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Applications of the Birkhoff–Hopf theorem to the spectral theory of positive linear operators

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Introduction

This paper may be regarded as a sequel to our earlier paper [**19**], where we give an elementary and self-contained proof of a very general form of the Hopf theorem on order-preserving linear operators in partially ordered vector spaces (reproduced here as Theorem 1·1).

Versions of this theorem and related ideas have been used by various authors to study both linear and nonlinear integral equations (Thompson[**41**], Bushell[**9**, **11**], Potter[**38**, **39**], Eveson[**16**, **17**], Bushell and Okrasiński[**12**, **13**]); the convergence properties of nonlinear maps (Nussbaum[**32**, **33**]); so-called DAD theorems (Borwein, Lewis and Nussbaum[**8**]) and in the proof of weak ergodic theorems (Fujimoto and Krause[**20**], Nussbaum[**34**]).

The original motivation for this theorem was, however, its application to the study of the spectral properties of a class of order-preserving linear maps. This class includes matrices with strictly positive entries and integral operators with strictly positive, continuous kernels (Birkhoff[**5**, **6**, **7**], Hopf[**22**, **23**]).

These applications were further developed by a number of authors (Ostrowski[**36**], Bauer[**1**], Bushell[**10**], Eveson[**18**]). Similar ideas were developed by Pokornyi[**37**], Zabreiko, Krasnosel'skii and Pokornyi[**42**] and Krasnosel'skii and Sobolev[**28**]; an exposition can be found in the recent book [**27**] by Krasnosel'skii, Lifshits and Sobolev. These authors were apparently unaware of the earlier, directly relevant work of Hopf, Birkhoff, Bauer and Ostrowski.

We shall show that these results follow by elementary arguments from a sufficiently general version of the Birkhoff–Hopf theorem, such as that proved in [**19**]. If L is a suitable operator, then L has a unique normalized positive eigenvector v, the corresponding eigenvalue is algebraically simple and is equal to the spectral radius $r(L)$ of L and for any positive vector x, $L^n x/\|L^n x\|$ converges at a geometric rate to v. We shall also give an exact formula for the so-called spectral clearance $q(L)$ of L defined by

$$q(L) = \sup\left\{\frac{|\lambda|}{r(L)} : \lambda \in \sigma(L), \lambda \neq r(L)\right\}$$

which for these operators is always less than 1. The same formula gives an estimate for the radius of the essential spectrum of L, $r_e(L)$; in fact $r_e(L) \leqslant q(L)\, r(L)$.

Related results in evolving degrees of precision were obtained in [37], [42], [28] and [27], but we believe that our methods, using the Birkhoff–Hopf theorem, provide a more natural approach to sharper theorems.

1. *Preliminary definitions and results*

We begin with some definitions. These ideas are discussed at greater length in [19].

Definition 1·1. A non-empty subset C of a real vector space V is called a wedge if it is closed under addition and non-negative scalar multiplication; that is, if for all x, $y \in C$ and real numbers $\lambda, \mu \geqslant 0$, $\lambda x + \mu y \in C$. If a wedge C has the additional property that the only vector x for which both $x \in C$ and $-x \in C$ is the zero vector, then C is called a cone (with vertex at 0). A cone C induces an order relation on V by the rule

$$x \leqslant y \quad \text{if} \quad y - x \in C.$$

If $x \in C$ and $y \in V$, x is said to dominate y if there exist real numbers α and β such that

$$\alpha x \leqslant y \leqslant \beta x.$$

If x dominates y and y dominates x (equivalently, if $\alpha x \leqslant y \leqslant \beta x$ for some $\alpha, \beta > 0$) then x and y are called comparable, written $x \sim y$ or, if we wish to emphasise the cone inducing the order relation, $x \sim_C y$.

If x dominates y and $x \neq 0$, define

$$m(y/x) = \sup\{\alpha \in \mathbb{R} : \alpha x \leqslant y\},$$
$$M(y/x) = \inf\{\beta \in \mathbb{R} : y \leqslant \beta x\},$$
$$\omega(y/x) = M(y/x) - m(y/x),$$

and if x is comparable to y, let

$$d(x, y) = \log \frac{M(x/y)}{m(x/y)}.$$

The quantity $\omega(y/x)$ is called the oscillation of y over x; d is called the Hilbert projective metric. We make the additional convention that $d(0,0) = 0$ and $\omega(0/0) = 0$.

The cone C is called almost Archimedean if whenever $x \in C$ and $y \in V$ are such that for all $\epsilon > 0$,

$$-\epsilon x \leqslant y \leqslant \epsilon x$$

it follows that $y = 0$. This is equivalent (see Jameson [25]) to the property that C intersects every two-dimensional subspace F of V in a set whose relative closure in F is a cone.

The reader may easily verify the following elementary properties of ω and d, or refer to [19] for the proofs of these and other results:

Proposition 1·1. *Let V be a real vector space partially ordered by a cone C. If $x \in C$, y, $z \in V$ are such that x dominates both y and z, and $\lambda, \mu, \nu \in \mathbb{R}$ with $\nu > 0$, then*

1. $\omega(y/x) \geqslant 0$;
2. $\omega(\lambda x + \mu y/\nu x) = |\mu|\,\nu^{-1}\omega(y/x)$;
3. $\omega(y + z/x) \leqslant \omega(y/x) + \omega(z/x)$.

Comparability is an equivalence relation on C; if x, y, z are comparable elements of C and $\lambda, \mu \in \mathbb{R}$, $\lambda, \mu > 0$ then

1. $d(x, y) \geqslant 0$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leqslant d(x, z) + d(z, y)$;
4. $d(\lambda x, \mu y) = d(x, y)$.

If, in addition, C is almost Archimedean then

1. *if x dominates y and $\omega(y/x) = 0$ then $x = \lambda y$ for some $\lambda \in \mathbb{R}$;*
2. *if x is comparable to y and $d(x, y) = 0$ then $x = \lambda y$ for some $\lambda \in \mathbb{R}$, $\lambda > 0$.*

Definition 1·2. Let C be a cone in a real vector space V. For $u \in C$, let P_u, the component of C containing u, be defined by

$$P_u = \{x \in V : x \text{ is comparable to } u\}.$$

Clearly, $K_u := P_u \cup \{0\}$ is a cone in V; if C is a closed cone in a Banach space, then K_u is not in general closed. We leave to the reader the proof of the following simple lemma:

Lemma 1·1. *Let C be a cone in a real vector space V. If $u \in C \backslash \{0\}$, let P_u be as in Definition 1·2 and let $K_u = P_u \cup \{0\}$. Then for all $x, y \in P_u$, x and y are comparable in both K_u and C and*

$$M(y/x; K_u) = M(y/x; C); \quad m(y/x; K_u) = m(y/x; C).$$

In particular, we have

$$d(x, y; K_u) = d(x, y; C); \quad \omega(y/x; K_u) = \omega(y/x; C).$$

If C is a cone in a real normed linear space E and $u \in C$, we shall call the component P_u normal if

$$\sup\{\|x\| : 0 \leqslant x \leqslant u\} < \infty.$$

This is easily seen to be independent of the choice of u; that is, if $P_u = P_v$ (equivalently, if $u \sim_C v$) then the sets $\{\|x\| : 0 \leqslant x \leqslant u\}$ and $\{\|x\| : 0 \leqslant x \leqslant v\}$ are either both bounded or both unbounded.

The cone C is called normal if there exists a constant γ such that for all $x, y \in C$ with $0 \leqslant x \leqslant y$ we have $\|x\| \leqslant \gamma\|y\|$. It may be shown that if C is a closed cone in a Banach space and all its components are normal, then C itself is normal. These concepts are discussed for general topological vector spaces in Schaefer[**40**, chapter 3, section 5].

A wedge C in E is called reproducing if

$$E = \{x - y : x, y \in C\}$$

or total if E is the closure of $\{x - y : x, y \in C\}$.

As usual, E^* denotes the Banach space of continuous, real-valued linear maps from E to $\mathbb{R}$. If C is a cone or wedge, its dual C^* is defined by

$$C^* = \{h \in E^* : h(x) \geqslant 0 \text{ for all } x \in C\}.$$

It is known (see Krasnosel'skii[**26**, chapter 1] or Deimling[**14**, chapter 19]) that if C is a closed cone in a Banach space then C is normal if and only if C^* is reproducing, and C is reproducing if and only if C^* is normal.

Definition 1·3. If $u \in C$, where C is a closed cone in a Banach space E, we define a normed linear space E_u by

$$E_u = \{x \in E : \text{there exists } \alpha \geqslant 0 \text{ with } -\alpha u \leqslant x \leqslant \alpha u\}.$$

The norm on E_u is defined by

$$|x|_u = \inf\{\alpha \geqslant 0\colon -\alpha u \leqslant x \leqslant \alpha u\}.$$

If the component P_u is normal, it is not hard to see that there exists a constant M such that for all $x \in E_u$, $\|x\| \leqslant M|x|_u$; thus, the inclusion map of E_u into E is continuous if P_u is normal.

If C is a closed cone in a Banach space E, $u \in C$ and P_u is the component of u in C, define

$$\Sigma_u = \{x \in P_u\colon \|x\| = 1\}.$$

If $d(x, y; C) = d(x, y)$ denotes Hilbert's projective metric, we know from Proposition 1·1 that (Σ_u, d) is a metric space. If C is the cone in a Banach lattice, Birkhoff proved that (Σ_u, d) is a complete metric space. More generally, if P_u is normal, it follows easily from the work of A. C. Thompson [**41**] that (Σ_u, d) is complete. Furthermore, the topology induced by d is the same as that induced by $|\cdot|_u$. (Thompson actually used a closely related metric $\bar{d}$. The reader should consult section 2 of [**32**] for more details on this point.)

