Global Continuation and Asymptotic Behaviour
for Periodic Solutions of a Differential-Delay Equation (*).

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Summary. – *The singularly perturbed differential-delay equation*

$$\varepsilon \dot{x}(t) = - x(t) + f(x(t-1))$$

is studied. Existence of periodic solutions is shown using a global continuation technique based on degree theory. For small ε these solutions are proved to have a square-wave shape, and are related to periodic points of the mapping $f: \mathbf{R} \to \mathbf{R}$. When f is not monotone the convergence of $x(t)$ to the square-wave typically is not uniform, and resembles the Gibbs phenomenon of Fourier series.

0. – Introduction.

In recent years the differential-delay equation

$$(0.1)_\varepsilon \qquad \varepsilon \dot{x}(t) = - x(t) + f(x(t-1))$$

with scalar variable $x \in \mathbf{R}$, nonlinearity $f: \mathbf{R} \to \mathbf{R}$, and parameter $\varepsilon > 0$, has been proposed as a mathematical model for several problems in different areas of science. In optics, for example, equation $(0.1)_\varepsilon$ with the trigonometric nonlinearity

$$(0.2) \qquad f(x) = \mu_1 + \mu_2 \sin(\mu_3 x + \mu_4)$$

where $\mu_j \in \mathbf{R}$, $j = 1, 2, 3, 4$, are parameters, arises in the study of an optically bistable device. See for example [13, 14, 21, 29, 31, 32, 33]. Equation $(0.1)_\varepsilon$ with a nonlinearity f of the form shown in Figure 1 has been proposed as a model for a variety of physiological processes and conditions including production of blood cells, respira-

tion, and cardiac arrhythmias. See for example [22, 26, 27, 37, 38, 39, 40, 61] where, in most cases, one of the model functions

$$(0.3) \qquad\qquad f(x) = \mu x^\nu \exp[-x]$$

or

$$(0.4) \qquad\qquad f(x) = \frac{\mu x}{1 + x^\nu}$$

with parameters $\mu > 0$ and $\nu > 0$ is considered. Equation $(0.1)_\varepsilon$ also arises in population models [2, 23, 30] where again f has the form depicted in Figure 1.
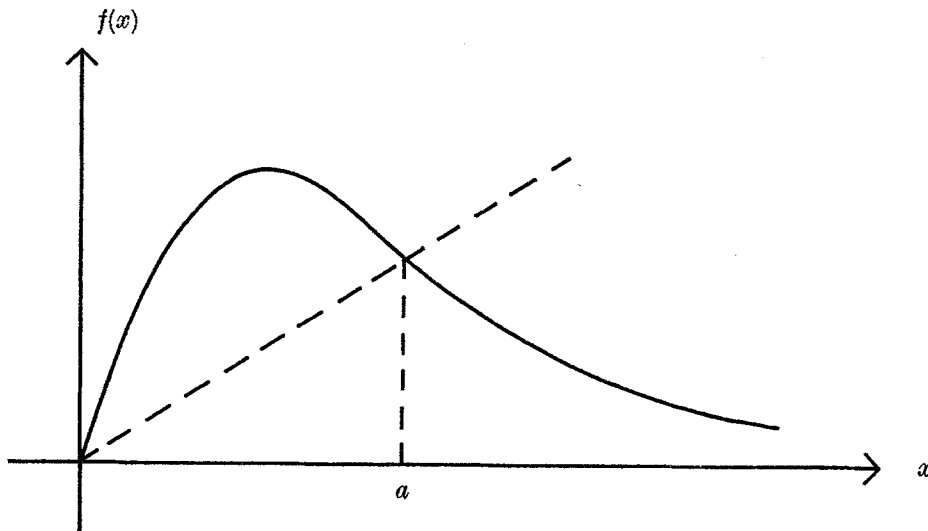

Fig. 1. Typical nonlinearity $f(x)$, with $f(a) = a$.

Note here that the assumption of a unit time delay $\lambda = 1$ in $(0.1)_\varepsilon$ is merely a normalization. In fact, if one rescales the time t by setting

$$t = \varepsilon s \quad \text{and} \quad x(t) = \bar{x}(s) \ ,$$

then equation $(0.1)_\varepsilon$ is equivalent to

$$(0.5)_\lambda \qquad\qquad \frac{d\bar{x}(s)}{ds} = -\bar{x}(s) + f\big(\bar{x}(s - \lambda)\big) \quad \text{where} \quad \lambda = \varepsilon^{-1} \ .$$

Thus one sees that the singular perturbation case $\varepsilon \to 0^+$ in $(0.1)_\varepsilon$ is equivalent to the case of large delay $\lambda \to \infty$ in equation $(0.5)_\lambda$. Most of the results of this paper concern this case.

Many authors have studied equation $(0.1)_\varepsilon$ and similar equations. Local Hopf bifurcations were analyzed in [46]; see also [7]. In [4, 8, 24, 35, 36, 53] periodic solutions were found through various topological and analytical techniques. See also [47, 48, 49, 50] where a variety of equations are treated in this spirit. Numerical studies can be found in many of the applied works referenced earlier, as well as in [6, 17, 56, 57]. Chaotic solutions were proved to exist in [28, 55, 60], for some classes of f. Results on global dynamical behavior are found in [58, 59]. Following this, a further description of the global picture as a Morse decomposition is given in [41, 42]. A variety of singularly perturbed delay equations is treated in [3, 5, 9, 20, 51]; in particular, the basic results of COOKE [11] and COOKE and MEYER [12] concern linear equations. A general reference for singular perturbations of all types is the book of O'MALLEY [54].

The results of this paper have been announced without proof by the same authors in several other papers [41, 43] (See also [9]). Further results are presented in [44]. The reader may find these shorter presentations helpful while reading this paper.

Formally taking the limit $\varepsilon \to 0^+$ in equation $(0.1)_\varepsilon$ leads to the difference equation

$$(0.6) \qquad x(t) = f(x(t-1))$$

which one may also write as a discrete system

$$(0.7) \qquad x_{n+1} = f(x_n) \ .$$

A fundamental problem, which we study in this paper, is to determine how the dynamics of the differential equation $(0.1)_\varepsilon$ mirror the dynamics of the (presumably simpler) discrete system (0.7) when ε is small. Indeed, while the discrete system has been extensively studied [10, 18, 19], not much is known about the differential equation as $\varepsilon \to 0^+$. More specifically, suppose $a_0 \in \mathbf{R}$ is such that $f^n(a_0) = a_0$ for some $n \geqslant 2$, but $f^j(a_0) \neq a_0$ if $1 \leqslant j < n$. Define a function

$$x_0(t) = a_j \quad \text{if } j \leqslant t < j+1$$

where the points a_j are given by the iterates

$$a_j = f^j(a_0) \quad \text{for } 0 \leqslant j < n \ .$$

If the function $x_0(t)$ is extended periodically, so that

$$x_0(t+n) = x_0(t)$$

for all t, then $x_0(t)$ is a solution of the difference equation (0.6) of period n. One can ask whether, for ε positive and small, the differential equation $(0.1)_\varepsilon$ has a periodic solution $x_\varepsilon(t)$ with period near n, such that $x_\varepsilon(t)$ approaches $x_0(t)$ in some sense as ε

approaches zero. In this paper, we shall study this question for the case $n = 2$, that is, when $f(a_0) = a_1$ and $f(a_1) = a_0$ for some points $a_0 \neq a_1$.

The situation we study could typically arise through a period-doubling bifurcation from a fixed point $a = a(\mu)$ of a parametrized function $f(x, \mu)$. If one has $f(a(\mu), \mu) = a(\mu)$ for some function $a(\mu)$ of the parameter μ varying in an interval, then consider the derivative

$$e(\mu) = f_x(x, \mu)|_{x=a(\mu)}$$

evaluated at the fixed point $a(\mu)$. If there exists a parameter value $\mu = \mu_*$ at which the function $1 + e(\mu)$ changes sign, say

$$e(\mu) > -1 \quad \text{if } \mu < \mu_*$$

$$e(\mu) < -1 \quad \text{if } \mu > \mu_* \,,$$

then it is known that a branch of period-two points $\{a_0, a_1\}$, as described above, must bifurcate from the point $(x, \mu) = (a(\mu_*), \mu_*)$. Indeed, under a generic condition on the function f the bifurcating points a_0 and a_1 form, locally, a smooth one-parameter family which lies on one side only of the critical parameter value μ_*. In such a case one has

$$a_1(\mu) < a(\mu) < a_0(\mu) \quad \text{where} \quad \lim_{\mu \to \mu_*} a_j(\mu) = a(\mu_*), \quad j = 0, 1$$

either for all $\mu \in (\mu_*, \mu_*^2 + \delta)$ (supercritical bifurcation) or else for all $\mu \in (\mu_* - \delta, \mu_*)$ (subcritical bifurcation), for some $\delta > 0$. The details of this bifurcation are given, for example, in the book of COLLET and ECKMANN [10]. In particular, the model nonlinearity

$$f(x, \mu) = \mu - x^2$$

is shown to undergo a supercritical period-two bifurcation at $(a(\mu_*), \mu_*) = (\frac{1}{2}, \frac{3}{4})$. (In fact, an infinite cascade of period-doubling bifurcations occurs, giving rise to points of period $n = 2, 4, 8, 16, \ldots$; see FEIGENBAUM [18, 19]. Our modest efforts here are devoted to studying those points of period $n = 2$.)

The basic hypotheses we impose on the function f are motivated by the situation to the right of the supercritical bifurcation above. (Note, however, that in our study here of equation $(0.1)_\varepsilon$ the function f is fixed: there is no bifurcation parameter μ.) A typical assumption on f is the existence of a fixed point $x = a$ at which $f'(a) < -1$; by means of a linear translation one has $a = 0$ without loss, and so

$$(0.8) \qquad\qquad f(0) = 0 \quad \text{and} \quad f'(0) < -1$$

are assumed. In addition, we generally assume the negative feedback condition

(0.9) $$xf(x) < 0 \;, \qquad x \neq 0$$

for x in some sufficiently large interval of interest about the origin. (Clearly (0.8) implies (0.9) near zero, at least.) Finally, the existence of period-two points

$$a_1 < 0 < a_0 \;, \qquad f(a_0) = a_1 \quad \text{and} \quad f(a_1) = a_0$$

and possibly some stability conditions on these points (for the discrete map f) is sometimes assumed. Monotonicity of f between a_0 and a_1 is *not* required; indeed, out most interesting results concern the non-monotone case. Figure 2 depicts the graph of a typical function f of interest.
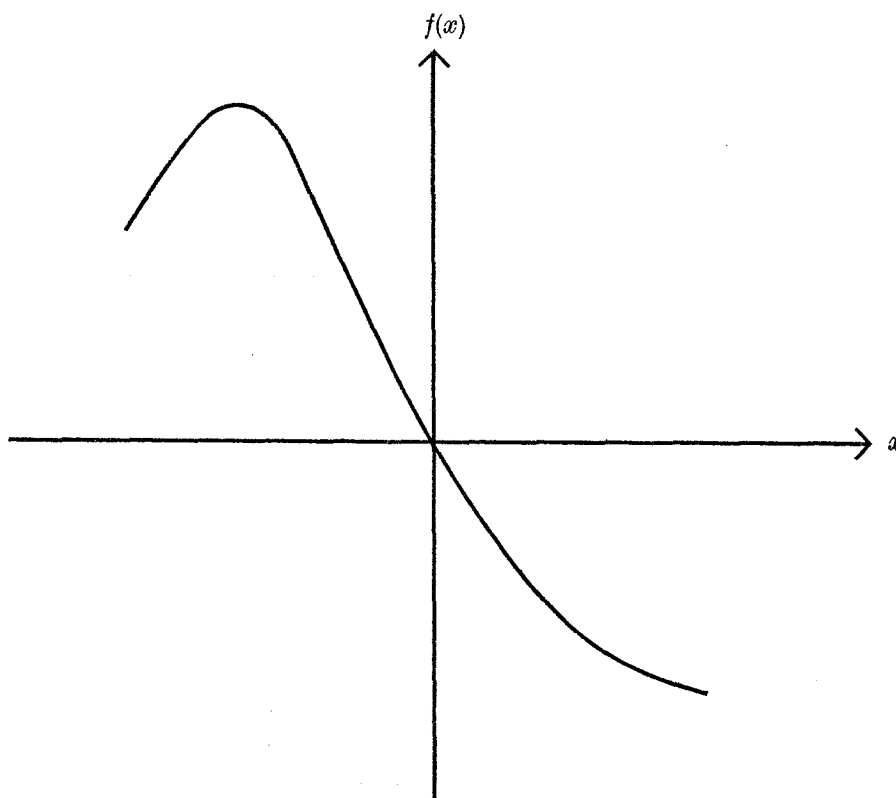

Fig. 2. Nonlinearity $f(x)$ normalized so $f(0) = 0$.

The main results of this paper occupy Sections 3 and 4, and describe the asymptotic behaviour of the periodic solutions $x_\varepsilon(t)$ as $\varepsilon \to 0^+$ under the assumptions on f outlined in the preceeding paragraph. (The existence of the periodic solutions

$x_\varepsilon(t)$ was proved by HADELER and TOMIUK [24]; see also CHOW [4] and KAPLAN and YORKE [35, 36] for some special cases and related equations. In Section 1 we show there is a continuum of such solutions extending from a Hopf bifurcation point $\varepsilon = \lambda_0^{-1} > 0$ to $\varepsilon = 0$.) Following a preliminary estimate on the period of $x_\varepsilon(t)$ given in Theorem 3.1, the nature of the convergence of $x_\varepsilon(t)$ to the period-two step function $x_0(t)$ (which we denote by sqw (t) for « square wave ») is investigated in detail in Section 4. In particular, our results confirm phenomena observed in numerical and experimental studies by other authors. Namely, when f is monotone between a_0 and a_1 the convergence of $x_\varepsilon(t)$ to $x_0(t)$ is very regular (in the sense that the graph of $x_\varepsilon(t)$ resembles a square wave as $\varepsilon \to 0^+$), but that if f is not monotone there, then $x_\varepsilon(t)$ often exhibits a non-uniform convergence to $x_0(t)$ reminiscent of the Gibbs phenomenon of Fourier series.

The nature of this Gibbs-like convergences is as follows. The solution $x_\varepsilon(t)$ converges to the step function $x_0(t)$ uniformly on compact subsets of $\boldsymbol{R} - \boldsymbol{Z}$ (where $\boldsymbol{Z}$ denotes the integers). However, near integer points $t = j$, where $x_0(t)$ jumps, $x_\varepsilon(t)$ can overshoot the values $x_0(t) = a_0, a_1$ by an amount which does *not* become small
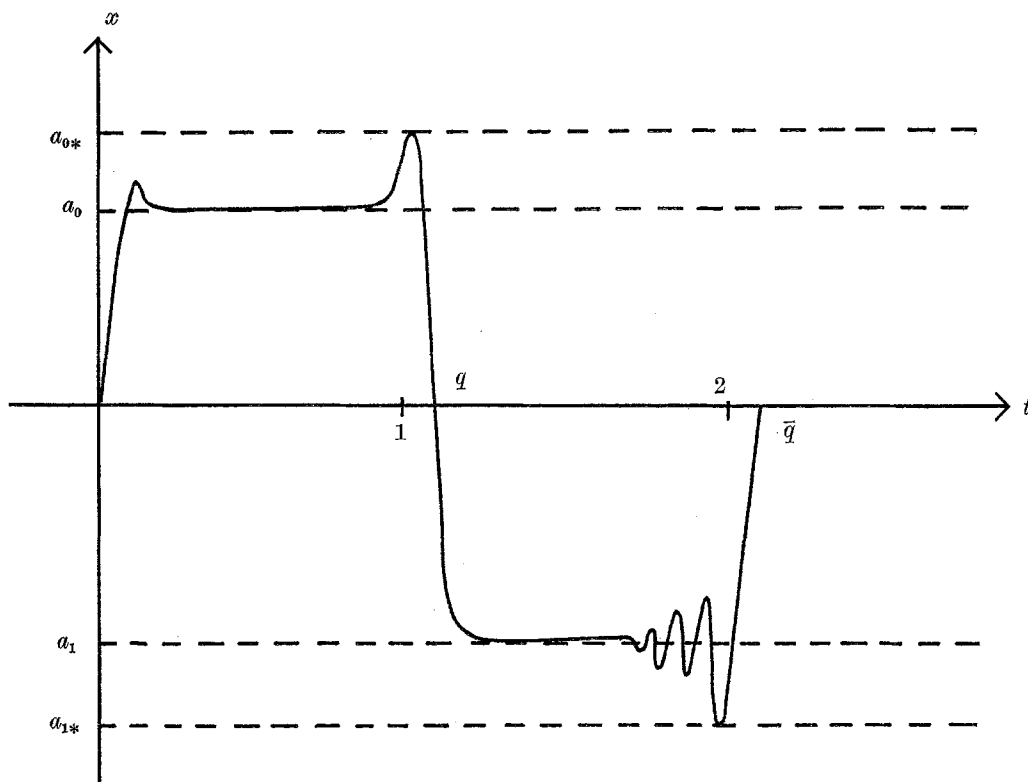

Fig. 3. The Gibbs phenomenon for small ε.

as $\varepsilon \to 0^+$: there exists either $a_{0*} > a_0$ such that for each $\delta < 1$

$$\limsup_{\varepsilon \to 0} \max_{|t-1| \leqslant \delta} x_\varepsilon(t) \geqslant a_{0*} > a_0$$

or else there exists $a_{1*} < a_1$ such that for each $\delta < 1$

$$\liminf_{\varepsilon \to 0} \min_{|t| \leqslant \delta} x_\varepsilon(t) \leqslant a_{1*} < a_1 \ .$$

Figure 3 depicts such a solution $x_\varepsilon(t)$. Theorem 4.2 describes the convergence on $\boldsymbol{R} - \boldsymbol{Z}$, while the more delicate convergence near integer points is described in Theorem 4.1 and Corollaries 4.1 and 4.2.

Although this Gibbs phenomenon does not occur if the function f satisfying conditions described earlier is monotone for $a_1 \leqslant x \leqslant a_0$, we prove in Proposition 4.1 that it must occur for many (non-monotone) functions f for which $f([a_1, a_0])$ properly contains $[a_1, a_0]$. The analytical device which allows one to describe the structure of $x_\varepsilon(t)$ near the jumps in $x_0(t)$, and so prove these results, is a pair of transition layer equations

$$(0.10) \qquad \dot{y}(t) = y(t) - f(z(t-r)) \ , \qquad \dot{z}(t) = z(t) - f(y(t-r)) \ .$$

For an appropriate choice of the parameter $r > 0$, there is a solution $(y(t), z(t))$ such that both $|x_\varepsilon(-\varepsilon t) - y(t)|$ and $|x_\varepsilon(1 + \varepsilon r - \varepsilon t) - z(t)|$ are small, uniformly on compact t-intervals, for sequences $\varepsilon_n \to 0^+$. In addition,

$$\lim_{t \to -\infty} (y(t), z(t)) = (a_0, a_1)$$

$$\lim_{t \to \infty} (y(t), z(t)) = (a_1, a_0)$$

that is, $(y(t), z(t))$ is a heteroclinic orbit joining the critical point (a_0, a_1) of the system (0.10) to the critical point (a_1, a_0). Such solutions of the transition layer equations thus describe the fine structure of $x_\varepsilon(t)$, in neighborhoods of width $O(\varepsilon)$, about the jump points of $x_0(t)$. Further studies of the transition layer system (0.10) are found in [52]. In [5] a transition layer equation was used to prove similar results for a nonlinear integral equation.

In Sections 1 and 2 we prove some general results valid for larger ranges of ε. In this case we shall typically write equation $(0.1)_\varepsilon$ in the equivalent form

$$\dot{x}(t) = -\lambda x(t) + \lambda f(x(t-1))$$

where $\lambda = \varepsilon^{-1}$. Most of our results in these sections concern the global Hopf bifurcations from the origin $x = 0$ at a sequence of parameter values $\lambda = \lambda_m$ satisfying

$$\ldots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots \ .$$

The local bifurcation from each of these points was described by MARTELLI, SCHMITT and SMITH [46]. We prove that the local Hopf branch emanating from the point $(x, \lambda) = (0, \lambda_m)$ lies on a continuum Σ_m of periodic solutions extending for all $\lambda > \lambda_m$ if $m \geqslant 0$, and all $\lambda < \lambda_m$ if $m < 0$. The sets Σ_m are pairwise disjoint and the integer m is related to the rate at which solutions oscillate. The solutions on Σ_0 have consecutive zeros spaced a distance greater than one apart, and repeat after their second zero: they are all « slowly oscillating » periodic solutions and in many cases are observed numerically. The solutions on Σ_m for $m \neq 0$, in contrast, oscillate more rapidly and seem generally to be unstable. Section 1 deals with the slowly oscillating periodic solutions. Σ_0, while in Section 2 we study Z_m for $m \neq 0$. The results of Section 2 will not be needed for the remainder of the paper.

The appendix contains proofs of several facts related to the location of eigenvalues of the linear problem.

One issue which we have not addressed in this paper, for reasons of space, is the problem of determining for which parameter ranges our results apply to the specific nonlinearities (0.2), (0.3) and (0.4) in the applied models. This is treated in a companion paper [45] by the same authors.

1. – The existence of a global continuum of periodic solutions.

We shall be interested in this section in finding a continuum of periodic solutions of a parametrized family of differential-delay equations of the form

$$(1.1)_\lambda \qquad \dot{x}(t) = -\lambda x(t) + \lambda f(x(t-1)) , \qquad \lambda > 0 .$$

We shall also explore some other aspects of equation $(1.1)_\lambda$ which relate the dynamical behaviour of solutions $(1.1)_\lambda$ to the orbits $\{x_n\}$ of the discrete dynamical system

$$(1.2) \qquad x_{n+1} = f(x_n)$$

obtained by iterating the map f.

It will often be necessary to consider an initial value problem for $(1.1)_\lambda$. If $\varphi \in C[0, 1]$ is a given continuous function, f is continuous, and $\lambda > 0$, one can easily prove there is a unique function $x(t)$ which is continuous on $[0, \infty)$, continuously differentiable on $[1, \infty)$, and satisfies

$$(1.3)_\lambda \qquad \begin{aligned} \dot{x}(t) &= -\lambda x(t) + \lambda f(x(t-1)) \qquad \text{for all } t \geqslant 1 , \\ x|[0, 1] &= \varphi . \end{aligned}$$

We will denote the unique solution $x(t)$ of $(1.3)_\lambda$ by $x(t; \lambda, \varphi)$. Of course $x(t; \lambda, \varphi)$

also depends on f, but f will usually be fixed. The solution $x(t; \lambda, \varphi)$ is obtained by step by step integration on intervals of length one.

Our first result concerns invariant intervals for equations $(1.1)_\lambda$ and (1.2).

PROPOSITION 1.1. – (i) *Let* $f: \mathbf{R} \to \mathbf{R}$ *be a continuous function and let* $\lambda > 0$. *Let* $I \subseteq \mathbf{R}$ *be a closed (possibly infinite) interval such that* $f(I) \subseteq I$. *If* $\varphi \in C[0, 1]$ *satisfies*

$$(1.4) \qquad\qquad \varphi(t) \in I \quad \text{for all } t \in [0, 1]$$

then the solution $x(t; \lambda, \varphi)$ *of* $(1.3)_\lambda$ *satisfies*

$$x(t; \lambda, \varphi) \in I \quad \text{for all } t \geqslant 1 .$$

If in addition $\varphi(1) \in \operatorname{int}(I)$, *where* «int» *denotes interior, then* $x(t; \lambda, \varphi) \in \operatorname{int}(I)$ *for all* $t \geqslant 1$.

(ii) *Further, define the set*

$$(1.5) \qquad\qquad I_\infty = \bigcap_{n=0}^{\infty} \overline{f^n(I)} ;$$

necessarily I_∞ *is a closed connected subset of* I. *If* $I_\infty \neq \emptyset$, *then the above solution of* $(1.3)_\lambda$ *with* (1.4) *satisfies*

$$\operatorname{dist}(x(t; \lambda, \varphi), I_\infty) \to 0 \quad \text{as } t \to \infty$$

where «dist» *denotes the distance from a point to a set.*

PROOF. – (i) With φ and $x(t) = x(t; \lambda, \varphi)$ as in the statement of the proposition, let

$$t_0 = \sup \{t \in [1, \infty) : x(s) \in I \text{ for all } s \in [1, t]\}$$

and suppose that $t_0 < \infty$. For definiteness suppose that $I = [D, C]$ is a compact interval. From $(1.3)_\lambda$, (1.4), the definition of t_0, and the invariance property $f(I) \subseteq I$ one obtains

$$(1.6) \qquad \frac{d}{ds}\left(\exp[\lambda s] x(s)\right) = \lambda \exp[\lambda s] f(x(s-1)) \leqslant \lambda \exp[\lambda s] C$$

for all $s \in [1, t_0 + 1]$. Integrating (1.6) from 1 to $t \in [1, t_0 + 1]$ and noting $x(1) = \varphi(1) \leqslant C$ gives

$$(1.7) \qquad\qquad x(t) \leqslant C + \exp[-\lambda(t-1)](x(1) - C) \leqslant C .$$

A similar argument shows $x(t) \geqslant D$ if $t \in [1, t_0 + 1]$ and so $x(t) \in I$ for this range of t. But this contradicts the definition of t_0. Hence $t_0 = \infty$.

If in addition $\varphi(1) \in \operatorname{int}(I)$, then the second inequality in (1.7) is strict for each $t \geqslant 1$. And similarly $x(t) > D$, so $x(t) \in \operatorname{int}(I)$ for each $t \geqslant 1$.

(ii) Again for definiteness assume $I = [D, C]$ is compact. The sets $f^n(I)$ form a nested decreasing sequence of compact intervals (or points), so one may write

$$f^n(I) = [D_n, C_n], \qquad n \geqslant 0,$$

where

$$D = D_0 \leqslant D_1 \leqslant D_2 \leqslant \dots \leqslant C_2 \leqslant C_1 \leqslant C_0 = C.$$

Note that

$$I_\infty = [D_\infty, C_\infty]$$

where

$$D_\infty = \lim_{n \to \infty} D_n \quad \text{and} \quad C_\infty = \lim_{n \to \infty} C_n.$$

Suppose it is shown that for some $n \geqslant 0$ one has

$(1.8)_n$ $\qquad\qquad$ dist $\big(x(t), f^n(I)\big) \to 0 \quad$ as $t \to \infty$,

or what is equivalent

$$D_n \leqslant \liminf_{t \to \infty} x(t) \leqslant \limsup_{t \to \infty} x(t) \leqslant C_n.$$

Then certainly

$$D_{n+1} \leqslant \liminf_{t \to \infty} f\big(x(t)\big) \leqslant \limsup_{t \to \infty} f\big(x(t)\big) \leqslant C_{n+1}$$

so that for each $\delta > 0$ there exists $T \geqslant 1$ such that

(1.9) $\qquad\qquad D_{n+1} - \delta \leqslant f\big(x(t-1)\big) \leqslant C_{n+1} + \delta \quad$ for all $t \geqslant T$.

Integrating the equality in (1.6) from T to t, using (1.9), and letting $t \to \infty$ gives $D_{n+1} - \delta \leqslant \liminf_{t \to \infty} x(t) \leqslant \limsup_{t \to \infty} x(t) \leqslant C_{n+1} + \delta$. But δ is arbitrary, hence

$$\text{dist } \big(x(t), f^{n+1}(I)\big) \to 0 \quad \text{as } t \to \infty.$$

Thus $(1.8)_n$ implies $(1.8)_{n+1}$. As $(1.8)_n$ already holds for $n = 0$ by part (i) above, by induction it holds for each $n \geqslant 0$. But this proves

$$\text{dist } \big(x(t), I_\infty\big) \to 0 \quad \text{as } t \to \infty$$

as required.

If I is an infinite interval (and also $I_\infty \neq \emptyset$ in part (ii)), then all the above arguments are valid with only minor changes. $\square$

REMARK 1.1. - If $I_\infty = \emptyset$ in Proposition 1.1, then either $\overline{f^n(I)} = [D_n, \infty)$ for large n, where $D_n \to \infty$, or else $\overline{f^n(I)} = (-\infty, C_n]$ where $C_n \to -\infty$. In the former case one can show $\lim_{t \to \infty} x(t) = \infty$ while in the latter case $\lim_{t \to \infty} x(t) = -\infty$ for any solution of $(1.3)_\lambda$ satisfying (1.4).

