist $\delta > 0$ and $c < 1$ such that Hypothesis Z' holds. Then there exists a constant $M > 0$ such that*

$$b_n \leq M r^n \tag{4.56}$$

for every $n \geq 1$, with b_n given by (4.4) and where $r = r(F)$.

Proof. Let $0 < \delta \leq \mu/2$ be small enough that whichever of Hypotheses Y' or Z' is assumed holds. If neither of these hypotheses is assumed then take any $0 < \delta \leq \mu/2$. Now take $n \geq 1$ and let $\sigma \in \mathcal{S}_n$ be such that $b_n = a_n(\sigma)$, as in (4.4). If $s_0 \in [\delta, \mu - \delta]$ then $b_n = b_n(\delta) = b_n^R(\delta) = b_n^L(\delta)$ by (4.45) and (4.46). As at least one of Hypotheses X, Y, or Z is assumed to hold, we have (4.56) with $M = M_\delta, M_\delta^R$, or M_δ^L , respectively, by whichever of (4.47), (4.48), or (4.49) holds.

Suppose therefore that $s_0 \notin [\delta, \mu - \delta]$. For definiteness assume $s_0 \in [0, \delta)$, the case $s_0 \in (\mu - \delta, \mu]$ being treated similarly. Now $b_n = b_n^R(\delta)$, so if Hypothesis Y holds we have (4.48) and hence (4.56) with $M = M_\delta^R$. Assume therefore that Hypothesis Y is false, and that consequently Hypothesis Y' is assumed to hold. Let $\pi \in \mathcal{S}_n$ be such that $p_0 = \delta$ and $\pi|\sigma$. Then by Lemma 4.14 we have (4.20) for some $0 \leq k \leq n$. With this k , and with the lipschitz properties of α and of a near 0 and $(0, 0)$ respectively,

one has that

$$\begin{aligned}
 0 &\leq \log a_n(\sigma) - \log a_n(\pi) \\
 &= \sum_{i=1}^k \left(\log a(s_{i-1}, s_i) - \log a(p_{i-1}, p_i) \right) + \log a(s_k, s_{k+1}) - \log a(p_k, p_{k+1}) \\
 &\leq 2C \sum_{i=0}^k |s_i - p_i| + \log a(s_k, s_{k+1}) - \log a(p_k, p_{k+1}) \\
 &\leq 2C \sum_{i=0}^k |s_i - p_i| + \log(A_-^{-1} A_+) \tag{4.57} \\
 &\leq 2C \sum_{i=0}^k |\alpha^i(s_0) - \alpha^i(p_0)| + \log(A_-^{-1} A_+) \\
 &\leq 2C \sum_{i=0}^k c^i \delta + \log(A_-^{-1} A_+) < \frac{2C\delta}{1-c} + \log(A_-^{-1} A_+),
 \end{aligned}$$

where $C > 0$ is a lipschitz constant for $\log a$ in \mathcal{S}_δ^L , and where we have used the fact that $\alpha^i : [0, \delta] \rightarrow [0, \delta]$ has lipschitz constant c^i and that $s_i, p_i \leq \alpha^i(p_0) \leq \delta$ for $0 \leq i \leq k$. As Hypothesis Y is false we have that either Hypothesis X or Z holds, and so

$$a_n(\pi) \leq \max\{b_n(\delta), b_n^L(\delta)\} \leq \max\{M_\delta, M_\delta^L\} r^n.$$

We conclude from this and from (4.57) that

$$\begin{aligned}
 b_n &= a_n(\sigma) = \frac{a_n(\sigma)}{a_n(\pi)} a_n(\pi) \leq \frac{a_n(\sigma)}{a_n(\pi)} \max\{M_\delta, M_\delta^L\} r^n \\
 &\leq \exp(2C\delta/(1-c)) A_-^{-1} A_+ \max\{M_\delta, M_\delta^L\} r^n.
 \end{aligned}$$

This gives (4.56) with $M = \exp(2C\delta/(1-c)) A_-^{-1} A_+ \max\{M_\delta, M_\delta^L\}$. \square

Lemma 4.27. *Assume that α and β are monotone increasing in $[0, \mu]$, that (1.5) holds, and that the function a is strictly positive in \mathcal{S} . Also assume that Hypothesis X holds and that*

$$a(0, 0) < a(\mu, \mu). \tag{4.58}$$

Then there exist $\delta > 0$ and $m \geq 1$ such that the following holds. Let $s \in [0, \mu]$ and $n \geq 1$ be given and suppose $\sigma \in \mathcal{S}_n$ maximizes $a_n(\sigma)$ among all elements of \mathcal{S}_n for which $s_0 = s$. Suppose further for some indices $0 \leq i_1 \leq i_2 \leq n$ that

$$s_i \leq \delta, \quad i_1 \leq i \leq i_2. \tag{4.59}$$

Then necessarily $i_2 - i_1 < m$.