Closely related results were obtained by Bauer and Bear [**3**, **4**], who seem to have been unaware of Thompson's work. Finally, the following theorem (see lemma 1 in [**42**]) was obtained by Zabreiko, Krasnosel'skii and Pokornyi, who apparently were unaware of the work of Birkhoff, Thompson, Bauer and Bear.

LEMMA 1·2. *Let C be a closed cone in a Banach space E and let d denote Hilbert's projective metric on C. For $u \in C\backslash\{0\}$, let P_u and E_u be as in Definitions* 1·2 *and* 1·3. *Let*

$$\Sigma_u = \{x \in P_u\colon \|x\| = 1\}.$$

Then the following conditions are equivalent:
1. *(Σ_u, d) is a complete metric space.*
2. *$(E_u, |\cdot|_u)$ is a Banach space.*
3. *P_u is normal, that is* $\sup\{\|z\|: 0 \leqslant z \leqslant u\} < \infty$.

It follows from this that if C is normal, then $|\cdot|_u$ dominates $\|\cdot\|$ in the subspace E_u, and $|\cdot|_u$ induces the same topology as d on the component $P_u \subseteq E_u$. Thus, d dominates $\|\cdot\|$ on P_u. It is useful to have an explicit estimate for $\|\cdot\|$ in terms of d, since we shall be proving d convergence of a sequence of iterates and wish to interpret this in terms of $\|\cdot\|$ convergence. There are a number of estimates in the literature; the following is from [**35**].

LEMMA 1·3. *Let C be a normal cone in a normed linear space X, so there exists a constant $\gamma > 0$ such that for all $x, y \in X$ with $0 \leqslant x \leqslant y$,*

$$\|x\| \leqslant \gamma\|y\|.$$

Then for any two comparable vectors $x, y \in C$ with $\|x\| = \|y\| = 1$,

$$\|x - y\| \leqslant 2\gamma \min\{m(x/y), m(y/x)\}(\exp(d(x, y)) - 1) \leqslant 2\gamma(\exp(d(x, y)) - 1).$$

We now give some preparatory definitions and the statement of the Birkhoff–Hopf Theorem itself. This result is proved in [**19**], which also includes some notes on the history of the theorem and many further references to the literature.

Definition 1·4. Let C be a cone in a real vector space V, D be a cone in a real vector space W and $L: V \to W$ be a linear map with $L(C) \subseteq D$. Define non-negative real numbers $N(L;C,D)$, $k(L;C,D)$, $\Delta(L;C,D)$ and $\chi(L;C,D)$ by

$$N(L;C,D) = \inf\{\mu \geqslant 0: \omega(Ly/Lx;D) \leqslant \mu\omega(y/x;C) \text{ for all } x\in C, y\in V \text{ such that } x \text{ dominates } y\}$$

$$k(L;C,D) = \inf\{\lambda \geqslant 0: d(Lx,Ly;D) \leqslant \lambda d(x,y;C) \text{ for all } x,y\in C \text{ such that } x \sim_C y\};$$

$$\Delta(L;C,D) = \sup\{d(Lx,Ly;D): x,y\in C \text{ and } Lx \sim_D Ly\};$$

$$\chi(L;C,D)^2 = \sup\left\{\frac{M(Ly/Lx)}{m(Ly/Lx)}: x,y\in C, Lx \sim_D Ly\right\}.$$

In the definition of χ, we adopt the convention that $M(Ly/Lx)/m(Ly/Lx) = 1$ if both $M(Ly/Lx)$ and $m(Ly/Lx)$ are zero.

$N(L;C,D)$ is called the Hopf oscillation ratio, $k(L;C,D)$ is called the Birkhoff contraction ratio and $\Delta(L;C,D)$ is called the projective diameter.

THEOREM 1·1 (The Birkhoff–Hopf Theorem). *With the notation and hypotheses of Definition* 1·4, *suppose that* $\Delta(L) < \infty$. *Then we have*

$$k(L;C,D) = N(L;C,D) = \tanh\tfrac{1}{4}\Delta(L;C,D) = \frac{\chi(L;C,D)-1}{\chi(L;C,D)+1}.$$

If, in addition, C is almost Archimedean then there is a component P_u of D such that $L(C) \subseteq P_u \cup \{0\}$.

If, on the other hand, $\Delta(L) = \infty$, *then*

$$k(L;C,D) = N(L;C,D) = 1; \quad \chi(L;C,D) = \infty.$$

Definition 1·5. If C is a closed cone in a Banach space X and $S \subseteq C^*$, then S is called sufficient for C (see [**42**]) if

$$C = \{x\in X: h(x) \geqslant 0 \text{ for all } h\in S\}.$$

The Hahn–Banach theorem implies that C^* is sufficient for C. Obviously, viewing C as a subset of X^{**} by the usual embedding, C is a sufficient set for C^*.

The following simple lemma is a more careful statement of formulae in [**42**], where equations (13) and (18) as stated involve possible division by zero.

LEMMA 1·4. *Let C be a closed cone in a Banach space X and let $S \subseteq C^*$ be a sufficient set for C.*

Given $x\in C\backslash\{0\}$ and $y\in X$, let

$$R(y/x) = \left\{\frac{h(y)}{h(x)}: h\in S \text{ and } h(x) \neq 0\right\}.$$

Then x dominates y if and only if $R(y/x)$ is bounded and for all $h\in S$, $h(x) = 0$ implies $h(y) = 0$, in which case

$$M(y/x;C) = \sup R(y/x); \quad m(y/x;C) = \inf R(y/x).$$

If $y\in C$, then y is comparable to x if and only if $R(y/x)$ is bounded above and is bounded away from zero, that is that

$$R(y/x) \subseteq (\alpha,\beta)$$

for some $\alpha, \beta > 0$, *and for all* $h \in S$, $h(x) = 0$ *if and only if* $h(y) = 0$, *in which case*

$$\exp(d(x,y)) = \sup\left\{\frac{g(x)\,h(y)}{g(y)\,h(x)} : g, h \in S, g(y)\,h(x) \neq 0\right\}.$$

Proof. Suppose first that $x \in C\backslash\{0\}$ dominates $y \in X$, so $m(y/x)$ and $M(y/x)$ are defined and for any $\epsilon > 0$,

$$(m(y/x) - \epsilon)\,x \leqslant y \leqslant (M(y/x) + \epsilon)\,x.$$

Then, for any positive linear functional f,

$$(m(y/x) - \epsilon) f(x) \leqslant f(y) \leqslant (M(y/x) + \epsilon) f(x).$$

It follows that if $f(x) = 0$ then $f(y) = 0$ and that if $f(x) \neq 0$ then $m(y/x) - \epsilon \leqslant f(y)/f(x) \leqslant M(y/x) + \epsilon$. Thus,

$$m(y/x) \leqslant \inf R(y/x) \leqslant \sup R(y/x) \leqslant M(y/x).$$

Now suppose that for all $f \in S$, $f(x) = 0$ implies that $f(y) = 0$ and that $R(y/x)$ is bounded. Let $\alpha = \inf R(y/x)$ and $\beta = \sup R(y/x)$. We shall show that $\alpha x \leqslant y \leqslant \beta x$. Let $f \in S$; we have that either $f(x) = f(y) = 0$ or $f(x) \neq 0$ and $\alpha \leqslant f(y)/f(x)$. In either case, $f(y - \alpha x) \geqslant 0$. Since S is sufficient for C, this implies that $y - \alpha x \geqslant 0$. A similar argument shows that $\beta x - y \geqslant 0$. Thus, x dominates y and

$$\inf R(y/x) \leqslant m(y/x) \leqslant M(y/x) \leqslant \sup R(y/x).$$

Combining these results shows that x dominates y if and only if for all $f \in S$, $f(x) = 0$ implies $f(y) = 0$, and the set $R(y/x)$ is bounded, and that in this case $m(y/x)$ and $M(y/x)$ are respectively the infimum and supremum of $R(y/x)$. The remaining assertions in the lemma are simple consequences of this.

If X and Y are Banach spaces and $A: X \to Y$, recall that the adjoint $A^*: Y^* \to X^*$ of A is defined by $A^*h = h \circ A$. It is immediate that if C and D are cones in X and Y, respectively, and $A(C) \subseteq D$ then $A^*(D^*) \subseteq C^*$.

With the aid of Lemma 1·4, it is easy to show that $\Delta(A; C, D)$ is equal to $\Delta(A^*; D^*, C^*)$; this is a simple corollary of the following representation of the projective diameter of a bounded linear operator.