COROLLARY 1.1. - *Let f, λ, I and I_∞ be as in Proposition 1.1. If $x(t)$ is a periodic solution of $(1.1)_\lambda$ satisfying*

$$x(t) \in I \quad \text{for all } t \in \mathbf{R}$$

then one has

$$x(t) \in I_\infty \quad \text{for all } t \in \mathbf{R}\,.$$

Further, if $x(t)$ is non-constant, then

$$(1.10) \qquad\qquad x(t) \in \mathrm{int}\,(I_\infty) \quad \text{for all } t \in \mathbf{R}\,.$$

PROOF. - That $x(t) \in I_\infty$ for all t is an immediate consequence of the periodicity of $x(t)$ and of (ii) of Proposition 1.1.

If $x(t)$ is non-constant, then $x(t_0) \in \mathrm{int}\,(I_\infty)$ for some $t_0 \in \mathbf{R}$. By replacing $x(t)$ with the solution $x(t + t_0 - 1)$ if necessary, we may assume without loss of generality that $t_0 = 1$. But then $\varphi(1) \in \mathrm{int}\,(I_\infty)$ where $\varphi = x|[0, 1]$, hence by (i) of Proposition 1.1, with I_∞ replacing I, one has $x(t) \in \mathrm{int}\,(I_\infty)$ for all $t \geqslant 1$. As $x(t)$ is periodic, (1.10) is proved. $\square$

REMARK 1.2. - Suppose $f: \mathbf{R} \to \mathbf{R}$ is a continuous function for which $f(I) \subseteq I$, where $I = [D, C]$ is a compact interval. Define a new function $\tilde{f}$ by

$$\tilde{f}(x) = \begin{cases} f(D) & \text{if } x \leqslant D\,, \\ f(x) & \text{if } D \leqslant x \leqslant C\,, \\ f(C) & \text{if } x \geqslant C\,. \end{cases}$$

By Corollary 1.1 any non-constant periodic solution $x(t)$ of

$$(1.11)_\lambda \qquad\qquad \dot{x}(t) = -\lambda x(t) + \lambda \tilde{f}(x(t - 1))\,, \quad \lambda > 0$$

takes values only in the interior of $I_\infty = \bigcap_{n=0}^{\infty} \overline{\tilde{f}^n(\mathbf{R})} \subseteq [D, C]$ and hence satisfies

$$D < x(t) < C \quad \text{for all } t\,.$$

But then $x(t)$ is also a solution of $(1.1)_\lambda$. Thus if one only studies periodic solutions of $(1.1)_\lambda$ taking values in $[D, C]$, it suffices to look for periodic solutions of $(1.11)_\lambda$.

REMARK 1.3. - Suppose $f: I \to I$ is continuous, where $I = [D, C]$ is a compact interval. Further suppose that $I_\infty = \{a\}$ for some $a \in I$. It is easy then to see that for any $x_0 \in I$ one has $x_n \in f^n(I)$ and hence $\lim_{n \to \infty} x_n = a$, where x_n is defined by (1.2). What is not so obvious is the following converse.

PROPOSITION 1.2. - *Suppose* $f: I \to I$ *is continuous where* I *is a compact interval. Suppose further that there exists* $a \in I$ *such that*

$$(1.12) \qquad \lim_{n \to \infty} x_n = a$$

whenever $x_0 \in I$ *and* $x_n = f^n(x_0)$ *is the n-th iterate of* x_0 *by the map* f. *Then the set* $I_\infty = \bigcap_{n=0}^{\infty} f^n(I)$ *is simply* $I_\infty = \{a\}$.

PROOF. - First note that f has a unique fixed point in I, namely $x = a$. The existence of a fixed point is clear, as $f(I) \subseteq I$; that any fixed point x of f must in fact equal a follows from (1.12) upon iterating that point: $x = f^n(x) = x_n \to a$ implies $x = a$. A similar argument shows further that $x = a$ is the unique fixed point of the composed function $f \circ f$ in I. Thus

$$f(x) = x \in I \iff x = a,$$

and

$$f(f(x)) = x \in I \iff x = a.$$

Next observe that f maps the set I_∞ onto itself; this is clear from the definition (1.5) of I_∞. As I_∞ is a nonempty connected compact set, it must contain a fixed point of f, that is, the point $a \in I_\infty$. As before, denote

$$I_\infty = [D_\infty, C_\infty] \quad \text{where} \quad D_\infty \leqslant a \leqslant C_\infty.$$

We wish to prove that $D_\infty = C_\infty$.

As $x = a$ is the unique fixed point of f in I_∞, one has that

$$(1.13) \qquad \begin{cases} f(x) > x & \text{if } D_\infty \leqslant x < a, \\ \text{and} \\ f(x) < x & \text{if } a < x \leqslant C. \end{cases}$$

In the same way, for the composed function one has

$$(1.14) \quad \begin{cases} f(f(x)) > x & \text{if } D_\infty \leqslant x < a , \\ \text{and} \\ f(f(x)) < x & \text{if } a < x \leqslant C_\infty . \end{cases}$$

From (1.13) one has $f(x) \neq C_\infty$ if $a < x \leqslant C_\infty$. But $C_\infty \in I_\infty = f(I_\infty)$, so there exists some $\xi \in [D_\infty, a]$ such that $f(\xi) = C_\infty$. Similarly, there exists $\eta \in [a, C_\infty]$ with $f(\eta) = D_\infty$. Thus

$$\eta \in [a, C_\infty] \subseteq f([\xi, a])$$

so there exists $\zeta \in [\xi, a]$ such that $f(\zeta) = \eta$. But then $f(f(\zeta)) = D_\infty \leqslant \zeta$, so $\zeta \in [a, C_\infty]$ by (1.14). Thus $\zeta \in [\xi, a] \cap [a, C_\infty]$, implying $\zeta = a$, and

$$D_\infty = f(f(\zeta)) = f(f(a)) = a .$$

In a similar fashion one has $a = C_\infty$, thus proving that $C_\infty = D_\infty$.   □

A consequence of Propositions 1.1 and 1.2 is that if the point a attracts all orbits x_n of the discrete system (1.2) with initial condition $x_0 \in I$ for a compact interval I satisfying $f(I) \subseteq I$, then $x = a$ is an equilibrium solution of the differential-delay equation (1.1)$_\lambda$ which attracts solutions with initial conditions taking values only in I. More precisely, the following result holds.

COROLLARY 1.2. – *Let f, $I = [D, C]$ and a be as in Proposition 1.2. If $\varphi \in C[0, 1]$ satisfies*

$$D \leqslant \varphi(t) \leqslant C \quad \text{for all } t \in [0, 1]$$

and $\lambda > 0$, then one has for the solution of (1.1)$_\lambda$

$$\lim_{t \to \infty} x(t; \lambda, \varphi) = a .$$

REMARK 1.4. – One is tempted to relax the condition (1.12) in Proposition 1.2, and replace it with a condition such as

$$(1.15) \qquad \lim_{n \to \infty} \text{dist} (x_n, K) = 0$$

where $K \subseteq I$ is a compact interval, and « dist » denotes the distance from a point to a set. However, the natural conclusion, that $I_\infty \subseteq K$, is unfortunately false in general. For example, if f is a function mapping $I = [0, 1]$ onto itself, and satisfying

in addition that $f(x) = \frac{1}{2}$ whenever $\frac{1}{2} \leqslant x \leqslant 1$, then one easily sees that (1.15) holds with $K = [0, \frac{1}{2}]$. But $I_\infty = I$ is not a subset of K.

Whether the corresponding generalization of Corollary 1.2 holds is another matter. That is, if $\varphi(t) \in I$ for all $t \in [0, 1]$, does

$$\lim_{t \to \infty} \text{dist}\left(x(t; \lambda, \varphi), K\right) = 0 ?$$

This question remains open.

REMARK 1.5. – One may ask whether a generalization of Proposition 1.2 holds with (1.12), but for compact sets I other than intervals. This, unfortunately, is also false in general. For example, if S^1 denotes the unit circle in the complex
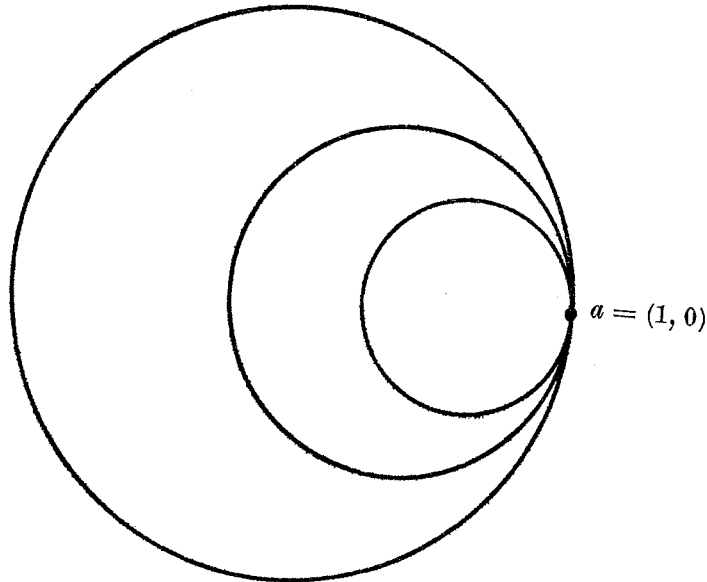
Fig. 4. The counterexample of Remark 1.5.

plane, define $f \colon S^1 \to S^1$ by $f(\exp [2\pi i\theta]) = \exp [2\pi i \sqrt{\theta}]$ for $0 \leqslant \theta \leqslant 1$. Then $f^n(x_0) \to$ $\to a = 1$ for any $x_0 \in S^1$, yet $\bigcap_{n=0}^{\infty} f^n(S^1) = S^1$ as f maps S^1 onto S^1.

A slightly more complicated example can be constructed on the closed unit disc D^2 in the plane. Note that D^2 can be written as a union of circles, any two of which have only the point $a = (1, 0)$ in common; see Figure 4. Map each circle onto itself by a map of the kind above. This gives a continuous map f of D^2 onto itself, for which $f^n(x_0) \to a$ for any $x_0 \in D^2$. Again $\bigcap_{n=0}^{\infty} f^n(D^2) = D^2$ so the conclusion of Proposition 1.2 fails.

Similar examples can be constructed on the sphere S^2 and many other spaces. The examples here and in Remark 1.4 thus indicate the hypotheses of Proposition 1.2 are, in some sense, best possible.

We shall be interested in periodic solutions of $(1.1)_\lambda$ which oscillate about a fixed point a of f. Without loss of generality we may take $a = 0$; for if $a \neq 0$ we may introduce $\bar{x} = x - a$ and define $\bar{f}(\bar{x}) = f(\bar{x} + a) - f(a)$. Then $\bar{f}(0) = 0$, and equation $(1.1)_\lambda$ is equivalent to the equation

$$\frac{d\bar{x}(t)}{dt} = -\lambda \bar{x}(t) + \lambda \bar{f}(\bar{x}(t-1)) .$$

Generally, then, we shall assume $f(0) = 0$. In contrast to Proposition 1.2 and Corollary 1.2 however, we shall *not* assume this fixed point is attractive. In fact, we will often require that $|f'(0)| > 1$, so that $x = 0$ repels iterates x_n. In addition, a *negative feedback* condition $xf(x) < 0$ will be assumed for certain values of x on either side of zero. Typically, this condition causes solutions of $(1.1)_\lambda$ to oscillate about $x = 0$.

We state precisely several hypotheses which f can satisfy. Conditions (H1) and (H2) will usually be assumed. In addition, condition (H3), which strengthens (H1), will occasionally be assumed.

(H1)   *The function* $f: \mathbf{R} \to \mathbf{R}$ *is continuous. There exist positive constants* A *and* B *such that*

$$f([-B, A]) \subseteq [-B, A]$$

and

$$xf(x) < 0 \quad \text{if } x \in [-B, A], \; x \neq 0 .$$

Further $f(x) = f(-B)$ *if* $x \leqslant -B$ *and* $f(x) = f(A)$ *if* $x \geqslant A$.

(H2)   *The function* $f: \mathbf{R} \to \mathbf{R}$ *is continuous, satisfies* $f(0) = 0$, *and is differentiable at* $x = 0$ *with*

$$-f'(0) \overset{\text{def}}{=} k > 1 .$$

Further, f *is monotone decreasing* [1] *on some neighborhood of* $x = 0$.

(H3)   *The function* $f: \mathbf{R} \to \mathbf{R}$ *satisfies* (H1). *In addition, there exist positive constants* a *and* b *satisfying* $a \leqslant A$ *and* $b \leqslant B$, *such that if* $x_0 \in (0, A]$ *and* $x_n = f^n(x_0)$ *is the n-th iterate of* x_0 *under* f, *for each* $n \geqslant 0$, *then*

$$\lim_{n \to \infty} x_{2n} = a \quad \text{and} \quad \lim_{n \to \infty} x_{2n+1} = -b .$$

---

[1] We say a function f is *monotone decreasing* on an interval I in case $x_1 < x_2$ implies $f(x_1) \geqslant f(x_2)$ and that f is *strictly decreasing* on I if $x_1 < x_2$ implies $f(x_1) > f(x_2)$, for $x_1, x_2 \in I$. We also make the analogous definitions of *monotone increasing* and *strictly increasing*.

If f satisfies all the conditions in (H1) except the last sentence (i.e., that $f(x)$ is constant on $(-\infty, -B]$ and on $[A, \infty)$), then we may define a new function $\tilde{f}$ by

$$\tilde{f}(x) = \begin{cases} f(-B) & \text{if } x \leqslant -B, \\ f(x) & \text{if } -B \leqslant x \leqslant A, \\ f(A) & \text{if } x \geqslant A. \end{cases}$$

Clearly $\tilde{f}$ does satisfy (H1). Remark 1.2 shows that every non-constant periodic solution of

$$\dot{x}(t) = -\lambda x(t) + \lambda \tilde{f}(x(t-1)), \quad \lambda > 0$$

satisfies $-B < x(t) < A$ for all t, hence is a periodic solution of (1.1)$_\lambda$. As these are the solutions we shall be interested in, we lose no generality by considering the modified function $\tilde{f}$ in place of f. That is, the final sentence of hypothesis (H1) essentially imposes no additional restriction on f for our purposes. To emphasize this, we state the following result.

PROPOSITION 1.3. – *Let f satisfy* (H1). *Then any periodic solution of equation* (1.1)$_\lambda$ *satisfies*

$$-B < x(t) < A \quad \text{for all } t.$$

Hypothesis (H3) implies that $f(a) = -b$ and $f(-b) = a$. Further, the set $\{-b, a\}$ attracts iterates of any nonzero $x_0 \in [-B, A]$ under the discrete dynamical system (1.2); compare this assumption with the hypotheses of Proposition 1.2. Observe that the iterates $x_n = f^n(x_0)$ alternate in sign, due to the negative feedback condition $xf(x) < 0$ of (H1). A function satisfying (H3) could typically arise in a parametrized family $f(x, \mu)$ in which the fixed point $x = 0$ underwent a period doubling bifurcation as the derivative $f_x(0, \mu)$ passed through -1, as described in the introduction.

We shall study a subclass of periodic solutions of (1.1)$_\lambda$.

DEFINITION 1.1. – A periodic solution $x(t)$ of (1.1)$_\lambda$ is called a *slowly oscillating periodic solution* if there exist numbers $q > 1$ and $\bar{q} > q + 1$ such that

$$x(0) = 0,$$

$$x(t) > 0 \quad \text{when } 0 < t < q,$$

$$x(t) < 0 \quad \text{when } q < t < \bar{q},$$

and

$$x(t + \bar{q}) = x(t) \quad \text{for all } t.$$
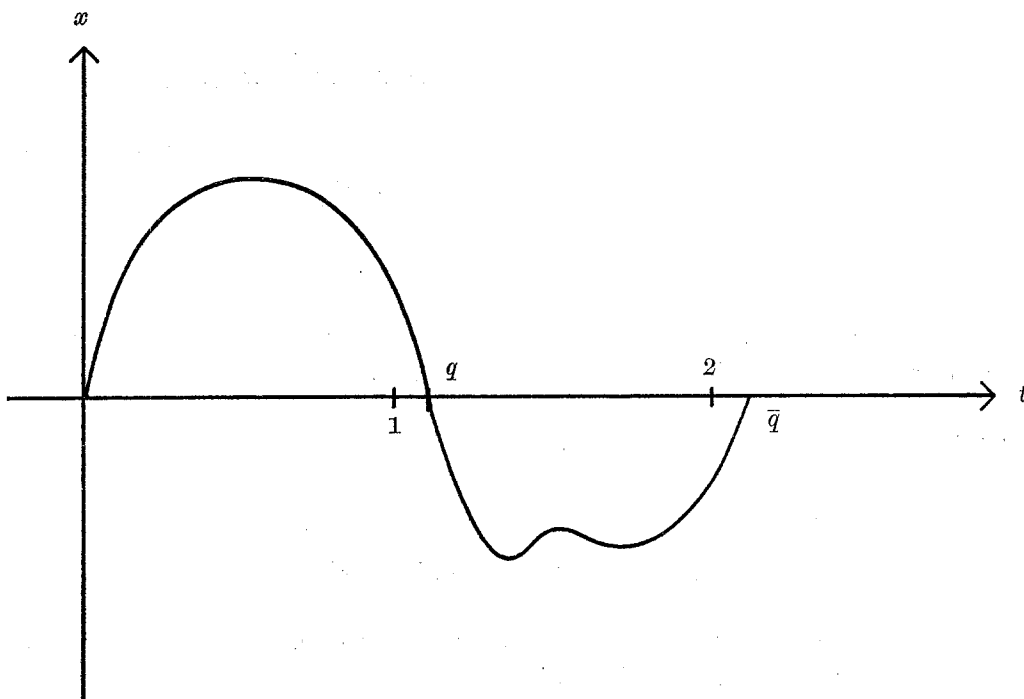
(Of course $x(q) = x(\bar{q}) = 0$.)

Fig. 5. A slowly oscillating periodic solution.

« Slowly » in Definition 1.1 refers to the fact that the separation of zeros of $x(t)$ is greater than the time lag. Note that $x(t)$ is assumed to repeat after two zeros. Also note that when f satisfies (H1), then all its zeros are simple, so that $\dot{x}(q) < 0$ and $\dot{x}(0) = \dot{x}(\bar{q}) > 0$. Figure 5 illustrates a slowly oscillating periodic solution.

We shall now convert the problem of finding slowly oscillating periodic solutions of equation $(1.1)_\lambda$ with hypothesis (H1) holding, to an equivalent problem of finding fixed points of a map Ψ_λ from a space K_λ to itself. Such fixed points will then be found by using topological techniques.

If $x(t)$ is a slowly oscillating periodic solution of $(1.1)_\lambda$ for some $\lambda > 0$, then one can see that $x(t)$ is uniquely determined by (λ, φ) where $\varphi = x|[0, 1]$. If the set K is defined by

$$K = \{\psi \in C[0, 1]: \psi(0) = 0 \text{ and } \psi(t) \geqslant 0 \text{ for all } t \in [0, 1]\}$$

then $(\lambda, \varphi) \in (0, \infty) \times K$. Actually, more can be said when (H1) holds. If $\bar{q}$ is the second positive zero of $x(t)$ (as in Definition 1.1), then $x(t + \bar{q}) = x(t)$, so that

$$\frac{d}{dt}\big(\exp[\lambda t]x(t)\big) = \lambda \exp[\lambda t]f\big(x(t-1)\big) = \lambda \exp[\lambda t]f\big(x(t+\bar{q}-1)\big) \geqslant 0 \quad \text{if } 0 \leqslant t \leqslant 1.$$

It follows that $\varphi \in K_\lambda$ where

(1.16) $K_\lambda = \{\psi \in K: \exp[\lambda t]\psi(t) \text{ is monotone increasing on } [0, 1]\}$.

Conversely, suppose $(\lambda, \varphi) \in (0, \infty) \times K_\lambda$ and let $x(t; \lambda, \varphi)$ be the solution of the initial value problem $(1.3)_\lambda$. If φ is not identically zero, then $\varphi(1) > 0$. In this case define

$$q = q(\lambda, \varphi) = \inf\{t > 1: x(t; \lambda, \varphi) = 0\}$$

to be the first zero of $x(t; \lambda, \varphi)$ in $(1, \infty)$, if such exists. From the negative feedback condition $xf(x) < 0$ of (H1), and because $\varphi \in K_\lambda$, one can easily show that

(1.17) $\dot{x}(q; \lambda, \varphi) < 0$.

If a set Γ is defined by

(1.18) $\Gamma = \{(\lambda, \varphi) \in (0, \infty) \times K: \varphi \in K_\lambda\}$

then (1.17) implies that the function $q(\lambda, \varphi)$ is defined on an open subset of Γ, and is continuous on this domain of definition.

If $q(\lambda, \varphi)$ is defined for some $(\lambda, \varphi) \in \Gamma$, then $(1.1)_\lambda$ and (H1) imply

$$\frac{d}{dt}\big(\exp[\lambda t]x(t; \lambda, \varphi)\big) \leqslant 0 \quad \text{if } q = q(\lambda, \varphi) \leqslant t \leqslant q + 1 .$$

Thus $x(t; \lambda, \varphi) < 0$ if $q < t \leqslant q + 1$. We may now define

$$\bar{q}(\lambda, \varphi) = \inf\{t > q(\lambda, \varphi) + 1: x(t; \lambda, \varphi) = 0\}$$

to be the next zero of $x(t; \lambda, \varphi)$ beyond $q(\lambda, \varphi)$, if such exists. As before, $\bar{q}$ is a simple zero of $x(t; \lambda, \varphi)$, and the function $\bar{q}(\lambda, \varphi)$ is defined on an open subset of Γ, and is continuous on this domain. Further, because $x(t; \lambda, \varphi) < 0$ if $q < t < \bar{q}$, one sees that

(1.19) $\frac{d}{dt}\big(\exp[\lambda t]x(t + \bar{q}; \lambda, \varphi)\big) \geqslant 0 \quad \text{if } 0 \leqslant t \leqslant 1$.

Equation (1.19) allows us to define a map $\Psi_\lambda: K_\lambda \to K_\lambda$ for each $\lambda > 0$ as follows. If $\varphi \in K_\lambda$ and $\bar{q}(\lambda, \varphi)$ is defined, then set

(1.20) $(\Psi_\lambda\varphi)(t) = x(t + \bar{q}(\lambda, \varphi); \lambda, \varphi) \quad \text{for } 0 \leqslant t \leqslant 1$.

If $\bar{q}(\lambda, \varphi)$ is undefined (in particular if φ is the zero function), then set

$$(\Psi_\lambda\varphi)(t) = 0 \quad \text{for } 0 \leqslant t \leqslant 1 .$$

More or less standard arguments show that Ψ_λ is a continuous compact map from K_λ into itself. In fact if we define $\Psi(\lambda, \varphi) = \Psi_\lambda(\varphi)$, then $\Psi: \Gamma \to K$ is continuous and compact. One sees that under hypothesis (H1), there is a one-to-one correspondence between slowly oscillating periodic solutions of $(1.1)_\lambda$ and nontrivial fixed points of Ψ_λ. That is, $x(t)$ is a slowly oscillating periodic solution if and only if $\Psi_\lambda(\varphi) = \varphi$ where $\varphi \in K_\lambda - \{0\}$ is the initial value $\varphi = x|[0, 1]$.

Let us define then

$$(1.21) \qquad \Sigma = \{(\lambda, \varphi) \in (0, \infty) \times K: \varphi \in K_\lambda - \{0\} \text{ and } \Psi_\lambda(\varphi) = \varphi\}$$

which represents the set of all such solutions. The main result of this section, Theorem 1.1 below, asserts that Σ contains a continuum Σ_0 which extends from a Hopf bifurcation point $(\lambda_0, 0)$ throughout the range $\lambda_0 < \lambda < \infty$. Before stating Theorem 1.1 we must define the bifurcation point λ_0.

If $f'(0)$ is assumed to exist, and equation $(1.1)_\lambda$ is linearized about $x = 0$, one obtains

$$(1.22)_\lambda \qquad \dot{x}(t) = - \lambda x(t) - \lambda k x(t - 1), \quad \text{where } k = - f'(0).$$

If one seeks a solution $x(t) = \exp[\zeta t]$ (for ζ complex) of $(1.22)_\lambda$, then one is led to the characteristic equation

$$(1.23) \qquad \zeta = - \lambda - \lambda k \exp[- \zeta].$$

If one assumes $k > 1$ (as in (H2) for example), then the results of the appendix (see also [46]) show that (1.23) has a solution

$$\zeta = i\nu_0$$

with real part zero, for some $\lambda = \lambda_0 > 0$. The values of ν_0 and the parameter λ_0 are unique under the condition

$$\nu_0 \in \left(\frac{\pi}{2}, \pi\right).$$

Indeed, ν_0 and λ_0 are obtained by solving

$$(1.24) \qquad \cos \nu_0 = \frac{-1}{k}, \qquad \nu_0 \in \left(\frac{\pi}{2}, \pi\right), \qquad \lambda_0 = \frac{\nu_0}{\sqrt{k^2 - 1}}.$$

Observe that because $\nu_0 < \pi$, the eigensolution $x(t) = \sin \nu_0 t$ of $(1.22)_{\lambda_0}$ has consecutive zeros spaced a distance $\pi/\nu_0 > 1$ apart, hence is a slowly oscillating periodic solution of $(1.22)_{\lambda_0}$. The parameter value λ_0 may be characterized as the only value of $\lambda > 0$ for which $(1.22)_\lambda$ has such a solution; this is a consequence of the results

of the appendix. Further, it is proved there that when $\lambda = \lambda_0$ the solutions $\zeta = \pm i\nu_0$ are simple roots of the characteristic equation $(1.23)_\lambda$ and they cross the imaginary ζ-axis transversally as λ increases; also, when $\lambda = \lambda_0$ equation $(1.23)_\lambda$ has no other roots on the imaginary axis. These results and the Hopf bifurcation theorem in [25] show that if $f(x)$ is C^1 near $x = 0$, then one obtains a local Hopf bifurcation of periodic solutions of $(1.1)_\lambda$ from $(\lambda, x) = (\lambda_0, 0)$. One expects that these solutions are slowly oscillating periodic solutions, and so the bifurcation point $(\lambda_0, 0)$ should belong to the closure of Σ in $(0, \infty) \times K$. This is indeed the case, as the following theorem shows. In fact there is a global Hopf bifurcation from $(\lambda_0, 0)$, giving rise to an unbounded continuum $\Sigma_0 \subseteq \Sigma$.

THEOREM 1.1. – *Assume that f satisfies* (H1) *and is differentiable at $x = 0$ with $f'(0) = -k$, where $k > 1$. Let the set $\Sigma \subseteq (0, \infty) \times K$ be defined by (1.21) and $\lambda_0 > 0$ be given by (1.24). Then one has the following.*

(i) *There exists $\delta > 0$ such that if $(\lambda, \varphi) \in \Sigma$, then $\lambda \geqslant \delta$ and $\|\varphi\| < A$ (with A as in hypothesis* (H1) *and $\| \ \|$ denoting the sup norm).*

(ii) *The closure $\overline{\Sigma}$ of Σ in $(0, \infty) \times K$ is*

$$\overline{\Sigma} = \Sigma \cup \{(\lambda_0, 0)\} \ .$$

(iii) *Let $\Sigma_0 \subseteq \overline{\Sigma}$ be the maximal connected component of $\overline{\Sigma}$ containing $(\lambda_0, 0)$. Then Σ_0 is an unbounded subset of $(0, \infty) \times K$.*

(iv) *For each $\lambda > \lambda_0$ there exists a slowly oscillating periodic solution $x(t)$ of $(1.1)_\lambda$ such that $(\lambda, \varphi) \in \Sigma_0$, where $\varphi = x|[0, 1]$, and $-B < x(t) < A$ for all t.*

REMARK 1.6. – As noted in the introduction, the fact that equation $(1.1)_\lambda$ has a slowly oscillating periodic solution for each $\lambda > \lambda_0$ was proved in [24], and the local Hopf bifurcation was studied in [46].

Before giving the proof of Theorem 1.1, we shall present several lemmas. The first of these is used to extend the domain of definition of Ψ from Γ to all of $(0, \infty) \times K$. As there are difficulties in trying to use directly the definition (1.20) of Ψ to do this, a different approach is needed.