The corresponding result when $a(0, 0) > a(\mu, \mu)$ holds.

Proof. Let δ be small enough that $\alpha(\mu) < \mu - \delta$ and $\beta(0) > \delta$ both hold, and in addition that

$$\min_{\substack{(p, \tilde{p}) \in \mathcal{S}_\delta^R \\ (s, \tilde{s}) \in \mathcal{S}_\delta^L}} \frac{a(p, \tilde{p})}{a(s, \tilde{s})} = C > 1, \tag{4.60}$$

where (4.60) defines the quantity C . Such δ exists by Hypothesis X and by (4.58). With δ fixed, let \tilde{m} be such that $\alpha^{\tilde{m}}(\mu) \leq \delta$ and $\beta^{\tilde{m}}(0) \geq \mu - \delta$. Finally let m be such that

$$(A_+^{-1}A_-)^{2\tilde{m}}C^{m-2\tilde{m}+1} > 1. \tag{4.61}$$

We keep m and \tilde{m} fixed for the remainder of this proof.

Now suppose $\sigma \in \mathcal{S}_n$ is such that (4.59) holds for some $0 \leq i_1 \leq i_2 \leq n$ and that $i_2 - i_1 \geq m$. Then it is enough for us to show that there exists $\pi \in \mathcal{S}_n$, with $p_0 = s_0$, such that

$$a_n(\pi) > a_n(\sigma). \tag{4.62}$$

Without loss we may suppose that either $i_2 = n$, or else that $i_2 < n$ and $s_{i_2+1} > \delta$. Furthermore we may suppose that $i_2 - i_1 = m$. We shall construct π by replacing some of the terms s_i in σ in the range $i_1 + 1 \leq i \leq i_2$ with new terms p_i , so that the new sequence belongs to \mathcal{S}_n and satisfies (4.62). Note that in this construction $p_0 = s_0$ as $i = 0$ is not in the replacement range. For ease of notation let us denote

$$P(j, k) = \prod_{i=j}^k \frac{a(p_{i-1}, p_i)}{a(s_{i-1}, s_i)}$$

in the calculations below.

Three cases arise. First suppose that $i_2 = n$. Then define π by

$$p_i = \beta^{i-i_1}(s_{i_1}), \quad i_1 + 1 \leq i \leq i_2, \tag{4.63}$$

with $p_i = s_i$ for all other values of i . Clearly $\pi \in \mathcal{S}_n$ and

$$\begin{aligned} \frac{a_n(\pi)}{a_n(\sigma)} &= P(i_1 + 1, i_1 + \tilde{m})P(i_1 + \tilde{m} + 1, i_2) \\ &\geq (A_+^{-1}A_-)^{\tilde{m}}C^{i_2-i_1-\tilde{m}} \geq (A_+^{-1}A_-)^{2\tilde{m}}C^{m-2\tilde{m}+1} > 1, \end{aligned}$$

where we observe for $i_1 + \tilde{m} \leq i \leq i_2$ that $p_i \geq \beta^{\tilde{m}}(s_{i_1}) \geq \beta^{\tilde{m}}(0) \geq \mu - \delta$ and that $s_i \leq \delta$, and where we use (4.60) and (4.61).