LEMMA 1·5. *Let X and Y be Banach spaces, C and D be closed, total cones in X and Y respectively and S be a sufficient subset of D^*. Assume that $A: C \to D$ is a bounded linear operator, not identically zero, with $A(C) \subseteq D$, and $\Delta(A) < \infty$. Then*

$$\Delta(A; C, D) = \inf\{M > 0 : f(Ax)\,g(Ay) \leqslant e^M f(Ay)\,g(Ax) \text{ for all } x, y \in C, f, g \in S\}.$$

Proof. We shall show that $\Delta(A; C, D) \leqslant M$ if and only if for all $x, y \in C$ and for all $f, g \in S$,

$$f(Ax)\,g(Ay) \leqslant e^M f(Ay)\,g(Ax). \tag{1}$$

Suppose $\Delta(A) \leqslant M$. Then (Theorem 1·1) for all $x, y \in C$ such that Ax, $Ay \neq 0$, Ax is comparable to Ay in D and $d(Ax, Ay) \leqslant M$. By Lemma 1·4, if $f \in S$ then $f(Ax) = 0$ if and only if $f(Ay) = 0$ and for all $f, g \in S$ such that $f(Ay)\,g(Ax) \neq 0$,

$$\frac{f(Ax)\,g(Ay)}{f(Ay)\,g(Ax)} \leqslant e^M.$$

Thus, (1) is true for all $x, y \in C$ with $Ax \neq 0$ and $Ay \neq 0$ and for all $f, g \in S$ such that $f(Ay)g(Ax) \neq 0$. We now claim that (1) holds for all $x, y \in C$ and for all $f, g \in S$. If Ax and Ay are non-zero but $f(Ay)g(Ax) = 0$, then either $f(Ay) = 0$ and hence $g(Ax) = 0$, or $g(Ay) = 0$ and hence $f(Ay) = 0$; in either case, (1) holds because both sides are zero. Finally, if $Ax = 0$ or $Ay = 0$ then both sides of (1) are identically zero, so it is also true in this case.

Now suppose that (1) holds for all $x, y \in C$ and for all $f, g \in S$, and that $Ax, Ay \neq 0$. By symmetry in the roles of x and y, we have

$$e^{-M}f(Ay)g(Ax) \leqslant f(Ax)g(Ay) \leqslant e^{M}f(Ay)g(Ax),$$

again for all $x, y \in C$ and for all $f, g \in S$. Choosing $g \in S$ such that $g(Ay) > 0$ (this is possible because S is sufficient), we see that if $f(Ay) = 0$ then $f(Ax) = 0$. A similar argument shows that if $f(Ax) = 0$ then $f(Ay) = 0$, and (1) shows that if $f(Ay)g(Ax) \neq 0$ then

$$\frac{f(Ax)g(Ay)}{f(Ay)g(Ax)} \leqslant e^{M}.$$

It follows from this inequality and Lemma 1·4 that Ax is comparable to Ay and $d(Ax, Ay) \leqslant M$.

Corollary 1·1. *With the same hypotheses and notation as Lemma* 1·5, $\Delta(A; C, D) = \Delta(A^*; D^*, D^*)$.

Proof. By Lemma 1·5 applied to the map $A: X \to Y$ and using as the sufficient set S the whole cone D^*,

$$\Delta(A; C, D) = \inf\{M > 0 : f(Ax)g(Ay) \leqslant e^{M}f(Ay)g(Ax) \textit{ for all } x, y \in C, f, g \in D^*\}.$$

However, by applying Lemma 1·5 to the map $A^*: Y^* \to X^*$ and using as the sufficient set S the image of C in X^{**} under the canonical embedding, we see that

$$\begin{aligned}\Delta(A^*) &= \inf\{M > 0 : \phi(A^*f)\psi(A^*g) \leqslant e^{M}\phi(A^*g)\psi(A^*f) \text{ for all } f, g \in D^*, \phi, \psi \in S\}\\ &= \inf\{M > 0 : (A^*f)(x)(A^*g)(y) \leqslant e^{M}(A^*g)(x)(A^*f)(y) \text{ for all } f, g \in D^*, x, y \in C\}\\ &= \inf\{M > 0 : f(Ax)g(Ay) \leqslant e^{M}g(Ax)f(Ay) \text{ for all } f, g \in D^*, x, y \in C\},\end{aligned}$$

so $\Delta(A) = \Delta(A^*)$, as claimed.

2. *Applications to spectral theory*

Before stating our first theorem, we recall a variant of the contraction mapping principle. Suppose that (Σ, ρ) is a complete metric space and that $f: \Sigma \to \Sigma$ is a map. Assume that there is an integer $m \geqslant 1$ and a constant $c < 1$ with

$$\rho(f^m(x), f^m(y)) \leqslant c\rho(x, y)$$

for all $x, y \in \Sigma$. Then f has a unique fixed point $x_0 \in \Sigma$. Moreover, if $x \in \Sigma$ and j is a positive integer with $j = pm + r$ for integers $p \geqslant 0$, $0 \leqslant r < m$ then

$$\rho(f^j(x), x_0) \leqslant \frac{c^p}{1-c}\rho(f^r(x), f^{m+r}(x))$$

so $f^j(x)$ converges to x_0 at a geometric rate.

Theorem 2·1. *Let C be a closed cone in a Banach space E and let $L: E \to E$ be a linear map such that $L(C) \subseteq C$. Assume that there exists $u \in C \backslash \{0\}$ such that*

1. *P_u, the component of C containing u, is normal;*
2. *Lu is comparable to u in C.*

Write $K_u = P_u \cup \{0\}$ and assume that there exists an integer $m \geq 1$ such that

$$\sup\{d(L^m x, L^m y; C): x, y \in P_u\} = \Delta(L^m; K_u, K_u) < \infty.$$

Then L has a unique eigenvector $v \in P_u$ with $\|v\| = 1$ and $Lv = \lambda v$ for some $\lambda > 0$, and for any $x \in P_u$

$$\lim_{j\to\infty} d(L^j x, v; C) = 0.$$

In fact, if we define

$$c = \tanh \tfrac{1}{4}\Delta(L^m; K_u, K_u),$$

so $c < 1$, and if $j = mp + r$, where $p \geq 0$ is an integer and $0 \leq r < m$, then

$$d(L^j(x), v) \leq \frac{c^p}{1-c} d(L^r(x), L^{m+r}(x)). \tag{2}$$

If, in addition, the cone C is normal, so there exists a constant $\gamma > 0$ such that $0 \leq x \leq y$ in C implies $\|x\| \leq \gamma\|y\|$, then

$$\left\| \frac{L^j x}{\|L^j x\|} - v \right\| \leq 2\gamma\{\exp(d(L^j x, v)) - 1\}. \tag{3}$$

Proof. K_u is a cone, and our hypotheses imply that $L(P_u) \subseteq P_u$ and $L(K_u) \subseteq K_u$. By using Lemma 1·1, we see that

$$\sup\{d(L^m x, L^m y; C): x, y \in P_u\} = \Delta(L^m; K_u, K_u).$$

Theorem 1·1 now implies that

$$d(L^m x, L^m y; K_u) \leq c d(x, y; K_u).$$

(By virtue of Lemma 1·1, if $\xi, \eta \in P_u$, we may write $d(\xi, \eta; K_u)$ and $d(\xi, \eta, C)$ interchangeably.) Let

$$\Sigma_u = \{x \in P_u : \|x\| = 1\}$$

and define $f: \Sigma_u \to \Sigma_u$ by

$$f(x) = \frac{Lx}{\|Lx\|}.$$

Linearity implies that

$$f^j(x) = \frac{L^j x}{\|L^j x\|}$$

so we obtain that

$$d(f^m(x), f^m(y); C) \leq c d(x, y; C)$$

for all $x, y \in \Sigma_u$. Because P_u is normal, Lemma 1·2 implies that (Σ_u, d) is a complete metric space. The existence and uniqueness of v and the estimate for d convergence now follow from the contraction mapping principle in the form stated above, since the fixed points of f are exactly the normalized eigenvalues of L. The estimate for norm convergence follows from Lemma 1·3.

This theorem implies that L has a unique, normalized eigenvector v in P_u. However, even in the matrix case, the assumptions made are not strong enough to imply that the eigenvalue λ corresponding to v is equal to the spectral radius of L (consider a diagonal matrix with distinct positive entries on the diagonal). We now begin to place stronger assumptions on L, so as to force $\lambda = r(L)$ and to obtain information about the spectral clearance $q(L)$ of L.

THEOREM 2·2. *Let C be a closed, normal cone in a Banach space E, so there exists a constant γ such that $0 \leqslant x \leqslant y$ in C implies that $\|x\| \leqslant \gamma\|y\|$, and let $L: E \to E$ be a linear map such that $L(C) \subseteq C$. Assume that there exists an integer $m \geqslant 1$ such that $\Delta(L^m; C, C) < \infty$ and $L^{m+1} | C$ is not identically zero. Then L has a unique normalized eigenvector v in C with $Lv = \lambda v$ for some $\lambda > 0$. Moreover, if c is defined by*

$$c = \tanh \tfrac{1}{4}\Delta(L^m; C, C)$$

then for every $x \in C$ such that $L^m x \neq 0$ (in particular for all $x \in L^m(C) \backslash \{0\}$) and for every positive integer $j = mp + r$ ($p \geqslant 0$ an integer, $0 \leqslant r < m$), estimates (2) *and* (3) *in Theorem* 2·1 *hold.*

Proof. Theorem 1·1 implies that all nonzero elements of $L^m(C)$ are comparable. Since we assume that $L^m | C$ is not identically zero, there exists $\xi \in C$ with $L^{m+1}\xi \neq 0$. Because $u := L^m\xi$ and $x = L^m(L\xi) = Lu$ are comparable, u and Lu are comparable in C. Define P_u to be the component of C containing u and $K_u = P_u \cup \{0\}$. It is easy to see, using Lemma 1·1, that

$$\Delta(L^m; K_u, K_u) \leqslant \Delta(L^m; C, C) < \infty. \tag{4}$$

Theorem 2·1 now implies that L has a unique normalized eigenvalue v in P_u and since $L^m(C) \subseteq K_u$, any eigenvector of L in C with a positive eigenvalue must be in P_u. Thus, v is the only normalized eigenvector of L in C with a positive eigenvalue.