LEMMA 1.1. – *There exists a continuous retraction ϱ of $(0, \infty) \times K$ onto Γ (where Γ is given by (1.18)). This retraction has the form*

$$(1.25) \qquad \varrho(\lambda, \varphi) = (\lambda, \varrho_\lambda(\varphi)) \qquad \text{for all } (\lambda, \varphi) \in (0, \infty) \times K$$

where $\varrho_\lambda^{-1}(0) = \{0\}$.

PROOF. - If K_λ is defined by (1.16), note that for any real numbers α and β the sets K_α and K_β are homeomorphic by a homeomorphism $h_{\alpha,\beta}: K_\alpha \to K_\beta$ defined by

$$(1.26) \qquad (h_{\alpha,\beta}\varphi)(t) = \exp\left[(\alpha - \beta)t\right]\varphi(t), \qquad 0 \leqslant t \leqslant 1.$$

The inverse of $h_{\alpha,\beta}$ is $h_{\beta,\alpha}$. Note that the formula (1.26) in fact defined an extension of $h_{\alpha,\beta}$ to a homeomorphism from K onto K; we denote the extended maps by $H_{\alpha,\beta}$.

One can easily check that the map $\varrho_0: K \to K_0$ defined by

$$(\varrho_0\varphi)(t) = \max_{0 \leqslant s \leqslant t} \varphi(s) \qquad \text{for } 0 \leqslant t \leqslant 1$$

is a continuous retraction of K onto K_0 and that $\varrho_0^{-1}(0) = \{0\}$. Define a continuous retraction of K onto K_λ by

$$\varrho_\lambda(\varphi) = h_{0,\lambda}\Big(\varrho_0\big(H_{\lambda,0}(\varphi)\big)\Big).$$

Then (1.25) defined a continuous retraction $\varrho: (0, \infty) \times K \to \Gamma$; observe that $\varrho_\lambda^{-1}(0) = \{0\}$. $\square$

In our case the existence of ϱ_0 is easy. More generally, the existence of a retraction from a Banach space Y onto an arbitrary non-empty closed convex subset of Y follows from a deep theorem of DUGUNDJI [15].

With the aid of Lemma 1.1 define an extension G of Ψ by

$$G(\lambda, \varphi) = \Psi(\varrho(\lambda, \varphi)),$$

and define $G_\lambda: K \to K_\lambda$ by

$$G_\lambda(\varphi) = G(\lambda, \varphi).$$

One easily sees that the set of fixed points of G_λ is precisely the same as the set of fixed points of Ψ_λ, and hence

$$\Sigma = \{(\lambda, \varphi) \in (0, \infty) \times K: G_\lambda(\varphi) = \varphi \text{ and } \varphi \text{ is not the zero function}\}.$$

To prove that Σ_0 is unbounded (in part (iii) of Theorem 1.1) we shall use Theorem 1.2 in [49]. First we need to recall some definitions. Recall that if Y is a topological space and $g: Y \to Y$ a continuous map with fixed point y_0, then y_0 is an *attractive fixed point* if there exists an open neighborhood U of y_0 such that for every open neighborhood V of y_0, there exists an integer $n = n(V)$ such that $f^j(y) \in V$ for all $j \geqslant n$ and $y \in U$. The fixed point y_0 is called an *ejective fixed point* if there exists an open neighborhood W of y_0 such that for every $y \in W - \{y_0\}$ there is an integer $n = n(y)$ such that $g^n(y) \notin W$.

Given the fact (proved in the appendix) that the characteristic equation (1.23) has only roots with negative real part if $0 < \lambda < \lambda_0$, the following lemma is a standard result (see [25]) in the stability theory of differential-delay equations. As a special case, it implies that $0 \in K$ is an attractive fixed point of $G_\lambda \colon K \to K$ if $0 < \lambda < \lambda_0$.

LEMMA 1.2. – *Assume that f satisfies* (H1) *and is differentiable at $x = 0$ with $f'(0) = -k$. Assume that $0 < \lambda < \lambda_0$ with λ_0 as in* (1.24) *if $k > 1$ and $\lambda_0 = \infty$ if $0 \leqslant k \leqslant 1$. Then there exists $\delta > 0$ such that for every $\varkappa > 0$ there is a number $T_\varkappa \geqslant 0$ such that*

$$\sup_{t \geqslant T_\varkappa} |x(t; \lambda, \varphi)| < \varkappa$$

whenever $\varphi \in C[0, 1]$ satisfies $\|\varphi\| < \delta$.

If, on the other hand $\lambda > \lambda_0$, then 0 is an ejective fixed point of $\Psi_\lambda \colon K_\lambda \to K_\lambda$: This is proved by HADELER and TOMIUK in Lemma 11 of [24]. Some checking is necessary because Hadeler and Tomiuk use different parameters from ours, but this is mostly a question of notation.

Next, recall that our retraction $\varrho_\lambda \colon K_\lambda \to K_\lambda$ is chosen so that $\varrho_\lambda^{-1}(0) = \{0\}$. Using this fact and the fact that for any $\varphi \in K$ one has $G_\lambda^n(\varphi) = \Psi_\lambda^n(\varrho_\lambda(\varphi))$, one sees that ejectivity of Ψ_λ implies ejectivity of G_λ, so one obtains the following result.

LEMMA 1.3 (See Lemma 11 of [24]). – *Assume that f satisfies* (H1) *and is differentiable at $x = 0$ with $f'(0) = -k$, where $k > 1$. Assume that $\lambda > \lambda_0$ with λ_0 as in* (1.24). *Then $0 \in K$ is an ejective fixed point of G_λ.*

The next lemma gives most of part (i) of Theorem 1.1.

LEMMA 1.4. – *Assume that f satisfies* (H1) *and is differentiable at $x = 0$. Then there exists $\delta > 0$ such that if $x(t)$ is a slowly oscillating periodic solution of* $(1.1)_\lambda$ *for some $\lambda > 0$, then $\lambda \geqslant \delta$. Hence*

$$\Sigma \subseteq [\delta, \infty) \times K .$$

PROOF. – We know from Proposition 1.3 that every slowly oscillating periodic solution $x(t)$ of $(1.1)_\lambda$ satisfies

$$-B < x(t) < A \quad \text{for all } t$$

where A and B are as in (H1). Because f is differentiable at the origin, there exists a constant Ω such that

$$|f(x)| \leqslant \Omega |x| \quad \text{if} \quad -B \leqslant x \leqslant A .$$

Let $x(t)$ be a slowly oscillating periodic solution of $(1.1)_\lambda$ for some $\lambda > 0$ with first zero q and second zero $\bar{q}$, and set

$$M_+ = \max_{0 \leqslant t \leqslant q} x(t) \quad \text{and} \quad M_- = \max_{q \leqslant t \leqslant \bar{q}} |x(t)| .$$

The assumptions of f imply from $(1.1)_\lambda$ that $\dot{x}(t) \leqslant 0$ on $[1, q]$ and $\dot{x}(t) \geqslant 0$ on $[q + 1, \bar{q}]$, so

(1.27) $$M_+ = \max_{0 \leqslant t \leqslant 1} x(t) \quad \text{and} \quad M_- = \max_{q \leqslant t \leqslant q+1} |x(t)| ,$$

and periodicity gives

$$M_+ = \max_{\bar{q} \leqslant t \leqslant \bar{q}+1} x(t) .$$

Integrating $(1.1)_\lambda$ from q to t gives

$$x(t) = \int_q^t \lambda \exp [\lambda(s - t)] f(x(s - 1)) \, ds$$

which implies

$$|x(t)| \leqslant (1 - \exp [\lambda(q - t)]) \Omega M_+ \quad \text{if } q \leqslant t \leqslant q + 1$$

so that

(1.28) $$M_- \leqslant (1 - \exp [-\lambda]) \Omega M_+ .$$

A similar argument shows $M_+ \leqslant (1 - \exp [-\lambda]) \Omega M_-$, and combining this with (1.28) gives

(1.29) $$M_+ \leqslant (1 - \exp [-\lambda])^2 \Omega^2 M_+ .$$

But if $\delta > 0$ is chosen so that $(1 - \exp [-\delta])^2 \Omega^2 \leqslant 1$, then (1.29) is impossible for $0 < \lambda < \delta$ and $M_+ > 0$. $\square$

It will be convenient to obtain an *a priori* bound on the minimal period of any slowly oscillating periodic solution; the following lemma does that.

LEMMA 1.5. – *Assume the hypotheses of Lemma 1.4. There exists a number $\bar{Q} > 0$ such that if $x(t)$ is a slowly oscillating periodic solution of $(1.1)_\lambda$ for some $\lambda > 0$, then the minimal period $\bar{q}$ of $x(t)$ satisfies*

$$\bar{q} \leqslant \bar{Q} .$$

The number $\bar{Q}$ depends only on f, and not on λ.

PROOF. – Let the notation be as in the proof of Lemma 1.4, and suppose $x(t)$ is a slowly oscillating periodic solution of $(1.1)_\lambda$ for some $\lambda > 0$. Lemma 1.4 therefore implies $\lambda \geqslant \delta$.

From $(1.1)_\lambda$ one has

$$\frac{d}{dt}\left(\exp\left[\lambda(t-1)\right]x(t)\right) \leqslant 0 \quad \text{if } 1 \leqslant t \leqslant q$$

and hence

$$(1.30) \qquad x(t) \leqslant \exp\left[-\lambda(t-1)\right]x(1) \leqslant \exp\left[-\delta(t-1)\right]M_+ \quad \text{if } 1 \leqslant t \leqslant q.$$

Equation (1.30) implies

$$0 \leqslant x(t) \leqslant \exp\left[-\delta(q-2)\right]M_+ \quad \text{if } q-1 \leqslant t \leqslant q;$$

using this fact and integrating $(1.1)_\lambda$ from q to $t \in [q, q+1]$ gives

$$\exp\left[\lambda(t-q)\right]|x(t)| = \int_q^t \lambda \exp\left[\lambda(s-q)\right]|f(x(s-1))|\, ds \leqslant$$
$$\leqslant \left(\exp\left[\lambda(t-q)\right]-1\right)\exp\left[-\delta(q-2)\right]\Omega M_+.$$

This and (1.27) imply

$$(1.31) \qquad M_- \leqslant \exp\left[-\delta(q-2)\right]\Omega M_+.$$

A similar argument shows $M_+ \leqslant \exp\left[-\delta(\bar{q}-q-2)\right]\Omega M_-$, and combining this with (1.31) gives

$$M_+ \leqslant \exp\left[-\delta(\bar{q}-4)\right]\Omega^2 M_+.$$

Because $M_+ > 0$ one has $\exp\left[-\delta(\bar{q}-4)\right]\Omega^2 \geqslant 1$, hence

$$\bar{q} \leqslant \bar{Q} = \frac{2\log\Omega}{\delta} + 4$$

as required.   □

Our last lemma shows that $\Sigma \subseteq (0,\infty) \times K$ remains bounded away from $(0,\infty) \times \{0\}$ except possibly at the point $(\lambda_0, 0)$.

LEMMA 1.6. – *Assume the hypotheses of Lemma 1.4, set* $k = -f'(0)$, *and let* $J \subseteq (0,\infty)$ *be any compact set. Further, if* $k > 1$ *let* λ_0 *be as in* (1.24) *and assume that* $\lambda_0 \notin J$. *Then there exists* $\varkappa = \varkappa(J) > 0$ *such that if* $x(t)$ *is a slowly oscillating periodic solution of* $(1.1)_\lambda$ *for some* $\lambda \in J$, *then* $\max_t |x(t)| > \varkappa$. *Consequently one has*

$$(1.32) \qquad \bar{\Sigma} - \Sigma \subseteq \{(\lambda_0, 0)\}.$$

PROOF. – Suppose for some J there is no $\varkappa$. Then it is easy to show there is a sequence of slowly oscillating periodic solutions $x_n(t)$ of $(1.1)_{\lambda_n}$ for some sequence $\lambda_n \in J$, such that $\max\limits_t |x_n(t)| \to 0$ as $n \to \infty$. Define $y_n(t)$ by

$$y_n(t) = \frac{x_n(t)}{\|x_n\|}$$

where $\|x_n\| = \max\limits_t |x_n(t)|$; then

$$(1.33) \qquad \dot{y}_n(t) = -\lambda_n y_n(t) - \lambda_n k y_n(t-1) + \lambda_n R(y_n(t-1), \|x_n\|)$$

where R is the continuous function defined by

$$(1.34) \qquad R(y, \xi) = \begin{cases} ky + \xi^{-1}f(\xi y) & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Also, let q_n and $\bar{q}_n$ denote the first and second zeros of $x_n(t)$.

By taking subsequences one can now assume

$$(1.35) \qquad \begin{cases} \lambda_n \to \lambda \in J \\ q_n \to q \quad \text{and} \quad \bar{q}_n \to \bar{q}, \\ y_n(t) \to y(t) \quad \text{and} \quad \dot{y}_n(t) \to \dot{y}(t) \quad \text{uniformly on compact sets} \end{cases}$$

for some λ, q and $\bar{q}$, and some continuously differentiable function $y \colon \boldsymbol{R} \to \boldsymbol{R}$. These claims follow from Lemma 1.5, from an application of Ascoli's theorem to $y_n(t)$ upon observing in (1.33) that $y_n(t)$ and $\dot{y}_n(t)$ are both uniformly bounded, and by taking the limit in (1.33) where one uses $\|x_n\| \to 0$. In fact, one further sees that

$$(1.36) \qquad \dot{y}(t) = -\lambda y(t) - \lambda k y(t-1) \quad \text{for all } t,$$

$$(1.37) \qquad y(t) \geqslant 0 \quad \text{on } (0, q) \qquad \text{and} \qquad y(t) \leqslant 0 \quad \text{on } (q, \bar{q}),$$

$$(1.38) \qquad y(t + \bar{q}) = y(t) \quad \text{for all } t,$$

$$(1.39) \qquad q \geqslant 1 \quad \text{and} \quad \bar{q} - q \geqslant 1,$$

and

$$\max\limits_t |y(t)| = 1.$$

The standard theory for linear differential-delay equations [1, 16, 25] implies that (1.36) cannot have a periodic solution unless the characteristic equation (1.23) has a root with real part zero. It is proved in the appendix that if this is so, and $\lambda > 0$ and $k \geqslant 0$ (as is the case here), then necessarily $k > 1$ and $\lambda = \lambda_m$ for some $m \geqslant 0$ where

$$\lambda_m = \frac{\nu_m}{\sqrt{k^2 - 1}}, \qquad \nu_m = \nu_0 + 2\pi m$$

with $\nu_0 \in (\pi/2, \pi)$ as in (1.24). And further, for $\lambda = \lambda_m$ the only roots of the characteristic equation (1.23) on the imaginary axis are a pair of simple complex conjugate roots

$$\zeta = \pm i\nu_m .$$

Thus, the theory of linear differential-delay equations implies that for $\lambda = \lambda_m$ the only periodic solutions of (1.36) are linear combinations of $\sin \nu_m t$ and $\cos \nu_m t$. And formulas (1.37), (1.38) and (1.39) imply that $y(t)$ is a multiple of $\sin \nu_m t$, that $\nu_m \leqslant \pi$, hence $m = 0$. But then $\lambda = \lambda_0 \notin J$, contradicting (1.35).

Because Σ consists of those (λ, φ) for which $\lambda \geqslant \delta$ and $G_\lambda(\varphi) = \varphi \neq 0$, it follows that $\bar{\Sigma} - \Sigma \subseteq (0, \infty) \times \{0\} \subseteq (0, \infty) \times K$. And the above results show further the inclusion (1.32). □

We now present the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. – (i) The claim that $\lambda \geqslant \delta$ is simply Lemma 1.4. If $(\lambda, \varphi) \in \Sigma$ then $\varphi = x|[0, 1]$ where $x(t) = x(t; \lambda, \varphi)$ is a slowly oscillating periodic solution of $(1.1)_\lambda$. By Proposition 1.3 one has $-B < x(t) < A$ for all t, hence $0 \leqslant \varphi(t) < A$ if $0 \leqslant t \leqslant 1$, hence $\|\varphi\| < A$.

(ii) and (iii) These parts of the theorem are a specialization of Theorem 1.2 of [49] to our situation. Let

$$B_\varkappa = \{\varphi \in K \colon \|\varphi\| < \varkappa\} .$$

By Lemma 1.6, for each $\lambda > 0$ with $\lambda \neq \lambda_0$ there exists $\varkappa(\lambda)$ such that the map G_λ has no nontrivial fixed points in $\bar{B}_{\varkappa(\lambda)}$. Hence the fixed point index $i_K(G_\lambda, B_{\varkappa(\lambda)})$ is defined (see [50] for a summary of the properties of the fixed point index). To apply Theorem 1.2 of [49] here it suffices to show

$$
\begin{array}{ll}
(1.40) \\
(1.41)
\end{array}
\qquad
i_K(G_\lambda, B_{\varkappa(\lambda)}) =
\begin{cases}
1 & \text{if } 0 < \lambda < \lambda_0 , \\
0 & \text{if } \lambda > \lambda_0 .
\end{cases}
$$

But Lemma 1.4 in [49] implies (1.40) because 0 is an attractive fixed point of G_λ (by Lemma 1.2 above); and Corollary 1.2 in [47] (also Theorem 1 in [50]) implies (1.41) because 0 is an ejective fixed point of G_λ (by Lemma 1.3 above).

(iv) As $\Sigma_0 \subseteq (0, \infty) \times K$ is unbounded, and each $(\lambda, \varphi) \in \Sigma_0$ satisfies (i) of this theorem, and $(\lambda_0, 0) \in \Sigma_0$, one sees there exists $(\lambda, \varphi) \in \Sigma_0$ for each $\lambda > \lambda_0$. And as noted, each $(\lambda, \varphi) \in \Sigma$ with $\varphi \neq 0$ corresponds to a slowly oscillating periodic solution of $(1.1)_\lambda$. □

If the function $f(x)$ is odd, it is natural to look for slowly oscillating periodic solutions $x(t)$ of $(1.1)_\lambda$ such that

$$(1.42) \qquad\qquad x(t + q) = - x(t) \qquad \text{for all } t$$

where q is the first zero of $x(t)$. Following SAUPE [56, 57], we shall call such a solution an *S-solution*. If f is odd and satisfies (H1), then one has $B = A$. For $\varphi \in K_\lambda$ and $\lambda > 0$, define $S_\lambda(\varphi)$ by

$$(S_\lambda \varphi)(t) = -x(t + q(\lambda, \varphi); \lambda, \varphi) \quad \text{for } 0 \leqslant t \leqslant 1$$

if $q(\lambda, \varphi)$ is defined, and

$$(S_\lambda \varphi)(t) = 0 \quad \text{for } 0 \leqslant t \leqslant 1$$

if $q(\lambda, \varphi)$ is undefined or if $\varphi = 0$. The S-solutions thus correspond to nonzero fixed points of S_λ. Define

$$S = \{(\lambda, \varphi) \in (0, \infty) \times K: \varphi \in K_\lambda - \{0\} \text{ and } S_\lambda(\varphi) = \varphi\}.$$

By suitably modifying the lemmas leading up to the proof of Theorem 1.1, one can prove the following analog of this theorem for S-solutions.

THEOREM 1.2. – *Assume that f is as in Theorem* 1.1. *In addition, assume that f is an odd function, so that $B = A$ in* (H1). *Then the conclusions of Theorem* 1.1 *hold with* Σ, Σ_0, *and « slowly oscillating periodic solution » being replaced with* S, S_0, *and « S-solution » respectively.*

2. – Global continua of rapidly oscillating periodic solutions.

Consider again the characteristic equation

$$(2.1) \qquad \qquad \zeta + \lambda + \lambda k \exp[-\zeta] = 0$$

of the linear equation $(1.22)_\lambda$; we assume that $k > 1$, and that $\lambda \neq 0$ is real but not necessarily positive. It is proved in the appendix (see also [46]) that (2.1) has a solution ζ with real part zero if and only if $\lambda = \lambda_m$ for some integer m, where

$$(2.2) \qquad \qquad \lambda_m = \frac{\nu_0 + 2\pi m}{\sqrt{k^2 - 1}}$$

and ν_0 satisfies (1.24). Moreover, if $\lambda = \lambda_m$ then equation (2.1) has exactly two solutions with zero real part, namely

$$\zeta = \pm i\nu_m \quad \text{where } \nu_m = \nu_0 + 2\pi m$$

and that both of these are simple roots of (2.1). Finally, it is shown that for λ near λ_m there is a unique solution $\zeta = \zeta_m(\lambda)$ of (2.1) near $i\nu_m$, that $\zeta_m(\lambda)$ varies analytically

as a function of λ, and that $\zeta_m(\lambda)$ together with its complex conjugate $\overline{\zeta_m(\lambda)}$ cross the imaginary axis transversally. In fact one has

$$\zeta_m(\lambda_m) = i\nu_m \quad \text{and} \quad \text{sgn } (\lambda_m) \text{ Re } \zeta_m'(\lambda_m) > 0$$

where $\zeta_m'(\lambda)$ denotes the derivative of $\zeta_m(\lambda)$ with respect to λ.

It follows from the above facts and from the Hopf bifurcation theorem for differential-delay equations (see [25]) that the nonlinear equation

$$\dot{x}(t) = -\lambda x(t) + \lambda f(x(t-1))$$

has a one-parameter family of periodic solutions near $(\lambda, x) = (\lambda_m, 0)$, where f is assumed to be C^1 in a neighborhood of $x = 0$ and satisfy $f(0) = 0$ and $f'(0) = -k$. A basic question is whether this family, which is given only locally by the Hopf bifurcation theorem, can be extended to an unbounded connected set of periodic solutions. For $m = 0$ such an extension is provided by the set Σ_0 obtained in Theorem 1.1. We shall reduce the problem for general m to the case $m = 0$ by employing a change of variables which can be found in [34] and which is attributed to K. L. Cooke.

Consider a parametrized family of differential-delay equations of the form

$$(2.3)_\lambda \qquad \dot{x}(t) = \lambda g\big(x(t-N_1), x(t-N_2), ..., x(t-N_n)\big)$$

where $\lambda \in R$ and each N_j is an integer. Suppose for some λ that $x(t)$ is a periodic solution of equation $(2.3)_\lambda$ of period p. Fix an integer m and define a function $\bar{x}(t)$ and real number $\bar{\lambda}$ by

$$\bar{x}(t) = x(\omega t) \quad \text{and} \quad \bar{\lambda} = \lambda\omega$$

where ω is given by

$$\omega = mp + 1.$$

A simple calculation now shows that $\bar{x}(t)$ satisfies

$$\frac{d\bar{x}(t)}{dt} = \bar{\lambda} g\big(\bar{x}(t-N_1), \bar{x}(t-N_2), ..., \bar{x}(t-N_n)\big).$$

That is, $\bar{x}(t)$ is a solution of the differential equation $(2.3)_{\bar{\lambda}}$ with the new parameter value $\bar{\lambda}$. Also, if $\omega \neq 0$ then $\bar{x}(t)$ is periodic with period $p/|\omega|$, and if p is the minimal period of $x(t)$ then $p/|\omega|$ is the minimal period of $\bar{x}(t)$.

We shall use this change of variables to obtain branches of « rapidly oscillating » periodic solutions from the branch Σ_0 of slowly oscillating periodic solutions. To do this we need the following lemma.

LEMMA 2.1. – *Assume that f satisfies* (H1) *and is differentiable at $x = 0$ with* $f'(0) = -k$, *where* $k > 1$. *Let* Σ *be as in Theorem* 1.1. *For each* $(\lambda, \varphi) \in \Sigma$ *define* $\bar{q}(\lambda, \varphi)$ *to be the minimal period of the slowly oscillating periodic solution* $x(t; \lambda, \varphi)$ *of equation* $(1.1)_\lambda$. *In addition, set*

$$\bar{q}(\lambda_0, 0) = \frac{2\pi}{\nu_0}$$

where λ_0 *and* ν_0 *are as in* (1.24), *so that* $\bar{q}$ *is a well-defined function*

$$\bar{q}: \bar{\Sigma} = \Sigma \cup \{(\lambda_0, 0)\} \to (0, \infty) .$$

Then $\bar{q}$ *is a continuous function on the space* $\bar{\Sigma}$.

PROOF. – As noted in Section 1, if $(\lambda, \varphi) \in \Sigma$ then the slowly oscillating periodic solution $x(t; \lambda, \varphi)$ satisfies $\dot{x}(\bar{q}(\lambda, \varphi); \lambda, \varphi) > 0$, and this implies the function $\bar{q}(\lambda, \varphi)$ is continuous at (λ, φ). Therefore, it remains to prove that $\bar{q}$ is continuous at the point $(\lambda_0, 0) \in \bar{\Sigma} - \Sigma$.

Suppose there exists a sequence $(\sigma_n, \varphi_n) \in \Sigma$ such that $(\sigma_n, \varphi_n) \to (\lambda_0, 0)$ but $\bar{q}(\sigma_n, \varphi_n)$ does not approach $2\pi/\nu_0$. By taking a subsequence and using Lemma 1.5 to bound $\bar{q}(\sigma_n, \varphi_n)$ one can assume without loss that

$$(2.4) \qquad q(\sigma_n, \varphi_n) \to q_0 \quad \text{and} \quad \bar{q}(\sigma_n, \varphi_n) \to \bar{q}_0 \neq \frac{2\pi}{\nu_0}$$

where, as in Section 1, $q(\lambda, \varphi)$ denotes the first positive zero of $x(t; \lambda, \varphi)$. The limits q_0 and $\bar{q}_0$ satisfy $q_0 \geqslant 1$ and $\bar{q}_0 \geqslant 1 + q_0$. Denote

$$x_n(t) = x(t; \sigma_n, \varphi_n) \quad \text{and} \quad \|x_n\| = \max_t |x_n(t)| ,$$

note that $\|x_n\| \to 0$, and set $y_n(t) = x_n(t)/\|x_n\|$. As in the proof of Lemma 1.6 one may take limits of some subsequence of $y_n(t)$ to obtain a C^1 function $y(t)$ satisfying

$$(2.5) \qquad \dot{y}(t) = -\lambda_0 y(t) - \lambda_0 k y(t-1) \quad \text{for all } t ,$$

$$y(t) \geqslant 0 \quad \text{on } (0, q_0) \quad \text{and} \quad y(t) \leqslant 0 \quad \text{on } (q_0, \bar{q}_0) ,$$

$$y(t + \bar{q}_0) = y(t) \quad \text{for all } t ,$$

and

$$\max_t |y(t)| = 1 .$$

Clearly, the minimal period of the function $y(t)$ is $\bar{q}_0$.

When $\lambda = \lambda_0$ the only roots of the characteristic equation (2.1) on the imaginary axis are the simple roots $\zeta = \pm i\nu_0$. Therefore, by the theory of linear differential-

delay equations the solution $y(t)$ of (2.5) must have the form $y(t) = K \sin (\nu_0 t + \theta)$ for some $K \neq 0$ and θ, and so the minimal period $\bar{q}_0$ of $y(t)$ equals $2\pi/\nu_0$. This, however, contradicts (2.4). $\square$

Assume now that f and $\Sigma_0 \subseteq \Sigma$ are as in Theorem 1.1. Fix an integer m (which may be negative) and for each $(\lambda, \varphi) \in \Sigma_0$ define, as above $\bar{x}(t) = x(\omega t)$ and $\bar{\lambda} = \lambda \omega$ where

$$\omega = m\bar{q}(\lambda, \varphi) + 1 .$$

Let $\tilde{\varphi} = \bar{x}|[0, 1]$ denote the initial condition of $\bar{x}(t)$ in $C[0, 1]$ and define a map

$$\Phi_m: \Sigma_0 \to \mathbf{R} \times C[0, 1]$$

by setting

$$\Phi_m(\lambda, \varphi) = (\bar{\lambda}, \tilde{\varphi}) .$$

Let Σ_m denote the image of the map Φ_m, namely the set

$$(2.6) \qquad\qquad \Sigma_m = \Phi_m(\Sigma_0) .$$

Lemma 2.1 implies that Φ_m is continuous, so Σ_m is a connected set. As noted above, each $(\bar{\lambda}, \tilde{\varphi}) \in \Sigma_m$ with $\tilde{\varphi} \neq 0$ gives rise to a solution $x(t; \bar{\lambda}, \tilde{\varphi})$ of equation $(1.1)_{\bar{\lambda}}$ of period $\bar{q}(\lambda, \varphi)/|m\bar{q}(\lambda, \varphi) + 1|$ (note that $\bar{q}(\lambda, \varphi) > 2$, so the denominator is not zero; also note that $\bar{\lambda} < 0$ is possible). One also sees that $(\lambda, 0) \in \Sigma_m$ if and only if $\lambda = \lambda_m$, the quantity given by (2.2), so that Σ_m must agree with the local Hopf bifurcation at $(\lambda_m, 0)$ for (λ, φ) near $(\lambda_m, 0)$. We leave to the reader to show that if $(\bar{\lambda}, \tilde{\varphi}) \in \Sigma_m$ then

$$(2.7) \qquad\qquad \bar{\lambda} < 0 \quad \text{if } m < 0 ,$$

$$(2.8) \qquad\qquad \bar{\lambda} > 0 \quad \text{if } m \geqslant 0 ,$$

and

$$(2.9) \qquad\qquad -B < \tilde{\varphi}(t) < A \quad \text{for all } t \in [0, 1]$$

and that Σ_m is an unbounded subset of $\mathbf{R} \times C[0, 1]$.

