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Integral Equations and Operator Theory

EXISTENCE, UNIQUENESS AND ANALYTICITY FOR PERIODIC SOLUTIONS OF A NON-LINEAR CONVOLUTION EQUATION.

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We study existence, uniqueness and analyticity for periodic solutions of
$u(x) = \Phi(\int_{I\!R} J(y)u(x-y)dy)$ for $x \in I\!R$.

1 Introduction

We study the periodic solutions u of the equation

$$\forall x \in I\!R, \quad u(x) = \Phi(\int_{I\!R} J(y)u(x-y)dy). \tag{1.1}$$

This problem is motivated by the case where $\Phi(x) = \tanh(x)$ and $J(x) = \beta\exp(-\pi x^2)$ with $\beta > 1$. Indeed, to study phase separation in a system where the total density is conserved, Lebowitz, Orlandi and Presutti [7] proposed the evolution law

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}u(x,t) - (1-u^2(x,t))\frac{\partial}{\partial x}\int_{I\!R} J(x-y)u(y,t)dy\right]. \tag{1.2}$$

In (1.2), u represents the density of magnetization and takes values in $[-1,1]$. Intuitively, the first term on the right hand side of (1.2) is a diffusive term which tends to homogenize the magnetization, while the second term corresponds to an interaction between particles countering the diffusive term with the net result, when $\beta := \int J > 1$, of favoring clumps of the "pure phases" $\{a, -a\}$, where $a = \tanh(\beta a) > 0$.

From the standpoint of statistical physics, the precise form of the "interaction" kernel J is not known, and we will restrict ourselves to the assumptions that J is even, non-negative and $J(x) \geq J(y)$ for $0 \leq x \leq y$. In other words, we deal with a symmetric, attractive interaction decaying with the distance. It is known [4], in the case where J has compact support and $\Phi = \tanh$, that there is a solution u_β of (1.1) odd and increasing from $-a$ to a unique in the class of functions with $\liminf_{+\infty} u > 0$ and $\limsup_{-\infty} u < 0$. The profile u_β represents coexistence between the two pure phases with a diffusive interface. Phenomenologically, we expect a conservative system to settle at low temperature (large β

here) in a crystal-like equilibrium state. Thus, periodic profiles, oscillating between the two pure phases should be stationary solutions of (1.2) for β large. Thus, we study here periodic solutions of (1.1). An interesting open problem is the stability of these periodic solutions.

When u is T-periodic and $x \in [-T/2, T/2]$ and some further assumptions are made,

$$J*u(x) := \int_{\mathbb{R}} J(y)u(x-y)dy = \int_{-T/2}^{T/2} J_T(y)u(x-y)dy, \quad \text{with} \quad J_T(x) = \sum_{n\in\mathbb{Z}} J(x+nT). \quad (1.3)$$

The fixed point problem for a given period on the circle, was studied by Comets, Eisele and Schatzmann [3]. However, they assumed that J was such that for some integer p, $\int J(x)\exp(i2\pi nx/T)dx$ vanishes for $n \in (2\mathbb{Z}+1)p\backslash\{p,-p\}$, and they looked for fixed points u such that $\int u(x)\exp(i2\pi nx/T)dx = 0$ for all $n \notin (2\mathbb{Z}+1)p$.

Besides the fact that we do not assume such features on J, our starting point is J on the whole line. Our goal is to go beyond existence results, to give some information about the fixed points and to show uniqueness in some natural classes of functions.

2 Notations and Results

For $T > 0$, we work in the Banach space, $(X_T, |\cdot|_\infty)$, of continuous functions of period T, odd with respect to 0 and even with respect to $T/4$

$$X_T = \{u \in C^o(\mathbb{R}) : u(-x) = -u(x), \quad \text{and} \quad u(x+T/2) = -u(x), \forall x\},$$

with the supremum norm $|\cdot|_\infty$. Also, we often consider the Hilbert space $\mathcal{H}_T$, obtained by completion of X_T under the scalar product

$$(\varphi, \psi) := \int_0^T \varphi(x)\psi(x)dx, \quad \text{for} \quad \varphi, \psi \in X_T. \quad (2.1)$$

We label the different properties of J. (Aa) $J : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, nonnegative, and even; (Ab) J is integrable; (Ac) J is bounded; (Ad) $J(x) \geq J(y)$ for $0 \leq x \leq y$. We say that A holds if (Aa), (Ab), (Ac) and (Ad) hold.

Similarly, we label the properties of Φ. (Ba) $\Phi : \mathbb{R} \to \mathbb{R}$ is odd, bounded, continuously differentiable and $\Phi'(0) = 1$; (Bb) Φ is increasing; (Bc) Φ is concave in $[0,\infty)$; (Bd) Φ is C^3 in a neighborhood of 0, and $\Phi'''(0) < 0$.

Remark 2.1 . With no less generality, we will assume that $\sup\{\Phi(x) : x \geq 0\} = 1$. Indeed, for any positive constant c, solving the equation $u = \Phi(J*u)$ is equivalent to solving $\tilde{u} = \tilde{\Phi}(J*\tilde{u})$, where $\tilde{\Phi}(y) := \Phi(cy)/c$. Also, we call $\beta := \int J(x)dx$

Our approach is based on the observation that when J_T is decreasing on $[0, T/2]$ and Φ satisfies (Ba) (respectively (Ba)-(Bc)), then the map $f(u) := \Phi(J*u)$ preserves the cone

$$C_T = \{u \in \mathcal{H}_T : u(x) \geq 0 \text{ a.e.} - \text{dx for } x \in [0, T/2]\},$$

(respectively $K_T = \{u \in X_T : u(x)$ concave and increasing in $[0, T/4]\}$).

If we do not assume that J_T is decreasing, no obvious cone is left invariant, and our main result is

Theorem 2.2 . *Assume A, B and that $\beta > 1$. Then, there are $T_0 > 0$ and $\epsilon_0 \in (0,1)$ such that for $T > T_0$, there is a fixed point, u, of f in $C_T \backslash \{0\}$, where f is defined by $f(u)(x) = \Phi(\int_{\mathbb{R}} J(y)u(x-y)dy)$. Moreover, f has no other fixed point $w \in X_T$ satisfying $|w(x) - v(x)| \leq \epsilon_0 |v(x)|$ for some $v \in K_T \backslash \{0\}$ and all $x \in \mathbb{R}$. Also, if Φ is real analytic, then u is real analytic.*

It is based on the weaker but more satisfactory result.

Theorem 2.3 . *Assume A, B, and $f(C_T) \subset C_T$. If T is such that*

$$\hat{J}(\frac{2\pi}{T}) := \int_{\mathbb{R}} J(x) \cos(\frac{2\pi}{T}x)dx > 1 \tag{2.2}$$

then, there is a unique fixed point, u, of f in $C_T \backslash \{0\}$, and $u \in K_T$. If G is a bounded, relatively open neighborhood of u in C_T and $0 \notin \bar{G}$, then $i_{C_T}(f, G) = 1$, where $i_{C_T}(f, G)$ denotes the fixed point index of $f : G \to C_T$ (see [8]). Also, if

$$D_T = \{v \in X_T : v(x) \geq 0, \text{ for } 0 \leq x \leq T/2\},$$

if H is a relatively open neighborhood of u in D_T with $0 \notin \bar{H}$ and if Θ is a relatively open neighborhood of $u \in K_T$ with $0 \notin \bar{\Theta}$, then $i_{D_T}(f, H) = 1$ and $i_{K_T}(f, \Theta) = 1$. Moreover, if Φ is real analytic, then u is real analytic.

Remark 2.4 . (i) We have stated Theorem 2.3 with assumptions A and B because our primary purpose is Theorem 2.2. However, we can treat more general cases than A. For instance, there are cases where $\int J = \infty$ (e.g. $J(y) = 1/\log(1 + \log(1 + |y|))$), which can be treated by the same method, when we use the oddness of u to interpret $J * u$ as

$$\int_0^\infty (J(x-y) - J(x+y))\, u(y)dy.$$

Thus, if

$$\sup_x \int |J(y-x) - J(y)|dy < \infty, \quad \text{and} \quad f(C_T) \subset C_T$$

then by analyzing the linear map $u \mapsto J * u$, we could obtain an analogue of Theorem 2.3.
(ii) For simplicity, in Theorem 2.3 we have made the hypotheses on Φ stronger than necessary. Assume A, B(a) and B(b) and suppose that Φ is C^k for some $k \geq 1$, $\Phi'(0) = 1$, Φ is concave in $[0, \infty)$ and $\Phi(x)/x$ is strictly decreasing on $(0, \infty)$. If equation (2.2) holds, then the argument we shall give proves that there is a unique fixed point u of f in $C_T \backslash \{0\}$ and that $u \in C^k$, $u'(0) > 0$ and $u(x) > 0$ for $0 < x \leq T/4$. Moreover, if $u \in K_T$ and G, H and Θ are as in Theorem 2.3, $i_{C_T}(f, G) = i_{D_T}(f, H) = i_{K_T}(f, \Theta) = 1$.

Furthermore, we have characterized some J's for which $f(C_T) \subset C_T$.

Lemma 2.5 . *Assume A, (Ba), and that there is $C > 0$ such that $J'(x)$ exists for all $x \geq C$ and J' is concave in $\{x \geq C\}$. Then, for $T > 4C$, $f(C_T) \subset C_T$.*

To give a complete picture, we recall a known result [3]

Lemma 2.6 . *Assume A and B. If T is such that*

$$\sup_{n\geq 0}\int_{\mathbb{R}} J(x)\cos(\frac{2\pi(2n+1)}{T}x)dx \leq 1$$

then, 0 is the only fixed point in $\mathcal{H}_T$.

Existence results for periodic solutions of (1.1) are simpler and do not require all these hypotheses for Φ. For the sake of completeness, we will provide a variational proof of the following.

Lemma 2.7 . *Assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is increasing, continuously differentiable, bounded, $\Phi(0) = 0$, and $\Phi'(0) = 1$. Also, assume that J satisfies A and for $T > 0$*

$$\sup_{n} \hat{J}(\frac{2\pi}{T}(2n+1)) > 1. \tag{2.3}$$

Then, there is a non-zero fixed point of f of period T.

If we drop the assumption that Φ is increasing, then it becomes unclear whether our problem can be put in a variational form. However, we have the following result. Define

$$\forall x \in [-T/2, T/2] \qquad \tilde{J}_T(x) = (J_T(x) - J_T(T/2 + x)). \tag{2.4}$$

Lemma 2.8 . *Assume (Ba) and that $\Phi(x) > 0$ for $x > 0$. Also, assume A and that $\tilde{J}_T$ is decreasing in $[0, T/2]$, $\tilde{J}_T$ is nonnegative a.e. on $[0, T/4]$, and $\hat{J}((2\pi)/T) > 1$. Then, there is a non-zero fixed point of f in D_T. There exist $\rho > 0$ and $R > \rho$ such that $f(u) \neq u$ for $0 < |u|_\infty \leq \rho$, $u \in D_T$ and $f(u) \neq u$ for $|u|_\infty \geq R$ and $u \in D_T$; and if $G_{\rho,R} = \{u \in D_T : \rho < |u|_\infty < R\}$, then $i_{D_T}(f, G_{\rho,R}) = 1$. Furthermore, if Φ is also increasing and concave on $[0, \infty)$, then $f(K_T) \subset K_T$; and if $H_{\rho,R} = \{u \in K_T : \rho < |u|_\infty < R\}$, then $i_{K_T}(f, H_{\rho,R}) = 1$.*

We now illustrate Theorem 2.3 with two examples. First, we consider $J_1(x)$ equals $\beta/2$ if $|x| \leq 1$ and equals 0 otherwise. We will see in section 9 that for Φ satisfying B

$$\begin{aligned} f(C_T) \subset -C_T, &\quad \text{for} \quad T \in \bigcup_{n\geq 0}(\frac{1}{2n+2}, \frac{1}{2n+1}), \\ f(C_T) \subset C_T, &\quad \text{for} \quad T \in \bigcup_{n\geq 0}(\frac{1}{2n+3}, \frac{1}{2n+2}) \cup (1, \infty), \\ \text{and,} \quad f(C_T) \subset \{0\}, &\quad \text{for} \quad T \in \bigcup_{n\geq 0}\{1/(n+1)\}. \end{aligned} \tag{2.5}$$

By Theorem 2.3, (and Lemma 2.6), f has a fixed point in $C_T \backslash \{0\}$ if and only if

$$\beta \sin(\frac{\pi}{T}) > \frac{\pi}{T} \quad \text{and} \quad f(C_T) \subset C_T.$$

Depending on β, there will be an alternation of intervals where the period is such that f has a unique fixed point in $C_T\backslash\{0\}$ with intervals with no fixed point in $C_T\backslash\{0\}$.

For the case $J_2(x) = \beta e^{-x^2/2}/(\sqrt{2\pi})$, we will see in section 9 (assuming B) that $f(C_T) \subset C_T$, for any T. Thus, if $T_0 = 2\pi/(\sqrt{2\log(\beta)})$ and $T > T_0$, f has a unique fixed point in $C_T\backslash\{0\}$, whereas if $T \leq T_0$, 0 is the only fixed point in $\mathcal{H}_T$.

An outline of the paper is as follows. We give in section 3 conditions on J equivalent to having $f(C_T) \subset C_T$. We show also that a large class of kernels satisfies this condition. In section 4, we give two types of complementary existence results: when Φ is increasing, we use a variational method, whereas when $f(C_T) \subset C_T$ but Φ not increasing, we use a fixed point index argument. In section 5, we establish that the fixed points are real analytic functions. In section 6, we deal with the problem of uniqueness in the case where $f(C_T) \subset C_T$. When only A and B hold, we approximate the map f with f_ϵ such that $f_\epsilon(C_T) \subset C_T$. Results of section 6 tell us then that f_ϵ has a unique fixed point u_ϵ in C_T. We show then that $||df_\epsilon(u_\epsilon)|| < 1$, uniformly in ϵ, in an appropriate Banach space: this is the content of Lemma 7.6 of section 7.3. Many results of section 8 rely on a priori estimates of u_ϵ that we have gathered in section 7. The implicit function argument is then developed in section 8.1. Finally, we illustrate our results on some concrete examples of J's in section 9.

3 Invariant Cones.

Our task in this section is to give conditions on J which guarantee that K_T is invariant under convolution with J. We emphasize that the relation between J and J_T is not trivial: see the first example of section 9.

Fix $T > 0$ and define J_T as in (1.3), allowing J_T to have the value $+\infty$. We state two Lemmas whose proofs are given in the appendix.

Lemma 3.1 . *If (Aa) holds, then J_T is nonnegative, even, Lebesgue measurable and T-periodic. If (Aa) and (Ab) hold, then, J_T is integrable on $[0,T]$ and*

$$\int_0^T J_T(x)dx = \int_{-\infty}^{\infty} J(x)dx.$$

If A holds, then J_T is bounded.

Lemma 3.2 . *Assume A. For $u \in \mathcal{H}_T$, we define*

$$L_T u(x) = \int_{\mathbb{R}} J(x-y)u(y)dy = \int_{-T/2}^{T/2} J_T(x-y)u(y)dy. \tag{3.1}$$

Then, $L_T u \in \mathcal{H}_T$ and L_T defines a compact linear map of $\mathcal{H}_T$ into X_T.

We note that $\tilde{J}_T$ (see (2.4) is even, and odd with respect to $x = T/4$, i.e. $\tilde{J}_T(T/2-x) = -\tilde{J}_T(x)$. Also, for $u \in \mathcal{H}_T$, $2J_T * u(x) = \tilde{J}_T * u(x)$. Indeed, it is enough to note that

$$\int_{-T/2}^{T/2} J_T(T/2+y)u(x-y)dy = \int_0^T J_T(y)u(x-y+T/2)dy$$

$$= -\int_0^T J_T(y)u(x-y)dy = -\int_{-T/2}^{T/2} J_T(y)u(x-y)dy.$$

Therefore,

$$\begin{aligned}(L_T u)(x) &= \frac{1}{2}\int_0^{T/2} u(y)(\tilde{J}_T(x-y) - \tilde{J}_T(x+y))dy \qquad \text{[by oddness of u]} \\ &= \int_0^{T/4} u(y)(\tilde{J}_T(x-y) - \tilde{J}_T(x+y))dy. \qquad (3.2)\end{aligned}$$

Lemma 3.3 . *Assume A. Then, $L_T(C_T) \subset C_T$ if and only if $\tilde{J}_T$ decreases a.e.-dx in $[0, T/4]$ and $\tilde{J}_T(x) \geq 0$ a.e.-dx for $x \in [0, T/4]$. If Φ is odd, continuous, $\Phi(x) > 0$ for $x > 0$ and $L_T(C_T) \subset C_T$, then $f(C_T) \subset C_T$.*

Proof. First, for $u \in X_T$,

$$(L_T u)(x) = \int_0^{T/4} u(y)[\tilde{J}_T(x-y) - \tilde{J}_T(x+y)]dy$$

For every $x \in [0, T/4]$ and almost all $y \in [0, T/4 - x]$, we have $|x-y| \leq x+y \leq T/4$, and our hypothesis implies that $\tilde{J}_T(x-y) \geq \tilde{J}_T(x+y)$. For almost all $y \in [T/4 - x, T/4]$, $\tilde{J}_T(x+y) = -\tilde{J}_T(T/2 - (x+y)) \leq 0$ because $\tilde{J}_T \geq 0$ a.e.-dx in $[0, T/4]$.