Estimates (2) and (3) in Theorem 2·1 now follow immediately from Theorem 2·1 and equation (4).

The earliest version of this central result is due to Birkhoff[**5**, theorem 3]. In our terminology, Birkhoff's theorem states that a strongly positive linear map L of finite projective diameter on a Banach lattice has a unique normalized positive eigenvector v, and that if x is any non-zero, non-negative vector then $L^n x/\|L^n x\|$ converges geometrically to v. Hopf[**22**, **23**] established a similar result for integral operators with non-negative kernels satisfying a cross-ratio condition equivalent to the operator having finite projective diameter (this condition is disussed in [**19**]). The estimates for the speed of convergence in both of these papers are weaker than those presented here. The existence and uniqueness of a positive eigenvector for an operator of finite projective diameter in a normal cone is obtained in [**42**], but the authors do not discuss the convergence of the sequence of iterates. Bushell[**9**] establishes the existence and uniqueness of positive eigenvectors of operators of finite projective diameter in a number of special cases, including $\mathbb{R}^n$, cones of positive definite matrices and cones of non-negative continuous functions. More recently, theorem 2·5·1 in [**18**] is a very similar result, but has somewhat stronger hypotheses and weaker convergence estimates.

Theorem 2·2 immediately yields a result about positive eigenvectors of the Banach space adjoint L^* of L. We label this result a proposition since we shall soon be able to give a much sharper version.

PROPOSITION 2·1. *Let C be a closed, normal, reproducing cone in a Banach space E. Let $L: E \to E$ be a bounded linear operator such that $L(C) \subseteq C$. Assume there exists an integer $m \geqslant 1$ such that $\Delta(L^m; C, C) < \infty$ and $L^{m+1} \neq 0$. Then L has a unique eigenvector $v \in C$ with $\|v\| = 1$, L^* has a unique eigenvector $h \in C^*$ with $\|h\| = 1$, $h(v) > 0$ and the corresponding eigenvalues are the same, so $Av = \lambda v$ and $A^*h = \lambda h$ for some $\lambda > 0$. Moreover, for any $x \in C$ with $L^{m+1}x \neq 0$ and for every $f \in C^*$ with $(L^*)^{m+1} f \neq 0$,*

$$\lim_{j\to\infty} \left\| \frac{L^j x}{\|L^j x\|} - v \right\| = 0; \quad \lim_{j\to\infty} \left\| \frac{(L^*)^j f}{\|(L^*)^j f\|} - h \right\| = 0.$$

Proof. Our previous remarks about cones imply that C^* is reproducing and normal. We may therefore apply Corollary 1·1 to $A = L^m$ to show that

$$\Delta((L^*)^m; C^*, C^*) = \Delta(L^m; C, C) < \infty.$$

Because $L^m \neq 0$, we must have $(L^*)^{m+1} \neq 0$ and because C and C^* are reproducing, it follows that $L^m | C$ and $(L^*)^{m+1} | C^*$ are not identically zero.

If we apply Theorem 2·2 to L and L^*, we conclude that L has a unique normalized eigenvector $v \in C$ with $Av = \lambda v$ and $\lambda > 0$, and that L^* has a unique normalized eigenvector $h \in C^*$ with $L^*h = \mu h$ and $\mu > 0$. Furthermore, the stated convergence results hold. It remains only to show that $\lambda = \mu$ and that $h(v) > 0$.

Suppose $h(v) = 0$. We know that for all $x \in C$, v dominates $L^m x$, so $h(L^m x) = 0$ for all $x \in C$. Since C is reproducing in E, we have $h(L^m z) = 0$ for all $z \in E$ and hence $(L^*)^m h = 0$, which is a contradiction.

The fact that $\lambda = \mu$ now follows from

$$(L^*h)(v) = \mu h(v); \quad (L^*h)(v) = h(Lv) = \lambda h(v).$$

Remark 2·1. Explicit estimates for the speed of convergence of $L_j x$ and $(L^*)^j f$ may of course be obtained from Theorem 2·2. For the sake of brevity we have omitted these estimates.

If X is a real Banach space, we shall denote by $\tilde{X}$ the complexification of X; formally,

$$\tilde{X} = \{x + iy : x, y \in X, i = \sqrt{-1}\}.$$

If $z = x + iy \in \tilde{X}$, the norm of z is defined by

$$\|z\| = \sup\{\|(\cos\theta)x + (\sin\theta)y\| : 0 \leqslant \theta \leqslant 2\pi\}.$$

Of course, $\tilde{X}$ is a complex Banach space. If $L: X \to X$ is a bounded linear operator, L defines a complex linear operator $\tilde{L}$ on $\tilde{X}$ by

$$\tilde{L}(x + iy) = Lx + iLy.$$

With this definition $\tilde{L}$ is a bounded linear operator and one may show that $\|\tilde{L}\| = \|L\|$. If $\sigma(\tilde{L})$ denotes the spectrum of $\tilde{L}$, we define $\sigma(L) = \sigma(\tilde{L})$. We also define $r(L)$, the spectral radius of L, in the usual way:

$$r(L) = \sup\{|z| : z \in \sigma(\tilde{L})\} = \lim_{n\to\infty} \|\tilde{L}^n\|^{1/n} = \lim_{n\to\infty} \|L^n\|^{1/n}.$$

We wish to prove that $\lambda = r(L)$ in Proposition 2·1 and give a variety of formulae for the spectral clearance $q(L)$ of L, defined by

$$q(L) = \sup\left\{\frac{|z|}{r(L)} : z \in \sigma(L), z \neq r(L)\right\}. \tag{5}$$

We shall also need a lemma which is part of the 'folklore' of elementary spectral theory. The following result was communicated to R.D.N. several years ago by Larry Corwin, a respected former colleague.

LEMMA 2·1. *Let X and Y be Banach spaces and $S: X \to Y$ and $T: Y \to X$ be linear maps (not necessarily continuous). Assume that $ST: Y \to Y$ and $TS: X \to X$ are bounded. Then $\sigma(ST)\backslash\{0\} = \sigma(TS)\backslash\{0\}$.*

For $z \neq 0$, $zI_X - TS$ is invertible if and only if $zI_Y - ST$ is invertible, and

$$(zI_Y - ST)^{-1} = z^{-1}I_Y + z^{-1}S(zI_X - TS)^{-1}T$$
$$(zI_X - TS)^{-1} = z^{-1}I_X + z^{-1}T(zI_Y - ST)^{-1}S$$

where I_X denotes the identity map on X and I_Y denotes the identity map on Y.

Proof. It suffices to prove the equations above. By symmetry, we may restrict attention to the first equation, so we assume that $zI_X - TS$ is a bijection on X and that $z \neq 0$. To verify the first equation, it suffices to show that

$$I_Y = (zI_Y - ST)[z^{-1}I_Y + z^{-1}S(zI_X - TS)^{-1}T] = [z^{-1}I_Y + z^{-1}S(zI_X - TS)^{-1}T](zI_Y - ST).$$

However, the above equation follows by simple algebraic manipulation, such as the observation that

$$(zI_Y - ST)(z^{-1}S) = (z^{-1}S)(zI_X - TS).$$

Lemma 2·1 of course implies that $r(ST) = r(TS)$. We shall use it in the following situation. Let X and Y be Banach spaces with $Y \subseteq X$ and let $T = i$, the inclusion map of Y into X, which is not assumed to be continuous. We suppose that S is a linear map from X to Y and that $L_1 = TS$ and $\Lambda_1 = ST$ are continuous. Lemma 2·1 then implies that $\sigma(\Lambda_1)\backslash\{0\}$ and $\sigma(L_1)\backslash\{0\}$ are equal and that $r(\Lambda_1) = r(L_1)$.

In order to state our next theorem, we also need the idea of the essential spectrum $\sigma_e(L)$ and the radius of the essential spectrum $r_e(L)$, where $L: X \to X$ is a bounded linear map of a complex Banach space X to itself. We shall only recall a few facts here and refer the reader to [**29**] and [**30**] for more details. The set $\sigma_e(L)$ is a closed subset of $\sigma(L)$ and $z - L$ is of Fredholm index zero for all $z \notin \sigma_e(L)$. There are several possible definitions of $\sigma_e(L)$ and these are, in general, inequivalent. If we use F. E. Browder's definition (see [**29, 30**]) and if f is an analytic map defined on some open neighbourhood of $\sigma(L)$, then

$$f(\sigma_e(L)) = \{f(z): z \in \sigma_e(L)\} = \sigma_e(f(L)).$$

If we define a seminorm p_K on the space of bounded linear maps $L: X \to X$ by

$$p_K(L) = \inf\{\|L+B\| : B: X \to X \text{ is a compact linear map}\}$$

then we have

$$r_e(L) = \sup\{|z| : z \in \sigma_e(L)\} = \lim_{j\to\infty} (p_K(L^j))^{1/j} = \inf_{j\geq 1} (p_K(L^j))^{1/j}$$

and this formula is valid for all definitions of the essential spectrum. For any fixed $\epsilon > 0$, there are only finitely many $z \in \sigma(L)$ with $|z| \geq r_e(L) + \epsilon$ and for each such z, z is an eigenvalue of finite algebraic multiplicity and $z - L$ is of Fredholm index zero. (Recall that if $z \in \sigma(L)$ then the dimension of the subspace

$$\{x \in X : (z-L)^k x = 0 \text{ for some } k \geq 1\}$$

is called the algebraic multiplicity of z and that z is called an algebraically simple eigenvalue of L if its algebraic multiplicity is equal to one.)