THEOREM 2.1. – *Assume that f satisfies* (H1) *and is differentiable at $x = 0$ with* $f'(0) = -k$, *where* $k > 1$. *For each integer m define the set* $\Sigma_m \subseteq \mathbf{R} \times C[0, 1]$ *by* (2.6). *Then Σ_m is an unbounded connected set, and* $(\lambda, 0) \in \Sigma_m$ *if and only if* $\lambda = \lambda_m$ *where* λ_m *is given by* (2.2). *For each* $(\bar{\lambda}, \tilde{\varphi}) \in \Sigma_m$ *one has* (2.7), (2.8), *and* (2.9), *and if* $\tilde{\varphi} \neq 0$,

then $x(t; \bar{\lambda}, \bar{\varphi})$ *is a solution of equation* $(1.1)_{\bar{\lambda}}$ *with minimal period* p *satisfying*

$$(2.10) \qquad \frac{1}{|m|} < p < \frac{1}{|m + \frac{1}{2}|} \qquad if \quad m < 0 \,,$$

$$(2.11) \qquad p > 2 \qquad if \quad m = 0 \,,$$

and

$$(2.12) \qquad \frac{1}{m + \frac{1}{2}} < p < \frac{1}{m} \qquad if \quad m > 0 \,.$$

The sets Σ_m *are pairwise disjoint. If* $m \geqslant 0$ *and* $\lambda > \lambda_m$, *then equation* $(1.1)_\lambda$ *has at least* $m + 1$ *distinct periodic solutions, while if* $m < 0$ *and* $\lambda < \lambda_m$, *then it has at least* $|m|$ *periodic solutions.*

PROOF. – The first part of the theorem has already been proved. The bounds (2.10), (2.11), and (2.12) on the minimal period p follow immediately from the formula $p = \bar{q}(\lambda, \varphi)/|m\bar{q}(\lambda, \varphi) + 1|$ noted above, and the fact that $\bar{q}(\lambda, \varphi) > 2$ for $(\lambda, \varphi) \in \Sigma_0$. These bounds also imply that the sets Σ_m are pairwise disjoint: one can easily check that the intervals of values p given in (2.10), (2.11), and (2.12) for various integers m are pairwise disjoint. Finally, the connectedness, unboundedness and disjointness of the Σ_m, the bounds (2.7), (2.8), and (2.9) for $(\bar{\lambda}, \bar{\varphi}) \in \Sigma_m$, and the ordering

$$... < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_0 < \lambda_1 < \lambda_2 < ...$$

imply the last sentence in the statement of the theorem. $\square$
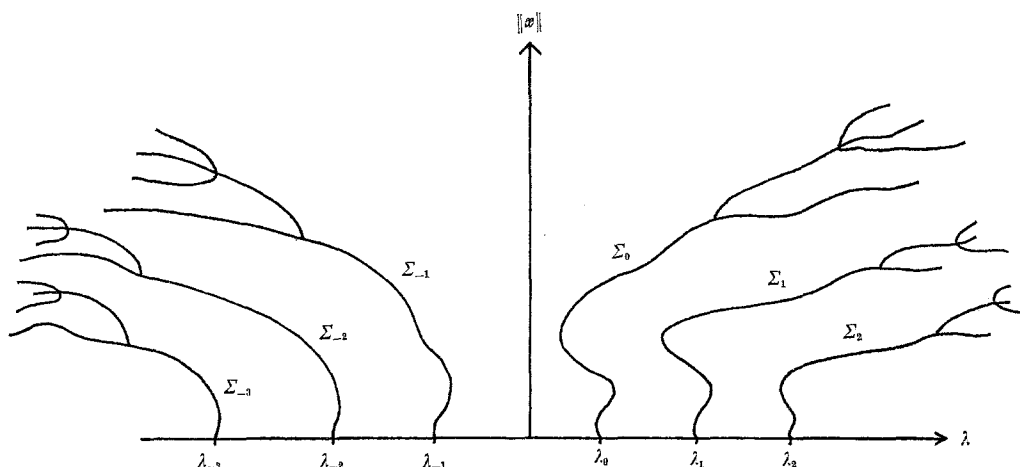
Figure 6 depicts schematically the branches Σ_m.

Fig. 6. The global Hopf branches Σ_m.

REMARK 2.1. - If $m \neq 0$, then any nontrivial solution $x(t; \lambda, \varphi)$ for $(\lambda, \varphi) \in \Sigma_m$ has period $p < 1$. Such a solution may be described as « rapidly oscillating », in contrast to the slowly oscillating solutions obtained when $m = 0$. Global families of rapidly oscillating periodic solutions, analogous to the families Σ_m, have been obtained for a class of differential-delay equations including many of the form

$$\dot{x}(t) = [\lambda_1 x(t-1) + \lambda_2 x(t-2)]f(x(t))$$

using the Fuller index. See CHOW and MALLET-PARET [8].

REMARK 2.2. - With only slightly more effort, Theorem 2.1 can be sharpened. We claim that Σ_m is a closed subset of $\mathbf{R} \times C[0, 1]$ and that $\Phi_m: \Sigma_0 \to \Sigma_m$ is a homeomorphism of Σ_0 onto Σ_m for each integer m. To see this, observe first that Φ_m is one-to-one when considered as a map from $\overline{\Sigma}$ into $\mathbf{R} \times C[0, 1]$. For suppose that

$$(2.13) \qquad \Phi_m(\lambda, \varphi) = \Phi_m(\sigma, \psi)$$

for some points (λ, φ) and (σ, ψ) in $\overline{\Sigma}$. If $\Phi_m(\lambda, \varphi) = 0$, then one easily concludes that $(\lambda, \varphi) = (\sigma, \psi) = (\lambda_0, 0)$, so assume that $\Phi_m(\lambda, \varphi) \neq 0$. Let $x(t) = x(t; \lambda, \varphi)$ and $y(t) = x(t; \sigma, \psi)$, extended to $\mathbf{R}$ as periodic functions. Then equation (2.13) implies that

$$\bar{\lambda} = \lambda\omega = \bar{\sigma} = \sigma\alpha$$

and

$$(2.14) \qquad \bar{x}(t) = x(\omega t) = \bar{y}(t) = y(\alpha t) \qquad \text{for all } t$$

where $\omega \neq 0$ and $\alpha \neq 0$ are given by

$$\omega = m\bar{q}(\lambda, \varphi) + 1 \qquad \text{and} \qquad \alpha = m\bar{q}(\sigma, \psi) + 1 .$$

Because $\bar{q}(\lambda, \varphi) > 2$ and $\bar{q}(\sigma, \psi) > 2$, one sees that $\omega \neq 0$ and $\alpha \neq 0$ have the same sign, so one may define the positive quantity $\varkappa = \alpha/\omega$. Indeed, by relabelling if necessary one can assume that

$$(2.15) \qquad 0 < \varkappa \leqslant 1 .$$

Also note that

$$(2.16) \qquad x(t) = y(\varkappa t) \qquad \text{for all } t ,$$

by (2.14).

By differentiating (2.16) and using the fact that $\lambda = \varkappa\sigma$, one obtains from the defining differential-delay equations for $x(t)$ and $y(t)$ that

$$\dot{x}(t) = -\lambda x(t) - \lambda f\big(x(t-1)\big) = \frac{dy(\varkappa t)}{dt} = -\lambda y(\varkappa t) - \lambda f\big(y(\varkappa t-1)\big)$$

and this yields

$$(2.17) \qquad\qquad f\big(x(t-1)\big) = f\big(y(\varkappa t-1)\big) \qquad \text{for all } t .$$

Setting $t = 1$ in (2.17), one finds

$$(2.18) \qquad\qquad f\big(y(\varkappa-1)\big) = 0 \qquad \text{hence} \qquad y(\varkappa-1) = 0 ;$$

and because $y(t)$ is a slowly oscillating periodic solution of equation $(1.1)_\sigma$ one concludes from (2.15) and (2.18) that $\varkappa = 1$. It follows immediately that $\lambda = \sigma$ and $x(t) = y(t)$ for all t. This proves that Φ_m is one-to-one.

To complete the proof of our claim, suppose that $\Phi_m(\sigma_n, \varphi_n) = (\bar{\sigma}_n, \bar{\varphi}_n)$ for some sequence $(\sigma_n, \varphi_n) \in \overline{\Sigma}$, and that $(\bar{\sigma}_n, \bar{\varphi}_n) \to (\bar{\sigma}, \bar{\varphi})$ for some $(\bar{\sigma}, \bar{\varphi}) \in \boldsymbol{R} \times C[0, 1]$. It suffices (because we know Φ_m is continuous on the closed set $\overline{\Sigma}$) to prove that there exists a convergent subsequence of (σ_n, φ_n). Let $x_n(t) = x(t; \sigma_n, \varphi_n)$ (extended periodically) and $\bar{x}_n(t) = x_n(\omega_n t)$, where

$$(2.19) \qquad\qquad \omega_n = m\bar{q}(\sigma_n, \varphi_n) + 1 ,$$

so $\bar{\sigma}_n = \sigma_n \omega_n$ and $\bar{\varphi}_n = \bar{x}_n|[0, 1]$. By using Lemma 1.5 to bound $\bar{q}(\sigma_n, \varphi_n)$ above, one may take a convergent subsequence in (2.19) to yield $\omega_n \to \omega$ for some ω. As $\bar{q}(\sigma_n, \varphi_n) > 2$, one has $\omega \neq 0$ and hence the limit $\sigma_n \to \bar{\sigma}/\omega$ exists. Proposition 1.3 now implies that $-B < x_n(t) < A$ for all t. As $x_n(t)$ is a solution of equation $(1.1)_{\sigma_n}$ and σ_n is a bounded sequence, it follows that $\dot{x}_n(t)$ is uniformly bounded. Thus Ascoli's theorem implies that for a further subsequence φ_n converges uniformly to some $\varphi \in C[0, 1]$. As remarked, this completes the proof.

There is another change of variables for periodic solutions of parametrized differential-delay equations which will prove extremely useful in studying asymptotic properties of equation $(1.1)_\lambda$ in Section 4. Consider a differential-delay equation of the form

$$(2.20) \qquad \varepsilon\dot{x}(t) = g\big(x(t), x(t-1), x(t-2), ..., x(t-(n-1))\big)$$

for some integer n. One could think of ε as a small positive parameter related to the parameter λ by $\varepsilon = \lambda^{-1}$; however, it is not essential here that ε be small. Let $x(t)$ be a periodic solution of equation (2.20) for some $\varepsilon \neq 0$, of period p, and define $r \in \boldsymbol{R}$ by

$$p = n(1 + \varepsilon r) .$$

Define functions $y_j(t)$, where the subscripts j are taken mod n, by

$$y_j(t) = x\left(\frac{jp}{n} - \varepsilon t\right).$$

Then one can verify that the $y_j(t)$ satisfy the system of n equations

(2.21) $\dot{y}_j(t) = -g\left(y_j(t), y_{j-1}(t-r), y_{j-2}(t-2r), \dots, y_{j-(n-1)}(t-(n-1)r)\right).$

Note that the parameter ε is absent from (2.21), having been replaced with the parameter r.

Applying this change of variables to equation (1.1)$_\lambda$ with $\varepsilon = \lambda^{-1}$ and $n = 2$, one finds

(2.22) $\dot{y}(t) = y(t) - f(z(t-r)), \quad \dot{z}(t) = z(t) - f(y(t-r))$

where $y(t) = y_0(t) = x(-\varepsilon t)$ and $z(t) = y_1(t) = x(\frac{1}{2}p - \varepsilon t)$. If f is an odd function and $x(t)$ is an S-solution, then $z(t) = -y(t)$ and the system (2.22) reduces to the single equation

$$\dot{y}(t) = y(t) + (y(t-r)).$$

If f satisfies the conditions of Theorem 1.1, then using this change of variables one can obtain from the set Σ_0 a connected set of periodic solutions of the system (2.22), bifurcating from the zero solution at the value $r_0 = \lambda_0(\pi/\nu_0 - 1)$.

3. – Asymptotic estimates for the period as $\varepsilon \to 0^+$ and derivation of the transition layer equations.

In this section we begin the study of the asymptotic behaviour of slowly oscillating periodic solutions of

(3.1)$_\varepsilon$ $\varepsilon\dot{x}(t) = -x(t) + f(x(t-1)), \quad \varepsilon > 0,$

as $\varepsilon \to 0^+$. Observe that this is just equation (1.1)$_\lambda$ with $\varepsilon = \lambda^{-1}$; we write our equation in this equivalent form to emphasize the fact that we are interested in the case when ε is small. In Theorem 3.2 we shall prove that if f satisfies hypotheses (H1) and (H2), then the distance between consecutive zeros of any slowly oscillating periodic solution is $1 + O(\varepsilon)$. Such solutions, therefore, have minimal period $2 + O(\varepsilon)$. An important role in our work will be played by Theorem 3.1, which provides information about the shape of slowly oscillating periodic solutions and (see Remark 3.4) other kinds of periodic solutions. In Proposition 3.1 we shall use our

estimate on the period to derive a pair of « transition layer equations » associated with equation $(3.1)_\varepsilon$. In Section 4 these transition layer equations will play a central role in obtaining very precise results about the asymptotic form of solutions of $(3.1)_\varepsilon$ for small ε.
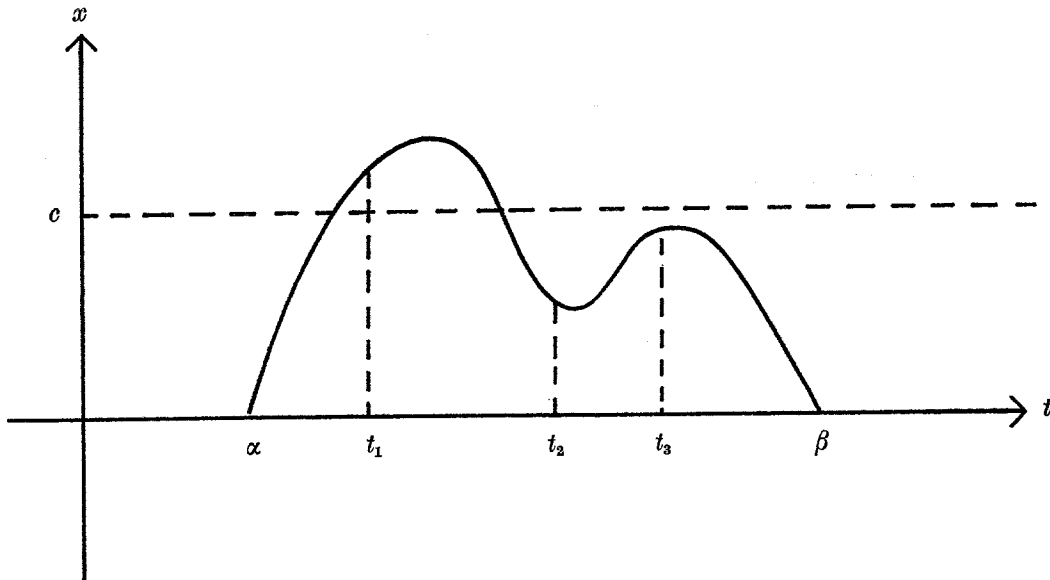
We start our work with a simple calculus lemma.

Fig. 7. The function $x(t)$ of Lemma 3.1.

LEMMA 3.1. – *Let* x: $[\alpha, \beta] \to \mathbf{R}$ *be a* C^1 *function such that* $x(\alpha) = x(\beta) = 0$ *and* $x(t) > 0$ *when* $\alpha < t < \beta$. *Suppose that* c *is a positive number and that there are numbers* $t_1 < t_2 < t_3$ *in* (α, β) *such that*

$$x(t_2) < x(t_j) \quad for \ j = 1, 3 ,$$

and

$$x(t_2) < c .$$

Then there exist numbers $t_1' < t_2' < t_3'$ *in* (α, β) *such that*

$$x(t_2') < x(t_j') \leqslant c \quad and \quad \dot{x}(t_j') \geqslant 0 \quad for \ j = 1, 3 ,$$

and

$$x(t_2') \leqslant x(t_2) \quad and \quad \dot{x}(t_2') = 0 .$$

PROOF. – This lemma is obvious if one graphs $x(t)$; see Figure 7. Alternatively, select $t_2' \in [t_1, t_3]$ such that

$$x(t_2') = \min \{x(t): t_1 \leqslant t \leqslant t_3\} .$$

Let $s_1 \in [\alpha, t_2']$ and $s_3 \in [t_2', \beta]$ satisfy

$$x(s_1) = \max \{x(t): \alpha \leqslant t \leqslant t_2'\} ,$$

and

$$x(s_3) = \max \{x(t): t_2' \leqslant t \leqslant \beta\} .$$

If $x(s_1) \leqslant c$ define $t_1' = s_1$; otherwise define

$$t_1' = \inf \{t \in [\alpha, t_2']: x(t) = c\} .$$

If $x(s_3) \leqslant c$ define $t_3' = s_3$; otherwise define

$$t_3' = \inf \{t \in [t_2', \beta]: x(t) = c\} .$$

We leave to the reader the verification that t_1', t_2' and t_3' so defined satisfy the conditions of the lemma. $\square$

Numerical studies (see the references in the introduction) have clearly suggested that slowly oscillating periodic solutions $x(t)$ of $(3.1)_\varepsilon$ may have complicated graphs with multiple relative extrema, especially when the nonlinearity $f(x)$ in the differential equation is not a monotone function of x. Nevertheless, throughout part of its cycle the solution $x(t)$ is often nicely behaved: there may be numbers $-d < 0 < c$ such that $x(t)$ oscillates about zero in a « nice » (i.e., monotone) way for those values of t such that $-d < x(t) < c$. The following definition and theorem make this concept precise.

DEFINITION 3.1. – Let $x(t)$ be a slowly oscillating periodic solution of equation $(3.1)_\varepsilon$ with (as in Definitions 1.1) first and second zeros q and $\bar{q}$. Fix numbers $-d < 0 < c$. We say that $x(t)$ *satisfies Property M between* $-d$ *and* c if

(1) x is monotone increasing on $[0, \sigma_1]$;

(2) $x(t) \geqslant c$ if $\sigma_1 < t < \tau_1$;

(3) x is monotone decreasing on $[\tau_1, \sigma_2]$;

(4) $x(t) \leqslant -d$ if $\sigma_2 < t < \tau_2$, and

(5) x is monotone increasing on $[\tau_2, \bar{q}]$,

where the numbers $\sigma_1, \tau_1, \sigma_2$ and τ_2 are defined as follows. Select $\varrho_1 \in (0, q)$ and $\varrho_2 \in (q, \bar{q})$ such that

$$x(\varrho_1) = \max \{x(t): 0 \leqslant t \leqslant q\} > 0$$

and

$$x(\varrho_2) = \min \{x(t): q \leqslant t \leqslant \bar{q}\} < 0 .$$

If $x(\varrho_1) > c$ define

$$\sigma_1 = \inf \ \{t \in [0, q]: x(t) = c\}$$

and

$$\tau_1 = \sup \ \{t \in [0, q]: x(t) = c\} \ ;$$

if $x(\varrho_1) \leqslant c$ define $\sigma_1 = \tau_1 = \varrho_1$. Similarly, if $x(\varrho_2) < - d$ define

$$\sigma_2 = \inf \ \{t \in [q, \bar{q}]: x(t) = - d\}$$

and

$$\tau_2 = \sup \ \{t \in [q, \bar{q}]: x(t) = - d\} \ ,$$

and if $x(\varrho_2) \geqslant - d$ define $\sigma_2 = \tau_2 = \varrho_2$. Note that even though ϱ_1 and ϱ_2 may not be uniquely determined, the concept of Property M is well defined. Also note that if $\sigma_1 = \tau_1$ (or $\sigma_2 = \tau_2$), then condition (2) (or (4)) is satisfied vacuously.
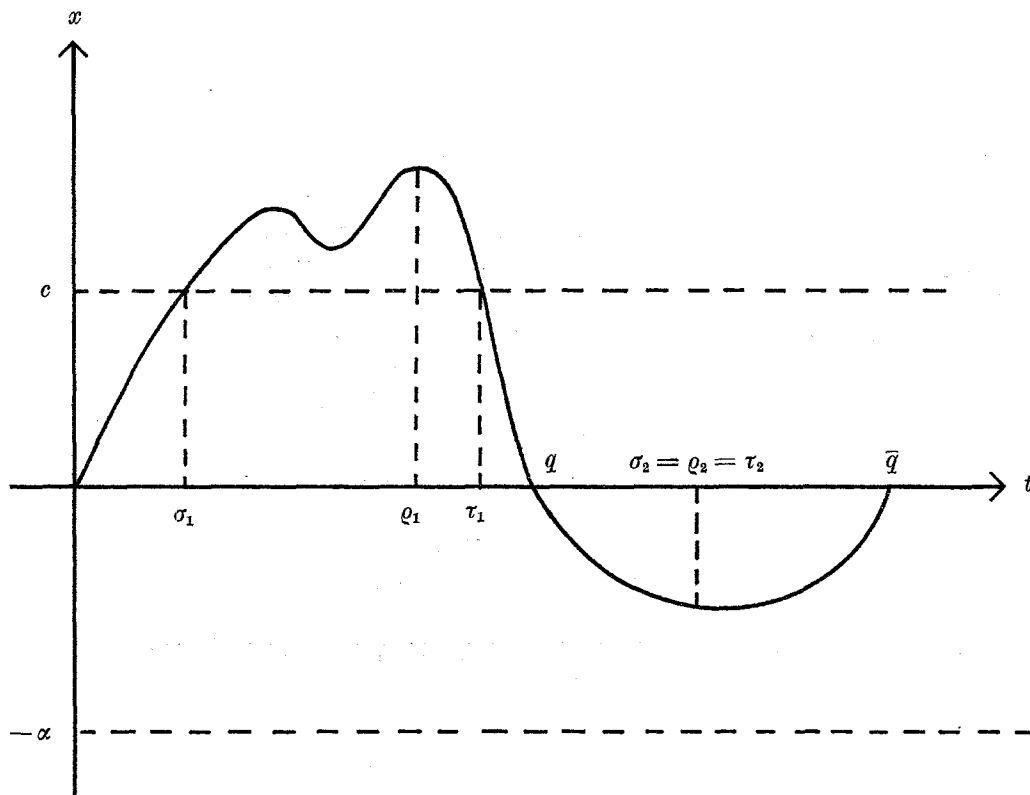

Fig. 8. A slowly oscillating periodic solution sarisfying Property M.

Figure 8 depicts a solution satisfying Property M between $-d$ and c. The following theorem implies that the slowly oscillating periodic solutions obtained in Section 1 satisfy Property M for some $-d$ and c *independent* of ε.

THEOREM 3.1. – *Suppose that A and B are positive numbers and f: $[-B, A] \to \boldsymbol{R}$ is a continuous function such that*

$$(3.2) \qquad\qquad xf(x) < 0 \quad \text{if} \quad x \in [-B, A] \quad \text{and} \quad x \neq 0 .$$

Suppose further that there are positive numbers $c \leqslant A$ and $d \leqslant B$ and a constant $\gamma > 1$ such that

(1) *f is monotone decreasing on $[-d, c]$;*

(2) *$f(x) \geqslant c$ if $-B \leqslant x \leqslant -d$, and $f(x) \leqslant -d$ if $c \leqslant x \leqslant A$, and*

(3) *$|f(f(x))| \geqslant \gamma |x|$ whenever $-d \leqslant x \leqslant c$ and the composition $f(f(x))$ is defined (i e, when $f(x) \in [-B, A]$).*

Let $\varepsilon > 0$ and suppose $x(t)$ is a slowly oscillating periodic solution of equation $(3.1)_\varepsilon$ which satisfies $-B \leqslant x(t) \leqslant A$ for all t. Then $x(t)$ satisfies Property M between $-d$ and c.

PROOF. – Assume, by way of contradiction, that some slowly oscillating periodic solution $x(t)$ does not satisfy Property M between $-d$ and c. For definiteness assume that the conditions of Property M are not satisfied on the interval $[0, q]$.

We first note that there exist triples of numbers $t_1 < t_2 < t_3$ in $(0, q)$ which satisfy the hypotheses of Lemma 3.1 with $[\alpha, \beta] = [0, q]$. This is a consequence of the assumption that Property M fails. If for example $x(t)$ is not monotone on $[0, \sigma_1]$, then there must exist $t \in (0, \sigma_1)$ such that $\dot{x}(t) < 0$. For small enough $\delta > 0$ one then has $0 < t - \delta < t + \delta < \sigma_1$ and $c \geqslant x(t - \delta) > x(t + \delta)$, so one can define $t_1 = t - \delta$, $t_2 = t + \delta$, and $t_3 = \sigma_1$. Similar choices of t_j work if $x(t)$ is not monotone on $[\tau_1, q]$. If, on the other hand, Property M is violated because $x(t) < c$ at some $t \in (\sigma_1, \tau_1)$, then one may set $t_1 = \sigma_1$, $t_3 = \tau_1$, and let t_2 be the location of the minimum of $x(t)$ in (σ_1, τ_1).

Having established the existence of such triples $t_1 < t_2 < t_3$, we define the set S to be the collection of all points like t_2. More precisely, let

$$S = \{t \in [0, q]\colon x(t) < c, \text{ and there exist } t_1 \text{ and } t_3 \text{ in } (0, q) \text{ such that } t_1 < t < t_3$$
$$\text{and } x(t_j) > x(t) \text{ for } j = 1, 3\}.$$

Of course the points t_1 and t_3 in the definition of S depend on t. Define

$$(3.3) \qquad\qquad \xi = \inf \{x(t)\colon t \in S\}$$

and note that $\xi > 0$ because $\dot{x}(0) > 0$ and $\dot{x}(q) < 0$. Also note that $\xi < c$. Now fix $t_2 \in S$ so that

$$(3.4) \qquad\qquad x(t_2) < \gamma\xi$$

and let $t_1 < t_2 < t_3$ be as in the definition of S. These points t_j, $j = 1, 2, 3$, will stay fixed for the remainder of this proof.

Lemma 3.1 now implies there are points $t_1' < t_2' < t_3'$ in $(0, q)$ such that

$$(3.5) \qquad\qquad x(t_2') < x(t_j') \leqslant c \quad \text{for } j = 1, 3 ,$$

$$(3.6) \qquad\qquad x(t_2') \leqslant x(t_2) ,$$

$$(3.7) \qquad\qquad \dot{x}(t_2') = 0 \quad \text{and} \quad \dot{x}(t_j') \geqslant 0 \quad \text{for } j = 1, 3 .$$

Equations $(3.1)_\varepsilon$, (3.5) and (3.7) imply

$$(3.8) \qquad 0 < x(t_2') = f(x(t_2'-1)) < x(t_j') \leqslant f(x(t_j'-1)) \quad \text{for } j = 1, 3$$

and hence $t_j' - 1 \in (-(\bar{q} - q), 0)$ for $j = 1, 2, 3$, because $x(t_j' - 1) < 0$ by (3.2). Set $s_j = t_j' - 1$. We now observe that the triple $s_1 < s_2 < s_3$ satisfies the hypotheses of Lemma 3.1, but for the function $-x(t)$ in the interval $[-(\bar{q} - q), 0]$, and with d in place of c. Indeed, assumptions (1) and (2) in the statement of this theorem together with (3.5) and (3.8) above easily imply that

$$x(s_j) < x(s_2) < 0 \quad \text{for } j = 1, 3 ,$$

and

$$(3.9) \qquad\qquad -d < x(s_2) .$$

Applying Lemma 3.1 a second time, this time to $-x(t)$, produces points $s_1' < s_2' < s_3'$ in $(-(\bar{q} - q), 0)$ satisfying analogs of (3.5), (3.6) and (3.7), namely

$$-d \leqslant x(s_j') < x(s_2') \quad \text{for } j = 1, 3 ,$$

$$(3.10) \qquad\qquad x(s_2) \leqslant x(s_2') < 0 ,$$

$$(3.11) \qquad\qquad \dot{x}(s_2') = 0 \quad \text{and} \quad \dot{x}(s_j') \leqslant 0 \quad \text{for } j = 1, 3 .$$

Setting $r_j = s_j' + \bar{q} - 1$ and arguing as before shows the points $r_1 < r_2 < r_3$ are in $(0, q)$ and satisfy the hypotheses of Lemma 3.1; in particular,

$$(3.12) \qquad\qquad x(r_2) < c .$$

Thus there are numbers $r_1' < r_2' < r_3'$ in $(0, q)$ such that

(3.13) $x(r_2') < x(r_j') \leqslant c$ for $j = 1, 3$,

and

(3.14) $x(r_2') \leqslant x(r_2)$

so $r_2' \in S$.