Next suppose that $i_2 < n$ and that $s_{i_2+1} \in (\delta, \mu - \delta]$. Define

$$p_i = \beta^{i-i_1}(s_{i_1}), \quad i_1 + 1 \leq i \leq i_2 - \tilde{m} + 1.$$

Note that (4.53), but with \tilde{m} in place of m , holds for every $s \in [0, \mu]$, and hence $s_{i_2+1} \in J^{\tilde{m}}(p_{i_2-\tilde{m}+1})$. By Lemma 4.24 there exist p_i for $i_2 - \tilde{m} + 2 \leq i \leq i_2 + 1$ such that $(p_{i_2-\tilde{m}+1}, p_{i_2-\tilde{m}+2}, \dots, p_{i_2}, p_{i_2+1}) \in \mathcal{S}_{\tilde{m}}$ and $p_{i_2+1} = s_{i_2+1}$. Set $p_i = s_i$ for i outside the range $i_1 + 1 \leq i \leq i_2$. Then

$$\begin{aligned} \frac{a_n(\pi)}{a_n(\sigma)} &= P(i_1 + 1, i_1 + \tilde{m})P(i_1 + \tilde{m} + 1, i_2 - \tilde{m} + 1)P(i_2 - \tilde{m} + 2, i_2 + 1) \\ &\geq (A_+^{-1}A_-)^{\tilde{m}}C^{i_2-i_1-2\tilde{m}+1}(A_+^{-1}A_-)^{\tilde{m}} = (A_+^{-1}A_-)^{2\tilde{m}}C^{m-2\tilde{m}+1} > 1, \end{aligned}$$

where for $i_1 + \tilde{m} \leq i \leq i_2 - \tilde{m} + 1$ we have that $p_i \geq \mu - \delta$ and $s_i \leq \delta$.

Finally suppose that $i_2 < n$ and $s_{i_2+1} \in (\mu - \delta, \mu]$. Let p_i be as in (4.63) in the indicated range. Let us now observe that $s_{i_2+1} \in J(p_{i_2})$. Indeed, $s_{i_2} \leq \delta < \beta(0) \leq$

$\beta(p_{i_2-1}) = p_{i_2}$ and so $s_{i_2+1} \leq \beta(s_{i_2}) \leq \beta(p_{i_2})$. And $s_{i_2+1} > \mu - \delta > \alpha(\mu) \geq \alpha(p_{i_2})$. Thus by setting $p_i = s_i$ outside the range $i_1 + 1 \leq i \leq i_2$, we have $\pi \in \mathcal{S}_n$, and

$$\begin{aligned} \frac{a_n(\pi)}{a_n(\sigma)} &= P(i_1 + 1, i_1 + \tilde{m})P(i_1 + \tilde{m} + 1, i_2)P(i_2 + 1, i_2 + 1) \\ &\geq (A_+^{-1}A_-)^{\tilde{m}}C^{i_2-i_1-\tilde{m}}(A_+^{-1}A_-) \geq (A_+^{-1}A_-)^{2\tilde{m}}C^{m-2\tilde{m}+1} > 1 \end{aligned}$$

as $p_i \geq \mu - \delta$ and $s_i \leq \delta$ in the range $i_1 + \tilde{m} \leq i \leq i_2$.

We see that in every case we have constructed $\pi \in \mathcal{S}_n$ with $p_0 = s_0$ and with (4.62) holding, as desired. \square

Proof of Theorem 4.1. The proof of the existence of an eigenfunction $x \in K \setminus \{0\}$ with eigenvalue r falls into three cases. For case one we assume the inequality (4.38), with a_+ as in (4.1). Then the existence of x follows from Corollary 4.22.

For cases two and three we assume that

$$a_- = \min\{a(0, 0), a(\mu, \mu)\} \leq \max\{a(0, 0), a(\mu, \mu)\} = a_+, \tag{4.64}$$

where (4.64) serves as the definition of a_- (the second equality in (4.64) is our assumption, not the definition of a_+). Here we shall use Corollary 3.11. Let us note by Corollary 4.9 that $a(s, s) \leq r$ for every $s \in [0, \mu]$, as every (s, s) is a 1-cycle, and thus

$$a_+ \leq r. \tag{4.65}$$

Letting $e \in K$ denote the function which is identically 1, we see by (4.4) of Theorem 4.3 and by (4.13) of Proposition 4.8, that $\|F^n(e)\| = b_n$. If we are able to apply Lemma 4.26 then it will follow that this sequence enjoys the estimate (4.56) which is the inequality (3.21) necessary for Corollary 3.11. Let us observe that the conditions of Lemma 4.26 hold. Either $a(0, 0) = a_+$ or $a(\mu, \mu) = a_+$ by (4.64) and so Hypothesis X is assumed. If Hypothesis Y fails then necessarily $a(0, 0) = r$, hence $a(0, 0) = a_+$ by (4.65), and we assume Hypothesis Y'. Similarly if Hypothesis Z fails then Hypothesis Z' holds. Let us also observe that $r > 0$, otherwise $\|F(e)\| = b_1 = 0$ by (4.56), contradicting the positivity of a .