THEOREM 2·3. *Let C be a closed, normal, reproducing cone in a Banach space E and let $L: E \to E$ be a bounded linear operator with $L(C) \subseteq C$ such that for some integer $m \geq 1$, $\Delta(L^m; C, C) < \infty$ and $L^{m+1} \neq 0$.*

Then the operator L has a unique normalised eigenvector $v \in C$ and the operator L^* has a unique normalised eigenvector $h \in C^*$. The corresponding eigenvalue in both cases is $r(L)$, the spectral radius of L, which is an algebraically simple eigenvalue of L. Moreover, for every $x \in C$ with $L^m x \neq 0$ and every $f \in C^*$ with $(L^*)^m (f) \neq 0$,

$$\lim_{j\to\infty} \left\| \frac{L^j x}{\|L^j x\|} - v \right\| = 0; \quad \lim_{j\to\infty} \left\| \frac{(L^*)^j f}{\|(L^*)^j f\|} - h \right\| = 0.$$

If a seminorm N_v is defined on E_v by

$$N_v(x) = \omega(x/v; C)$$

and if for any continuous linear operator $A: E_v \to E_v$ we define $N_v(A)$ by

$$N_v(A) = \inf\{\mu \geq 0 : N_v(Ax) \leq \mu N_v(x) \text{ for all } x \in E_v\},$$

then we have the following estimate for the spectral clearance of L defined in equation (5):

$$q(L) = r(L)^{-1} \lim_{j\to\infty} (N_v(L^j))^{1/j} = r(L)^{-1} \inf_{j\geq 1} (N_v(L^j))^{1/j} \leq \inf_{j\geq 1} (N(L^j))^{1/j}. \tag{6}$$

Finally,

$$\frac{r_e(L)}{r(L)} \leq q(L) < 1. \tag{7}$$

Proof. Proposition 2·1 already establishes the existence of unique normalized eigenvectors $v \in C$ and $h \in C^*$ for L and L^* respectively, with the same eigenvalue λ, such that $h(v) > 0$.

Let Λ denote L as a map of E_v into E_v. Since $L(C) \subseteq C$ and $Lv = \lambda v$, it is easy to see that Λ is a bounded linear map of norm λ. If j is any integer with $j \geq m$, let $L_1 = L^j$ and $\Lambda_1 = \Lambda^j$. Note that $L_1(E) \subseteq E_v$ and $L_1 | E_v = \Lambda_1$. If S denotes L_1 as a map from E to E_v and T denotes the inclusion of E_v in E then $L_1 = TS$ and $\Lambda_1 = ST$ and Lemma 2·1 implies that

$$\sigma(L^j)\backslash\{0\} = \sigma(L_1)\backslash\{0\} = \sigma(\Lambda_1)\backslash\{0\} = \sigma(\Lambda^j)\backslash\{0\}.$$

On the other hand, the spectral mapping theorem implies that

$$\sigma(L^j) = (\sigma(L))^j; \quad \sigma(\Lambda^j) = (\sigma(\Lambda))^j$$

so we obtain that for all $j \geq m$,

$$(\sigma(L))^j\backslash\{0\} = (\sigma(\Lambda))^j\backslash\{0\}.$$

This implies that $r(L) = r(\Lambda)$ and $q(L) = q(\Lambda)$.

Because $h \in C^*$, it is easy to show that h defines a continuous linear functional on E_v. If we define

$$W_v = \{x \in E_v : h(x) = 0\},$$

we note that $\Lambda(W_v) \subseteq W_v$, for if $h(x) = 0$ for some $x \in E_v$ then

$$h(\Lambda x) = h(Lx) = (L^* h)(x) = \lambda h(x) = 0$$

so $\Lambda x \in W_v$. If $\langle v \rangle$ denotes the linear span of v, then

$$E_v = W_v \oplus \langle v \rangle,$$

so standard spectral theory results imply that

$$\sigma(\Lambda) = \sigma(\Lambda \mid W_v) \cup \{\lambda\}.$$

We now claim that the restriction of the seminorm N_v to W_v gives a norm on W_v which is equivalent to $|\cdot|_v$ on W_v. In fact, for fixed $x \in W_v$ define $\alpha_0 = m(x/v; C)$ and $\beta_0 = M(x/v; C)$, so

$$\alpha_0 v \leqslant x \leqslant \beta_0 v$$

and

$$\alpha_0 h(v) \leqslant 0 \leqslant \beta_0 h(v).$$

Since $h(v) > 0$, we have $\alpha_0 \leqslant 0 \leqslant \beta_0$ and the definition of $|x|_v$ gives

$$|x|_v = \max\{\beta_0, -\alpha_0\} \leqslant \beta_0 - \alpha_0 = N_v(x) \leqslant 2|x|_v.$$

Thus, if we restrict attention to W_v, we can use the norm N_v and the formula for the spectral radius gives

$$r(\Lambda \mid W_v) = \lim_{j\to\infty} (N_v(L^j))^{1/j} = \inf_{j\geqslant 1} (N_v(L^j))^{1/j}.$$

We are also using here the facts that every element x of E_v is of the form $x = y + \alpha v$ for some $y \in W_v$ and that $\omega(y+\alpha v/v) = \omega(y/v)$, from which one obtains that

$$N_v(L^j) = \inf\{\mu \geqslant 0 : \omega(L^j y/v) \leqslant \mu\omega(y/v) \text{ for all } y \in W_v\}.$$

We must relate $N_v(L^j)$ to $N(L^j)$. For any continuous linear map $A : X \to X$ such that $A(C) \subseteq C$, define $N'_v(A)$ by

$$N'_v(A) = \inf\{\mu \geqslant 0 : \omega(Ay/Av) \leqslant \mu\omega(y/v) \text{ for all } y \in C \text{ such that } v \text{ dominates } y\}.$$

If $\mu = N'_v(A)$ and $y \in E_v$, there exists $\alpha > 0$ with $-\alpha v \leqslant y \leqslant \alpha v$. It follows that $y + \alpha v \in C$ and

$$\omega(A(y+\alpha v)/Av) = \omega(Ay/Av) \leqslant \mu\omega(y+\alpha v/v) = \mu\omega(y/v).$$

This proves that

$$N'_v(A) = \inf\{\mu \geqslant 0 : \omega(Ay/Av) \leqslant \mu\omega(y/v) \text{ for all } y \in E_v\}$$

and recalling that $Av = \lambda^j v$ if $A = L^j$, we obtain

$$(N_v(L^j))^{1/j} = \lambda(N'_v(L^j))^{1/j}.$$

By the definition of $N(A)$ we certainly have $N'_v(A) \leqslant N(A)$, so we conclude that

$$\sup\left\{\frac{|z|}{\lambda} : z \in \sigma(\Lambda \mid W_v)\right\} = \lambda^{-1} \lim_{j\to\infty} (N_v(L^j))^{1/j} = \lambda^{-1} \inf_{j\geqslant 1} (N_v(L^j))^{1/j} < 1.$$

In particular, this proves that $q(\Lambda) < 1$ and that λ is the spectral radius of Λ and hence of L. Since $q(\Lambda) = q(L)$, we have proved (6).

If $W = \{x \in E : h(x) = 0\}$, note that $L(W) \subseteq W$ and $L^j(W) \subseteq W_v$ for all $j \geqslant m$. If $L \mid W$ denotes the restriction of L to W, so $L \mid W : W \to W$, and $r(L \mid W)$ denotes the spectral radius of $L \mid W$, the same reasoning used before for $L : E \to E$ shows that for $j \geqslant m$,

$$(\sigma(L \mid W))^j \backslash \{0\} = (\sigma(\Lambda \mid W_v))^j \backslash \{0\}.$$

These equations imply that $r(L \mid W) = r(\Lambda \mid W_v)$.

We already know that $r(\Lambda \mid W_v) < \lambda = r(L)$, so $(\lambda - L) \mid W$ is a bijection on W. Now suppose that $x \in \tilde{E}$, the complexification of E, and $(\lambda - L)^j (x) = 0$ for some $j \geqslant 1$. We can write $x = w + \alpha v$ where $w \in \tilde{W}$, the complexification of W, and $\alpha \in \mathbb{C}$. It follows that $0 = (\lambda - L)^j w$ and since $(\lambda - L) \mid W$ is one-one, $w = 0$ and $x = \alpha v$. Thus, λ has algebraic multiplicity 1.

It remains to prove (7). If $j \geqslant m$, define $A : X \to X$ by

$$Ax = L^j x - \lambda^j \frac{h(x)}{h(v)} v.$$

Because A differs from L^j by a compact linear map, we have

$$r_e(A) = r_e(L^j).$$

The formula for essential spectral radius implies that

$$r_e(L^j) = (r_e(L))^j.$$

Notice that $A(X) \subseteq W_v$ and $A \mid W_v = \Lambda^j \mid W_v$. It follows from Lemma 2·1 that

$$\sigma_e(A) \backslash \{0\} \subseteq \sigma(A) \backslash \{0\} = \sigma((\Lambda \mid W_v)^j) \backslash \{0\} = (\sigma(\Lambda \mid W_v))^j \backslash \{0\}.$$

We have already seen that $\sigma(\Lambda \mid W_v)$ is contained inside a ball of radius $q(L)\, r(L)$, so we conclude that

$$(r_e(L))^j = \sup \{|z| : z \in \sigma_e(L)\} \leqslant (q(L)\, r(L))^j$$

which yields equation (7).