We are now in a position to obtain a contradiction. Because $r_2' \in S$, we know from (3.3) that

(3.15) $x(r_2') \geqslant \xi$.

On the other hand, by using the monotonicity property (1) of f in the statement of this theorem, and the properties of t_2, t_2', s_2, s_2', r_2 and r_2' in equations (3.6) through (3.12), and (3.14), we obtain

(3.16) $x(t_2) \geqslant x(t_2') = f(x(t_2' - 1)) = f(x(s_2)) \geqslant f(x(s_2')) =$

$$= f\big(f(x(s_2' - 1))\big) = f\big(f(x(r_2))\big) \geqslant f\big(f(x(r_2'))\big) \, .$$

However, property (3) in the statement of this theorem, and equations (3.13) and (3.15), imply

(3.17) $f\big(f(x(r_2'))\big) \geqslant \gamma x(r_2') \geqslant \gamma \xi$

so from (3.16) and (3.17) we obtain $x(t_2) \geqslant \gamma \xi$. This contradicts (3.4). $\square$

REMARK 3.1. – If the function f satisfies hypotheses (H1) and (H2), then the conditions for Theorem 3.1 are fulfilled for some numbers $c \leqslant A$ and $d \leqslant B$.

REMARK 3.2. – If condition (1) in Theorem 3.1 is strengthened to assume that

(3.18) f is strictly decreasing on $[-d, c]$,

then one obtains the stronger conclusion that in addition to $x(t)$ satisfying Property M between $-d$ and c,

   $x(t)$ is strictly increasing on $[0, \sigma_1]$, strictly decreasing on $[\tau_1, \sigma_2]$, and strictly increasing on $[\tau_2, \bar{q}]$.

We shal prove this stronger conclusion for the intervals $[0, \sigma_1]$ and $[\tau_1, q]$, as the

proof for $[q, \sigma_2]$ and $[\tau_2, \bar{q}]$ is analogous. Suppose the conclusion is false; define sets

$$T_1 = \{t \in [0, \sigma_1]: \text{ there exists } t_0 \in [0, \sigma_1] \text{ with } t_0 \neq t \text{ and } x(t_0) = x(t)\} ,$$

$$T_2 = \{t \in [\tau_1, q]: \text{ there exists } t_0 \in [\tau_1, q] \text{ with } t_0 \neq t \text{ and } x(t_0) = x(t)\} ,$$

and

$$T = T_1 \cup T_2 .$$

By assumption $T \neq \emptyset$. Let

(3.19) $$\xi = \inf \{x(t): t \in T\}$$

and note that $\xi > 0$ because $\dot{x}(0) > 0$ and $\dot{x}(q) < 0$. Select numbers $t_1, t_2 \in T$, with $t_1 < t_2$, such that

(3.20) $$x(t) = x(t_1) \quad \text{if} \quad t_1 \leqslant t \leqslant t_2 ,$$

and

(3.21) $$x(t_1) < \gamma \xi$$

where γ is as in the statement of Theorem 3.1. Note that Theorem 3.1 and the definition of T imply the existence of t_1 and t_2, and the definitions of σ_1 and τ_1 imply that

(3.22) $$x(t_1) < c .$$

The differential equation $(3.1)_\varepsilon$, and equation (3.20) imply that

(3.23) $$\varepsilon \dot{x}(t) = 0 = - x(t) + f(x(t-1)) \quad \text{if } t_1 \leqslant t \leqslant t_2 .$$

Equations (3.22) and (3.23), and condition (2) in the statement of Theorem 3.1, and the strict monotonicity condition (3.18) imply that $x(t-1)$ is constant for $t_1 \leqslant t \leqslant t_2$ and satisfies $- d < x(t-1) < 0$ there. Therefore one obtains

$$\varepsilon \dot{x}(t-1) = 0 = - x(t-1) + f(x(t-2)) \quad \text{if } t_1 \leqslant t \leqslant t_2 .$$

A repetition of the argument just given implies further that $x(t-2) = x(t + \bar{q} - 2)$ is constant for $t_1 \leqslant t \leqslant t_2$, with a value $0 < x(t-2) < c$. One concludes that $t_1 + \bar{q} - 2 \in T$, and so

(3.24) $$x(t_1 - 2) = x(t_1 + \bar{q} - 2) \geqslant \xi$$

by equation (3.19). However, using (3.24) and condition (3) in Theorem 3.1 gives

$$x(t_1) = f(x(t_1 - 1)) = f(f(x(t_1 - 2))) \geqslant \gamma x(t_1 - 2) \geqslant \gamma \xi .$$

This contradicts (3.21).

Remark 3.3. - If condition (1) in Theorem 3.1 is further strengthened to assume that

f is differentiable on $(-d, c)$, with $f'(x) < 0$ there ,

then the corresponding stronger result that

$$\dot{x}(t) > 0 \qquad \text{on} \ [0, \sigma_1) \cup (\tau_2, \bar{q}]$$

and

$$\dot{x}(t) < 0 \qquad \text{on} \ (\tau_1, \sigma_2)$$

s obtained. We omit the proof of this.

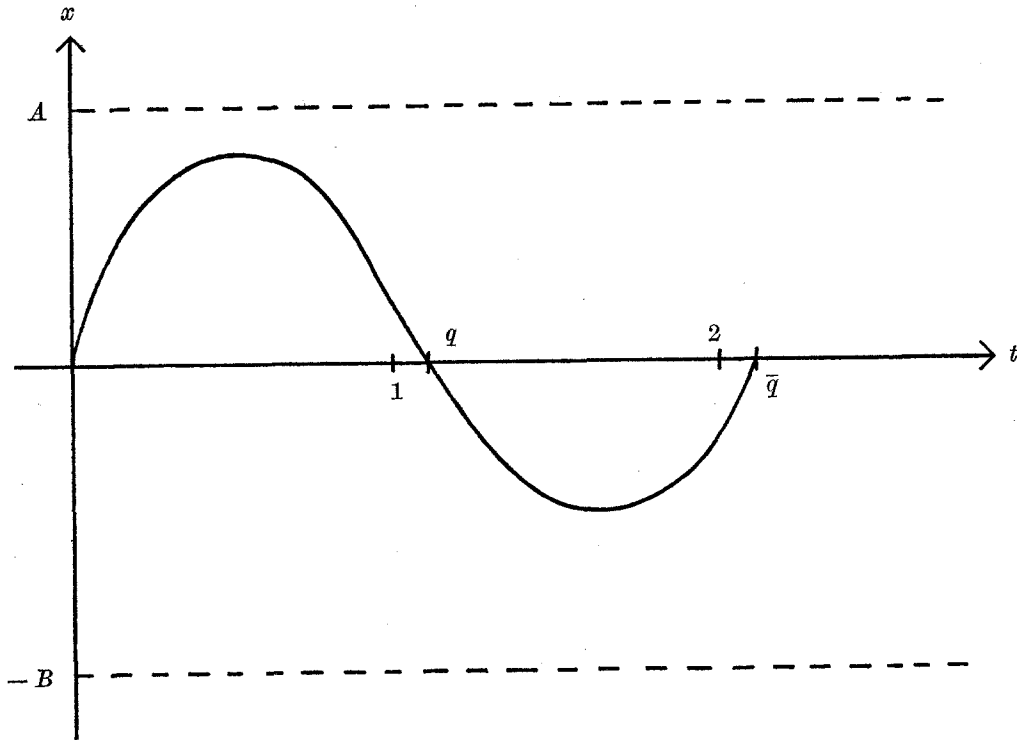

Fig. 9. A slowly oscillating periodic solution with monotone f.

The next result describes an important special case of Theorem 3.1 which occurs when the nonlinearity f is monotone throughout the range of a slowly oscillating periodic solution. Such a periodic solution then possesses a monotonicity property. See Figure 9.

COROLLARY 3.1. – *Suppose that A and B are positive numbers and f: $[-B, A] \to R$ is a continuous function such that $xf(x) < 0$ if $x \in [-B, A]$ and $x \neq 0$. Suppose further that f is nonincreasing on $[-B, A]$ and satisfies $|f(f(x))| > |x|$ whenever $x \in (-B, A)$, $x \neq 0$, and $f(x) \in [-B, A]$. Also suppose that*

$$(3.25) \qquad \liminf_{|x| \to 0} \frac{|f(f(x))|}{|x|} > 1 .$$

Then if $x(t)$ is any slowly oscillating periodic solution of equation $(3.1)_\varepsilon$ for some $\varepsilon > 0$, and if $-B \leqslant x(t) \leqslant A$ for all t, there must exist $\varrho_1 \in (0, q)$ and $\varrho_2 \in (q, \bar{q})$ such that $x(t)$ is monotone increasing on $[0, \varrho_1]$, is monotone decreasing on $[\varrho_1, \varrho_2]$ and is monotone increasing on $[\varrho_2, \bar{q}]$. (As before, q and $\bar{q}$ are the first and second zeros respectively of $x(t)$.)

This result holds in particular if f satisfies hypotheses (H1) and (H2), is monotone decreasing on $[-B, A]$, and satisfies $|f(f(x))| > |x|$ if $x \in (-B, A)$ and $x \neq 0$.

PROOF. – Let c_n and d_n be increasing sequences of positive numbers satisfying

$$(3.26) \qquad c_n \to M_+, \qquad d_n \to M_-,$$

$$(3.27) \qquad f(c_n) \leqslant -d_n \qquad \text{and} \qquad f(-d_n) \geqslant c_n$$

where $M_+ \leqslant A$ and $M_- \leqslant B$ are the positive numbers

$$(3.28) \qquad M_+ = \max_t x(t) \qquad \text{and} \qquad M_- = -\min_t x(t) .$$

Such sequences are easily constructed; for example let $\{c_n\}$ be any strictly increasing sequence which approaches M_+. If each d_n is then chosen less than, but sufficiently near the quantity $\min \{-f(c_n), M_-\}$, then one can verify that the required conditions (3.26) and (3.27) hold. In so doing the inequalities

$$(3.29) \qquad f(M_+) \leqslant -M_- \qquad \text{and} \qquad f(-M_-) \geqslant M_+$$

are needed. They are easily proved. For example, if $x(t)$ attains its minimum at ϱ, then $\dot{x}(\varrho) = 0$, so we have from the differential equation and monotonicity of f that $-M_- = x(\varrho) = f(x(\varrho - 1)) \geqslant f(M_+)$. The second inequality in (3.29) is proved similarly.

Having established the existence of c_n and d_n, one checks that the hypotheses of Theorem 3.1 hold for these quantities. In particular, (3.25) is needed to show the quantity $\gamma = \gamma_n$ of condition (3) satisfies $\gamma_n > 1$. The solution $x(t)$ thus satisfies Property M between $-d_n$ and c_n for each n. This is easily seen to imply our result. $\square$

REMARK 3.4. – The definition of Property M can be extended to include functions other than slowly oscillating periodic solutions. Let $x(t)$ be a continuous real-valued function defined for all $t \in \mathbf{R}$, and let c and d be positive constants. Let us understand by an interval J of real numbers to be either the empty set, a single point, or an interval (of either finite or infinite length) in the usual sense. We say that $x(t)$ satisfies Property M between $-d$ and c if it is possible to write

$$\mathbf{R} = \bigcup_{n=-\infty}^{\infty} J_n$$

where each J_n is an interval in the above sense, the sets J_n are pairwise disjoint, J_m lies to the left of J_n whenever $m < n$, and such that whenever $J_n \neq \emptyset$ exactly one of the following occurs:

(1) $x(t) \geqslant c$ for all $t \in J_n$;

(2) $x(t)$ is monotone increasing in J_n and $-d < x(t) < c$ for all $t \in J_n$;

(3) $x(t)$ is monotone decreasing in J_n and $-d < x(t) < c$ for all $t \in J_n$, or

(4) $x(t) \leqslant -d$ for all $t \in J_n$;

and that further, if J_n is of type (2) or (3) above, then either J_n has infinite length, or else $x(t)$ assumes both positive and negative values for $t \in J_n$.

It is not hard to see that this definition of Property M is the same as the one given in Definition 3.1 for slowly oscillating periodic solutions. Numerical studies [2, 6, 14, 17, 22, 33, 39, 40] suggest that for certain functions f satisfying the hypotheses of Theorem 3.1, equation $(3.1)_\varepsilon$ has solutions $x(t)$ which are not slowly oscillating periodic solutions, but which do satisfy the property that any two consecutive zeros q_n and q_{n+1} of $x(t)$ satisfy $q_{n+1} - q_n > 1$; and in addition, $-B \leqslant x(t) \leqslant A$ for all real t. Although we do not prove the existence of such solutions in this paper, a slight extension of the proof of Theorem 3.1 shows that such solutions, if they exist, satisfy Property M between $-d$ and c.

Solutions satisfying Property M also occur for the «transition layer equations» associated with equation $(3.1)_\varepsilon$. These are important in describing the asymptotic behaviour of solutions of $(3.1)_\varepsilon$ as $\varepsilon \to 0^+$, and will be discussed in Proposition 3.1 and in Section 4.

Before establishing another main result of this section it will be convenient to give a lemma.

LEMMA 3.2. – *Suppose that A and B are positive numbers and* $f: [-B, A] \to \mathbf{R}$ *is a continuous function such that* $xf(x) < 0$ *if* $x \in [-B, A]$ *and* $x \neq 0$. *Assume that there exists* $\Omega > 0$ *such that*

$$|f(x)| \leqslant \Omega|x| \quad \text{if} \quad -B \leqslant x \leqslant A .$$

If $x(t)$ is a slowly oscillating periodic solution of equation $(3.1)_\varepsilon$ *for some* $\varepsilon > 0$, *and* $-B \leqslant x(t) \leqslant A$ *for all* t, *then*

(3.30) $$M_- \leqslant \Omega M_+ \quad \text{and} \quad M_+ \leqslant \Omega M_-$$

where $-M_- < 0$ *and* $M_+ > 0$ *are respectively the minimum and maximum values of $x(t)$, as in* (3.28).

PROOF. – Let $x(t)$ achieve its minimum at ϱ; then the differential equation $(3.1)_\varepsilon$ implies that

$$M_- = |x(\varrho)| = |f(x(\varrho - 1))| \leqslant \Omega|x(\varrho - 1)| \leqslant \Omega M_+ .$$

A similar argument gives the other part of (3.30).   □

We want to prove that under hypotheses (H1) and (H2), the minimal period $\bar{q}$ of any slowly oscillating periodic solution of (3.1) is less that $2(1 + \bar{C}\varepsilon)$ for some constant $\bar{C} > 0$ *independent* of ε. Once we have this, it follows immediately that the separation between consecutive zeros of $x(t)$ is $1 + O(\varepsilon)$: just use the facts that their separation is greater than one, and that $x(t)$ has two zeros per period. More precise information about the asymptotic forms as $\varepsilon \to 0^+$ of the zeros q and $\bar{q}$, and of the solution $x(t)$ itself, will be established in Section 4.

THEOREM 3.2. – *Assume f satisfies hypotheses* (H1) *and* (H2). *Then there exist constants $C > 0$ and $\bar{C} > 0$ such that if $x(t)$ is a slowly oscillating periodic solution of equation* $(3.1)_\varepsilon$ *for some* $\varepsilon > 0$, *then its first two positive zeros q and $\bar{q}$ satisfy*

(3.31) $$1 < q < 1 + C\varepsilon ,$$

and

(3.32) $$2 < \bar{q} < 2(1 + \bar{C}\varepsilon) .$$

The constants C and $\bar{C}$ depend only on f, and not on ε nor on the solution $x(t)$.

PROOF. – The lower bounds $q > 1$ and $\bar{q} > 2$ follow from the definition of slowly oscillating periodic solution. If the upper bound (3.32) for $\bar{q}$ holds, then the inequality $\bar{q} - q > 1$ immediately implies the upper bound (3.31) for q, with $C = 2\bar{C}$. Thus, all our effort will be devoted to obtaining the upper bound (3.32) for $\bar{q}$.

From the assumptions (H1) and (H2) one sees there exist positive constants c_0, γ, and Ω depending only on f, such that

$$f(x) \quad \text{is monotone decreasing in} \quad [-c_0, c_0] \,,$$

$$|f(x)| \geqslant \gamma |x| \quad \text{and} \quad |f(f(x))| \geqslant \gamma |x| \quad \text{if} \quad |x| \leqslant c_0 \,,$$

$$|f(x)| \geqslant \gamma c_0 \quad \text{if} \quad |x| \geqslant c_0 \,,$$

(3.33) $$|f(x)| \leqslant \Omega |x| \quad \text{for all} \quad x \,,$$

$$\gamma > 1 \quad \text{and} \quad \Omega > 1 \,.$$

Now suppose that $x(t)$ is a slowly oscillating periodic solution of $(3.1)_\varepsilon$ for some $\varepsilon > 0$. Let $\|x\| = \max\limits_t |x(t)|$ and set

(3.34) $$\Lambda = \frac{\|x\|}{\Omega}$$

and

(3.35) $$c = \min\left\{c_0, \frac{\Lambda}{2\Omega}\right\} .$$

One easily sees that the hypotheses of Theorem 3.1 hold with $d = c$, hence the solution $x(t)$ satisfies Property M between $-c$ and c. Note also that

(3.36) $$|f(x)| \geqslant \gamma c \quad \text{if} \quad |x| \geqslant c \,.$$

Furthermore, one sees that Lemma 3.2 implies

(3.37) $$\max\limits_t x(t) \geqslant \Lambda > c \quad \text{and} \quad \min\limits_t x(t) \leqslant -\Lambda < -c$$

so that the range of $x(t)$ contains the interval $[-c, c]$ (see Figure 10). Finally, recall from Proposition 1.3 that

(3.38) $$-B < x(t) < A \quad \text{for all} \quad t \,.$$

Define σ_1 and τ_1 to be the first and last times respectively in $(0, q)$ such that $x(t) = c$; similarly let σ_2 and τ_2 be the first and last times in $(q, \bar{q})$ such that $x(t) = -c$. Throughout this proof we shall repeatedly use the fact that $x(t)$ satisfies Property M between $-c$ and c. The quantities σ_j and τ_j will also play a prominent role.

We now claim the following: there exists a constant C_*, independent of ε and of the solution $x(t)$, such that

(3.39)   if $\tau_1 - \sigma_1 \geqslant C_* \varepsilon$ then $[1 + \sigma_1 + C_* \varepsilon, 1 + \tau_1] \subseteq [\sigma_2, \tau_2]$,
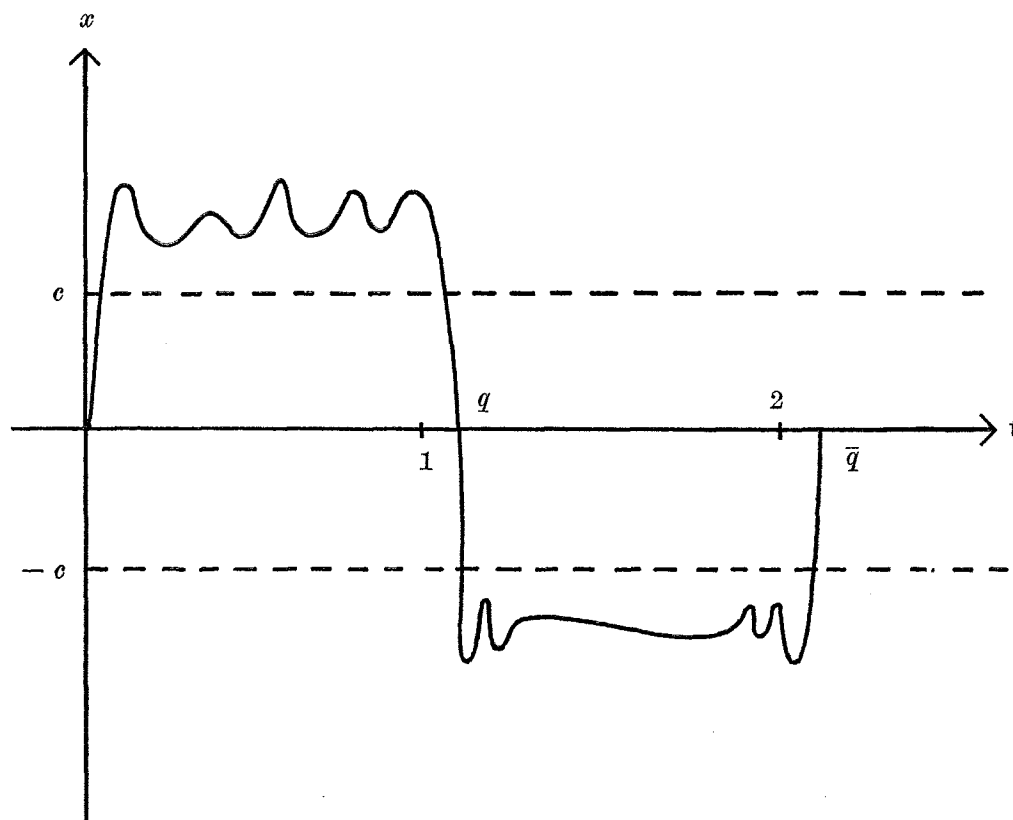
Fig. 10. A slowly oscillating periodic solution with small ε.

and similarly

(3.40)   if $\tau_2 - \sigma_2 \geqslant C_* \varepsilon$ then $[1 + \sigma_2 + C_* \varepsilon, 1 + \tau_2] \subseteq [\sigma_1 + \bar{q}, \tau_1 + \bar{q}]$.

We shall only prove (3.39) as the proof of (3.40) is analogous.

We first give the value of the constant C_*. Set

$$C_* = \max \left\{ \log \frac{A + \gamma c_0}{(\gamma - 1) c_0}, \ \log \frac{B + \gamma c_0}{(\gamma - 1) c_0}, \ \log \frac{2\Omega^2 + \gamma}{\gamma - 1} \right\}.$$

Now suppose that $\tau_1 - \sigma_1 \geqslant C_* \varepsilon$. Then for $1 + \sigma_1 \leqslant t \leqslant 1 + \sigma_1 + C_* \varepsilon$ one has $x(t-1) \geqslant c$, and hence

(3.41) $\varepsilon \dot{x}(t) \leqslant - x(t) - \gamma c$

from (3.36). Integrating the inequality (3.41) over this range of t gives, after some

calculation involving (3.34), (3.35) and (3.38), the inequalities

$$x(1 + \sigma_1 + C_* \varepsilon) \leqslant x(1 + \sigma_1) \exp \left[-C_*\right] - (1 - \exp \left[-C_*\right]) \gamma c \leqslant$$

$$\leqslant \|x\| \exp \left[-C_*\right] - (1 - \exp \left[-C^*\right]) \gamma c \leqslant -c ,$$

and this implies (by Property M and the definition of σ_2) that

$$(3.42) \qquad\qquad 1 + \sigma_1 + C_* \varepsilon \geqslant \sigma_2 .$$

Next we show that $1 + \tau_1 < \tau_2$, which together with equation (3.42) will establish (3.39). We first claim that

$$(3.43) \qquad\qquad 1 + \sigma_1 < \tau_2 .$$

If (3.43) were false, we would have

$$\tau_2 \leqslant 1 + \sigma_1 < 1 + q < \bar{q}$$

which implies (by the definition of τ_2) that $x(1 + \sigma_1) \geqslant -c$ and

$$(3.44) \qquad\qquad \dot{x}(1 + \sigma_1) \geqslant 0 .$$

As $x(\sigma_1) = c$, were have from equation $(3.1)_\varepsilon$ and from (3.36) that

$$\varepsilon \dot{x}(1 + \sigma_1) = -x(1 + \sigma_1) + f(x(\sigma_1)) \leqslant c - \gamma c < 0 .$$

This contradicts (3.44) and thereby proves (3.43).

By Property M one has $\dot{x}(\tau_2) \geqslant 0$ and so

$$-c = x(\tau_2) \leqslant f(x(\tau_2 - 1))$$

which implies, by (3.36), that

$$(3.45) \qquad\qquad x(\tau_2 - 1) < c .$$

The inequalities (3.43) and (3.45) together imply that $\tau_2 - 1 > \tau_1$, that is,

$$(3.46) \qquad\qquad 1 + \tau_1 < \tau_2 .$$

Equations (3.42) and (3.46) now give the desired inclusion in (3.39).

Having now established both the implications (3.39) and (3.40), we observe that our theorem is easily proved if either $\tau_1 - \sigma_1 \geqslant 2C_* \varepsilon$ or if $\tau_2 - \sigma_2 \geqslant 2C_* \varepsilon$. For suppose

that $\tau_1 - \sigma_1 \geqslant 2C_* \varepsilon$. Then the inclusion in (3.39) holds, and comparing the lengths of the intervals there shows that

$$\tau_2 - \sigma_2 \geqslant \tau_1 - \sigma_1 - C_* \varepsilon \geqslant C_* \varepsilon \, ,$$

hence the inclusion in (3.40) also holds. From (3.39) and (3.40) one now has

$$\sigma_1 + \bar{q} \leqslant 1 + \sigma_2 + C_* \varepsilon \leqslant 1 + (1 + \sigma_1 + C_* \varepsilon) + C_* \varepsilon$$

and so

$$(3.47) \qquad \bar{q} \leqslant 2(1 + C_* \varepsilon)$$

which proves the theorem in this case.

Thus, it suffices to prove our theorem under the additional assumptions

$$(3.48) \qquad \tau_1 - \sigma_1 < 2C_* \varepsilon \quad \text{and} \quad \tau_2 - \sigma_2 < 2C_* \varepsilon \, .$$

Furthermore, we may also assume that ε is sufficiently small, to be specific, that

$$(3.49) \qquad 2C_* \varepsilon \leqslant 1 \, .$$

(The case when $2C_* \varepsilon > 1$ is easily handled using the bound $\bar{q} \leqslant \bar{Q}$ of Lemma 1.5, for then one has

$$(3.50) \qquad \bar{q} \leqslant \bar{Q} < 2 + 2C_*(\bar{Q} - 2)\varepsilon.)$$

We first show that

$$(3.51) \qquad \sigma_2 - \tau_1 \leqslant 1 \quad \text{and} \quad \sigma_1 + \bar{q} - \tau_2 \leqslant 1 \, ,$$

assuming that the conditions (3.48) and (3.49) hold. We shall prove only the first inequality in (3.51). Assume, by way of contradiction, that $1 + \tau_1 < \sigma_2$ and select $\varrho \in [\sigma_2, \tau_2]$ to be a point at which $x(t)$ attains its minimum:

$$x(\varrho) = \min_t x(t) < 0 \, .$$

Equations (3.48) and (3.49) imply that the length of the interval (σ_2, τ_2) is less than one, hence $\varrho - 1 < \sigma_2$. Also, $\varrho - 1 \geqslant \sigma_2 - 1 > \tau_1$ and so $|x(t-1)| \leqslant c$ for $1 + \tau_1 \leqslant t \leqslant \varrho$. Therefore, from (3.33) one has for this range of t that

$$\varepsilon \dot{x}(t) \geqslant - x(t) - \Omega c \, .$$

Integrating this inequality from $1 + \tau_1$ to ϱ gives

$$0 > x(\varrho) \geqslant \exp\left[-\varkappa\right]x(1 + \tau_1) - (1 - \exp\left[-\varkappa\right])\Omega c \geqslant -(1 + \Omega)c$$

where $\varkappa = (\varrho - \tau_1 - 1)/\varepsilon \geqslant 0$. Therefore

$$|x(\varrho)| \leqslant (1 + \Omega)c .$$

On the other hand, (3.37) implies that $|x(\varrho)| \geqslant \Lambda$ and so

$$(3.52) \qquad\qquad \Lambda \leqslant (1 + \Omega)c .$$

But (3.52) and the definition (3.35) of c are inconsistent with the fact that $\Omega > 1$. With this contradiction the inequalities (3.51) are proved.