Conversely, suppose A holds and $L(C_T) \subset C_T$. For any $\alpha < \beta$ in $(0, T/4)$ and $\epsilon \in (0, \min(\alpha, (\beta - \alpha)/2, T/4 - \beta)$ we choose $u'(x) = 1/\epsilon$ if $|x - \alpha| \leq \epsilon$ and $u'(x) = -1/\epsilon$ if $|x - \beta| \leq \epsilon$ and $u'(x) = 0$ for other $x \in [0, T/4]$. This insures that $(L_T u)(x) \geq 0$ for $x \in [0, T/4]$. We rewrite $(L_T u)(x)$ as

$$(L_T u)(x) = \frac{1}{2}\int_0^{T/2} \tilde{J}_T(y)(u(x+y) - u(y-x))dy. \qquad (3.3)$$

As, $\tilde{J}_T$ is bounded, the Lebesgue dominated convergence theorem implies that

$$\lim_{x\to 0} \frac{(L_T u)(x)}{x} = \int_0^{T/2} u'(y)\tilde{J}_T(y)dy = 2\int_0^{T/4} u'(y)\tilde{J}_T(y)dy \geq 0, \qquad (3.4)$$

Therefore,

$$\frac{1}{\epsilon}\int_{|y-\alpha|\leq\epsilon} \tilde{J}_T(y)dy \geq \frac{1}{\epsilon}\int_{|y-\beta|\leq\epsilon} \tilde{J}_T(y)dy.$$

Now taking $\epsilon \to 0$ and invoking the Lebesgue differentiation theorem, for α and β outside a subset of measure 0, $\tilde{J}_T(\alpha) \geq \tilde{J}_T(\beta)$.

Finally, for any $\gamma \in (0, T/4)$, and $\epsilon < \min(\gamma, T/4 - \gamma)$, we can define a piecewise linear continuous function $w \in C_T$ such that $w'(x) = 1/\epsilon$ if $|x - \gamma| < \epsilon$ and $w'(x) = 0$ if $|x - \gamma| > \epsilon$. Thus,

$$\lim_{x\to 0+} \frac{(L_T w)(x)}{x} = 2\int_0^{T/4} w'(y)\tilde{J}_T(y)dy \geq 0 = \frac{2}{\epsilon}\int_{|y-\gamma|\leq\epsilon} \tilde{J}_T(y)dy \geq 0.$$

Therefore, for almost all $\gamma \in [0, T/4]$ $\tilde{J}_T(\gamma) \geq 0$.

We prove in the Appendix that $L_T(C_T)$ comprises continuous functions. It follows then, under our assumptions, that $\Phi(L_T(C_T)) \subset C_T$. ∎

Corollary 3.4 . *Assume A. Then, $L_T(K_T) \subset K_T$ is equivalent to $\tilde{J}_T$ decreasing a.e.-dx in $[0, T/4]$, and $\tilde{J}_T(x) \geq 0$ a.e.-dx in $[0, T/4]$. If Φ is odd, continuous, increasing and concave and $L_T(K_T) \subset K_T$, then $f(K_T) \subset K_T$.*

Proof. (i) We first show that $\tilde{J}_T$ decreasing a.e.-dx in $[0, T/4]$, and $\tilde{J}_T(x) \geq 0$ a.e.-dx in $[0, T/4]$, implies that $L_T(K_T) \subset K_T$. It is enough to show that $L_T(K_T \cap C^2) \subset K_T \cap C^2$. Indeed, $K_T \cap C^2$ functions are dense in $(K_T, | \, |_\infty)$ and L_T is continuous on $(X_T, | \, |_\infty)$ (see Lemma 10.2 of the Appendix). Now, it is easy to see that $L_T(C^2) \subset C^2$ and that $u \in K_T \cap C^2$ is equivalent to $u \in X_T \cap C^2$ and $-u'' \in C_T$. By Lemma 3.3, $L_T(u) \in X_T \cap C^2$ and $-L_T(u)'' \in C_T$, so that $L_T(u) \in K_T$.

(ii) Assume $L_T(K_T) \subset K_T$. For any continuous $\phi \in C_T$ one can find $u \in K_T \cap C^2$ such that $u'' = -\phi$. Then $L_T(u) \in K_T \cap C^2$. Thus, $-L_T(u)'' = L_T(\phi) \in C_T$. $L_T(C_T) \subset C_T$ because continuous functions are dense in C_T in the $\mathcal{H}_T$ topology and L_T is continuous. Lemma 3.3 implies that $\tilde{J}_T$ is decreasing in $[0, T/4]$ and $\tilde{J}_T(x) \geq 0$ a.e.-dx in $[0, T/4]$.

The proof is concluded by noting that the composition of two increasing concave functions is still increasing and concave. ∎

Remark 3.5 . In general $\tilde{J}_T$ is not decreasing in $[0, T/4]$ (see example 9.1). However, if we choose $T \geq 2M(\epsilon)$ and define $J^\epsilon = J I_{[-M(\epsilon), M(\epsilon)]}$, then $(J^\epsilon)_T = J^\epsilon$ on $[-T/2, T/2]$, and Lemma 3.3 applies to J^ϵ. Here, for $S \subset \mathbb{R}$, we use $I_S(x)$ to denote the characteristic function of S, so $I_S(x) = 1$ for $x \in S$ and $I_S(x) = 0$ for $x \notin S$. A natural class of J leaving K_T invariant are those of Lemma 2.5. This class is natural in most applications in physics where such a fixed point problem arises.

Proof of Lemma 2.5. We write

$$\sum_{n \in \mathbb{Z}} J(nT + T/2 + x) = \sum_{n=0}^{\infty} J(nT + T/2 + x) + \sum_{n=1}^{\infty} J(-nT + T/2 + x).$$

Then, using that J is even

$$\sum_{n=1}^{\infty} J(-nT + T/2 + x) = \sum_{n=0}^{\infty} J(-nT - T/2 + x) = \sum_{n=0}^{\infty} J(nT + T/2 - x),$$

and therefore, for $x \in [0, T/4]$, $\tilde{J}_T = J(x) + \sum_{n \geq 0} j_n(x)$, with

$$j_n(x) = J(nT + T + x) + J(nT + T - x) - J(nT + T/2 + x) - J(nT + T/2 - x). \quad (3.5)$$

Now, for $n \geq 0$, j_n is decreasing in $[0, T/4]$, as one can see by taking the derivative of j_n and using the concavity of J' on $[C, \infty)$. As J is decreasing on $[0, T/4]$, we conclude that $\tilde{J}_T$ is decreasing on $[0, T/4]$. Also, $\tilde{J}_T(T/4) = 0$ implies that $\tilde{J}_T(x) \geq 0$ for all $x \in [0, T/4]$. ∎

Remark 3.6 . Assume that (Aa), (Ac) and (Ad) hold and that $\int |J(y-x)-J(y+x)|dy < \infty$ for all x. Then, $\tilde{J}(z) = J(z) - J(z - T/2)$ is an integrable function and we can define $\tilde{J}_T(z) = \sum_{\mathbb{Z}} \tilde{J}(z + nT)$. Now, for $u \in X_T$, we interpret $f(u)$ as

$$f(u) = \Phi\left(\int_0^{T/4} [\tilde{J}_T(y-x) - \tilde{J}_T(y+x)]u(y)dy\right).$$

In case where J' is concave in $\{x \geq C\}$, then for $T > 4C$, $f(C_T) \subset C_T$ because the formula $\tilde{J}_T = J(x) + \sum_{n\geq 0} j_n(x)$, with (3.5) still holds.

4 Existence Results.

Proof of Lemma 2.7. Because Φ is increasing, we can define its inverse ψ. For notational convenience, we will assume in the proof that Φ is odd and $\lim_\infty \Phi = 1$; these features play no role. Also, for $x \in [-1, 1]$, we define

$$\Psi(x) = \int_0^x \psi(y)dy. \tag{4.1}$$

For $u \in G_T := \{u \in \mathcal{H}_T : |u(x)| \leq 1, \text{ a.e.}\}$, a closed bounded convex set in $\mathcal{H}_T$, we define the energy as

$$\mathcal{F}[u] := \int_0^T \Psi(u(x))dx - \frac{1}{2}\int_0^T\int_0^T J_T(x-y)u(x)u(y)dxdy \equiv (\Psi(u), 1) - \frac{1}{2}(J_T * u, u). \tag{4.2}$$

If $|u|_\infty = 1$, we set $\mathcal{F}[u] = \infty$. We first claim that there is $u \in G_T$ such that $\mathcal{F}[u] < 0$ (note that $\mathcal{F}[0] = 0$). We assume that the supremum in (2.3) is achieved for n_0 and set $u_0 = \sin(2\pi(2n_0 + 1)x/T) \in G_T$ so that $J_T * u_0 = \lambda u_0$, and $\lambda > 1$. Then, the claim follows by choosing ϵ small enough and noticing that $\Psi(x) \sim x^2/2$ close to 0.

$$\mathcal{F}[\epsilon u_0] = (\Psi(\epsilon u_0), 1) - \frac{1}{2}\epsilon^2(J_T * u_0, u_0) = \frac{\epsilon^2}{2}(1-\lambda)(u_0, u_0) + o(\epsilon^2). \tag{4.3}$$

We now show that we can always choose a minimizing sequence in $\{u : |u|_\infty \leq 1-\delta\}$ for δ small enough. For, $\delta \in (0, 1/2)$, let u_δ be a truncation of u

$$u_\delta = uI_{\{|u|\leq 1-\delta\}} + (1-\delta)I_{\{u>1-\delta\}} - (1-\delta)I_{\{-u>1-\delta\}}. \tag{4.4}$$

It follows from a simple computation that

$$\begin{aligned} |(J * u, u) - (J * u_\delta, u_\delta)| \leq \quad & C\int_0^T ||u| - (1-\delta)|I_{|u|>1-\delta}dx \\ = \quad & C\int_{1-\delta}^1 |\{x \in [0,T] : |u(x)| > s\}|ds. \end{aligned} \tag{4.5}$$

Now, $\sup|\Phi| = 1$ implies that $\lim_1 \psi = \infty$. Thus, it is always possible to choose δ such that $\psi(1-\delta) = C$. Now,

$$\mathcal{F}[u] - \mathcal{F}[u_\delta] \geq \int_{|u|>1-\delta} (\Psi(u(x)) - \Psi(u_\delta(x)))dx - \frac{1}{2}C\int_{1-\delta}^1 |\{x : |u(x)| > s\}|ds. \tag{4.6}$$

Note that

$$\int_{|u|>1-\delta}(\Psi(u)-\Psi(u_\delta)) = \int_{1-\delta}^{1}\psi(s)|\{x:\ |u(x)|>s\}|ds$$
$$\geq \psi(1-\delta)\int_{1-\delta}^{1}|\{x:\ |u(x)|>s\}|ds. \tag{4.7}$$

Thus,

$$\mathcal{F}[u]-\mathcal{F}[u_\delta] \geq \int_{1-\delta}^{1}|\{x:\ |u(x)|>s\}|ds\left(\psi(1-\delta)-\frac{C}{2}\right)>0. \tag{4.8}$$

Now, let (u_n) be a sequence in $\{u\in G_T:\ |u|_\infty\leq 1-\delta\}$, such that $\lim\mathcal{F}[u_n]=\inf\mathcal{F}$. There is a subsequence converging weakly to $u^*\in G_T$ with $|u^*|_\infty\leq 1-\delta$. Now, it is well known [2] that $\mathcal{F}$ is weakly lower semi-continuous so that $\mathcal{F}[u^*]=\inf\mathcal{F}$. Also, it is easy to see that $\psi(u^*)-J_T*u^*=0$. ∎

If we drop the assumption that Φ is increasing, but demand that Φ be odd, then there is a case, natural from our point of view, where we can still have an existence result through a fixed point index theorem. First, we recall that a closed cone C with vertex at 0 in a Banach space Y is a closed convex set such that (a) $\lambda C\subset C$ for all $\lambda\geq 0$ and (b) $C\cap(-C)=\{0\}$. We say that $g:C\to C$ is Fréchet differentiable at 0 with respect to C if there exists a bounded linear map $L=dg_0:Y\to Y$ such that $g(x)=g(0)+L(x)+R(x)$ for all $x\in C$, where $||R(x)||\leq\eta(\rho)\rho$ for all $x\in C$ with $||x||\leq\rho$, and $\lim_{r\to 0^+}\eta(r)=0$. The following theorem can be found in [8].

Theorem 4.1 . *Assume that C is a closed cone in a Banach space $(Y,||\ ||)$. If (o) $f:C\to C$ is a continuous map with $f(0)=0$, (i) f is compact, (ii) f is Fréchet differentiable at 0 with respect to C and there exist $v\in C\backslash\{0\}$ and $\lambda>1$ with $df_0(v)=\lambda v$, (iii) $df_0(x)\neq x$ for $x\in C\backslash\{0\}$, (iv) there is $\alpha>0$ such that $tf(x)\neq x$ $\forall t\in[0,1]$ for $x\in C\backslash\{0\}$, $||x||=\alpha$, then f has a fixed point $u\in C$ with $0<||u||<\alpha$.*

Furthermore, if $B_\epsilon=\{x\in C:\ ||x||<\epsilon\}$ and $U_{\epsilon,\alpha}=\{x\in C:\ \epsilon<||x||<\alpha\}$, there exists ϵ_0 such that 0 is the only fixed point of f in B_{ϵ_0}; for $0<\epsilon<\epsilon_0$, $i_C(f,B_\epsilon)=0$ and $i_C(f,U_{\epsilon,\alpha})=1$.

Proof of Lemma 2.8. We need to verify the hypotheses of Theorem 4.1. Our Banach space is $(X_T,|\ |_\infty)$ and our closed cone is D_T. Lemma 7 and Corollary 1 imply that $f(D_T)\subset D_T$. Lemma 6 implies that L_T is continuous and compact as a map fro $\mathcal{H}_T$ to X_T, so $f:D_T\to D_T$ is compact. Using these observations, it is easy to see that $f:D_T\to D_T$ satisfies conditions (0) and (i) of Theorem 4.1. We can also consider f as a map from X_T to X_T, and it is easy to check that f is Fréchet differentiable at 0 with Fréchet derivative $df_0:X_T\to X_T$ given by $df_0=L_T$. Also, for $v(x)=\sin(2\pi x/T)$, $L_T(v)=\lambda v$ with $\lambda=\hat{J}(2\pi/T)>1$. Thus, condition (ii) is satisfied.

One can consider L_T as a bounded linear map of $\mathcal{H}_T$ into itself; since $L_T:\mathcal{H}_T\to X_T\subset\mathcal{H}_T$ is compact, the spectrum of $L_T:\mathcal{H}_T\to\mathcal{H}_T$ is the same as the spectrum of $L_T:X_T\to X_T$. Using Fourier series in $\mathcal{H}_T$, one can see that $\sigma(df_0)$ is given by

$$\sigma(df_0)=\{\hat{J}(\frac{2\pi(2n+1)}{T}),\ n=0,1,\ldots\}\cup\{0\},$$

and that the eigenvector corresponding to $\lambda_n = \hat{J}(\frac{2\pi(2n+1)}{T})$ is $u_n(x) = \sin(\frac{2\pi(2n+1)x}{T})$.

Since $u_n \notin D_T$ for $n \geq 1$ and $\lambda_0 > 1$, we conclude that $df_0(u) \neq u$ for $u \in D_T\backslash\{0\}$, and condition (iii) holds.

Condition (iv). If $R > |\Phi|_\infty$, then for any $x \in D_T$ $|f(x)|_\infty < R$. Thus, for any $t \in [0,1]$, $tf(x) \neq x$ when $x \in D_T$ with $|V|_\infty = R$.

If ϵ is chosen as in Theorem 3 and R is as above and $G_{\epsilon,R}$ is as in Lemma 4, then Theorem 3 implies that $i_{D_T}(f, G_{\epsilon,R}) = 1$. The case for general ρ and R follows from the additivity property of the fixed point index. Since $i_{D_T}(f, G_{\epsilon,R}) = 1$, the properties of the fixed point index imply that f has a fixed point in $G_{\epsilon,R}$. If Φ is also increasing and concave on $[0,\infty)$, we have seen that $f(K_T) \subset K_T$ and the same argument given above, with K_T replacing D_T, shows that $i_{K_T}(f, H_{\epsilon,R}) = 1$. ∎

Finally, for the sake of completeness, we prove here Lemma 2.6 (compare with [3]).

Proof of Lemma 2.6. Assume that $u \in C_T\backslash\{0\}$ is such that $f(u) = u$. Thus,

$$\Phi(L_T u) = u \quad \Longrightarrow \quad u(x)L_T u(x) \geq u(x)^2 \tag{4.9}$$

with equality only for x such that $u(x) = 0$. Because u is continuous (Lemma 3.2) and not identically zero, we see that $(L_T u, u)_T > (u,u)_T$. As L_T is self-adjoint, we have $r(L_T) > 1$, which is a contradiction. ∎

5 Regularity.

Theorem 5.1 . *Assume A and suppose that Φ is real analytic. If $u : \mathbb{R} \to \mathbb{R}$ is measurable and bounded and $\Phi(J * u) = u$, then u is real analytic on $\mathbb{R}$.*

Proof. The proof will proceed in 3 steps. In step 1, we show that u is Lipschitz; in step 2 we show that u is C^∞ and in step 3 that u is real analytic.