Remark 2·2. Assume that C is a closed total cone in a Banach space E and that $L: E \to E$ is a bounded linear operator with $L(C) \subseteq C$ and $r_e(L) < r(L)$. Then it is proved in [**31**] that there exists $v \in C$, $\|v\| = 1$, and $h \in C^*$, $\|h\| = 1$, with $Lv = rv$, $L^* h = rh$ and $r = r(L)$. Note that no assumption of normality is needed. If, in addition, there exists an integer $m \geqslant 1$ with $\Delta(L^m; C, C) < \infty$ then v is the only normalized eigenvector in C with a corresponding positive eigenvalue and similarly for h. Furthermore, if $x \in C$ and $L^m x \neq 0$ then

$$\lim_{j \to \infty} \omega(L^j x / v) = 0; \quad \lim_{j \to \infty} d(L^j x, v) = 0$$

with a similar statement for h. Once one has existence of h and v, the proofs of the other results mentioned do not require normality and follow with the aid of Theorem 1·1.

It is less clear what should be the analogue of (6) in this generality. If $\alpha \in \sigma(L)$ and $r_e(L) < |\alpha|$, then it is known that α is an eigenvalue of L. A simple argument shows that

$$\frac{|\alpha|}{r} \leqslant \frac{1}{r} \lim_{j \to \infty} (N_v(L_j))^{1/j} = \frac{1}{r} \inf_{j \geqslant 1} (N_v(L^j))^{1/j} \leqslant \inf_{j \geqslant 1} (N(L^j))^{1/j}.$$

Using this fact we conclude that

$$q(L) = \max \left\{ \frac{r_e(L)}{r(L)}, \frac{1}{r(L)} \inf_{j \geqslant 1} N_v(L^j)^{1/j} \right\} \leqslant \max \left\{ \frac{r_e(L)}{r(L)}, \inf_{j \geqslant 1} N(L^j)^{1/j} \right\} < 1.$$

Returning now to the case of a closed, normal, reproducing cone in a Banach space, Equation (6) may be sharpened. We shall prove that

$$r(L)^{-1} \inf_{j \geqslant 1} (N_v(L^j))^{1/j} = \inf_{j \geqslant 1} (N(L^j))^{1/j} \tag{8}$$

and also establish a variety of closely related results. Note that (6) and (8) apply even though L may not be compact and $\sigma(L)$ may contain elements which are not eigenvalues.

We begin by recalling a classical calculus lemma which is often used in deriving the formula for the spectral radius of a linear operator.

LEMMA 2·2. *Let $(c_m)_{m\geqslant 1}$ be a sequence of non-negative real numbers with $c_{m+n} \leqslant c_m c_n$ for all $m, n \geqslant 1$. Then it follows that $\lim_{m\to\infty} c_m^{1/m}$ exists and that*

$$\lim_{m\to\infty} c_m^{1/m} = \inf_{m\geqslant 1} c_m^{1/m}.$$

With the aid of Theorem 1·1, we may easily show the relevance of this lemma to proving (8).

PROPOSITION 2·2. *Let C_j be a cone in a real vector space E_j for $1 \leqslant j \leqslant 3$. Suppose that $A: E_1 \to E_2$ is a linear map with $A(C_1) \subseteq C_2$ and $B: E_2 \to E_3$ is a linear map with $B(C_2) \subseteq C_3$. Let $k(A) = k(A; C_1, C_2)$, $N(A) = N(A; C_1, C_2)$ etc. Then we have*

$$k(BA) = N(BA) \leqslant k(A)\,k(B) = N(A)\,N(B). \tag{9}$$

If $E_1 = E_2$ and $C_1 = C_2$, so $A: C_1 \to C_1$, we also have

$$\lim_{j\to\infty} (k(A^j))^{1/j} = \inf_{j\geqslant 1} (k(A^j))^{1/j} = \lim_{j\to\infty} (N(A^j))^{1/j} = \inf_{j\geqslant 1} (N(A^j))^{1/j}. \tag{10}$$

If $\Delta(A^p) < \infty$ for some $p \geqslant 1$, then $\lim_{j\to\infty} \Delta(A^j) = 0$ and

$$\lim_{j\to\infty} (k(A^j))^{1/j} = \lim_{j\to\infty} (\Delta(A^j))^{1/j}.$$

Proof. Theorem 1·1 implies that for $L = A$, B or BA we have $k(L) = N(L)$. Thus, to prove (9) it suffices to prove that $k(BA) \leqslant k(A)\,k(B)$.

To see this, let $x, y \in C_1$ with x comparable to y, so Ax and Ay are comparable in C_2 and by definition of $k(A)$, $d(Ax, Ay) \leqslant k(A)\,d(x,y)$. Now, $B(Ax)$ and $B(Ay)$ are comparable in C_2 and

$$d(B(Ax), B(Ay)) \leqslant k(B)\,d(Ax, Ay) \leqslant k(A)\,k(B)\,d(x,y).$$

If $E_1 = E_2$ and $C_1 = C_2$, applications of (9) imply that for all positive integers m and n we have

$$k(A^{m+n}) \leqslant k(A^m)\,k(A^n).$$

If we define $c_m = k(A^m)$ and apply Lemma 2·2 and Theorem 1·1, we obtain (10).

If $\Delta(A^p) < \infty$ for some $p \geqslant 1$, we have $k(A^p) = c_p < 1$ and

$$0 \leqslant \lim_{m\to\infty} k(A^{pm}) \leqslant \lim_{m\to\infty} c_p^m = 0.$$

It follows from our previous remarks and Theorem 1·1 that for $j \geqslant p$,

$$k(A^j) = \tanh \tfrac{1}{4}\Delta(A^j) < \infty; \quad \lim_{j\to\infty} \Delta(A^j) = 0.$$

These two equations imply that

$$\lim_{j\to\infty} k(A^j) = 0.$$

It is now a simple calculus exercise, which we leave to the reader, to prove that

$$\lim_{j\to\infty} (k(A^j))^{1/j} = \lim_{j\to\infty} [\tanh(\tfrac{1}{4}\Delta(A^j))]^{1/j} = \lim_{j\to\infty} (\Delta(A^j))^{1/j}.$$

Remark 2·3. This proposition, if rewritten in terms of $\chi(L)$, is a generalization of Theorem 2 in [**28**]. From our point of view it is a simple consequence of Theorem 1·1, but the authors of [**28**] were unaware of this theorem.

In order to prove Equation (8) we shall need some variants of $k(L)$ and $N(L)$.

Definition 2·1. Let C be a cone in a real vector space E and $L: E \to E$ be a linear map with $L(C) \subseteq C$. Assume there exists $v \in C\backslash\{0\}$ with $Lv = \lambda v$. Define E_v by

$$E_v = \{x \in E:\ -\alpha v \leqslant x \leqslant \alpha v \text{ for some } \alpha > 0\}.$$

Now for $R > 0$ define numbers $k_v(L)$, $k_{v,R}(L)$, $N_v(L)$, $N'_v(L)$ and $N'_{v,R}(L)$ as follows:

$$k_v(L) = \inf\{\mu > 0: d(Lx, Lv) \leqslant \mu d(x,v) \text{ for all } x \in C \text{ with } x \sim v\},$$
$$k_{v,R}(L) = \inf\{\mu > 0: d(Lx, Lv) \leqslant \mu d(x,v) \text{ for all } x \in C \text{ with } x \sim v \text{ and } d(x,v) \leqslant R\},$$
$$N_v(L) = \inf\{\mu > 0: \omega(Lx/v) \leqslant \mu\omega(x/v) \text{ for all } x \in E_v\},$$
$$N'_v(L) = \inf\{\mu > 0: \omega(Lx/Lv) \leqslant \mu\omega(x,v) \text{ for all } x \in C \text{ such that } v \text{ dominates } x\},$$
$$N'_{v,R}(L) = \inf\{\mu > 0: \omega(Lx/Lv) \leqslant \mu\omega(x,v) \text{ for all } x \in C \text{ such that } v \text{ dominates } x \text{ and } \omega(x/v) \leqslant R\}.$$

LEMMA 2·3. *With the notation and hypotheses of Definition* 2·1, *we have*

$$N'_v(L) = N'_{v,R}(L) \leqslant k_{v,R}(L) \leqslant e^R N'_v(L) \tag{11}$$

and

$$\lambda^{-1} N_v(L) = N'_v(L). \tag{12}$$

Proof. The argument given in the proof of Theorem 2·3 proves (12). To prove (11) we can assume that $\lambda = 1$, since all the quantities in (11) are unchanged if L is replaced by $\lambda^{-1}L$.

It is immediate that for any $R > 0$, $N'_{v,R}(L) \leqslant N'_v(L)$. To prove the reverse inequality, let $R > 0$ and notice that for any x dominated by v there exists $t > 0$ such that $\omega(tx/v) \leqslant R$ (because $\omega(tx/v) = t\omega(x/v)$). Now,

$$\omega(Lx/Lv) = t^{-1}\omega(L(tx)/Lv) \leqslant t^{-1}N'_{v,R}(L)\,\omega(tx/v) = N'_{v,R}(L)\,\omega(x/v)$$

which shows that $N'_v(L) \leqslant N'_{v,R}(L)$.