To complete the proof of Theorem 3.2, add together the four inequalities in (3.48) and (3.51). This yields the desired estimate

$$(3.53) \qquad\qquad \bar{q} < 2(1 + 2C_* \varepsilon) .$$

In summary, the various bounds (3.47), (3.50) and (3.53) for $\bar{q}$ imply that $\bar{q} < 2(1 + \bar{C}\varepsilon)$ holds in any case, if $\bar{C} \geqslant \max\{2C_*, C_*(\bar{Q} - 2)\}$. $\square$

As a corollary of Theorem 3.2 we obtain information about the possible periods of slowly oscillating periodic solutions of equation $(3.1)_\varepsilon$.

COROLLARY 3.2. - *Let f be as in Theorem 3.2 and let v_0 be defined as in equation (1.24) (recall that $k = -f'(0)$ there). Then for every number p such that $2 < p < 2\pi/v_0$ there exists $\varepsilon > 0$ and a slowly oscillating periodic solution $x(t)$ of equation $(3.1)_\varepsilon$, such that the minimal period of $x(t)$ is p.*

PROOF. - Let $\Sigma_0 \subseteq (0, \infty) \times K$ be the continuum obtained in Theorem 1.1, with λ_0 as in that theorem, and let $\bar{q}(\lambda, \varphi)$ denote the (minimal) period of the slowly oscillating periodic solution $x(t; \lambda, \varphi)$ for each $(\lambda, \varphi) \in \Sigma_0 - \{(\lambda_0, 0)\}$. It is proved in Lemma 2.1 that if one defines $\bar{q}(\lambda_0, 0) = 2\pi/v_0$ then $\bar{q}$ is a continuous function on Σ_0. As $\bar{q}(\lambda, \varphi) > 2$ always holds, Theorems 1.1 and 3.2 imply (recall $\lambda = \varepsilon^{-1}$) that

$$\inf\{\bar{q}(\lambda, \varphi): (\lambda, \varphi) \in \Sigma_0\} = 2 .$$

Corollary 3.2 thus follows from the connectedness of Σ_0 and continuity of $\bar{q}$. $\square$

In analyzing equation $(3.1)_\varepsilon$ for small values of ε we shall make use of a system of associated differential equations, namely

$$(3.54)_r \qquad\qquad \dot{y}(t) = y(t) - f(z(t - r))$$

$$(3.55)_r \qquad\qquad \dot{z}(t) = z(t) - f(y(t - r))$$

where the number $r \geqslant 0$ is a parameter. This system was introduced in Section 2.

DEFINITION 3.2. - Equations $(3.54)_r$ and $(3.55)_r$ are known as the *transition layer equations* associated with equation $(3.1)_\varepsilon$.

The reason for the name « transition layer equation » will become clearer in Section 4. In essence, slowly oscillating periodic solutions of equation $(3.1)_\varepsilon$ will satisfy equations $(3.54)_r$ and $(3.55)_r$ after a time scaling by a factor ε is made. This time scaling causes the parameter ε to be absent from the new equations, but introduces a parameter r which is related to the period $\bar{q}$. The properties of solutions of the transition layer equations are intimately related to those of equation $(3.1)_\varepsilon$ for small ε.

The following result clarifies the relation between equation $(3.1)_\varepsilon$ and equations $(3.54)_r$ and $(3.55)_r$.

PROPOSITION 3.1. - *Assume that f satisfies hypotheses* (H1) *and* (H2).

(i) *Let $\varepsilon > 0$, let $x(t)$ be a slowly oscillating periodic solution of equation $(3.1)_\varepsilon$ of minimal period $\bar{q}$, and let $\theta \in \mathbf{R}$. Define*

$$y(t) = x(\theta - \varepsilon t) \quad and \quad z(t) = x(\theta + \tfrac{1}{2}\bar{q} - \varepsilon t) .$$

Then $(y(t), z(t))$ satisfies the transition layer equations $(3.54)_r$ and $(3.55)_r$ where $r > 0$ is given by

$$\bar{q} = 2(1 + \varepsilon r) .$$

(ii) *Let ε_n, $x_n(t)$, $\bar{q}_n$ and $\theta_n \in \mathbf{R}$ be sequences such that $\varepsilon_n \to 0^+$, and*

$x_n(t)$ *is a slowly oscillating periodic solution of equation $(3.1)_{\varepsilon_n}$ of minimal period $\bar{q}_n$.*

Let $r_n > 0$, and $y_n(t)$ and $z_n(t)$ be defined by

$$(3.56) \quad \bar{q}_n = 2(1 + \varepsilon_n r_n) , \, y_n(t) = x_n(\theta_n - \varepsilon_n t) \, and \, z_n(t) = x_n(\theta_n + \tfrac{1}{2}\bar{q}_n - \varepsilon_n t) .$$

Then r_n is a bounded sequence, and the sequences of functions $y_n(t)$ and $z_n(t)$ are equicontinuous and uniformly bounded. If one takes limits

$$r_n \to r \geqslant 0 , \quad y_n(t) \to y(t) \quad and \quad z_n(t) \to z(t)$$

uniformly on compact intervals for some subsequence $n = n_j \to \infty$, then the functions $y(t)$ and $z(t)$ are C^1 and satisfy the transition layer equations $(3.54)_r$ and $(3.55)_r$ for all $t \in \mathbf{R}$. For all t one has

$$- B \leqslant y(t) \leqslant A \quad and \quad - B \leqslant z(t) \leqslant A .$$

Also, $\dot{y}_n(t)$ and $\dot{z}_n(t)$ converge respectively to $\dot{y}(t)$ and $\dot{z}(t)$ uniformly on compact intervals.

(iii) *If $y(t)$ and $z(t)$ are obtained by the limiting procedure in* (ii), *then $y(t)$ has at most one sign change on the real line. That is, there do not exist $t_1 < t_2 < t_3$ such that $y(t_1)y(t_2) < 0$ and $y(t_2)y(t_3) < 0$. The same holds true of $z(t)$.*

(iv) *If $y(t)$ and $z(t)$ are as in* (ii) *then they have opposite signs for large t. More precisely, if $y(t) \geqslant 0$ for large t, then $z(t) \leqslant 0$ for large t, and if $y(t) \leqslant 0$ for large t then $z(t) \geqslant 0$ for large t. Similarly, $y(t)$ and $z(t)$ have opposite signs as $t \to -\infty$, that is, for $-t$ large. (Note that the signs of $y(t)$ as $t \to \infty$ and $t \to -\infty$ may or may not be different, depending on whether or not $y(t)$ undergoes a sign change.)*

(v) *If $y(t)$ and $z(t)$ are as in* (ii) *and if $\lim_{t \to \infty} y(t) = L$ exists for some real number L, then $\lim_{t \to \infty} z(t) = f(L)$ and $f(f(L)) = L$. The corresponding result holds as $t \to -\infty$. Also, the roles of $y(t)$ and $z(t)$ may be exchanged in these results.*

PROOF. – (i) From the definitions of $y(t)$, $z(t)$, and r, and the $\bar{q}$-periodicity of $x(t)$, one has

$$\dot{y}(t) = -\varepsilon \dot{x}(\theta - \varepsilon t) = x(\theta - \varepsilon t) - f(x(\theta - \varepsilon t - 1)) = y(t) - f(z(t - r))$$

to give equation $(3.54)_r$. The derivation of equation $(3.55)_r$ is similar.

(ii) Theorem 3.2 implies the sequence r_n is bounded. The functions $y_n(t)$ and $z_n(t)$ are uniformly bounded between $-B$ and A because $x_n(t)$ also is, by Proposition 1.3. The equicontinuity of $y_n(t)$ and $z_n(t)$ follows from the uniform boundedness of $\dot{y}_n(t)$ and $\dot{z}_n(t)$, by the transition layer equations $(3.54)_{r_n}$ and $(3.55)_{r_n}$. The remainder of (ii) involves standard limiting arguments using Ascoli's theorem and the integrated forms of equations $(3.54)_{r_n}$ and $(3.55)_{r_n}$.

(iii) If there exist $t_1 < t_2 < t_3$ as in the statement of the theorem, then $y(t)$, and hence $y_n(t)$ for large n, vanish at least twice in the interval (t_1, t_3). Thus equation (3.56) implies $x_n(t)$ vanishes at least twice in an interval of length $(t_3 - t_1)\varepsilon_n$; but $(t_3 - t_1)\varepsilon_n < 1$ for large n, and this contradicts the fact that $x_n(t)$ is slowly oscillating.

(iv) Suppose that $y(t) \geqslant 0$ for large t, say for all $t \geqslant T$. From equation $(3.55)_r$ one has

$$\dot{z}(t) \geqslant z(t) \quad \text{if } t \geqslant T + r,$$

so if $z(t_0) > 0$ for some $t_0 \geqslant T + r$, then $z(t) \geqslant z(t_0) \exp [t - t_0]$ for $t \geqslant t_0$. This contradicts the boundedness of $z(t)$. Thus it follows that $z(t) \leqslant 0$ if $t \geqslant T + r$. A similar agreement shows that if $y(t) \leqslant 0$ for large t, then $z(t) \geqslant 0$ for large t.

The proof of (iv) as $t \to -\infty$ is slightly different. Suppose that $y(t) \geqslant 0$ for all $t \leqslant T$ but that (iv) fails in this case. Because $z(t)$ has at most one sign change, we may assume (possibly by decreasing T) that $z(t) \geqslant 0$ for all $t \leqslant T$, and that $z(t_0) > 0$

for some $t_0 \leqslant T$. Define

$$t_* = \sup \{t \geqslant t_0 : y(s) \geqslant 0 \text{ and } z(s) \geqslant 0 \text{ for all } s \in [t_0, t]\} .$$

If $t_* < \infty$, then on $[t_0, t_* + r]$ one has $\dot{y}(t) \geqslant y(t)$ and $\dot{z}(t) \geqslant z(t)$ by equations $(3.54)_r$ and $(3.55)_r$, and hence $y(t) \geqslant 0$ and $z(t) \geqslant 0$ there. This contradicts the definition of t_*, so therefore $t_* = \infty$. But then $\dot{z}(t) \geqslant z(t)$ for all $t \geqslant t_0$, and so $z(t) \geqslant z(t_0) \exp[t - t_0] > 0$ for such t, which implies $z(t) \to \infty$ as $t \to \infty$. This contradicts the boundedness of $z(t)$ and thereby proves (iv).

(v) Suppose $\lim_{t \to \infty} y(t) = L$. Rewriting equation $(3.46)_r$ gives

$$(3.57) \qquad \frac{d}{ds} \left(\exp[t - s]z(s) \right) = - \exp[t - s]f(y(s - r))$$

for any t. Integrating equation (3.57) with respect to s from t to ∞, and using the boundedness of the solutions, gives

$$(3.58) \qquad z(t) = \int_t^\infty \exp[t - s]f(y(s - r)) \, ds .$$

It is now a simple exercise to show that equation (3.58) implies that $\lim_{t \to \infty} z(t) = f(L)$. Repeating this argument with equation $(3.45)_r$ shows that $\lim_{t \to \infty} y(t) = f(f(L))$, hence $f(f(L)) = L$.

If, on the other hand, one has $\lim_{t \to -\infty} y(t) = L$, then upon rewriting (3.58) as

$$z(t) = \int_0^\infty \exp[-s]f(y(t + s - r)) \, ds$$

one easily sees that $\lim_{t \to -\infty} z(t) = f(L)$. As before, $f(f(L)) = L$. $\qquad \square$

DEFINITION 3.3. – Let $\varepsilon_n \to 0^+$, let $x_n(t)$ be a slowly oscillating periodic solution of equation $(3.1)_{\varepsilon_n}$, let $\theta_n \in R$, and assume f satisfies hypotheses (H1) and (H2). Let $y(t)$, $z(t)$, and r be obtained by the limiting procedure in part (ii) of Proposition 3.1. Then $(y(t), z(t), r)$ is called a *transition layer solution associated with the sequence* $(\varepsilon_n, x_n(t), \theta_n)$. Note that because one takes the limit of a subsequence, this definition allows the possibility that $(y(t), z(t), r)$ might not be unique.

REMARK 3.5. – Observe that *a priori* it is possible that a transition layer solution $(y(t), z(t))$ might be identically zero. Part of our effort when we use Proposition 3.1 in the next section will be to exclude this possibility.

We end this section with a technical lemma about the transition layer equations; this lemma will be used later.

LEMMA 3.3. – *Assume that f satisfies hypothesis* (H3). *Let $r \geqslant 0$, and let $y(t)$ and $z(t)$ be solutions to the transition layer equations* $(3.54)_r$ *and* $(3.55)_r$ *for $t \in \mathbf{R}$. Suppose that for some $c > 0$ these solutions satisfy*

$$c \leqslant y(t) \leqslant A \quad and \quad -B \leqslant z(t) \leqslant -c \quad for\ all\ t \in \mathbf{R}.$$

Then in fact

$$y(t) = a \quad and \quad z(t) = -b \quad for\ all\ t \in \mathbf{R}$$

where a and b are as in (H3).

PROOF. – Define two nonempty compact connected sets I and J by

$$I = \text{closure } \{y(t): t \in \mathbf{R}\} \quad and \quad J = \text{closure } \{z(t): t \in \mathbf{R}\} \,,$$

and denote them by

$$(3.59) \qquad I = [a_1, a_2] \subseteq [c, A] \quad and \quad J = [-b_2, -b_1] \subseteq [-B, -c] \,.$$

We wish to show that $I = \{a\}$ and $J = \{-b\}$.

Clearly $a_2 = \sup_t y(t)$, so there exists a sequence $t_n \in \mathbf{R}$ such that $y(t_n) \to a_2$ as $n \to \infty$. Moreover, the differential equation $(3.54)_r$ and the boundedness of $y(t)$ and $z(t)$ and their derivatives imply that the derivative $\dot{y}(t)$ is uniformly continuous on $\mathbf{R}$; from this it easily follows that $\dot{y}(t_n) \to 0$ as $n \to \infty$. Taking this limit in equation $(3.54)_r$ implies $f(z(t_n - r)) \to a_2$, and consequently $a_2 \in f(J)$. In the same fashion one has $a_1 \in f(J)$, and therefore $I = [a_1, a_2] \subseteq f(J)$ holds because J is connected. Similarly $J \subseteq f(I)$, and hence

$$(3.60) \qquad\qquad\qquad I \subseteq f(f(I)) \,.$$

Now consider the map $f^2 = f \circ f$ on the interval $[c, A]$. Without loss of generality we may assume $f^2([c, A]) \subseteq [c, A]$; this inclusion holds if c is chosen sufficiently small, because (H3) implies that $f^2(x) > x$ if $x \in (0, a)$. Therefore, Proposition 1.2 and hypothesis (H3) together imply that

$$(3.61) \qquad\qquad\qquad \bigcap_{n=0}^{\infty} f^{2n}([c, A]) = \{a\} \,.$$

However, the inclusions (3.59) and (3.60) yield

$$(3.62) \qquad\qquad\qquad \bigcap_{n=0}^{\infty} f^{2n}([c, A]) \supseteq \bigcap_{n=0}^{\infty} f^{2n}(I) = I \,,$$

and combining (3.61) and (3.62) proves that $I = \{a\}$. A similar argument shows that $J = \{-b\}$. $\square$

4. – The asymptotic shape of periodic solutions as $\varepsilon \to 0^+$.

The work of the previous section provides only weak information about the behaviour of slowly oscillating periodic solutions of equation $(3.1)_\varepsilon$ as $\varepsilon \to 0^+$. For example, nothing proved so far precludes the norms of such solutions approaching zero as $\varepsilon \to 0^+$. In this section, we shall obtain sharp information about the asymptotic shape of slowly oscillating periodic solutions of $(3.1)_\varepsilon$; in particular, one of our lemmas below proves that the norms of such solutions are bounded away from zero as $\varepsilon \to 0^+$. Further results imply that $|x(t)|$ remains uniformly bounded away from zero, by an amount independent of ε, everywhere on $[0, \bar{q}]$ except on small neighborhoods of $t = 0$, q and $\bar{q}$ of size $O(\varepsilon)$. (As before, q and $\bar{q}$ are the first two zeros of the slowly oscillating periodic solution $x(t)$.) The behaviour of $x(t)$ within these $O(\varepsilon)$-neighborhoods is described by certain solutions of the transition layer equations $(3.54)_r$ and $(3.55)_r$.

If f satisfies hypotheses (H2) and (H3), then we will show that slowly oscillating periodic solutions $x(t)$ converge, as $\varepsilon \to 0^+$, to the discontinuous square wave function

$$\operatorname{sqw}(t) = \begin{cases} a & \text{if } 2n < t < 2n + 1 \\ -b & \text{if } 2n + 1 < t < 2n + 2 \end{cases}$$

uniformly on compact subsets of $\mathbf{R} - \mathbf{Z}$ where $\mathbf{Z}$ is the set of integers. Near the integers, where sqw (t) is discontinuous, the behaviour of $x(t)$ can be very complicated. Typically, a « non-uniform » convergence to sqw (t), with features similar to that of the Gibbs phenomenon of classical Fourier series, can be proved to occur. On the other hand, if f is monotone throughout $[-b, a]$, then the convergence of $x(t)$ to sqw (t) is nice.

We begin by proving that the norms of slowly oscillating periodic solutions do not approach zero as $\varepsilon \to 0^+$.

LEMMA 4.1. – *Assume that f satisfies hypothesis* (H2). *Then there do not exist sequences $\varepsilon_n \to 0^+$ and $x_n(t)$ such that $x_n(t)$ is a slowly oscillating periodic solution of equation $(3.1)_{\varepsilon_n}$, with norm $\|x_n\| = \max_t |x_n(t)|$ satisfying $\|x_n\| \to 0$ as $n \to \infty$.*

PROOF. – Suppose to the contrary that such sequences ε_n and $x_n(t)$ exist. Let $\bar{q}_n$ denote the period of $x_n(t)$, define $r_n > 0$ by $\bar{q}_n = 2(1 + \varepsilon_n r_n)$, and let $\varrho_n \in [0, \bar{q}_n]$ be a point where $|x_n(t)|$ achieves its maximum $\|x_n\|$. By Theorem 3.2 the sequence $r_n > 0$ is bounded (note that the hypotheses of Theorem 3.2 are satisfied once f is modified appropriately outside a small neighborhood of zero). Define the functions

$$(4.1) \qquad y_n(t) = \frac{x_n(\varrho_n - \varepsilon_n t)}{\|x_n\|} \qquad \text{and} \qquad z_n(t) = \frac{x_n(\varrho_n + \frac{1}{2}\bar{q}_n - \varepsilon_n t)}{\|x_n\|}$$

and observe that they satisfy the system of equations

$$\dot{y}_n(t) = y_n(t) + kz_n(t - r_n) - R(z_n(t - r_n), \|x_n\|)$$

(4.2)

$$\dot{z}_n(t) = z_n(t) + ky_n(t - r_n) - R(y_n(t - r_n), \|x_n\|) ,$$

where R is the continuous function (1.34) considered in the proof of Lemma 1.6, and $k = -f'(0) > 1$ as in (H2). Also note that

$$|y_n(0)| = 1 \quad \text{and} \quad |y_n(t)| , \quad |z_n(t)| \leqslant 1 \quad \text{for all } t \in \boldsymbol{R} .$$

Using the fact that $R(y, 0) = 0$, one sees that the terms involving R in the right hand sides of (4.2) tend to zero uniformly for $t \in \boldsymbol{R}$. As in the proof of Lemma 1.6, one may use Ascoli's theorem to take limits in equations (4.2) after passing to a subsequence. One thus obtains C^1 functions $y(t)$ and $z(t)$ which satisfy the linear system

(4.3) $$\dot{y}(t) = y(t) + kz(t - r) , \quad \dot{z}(t) = z(t) + ky(t - r)$$

for all real t, and also satisfy

$$|y(0)| = 1 \quad \text{and} \quad |y(t)| , \quad |z(t)| \leqslant 1 \quad \text{for all } t \in \boldsymbol{R} .$$

In addition, because $x_n(t)$ is a slowly oscillating periodic solution and because $\varepsilon_n \to 0^+$ one has from (4.1) that $y(t)$ has at most one sign change on the real line; that is, there do not exist $t_1 < t_2 < t_3$ such that $y(t_1)y(t_2) < 0$ and $y(t_2)y(t_3) < 0$.

We note (see the appendix) that if $W(t)$ is a nontrivial solution, for all $t \in \boldsymbol{R}$, of an n-dimensional linear autonomous retarded functional differential equation, and if $\sup_{t \in \boldsymbol{R}} |W(t)| < \infty$, then $W(t)$ has the form

(4.4) $$W(t) = \sum_{j=1}^{p} w_j \exp [i\nu_j t]$$

where for each j the coefficient $w_j \in \boldsymbol{C}^n$ is a nonzero constant vector, where $\nu_j \in \boldsymbol{R}$, and where the exponent $\zeta_j = i\nu_j$ is a root of the characteristic equation of the differential equation. Also, it is proved in the appendix that for our system (4.3) the characteristic equation

(4.5) $$\begin{vmatrix} \zeta - 1 & - k \exp [- \zeta r] \\ - k \exp [- \zeta r] & \zeta - 1 \end{vmatrix} = 0$$

has at most two roots ζ (counting multiplicity) with real part zero, and that $\zeta = 0$ is not a root of (4.5). Therefore, the solution $W(t) = (y(t), z(t))$ obtained above
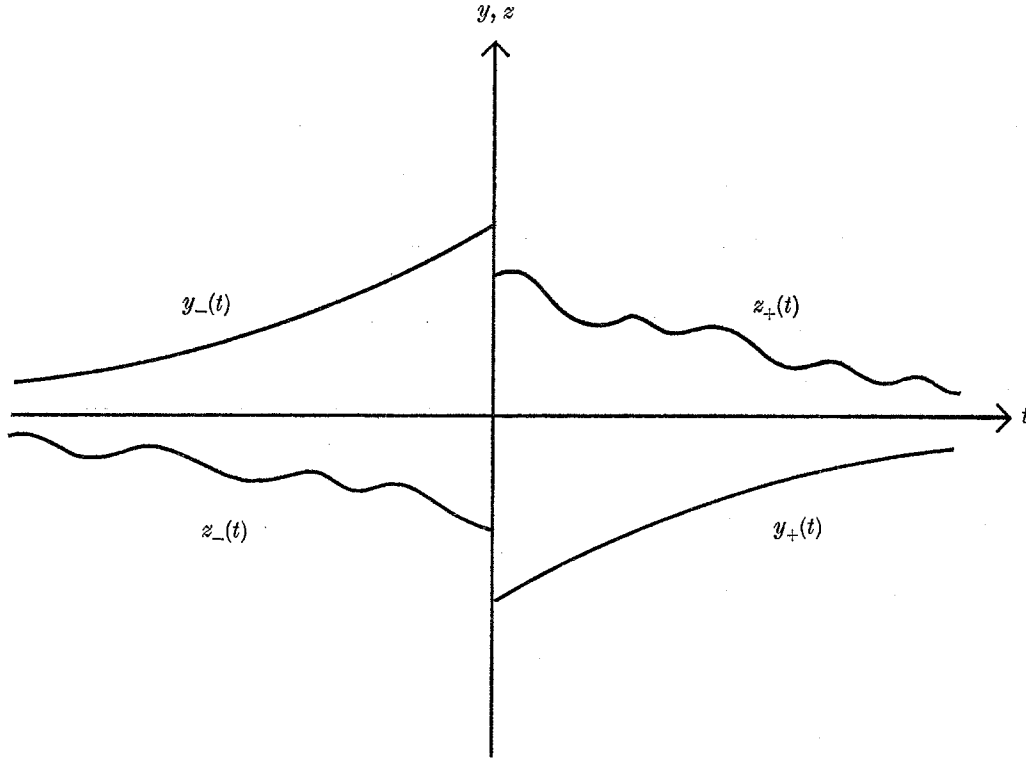
Fig. 11. Solutions $y_{\pm}(t)$, $z_{\pm}(t)$ of Lemma 4.2.

must have the form (4.4) for $p = 2$, where $\nu_2 = -\nu_1 \neq 0$. Because $y(t)$ is real valued, one may take the real part of (4.4) to obtain $y(t) = K \sin(\nu_1 t + \theta)$ for some real numbers $K \neq 0$ and θ. This contradicts the fact, noted earlier, that $y(t)$ has at most one sign change. $\quad\square$

The next lemma is essentially a linear perturbation result which describes the behaviour of solutions on the stable and unstable manifolds of zero for the transition layer equations $(3.54)_r$ and $(3.55)_r$. Figure 11 depicts nontrivial solution $(y_+(t)$, $z_+(t))$ and $(y_-(t), z_-(t))$ of these equations which satisfy the hypotheses of this lemma. The conclusion of the lemma is that such solutions cannot, in fact, simultaneously exist for these equations.

LEMMA 4.2. – *Assume that* $f: \mathbf{R} \to \mathbf{R}$ *is a continuous function which satisfies* $f(0) = 0$, *and is differentiable at* $x = 0$ *with* $-f'(0) = k > 1$. *Suppose also that* $r \geqslant 0$, *that* $y_+, z_+ : [-r, \infty) \to \mathbf{R}$ *are* C^1 *functions satisfying the transition layer equations* $(3.54)_r$ *and* $(3.55)_r$ *for all* $t \geqslant 0$, *and that* $y_-, z_- : (-\infty, 0] \to \mathbf{R}$ *are* C^1 *functions satisfying* $(3.54)_r$ *and* $(3.55)_r$ *for the same* r, *for all* $t \leqslant 0$. *Finally, suppose there exist constants*

$T_+ \geqslant 0$ and $T_- \geqslant 0$ such that

(4.6) $y_+(t) \leqslant 0$, $\dot{y}_+(t) \geqslant 0$,   and $z_+(t) \geqslant 0$ if $t \geqslant T_+$,

(4.7) $y_-(t) \geqslant 0$, $\dot{y}_-(t) \geqslant 0$,   and $z_-(t) \leqslant 0$ if $t \leqslant -T_-$,

and

(4.8) $\lim_{t \to \infty} \big(y_+(t), z_+(t) \big) = \lim_{t \to -\infty} \big(y_-(t), z_-(t) \big) = (0, 0)$.

Then either $(y_+(t), z_+(t)) = (0, 0)$ for all $t \geqslant -r$, or else $(y_-(t), z_-(t)) = (0, 0)$ for all $t \leqslant 0$.

Rather than prove Lemma 4.2 directly, it will be convenient first to establish various properties of the solutions $(y_+(t), z_+(t))$ and $(y_-(t), z_-(t))$.

LEMMA 4.2A. – *Let the assumptions and notation be as in Lemma 4.2 and assume the strict inequality $r > 0$. Fix a constant $\varkappa > 1$. Then there exists a constant $T_0 = T_0(\varkappa) \geqslant \max \{T_+, T_-\}$ such that if $T_0 \leqslant t \leqslant s$, then*

(4.9) $z_+(s) \leqslant \varkappa z_+(t)$,

and if $s \leqslant t \leqslant -T_0$, then

(4.10) $|z_-(s)| \leqslant \varkappa |z_-(t)|$.

If there exists $t_0 \geqslant T_0$ such that either $y_+(t_0) = 0$ or $z_+(t_0) = 0$, then $(y_+(t), z_+(t)) = (0, 0)$ for all $t \geqslant -r$; and if either $y_-(t_0) = 0$ or $z_-(t_0) = 0$ for some $t_0 \leqslant -T_0$ then $(y_-(t), z_-(t)) = (0, 0)$ for all $t \leqslant 0$.