Step 1. Our first claim is that for $x < z$,

$$|J * u(z) - J * u(x)| \leq 3J(0)|u|_\infty (z - x). \tag{5.1}$$

Indeed,

$$\begin{aligned} &\Big|\int_{-\infty}^{\infty} u(y)(J(x-y) - J(z-y))dy\Big| \leq |u|_\infty \Big(\int_{-\infty}^{\infty} |J(x-y) - J(z-y)|dy\Big) \\ &\leq |u|_\infty \Big(\int_{-\infty}^{x} [J(x-y) - J(z-y)]dy \\ &\qquad + \int_x^z |J(x-y) - J(z-y)|dy + \int_z^\infty [J(y-z) - J(y-x)]dy\Big) \\ &\leq |u|_\infty \Big(2\int_0^{z-x} J + \int_x^z |J(x-y) - J(z-y)|dy\Big) \\ &\leq 3J(0)|u|_\infty (z-x). \end{aligned} \tag{5.2}$$

Note that $|J * u|_\infty \le |J|_1 |u|_\infty$ and if we call

$$M = 3J(0), \quad \text{then} \quad |u(z) - u(x)| \le \left(\sup_{|y| \le |J|_1 |u|_\infty} |\Phi'(y)| \right) M |z - x| |u|_\infty.$$

Step 2. We show that u is k times continuously differentiable (C^k) for all $k \ge 1$, and

$$|D^k(J * u)|_\infty \le M |D^{k-1} u|_\infty. \tag{5.3}$$

First, we show that u is C^1 and

$$|D(J * u)|_\infty \le M |u|_\infty. \tag{5.4}$$

Indeed, u Lipschitz implies that $u'(x)$ exists almost everywhere, $|u'|_\infty < \infty$, and

$$\forall x, \qquad u(x) = u(0) + \int_0^x u'(y) dy.$$

Thus, $|J * u'(z) - J * u'(x)| \le M |u'|_\infty |z - x|$ and

$$\left|\frac{J * u(x + \epsilon) - J * u(x)}{\epsilon} - J * u'(x)\right| = \left|\frac{1}{\epsilon} \int_0^\epsilon (J * u'(z + x) - J * u'(x))\, dz\right| \le \frac{M |\epsilon| |u'|_\infty}{2}.$$

This implies that $J * u$ is differentiable and $(J * u)' = J * (u')$. It follows that (5.4) holds, $u \in C^1$ and

$$|u'|_\infty = |\Phi'(J * u) J * u'|_\infty \le M \sup_{|x| \le |J|_1 |u|_\infty} |\Phi'(x)| |u|_\infty. \tag{5.5}$$

Now, we assume, by way of induction, that $u \in C^k$ and $D^i u$ is bounded for $i \le k$. We note that $D^k(J * u) = J * D^k u$, and by arguing similarly as above,

$$|D^k J * u(z) - D^k J * u(x)| \le M |D^k u|_\infty |z - x|, \quad \text{and} \quad D^k u = \Phi'(J * u) D^k(J * u) + R_k, \tag{5.6}$$

where $R_k \in C^1$ and DR_k is bounded. Thus, (5.6) shows that $D^k u \in C^1$ and $D^{k+1} u$ is bounded. The first inequality of (5.6) implies (5.3).

Step 3. We show by induction that there are positive numbers $\{p_n,\ n \ge 0\}$ such that

$$\frac{|u^{(i)}|_\infty}{i!} \le p_i, \quad \text{with} \quad \sum_{n=0}^{\infty} p_n x^n < \infty, \tag{5.7}$$

for x positive and small enough.

We define for any $b > 0$ a sequence $\{q_n(b)\}$, with $q_1(b) = b$ and

$$\forall n > 1, \qquad q_n(b) = \frac{1 \cdot 3 \ldots (2n - 3)}{n!} b^n 2^{n-1}.$$

It is known [6] (p.343-344) that $\{q_n(b)\}$ satisfies

$$\sum_{i=1}^{n-1} q_i(b) q_{n-i}(b) = q_n(b).$$

We choose a $\delta > 0$ to be specified later, and $\gamma = 2p_0 + \delta$. We take $p_0 = |u|_\infty$, $p_1 = |u'|_\infty$ and for $n > 1$

$$p_n = \delta q_n(\frac{p_1}{\delta}) \tag{5.8}$$

Thus, the sequence $\{p_n, n \geq 0\}$ satisfies

$$\sum_{i=0}^{n} p_i p_{n-i} \leq \gamma p_n.$$

Furthermore,

$$\sum_{j \in S_m(n)} p_{j_1} \cdots p_{j_m} \leq \gamma^{m-1} p_n, \tag{5.9}$$

where the summation is taken over $S_m(n) \equiv \{(j_1, \ldots, j_m) : j_1, \ldots, j_m \geq 0, \sum j_i = n\}$. First, (5.9) holds for $m = 2$. By induction, assume that (5.9) is true for some $m \geq 2$ (and any n). Then, for any n,

$$\sum_{j \in S_{m+1}(n)} p_{j_1} \cdots p_{j_{m+1}} = \sum_{i=0}^{n} p_i \left(\sum_{j \in S_m(n-i)} p_{j_1} \cdots p_{j_m} \right) \leq \sum_{i=0}^{n} p_i (\gamma^{m-1} p_{n-i}) \leq \gamma^m p_n.$$

We assume now that (5.7) holds up to order n. We define $h = J * u$ and $g = h'$ and start with the equation

$$u'(x) = \Phi'(h(x))g(x). \tag{5.10}$$

We fix $x_0 \in I\!R$. Taylor's theorem implies that for ξ near $h(x_0)$,

$$\Phi'(\xi) = \sum_{j \geq 0} c_j (\xi - h(x_0))^j, \quad \text{where} \quad |c_j| \leq C^j \quad \text{for} \quad j \geq 1. \tag{5.11}$$

Indeed, as $h : I\!R \to I\!R$ is bounded, C can be chosen independently of x_0. Using (5.11),

$$D^k \Phi'(h(x))\Big|_{x=x_0} = \sum_{m=1}^{k} c_m \sum_{j \in S_m^*(k)} \frac{k!}{j_1! j_2! \ldots j_m!} [D^{j_1} h(x_0) \ldots D^{j_m} h(x_0)], \tag{5.12}$$

where we have called $S_m^*(k) = S_m(k) \cap \{j_1 > 0, \ldots, j_m > 0\}$. Starting from (5.10) and using Leibnitz formula, we obtain

$$|u^{(n+1)}(x_0)| \leq \sum_{k=0}^{n} \binom{n}{k} \Big| D^k \Phi'(h(x))\Big|_{x=x_0} \Big| D^{n-k} g(x_0) \Big|. \tag{5.13}$$

This gives, using (5.3)

$$\frac{|u^{(n+1)}(x_0)|}{(n+1)!} \leq \frac{CM|u^{(n)}|_\infty}{(n+1)!} + \frac{M}{(n+1)!} \sum_{k=1}^{n} \binom{n}{k} \Big| D^k \Phi'(h(x))\Big|_{x=x_0} |u^{(n-k)}|_\infty. \tag{5.14}$$

Combining (5.12) and (5.14), we obtain

$$\frac{|u^{(n+1)}(x_0)|}{(n+1)!} \leq \frac{CM|u^{(n)}|_\infty}{(n+1)!} + \frac{M}{n+1} \sum_{k=1}^{n} \left(\sum_{m=1}^{k} c_m \sum_{j \in S_m^*(k)} \prod_{i=1}^{m} (\frac{M|u^{(j_i-1)}|_\infty}{j_i!}) \right) \frac{|u^{(n-k)}|_\infty}{(n-k)!}. \tag{5.15}$$

Using the inductive hypothesis and (5.9), we obtain from (5.15) (and $|c_n| \leq C^n$) that

$$\begin{aligned}\frac{|u^{(n+1)}(x_0)|}{(n+1)!} &\leq \frac{CM|u^{(n)}|_\infty}{(n+1)!} + \frac{M}{n+1}\sum_{m=1}^{n} c_m \sum_{k=m}^{n}\left(p_{n-k}\sum_{j\in S_m^*(k)} M^m \prod_{i=1}^{m} p_{j_i-1}\right)\\ &\leq \frac{CM|u^{(n)}|_\infty}{(n+1)!} + \frac{M}{n+1}\sum_{m=1}^{n} c_m \sum_{k=m}^{n} M^m\gamma^{m-1}p_{n-k}p_{k-m}\\ &\leq \frac{CM|u^{(n)}|_\infty}{(n+1)!} + \frac{M}{n+1}\sum_{m=1}^{n} C^m M^m \gamma^m p_{n-m}. \qquad (5.16)\end{aligned}$$

If $0 < \delta \leq 1/2$ and $0 \leq k < n$, we obtain from (5.8) that $p_{n+1} \geq (\frac{2p_1}{\delta})^{k+1}p_{n-k}$. Thus, assuming δ so small that $\epsilon = (CM\gamma\delta)/2p_1 < 1$, we obtain

$$\frac{|u^{(n+1)}(x_0)|}{(n+1)!} \leq \frac{1}{(n+1)\gamma}(\sum_1^\infty \epsilon^m)p_{n+1} + (\frac{M}{n+1})C^n M^n \gamma^n p_0.$$

Also, it is easy to see that

$$(CM\gamma)^n p_0 \leq 2\epsilon^n(\frac{\delta}{p_1})p_0 p_{n+1}.$$

Thus, we obtain

$$\frac{|u^{(n+1)}(x_0)|}{(n+1)!} \leq \left[\frac{\epsilon}{(n+1)\gamma(1-\epsilon)} + M\epsilon\frac{\delta}{p_1}p_0\right]p_{n+1}.$$

Thus, for δ small enough, independently of n and x_0, we have $|u^{(n+1)}(x_0)| \leq p_{n+1}(n+1)!$. Taking the supremum over all x_0, we have proved that the induction is correct. The fact that the power series in (5.7) has a positive radius of convergence follows now from the explicit expression for the p_n and Stirling's formula for $n!$. ∎

6 Case where $f(C_T) \subset C_T$.

We assume in this section that A and B hold and that $f(C_T) \subset C_T$.

If there is $C > 0$ such that J is continuously differentiable in $[C, \infty)$ and J' is concave in $[C, \infty)$, then for $T \geq 4C$, Lemma 2.5 shows that $f(C_T) \subset C_T$.

Remark 6.1 . If we make the assumption that J_T itself is decreasing in $[0, T/2]$, then it is easy to see that a cone larger than C_T is invariant, namely

$$\tilde{C}_T = \{u:\ \text{odd and periodic and } u(x) \geq 0 \text{ a.e.} - \text{dx for } x \in [0, T/2]\}.$$

All the results of this section hold in this larger setting (i.e. without the symmetry with respect to $x = T/4$) with trivial modifications.

Now, Lemma 3.3 implies that $\tilde{J}_T$ is a.e.-dx decreasing and nonnegative on $[0, T/4]$. Let $\Gamma \subset [0, T/4]$ be a measurable set of measure $T/4$ such that $\tilde{J}_T$ is decreasing and nonnegative on Γ. We define a right continuous, increasing function on $[0, T/4)$ by

$$F(x) = \lim_{y\in\Gamma\to x+} -\tilde{J}_T(y), \quad x \in [0, T/4).$$

We define $F(T/4) = \tilde{J}(T/4) = 0$ and $F(x) = -F(T/2 - x)$ for $T/4 < x \leq T/2$. Finally, we define $F(x) = F(-x)$ for $-T/2 \leq x \leq 0$ and we extend F to be T-periodic. The map F is bounded and increasing on $[0, T/4]$, so F is of bounded variation on $[0, T/4]$. It follows that for $\varphi \in C([0, T/4])$ the Riemann-Stieltjies integral

$$\int_0^{T/4} \varphi(x) dF(x) = \Lambda(\varphi)$$

is defined, and Λ is a bounded linear functional on $C([0, T/4])$. The Riesz representation theorem implies that there is a regular Borel measure ν on $[0, T/4]$ such that

$$\nu((0, x]) = F(x) - F(0), \qquad \forall\, x \in [0, T/4].$$

It is important to note that $F = -\tilde{J}_T$ a.e.-dx, and that as $\tilde{J}_T$ always appears integrated against some function, we can replace $\tilde{J}_T$ by $-F$.

The measure ν may be singular with respect to Lebesgue measure, but we know that if $\tilde{J}_T$ is not equal to zero a.e.-dx on $[0, T/4]$: there is a point of increase. In other words, either (a) there is $\alpha \in (0, T/4)$ such that

$$\forall x \in [0, \alpha),\ \forall y \in (\alpha, T/4], \qquad F(x) < F(y),$$

or (b) F is constant on $[0, T/4)$ and $F(x) < F(T/4) = 0$ for $0 \leq x < T/4$. Here is an illustration of case (b): if $J(x) = 1$ for $|x| \leq T/4$ and $J(x) = 0$ for $|x| > T/4$, then $\tilde{J}_T(x) = 1$ for $0 \leq x < T/4$, $\tilde{J}_T(x) = -1$ for $T/4 < x \leq T/2$ and $\tilde{J}_T(T/4) = 0$.

In case (a), $\forall \epsilon > 0$ small, $\nu((\alpha - \epsilon, \alpha + \epsilon]) > 0$; in case (b) $\forall \epsilon > 0$ small, $\nu((T/4 - \epsilon, T/4]) > 0$. We recall also a standard fact that we will use repeatedly. If $u \in X_T$ is continuously differentiable, and F is as above,

$$\lim_{x \to 0+} \int_0^{T/4} u(y) \frac{F(y + x) - F(y - x)}{2x} dy = \int_0^{T/4} u(y) d\nu(y). \tag{6.1}$$

We are assuming here that $L_T(C_T) \subset C_T$. Observe that $\mathcal{H}_T = C_T - C_T$ and $L_T : \mathcal{H}_T \to \mathcal{H}_T$ is compact. If $r(L_T) > 0$, the Krein-Rutman theorem implies that there exists $\varphi \in C_T \backslash \{0\}$ with $L_T(\varphi) = r(L_T)\varphi$. However the spectrum of L_T is

$$\{0\} \cup \{\hat{J}(\frac{2\pi}{T}(2k + 1),\ k = 0, 1, \ldots\}. \tag{6.2}$$

Furthermore, there is a unique eigenvector $\varphi_k(x) = \sin(2\pi(2k + 1)x/T)$ corresponding to $\hat{J}(2\pi(2k + 1)/T)$. Clearly, $\varphi_k \in C_T$ if and only if $k = 0$. Thus, if

$$r(L_T) = \sup\{|\hat{J}(\frac{2\pi}{T}(2k + 1)|,\ k = 0, 1, \ldots\}\} > 0,$$

then $r(L_T) = \hat{J}(2\pi/T)$. Now, if $r(L_T) = 0$, we also have $r(L_T) = \hat{J}(2\pi/T)$. Also, because L_T is self-adjoint,

$$r(L_T) := \lim_{n \to \infty} ||L_T^n||_T^{1/n} = \sup_{u \in \mathcal{H}_T \backslash \{0\}} \frac{(L_T u, u)_T}{(u, u)_T}. \tag{6.3}$$

Our next lemma is a special case of Lemma 4, but we prefer to give the simpler constructive proof which is available when A and B hold. If $v_1, v_2 \in \mathcal{H}_T$, or $v_1, v_2 \in X_T$, we shall write $v_1 \succ v_2$ if $v_1(x) - v_2(x) \geq 0$ for almost all $0 \leq x \leq T/4$.