The other condition of Corollary 3.11 which we verify is that $\overline{\gamma_G^+(e)}$ is compact, where $G(x) = r^{-1}F(x)$. The set $\gamma_G^+(e)$ is certainly bounded, as $\|G^n(e)\| \leq M$ by (4.56), and so we must show the sequence of functions $x_n = G^n(e)$ is equicontinuous. It is enough to show the sequence of functions y_n defined by $y_n(s) = \log x_n(s)$ is equicontinuous, and indeed

$$y_n(s) = \max\{w_n(\sigma) \mid \sigma \in \mathcal{S}_n \text{ and } s_0 = s\}$$

by (4.13), where

$$w_n(\sigma) = \log(r^{-n}a_n(\sigma))$$

for $\sigma \in \mathcal{S}_n$.

We distinguish cases two and three based upon a strict inequality, or an equality, in (4.64). For case two we assume (4.64) together with the strict inequality $a_- < a_+$. Thus either

$$a(\mu, \mu) < a(0, 0) = a_+, \tag{4.66}$$

in which case we assume that Hypothesis Y' holds, or else

$$a(0, 0) < a(\mu, \mu) = a_+, \tag{4.67}$$

in which case Hypothesis Z' is assumed. Let δ be as in whichever of Hypotheses Y' or Z' is taken here. We assume further that δ is small enough that the conclusions of Lemma 4.27 hold, and we let m be as in that result. Also, for any $\eta > 0$ define

$$\varphi(\eta) = \max\{|w(s, t) - w(\tilde{s}, \tilde{t})| \mid (s, t), (\tilde{s}, \tilde{t}) \in \mathcal{S} \text{ with } |s - \tilde{s}| + |t - \tilde{t}| \leq \eta\},$$

where $w(s, t) = w_1(s, t) = \log(r^{-1}a(s, t))$. Finally, let $\tilde{m} \geq 1$ be such that $\alpha^{\tilde{m}}(\mu) \leq \delta$ and $\beta^{\tilde{m}}(0) \geq \mu - \delta$, where such \tilde{m} exists by (1.5) and Hypothesis X.

Now let $\eta > 0$ and take distinct points $s_0, p_0 \in [0, \mu]$ with $|s_0 - p_0| < \eta$. Assuming without loss that $y_n(s_0) \geq y_n(p_0)$, let $\sigma \in \mathcal{S}_n$ be such that $y_n(s_0) = w_n(\sigma)$ and let $\pi \in \mathcal{S}_n$ through p_0 be such that $\pi|_\sigma$, and so $\pi|_\eta\sigma$. Now assume further that $s_0 < p_0$. The proof when $s_0 > p_0$, which will be left to the reader, is analogous. By Lemma 4.14 one has (4.20) for some $0 \leq k \leq n$, and much as in (4.57) we have

$$0 \leq y_n(s_0) - y_n(p_0) \leq w_n(\sigma) - w_n(\pi) = \sum_{i=1}^{k+1} \left(w(s_{i-1}, s_i) - w(p_{i-1}, p_i) \right). \quad (4.68)$$

If $k < m + \tilde{m}$ then

$$|y_n(s_0) - y_n(p_0)| \leq \sum_{i=1}^{k+1} \varphi(\psi_{i-1}(\eta) + \psi_i(\eta)) \leq \sum_{i=1}^{m+\tilde{m}} \varphi(\psi_{i-1}(\eta) + \psi_i(\eta)), \quad (4.69)$$

where we have used the inequality $|s_i - p_i| \leq \psi_i(\eta)$ from Lemma 4.10. Suppose on the other hand that $m + \tilde{m} \leq k$. We claim that necessarily (4.66) holds. Assuming to the contrary that we have (4.67), we have by (4.20) that $s_i \leq \alpha^i(p_0) \leq \alpha^{\tilde{m}}(\mu) \leq \delta$ for $\tilde{m} \leq i \leq k$, so by Lemma 4.27 necessarily $k - \tilde{m} < m$, a contradiction. Thus (4.67) is false and (4.66) holds, so as noted earlier we have Hypothesis Y' .