The functions $y_+(t)$ and $z_+(t)$ are integrable on $[-r, \infty)$ and the functions $y_-(t)$ and $z_-(t)$ are integrable on $(-\infty, 0]$. There exists a number $T_* \geqslant \max \{T_+, T_-\}$ and a constant $\beta > 0$ such that

(4.11) $|y_+(t)| + |z_+(t)| \geqslant \beta \int_t^\infty |y_+(s)| + |z_+(s)| \, ds$ if $t \geqslant T_*$,

and

(4.12) $|y_-(t)| + |z_-(t)| \geqslant \beta \int_{-\infty}^t |y_-(s)| + |z_-(s)| \, ds$ if $t \leqslant -T_*$.

PROOF. – It will be convenient in this proof and in the proof of Lemma 4.2 to introduce functions R_1 and η defined by the formuls

(4.13) $f(y) = -ky + R_1(y)$

and

$$(4.14) \qquad \eta(\delta) = \sup_{0<|v|\leqslant\delta} \left| \frac{R_1(y)}{y} \right| .$$

The assumptions on f insure that

$$(4.15) \qquad \lim_{v\to 0} \frac{R_1(y)}{y} = \lim_{\delta\to 0} \eta(\delta) = 0 .$$

(Compare R_1 with the function R given by (1.34).) Also, note that η is monotone increasing.

Suppose these exists a constant $\varkappa > 1$ such that the first part of the lemma is false. For definiteness we shall assume that for every real number T, there exist t and s with $T\leqslant t < s$ such that (4.9) fails, so that

$$\varkappa z_+(t) < z_+(s) .$$

We shall obtain a contradiction. The analogous argument for the function $z_-(t)$ will be omitted.

We shall construct a sequence

$$T_+ \leqslant r_1 \leqslant t_1 < s_1 \leqslant r_2 \leqslant \ldots \leqslant r_n \leqslant t_n < s_n \leqslant r_{n+1} \leqslant \ldots$$

such that for each n one has

$$(4.16) \qquad \varkappa z_+(t_n) < z_+(s_n) ,$$

$$(4.17) \qquad \dot{z}_+(r_n) \leqslant 0 ,$$

and

$$(4.18) \qquad \dot{z}_+(t_n) = \dot{z}_+(s_n) = 0$$

and that moreover

$$(4.19) \qquad \lim_{n\to\infty} t_n = \infty .$$

We proceed by induction. If r_n has been defined and satisfies (4.17), then by assumption there exist t'_n and s'_n such that $r_n\leqslant t'_n < s'_n$ and $\varkappa z_+(t'_n) < z_+(s'_n)$. Select $t_n\in [r_n, s'_n]$ such that

$$z_+(t_n) = \min \{z_+(t)\colon r_n\leqslant t\leqslant s'_n\}$$

and select $s_n\geqslant t_n$ such that

$$z_+(s_n) = \max \{z_+(t)\colon t\geqslant t_n\} .$$

Our construction insures that $r_n \leqslant t_n < s_n$ and that (4.16) and (4.18) hold; this follows from the assumptions (4.6) and (4.8) on $z_+(t)$ and because $\dot{z}(r_n) \leqslant 0$. Now select $r_{n+1} \geqslant 1 + s_n$ such that $\dot{z}_+(r_{n+1}) \leqslant 0$ (such a point must exist). To begin the induction select $r_1 \geqslant T_+$ such that $\dot{z}_+(r_1) \leqslant 0$. It is clear that the sequence t_n now defined satisfies (4.19).

The transition layer equation $(3.55)_r$ now gives, using (4.18),

$$z_+(t_n) = f(y_+(t_n - r)) = -ky_+(t_n - r) + R_1(y_+(t_n - r))$$

and

$$z_+(s_n) = f(y_+(s_n - r)) = -ky_+(s_n - r) + R_1(y_+(s_n - r)) ,$$

and it follows, using (4.16), that

$$(4.20) \qquad \varkappa[- ky_+(t_n - r) + R_1(y_+(t_n - r))] < - ky_+(s_n - r) + R_1(y_+(s_n - r)) .$$

From (4.20), from the inequality $y_+(t_n - r) \leqslant y_+(s_n - r) \leqslant 0$ (which follows from the monotonicity (4.6) of $y_+(t)$), and from the definition (4.14) of $\eta(\delta)$ one obtains

$$(4.21) \qquad (\varkappa - 1)k|y_+(t_n - r)| < (\varkappa + 1)\eta(|y_+(t_n - r)|)|y_+(t_n - r)| .$$

The strict inequality in (4.21) implies that $y_+(t_n - r) \neq 0$, and so cancelling $|y_+(t_n - r)|$ gives

$$(4.22) \qquad (\varkappa - 1)k < (\varkappa + 1)\eta(|y_+(t_n - r)|) .$$

Because $\lim_{n \to \infty} \eta(|y_+(t_n - r)|) = 0$ (by (4.8), (4.15), and (4.19)), the inequality (4.22) gives a contradiction for large n.

Next, we wish to prove that if $y_+(t_0) = 0$ or $z_+(t_0) = 0$ for some $t_0 \geqslant T_0$, then $(y_+(t), z_+(t)) = (0, 0)$ for all $t \geqslant - r$. The corresponding proof for $(y_-(t), z_-(t))$ will be treated later. Suppose without loss of generality that $z_+(t_0) = 0$ for some $t_0 \geqslant T_0$; then the first part of the lemma implies that $z_+(t) = 0$ for all $t \geqslant t_0$, so from equation $(3.55)_r$ one has

$$(4.23) \qquad f(y_+(t - r)) = 0 \quad \text{if } t \geqslant t_0 .$$

Because there exists $\delta > 0$ such that $f(x) \neq 0$ if $0 < |x| \leqslant \delta$, and because $\lim_{t \to \infty} y_+(t - r) = 0$, it follows from (4.23) that there exists $t_1 \geqslant t_0 - r$ such that $y_+(t) = 0$ if $t \geqslant t_1$. We thus conclude that $(y_+(t), z_+(t)) = (0, 0)$ for all $t \geqslant \max\{t_0, t_1\}$.

To complete the proof $\left(\text{in the case of } (y_+(t), z_+(t))\right)$ define t_* by

$$(4.24) \qquad t_* = \inf \{t \in [- r, \infty): (y_+(s), z_+(s)) = (0, 0) \text{ for all } s \geqslant t\} .$$

Thus $t_* \leqslant \max \{t_0, t_1\}$, and one wants $t_* = -r$. Because $(\dot{y}_+(t), \dot{z}_+(t)) = (0, 0)$ if $t \geqslant \max \{t_*, 0\}$, the transition layer equations $(3.54)_r$ and $(3.55)_r$ imply that

$$(4.25) \qquad f(y_+(t-r)) = f(z_+(t-r)) = 0 \qquad \text{if } t \geqslant \max \{t_*, 0\} .$$

By using equation (4.25) and the fact that $x = 0$ is an isolated solution of $f(x) = 0$, it is not hard to see that the set E defined by

$$(4.26) \quad E = \{t \in [t_{**}, \infty): (y_+(t), z_+(t)) = (0, 0)\} \qquad \text{where } t_{**} = \max \{t_*, 0\} - r$$

is both open and closed in the relative topology on $[t_{**}, \infty)$. Connectedness then implies $E = [t_{**}, \infty)$, so one concludes

$$t_* \leqslant t_{**}$$

from the definition (4.24) of t_*. From this, from (4.26), and from the inequalities $t_* \geqslant -r$ and $r > 0$, one concludes that $t_* = -r$, as desired.

If $y_-(t_0) = 0$ or $z_-(t_0) = 0$ for some $t_0 \leqslant -T_0$, essentially the same argument used above shows that there exists t_2 such that $(y_-(t), z_-(t)) = (0, 0)$ if $t \leqslant t_2$. Note, however, the remainder of the proof is much easier than in the case of $(y_+(t), z_+(t))$. The transition layer equations $(3.54)_r$ and $(3.55)_r$ give

$$\dot{y}_-(t) = y_-(t) \qquad \text{and} \qquad \dot{z}_-(t) = z_-(t) \qquad \text{if } t_2 \leqslant t \leqslant t_2 + r .$$

As $(y_-(t_2), z_-(t_2)) = (0, 0)$, it follows that $(y_-(t), z_-(t)) = (0, 0)$ if $t_2 \leqslant t \leqslant t_2 + r$. Continuing in this way, one obtains $(y_-(t), z_-(t)) = (0, 0)$ for all $t \leqslant 0$.

It remains to prove the integrability of $y_\pm(t)$ and $z_\pm(t)$, and to establish the inequalities (4.11) and (4.12). We shall restrict attention to the case of $(y_+(t), z_+(t))$ since the proof of the corresponding result for $(y_-(t), z_-(t))$ is analogous, but easier.

Fix $\delta > 0$ and $\gamma > 1$ so that

$$(4.27) \qquad\qquad |f(x)| \geqslant \gamma |x| \qquad \text{if } |x| \leqslant \delta$$

and fix $T_* \geqslant T_+$ so that $|y_+(t)| \leqslant \delta$ and $|z_+(t)| \leqslant \delta$ for all $t \geqslant T_*$. Recalling from (4.6) that $y_+(t) \leqslant 0 \leqslant z_+(t)$ for all $t \geqslant T_+$, one obtains from equations $(3.54)_r$ and $(3.55)_r$ that for any numbers t and t' satisfying $T_* \leqslant t < t'$

$$(4.28) \quad |y_+(t)| + |z_+(t)| + y_+(t') - z_+(t') = \int_t^{t'} \dot{y}_+(s) - \dot{z}_+(s) \, ds =$$

$$= \int_t^{t'} [y_+(s) + f(y_+(s))] - [z_+(s) + f(z_+(s))] \, ds + \int_{t-r}^{t} f(y_+(s)) - f(z_+(s)) \, ds +$$

$$+ \int_{t'-r}^{t'} y_+(s) - z_+(s) \, ds \geqslant (\gamma - 1) \int_t^{t'-r} |y_+(s)| + |z_+(s)| \, ds + \int_{t'-r}^{t'} y_+(s) - z_+(s) \, ds .$$

Letting $t' \to \infty$ in (4.28) and using the fact that $(y_+(t'), z_+(t')) \to (0, 0)$, one obtains

$$|y_+(t)| + |z_+(t)| \geqslant (\gamma - 1) \int_t^\infty |y_+(s)| + |z_+(s)|\, ds$$

which is the desired estimate.   □

Remark 4.1. – Formulas (4.9) and (4.10) of Lemma 4.2A say that the solutions $z_+(t)$ and $z_-(t)$ are « almost monotone » as $t \to \infty$ and $t \to -\infty$ respectively. If in addition to the hypotheses of this lemma one assumes that the function f is monotone on some neighborhood of the origin (which is the case if f is C^1), then $z_+(t)$ and $z_-(t)$ are indeed monotone for sufficiently large t and $-t$. Moreover, the proof in this case is considerably simpler. Assume, to be definite, that f is monotone (decreasing) on an open interval I containing the origin. We claim that one can select T_0 so that $\dot{z}_+(t) \leqslant 0$ for all $t \geqslant T_0$, and $\dot{z}_-(t) \leqslant 0$ for all $t \leqslant -T_0$. To prove this result for $z_+(t)$, choose $T_0 \geqslant T_+ + r$ so that $y_+(t)$ and $z_+(t)$ lie in I for all $t \geqslant T_0 - r$, and so that $\dot{z}_+(T_0) \leqslant 0$. If $z_+(t)$ were not monotone decreasing in the interval $[T_0, \infty)$, the same argument used in the first part of the proof of Lemma 4.2A would show that there exist numbers t_0 and s_0 with $T_0 \leqslant t_0 < s_0$ such that $z_+(t_0) < z_+(s_0)$ and $\dot{z}_+(t_0) = \dot{z}_+(s_0) = 0$. The transition layer equations now imply that

$$z_+(t_0) = f(y_+(t_0 - r)) < z_+(s_0) = f(y_+(s_0 - r)) .$$

However, this contradicts the fact that $y_+(t)$ is monotone increasing in $[T_+, \infty)$ and f is monotone decreasing in I. The proof of the corresponding result for $z_-(t)$ is analogous and will be omitted.

Remark 4.2. – The assumption that $r > 0$ was needed in the proof of Lemma 4.2A, although this result is still true when $r = 0$ (however, we will not need to know this fact to establish our later results). In the case $r = 0$, observe that the transition layer equations (3.54)$_r$ and (3.55)$_r$ become the system of ordinary differential equations

$$(4.29) \qquad\qquad \dot{y} = y - f(z) , \quad \dot{z} = z - f(y) .$$

The only change in the proof of Lemma 4.2A that must be made when $r = 0$ is in showing that if $y_+(t_0) = 0$ or $z_+(t_0) = 0$ for some $t_0 \geqslant T$, then $(y_+(t), z_+(t)) = (0, 0)$ for all $t \geqslant -r$, and in proving the corresponding result for $(y_-(t), z_-(t))$. If, say, $z_+(t_0) = 0$ for some $t_0 \geqslant T_0$, then as before one sees that $(y_+(t), z_+(t)) = (0, 0)$ for all sufficiently large t, say $t \geqslant t_*$. To prove now that $(y_+(t), z_+(t)) = (0, 0)$ for all $t \geqslant -r$, one needs to know that the initial value problem

$$(4.30) \qquad\qquad (y(t_*), z(t_*)) = (0, 0)$$

for the system (4.29) has a unique solution. If the function f were lipschitz in a neighborhood of the origin, then the vector field for (4.29) would satisfy a lipschitz condition at (0, 0) so the uniqueness theorem for ordinary differential equation would apply. However, there are pathological examples of functions f for which $f'(0)$ exists but which are not lipschitz near zero. Nevertheless, by using the fact that $f(0) = 0$ and $f'(0)$ exists, one can still conclude that the zero solution is the unique solution of the initial value problem (4.29) and (4.30), and hence that $(y_+(t), z_+(t)) = (0, 0)$ for all $t \geqslant -r$. The proof of this is similar to that of the standard uniqueness theorem; details are left to the reader. The corresponding result for $(y_-(t), z_-(t))$ also follows from this uniqueness argument.

As noted earlier, Lemma 4.2 can be viewed as a perturbation result for the linearization (4.3) of the transition layer equations $(3.54)_r$ and $(3.55)_r$ about the origin. The basic idea of the proof is simple, but there are some technical complications, and it may be useful to discuss its outline before giving details. If one seeks a solution of (4.3) of the form $(y(t), z(t)) = w \exp [\zeta t]$ for a nonzero complex vector $w \in \mathbf{C}^2$ and $\zeta \in \mathbf{C}$, one is led to the characteristic equation (4.5) for ζ. This equation can be rewritten as

$$(\zeta - 1)^2 - k^2 \exp [-2\zeta r] = 0$$

and then factored, so that each root ζ satisfies either

(4.31)
$$\zeta = 1 + k \exp [-\zeta r]$$

or

(4.32)
$$\zeta = 1 - k \exp [-\zeta r].$$

If ζ satisfies equation (4.31) then one can verify that

$$(y(t), z(t)) = (1, 1) \exp [\zeta t]$$

is a solution of the linearized system (4.3), whereas if ζ satisfies (4.32), then

(4.33)
$$(y(t), z(t)) = (1, -1) \exp [\zeta t]$$

is a solution.

From the sign conditions (4.6) and (4.7) one expects that if neither $(y_+(t), z_+(t))$ nor $(y_-(t), z_-(t))$ is identically zero, then the asymptotic behavior of these solutions should be similar to the behavior of the eigensolution (4.33) for certain eigenvalues $\zeta_+ < 0$ and $\zeta_- > 0$, both of which are roots of equation (4.32). However, it is not difficult to see that since $k > 1$, equation (4.32) cannot simultaneously have solutions $\zeta_+ \in (-\infty, 0)$ and $\zeta_- \in (0, \infty)$. Hence one expects to conclude that either $(y_+(t), z_+(t))$ or $(y_-(t), z_-(t))$ is identically zero.

Although one could probably prove Lemma 4.2 directly with the above argument if $f(x)$ dependend linearly on x, the general nonlinear situation is not quite as simple. For example, it is not *a priori* clear that if the solution $(y_+(t), z_+(t))$ of the nonlinear equation is not identically zero, then it must be asymptotic to some eigensolution. One has to consider the possibility that

$$\lim_{n \to \infty} \frac{y_+(t_n)}{y_+(t_n - r)} = 0 \quad \text{or} \quad \lim_{n \to \infty} \frac{z_+(t_n)}{z_+(t_n - r)} = 0$$

for some sequence $t_n \to \infty$, so that the solution would decay in a « super-exponential » fashion in contrast to the exponential decay of (4.33).

Rather than deal with such matters, the proof of Lemma 4.2 given below follows a somewhat different outline. The transition layer equations $(3.54)_r$ and $(3.55)_r$ are viewed as a non-autonomous perburbation of the linear system (4.3), that is, as

$$(4.34) \qquad \dot{W}(t) = L(W_t) + \varrho(t)$$

where

$$W(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}, \quad W_t(s) = W(t + s) \quad \text{for} \ -r \leqslant s \leqslant 0,$$

$L: C([-r, 0], \mathbf{R}^2) \to \mathbf{R}^2$ is the linear operator

$$(4.35) \qquad L \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \varphi(0) + k\psi(-r) \\ \psi(0) + k\varphi(-r) \end{pmatrix} \quad \text{for} \ \varphi, \psi \in C[-r, 0],$$

and where $\varrho(t)$ is given by

$$\varrho(t) = \begin{pmatrix} -R_1(z(t-r)) \\ -R_1(y(t-r)) \end{pmatrix}.$$

(Recall the definition (4.13) of the remainder function R_1.)

Following [25] one may associate with any eigensolution $W(t) = w \exp [\zeta t]$ of a linear autonomous differential-delay equation in $\mathbf{R}^n$ a canonical projection onto the eigenspace in $C([-r, 0], \mathbf{R}^n)$. Applying such a projection to the solution W_t of an inhomogeneous linear equation of the form (4.34) yields a linear constant coefficient ordinary differential equation, with an inhomogeneous forcing term related to $\varrho(t)$. In the particular case of the operator L given by (4.35) and eigensolution (4.33) above (where ζ satisfies equation (4.32)), the canonical projection has a one-dimensional range. With respect to a conveniently chosen basis element in this range, the coordinate value $v(t)$ of this projection is given by the formula

$$v(t) = z(t) - y(t) - k \int_{t-r}^{t} \exp [\zeta(t - s - r)](z(s) - y(s)) \, ds.$$

Indeed, one easily sees that if ζ is a root of equation (4.32) and $(y(t), z(t))$ satisfies the transition layer equations (3.54)$_r$ and (3.55)$_r$ on some interval, then $v(t)$ satisfies the ordinary differential equation

(4.36) $\dot{v}(t) = \zeta v(t) + \sigma(t)$ where $\sigma(t) = R_1(z(t-r)) - R_1(y(t-r))$.

The idea of the following proof is that because $z(t) - y(t)$ is «almost monotonic» and of one sign $\big($where $(y(t), z(t)) = (y_+(t), z_+(t))$ or $(y_-(t), z_-(t))\big)$, it is possible to choose ζ so that the magnitude of $v(t)$ is comparable to the value $|z(t) - y(t)| = |z(t)| + |y(t)|$. However, the forcing term $\sigma(t)$ is of smaller order due to the estimate (4.15). Using the variation of constants formula to equate these two quantities yields a contradiction, unless $(y(t), z(t))$ is in fact the zero solution. Figure 12 schematically illustrates the situation in the phase space $C([-r, 0], \mathbf{R}^2)$
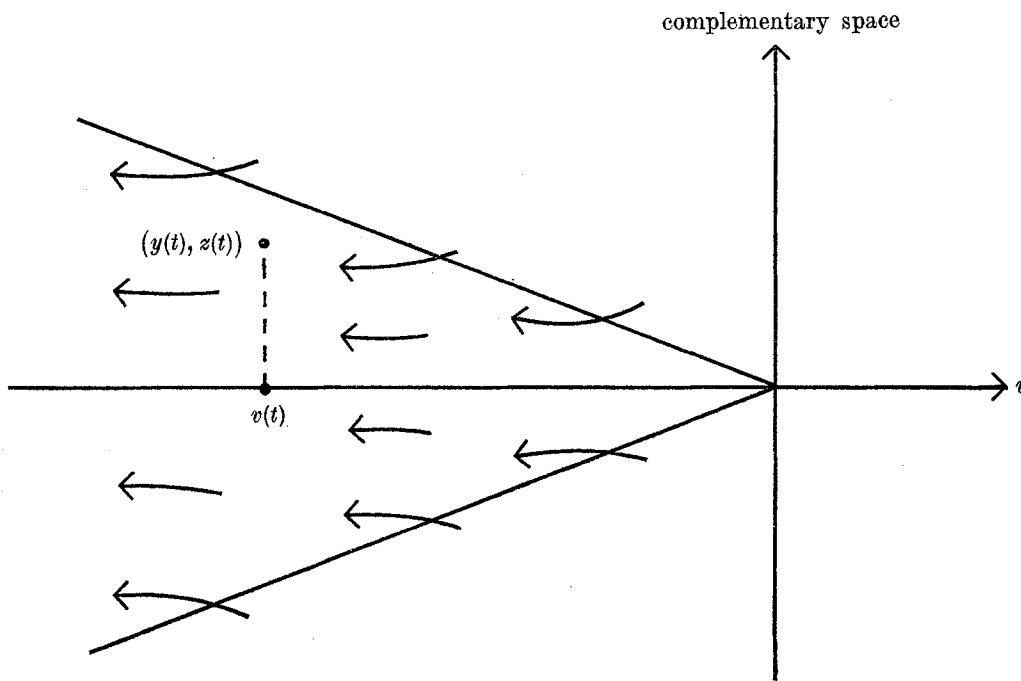

Fig. 12. The phase space setting in the proof of Lemma 4.2.

for the first case considered below, namely $\zeta > 0$. The horizontal axis represents the coordinate $v(t)$ in the range of the projection while the vertical axis is an infinite dimensional complement. Let $W(t) = (y_+(t), z_+(t))$. Then the sign and monotonicity conditions on $y_+(t)$ and $z_+(t)$ insure that W_t lies in the shaded cone. The arrows in the cone represent the vector field, and point outward because $\zeta > 0$. Therefore,

the only way that W_t can lie in the cone and follow the vector field to the origin as $t \to \infty$, is for W_t to lie at the vertex of the cone, that is, for $W(t) = (y_+(t), z_+(t))$ to be identically zero.

We make one final remark. Because $r \geqslant 0$, the roles of $(y_+(t), z_+(t))$ and $(y_-(t), z_-(t))$ are *not* entirely symmetric. Typically, the corresponding argument involving $(y_-(t), z_-(t))$ is simpler.

Proof of Lemma 4.2. — Consider the equation (4.32), each of whose solutions also satisfies the characteristic equation (4.5). We shall need the following fact whose proof is given in the appendix. If $r > 0$ and $k > 1$ then equation (4.32) has a (possibly complex) solution ζ satisfying

$$(4.37) \qquad\qquad 0 \leqslant \operatorname{Im} \zeta < \frac{\pi}{r}$$

and

$$(4.38) \qquad\qquad \zeta \neq 0 .$$

Assume that $r > 0$ (the case $r = 0$ will be considered later) and fix a root ζ of (4.32) satisfying (4.37) and (4.38). Several cases must now be considered, depending on whether ζ is real or complex, and depending on the sign of $\operatorname{Re} \zeta$. As the arguments in these cases are quite similar, we shall only present one of them fully, and merely outline the modifications needed to handle the remaining cases.

Let us consider the case that the root ζ of (4.32) chosen above is real and positive; assume that

$$\zeta > 0 .$$

We shall work with the solutions $(y_+(t), z_+(t))$ of the transition layer equations defined for $t \geqslant -r$, and prove that

$$(4.39) \qquad\qquad (y_+(t), z_+(t)) = (0, 0) \quad \text{for all } t \geqslant -r .$$

Suppose that (4.39) is false; then (4.6) and Lemma 4.2A imply that one has the strict inequalities

$$y_+(t) < 0 < z_+(t) \quad \text{if } t \geqslant T_0$$

where $T_0 = T_0(\varkappa)$ and $\varkappa > 1$ is fixed but arbitrary. (We use the notation of Lemma 4.2A throughout this proof.) To be specific, fix $\varkappa$ so that it satisfies the inequalities

$$(4.40) \qquad\qquad 1 < \varkappa < 1 + \zeta^{-1}(k - 1) .$$

Following the discussion above, introduce the functions

$$(4.41) \quad u(t) = z_+(t) - y_+(t) \quad \text{and} \quad v(t) = u(t) - k\int_{t-r}^{t} \exp\left[\zeta(t-s-r)\right]u(s)\,ds$$

and observe that for $t \geqslant 0$, $v(t)$ satisfies the linear ordinary differential equation (4.36) with the inhomogeneous term $\sigma(t)$ given by

$$\sigma(t) = R_1(z_+(t-r)) - R_1(y_+(t-r)) \,.$$

One also sees the following properties of $u(t)$ and $\sigma(t)$, which follow from Lemma 4.2A and from the definition (4.14) of $\eta(\delta)$:

$$(4.42) \qquad\qquad 0 < u(s) \leqslant \varkappa u(t) \qquad\qquad \text{if } s \geqslant t \geqslant T_0 \,,$$

and

$$(4.43) \qquad\qquad |\sigma(t)| \leqslant \eta(u(t-r))\,u(t-r) \qquad \text{if } t \geqslant T_0 + r \,.$$

In addition, $u(t)$ and hence $|\sigma(t)|$ are integrable on $[T_*, \infty)$, satisfying

$$(4.44) \qquad\qquad \int_{t}^{\infty} u(s)\,ds \leqslant \beta^{-1}u(t) \qquad \text{if } t \geqslant T_* \,.$$

The variation of constants formula applied to equation (4.36) yields

$$(4.45) \qquad\qquad v(t) = \exp\left[\zeta(t-t')v(t') + \int_{t'}^{t} \exp\left[\zeta(t-s)\right]\sigma(s)\,ds\right.$$

whenever $t, t' \geqslant 0$. Because $\exp\left[-\zeta t'\right]v(t') \to 0$ as $t' \to \infty$ and because $\exp\left[-\zeta s\right]\sigma(s)$ is integrable, one may let $t' \to \infty$ in (4.45) to give

$$(4.46) \qquad\qquad v(t) = -\int_{t}^{\infty} \exp\left[\zeta(t-s)\right]\sigma(s)\,ds \,.$$

The inequalities (4.42), (4.43), and (4.44) can now be used to bound the right hand side of (4.46) below; by doing this one obtains for $t \geqslant \max\{T_0, T_*\} + r$

$$v(t) \geqslant -\int_{t}^{\infty} \exp\left[\zeta(t-s)\right]\eta(u(s-r))\,u(s-r)\,ds \geqslant -\int_{t+r}^{\infty} \eta(\varkappa u(t))\,u(s-r)\,ds -$$

$$-\int_{t}^{t+r} \exp\left[\zeta(t-s)\right]\eta(\varkappa u(t-r))\,u(s-r)\,ds \geqslant -\eta(\varkappa u(t))\,\beta^{-1}u(t) -$$

$$-\eta(\varkappa u(t-r))\int_{t}^{t+r} \exp\left[\zeta(t-s)\right]u(s-r)\,ds \,.$$

Combining the above with (4.41) and using (4.9) to make further estimates, one obtains

$$(4.47) \quad [1 + \eta(\varkappa u(t))\beta^{-1}]u(t) \geqslant [k - \eta(\varkappa u(t-r))] \int_t^{t+r} \exp[\zeta(t-s)]u(s-r)\, ds \geqslant$$

$$\geqslant [k - \eta(\varkappa u(t-r))]\varkappa^{-1}u(t) \int_t^{t+r} \exp[\zeta(t-s)]\, ds =$$

$$= [k - \eta(\varkappa u(t-r))]\varkappa^{-1}\zeta^{-1}(1 - \exp[-\zeta r])u(t) .$$

Note that one requires

$$(4.48) \qquad\qquad k - \eta(\varkappa u(t-r)) > 0$$

in order to obtain the second inequality in (4.47); this certainly holds if t is sufficiently large. Assuming this to be so, upon cancelling $u(t) > 0$ in (4.47) and letting $t \to \infty$ one obtains the inequality

$$(4.49) \qquad\qquad 1 \geqslant k\varkappa^{-1}\zeta^{-1}(1 - \exp[-\zeta r]) ;$$

and substituting

$$(4.50) \qquad\qquad \exp[-\zeta r] = k^{-1}(1 - \zeta)$$

(obtained from (4.32)) into (4.49) and rearranging terms gives

$$(4.51) \qquad\qquad \varkappa \geqslant 1 + \zeta^{-1}(k-1) .$$

However, (4.51) contradicts (4.40). This completes the proof of the lemma in the case $\zeta > 0$.