Lemma 6.2 . *If $r(L_T) > 1$, there is $u \in K_T\backslash\{0\}$ such that $f(u) = u$.*

Proof. The eigenfunction corresponding to $r(L_T)$ is $\varphi(x) = \sin(2\pi x/T)$. We note that $\varphi \in K_T$ and that for ϵ small enough, $f(\epsilon\varphi) = \Phi(r(L_T)\epsilon\varphi) \succ \epsilon\varphi$. Thus, $\{f^n(\epsilon\varphi),\ n = 1, 2, \ldots\}$ is an increasing sequence in K_T. Let u^* be the pointwise limit of $f^n(\epsilon\varphi)$. Because f is continuous from $(\mathcal{H}_T, |\cdot|_T)$ to $(X_T, |\cdot|_\infty)$, $u^* \in K_T\backslash\{0\}$ and $f(u^*) = u^*$. ∎

Lemma 6.3 . *If $u \in C_T\backslash\{0\}$ is such that $f(u) = u$, then $u > 0$ in $(0, T/4]$, and $u'(0) > 0$.*

Proof. Case a. There exists $\alpha \in (0, T/4)$ point of increase of F. We recall that u is continuously differentiable. We choose $b \in [0, T/4]$ such that $u(b) > 0$. Now, let a be the smallest number such that $u > 0$ in (a, b). By definition of $u \in X_T$, $u(0) = 0$, thus $a \geq 0$ and $u(a) = 0$. We claim that $a < \alpha$. Indeed, if we assume that $a \geq \alpha$ we reach a contradiction in

$$\Phi^{-1}(u(a)) = \tilde{J}_T * u(a) \geq \int_a^b u(y)\left(\tilde{J}_T(y-a) - \tilde{J}_T(y+a)\right) dy > 0. \tag{6.4}$$

The last inequality of (6.4) follows because for $y \in (a, a+\alpha)$, we have $y + a > \alpha > y - a$. Now, we rewrite (6.4) at α

$$\Phi^{-1}(u(\alpha)) = \tilde{J}_T * u(\alpha) \geq \int_a^b u(y)\left(\tilde{J}_T(y-\alpha) - \tilde{J}_T(y+\alpha)\right) dy > 0, \tag{6.5}$$

because $|\{y \in (a,b):\ y + \alpha > \alpha > |y - \alpha|\}| > 0$. Thus, we can actually choose (a, b) to be the maximal interval in $[0, T/4]$, containing α, such that $u > 0$ in (a, b). Now, by (6.4), $u(a) = 0$ only if

$$|\{y \in (a,b):\ y + a > \alpha > y - a\}| = 0, \tag{6.6}$$

i.e., if $|(\alpha - a, \alpha + a)| = 0$, which implies that $a = 0$. On the other hand, $u(b) \neq 0$, because

$$|\{y \in (a,b):\ y + b > \alpha > b - y\}| = |[b - \alpha, T/4]| > 0.$$

Thus, $b = T/4$. This proves the first claim of Lemma 6.3. We prove now the last claim. We write

$$\frac{u(x)}{x} = \int_0^{T/4} u(y)\frac{(F(y+x) - F(y-x))}{x} dy,$$

and, by (6.1), we have for $\epsilon > 0$ small

$$\lim_{x\to 0}\frac{u(x)}{x} = 2\int_0^{T/4} u(y)d\nu(y) \geq 2(\inf_{(\alpha-\epsilon,\alpha+\epsilon]} u)\nu(\alpha - \epsilon, \alpha + \epsilon] > 0.$$

Case b. Because f has a nonzero fixed point in C_T, $\tilde{J}_T$ cannot equal zero a.e.-dx on $[0, T/4]$. However, $\tilde{J}_T$ may be constant a.e. on $[0, T/4]$, say $\tilde{J}_T(x) = c > 0$ a.e. on $[0, T/4]$ and $\tilde{J}_T(x) = -c$ a.e. on $[T/4, T/2]$. If this is the case and $u \in C_T\backslash\{0\}$ is a fixed point of f we find that

$$(L_T u)(x) = \int_{T/4-x}^{T/4} (2c)u(y)dy, \quad 0 \leq x \leq T/4,$$

and,

$$u(x) = \Phi\left(2c\int_{T/4-x}^{T/4} u(y)dy\right), \quad 0 \leq x \leq T/4. \tag{6.7}$$

It follows that u is increasing in $[0, T/4]$, and since $u \not\equiv 0$, we must have $u(T/4) > 0$. It follows easily from (6.7) that $u(x) > 0$ for $0 < x \leq T/4$. Also, (6.7) implies that

$$u'(0) = 2c\Phi'(0)u(\frac{T}{4}) > 0.$$

∎

Theorem 6.4 . *Assume that $r(L_T) > 1$. If u is any fixed point of f in $C_T\backslash\{0\}$ and if L denotes the Fréchet derivative of f at u in $\mathcal{H}_T$, then $r(L) < 1$.*

Proof. We denote by $g(x) = \tilde{J}_T * u(x)$. Then, for $v \in X_T$, the Fréchet derivative of f at u is

$$L : \mathcal{H}_T \to \mathcal{H}_T, \qquad Lv = \Phi'(g)\tilde{J}_T * v. \tag{6.8}$$

Thus, $L(C_T) \subset C_T$ and L is compact (as a consequence of Lemma 3.2). Thus, by the Krein-Rutman Theorem [8], if $r(L) \neq 0$, there is $w \in C_T\backslash\{0\}$ with

$$Lw = r(L)w. \tag{6.9}$$

Note that $w'(0)$ exists, and by Lemma 6.3, $g(x) > 0$ for $x \in (0, T/4]$. Thus we define the linear operator $L_1 : \mathcal{H}_T \to \mathcal{H}_T$ with

$$\forall x \in (0, T/4], \quad L_1 v(x) = \frac{\Phi(g(x))}{g(x)}\tilde{J}_T * v(x) \quad \text{and,} \quad L_1 v(0) = 0. \tag{6.10}$$

L_1 is well defined, for $\Phi(g(x))/g(x)$ is continuous and bounded. L_1 is such that $L_1 u = u$ and

$$Lv = \lambda L_1 v, \qquad \text{with} \qquad \lambda(x) = \frac{\Phi'(g(x))g(x)}{\Phi(g(x))}\ \forall x \in (0, T/4] \text{ and,} \quad \lambda(0) = 1. \tag{6.11}$$

As $\Phi'''(0) < 0$ and $g(x) > 0$ for $x \in (0, T/4]$, we have that

$$\lambda(x) \in [0, 1), \text{ for } x \in (0, T/4] \quad \text{and } \lambda(0) = 1. \tag{6.12}$$

Suppose we could prove that $\kappa u \succ L^2 u$ with a positive constant $\kappa < 1$, and that there is $M > 0$ such that $Mu \succ w$. Then, by applying L^2 k-times,

$$M\kappa^k u \succ ML^{2k}u \succ L^{2k}w = r(L)^{2k}w. \tag{6.13}$$

This would imply that $r^2(L) \leq \kappa < 1$ which is the desired result.

To prove that $\kappa u \succ L^2 u$ with $\kappa < 1$, we write

$$L^2 u = L(Lu) = L(\lambda u) = \lambda L_1(\lambda u) = \lambda\frac{\tilde{J}_T * (\lambda u)}{\tilde{J}_T * u}u,$$

and κ is given by

$$\kappa = \sup_{x\in[0,T/4]} \lambda(x)\frac{\tilde{J}_T * (\lambda u)}{\tilde{J}_T * u}(x) \tag{6.14}$$

Now, u and λ are in X_T, thus for $x \in [0, T/4]$

$$\frac{\tilde{J}_T * (\lambda u)}{\tilde{J}_T * u}(x) = \frac{\int_0^{T/4} \lambda(y)u(y)(\tilde{J}_T(x-y) - \tilde{J}_T(x+y))dy}{\int_0^{T/4} u(y)(\tilde{J}_T(x-y) - \tilde{J}_T(x+y))dy} \leq 1. \tag{6.15}$$

Because $\lambda(x) < 1$ for $x \in (0, T/4]$, it remains to show that the l.h.s. of (6.15) is strictly less than 1 at 0. After dividing numerator and denominator of (6.15) by x

$$\lim_{x\to 0} \frac{\tilde{J}_T * (\lambda u)}{\tilde{J}_T * u}(x) = \frac{\int_0^{T/4} \lambda(y)u(y)d\nu(y)}{\int_0^{T/4} u(y)d\nu(y)} \leq \frac{\int_0^{\alpha/2} ud\nu + (\sup_{[\alpha/2,T/4]} \lambda) \int_{\alpha/2}^{T/4} ud\nu}{\int_0^{\alpha/2} ud\nu + \int_{\alpha/2}^{T/4} ud\nu} \tag{6.16}$$

which is strictly less than 1 because

$$\int_{\alpha/2}^{T/4} ud\nu > (\inf_{[\alpha/2,T/4]} u)\nu(\alpha/2, T/4] > 0.$$

To prove that there is $M > 0$ such that $Mu \succ w$, just recall that on one hand $u'(0) > 0$ and $u(x) > 0$ for $x \in (0, T/4]$ by Lemma 6.3 and on the other hand, $w'(0) < \infty$, and w is bounded so that this last claim follows easily. ∎

Proof of Theorem 2.3. The spectral radius of L_T is $r(L_T) = \hat{J}(2\pi/T)$, and we assume $\hat{J}(2\pi/T) > 1$. Thus, by Lemma 6.2, there is at least one $u \in K_T\backslash\{0\}$, such that $f(u) = u$. Assume that there is $u' \in C_T\backslash\{0, u\}$ such that $f(u') = u'$. We define $\bar{u} \in X_T$ as

$$\bar{u}(x) := \max(u(x), u'(x)), \quad \forall x \in [0, T/4]. \tag{6.17}$$

By definition, $\bar{u} \succ u$ and $\bar{u} \succ u'$, thus $f(\bar{u}) \succ \bar{u}$. Also, we can assume that $\bar{u} \neq u$. For $\lambda \in [0, 1]$ we define $u_\lambda = \lambda\bar{u} + (1-\lambda)u$, by concavity of Φ, we have

$$f(u_\lambda) \succ \lambda f(\bar{u}) + (1-\lambda)f(u) \succ \lambda\bar{u} + (1-\lambda)u = u_\lambda. \tag{6.18}$$

Now, by Theorem 6.4,

$$r(df(u)) = \lim_{n\to\infty} ||df(u)^n||^{1/n} < 1. \tag{6.19}$$

Thus, there is n_0 such that $||df(u)^{n_0}|| < 1$. Now, by definition

$$f^{n_0}(v) = u + df(u)^{n_0}(v-u) + R(v,u) \qquad \text{and} \qquad \lim_{v\to u} \frac{R(v,u)}{||v-u||} = 0. \tag{6.20}$$

Thus, there is a neighborhood U of u such that

$$\forall v \in U, \quad \lim_{k\to\infty} f^{kn_0}(v) = u. \tag{6.21}$$

However, we can always take λ small enough so that $u_\lambda \in U$ and (6.18) contradicts (6.21).

If G, H and Θ are as in Theorem 2, the additivity property of the fixed point index and the uniqueness of the nonzero fixed point u of f imply that the value of the fixed point index is independent of the particular relatively open sets G, H and Θ. Furthermore, because $F(C_T) \subset D_T$, the commutativity property of the fixed point index implies that $i_{C_T}(f, G) = i_{D_T}(f, H)$.

If we take $\rho < |u|_\infty < R$ and define

$$H = \{v \in D_T : \rho < |v|_\infty < R\} \quad \text{and} \quad \Theta = \{v \in K_T : \rho < |v|_\infty < R\},$$

Lemma 4 implies that $i_{D_T}(f, H) = i_{K_T}(f, \Theta)$, which proves the result for general G, H and Θ. ∎

Counter-example to uniqueness.

Assume A and B. If we suppose that $\hat{J}(2\pi/T) > 1$, then Theorem 2 establishes that $f(v)(x) := \Phi(J * v(x))$ has a unique fixed point $u \in D_T \backslash \{0\}$. Furthermore, if G is any bounded, relatively open subset of D_T with $u \in G$ and $0 \notin \bar{G}$, then $i_{D_T}(f, G) = 1$. Because $f : \bar{G} \to D_T$ is compact and $f(v) \neq v$ for $v \in \bar{G} - G$, there exists $\delta > 0$ such that $||v - f(v)|| \geq \delta$ for all $v \in \bar{G} - G$.

Now, recall that $a > 0$, is such that $\Phi(\beta a) = a$ and define

$$\Phi_\epsilon(x) = \Phi(\frac{2ax}{\epsilon})(\frac{\epsilon}{2a}), \text{ for } |x| \leq \beta\epsilon/2, \quad \text{and} \quad \Phi_\epsilon(x) = \Phi(x) \text{ for } |x| \geq \beta\epsilon.$$

Complete the definition of Φ_ϵ for $\beta\epsilon/2 \leq |x| \leq \beta\epsilon$ so as Φ_ϵ is odd, increasing, and C^1. Note that $\Phi_\epsilon(\beta\epsilon/2) = \epsilon/2$, and that Φ_ϵ is concave on $[0, \beta\epsilon/2]$ but not on $[0, \infty)$. Define $f_\epsilon(v(x)) = \Phi_\epsilon(J * v(x))$. Thus, the same argument used in Lemma 8 shows that f_ϵ has a fixed point v_ϵ with $0 < |v_\epsilon|_\infty < \epsilon/2$. Now, notice that

$$\forall y, \quad |\Phi_\epsilon(y) - \Phi_0(y)| \leq \epsilon, \quad \text{so} \quad \forall v \in D_T, \quad ||f_\epsilon(v) - f(v)|| \leq \epsilon.$$

If G is as above, we can arrange that $||v|| \geq \eta > 0$ for all $v \in G$. For $0 < \epsilon < \min(\delta, \eta)$, consider the homotopy $(1-t)f(v) + tf_\epsilon(v)$, $0 \leq t \leq 1$, $v \in \bar{G}$. If $v \in \bar{G} - G$, we have

$$||v - (1-t)f(v) - tf_\epsilon(v)|| \geq ||v - f(v)|| - t||f(v) - f_\epsilon(v)|| \geq \delta - t\epsilon > 0.$$

It follows from the homotopy property for the fixed point index that $i_{D_T}(f_\epsilon, G) = i_{D_T}(f, G) = 1$, (Theorem 2 implies that $i_{D_T}(f, G) = 1$). Thus, f_ϵ has a second fixed point in G ($v_\epsilon \notin G$, because $||v_\epsilon|| < \eta$).

7 A priori estimates for the general case.

We do not assume here that $f(C_T) \subset C_T$, but only A and B.

It will be convenient to modify notations. Henceforth, we normalize J to have integral one, and we set $\Phi_\beta(x) = \Phi(\beta x)$ (recall that $\beta > 1$).

7.1 Truncation and Setting.

For any $\epsilon \in (0,1)$, we can find $M(\epsilon)$ such that

$$\int_{|x|\leq M(\epsilon)} J(x)dx = 1-\epsilon. \tag{7.1}$$

We denote by $J^\epsilon(x) := J(x)I_{[-M(\epsilon),M(\epsilon)]}$ and by f_ϵ the corresponding map. We note that for $T > 4M(\epsilon)$, $(J^\epsilon)_T(x) = J^\epsilon(x)$ and $(J^\epsilon)_T(x) - (J^\epsilon)_T(x+T/2) = J^\epsilon(x)$ for $|x| \leq T/4$, so our previous lemmas imply that $f_\epsilon(C_T) \subset C_T$.

If the support of J is compact, say $supp(J) \subset [-C,C]$, and $T > 4C$, then our previous results imply that the equation $f(u) := \Phi_\beta(J * u) = u$ has a unique nonzero solution $u \in K_T$. Thus, we shall also assume that J does not have compact support. Under our hypotheses, the map $M \mapsto \psi(M) := \int I_{|y|\leq M} J(y)dy$ is a strictly increasing, continuous function from $[0,\infty)$ onto $[0,1)$, so $M(\epsilon) := \psi^{-1}(1-\epsilon)$ is a continuous function of ϵ for $0 < \epsilon \leq 1$. Now, we fix a period T and denote by ϵ_T the positive number such that $T = 4M(\epsilon_T)$.

We fix a number $\epsilon_0 > 0$ and consider two cases:

$$(i) \quad f(u) = u \quad \text{and} \quad \exists v \in K_T\backslash\{0\},\ |u-v| < \epsilon_0|v|.$$

and,

$$(ii) \quad f_{\epsilon_T}(u) = u \quad \text{and} \quad u \in K_T\backslash\{0\}.$$

We want to obtain a priori estimates of $||df(u)||$ and $||df_{\epsilon_T}(u)||$ independent of T, in appropriate Banach spaces. To avoid repetition, we will treat a case which is more general than both. We say that u satisfies (E_T) if for some $\epsilon \in [0,\epsilon_T]$,

$$f_\epsilon(u) = u \quad \text{and} \quad \exists v \in K_T\backslash\{0\},\ |u-v| < \epsilon_0|v|.$$

Then, we need estimates independent of ϵ when T is large. The larger ϵ_0, the larger the class of functions in which we can prove uniqueness. We will assume that $\epsilon_0 < 1$ to ensure that $u \in C_T$.

7.2 A priori bounds for nonzero solutions of $f(u) = u$.

An elementary but crucial observation is that if we assume that a solution of $f_\epsilon(u) = u$ is "close enough" to an element of K_T, say v, then there is α, independent of T and ϵ, such that $u(\alpha)$ is "large enough".

First, we need some simple facts about Φ_β. Define $a, \bar{a}^* \geq \underline{a}^* > 0$ with

$$\Phi_\beta(a) = a, \quad \bar{a}^* := \inf\{x \geq 0 : \ \Phi'_\beta(x) < 1\}, \text{ and,} \quad \underline{a}^* := \sup\{x \geq 0 : \ \Phi'_\beta(x) > 1\}.$$

We claim that $a > \bar{a}^*$. Indeed, suppose by contradiction that $\Phi'_\beta(a) \geq 1$. Then, concavity implies that for all $x \in [0,a]$, $\Phi'_\beta(x) \geq 1$ and as $\Phi'_\beta(0) > 1$, we find that $\Phi_\beta(x) > x$ for any $x \in]0,a]$ which contradicts $\Phi_\beta(a) = a$. We note also that

$$\Phi'_\beta(\bar{a}^*) - \Phi'_\beta(\frac{\bar{a}^* + a}{2}) > 0.$$

Finally, we define

$$a_0 := \min(a - \bar{a}^*, \underline{a}^*)/2. \tag{7.2}$$

Proposition 7.1 . *There is T_1 such that for each $T > T_1$, there exists α, $0 < \alpha < T/4$, with the following property: if u satisfies (E_T) for ϵ_0 sufficiently small, then*

$$u(\alpha) = a - a_0 > \frac{a + \bar{a}^*}{2},$$

and,

$$\alpha \le M(\xi)\frac{1+\xi}{\xi}, \quad \text{with} \quad \xi \ge \frac{1}{6}\left(\Phi'_\beta(\bar{a}^*) - \Phi'_\beta(\frac{\bar{a}^* + a}{2})\right)\frac{a_0}{\beta}. \tag{7.3}$$

To be more precise, we shall define a number $\gamma > 0$ in the proof of Lemma 10 below, and we shall need $0 < \epsilon_0 < 1/2$ such that

$$\frac{\epsilon_0}{(1-\epsilon_0)} < \frac{\gamma a_0^2}{5}, \ \frac{1+\epsilon_0}{1-\epsilon_0} < \Phi'_\beta(\frac{\underline{a}^*}{2}), \ \frac{\epsilon_0}{1-2\epsilon_0} < \frac{1}{4}\left(\Phi'_\beta(\bar{a}^*) - \Phi'_\beta(\frac{\bar{a}^* + a}{2})\right)\frac{a_0}{\beta}. \tag{7.4}$$

The proof of Proposition 7.1 will rely on the following lemma, which we prove first.