To show that $N'_v(L) \leqslant k_{v,R}(L)$, take any $x \in C$ with $\omega(x/v) > 0$ and write $\alpha = m(x/v)$, $\beta = M(x/v)$, $\alpha' = m(Lx/v)$ and $\beta' = M(Lx/v)$. Because $Lv = v$, we have

$$\alpha \leqslant \alpha' \leqslant \beta' \leqslant \beta; \quad \frac{\omega(Lx/Lv)}{\omega(x/v)} = \frac{\beta' - \alpha'}{\beta - \alpha}.$$

If we define $x_t = x + tv$ for $t > 0$, we have

$$m(x_t/v) = \alpha + t; \quad M(x_t/v) = \beta + t; \quad m(Lx_t/v) = \alpha' + t; \quad M(Lx_t/v) = \beta' + t.$$

It follows that

$$\frac{d(Lx_t, v)}{d(x_t, v)} = \frac{\log(\beta' + t)/(\alpha' + t)}{\log(\beta + t)(\alpha + t)}.$$

Taking limits as $t \to \infty$ gives

$$\lim_{t\to\infty} \frac{d(Lx_t, v)}{d(x_t, v)} = \frac{\beta' - \alpha'}{\beta - \alpha} = \frac{\omega(Lx/Lv)}{\omega(x/v)}.$$

Now, since $\lim_{t\to\infty} d(x_t, v) = 0$, we have

$$\frac{\omega(Lx/Lv)}{\omega(x/v)} \leqslant k_{v,R}(L)$$

which, since x was an arbitrary element of C with $\omega(x/v) > 0$, shows that $N'_v(L) \leqslant k_{v,R}(L)$.

It remains to be shown that $k_{v,R}(L) \leqslant e^R N'_v(L)$. Suppose $x \in C$ and $0 < d(x,v) \leqslant R$. Let α, β, α' and β' be as above, so

$$\frac{d(Lx, Lv)}{d(x,v)} = \frac{\log \beta'/\alpha'}{\log \beta/a}.$$

Because $\alpha \leqslant \alpha' \leqslant \beta' \leqslant \beta$, we have

$$\frac{\beta' - \alpha'}{\beta - \alpha} = \frac{\alpha'/\alpha(\beta'/\alpha' - 1)}{\beta/\alpha - 1} \geqslant \frac{\beta'/\alpha' - 1}{\beta/\alpha - 1}.$$

On the other hand, the mean value theorem gives

$$\frac{\log \beta'/\alpha'}{\log \beta/\alpha} = \frac{(\xi')^{-1}(\beta'/\alpha' - 1)}{\xi^{-1}(\beta/\alpha - 1)}$$

where $1 \leqslant \xi' \leqslant \beta'/\alpha'$ and $1 \leqslant \xi \leqslant \beta/\alpha \leqslant e^R$. It follows that

$$\frac{d(Lx, v)}{d(x,v)} \leqslant e^R \frac{\beta'/\alpha' - 1}{\beta/\alpha - 1} \leqslant e^R \frac{\omega(Lx/Lv)}{\omega(x/v)} \leqslant e^R N'_v(L)$$

which is the required estimate.

We also need to relate $k_v(L)$ and $k_{v,R}(L)$.

LEMMA 2·4. *With the assumptions and notation of Definition* 2·1, *let*

$$P_v = \{x \in C : x \sim v\}; \quad K_v = P_v \cup \{0\}$$

so $L(K_v) \subseteq K_v$. *Assume there exists an integer* $m \geqslant 1$ *with*

$$\Delta(L^m; K_v, K_v) := \sup\{d(L^m x, L^m y; C) : x, y \in P_v\} < \infty.$$

Then for any $R > 0$, $\lim_{j\to\infty}(k_v(L^j))^{1/j}$ *and* $\lim_{j\to\infty}(k_{v,R}(L^j))^{1/j}$ *both exist and are equal.*

Proof. If we define $c_j = k_v(L^j)$ and $d_j = k_{v,R}(L^j)$, a simple argument like that used in Proposition 2·2 shows that for all $j, n \geqslant 1$ we have

$$c_{j+n} \leqslant c_j c_n; \quad d_{j+n} \leqslant d_j d_n.$$

It follows from Lemma 2·2 that the two limits mentioned both exist and that

$$\lim_{j\to\infty}(k_v(L^j))^{1/j} = \inf_{j\geqslant 1}(k_v(L^j))^{1/j}; \quad \lim_{j\to\infty}(k_{v,R}(L^j))^{1/j} = \inf_{j\geqslant 1}(k_{v,R}(L^j))^{1/j}.$$

This part of the argument is independent of the assumption that $\Delta(L^m; K_v, K_v) < \infty$.

By applying Proposition 2·2 to $L: K_v \to K_v$, we see that $\lim_{j\to\infty} \Delta(L^j; K_v, K_v) = 0$. Thus, by selecting sufficiently large m, we may assume that $d(L^m x, v) \leqslant R$ for all $x \in P_v$. If $d(L^m x, v) = 0$ for all $x \in P_v$, we are done. Otherwise, for all $j \geqslant m$ we have

$$\begin{aligned} k_v(L^j) &= \sup\left\{\frac{d(L^j x, v)}{d(x,v)} : x \in P_v \text{ and } d(L^m x, v) > 0\right\} \\ &= \sup\left\{\left[\frac{d(L^{j-m}(L^m x), v)}{d(L^m x, v)}\right]\left[\frac{d(L^m x, v)}{d(x,v)}\right] : x \in P_v \text{ and } d(L^m x, v) > 0\right\} \\ &= k_{v,R}(L^{j-m})\, k_v(L^m). \end{aligned}$$

It now follows that

$$(k_v(L^j))^{1/j} \leqslant (k_{v,R}(L^{j-m}))^{1/j}\,(k_v(L^m))^{1/j}.$$

Taking the limit as $j \to \infty$ we see that

$$\lim_{j\to\infty} (k_v(L^j))^{1/j} \leqslant \lim_{j\to\infty} (k_{v,R}(L^j))^{1/j}.$$

The opposite inequality is obvious, so the lemma is proved.

In our final lemma, we need to relate $\Delta(L^j)$ to $k_{v,R}(L^j)$.

LEMMA 2·5. *Let C be an almost Archimedean cone in a real vector space E and let $L: E \to E$ be a linear map with $L(C) \subseteq C$ and $\Delta(L^m; C, C) < \infty$ for some $m \geqslant 1$. Assume there exist $\lambda > 0$ and $v \in C\backslash\{0\}$ with $Lv = \lambda v$. Then one has, for any $R > 0$,*

$$\lim_{j\to\infty} (\Delta(L^j))^{1/j} = \lim_{j\to\infty} (k_{v,R}(L^j))^{1/j}.$$

Proof. By Lemma 2·4, we may assume that $R < 1$. Proposition 2·2 implies that $\Delta(L^j) \to 0$ as $j \to \infty$; using Theorem 1·1, we see that $L^j x$ is comparable to v if $x \in C$ and $L^j x \neq 0$. It follows that by increasing m we can assume that $d(L^m x, v) \leqslant R$ for all $x \in C$ with $L^m x \neq 0$. If $j > m$ and $x, y \in C$ are such that $L^m x \neq 0$ and $L^m y \neq 0$, we obtain

$$\begin{aligned} d(L^j x, L^j y) &\leqslant d(L^{j-m}(L^m x), v) + d(v, L^{j-m}(L^m y)) \\ &\leqslant \frac{d(L^{j-m}(L^m x), v)}{d(L^m x, v)} + \frac{d(v, L^{j-m}(L^m y))}{d(v, L^m y)} \\ &\leqslant 2k_{v,R}(L^{j-m}). \end{aligned}$$

It follows that

$$\Delta(L^j) \leqslant 2k_{v,R}(L^{j-m}).$$

Using this inequality, Proposition 2·2 and Lemma 2·4 we see that

$$\lim_{j\to\infty} (\Delta(L^j))^{1/j} \leqslant \lim_{j\to\infty} (k_{v,R}(L^j))^{1/j}.$$

Proposition 2·2 already implies that

$$\lim_{j\to\infty} (\Delta(L^j))^{1/j} = \lim_{j\to\infty} (k(L^j))^{1/j} \geqslant \lim_{j\to\infty} (k_{v,R}(L^j))^{1/j}$$

so the lemma is proved.

Our next proposition describes some connections between the various quantities we have defined.