We now estimate (4.68), where we are still assuming that $m + \tilde{m} \leq k$ and so Hypothesis Y' holds. From the fact that $\alpha : [0, \delta] \rightarrow [0, \delta]$ has lipschitz constant $c < 1$, and from the fact that $s_i, p_i \leq \alpha^i(p_0) \leq \delta$ for $\tilde{m} \leq i \leq k$ while $s_{k+1} = p_{k+1}$, we obtain as an upper bound for the absolute value of the summation in (4.68)

$$\begin{aligned} & \sum_{i=1}^{\tilde{m}} \varphi(\psi_{i-1}(\eta) + \psi_i(\eta)) + 2C \sum_{i=\tilde{m}}^k |s_i - p_i| + |w(s_k, s_{k+1}) - w(p_k, p_{k+1})| \\ & \leq \sum_{i=1}^{\tilde{m}} \varphi(\psi_{i-1}(\eta) + \psi_i(\eta)) + 2C \sum_{i=\tilde{m}}^k c^{i-\tilde{m}} |s_{\tilde{m}} - p_{\tilde{m}}| + \varphi(|s_k - p_k|) \\ & \leq \sum_{i=1}^{\tilde{m}} \varphi(\psi_{i-1}(\eta) + \psi_i(\eta)) + \left(\frac{2C}{1-c} \right) \psi_{\tilde{m}}(\eta) + \varphi(\alpha^{\tilde{m}}(p_0) - \alpha^{\tilde{m}}(s_0)) \\ & \leq \sum_{i=1}^{\tilde{m}} \varphi(\psi_{i-1}(\eta) + \psi_i(\eta)) + \left(\frac{2C}{1-c} \right) \psi_{\tilde{m}}(\eta) + \varphi(\psi_{\tilde{m}}(\eta)), \end{aligned} \quad (4.70)$$

where $C > 0$ is a lipschitz constant for w in \mathcal{S}_δ^L . We thus have the upper bounds (4.69) and (4.70) for $|y_n(s_0) - y_n(p_0)|$ in the cases that $k < m + \tilde{m}$ and $m + \tilde{m} \leq k$, respectively. Letting $\xi(\eta)$ denote the maximum of the final sum in (4.69) and the expression in the last line of (4.70), and noting that m and \tilde{m} are independent of n , we see that $|y_n(s_0) - y_n(p_0)| \leq \xi(\eta)$ whenever $|s_0 - p_0| \leq \eta$. Thus the sequence

of functions y_n is equicontinuous, as desired. This completes the proof of existence for case two.

For case three we have that

$$a(0, 0) = a(\mu, \mu) = a_+.$$

The proof here is similar to that of case two, although somewhat easier in that both Hypotheses Y' and Z' hold. We take \tilde{m} , σ , π , and k as before, with $|s_0 - p_0| \leq \eta$, and obtain (4.68). If $k < \tilde{m}$ then (4.69) holds with \tilde{m} in place of $m + \tilde{m}$. If $\tilde{m} \leq k$ then there is no need to establish Hypothesis Y' as we did in case two, as this condition is assumed. Thus we may proceed directly to obtain (4.70) which is an upper bound for the absolute value of the summation in (4.68). This completes case three.

The existence of an eigenfunction x having been established, the claims about the strict positivity of x and of r follow from Proposition 4.12. \square

Proof of Theorem 4.2. Let $K_0 \subseteq C[0, \beta_0]$ denote the cone of nonnegative functions and consider the operator $F_0 : K_0 \rightarrow K_0$ given by (1.2) but with s restricted to the interval $[0, \beta_0]$. Then Theorem 4.1 applied to the operator F_0 implies the existence of $x \in K_0 \setminus \{0\}$ with $F_0(x) = r_0 x$, where $r_0 = r_{K_0}(F_0) > 0$ is the cone spectral radius for the operator F_0 . Moreover, x is strictly positive in $[0, \beta_0]$.

Extend x to the interval $[0, \mu]$ by letting equation (1.10) with $\lambda = r_0$ define $x(s)$ for $s \in (\beta_0, \mu]$, and observe that x so defined is strictly positive and continuous in $[0, \mu]$. Thus $x \in \text{int}(K) \subseteq K \setminus \{0\}$, and so by (2.18) of Theorem 2.2 we have $r(F) = \mu(x)$, where clearly $\mu(x) = r_0$. This gives the result. \square

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Received March 2001; revised March 2002.