Lemma 7.2 . *There is T_1 such that if $T > T_1$, and u satisfies (E_T) for ϵ_0 small enough, then*

$$|u|_\infty \ge a - a_0. \tag{7.5}$$

Proof. Recall that Φ_β is bounded by 1 so that $|u|_\infty \le 1$. If u, v and ϵ_0 are as in condition (E_T), we deduce that, for all x,

$$|v(x)| \le |u(x)| + |v(x) - u(x)| \le 1 + \epsilon_0 |v(x)|,$$

which implies that $|v|_\infty \le 1/(1-\epsilon_0)$, an estimate we shall need below.

We now argue in two steps. In Step 1 we show that $|u|_\infty \notin [a_0, a - a_0]$, and in Step 2 we show that $|u|_\infty > \underline{a}^*/2$.

Step 1. We define $\gamma > 0$ by

$$\gamma := \min(\Phi'_\beta(a_0) - 1, 1 - \Phi'_\beta(a - a_0)).$$

The concavity of Φ_β in $(0, a)$ implies that

$$\inf_{y \in [a_0, a - a_0]} [\Phi_\beta(y) - y] = \min(\Phi_\beta(a_0) - a_0, \Phi_\beta(a - a_0) - (a - a_0)) \ge \gamma a_0.$$

Set $\eta := \gamma a_0^2/5$ and assume that $0 < \epsilon < \eta$. We claim that if $\Phi_\beta(y - 5\epsilon) \le y$, then $y \notin [a_0, a - a_0]$. Indeed, assume $y \in [a_0, a - a_0]$; the concavity of Φ_β implies that

$$\frac{\Phi_\beta(y) - \Phi_\beta(y - 5\epsilon)}{5\epsilon} \le \frac{\Phi_\beta(a_0) - \Phi_\beta(0)}{a_0}.$$

So,

$$\Phi_\beta(y-5\epsilon) - y \geq \Phi_\beta(y) - y - \frac{\Phi_\beta(a_0)}{a_0}5\epsilon \geq \gamma a_0 - \frac{5\epsilon}{a_0} > 0.$$

Now let $T_0 = 4M(\eta_1)(1+1/\eta_1)$, where $\eta_1 := (1-\epsilon_0)\eta$, and assume that $T > T_0$. The reader can verify that $\epsilon_T < \eta_1$. Suppose that u, v and ϵ are as in condition (E_T). Our initial remarks and the fact that $v \in K_T$ give $v(T/4) = |v|_\infty \leq 1/(1-\epsilon_0)$; and using the concavity of v on $[0, T/2]$, we obtain for $0 \leq y \leq M(\eta_1)$

$$v'(T/4-y) \leq \frac{v(T/4-y)-v(0)}{(T/4-y)} \leq \frac{(1-\epsilon_0)^{-1}}{T/4-M(\eta_1)} \leq \frac{(1-\epsilon_0)^{-1}\eta_1}{M(\eta_1)}.$$

The symmetry of v implies that for $|y| \leq M(\eta_1)$,

$$|v(T/4-y) - v(T/4)| \leq |y||v'(T/4-y)| \leq (1-\epsilon_0)^{-1}\eta_1. \tag{7.6}$$

It follows that

$$\begin{aligned} J^\epsilon * v(T/4) &= \int_{[|y|\leq M(\epsilon)]} J(y)v(T/4-y)dy \\ &= \int_{[|y|\leq M(\eta_1)]} J(y)v(T/4-y)dy + \int_{[M(\eta_1)\leq|y|\leq M(\epsilon)]} J(y)v(T/4-y)dy \\ &\geq v(T/4)(1-\eta_1) - \int_{[|y|\leq M(\eta_1)]} J(y)(v(T/4-y)-v(T/4))dy - \eta_1/(1-\epsilon_0) \\ &\geq v(T/4) - 3\eta_1(1-\epsilon_0)^{-1} \geq u(T/4) - 4\eta^*, \end{aligned} \tag{7.7}$$

where $4\eta^* := (3\eta_1+\epsilon_0)(1-\epsilon_0)^{-1} < 4\eta$. We have used (7.4), (7.6) and the estimate $|v|_\infty \leq (1-\epsilon_0)^{-1}$ to obtain (7.7). Using (7.7), we see that

$$\begin{aligned} u(T/4) = \Phi_\beta(J^\epsilon * u(T/4)) &\geq \Phi_\beta(J^\epsilon * v(T/4) + J^\epsilon * (u-v)(T/4)) \\ &\geq \Phi_\beta(J^\epsilon * v(T/4) - \epsilon_0(1-\epsilon_0)^{-1}). \end{aligned} \tag{7.8}$$

Combining (7.7) and (7.8), we obtain

$$u(T/4) \geq \Phi_\beta(u(T/4) - 5\eta^*).$$

Since $\eta^* < \eta$, our earlier remarks imply that $u(T/4) \notin [a_0, a-a_0]$.

Step 2. Choose $\epsilon_0 > 0$ and $\delta > 0$ so small that

$$\gamma_0 := \frac{1+\epsilon_0}{1-\epsilon_0} < \Phi'_\beta(\underline{a}^*/2), \quad \text{and} \quad B := 1 - \frac{\gamma_0}{\Phi'_\beta(\underline{a}^*/2)} - \delta(1+\gamma_0) > 0. \tag{7.9}$$

For this δ, and T_0 as in Step 1, choose $T_1 > \min(4M(\delta), T_0)$ such that

$$3M(\delta) < \left[\frac{T_1}{2} - 2M(\delta)\right] B. \tag{7.10}$$

Suppose now that $T > T_1$, u satisfies condition (E_T).

For notational convenience, define $\bar{u} = |u|_\infty$, and $\bar{v} = |v|_\infty$. Because Φ_β is concave on $[0, \infty)$, we obtain

$$\forall x \in [0, T/2], \quad \Phi_\beta(J^\epsilon * u(x)) \geq \Phi'_\beta(\bar{u}) J^\epsilon * u(x). \tag{7.11}$$

Let $A = [0, T/2]$ and $A_+ = [-M(\delta), T/2 + M(\delta)]$. By using (7.11) and condition (E_T) we obtain,

$$(1+\epsilon_0)\int_A v \geq \int_A u = \int_A \Phi_\beta(J^\epsilon * u) \geq \Phi'_\beta(\bar{u}) \int_A J^\epsilon * u.$$

Because $u \in C_T$, we obtain that for any interval I of length $T/2$

$$\int_I |u(z)|dz = \int_A u(z)dz.$$

Exploiting this fact, we obtain

$$\begin{aligned} |\int_A (J^\epsilon * u - J^\delta * u)| &= |\int_A \int_{I\!R} \left(J^\epsilon(y) - J^\delta(y)\right) u(x-y) dy dx| \\ &\leq \int_{I\!R} \left(J^\epsilon(y) - J^\delta(y)\right) \left(\int_A |u(x-y)| dx\right) dy \\ &\leq \int_A u \int_{I\!R} \left(J^\epsilon(y) - J^\delta(y)\right) = (\delta - \epsilon) \int_A u. \end{aligned}$$

Using this estimate, we find that

$$(1+\epsilon_0)\int_A v \geq \Phi'_\beta(\bar{u}) \left(\int_A J^\delta * u - \delta \int_A u\right).$$

Because $(1+\epsilon_0)v(y) \geq u(y) \geq (1-\epsilon_0)v(y)$ for $0 \leq y \leq T/2$, $u, v \in C_T$ and $T > 4M(\delta)$, Lemma 1 implies that

$$(1+\epsilon_0)\int_A v \geq \Phi'_\beta(\bar{u}) \left(\int_A J^\delta * u - \delta \int_A u\right) \geq (1-\epsilon_0)\Phi'_\beta(\bar{u}) \left(\int_A J^\delta * v - \delta\gamma_0 \int_A v\right).$$

It is easy, using that $v \in K_T$, to see that

$$\int_A J^\delta * v \geq \int_A J^\delta * [v I_A - v(M(\delta)) I_{A_+ \setminus A}].$$

We leave to the reader this simple check. Thus, we see that

$$(1+\epsilon_0)\int_A v \geq (1-\epsilon_0)\Phi'_\beta(\bar{u}) \left(\int_A J^\delta * \left(v I_A - v(M(\delta)) I_{A_+ \setminus A}\right)(x) dx - \delta\gamma_0 \int_A v\right).$$

Simple estimates give

$$\begin{aligned} \int_A J^\delta * (v(M(\delta)) I_{A_+ \setminus A}) &= v(M(\delta)) \int_A \int_{-M(\delta)}^{M(\delta)} J(y) I_{A_+ \setminus A}(x-y) dy dx \\ &= v(M(\delta)) \int_{-M(\delta)}^{M(\delta)} J(y) \left(\int_A I_{A_+ \setminus A}(x-y) dx\right) dy \\ &\leq M(\delta) v(M(\delta)) \int_{-M(\delta)}^{M(\delta)} J \leq M(\delta) v(M(\delta)). \end{aligned}$$

It follows that

$$(1+\epsilon_0)\int_A v \geq (1-\epsilon_0)\Phi'_\beta(\bar{u})\left(\int_{A_+} J^\delta * (vI_A) - \int_{A_+\backslash A} J^\delta * (vI_A) - M(\delta)v(M(\delta)) - \delta\gamma_0 \int_A v\right).$$

If $x \in A_+\backslash A$ and $|y| \leq M(\delta)$, then using that $v \in K_T$ one sees that $v(M(\delta)) \geq v(x-y)I_A(x-y) \geq 0$. Using this inequality, one finds that

$$\int_{A_+\backslash A} J^\delta * (vI_A) = \int_{-M(\delta)}^{M(\delta)} J(y)\left(\int_{A_+\backslash A} v(x-y)I_A(x-y)dx\right)dy \leq 2v(M(\delta))M(\delta).$$

It follows that

$$(1+\epsilon_0)\int_A v \geq (1-\epsilon_0)\Phi'_\beta(\bar{u})\left(\int_{A_+} J^\delta * (vI_A) - 3v(M(\delta))M(\delta) - \delta\gamma_0 \int_A v\right).$$

One can easily verify that

$$\int_{A_+} J^\delta * (vI_A) = (\int_{I\!R} J^\delta)(\int_A v) = (1-\delta)\int_A v,$$

so one obtains that

$$3v(M(\delta))M(\delta) \geq (\int_A v)[1-\delta-\delta\gamma_0 - \frac{\gamma_0}{\Phi'_\beta(\bar{u})}]. \tag{7.12}$$

Because $v(x) \geq v(M(\delta))$ in $[M(\delta), T/2 - M(\delta)]$, we see that

$$\int_A v \geq v(M(\delta))[T/2 - M(\delta)]. \tag{7.13}$$

Assume, by way of contradiction, that $\bar{u} \leq (\underline{a}^*/2)$, so $\Phi'_\beta(\bar{u}) \geq \Phi'_\beta(\underline{a}^*/2) > 1$ and (using eq. (7.9))

$$1 - \frac{\gamma_0}{\Phi'_\beta(\bar{u})} - \delta - \delta\gamma_0 \geq B > 0.$$

We can use eq. (7.12) and (7.13) and divide by $v(M(\delta))$ to obtain

$$3M(\delta) \geq [T/2 - 2M(\delta)]B, \tag{7.14}$$

which contradicts inequality (7.10) and completes Step2. The inequality (7.5) is obtained by combining Step1 and Step2. ∎

Proof of Proposition 7.1. By Lemma 7.2, there is T_1 such that for $T > T_1$, $a - |u|_\infty \leq a_0$. Thus, by continuity of u, there is $\alpha < T/4$ such that $a - u(\alpha) = a_0$. Now,

$$\begin{aligned}\Phi_\beta(u(\alpha)) - u(\alpha) &= a - \Phi_\beta(a) - (u(\alpha) - \Phi_\beta(u(\alpha))) \geq (1 - \Phi'_\beta(u(\alpha)))(a - u(\alpha)) \\ &\geq \left(\Phi'_\beta(\bar{a}^*) - \Phi'_\beta(u(\alpha))\right)a_0 \geq \left(\Phi'_\beta(\bar{a}^*) - \Phi'_\beta(\frac{\bar{a}^*+a}{2})\right)a_0.\end{aligned} \tag{7.15}$$

Now,

$$u(\alpha) = \Phi_\beta\left(J^\epsilon * u(\alpha)\right) \geq \Phi_\beta(J^\epsilon * v(\alpha) + J^\epsilon * (u-v)(\alpha))$$

Recall that $|v|_\infty \leq |u|_\infty/(1-\epsilon_0) \leq 1/(1-\epsilon_0)$, so we obtain

$$|J^\epsilon * (u-v)(\alpha)| \leq \epsilon_0 J^\epsilon * |v|(\alpha) \leq \epsilon_0 |v|_\infty \leq \epsilon_0/(1-\epsilon_0).$$

Combining these inequalities yields

$$u(\alpha) \geq \Phi_\beta(J^\epsilon * v(\alpha) - \epsilon_0/(1-\epsilon_0)).$$

We can always choose $\alpha = M(\xi)(1+1/\xi)$. Because v is concave on $[\alpha - M(\xi), \alpha + M(\xi)]$, we have for $y \in [\alpha - M(\xi), \alpha + M(\xi)]$,

$$v'(y) \leq \frac{v(\alpha - M(\xi)) - v(0)}{\alpha - M(\xi)} \leq \frac{1}{(1-\epsilon_0)(\alpha - M(\xi))}.$$

We now argue as in the inequalities (7.7), in the proof of Lemma 10:

$$J^\epsilon * v(\alpha) \geq v(\alpha) - 3\xi/(1-\epsilon_0).$$

Recalling that Φ_β is increasing, the previous inequality gives

$$\begin{aligned} u(\alpha) &\geq \Phi_\beta(v(\alpha) - (3\xi + \epsilon_0)/(1-\epsilon_0)) \\ &\geq \Phi_\beta(u(\alpha) - (3\xi + 2\epsilon_0)/(1-\epsilon_0)) \geq \Phi_\beta(u(\alpha)) - \beta(2\epsilon_0 + 3\xi)/(1-\epsilon_0). \end{aligned} \tag{7.16}$$

We have used that $|\Phi'_\beta(y)| \leq \beta$ for all y. Combining (7.15) and (7.16) gives

$$\beta(2\epsilon_0 + 3\xi)/(1-\epsilon_0) \geq \left(\Phi'_\beta(\bar{a}^*) - \Phi'_\beta(\frac{\bar{a}^* + a}{2})\right) a_0. \tag{7.17}$$

Inequalities (7.17) and (7.4) now yield (7.3). ∎

7.3 Norm of $df(u)$.

Let $(Y, ||.||)$ be a real Banach space and suppose that K is a closed cone in Y. K induces a partial ordering on Y: $x \prec y$ if and only if $y - x \in K$. Recall that K is called "normal" if there exists a constant C such that $||x|| \leq C||y||$ for all $x, y \in Y$ with $0 \prec x \prec y$. If $u \in K\backslash\{0\}$, we define a set Y_u by

$$Y_u = \{v \in Y: \ \ \exists M > 0, \text{ with } -Mu \prec v \prec Mu\}, \tag{7.18}$$

and for $v \in Y_u$, we define $||v||_u = \inf\{M: \ -Mu \prec v \prec Mu\}$.

The following lemma is well known (see [10]). For the reader's convenience, a proof is given in the appendix.

Lemma 7.3 . *Let $(Y, ||.||)$ be a real Banach space and K a closed cone in Y. If $u \in K\backslash\{0\}$, then $(Y_u, ||\cdot||_u)$ is a normed linear space; if K is normal, then $(Y_u, ||\cdot||_u)$ is a Banach space. If $L: Y \to Y$ is a linear map such that $L(K) \subset K$, and if there exists $N > 0$ such that $-Nu \prec L(u) \prec Nu$, then $L(Y_u) \subset Y_u$ and L induces a bounded linear map $\mathcal{L}: Y_u \to Y_u$ with $||\mathcal{L}||_u \leq N$.*

Suppose now that u satisfies (E_T) and that ϵ, ϵ_T, $0 \le \epsilon \le \epsilon_T$, are as in (E_T), so that $f_\epsilon(u) = u$. Let $L = df_\epsilon(u) : X_T \to X_T$ denote the Fréchet derivative of f_ϵ at u. We shall use Lemma 7.3 with $Y := X_T$ and $K := D_T = \{w \in X_T : \; w(x) \ge 0, \; \forall x \in [0, T/2]\}$. For T large, we shall show that $L(Y_u) \subset Y_u$ and L induces a bounded linear map $\mathcal{L} : Y_u \to Y_u$. Also, for $w, \tilde{w} \in X_T$ $w \prec \tilde{w}$ is equivalent to $\tilde{w}(x) - w(x) \ge 0$ for $0 \le x \le T/4$. Notice that if $\tilde{w} \in D_T \backslash \{0\}$ and $w \in X_T$, then $-M\tilde{w} \prec w \prec M\tilde{w}$ is equivalent to $|w(x)| \le M|\tilde{w}(x)|$ for all x.

We now state the main result of this section.