PROPOSITION 2·3. *Let C be an almost Archimedean cone in a real vector space E and $L: E \to E$ be a linear map with $L(C) \subseteq C$ and $\Delta(L^m) < \infty$ for some integer $m \geqslant 1$. Assume that there exist $\lambda > 0$ and $v \in C \backslash \{0\}$ with $Lv = \lambda v$. Then it follows that for any $R > 0$,*

$$\lambda^{-1} \lim_{j\to\infty} (N_v(L^j))^{1/j} = \lim_{j\to\infty} (N'_v(L^j))^{1/j} = \lim_{j\to\infty} (k_{v,R}(L^j))^{1/j} = \lim_{j\to\infty} (\Delta(L^j))^{1/j}.$$

Furthermore, we have

$$\lim_{j\to\infty} (\Delta(L^j))^{1/j} = \lim_{j\to\infty} (k(L^j))^{1/j} = \inf_{j\geqslant 1} (k(L^j))^{1/j} = \lim_{j\to\infty} (N(L^j))^{1/j} = \inf_{j\geqslant 1} (N(L^j))^{1/j}.$$

Proof. If we apply Lemma 2·3 to L^j we obtain

$$N'_v(L^j) \leqslant k_{v,R}(L^j) \leqslant e^R N'_v(L^j).$$

If we now use Lemma 2·4 we find that

$$\lim_{j\to\infty} (N'_v(L^j))^{1/j} = \lim_{j\to\infty} (k_{v,R}(L^j))^{1/j}.$$

Applying (12) to L^j gives

$$\lambda^{-j} N_v(L^j) = N'_v(L^j)$$

so the first equality in the above proposition is easy. The remaining assertions follow directly from Lemma 2·5 and Proposition 2·2.

COROLLARY 2·1. *Let assumptions and notation be as in Theorem 2·3. Then Equation* (6) *can be sharpened*:

$$q(L) = r^{-1} \lim_{j\to\infty} (N_v(L^j))^{1/j} = r^{-1} \inf_{j\geqslant 1} (N_v(L^j))^{1/j} = \inf_{j\geqslant 1} (N(L^j))^{1/j},$$

where $r = r(L) > 0$. Moreover, we have for all $j \geqslant 1$,

$$N(L^j) = k(L^j) = \tanh \tfrac{1}{4}\Delta(L^j) = \frac{\chi(L^j) - 1}{\chi(L^j) + 1}$$

and

$$\inf_{j\geqslant 1} (N(L^j))^{1/j} = \lim_{j\to\infty} (N(L^j))^{1/j} = \lim_{j\to\infty} (\Delta(L^j))^{1/j}.$$

Proof. This follows immediately from Theorem 2·3 and Proposition 2·3. Theorem 2·3 and Corollary 2·1 generalize results in [**28**], [**27**] and [**42**]. However, our real theme is that all the results of this section follow without great difficulty from Theorem 1·1.

With the aid of Theorem 2·3 and Corollary 2·1 one can obtain estimates for $q(L)$ which are of interest even in the case of non-negative matrices.

COROLLARY 2·2. (*Compare* [2]) *Let (S, μ) be a σ-finite measure space. Assume that $k: S \times S \to \mathbb{R}$ is a non-negative measurable function and that for almost all $s \in S$, $\int_S k(s,t)\,\mu(dt) = 1$. Let $A: L^\infty \to L^\infty$ be defined by*

$$(Ax)(s) = \int_S k(s,t)\, x(t)\, \mu(dt)$$

and let v denote the function identically equal to 1, so $Av = v$. If C is the cone of non-negative functions in $L^\infty(S)$ and $N_v(A)$ is defined as in Theorem 2·3 we have

$$N_v(A) \leqslant \tfrac{1}{2}\gamma := \tfrac{1}{2}\operatorname{ess.sup}\left\{ \int_S |k(s_1,t) - k(s_2,t)|\, \mu(dt) : (s_1, s_2) \in S \times S \right\}.$$

If S is a Hausdorff topological space and the map

$$(s_1, s_2) \mapsto \theta(s_1, s_2) := \int_S |k(s_1, t) - k(s_2, t)|\, \mu(dt)$$

is continuous on $S \times S$, we have

$$N_v(A) = \gamma/2.$$

If there exists an integer $m \geqslant 1$ so that $\Delta(A^m; C, C) < \infty$, we have that

$$q(A) = \sup\{|z| : z \in \sigma(A)\backslash\{1\}\} = \lim_{n\to\infty} (N_v(A^n))^{1/n} = \inf_{n \geqslant 1} (N_v(A^n))^{1/n}.$$

Proof. If $x \in L^\infty(S)$ and $\zeta = \tau x + \alpha v$ for $\tau > 0$ and $\alpha \in \mathbb{R}$, we have for $\omega(x/v) > 0$,

$$\frac{\omega(Ax/v)}{\omega(x/v)} = \frac{\omega(A\zeta/v)}{\omega(\zeta/v)}.$$

If $\omega(x/v) > 0$, we can arrange by appropriate choice of $\tau > 0$ and α that

$$M(\zeta/v) = 1; \quad m(\zeta/v) = -1.$$

Using this observation we see that

$$N_v(A) = \inf\{\rho > 0 : \omega(Ax/v) \leqslant \rho\omega(x/v) \text{ for all } x \in L^\infty \text{ with } M(x/v) = 1 \text{ and } m(x/v) = -1\}.$$

For x with $M(x/v) = 1$ and $m(x/v) = -1$, we have $|x(t)| \leqslant 1$ a.e. and

$$\begin{aligned}\omega(Ax/v) &= \text{ess.}\sup\{(Ax)(s_1) - (Ax)(s_2) : (s_1, s_2) \in S \times S\} \\ &= \text{ess.}\sup\left\{\int_S (k(s_1, t) - k(s_2, t))\, x(t)\, \mu(dt) : (s_1, s_2) \in S \times S\right\} \\ &= \text{ess.}\sup\left\{\int_S |k(s_1, t) - k(s_2, t)|\, \mu(dt) : (s_1, s_2) \in S \times S\right\} \\ &=: \gamma.\end{aligned}$$

Since $\omega(x/v) = 2$, this shows that $N_v(A) \leqslant \gamma/2$.

To prove equality when θ is continuous, note first that we can assume $\gamma > 0$, since equality is obvious for $\gamma = 0$. Since θ is continuous, the essential supremum in the formula for γ can be replaced by a supremum and we can select $\gamma_j \to \gamma$, $\gamma_j > 0$ and points $(s_{1j}, s_{2j}) \in S \times S$ with

$$\gamma_j = \int_S |k(s_{1j}, t) - k(s_{2j}, t)|\, \mu(dt).$$

Let $E_{1j} = \{t \in S : k(s_{1j}, t) - k(s_{2j}, t) \geqslant 0\}$ and $E_{2j} = S \backslash E_{1j}$. If $\mu(E_{2j}) = 0$, we find that

$$0 = 1 - 1 = \int_S (k(s_{1j}, t) - k(s_{2j,t}))\, \mu(dt) = \int_S |k(s_1, t) - k(s_{2j,t})|\, \mu(dt) = \gamma_j$$

which is a contradiction, so $\mu(E_{2j}) > 0$. Similarly, $\mu(E_{1j}) > 0$.

If we define $x_j \in L^\infty$ by $x_j(t) = 1$ for $t \in E_{1j}$ and $x_j(t) = -1$ for $t \in E_{2j}$, we have that $\omega(x_j/v) = 2$ and

$$\omega(Ax_j/v) \geqslant \int_S (k(s_{1j}, t) - k(s_{2j}, t))\, x_j(t)\, \mu(dt) = \int_S |k(s_{1j}, t) - k(s_{2j}, t)|\, \mu(dt) = \gamma_j.$$

It follows that $N_v(A) \geqslant \gamma_j/2$ for all $j \geqslant 1$, so $N_v(A) \geqslant \gamma/2$.

Remark 2·4. Suppose that (S, μ) is a σ-finite measure space, that $k: S \times S \to \mathbb{R}$ is a non-negative measurable function and that there exists $M > 0$ with $\int_S k(s,t)\,\mu(dt) \leqslant M$ almost everywhere. If A is defined as in Corollary 2·2 then $v = 1$ is not necessarily an eigenvector of A. Suppose that $w \in \mathring{C}$ is an eigenvector of A with positive eigenvalue λ and define $D: L^\infty(S) \to L^\infty(S)$ by $Dx = wx$. If we define $B = \lambda^{-1}D^{-1}AD$ then $\sigma(B) = \lambda^{-1}\sigma(A)$ and $Bv = v$, so Corollary 2·2 applies to B. Thus, if λ and w are known explicitly, $q(A)$ can be computed with the aid of Corollary 2·2.

Remark 2·5. If $S = \{j: 1 \leqslant j \leqslant n\}$ is a set with n elements, then A in Corollary 2·2 becomes an $n \times n$ non-negative matrix $A = (a_{ij})$ with

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \text{for} \quad 1 \leqslant 1 \leqslant n.$$

The formula for $N_v(A)$ becomes

$$N_v(A) = \tfrac{1}{2} \max_{p,q} \sum_{j=1}^{n} |a_{pj} - a_{qj}|. \tag{13}$$

In this case, results like Corollary 2·2 can be found in [**2**] and [**21**].

If A is an $n \times n$ non-negative matrix acting on column vectors in $\mathbb{R}^n$ and $\sum_{i=1}^n a_{ij} = 1$ for $1 \leqslant j \leqslant n$ then A maps $Y = \{y \in \mathbb{R}^n: \sum_{j=1}^n y_j = 0\}$ into itself. If we give $\mathbb{R}^n$ the l^1 norm $\|x\|_1 = \sum_{i=1}^n |x_i|$, Dobrushin has shown (see Lemma 1 in Section 3 of [**15**]) that

$$\|A \mid Y\|_1 = \tfrac{1}{2} \max_{p,q} \sum_{i=1}^{n} |a_{ip} - a_{iq}|. \tag{14}$$

Generalizations and further remarks in this direction are given in Section 3 of [**24**]. By considering adjoints it is not hard to see that (13) implies (14) and vice-versa. In particular, we could prove Corollary 2·2 in the matrix case directly by starting from (14).

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