Theorem 7.4 . *Assume that u satisfies (E_T) and that ϵ, $0 \le \epsilon \le \epsilon_T$, is as in (E_T). Let $df_\eta(u) : X_T \to X_T$ denote the Fréchet derivative of f_η at u. There is a number T_2, such that if $T > T_2$ and $0 \le \eta \le \epsilon_T$, then $df_\eta(u)$ induces a bounded linear map $\mathcal{L} : Y_u \to Y_u$. Furthermore, there exists $\kappa = \kappa(T_2) < 1$ such that $||\mathcal{L}^2||_u \le \kappa < 1$.*

The proof of Theorem 7.4 will require several lemmas.

Lemma 7.5 . *Let T_1 be as in Proposition 7.1, and assume that u satisfies (E_T) for some $T > T_1$. Then, there is a function $D(\rho)$ for $0 \le \rho \le 1$ such that $\lim_{\eta \to 0+} D(\eta) = 0$ and for any $w \in Y_u$ and η, $0 \le \eta \le 1$*

$$||J * w - J^\eta * w||_u \le D(\eta) ||w||_u.$$

Furthermore, for any η, $0 \le \eta \le 1$, the bounded linear operator $w \mapsto J^\eta * w$ on X_T induces a bounded linear operator on Y_u.

Proof. We rewrite, for $x \ge 0$ and any $w \in Y_u$

$$\begin{aligned} J * w(x) \; - J^\eta * w(x) &= \int_{I\!R} J(y) w(x-y) dy - \int_{|y| \le M(\eta)} J(y) w(x-y) dy \\ &= \int_{M(\eta)}^{\infty} J(y)(w(y-x) - w(y+x)) dy \\ &= \int_{M(\eta)+x}^{\infty} w(y) \left(J(y+x) - J(y-x) \right) dy + \int_{M(\eta)-x}^{M(\eta)+x} w(y) J(y+x) dy \end{aligned} \tag{7.19}$$

Exploiting the fact the J is decreasing and nonnegative on $[0, \infty)$, we obtain from eq.(7.19) that for $x \ge 0$

$$\begin{aligned} |J * w(x) \; - J^\eta * w(x)| &\le \int_{M(\eta)+x}^{\infty} |w(y)| \left(J(y-x) - J(y+x) \right) dy + \int_{M(\eta)-x}^{M(\eta)+x} |w(y)| J(y+x) dy \\ &\le \left(\sup_z |w(z)| \right) \left(\int_{M(\eta)+x}^{\infty} \left(J(y-x) - J(y+x) \right) dy + \int_{M(\eta)-x}^{M(\eta)+x} J(y+x) dy \right) \\ &\le \left(\sup_z |w(z)| \right) \min \left(4J(M(\eta))x, 2\eta \right). \end{aligned} \tag{7.20}$$

Let $v \in K_T$ and ϵ, $0 \le \epsilon \le \epsilon_T$, be as in (E_T). Let α be as in Proposition 7.1, so $u(\alpha) = a - a_0 > \bar{a}^*$. The concavity of v implies that for $0 \le x \le \alpha$, $\alpha v(x)/v(\alpha) \ge x$. Condition

(E_T) implies that $(1-\epsilon_0)v \prec u \prec (1+\epsilon_0)v$, so setting $\gamma_0 := (1+\epsilon_0)/(1-\epsilon_0)$, one derives that

$$\frac{\alpha\gamma_0}{\bar{a}^*}u(x) \geq \frac{\alpha\gamma_0 u(x)}{u(\alpha)} \geq \frac{\alpha v(x)}{v(\alpha)} \geq x, \quad 0 \leq x \leq \alpha.$$

Using this estimate, we derive from eq.(7.20) that

$$\forall x \in [0,\alpha],\ |J * w(x) - J^\eta * w(x)| \leq 4J(M(\eta))||w||_u \frac{\alpha\gamma_0}{\bar{a}^*}u(x), \tag{7.21}$$

If $\alpha \leq x \leq T/4$, $v(x)/v(\alpha) \geq 1$ and one concludes that

$$\frac{\gamma_0}{\bar{a}^*}u(x) \geq \frac{\gamma_0 u(x)}{u(\alpha)} \geq \frac{v(x)}{v(\alpha)} \geq 1.$$

It follows that for $\alpha \leq x \leq T/4$,

$$|J * w(x) - J^\eta * w(x)| \leq 2\eta \sup |w(x)| \leq \frac{2\eta\gamma_0}{\bar{a}^*}||w||_u u(x). \tag{7.22}$$

Inequality (7.3) implies that there is a constant $\tilde{\alpha}$, independent of $T > T_1$, ϵ, v, u and η such that $\alpha \leq \tilde{\alpha}$. Combining (7.20) and (7.22), we see that

$$||J * w - J^\eta * w||_u \leq \frac{\gamma_0}{\bar{a}^*}\max(4J(M(\eta))\tilde{\alpha}, 2\eta)||w||_u := D(\eta)||w||_u.$$

If $\eta = 1$, $J^\eta * w = 0$ and we see that $||J * w||_u \leq D(1)||w||_u$, for $w \in Y_u$, so $w \mapsto J * w$ induces a bounded linear map on Y_u. Our estimates also show that $w \mapsto J * w(x) - J^\eta * w$ gives a bounded linear map on Y_u, so does $w \mapsto J^\eta * w$ for $0 \leq \eta \leq 1$. ∎

Lemma 7.6 . *Assume A and B. Let T_1 be as in Proposition 7.1, assume that u satisfies (E_T) for $T > T_1$, and let ϵ, $0 \leq \epsilon \leq \epsilon_T$, and $v \in K_T$ be as in (E_T). Define*

$$g(x) = J^\epsilon * u(x), \qquad \lambda(x) = \frac{\Phi'_\beta(g(x))g(x)}{\Phi_\beta(g(x))}, \quad \text{and} \quad L_1 w = \frac{\Phi_\beta(g)}{g}J^{\epsilon_T} * w. \tag{7.23}$$

(Also, we define $\Phi_\beta(g(x))/g(x) = \Phi'_\beta(0) = \beta$ and $\lambda(x) = 1$ for $g(x) = 0$). Then, there is $\kappa_0 < 1$ independent of u, ϵ and T, such that for $T > T_1$

$$c := \sup_{x\in[0,T/4]} \lambda(x)\frac{L_1(\lambda u)(x)}{L_1(u)(x)} \leq \kappa_0 < 1. \tag{7.24}$$

Proof. For any $\delta > 0$, set $\bar{\lambda}(\delta) = \sup\{\lambda(x) : x \in [\delta, T/4]\}$. Because $\Phi_\beta(g(x)) = u(x)$ and $0 \prec (1-\epsilon_0)v \prec u$, we see that $0 \prec g$. Because of (Bc) and (Bd), we have that $\Phi'_\beta(y)y/\Phi_\beta(y) < 1$ for all $y > 0$. Combining these facts, we conclude that $\bar{\lambda}(\delta) < 1$ for $0 < \delta \leq T/4$. Because $\lim_{x\to 0+}\lambda(x) = 1$, we also see that $\lim_{\delta\to 0+}\bar{\lambda}(\delta) = 1$. Define for $x \neq 0$, $x, y \in I\!R$

$$K(x,y) := \frac{1}{x}(\tilde{J}_T^{\epsilon_T}(y-x) - \tilde{J}_T^{\epsilon_T}(y+x)). \tag{7.25}$$

If $x, y \in [0, T/4]$, then $K(x,y) \geq 0$. In fact, recalling that $J^{\epsilon_T}(z) = 0$ for $|z| \geq T/4$, $\tilde{J}_T^{\epsilon_T}(z) = J^{\epsilon_T}(z)$ for $0 \leq z \leq T/4$, $\tilde{J}_T^{\epsilon_T}(z) = -J^{\epsilon_T}(T/2 - z)$ for $T/4 \leq z \leq T/2$, and for $x, y \in [0, T/4]$, we have that

$$xK(x,y) = J_T^{\epsilon_T}(y-x) - J_T^{\epsilon_T}(y+x) \text{ for } 0 < y+x \leq T/4,$$

$$xK(x,y) = J_T^{\epsilon_T}(y-x) + J_T^{\epsilon_T}(T/2 - y - x) \text{ for } y + x > T/4.$$

Now, notice that

$$c = \sup_{0<x\leq T/4} \lambda(x) \frac{(J^{\epsilon_T} * \lambda u)(x)}{(J^{\epsilon_T} * u)(x)} = \sup_{0<x\leq T/4} \lambda(x) \frac{\int_0^{T/4} xK(x,y)\lambda(y)u(y)dy}{\int_0^{T/4} xK(x,y)u(y)dy}.$$

We know from Lemma 1 or Lemma 7 that $v \mapsto J^{\epsilon_T} * v$ preserves the partial ordering, so $J^{\epsilon_T} * (\lambda u) \prec J^{\epsilon_T} * u$ and $c \leq 1$. Also,

$$\begin{aligned} c &\leq \max\left(\bar{\lambda}(\delta), \sup_{0<x\leq\delta}\left[\left(\int_0^\delta uK(x,\cdot) + \bar{\lambda}(\delta)\int_\delta^{T/4} uK(x,\cdot)\right)\left(\frac{J^{\epsilon_T} * u(x)}{x}\right)^{-1}\right]\right) \\ &\leq \bar{\lambda}(\delta) + (1 - \bar{\lambda}(\delta)) \sup_{0\leq x\leq\delta}\left(\int_0^\delta uK(x,\cdot)\left(\frac{J^{\epsilon_T} * u(x)}{x}\right)^{-1}\right), \end{aligned} \tag{7.26}$$

We will estimate separately each term of (7.26).

Step 1: We will show that there is $\delta_1 > 0$, $C_1 > 0$ and $C_2 > 0$ independent of u, ϵ and $T > T_1$ such that for $0 < \delta < \delta_1$,

$$1 - C_2\delta^2 \leq \bar{\lambda}(\delta) < 1 - C_1\delta^2. \tag{7.27}$$

Since $g(x) = \Phi_\beta^{-1}(u(x))$, we introduce the function ψ

$$\psi(x) := \frac{\Phi'_\beta(\Phi_\beta^{-1}(x))\Phi_\beta^{-1}(x)}{x}, \quad \text{such that} \quad \lambda(x) = \psi(u(x)).$$

We claim that there exists $\delta_2 > 0$ such that ψ is strictly decreasing on $[0, \delta_2]$. Since $y \mapsto \Phi_\beta^{-1}(y)$ is strictly increasing, it suffices to prove that there exists $\delta_3 > 0$ such that $y \mapsto \Phi'_\beta(y)y/\Phi_\beta(y) := \theta(y)$ is strictly decreasing on $[0, \delta_3]$. A calculation gives, for $x > 0$,

$$\theta'(x) = \frac{x\Phi''_\beta(x)\Phi_\beta(x) + \Phi'_\beta(x)(\Phi_\beta(x) - x\Phi'_\beta(x))}{\Phi_\beta^2(x)}.$$

If we call $\theta_1(x)$ the numerator, then we need to show that $\theta_1(x) < 0$ on $(0, \delta_3]$, $\delta_3 > 0$. Since $\theta_1(0) = 0$, it suffices to find $\delta_3 > 0$ such that $\theta'_1(x) < 0$ for $0 < x < \delta_3$. A calculation gives

$$\theta'_1(x) = \Phi_\beta(x)\left(\Phi''_\beta(x) + x\Phi'''_\beta(x)\right) + \Phi''_\beta(x)\left(\Phi_\beta(x) - x\Phi'_\beta(x)\right).$$

The concavity of Φ_β on $[0, \infty)$ implies that for $x \geq 0$

$$\Phi''_\beta(x)\left(\Phi_\beta(x) - x\Phi'_\beta(x)\right) \leq 0 \quad \text{and,} \quad \theta'_1(x) \leq \Phi_\beta(x)\left(\Phi''_\beta(x) + x\Phi'''_\beta(x)\right). \tag{7.28}$$

It follows that if $\Phi'''_\beta(z) < 0$ on $[0, \delta_3]$, then $\theta'(x) < 0$ for $0 < x < \delta_3$. Our hypotheses imply that such a δ_3 exists. We recall now that u' satisfies $u' = \Phi'_\beta(u)(J^\epsilon * u)'$ and therefore by the inequality (5.3) of Theorem 5.1 in section 5, we have

$$v'(0) \le \frac{u'(0)}{1-\epsilon_0} \le \frac{M|u|_\infty}{1-\epsilon_0} \le \frac{M}{1-\epsilon_0}, \quad M := 3J(0). \tag{7.29}$$

The concavity of v in $[0, T/4]$, implies that

$$v(\delta) \le \frac{M}{1-\epsilon_0}\delta. \tag{7.30}$$

On the other hand, for α as in Proposition 7.1, the concavity of v implies that for $0 < \delta \le \alpha$, imply that for $\delta < \alpha$,

$$\frac{v(\delta)}{\delta} \ge \frac{v(\alpha)}{\alpha} \ge \frac{u(\alpha)}{1+\epsilon_0}\frac{1}{\alpha} \ge \frac{\bar{a}^*}{1+\epsilon_0}\frac{1}{\alpha}. \tag{7.31}$$

It follows from (7.30) and (7.31) that, writing $\gamma_0 = (1+\epsilon_0)/(1-\epsilon_0)$,

$$\frac{\bar{a}^*}{\gamma_0}\frac{1}{\alpha}\delta \le u(\delta) \le M\gamma_0\delta, \quad \text{for } 0 \le \delta \le \alpha. \tag{7.32}$$

If we define $\delta_4 = \min(\alpha, \delta_2/(M\gamma_0))$, we derive from (7.32) that $0 \le u(x) \le \delta_2$ for $0 \le x \le \delta_4$. It follows that for $0 < \delta < \delta_4$,

$$\psi(M\gamma_0\delta) \le \sup_{\delta \le x \le \delta_4} \psi(u(x)) \le \psi(\frac{\bar{a}^*}{\gamma_0}\frac{1}{\alpha}\delta) \tag{7.33}$$

We easily derive from (7.31) and (7.32) that $u(x) \ge (\bar{a}^*/\gamma_0)(\delta_4/\alpha)$ for $\delta_4 \le x \le T/4$, so

$$\sup_{\delta_4 \le x \le T/4} \psi(u(x)) \le \sup\{\psi(w) : \frac{\bar{a}^*}{\gamma_0}\frac{\delta_4}{\alpha} \le w \le 1\} := \rho < 1.$$

There exists $\delta_5 > 0$, independent of u, ϵ and T, such that for $0 \le \delta \le \delta_5$, $\psi(M\gamma_0\delta) > \rho$, and so that

$$\bar{\lambda}(\delta) = \max\{\sup_{\delta \le x \le \delta_4} \psi(u(x)), \sup_{\delta_4 \le x \le T/4} \psi(u(x))\} = \sup_{\delta \le x \le \delta_4} \psi(u(x)).$$

Thus we derive from (7.33) that for $0 \le \delta \le \delta_5$,

$$\psi(M\gamma_0\delta) \le \bar{\lambda}(\delta) \le \psi(\frac{\bar{a}^*}{\gamma_0}\frac{\delta}{\alpha}) \tag{7.34}$$

To estimate (7.34), we use Taylor's theorem to estimate $\psi(x)$ near 0, and we obtain

$$\psi(x) = 1 + \frac{\Phi'''(0)}{3}x^2 + o(x^2), \tag{7.35}$$

and from (7.35) and (7.34), we find that there is δ_1 independent of ϵ, u, v and $T > T_1$ (but which depends on ϵ_0 and J and Φ) such that for $0 < \delta < \delta_1$

$$1 + \frac{2\Phi'''(0)}{3}(\gamma_0 M)^2\delta^2 \le \bar{\lambda}(\delta) \le 1 + \frac{\Phi'''(0)}{6}(\frac{\bar{a}^*}{\gamma_0\alpha})^2\delta^2. \tag{7.36}$$

If we recall the estimates on α in Proposition 7.1, we obtain (7.27).

Step 2: We show that there is $A > 0$ independent of T such that for $\delta < \alpha$

$$\inf_{x\in[0,\delta)} \frac{J^{\epsilon_T} * u(x)}{x} \geq A. \tag{7.37}$$

First, if we use Lemma 7.5, then for $x \in [0, \delta)$

$$\begin{aligned} \frac{J^{\epsilon_T} * u(x)}{x} &= \frac{J^{\epsilon} * u(x) + (J^{\epsilon_T} - J^{\epsilon}) * u(x)}{x} \geq \frac{\Phi_\beta^{-1}(u(x)) - 2D(\epsilon_T)u(x)}{x} \\ &\geq \frac{u(x)}{\beta x}(1 - 2\beta D(\epsilon_T)). \end{aligned} \tag{7.38}$$

Now, equation (7.31) implies that for $x \leq \alpha$,

$$\frac{u(x)}{\beta x} \geq \frac{(1-\epsilon_0)v(x)}{\beta x} \geq \frac{\bar{a}^*}{\beta\alpha}\gamma_0(1 - 2\beta D(\epsilon_T)).$$

and Step 2 follows.

Step 3: Here, we estimate $\int_0^\delta K(x,y)u(y)dy$.

$$\int_0^\delta K(x,y)u(y)dy = \frac{1}{x}\int_0^\delta \left(\tilde{J}_T^{\epsilon_T}(y-x) - \tilde{J}_T^{\epsilon_T}(y+x)\right)u(y)dy.$$

Because $u(x) \leq \sup\{u(z)/z : 0 \leq z \leq \delta\}$ for $0 \leq x \leq \delta$, we find that

$$\int_0^\delta K(x,y)u(y)dy \leq \left(\sup_{z\leq\delta}(\frac{u(z)}{z})\cdot\delta\right)\left(\frac{1}{x}\int_0^\delta \left(\tilde{J}_T^{\epsilon_T}(y-x) - \tilde{J}_T^{\epsilon_T}(y+x)\right)dy\right). \tag{7.39}$$

We have already noted (Step 1) that $K(x,y) \geq 0$ for $x, y \in [0, T/4]$, so

$$0 \leq \int_0^\delta K(x,y)dy \leq \int_0^{T/4} K(x,y)dy.$$

Using properties of $J_T^{\epsilon_T}$ as noted in Step 1,

$$\begin{aligned} \int_0^{T/4} K(x,y)dy &= \frac{1}{x}\int_{-x}^{x} \tilde{J}_T^{\epsilon_T}(z)dz - \frac{1}{x}\int_{T/4-x}^{T/4+x} \tilde{J}_T^{\epsilon_T}(z)dz \\ &= \frac{1}{x}\int_{-x}^{x} J_T^{\epsilon_T}(z)dz - \frac{1}{x}\int_{T/4-x}^{T/4} J_T^{\epsilon_T}(z)dz - \frac{1}{x}\int_{T/4}^{T/4+x} J_T^{\epsilon_T}(z)dz \\ &\leq 3J(0). \end{aligned}$$

Recalling that $\sup_{z\leq\delta}(u(z)/z) \leq M := 3J(0)$, and combining these estimates gives

$$0 \leq \int_0^\delta K(x,y)u(y)dy \leq 3J(0)M\delta.$$

Step 4: If we choose $\delta \leq \min(\delta_1, \alpha)$ in inequality (7.26) and use Steps 1,2, and 3, it follows that

$$\kappa_0 \leq 1 - C_1\delta^2 + \frac{C_2}{A}(3J(0)M)\delta^3 \tag{7.40}$$

and the result follows by taking δ small enough. ∎

Proof of Theorem 7.4. Let L_1, λ and g be as defined in Lemma 7.6, and $\Lambda v = \Phi'_\beta(g)J^{\epsilon_T} * v$. We note that both L_1 and Λ preserve the order $\prec$.

First, we claim that

$$L_1 u \prec (1 + 2D(\epsilon_T)\beta)u. \tag{7.41}$$

Indeed, by using Lemma 7.5, we have

$$L_1 u = \frac{\Phi_\beta(g)}{g}\left(J^\epsilon * u + (J^{\epsilon_T} - J^\epsilon) * u\right) \prec (1 + 2\beta D(\epsilon_T))u. \tag{7.42}$$

Since $0 \prec \Lambda u \prec L_1 u \prec (1 + 2D(\epsilon_T)\beta)u$, we have

$$||\Lambda||_u \leq (1 + 2\beta D(\epsilon_T)). \tag{7.43}$$

By (7.42) and Lemma 7.6, we have

$$\begin{aligned} \Lambda^2(u) = \lambda L_1(\lambda L_1 u) \prec\ & (1 + 2\beta D(\epsilon_T))\lambda L_1(\lambda u) \\ \prec\ & (1 + 2\beta D(\epsilon_T))\kappa(L_1 u) \prec (1 + 2\beta D(\epsilon_T))^2 \kappa_0 u. \end{aligned}$$

Because Λ preserves the partial ordering $\prec$, we have

$$||\Lambda^2||_u \leq (1 + 2\beta D(\epsilon_T))^2 \kappa_0, \tag{7.44}$$

where κ_0 is given in (7.24) and $D(\epsilon)$ tends to 0 as ϵ goes to 0.

A consequence of Lemma 7.5 and $0 \leq \Phi'_\beta(g(x)) \leq \beta$ for all x is that

$$||\mathcal{L} - \Lambda||_u \leq 2\beta D(\epsilon_T). \tag{7.45}$$

It follows, from (7.43) and (7.45) that

$$||\mathcal{L}||_u \leq ||\Lambda||_u + ||\mathcal{L} - \Lambda||_u \leq (1 + 4\beta D(\epsilon_T)).$$

We conclude that

$$\begin{aligned} ||\mathcal{L}^2||_u \leq\ & ||\Lambda^2||_u + ||\Lambda(\mathcal{L} - \Lambda)||_u + ||(\mathcal{L} - \Lambda)\mathcal{L}||_u \\ \leq\ & ||\Lambda^2||_u + ||\Lambda||_u ||\mathcal{L} - \Lambda||_u + ||(\mathcal{L} - \Lambda)||_u ||\mathcal{L}||_u \\ \leq\ & (1 + 2\beta D(\epsilon_T))^2\kappa_0 + 2\beta D(\epsilon_T)[(1 + 2\beta D(\epsilon_T)) + (1 + 4\beta D(\epsilon_T))] := \kappa. \end{aligned}$$

The result follows for $T > T_2$, for T_2 large enough. ∎

8 Proof of Theorem 2.2.

8.1 Proof of existence.

We note that, with the notations of section 7, $f_{\epsilon_T}(C_T) \subset C_T$. By Theorem 2.3, there is T_3, such that for $T > T_3$, f_{ϵ_T} has a unique fixed point in C_T, say u_{ϵ_T}. Moreover, $u_{\epsilon_T} \in K_T$, and by Theorem 6.4, the spectral radius of $df_{\epsilon_T}(u_{\epsilon_T})$ in $\mathcal{H}_T$ is strictly less than 1. On the other hand, once we know that $u_{\epsilon_T} \in K_T$, Theorem 7.4 of section 7, establishes that there is $\kappa < 1$ independent of $T > T_2$ such that

$$||df^2_{\epsilon_T}(u_{\epsilon_T})||_{(u_{\epsilon_T})} \leq \kappa.$$

We will consider $T \geq \max(T_2, T_3)$ and for simplicity we henceforth drop the index T in f_{ϵ_T} and u_{ϵ_T}, and we denote the operator norm in $(Y_{u_\epsilon}, ||.||_{u_\epsilon})$ by $||.||$.

For $v \in Y_{u_\epsilon}$, define

$$F(J,v) = v - \Phi_\beta(J * v). \tag{8.1}$$

The Fréchet derivative of $F(J^\epsilon, v)$ with respect to v around u_ϵ is

$$dF_{(J^\epsilon,u_\epsilon)}(\xi) = \xi - \Phi'_\beta(J^\epsilon * u_\epsilon)J^\epsilon * \xi = \xi - df_\epsilon(u_\epsilon)\xi. \tag{8.2}$$

Lemma 7.6 establishes that $dF_{(J^\epsilon,u_\epsilon)}$ is invertible in Y_{u_ϵ}. In fact, let Λ denote $df_\epsilon(u_\epsilon)$, viewed as a map from Y_{u_ϵ} to Y_{u_ϵ}, so $dF_{(J^\epsilon,u_\epsilon)} = I - \Lambda$, and

$$A := \left(dF_{(J^\epsilon,u_\epsilon)}\right)^{-1} = (I-\Lambda)^{-1} = \sum_{j=0}^{\infty} \Lambda^j = (1+\Lambda)\sum_{j=0}^{\infty}\Lambda^{2j},$$

so,

$$||A(\xi)||_{(u_\epsilon)} \leq ||I+\Lambda|| \sum_{j=0}^{\infty} ||\Lambda^{2j}\xi||_{(u_\epsilon)} \leq 2\sum_{j=0}^{\infty}\kappa^j ||\xi||_{(u_\epsilon)} \leq \frac{2}{1-\kappa}||\xi||_{(u_\epsilon)}. \tag{8.3}$$

Before stating the main result, we make a simple remark.

Remark 8.1 . Assume A,B and $u = \Phi_\beta(J * u)$ with $u \in C_T\backslash\{0\}$. Then, $J * u \prec u$. First, we show that $|u|_\infty \leq a$. Because of properties B, $x > a$ implies that $\Phi_\beta(x) < x$. Now, by contradiction, assume there is $0 \leq x_0 \leq T/4$ such that $u(x_0) = |u|_\infty > a$. Then, $J * u(x_0) \leq u(x_0)$, so that $\Phi_\beta(J * u(x_0)) \leq \Phi_\beta(u(x_0)) < u(x_0)$, which is absurd. Thus, $|u|_\infty \leq a$. This in turn implies that $0 \leq J * u(x) \leq a$ for $0 \leq x \leq T/4$ so that $J * u(x) \leq \Phi_\beta(J * u(x)) = u(x)$.

Proposition 8.2 . *Assume A and B. There is* $\epsilon_1 > 0$ *such that for* $T \geq \max(T_2, T_3)$ *and* $\epsilon_T < \epsilon_1$, f *has a non trivial fixed point in* Y_{u_ϵ}, *say* u. *Moreover,* $u \in C_T$.

Proof. We recall that $|u_\epsilon|_\infty \leq 1$, and so is $|J^\epsilon * u_\epsilon|_\infty \leq 1$. Let α be as in Proposition 7.1, and let $\tilde{\alpha}$ be a uniform upper bound for α as given by equation 7.3. Let ϵ_0 be as in Proposition 7.1. By remark 8.1 $0 \prec J^\epsilon * u_\epsilon \prec u_\epsilon$. We define

$$\gamma = \sup_{|x|\leq 2} |\Phi''(x)|, \quad \text{and} \quad D(\epsilon) = \frac{(1+\epsilon_0)}{(1-\epsilon_0)a^*}\max(\epsilon, 4\tilde{\alpha}J(M(\epsilon))).$$

So $D(\epsilon)$ can serve as the function in the statement of Lemma 7.5. We can always choose $\epsilon_1 > 0$, $\epsilon_1 \leq \epsilon_0$, and $R < 1$ such that

$$\frac{2}{1-\kappa}\left(\frac{\gamma}{2}R^2 + 2\beta D(\epsilon_1)\right) \leq R, \tag{8.4}$$

and,

$$\frac{2}{1-\kappa}\left(\gamma R+(2\gamma+\beta)D(\epsilon_1)\right)<1. \tag{8.5}$$

We define $V: Y_{u_\epsilon} \to Y_{u_\epsilon}$, by

$$V(u)=u-A(F(J,u)). \tag{8.6}$$

We will show that there is a closed ball around u_ϵ, say B_R such that $V(B_R) \subset B_R$, and then that V is a contraction in B_R in Step 1 and Step 2, respectively. It is then routine to complete the proof. The fact that $u \in C_T$ follows simply because $R<1$:

$$||u-u_\epsilon||_{u_\epsilon} \le R \Longrightarrow (1-R)u_\epsilon \prec u.$$

Step 1. If $B_R=\{u \in Y_{u_\epsilon}\colon ||u-u_\epsilon||_{(u_\epsilon)} \le R\}$, then $V(B_R)\subset B_R$. First, by using that $u-u_\epsilon = AdF_{(J^\epsilon,u_\epsilon)}(u-u_\epsilon)$, we rewrite (8.6) as

$$V(u)-u_\epsilon = A[F(J^\epsilon,u)-F(J,u)]-A[F(J^\epsilon,u)-F(J^\epsilon,u_\epsilon)-dF_{(J^\epsilon,u_\epsilon)}(u-u_\epsilon)]. \tag{8.7}$$

A pointwise estimate of the last term and an application of Taylor's theorem yields

$$\begin{aligned}|F(J^\epsilon,u)-F(J^\epsilon,u_\epsilon)-dF_{(J^\epsilon,u_\epsilon)}(u-u_\epsilon)| &= \\ |\Phi_\beta(J^\epsilon*u)-\Phi_\beta(J^\epsilon*u_\epsilon)-\Phi'_\beta(J^\epsilon*u_\epsilon)J^\epsilon*(u-u_\epsilon)| &\le \frac{\gamma}{2}|(J^\epsilon*(u-u_\epsilon))^2|.\end{aligned} \tag{8.8}$$

To bound this term in norm, we establish that $||(J^\epsilon*v)^2||_{(u_\epsilon)} \le ||v||^2_{(u_\epsilon)}$. Assume that there is M such that $|v| \prec Mu_\epsilon$, then $|J^\epsilon*v| \prec MJ^\epsilon*u_\epsilon \prec Mu_\epsilon$. Thus, recalling that $|u_\epsilon(x)|\le 1$, we obtain

$$(J^\epsilon*v(x))^2 \le M^2u_\epsilon(x)^2 \le M^2u_\epsilon(x), \qquad \text{for} \quad x\in[0,T/4]. \tag{8.9}$$

Thus,

$$||A[F(J^\epsilon,u)-F(J^\epsilon,u_\epsilon)-dF_{(J^\epsilon,u_\epsilon)}(u-u_\epsilon)]||_{(u_\epsilon)} \le \frac{\gamma}{2}||A||\,||u-u_\epsilon||^2_{(u_\epsilon)}. \tag{8.10}$$

We now consider the first term of (8.7).

$$||A[F(J^\epsilon,u)-F(J,u)]||_{(u_\epsilon)} \le ||A||\cdot||\Phi_\beta(J^\epsilon*u)-\Phi_\beta(J*u)||_{(u_\epsilon)}. \tag{8.11}$$

Now, a pointwise estimate yields

$$|\Phi_\beta(J^\epsilon*u)-\Phi_\beta(J*u)| \le \beta|(J-J^\epsilon)*u|. \tag{8.12}$$

Thus, by Lemma 7.5,

$$||J*u-J^\epsilon*u||_{(u_\epsilon)} \le D(\epsilon)||u||_{(u_\epsilon)}. \tag{8.13}$$

Finally, (8.10) and (8.13) imply that

$$||V(u)-u_\epsilon||_{(u_\epsilon)} \le \frac{\gamma}{2}||A||\,||u-u_\epsilon||^2_{(u_\epsilon)}+\beta||A||D(\epsilon)||u||_{(u_\epsilon)}. \tag{8.14}$$

Now $||u||_{(u_\epsilon)} \le ||u-u_\epsilon||_{(u_\epsilon)}+||u_\epsilon||_{(u_\epsilon)} \le 1+R \le 2$, so eq.(8.14) gives

$$||V(u)-u_\epsilon||_{(u_\epsilon)} \le \frac{2}{1-\kappa}[(\frac{\gamma}{2})R^2+2\beta D(\epsilon)],$$

and we obtain inequality (8.4).

Step 2. We show that V is a contraction in B_R.

For u and u' in B_R, let $u_t = u' + t(u-u') \in B_R$. Also,

$$V(u) - V(u') = \int_0^1 dV_{u_t}(u-u')dt. \tag{8.15}$$

Thus, we obtain

$$||V(u) - V(u')||_{(u_\epsilon)} \leq \int_0^1 ||dV_{u_t}(u-u')||_{(u_\epsilon)}dt \leq (\sup_{B_R} ||dV_u||)||u-u'||_{(u_\epsilon)}. \tag{8.16}$$

Now, for $\xi \in Y_{u_\epsilon}$, and any $u \in B_R$,

$$dV_u(\xi) = \xi - A(dF_{(J,u)}(\xi)) = A\left(dF_{(J^\epsilon,u_\epsilon)}(\xi) - dF_{(J,u)}(\xi)\right). \tag{8.17}$$

Thus,

$$||dV_u|| \leq ||A|| \cdot ||dF_{(J^\epsilon,u_\epsilon)} - dF_{(J,u)}||. \tag{8.18}$$

Now,

$$dF_{(J,u)}(\xi) - dF_{(J^\epsilon,u_\epsilon)}(\xi) = (\Phi'_\beta(J*u) - \Phi'_\beta(J^\epsilon * u_\epsilon))J^\epsilon * \xi + \Phi'_\beta(J*u)((J-J^\epsilon)*\xi).$$

Thus,

$$\begin{aligned} ||(dF_{(J,u)} - dF_{(J^\epsilon,u_\epsilon)})\xi||_{(u_\epsilon)} \leq\; & \gamma||(J*u - J^\epsilon * u_\epsilon)J^\epsilon * \xi||_{(u_\epsilon)} + \beta||(J-J^\epsilon)*\xi||_{(u_\epsilon)} \\ \leq\; & \gamma\left(||(J-J^\epsilon)*u||_{(u_\epsilon)} + ||J^\epsilon * (u-u_\epsilon)||_{(u_\epsilon)}\right)||J^\epsilon * \xi||_{(u_\epsilon)} \\ & +\beta||(J-J^\epsilon)*\xi||_{(u_\epsilon)}. \end{aligned} \tag{8.19}$$

Because $0 \prec J^\epsilon * u_\epsilon \prec u_\epsilon$ and J^ϵ is order preserving, we see that $||J^\epsilon * \xi||_{(u_\epsilon)} \leq ||\xi||_{(u_\epsilon)}$, for all $\xi \in Y_{u_\epsilon}$. Using this fact, (8.13) (8.16), (8.19) and $||u||_{(u_\epsilon)} \leq 2$, we obtain

$$\begin{aligned} \sup_{u \in B_R} ||dV_u|| \leq\; & \gamma||A|| \cdot \left(\left(||(J-J^\epsilon)*u||_{(u_\epsilon)} + ||J^\epsilon * (u-u_\epsilon)||_{(u_\epsilon)}\right) + \beta||A||D(\epsilon)\right. \\ \leq\; & ||A||\left(\gamma D(\epsilon)||u||_{(u_\epsilon)} + \gamma R + \beta D(\epsilon)\right) \\ \leq\; & \frac{2}{1-\kappa}\left(\gamma R + (2\gamma+\beta)D(\epsilon)\right). \end{aligned} \tag{8.20}$$

Now, the estimates of Step 1, $||u||_{(u_\epsilon)} \leq 2$, and the conditions (8.4) and (8.5) on R and ϵ imply that V is a contraction in $B(R)$. ∎

8.2 Proof of uniqueness.

We first show, as a corollary of Lemma 7.6, that around any fixed point u of f, which is close enough to K_T, there is a neighborhood attracted to u whose width is independent of u and T. In other words, let u, ϵ, ϵ_T as in (E_T), then there is $\rho > 0$, such that if $T > T_2$

$$\forall w\colon\ ||w||_u \leq \rho \Longrightarrow \lim_{n\to\infty}(f_\epsilon)^n(u+w) = u. \tag{8.21}$$

Here, it is crucial that ρ be independent of u, T and ϵ.

If we set for any $w \in X_T$,

$$N(w) := f_\epsilon(w+u) - f_\epsilon(u) - df_\epsilon(u)(w),$$

then, if $\gamma = \sup\{|\Phi''_\beta(x)| : |x| \le 1\}$, it is a simple calculation to see that

$$|N(w)| \le \gamma(J^\epsilon * w)^2.$$

Now, as in (8.9), we first show that

$$||(J^\epsilon * w)^2||_u \le (1 + D(\epsilon_T))^2 ||w||_u^2.$$

We take $M > ||w||_u$, and recall that J^{ϵ_T} preserves the order $\prec$,

$$|J^{\epsilon_T} * w| \prec M J^{\epsilon_T} * u.$$

Thus, by invoking Lemma 7.5 and Remark 8.1

$$\begin{aligned} |J^\epsilon * w| &\prec MJ^\epsilon * u + (J^\epsilon - J^{\epsilon_T}) * w - M * (J^\epsilon - J^{\epsilon_T}) * u \\ &\prec MJ^\epsilon * u + D(\epsilon_T)||w||_u u + MD(\epsilon_T)u \prec M(1 + 2D(\epsilon_T))u \end{aligned}$$

and therefore

$$(J^\epsilon * w)^2 \prec M^2(1 + 2D(\epsilon_T))^2 u, \tag{8.22}$$

where we have used that $u \in C_T$ and $|u|_\infty < 1$. Thus,

$$||N(w)||_u \le \gamma(1 + 2D(\epsilon_T))^2 ||w||_u^2. \tag{8.23}$$

Using Lemma 7.6 and (8.23), we have for ϵ_T small enough

$$\begin{aligned} ||f_\epsilon(u+w) - f_\epsilon(u)||_u &\le ||df_\epsilon(u)||_u ||w||_u + ||N(w)||_u \\ &\le \kappa ||w||_u + 2\gamma ||w||_u^2. \end{aligned} \tag{8.24}$$

Hence, if $\rho = (1-\kappa)/4\gamma$, then $||w||_u \le \rho$ implies that

$$||f_\epsilon(u+w) - u||_u \le \frac{1+\kappa}{2} ||w||_u, \tag{8.25}$$

and the claim (8.21) follows easily from (8.25).

Assume that there is u and $\tilde{u}$ two distinct fixed points of f with

$$v \in K_T \backslash \{0\} : \ |u - v| < \epsilon_0 |v|, \quad \text{and} \quad \tilde{v} \in K_T \backslash \{0\} : \ |\tilde{u} - \tilde{v}| < \epsilon_0 |\tilde{v}|.$$

Theorem 7.4 tells us that for $T > T_2$,

$$\eta := \max(||df(u)^2||_u, ||df(\tilde{u})^2||_{\tilde{u}}) < 1$$

independently of T. To be able to use the same proof as that of Proposition 8.2 to obtain fixed point of f_{ϵ_T} in a neighborhood of u and $\tilde{u}$, we only need that ϵ_T be small enough so that we can define R which satisfies conditions identical to (8.4) and (8.5) but where η replaces κ. Thus, R is much smaller than ρ appearing in (8.21) when T is large. The same proof as that of Proposition 8.2 —which we omit— implies that f_{ϵ_T} has fixed points in B_R and say $\tilde{B}_R$. However,

$$B_R \subset \{v: ||u-v||_u \leq \rho\}, \qquad \tilde{B}_R \subset \{v: ||\tilde{u}-v||_{\tilde{u}} \leq \rho\},$$

and by (8.21)

$$\{v: ||u-v||_u \leq \rho\} \cap \{v: ||\tilde{u}-v||_{\tilde{u}} \leq \rho\} = \emptyset.$$

Furthermore, by Proposition 7.1, $0 \notin B_R \cup \tilde{B}_R$. Indeed, $|u|_\infty \geq a^*/2$, and $|\tilde{u}|_\infty \geq a^*/2$. Thus, this implies that f_{ϵ_T} has two non trivial fixed points in C_T, which is a contradiction to Theorem 2.3. ∎

9 Examples.

Our first example illustrates the non intuitive fact that J_T may not be decreasing even though J is decreasing in $[0,\infty)$.

Example 9.1 . Let $a > 0$ and suppose that J satisfies condition A, J is constant on $[-a,a]$ and J is strictly convex on $[a,\infty)$. A simple example of such a function is

$$J(x) = \frac{1}{2(1+a)} \quad \text{for } |x| \leq a, \qquad \text{and} \qquad J(x) = \frac{e^{a-|x|}}{2(1+a)} \quad \text{for } |x| \geq a.$$

Then, for $T > 2a$, $J_T(x)$ is strictly increasing on $[0,a]$. For our specific example, J_T is strictly decreasing on $[a, T/2]$.

Proof. For $0 \leq x \leq y \leq T/2$ and $n \geq 1$, we claim that

$$J(nT+y) + J(nT-y) > J(nT+x) + J(nT-x). \tag{9.1}$$

Inequality (9.1) is equivalent to proving that

$$J(nT-y) - J(nT-x) > J(nT+x) - J(nT+y).$$

Because $a \leq nT - y < nT - x$, the latter inequality follows from the strict convexity of J on $[a,\infty)$. Using the evenness of J, we see that

$$J_T(x) = J(x) + \sum_{n\geq 1} \left(J(nT+x) + J(nT-x)\right),$$

and,

$$J_T(y) = J(y) + \sum_{n\geq 1} \left(J(nT+y) + J(nT-y)\right).$$

For $0 \leq x < y \leq a$, $J(x) = J(y)$, so (9.1) implies that $J_T(x) < J_T(y)$.

For the explicit example given above, one can compute $J_T(x)$ (assuming $T > 2a$) for $0 \leq x \leq T/2$ and obtain

$$J_T(x) = \frac{1}{2(1+a)}[\theta(x) + Ce^{-x} + Ce^{x}], \quad \text{with} \quad C = e^a \left(\frac{e^{-T}}{1-e^{-T}}\right),$$

$\theta(x) = 1$ for $0 \leq x \leq a$, and $\theta(x) = \exp(a - x)$ for $a \leq x \leq T/2$. Using this formula, one verifies that J_T is strictly decreasing on $[a, T/2]$. ∎

Example 9.2 . Assume Φ satisfies B and define $\Phi_\beta(x) = \Phi(\beta x)$. If

$$J(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2),$$

then, for any $T > 0$, $\Phi_\beta(J_T * K_T) \subset K_T$.

Also, let $T_0 := 2\pi/(\sqrt{2\log(\beta)})$. For $T \leq T_0$, f has no fixed point in C_T, while for $T > T_0$, f has a unique fixed point in C_T.

Proof. First, we note that if $J_k(x) = \sqrt{k}J(\sqrt{k}x)$, then

$$(J_k)^{*k} = J.$$

Thus, if $J_k * K_T \subset K_T$, then $J * K_T \subset K_T$. Also, $J_k'(x)$ is concave for $x \geq \sqrt{2/k}$ and by Lemma 2.5, $J_k * K_T \subset K_T$ for $T \geq 4\sqrt{2/k}$. This means that for any $T > 0$, $J * K_T \subset K_T$. The value of T_0 is obtained from

$$T_0 = \inf\{T > 0 : \ \hat{\beta}J(\frac{2\pi}{T}) := \beta \exp(-\frac{1}{2}(\frac{2\pi}{T})^2) > 1, \}$$

∎

Example 9.3 . Let Φ_β be as in Example 2 and let $J(x) = \exp(-|x|)/2$. Here also, Lemma 2.5 implies that for any $T > 0$, $\Phi_\beta(J_T * K_T) \subset K_T$. Thus, the results of section 6 imply that if T_0 is defined as

$$(\frac{2\pi}{T_0})^2 = \beta - 1,$$

then, for $T \leq T_0$, f has no fixed point in C_T, while for $T > T_0$, f has a unique fixed point in C_T.

We look now at an example where J is not continuous.

Example 9.4 . Let Φ_β be as above and let $J(x) = I_{[-1/2,1/2]}$. Then, we have the relations (2.5).

Proof. Let $k \in N$ and $r \geq 0$ such that

$$\frac{T}{2} + kT \leq \frac{1}{2} < \frac{T}{2} + (k+1)T, \quad \text{and} \quad r = 1/2 - (T/2 + kT) \leq T.$$

We consider the three cases. (i) $0 < r < T/2$. A calculation gives, for $x \in [0, T/2]$,

$$J_T(x) = I_{[T/2-r,T/2]} + 2k + 1,$$

so that J_T is increasing and $J_T(x+T/2)$ is decreasing. Thus, by symmetry, $J_T * K_T \subset -K_T$. This case corresponds to $T \in (1/(2n+2), 1/(2n+1))$.

(ii) If $r \geq T/2$ and $x \in [0, T/2]$, then

$$J_T(x) = I_{[0,r-T/2]} + 2k + 2,$$

and $J_T(x)$ is decreasing in $\{x > 0\}$, and $J_T * K_T \subset K_T$.

(iii) If $T = 1/(n+1)$, then J_T is constant on $[-T/2, T/2]$ so that $J_T * u = \int u$.

If $T \geq 2$, we derive from Lemma 1 that $f(K_T) \subset K_T$. If $1 < T < 2$, a direct calculation gives that $\tilde{J}_T(x) = 1$ for $0 \leq x < T/2 - 1/2$, $\tilde{J}_T(x) = 0$ for $T/2 - 1/2 < x \leq 1/2$ and $\tilde{J}_T(x) = -1$ for $1/2 < x \leq T/2$, so Corollary 1 implies that $f(K_T) \subset K_T$. Also, the spectrum is $\{\beta g(2\pi n/T),\ n \in 2\mathbb{Z} + 1\}$ with $g(x) = \sin(x/2)/x/2$.

We define $T_0 = \sup\{T \geq 0 \colon \beta \sin(\pi/T)/(\pi/T) \geq 1\}$. The results of section 6 imply that if $T > T_0$, and $T \in (1/2n+3, 1/2n+2)$, $n \geq 0$, then the equation $f(u) = u$ has a unique nonzero fixed point in C_T. however, if $T \in [1/2n+2, 1/2n+1]$, $n \geq 0$, then there is no nontrivial fixed point in C_T. ∎

10 Appendix

Proof of Lemma 3.1. (i) Define the measurable function $J_T^N(\xi) = \sum_{|n| \leq N} J(\xi + nT)$. Because J is non–negative, for each ξ, $\{J_T^N(\xi), N = 1, 2, \ldots\}$ is an increasing sequence and we can define $J_T(\xi)$ as its pointwise limit. J_T is thus measurable and it follows easily that J_T is even and non–negative. Assume that $J_T(\xi) < \infty$, then it is easy to see that $J_T(\xi + T) < \infty$ and that

$$J_T(\xi + T) - J_T(\xi) = J(\xi + T + NT) - J(NT - \xi),$$

goes to 0 as N goes to infinity. Thus, J_T is periodic. (ii) follows by the monotone convergence theorem. For (iii), let $\xi \in [0, T/2]$ and write

$$\begin{aligned} J_T(\xi) = &\ J(\xi) + \sum_{n=1}^{\infty} [J(nT+\xi) + J(nT-\xi)] \\ \leq &\ J(\xi) + \frac{2}{T} \sum_{n \geq 1} [\int_{nT+\xi-T/2}^{nT+\xi} J + \int_{nT-\xi-T/2}^{nT-\xi}] \\ \leq &\ J(\xi) + \frac{4}{T} [\int_{T+\xi}^{\infty} J + \int_{T-\xi}^{\infty} J] < \infty. \end{aligned}$$

∎

Remark 10.1 . If J is even, nonnegative and bounded and $J|_{[0,\infty)}$ is decreasing, then J is automatically measurable. Furthermore, after modification on a countable set, we can assume that J is continuous at 0, right continuous on $(0, \infty)$ and left continuous on $(-\infty, 0)$. To see this, note that because $J|_{[0,\infty)}$ is decreasing, we know that J is continuous except possibly at countable many points. We define

$$\tilde{J}(\xi) = \lim_{x \to \xi+} J(x), \text{ for } \xi > 0, \quad \tilde{J}(\xi) = \lim_{x \to \xi-} J(x), \text{ for } \xi < 0 \quad \text{and} \quad \tilde{J}(0) = \lim_{x \to 0} J(x).$$

$\tilde{J}$ agrees with J except possibly at countable many points, $\tilde{J}$ is continuous at 0, and right continuous on $(0, \infty)$. It is an elementary exercise (See Rudin,[9]) to show that $\tilde{J}|_{[0,-\infty)}$ is Lebesgue measurable. Similarly, $\tilde{J}|_{(-\infty,0)}$ is Lebesgue measurable, and therefore $\tilde{J}$ and J are measurable.

Proof of Lemma 3.2. Let θ_x be the translation shift by x. Recall that $\theta_x : L^1 \to L^1$ is a continuous operator. Now,

$$\begin{aligned} |L_T u(x_1) - L_T u(x_2)| &\leq | \int_0^T (\theta_{x_1 - x_2} J_T(y) - J_T(y))\, u(y + x_2) dy| \\ &\leq (|J_T|_\infty \cdot |\theta_{x_1 - x_2} J_T - J_T|_1)^{1/2} \, |u|_T \end{aligned} \tag{10.1}$$

If we take $x_2 = -x_1$, then (10.1) shows that $|L_T u(x_1)| \leq \sqrt{|J_T|_\infty} |u|_T$ and that $L_T(\mathcal{H}_T) \in X_T$. Actually, L_T defines a bounded linear map of $\mathcal{H}_T \to X_T$ with norm less or equal to $\sqrt{|J_T|_\infty}$ and $L_T(\{u : u \in \mathcal{H}_T \text{ and } |u|_{\mathcal{H}_T} \leq 1\})$ is a bounded, equicontinuous family in X_T. This last fact shows that $L_T : \mathcal{H}_T \to X_T$ is compact. ∎

Lemma 10.2 . *$K_T \cap C^2$ is dense in K_T in the supremum norm topology.*

Proof. Let $u \in K_T$ and for each n integer let v_n be a piecewise linear function in K_T approximating u; i.e. if $x_i = T/(4n).i$ for $i = 0, \ldots, n$, we set $v_n(x_i) = u(x_i)$ and complete v_n on $[0, T/4]$ by straight lines joining the $\{v_n(x_i)\}$. We complete v_n on $[0, T]$ by symmetry. Note that $|u - v_n|_\infty \to 0$. Now let φ be a C^∞ function with support in $[-1/2, 1/2]$ and $\varphi_n(x) = n\varphi(nx)$. We set $w_n = \varphi_n * v_n$. Note that $w_n \in K_T \cap C^\infty$. Indeed, for $x \in [0, T/4]$,

$$w_n(x) = \int_{-T/2}^{T/2} \varphi_n(y) \tilde{v}_n(x - y) dy,$$

where $\tilde{v}_n(x) = v_n(x)$ for $x \in [0, T/2]$, and $\tilde{v}_n(x) = -x v_n(x_1)/x_1$ for $x < 0$. We build $\tilde{v}_n$ so as to be concave on $[-T/2, T/2]$, and so is $\varphi_n * \tilde{v}_n$ on $[0, T/4]$. Also, it is easy to see that $|u - w_n|_\infty \to 0$. ∎

Proof of Lemma 7.3. We leave to the reader the proof that $(Y_u, || \cdot ||_u)$ is a normed linear space when K is a closed cone. Suppose that K is also normal. If $||y||_u = M$, we have $-Mu \prec y \prec Mu$. Since $0 \prec y + Mu \prec 2Mu$, the normality of K implies that there exists a constant C, independent of y, such that

$$||y + Mu|| \leq C||2Mu|| = 2MC||u||.$$

If follows that $||y|| \leq ||y + Mu|| + || - Mu|| = (2C+1)||u||M := C_1||y||_u$, where C_1 is independent of y.

Now suppose that $\{v_n, n \geq 1\}$ is a Cauchy sequence in Y_u. Since $||y|| \leq C_1||y||_u$ for all $y \in Y_u$, it follows that $\{v_n, n \geq 1\}$ is a Cauchy sequence in Y, and that there exists $v \in Y$ with $\lim ||v_n - v|| = 0$. Given $\epsilon > 0$, select n_0 so that $||v_n - v_m|| \leq \epsilon$ for all $n, m \geq n_0$. Given $n \geq n_0$, this implies that for all $m \geq n_0$, $-\epsilon u \prec v_n - v_m \prec \epsilon u$. Since K is closed in Y, it follows by taking limits in Y as $m \to \infty$ that $-\epsilon u \prec v_n - v \prec \epsilon u$. Thus we see that $v \in Y_u$, and $||v_n - v||_u \leq \epsilon$ for all $n \geq n_0$, so Y_u is complete.

If L is as in the statement of Lemma 7.3 and $v \in Y_u$ with $||v||_u = M$, we have $-Mu \prec v \prec Mu$. Because L preserves the partial ordering, $-MNu \prec Lv \prec MNu$, so $Lv \in Y_u$ and $||Lv||_u \leq N||v||_u$. It follows that $\mathcal{L} : Y_u \to Y_u$ is a bounded linear map with $||\mathcal{L}||_u \leq N$. ∎

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