A few minor modifications must be made in the above argument to handle other ranges of the eigenvalue ζ, but for the most part the analysis in these cases is the same. For example, if $\zeta = \mu + i\nu$ is complex and satisfies

$$(4.52) \qquad\qquad \mathrm{Re}\,\zeta = \mu \geqslant 0 \quad \text{and} \quad 0 < \mathrm{Im}\,\zeta = \nu < \frac{\pi}{r}$$

then the proof is almost the same as the one above up to and including the integral formula (4.46) for $v(t)$; the only change to be made up to this point is that one may choose $\varkappa > 1$ arbitrarily rather than restricting it as in (4.40). Before making further estimates in (4.46), however, one first takes the negative of the imaginary parts

of this equation. Doing so and then estimating as before yields

$$- \operatorname{Im} v(t) = \int_{t}^{\infty} \exp \left[\mu(t-s)\right] \sin \left(\nu(t-s)\right) \sigma(s) \, ds \geqslant$$

$$\geqslant - \int_{t+r}^{\infty} \eta\big(\varkappa u(t)\big) u(s-r) \, ds + \int_{t}^{t+r} \exp \left[\mu(t-s)\right] \sin \left(\nu(t-s)\right) \eta\big(\varkappa u(t-r)\big) u(s-r) \, ds \geqslant$$

$$\geqslant - \eta\big(\varkappa u(t)\big) \beta^{-1} u(t) + \eta\big(\varkappa u(t-r)\big) \int_{t}^{t+r} \exp \left[\mu(t-s)\right] \sin \left(\nu(t-s)\right) u(s-r) \, ds \ .$$

Observe that the inequality $\sin \left(\nu(t-s)\right) \leqslant 0$ for $t \leqslant s \leqslant t+r$, which follows from (4.52), is used in deriving the above estimate. Proceeding much as before, one substitutes the equation

$$- \operatorname{Im} v(t) = k \int_{t-r}^{t} \exp \left[\mu(t-s-r)\right] \sin \left(\nu(t-s-r)\right) u(s) \, ds$$

obtained from (4.41) into the above inequality and obtains (after a short calculation, in which (4.48) is assumed)

$$\eta\big(\varkappa u(t)\big) \beta^{-1} u(t) \geqslant - \big[k - \eta\big(\varkappa u(t-r)\big)\big] \int_{t}^{t+r} \exp \left[\mu(t-s)\right] \sin \left(\nu(t-s)\right) u(s-r) \, ds \geqslant$$

$$\geqslant - \big[k - \eta\big(\varkappa u(t-r)\big)\big] \varkappa^{-1} u(t) \int_{t}^{t+r} \exp \left[\mu(t-s)\right] \sin \left(\nu(t-s)\right) \, ds =$$

$$= - \big[k - \eta\big(\varkappa u(t-r)\big)\big] \varkappa^{-1} \operatorname{Im} \big[\zeta^{-1}(1 - \exp \left[-\zeta r\right])\big] u(t) \ .$$

Cancelling $u(t) > 0$ in the above inequality and letting $t \to \infty$ gives

(4.53) $$0 \geqslant - k\varkappa^{-1} \operatorname{Im} \big[\zeta^{-1}(1 - \exp \left[-\zeta r\right])\big] \ ;$$

however, using (4.50) and (4.52) one obtains

$$- k\varkappa^{-1} \operatorname{Im} \big[\zeta^{-1}(1 - \exp \left[-\zeta r\right])\big] = - \varkappa^{-1}(k-1) \operatorname{Im} (\zeta^{-1}) = \varkappa^{-1}(k-1) \frac{\nu}{\mu^2 + \nu^2} > 0$$

which contradicts (4.53) and completes the proof in this case.

If the eigenvalue ζ satisfies $\operatorname{Re} \zeta \leqslant 0$, and if $r > 0$, then the lemma is proved by considering the solution $(y_-(t), z_-(t))$ defined for $t \leqslant 0$, rather than $(y_+(t), z_+(t))$ as above; otherwise the proof follows the scheme of the preceeding arguments, but is simpler. We leave this case to the reader.

The final case to be considered is when $r = 0$, so the transition layer equations are in fact the system of ordinary differential equations (4.29). Here one shows that

(4.54) $$(y_-(t), z_-(t)) = (0, 0) \quad \text{for all } t \leqslant 0$$

whenever $(y_-(t), z_i(t))$ satisfies (4.7) and (4.8)   Fix $\delta > 0$ and $\gamma > 1$ so that (4.27) holds, and fix $T_* \geqslant T_-$ so that $|y_-(t)| \leqslant \delta$ and $|z_-(t)| \leqslant \delta$ for all $t \leqslant -T_*$. Define

$$u(t) = y_-(t) - z_-(t)$$

From (4.7), (4.27) and (4.29) one sees that

(4.55) $u(t) \geqslant 0$ and $\dot{u}(t) \leqslant -(\gamma - 1)u(t)$ for all $t \leqslant -T_*$

and hence

(4.56) $\dfrac{d}{dt}\big(\exp[(\gamma - 1)t]u(t)\big) \leqslant 0$ for all $t \leqslant -T_*$.

Because $\gamma > 1$ one concludes from (4.56) that

$$u(-T_*) \leqslant \lim_{t \to -\infty} \exp[(\gamma - 1)t]u(t) = 0 ,$$

so from (4.55) and then (4.7) one has $u(-T_*) = 0$ and hence

(4.57) $(y_-(-T_*), z_-(-T_*)) = (0, 0)$.

As noted in Remark 4.2, the unique solution of initial value problem (4.57) for the system (4.29) is the trivial solution. This completes the proof of (4.54).   □

Before presenting a main result of this section, we must introduce a bit of notation. If $q, \bar{q}$ and ε are positive numbers, and if $\bar{q} > q$, define the set

$$N(q, \bar{q}, \varepsilon) = [0, \varepsilon] \cup [q - \varepsilon, q + \varepsilon] \cup [\bar{q} - \varepsilon, \bar{q}] .$$

THEOREM 4.1. – *Assume that f satisfies* (H1) *and* (H2), *let c and d be positive constants satisfying the conditions in the statement of Theorem* 3.1 *for some $\gamma > 1$, and let c_0 and d_0 be positive constants satisfying*

(4.58) $c_0 < c$ and $d_0 < d$.

Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon \leqslant \varepsilon_0$ and $x(t)$ is a slowly oscillating periodic solution of equation (3.1)$_\varepsilon$, *then*

$$\max_t x(t) > c_0   \text{ and }   \min_t x(t) < -d_0 .$$

Furthermore, there exists a constant $K > 0$ such that if $x(t)$ is a slowly oscillating periodic solution of (3.1)$_\varepsilon$ *for some $\varepsilon > 0$, then*

(4.59) $\{t \in [0, \bar{q}]: -d_0 \leqslant x(t) \leqslant c_0\} \subseteq N(q, \bar{q}, K\varepsilon)$

where q and $\bar{q}$ are the first two zeros of $x(t)$. The constants ε_0 and K depend on c_0 and d_0, but not on the solution $x(t)$; in addition, K does not depend on ε.

PROOF. – Fix a number $\varkappa$ satisfying

$$c_0 < \varkappa c, \quad d_0 < \varkappa d \quad \text{and} \quad 0 < \varkappa < 1.$$

If $x(t)$ is any slowly oscillating periodic solution of equation $(3.1)_\varepsilon$, define quantities c_* and d_* depending on the solution $x(t)$ by

$$(4.60) \qquad c_* = \min\{c_0, \max_t \varkappa x(t)\} > 0 \quad \text{and} \quad d_* = \min\{d_0, -\min_t \varkappa x(t)\} > 0.$$

In order to prove the theorem, it is sufficient to show that there exist $\varepsilon_0 > 0$ and $K_0 > 0$ not depending on the solution, such that if $\varepsilon \leqslant \varepsilon_0$ then

$$(4.61) \qquad \{t \in [0, \bar{q}]: -d_* \leqslant x(t) \leqslant c_*\} \subseteq N(q, \bar{q}, K_0\varepsilon)$$

and

$$(4.62) \qquad \max_t x(t) > \varkappa^{-1} c_0 \quad \text{and} \quad \min_t x(t) < -\varkappa^{-1} d_0$$

whenever $x(t)$ is a slowly oscillating periodic solution of $(3.1)_\varepsilon$ with first two zeros q and $\bar{q}$. If (4.62) holds, then $c_* = c_0$ and $d_* = d_0$, so (4.61) implies (4 59) with $K = K_0$. To obtain (4.59) for values of the parameter $\varepsilon \geqslant \varepsilon_0$ it may be necessary to increase K; this can be done independent of the solution $x(t)$ because by Lemma 1.5 one has $\bar{q} \leqslant \bar{Q}$, so (4.59) always holds if $\varepsilon \geqslant \varepsilon_0$ and $K\varepsilon_0 \geqslant \bar{Q}$.

We shall now prove the inclusion (4.61) for some K_0 and all sufficiently small ε. Noting that (4.61) holds for any fixed ε provided K_0 is large enough (specifically, if $K_0\varepsilon \geqslant \bar{Q}$), one sees that if this part of the theorem is false then there exist sequences $\varepsilon_n \to 0^+$ and $K_n \to \infty$, and a sequence $x_n(t)$ of slowly oscillating periodic solutions of $(3.1)_{\varepsilon_n}$, such that

$$(4.63) \qquad \{t \in [0, \bar{q}_n]: -d_{n*} \leqslant x_n(t) \leqslant c_{n*}\} \nsubseteq N(q_n, \bar{q}_n, K_n\varepsilon_n),$$

where c_{n*} and d_{n*} are given by (4.60) with $x_n(t)$ replacing $x(t)$. Suppose this to be so. Noting that $\varkappa < 1$ in (4.60), one may define quantities σ_{n1} and τ_{n1} to be respectively the smallest and largest $t \in [0, q_n]$ such that $x_n(t) = c_{n*}$. Similarly, let σ_{n2} and τ_{n2} be respectively the smallest and largest $t \in [q_n, \bar{q}_n]$ such that $x_n(t) = -d_{n*}$. By Theorem 3.1, $x_n(t)$ satisfies Property M between $-d$ and c, and therefore by (4.58) and (4.60) it also satisfies Property M between $-d_{n*}$ and c_{n*}. This implies that $x_n(t)$ is monotone increasing in $[\tau_{n2} - \bar{q}_n, \sigma_{n1}]$ and monotone decreasing in $[\tau_{n1}, \sigma_{n2}]$ as shown in Figure 13. In addition, the relation (4.63) implies that for each n either
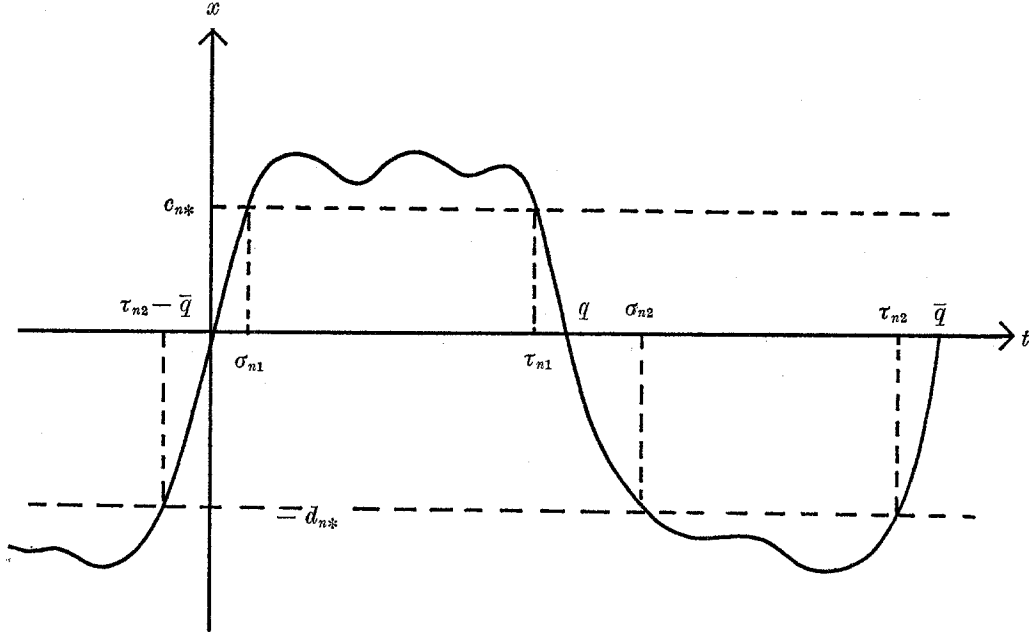
Fig. 13. The function $x_n(t)$ in the proof of Theorem 4.1.

$\sigma_{n1} + \bar{q}_n - \tau_{n2} > K_n \varepsilon_n$ or else $\sigma_{n2} - \tau_{n1} > K_n \varepsilon_n$. By taking a subsequence one may assume one of these inequalities holds for all n. For definiteness, assume that

$$(4.64) \qquad\qquad \sigma_{n2} - \tau_{n1} > K_n \varepsilon_n \quad \text{for all } n \ .$$

Now let $(y_+(t), z_+(t), r)$ and $(y_-(t), z_-(t), r)$ be transition layer solutions associated with the sequences $(\varepsilon_n, x_n(t), \sigma_{n2})$ and $(\varepsilon_n, x_n(t), \tau_{n1})$ respectively, as described in Proposition 3.1 and Definition 3.3. The monotonicity of $x_n(t)$ in $[\tau_{n1}, \sigma_{n2}]$, the definitions of τ_{n1} and σ_{n2}, and the inequality (4.64) imply that $-d \leqslant y_\pm(t) \leqslant c$ if $\pm t \geqslant 0$, and that

$$\dot{y}_+(t) \geqslant 0 \quad \text{if } t \geqslant 0$$

(4.65) \qquad\qquad and

$$\dot{y}_-(t) \geqslant 0 \quad \text{if } t \leqslant 0 \ .$$

Therefore the limits $\lim_{t \to \pm \infty} y_\pm(t) = L_\pm$ exist for some quantities $L_+, L_- \in [-d, c]$. From part (v) of Proposition 3.1 one sees that L_+ and L_- are both fixed points of the composed function $f \circ f$. The inequality (3) in the statement of Theorem 3.1 and the

fact that $\gamma > 1$, both of which are assumed here, imply that the only such fixed point is zero; thus $L_+ = L_- = 0$ so that

$$-d \leqslant y_+(t) \leqslant 0 \quad \text{if } t \geqslant 0$$

(4.66) and

$$0 \leqslant y_-(t) \leqslant c \quad \text{if } t \leqslant 0 .$$

In addition, parts (iv) and (v) of Proposition 3.1 imply that there exist $T_+ \geqslant 0$ and $T_- \geqslant 0$ such that

$$z_+(t) \geqslant 0 \quad \text{if } t \geqslant T_+$$

(4.67) and

$$z_-(t) \leqslant 0 \quad \text{if } t \leqslant - T_- ,$$

and that

(4.68) $$\lim_{t \to \infty} \big(y_+(t), z_+(t)\big) = \lim_{t \to -\infty} \big(y_-(t), z_-(t)\big) = (0, 0) .$$

Finally, one notes that neither $y_+(t)$ nor $y_-(t)$ is identically zero: by definition one has

$$y_+(0) = \lim_{n \to \infty} x_n(\sigma_{n2}) = \lim_{n \to \infty} - d_{n*}$$

(4.69) and

$$y_-(0) = \lim_{n \to \infty} x_n(\tau_{n1}) = \lim_{n \to \infty} c_{n*}$$

for some subsequences, and from Lemmas 3.2 and 4.1 it follows that for some $\Omega > 1$ one has

(4.70) $$\liminf_{n \to \infty} c_{n*} \geqslant \min \left\{ c_0, \ \liminf_{n \to \infty} \frac{\varkappa}{\Omega} \|x_n\| \right\} > 0$$

and similarly

(4.71) $$\liminf_{n \to \infty} d_{n*} > 0 .$$

One concludes from (4.69), (4.70) and (4.71) that

(4.72) $$y_+(0) < 0 \quad \text{and} \quad y_-(0) > 0 .$$

We are now able to obtain a contradiction. One sees from formulas (4.65), (4.66), (4.67), and (4.68) that the solutions $(y_\pm(t), z_\pm(t))$ satisfy the hypotheses of Lemma 4.2,

and that Lemma 4.2 implies either

$$(y_+(t), z_+(t)) = (0, 0) \quad \text{if } t \geqslant -r ,$$

or else

$$(y_-(t), z_-(t)) = (0, 0) \quad \text{if } t \leqslant 0 .$$

This conclusion immediately contradicts (4.72), and one concludes from this contradiction that inclusion (4.61) indeed holds.

New now prove the inequalities in (4.62) for small ε. Suppose that at least one of these inequalities is violated for a sequence $x_n(t)$ of slowly oscillating periodic solutions of $(1.3)_{\varepsilon_n}$, where $\varepsilon_n \to 0^+$; for definiteness suppose

$$(4.73) \qquad \max_t x_n(t) \leqslant \varkappa^{-1} c_0 < c \quad \text{for each } n .$$

Let $\varrho_n \in [0, q_n]$ be the location of the maximum of $x_n(t)$; then, as $x_n(t)$ satisfies property M between $-d$ and c, it follows that

$$\dot{x}_n(t) \geqslant 0 \quad \text{if } 0 \leqslant t \leqslant \varrho_n$$

(4.74)   and

$$\dot{x}_n(t) \leqslant 0 \quad \text{if } \varrho_n \leqslant t \leqslant q_n .$$

As before, consider the first and last points $t \in [0, q_n]$ such that $x_n(t) = c_{n*}$; denote these points by σ_n and τ_n, and observe that $x_n(t) \geqslant c_{n*}$ in $[\sigma_n, \tau_n]$, and that by the result (4.61) just proved

$$(4.75) \qquad \frac{\sigma_n}{\varepsilon_n} \quad \text{and} \quad \frac{q_n - \tau_n}{\varepsilon_n} \quad \text{are bounded sequences} .$$

Let $(y(t), z(t), r)$ be a transition layer solution associated with $(\varepsilon_n, x_n(t), \sigma_n)$. From the definition of $y(t)$, and from conditions (4.73), (4.74) and (4.75) one sees that $y(0) \leqslant y(t) < c$ if $t \leqslant 0$ and also that $y(t)$ is eventually monotone as $t \to -\infty$. Using (4.70) one sees that

$$y(0) \geqslant \liminf_{n \to \infty} x_n(\sigma_n) = \liminf_{n \to \infty} c_{n*} > 0$$

and so $\lim_{t \to -\infty} y(t) = L$ exists and satisfies $0 < L \leqslant c$. By (v) of Proposition 3.1 the number L is a fixed point of $f \circ f$; but as noted above, $f \circ f$ has no fixed point in $(0, c]$. This contradiction completes the proof of the theorem. $\quad \square$

The following corollary describes the behaviour of slowly oscillating solutions in the transition layers as $\varepsilon \to 0^+$. Figure 14 illustrates solutions $y(t)$ and $z(t)$ of the transition layer equations obtained in this result.
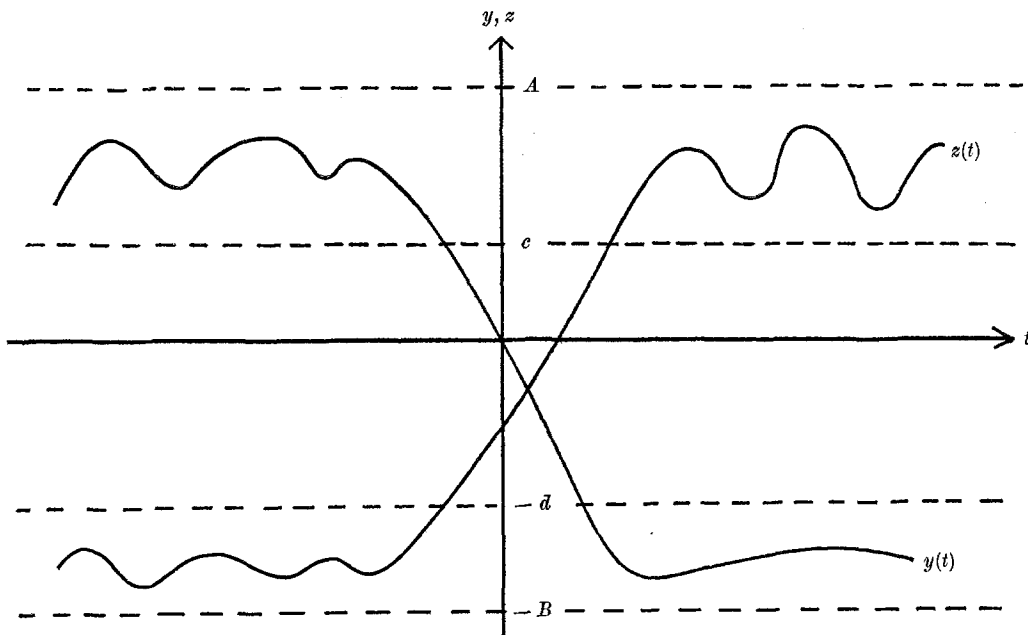
Fig. 14. Solutions of the transition layer equations in Corollary 4.1.

COROLLARY 4.1. – *Assume that f satisfies* (H1) *and* (H2), *and let c and d be positive constants satisfying the conditions in the statement of Theorem* 3.1 *for some $\gamma > 1$. Let $x_n(t)$ be a slowly oscillating periodic solution of equation* $(3.1)_{\varepsilon_n}$ *for some sequence $\varepsilon_n \to 0^+$, and define $r_n > 0$ by $\bar{q}_n = 2(1 + \varepsilon_n r_n)$ where $\bar{q}_n$ is the minimal period of $x_n(t)$. Then, upon taking a subsequence, one has*

$$(4.76) \qquad x_n(-\varepsilon_n t) \to y(t) \qquad and \qquad x_n(\tfrac{1}{2}\bar{q}_n - \varepsilon_n t) \to z(t)$$

uniformly on compact subsets of $\boldsymbol{R}$ for some C^1 functions $y(t)$ and $z(t)$. In addition, one has convergence of the first derivatives with respect to t in (4.76), *and the functions $(y(t), z(t))$ satisfy the transition layer equations*

$$(4.77)_r \qquad \dot{y}(t) = y(t) - f(z(t - r))$$

$$(4.78)_r \qquad \dot{z}(t) = z(t) - f(y(t - r))$$

for $t \in \boldsymbol{R}$, where

$$(4.79) \qquad r = \lim_{n \to \infty} r_n \quad is \ strictly \ positive \ .$$

Further, the solutions $y(t)$ and $z(t)$ satisfy the bounds

$$(4.80) \qquad -B \leqslant y(t), \ z(t) \leqslant A \qquad for \ all \ t \ ,$$

and have the asymptotic properties

(4.81)
$$\liminf_{t\to-\infty} y(t) \quad and \quad \liminf_{t\to\infty} z(t) \geqslant c,$$
$$\limsup_{t\to\infty} y(t) \quad and \quad \limsup_{t\to-\infty} z(t) \leqslant -d,$$

and

(4.82)
$$\limsup_{t\to-\infty} y(t) \quad and \quad \limsup_{t\to\infty} z(t) > c,$$
$$\liminf_{t\to\infty} y(t) \quad and \quad \liminf_{t\to-\infty} z(t) < -d.$$

Finally, $y(t)$ and $z(t)$ satisfy the monotonicity conditions

(4.83)
$$-d < y(t) < c \Rightarrow \dot{y}(t) \leqslant 0$$
and
$$-d < z(t) < c \Rightarrow \dot{z}(t) \geqslant 0.$$

PROOF. – One sees that the limits (4.76) exist for some subsequence, as $(y(t)$, $z(t), r)$ is simply a transition layer solution associated with the sequence $(\varepsilon_n, x_n(t), 0)$. Thus $y(t)$ and $z(t)$ satisfy the transition layer equations $(4.77)_r$ and $(4.78)_r$ where $r \geqslant 0$. The bound (4.80) follows from Proposition 1.3. The inclusion (4.59) (where $c_0 < c$ and $d_0 < d$ are arbitrary positive numbers), the fact that $x_n(t)$ satisfies Property M between $-d$ and c, the fact that $q_n = \frac{1}{2}\bar{q}_n + O(\varepsilon_n)$ where q_n is the first zero of $x_n(t)$, and the definitions of $y(t)$ and $z(t)$ together imply the inequalities in (4.81) but with c_0 and d_0 in place of c and d. However, as c_0 and d_0 can be chosen arbitrarily near c and d, one concludes that (4.81) holds as stated. The same properties of $x_n(t)$, in particular Property M, also imply the monotonicity conditions (4.83).

Now suppose that (4.82) is false. In view of (4.81) one may assume that one of the limits in (4.82) actually exists and is an equality rather than a strict inequality. For definiteness suppose that $\lim_{t\to-\infty} y(t) = c$. By (v) of Proposition 3.1 one has $f(f(c)) = c$. However, this contradicts $f(f(c)) \geqslant \gamma c > c$ obtained from our assumption of condition (3) in the statement of Theorem 3.1.

Finally, to prove that $r \geqslant 0$ is strictly positive, suppose to the contrary that $r = 0$ so that $y(t)$ and $z(t)$ satisfy the ordinary differential equations (4.29) obtained from the transition layer equations. Consider the subset

$$Q_2 = \{(y, z) \in \mathbf{R}^2 : -B \leqslant y \leqslant 0 \text{ and } 0 \leqslant z \leqslant A\}$$

of the second quadrant of the plane and observe that there exists some first time t_0 for which $(y(t), z(t)) \in Q_2$; in fact,

$$(y(t), z(t)) \notin Q_2 \quad \text{if } t < t_0,$$

and

$$(y(t_0), z(t_0)) \in ([- B, 0] \times \{0\}) \cup (\{0\} \times [0, A]) \subseteq \partial Q_2 .$$

The existence of t_0 and the above properties follow immediately from (4.80) and (4.81). One now sees, using the negative feedback condition $xf(x) < 0$ in $[- B, A] -$ $- \{0\}$, that the vector field (4.29) for $(y(t), z(t))$ points strictly outward from Q_2 at all points (y, z) in the set $([- B, 0] \times \{0\}) \cup (\{0\} \times [0, A])$ except at the stationary point $(0, 0)$. Therefore it follows from this simple phase plane consideration that

$$(y(t_0), z(t_0)) = (0, 0) .$$

However, as noted in Remark 4.2, this forces $(y(t), z(t)) = (0, 0)$ for all t, which is a contradiction. $\square$

REMARK 4.3. – Observe that Corollary 4.1 does not assert that the limits (4.76) and (4.79) exist for the full sequence ε_n, but only for some subsequence; we have not ruled out the possibility of multiple limit points. If one could show for some f that the parameter $r > 0$ and the solution $(y(t), z(t))$ of the transition layer equations were unique among those functions satisfying conditions (4.80), (4.81), (4.82), and (4.83), then it would follow that the limits (4.76) and (4.79) would exist for the full sequence ε_n.

The following result illustrates the interplay between the differential equation $(3.1)_\varepsilon$ and its transition layer equations $(4.77)_r$ and $(4.78)_r$: an analysis of the transition layer equations is used to obtain results about equation $(3.1)_\varepsilon$ which refine those of Theorem 4.1.

COROLLARY 4.2. – *Assume that f satisfies* (H1) *and* (H2), *and let c and d be positive constants satisfying the conditions in the statement of Theorem* 3.1 *for some $\gamma > 1$. Let I be an interval containing the origin, such that f is differentiable at each $x \in I$ and satisfies*

$$(4.84) \qquad x \in I \Rightarrow f'(x) < 0 .$$

Let J be any compact interval such that

$$(4.85) \qquad 0 \in J \subseteq (- d, c) \cap f(I) \cap f(f(I)) .$$

Then there exist positive constants ε_0 and K such that if $x(t)$ is a slowly oscillating periodic solution of equation $(3.1)_\varepsilon$ for some positive $\varepsilon \leqslant \varepsilon_0$, then

$$x(t) \in J \Rightarrow |\dot{x}(t)| \geqslant \frac{1}{K\varepsilon} .$$

PROOF. – We first establish some elementary consequences of the assumptions (4.84) and (4.85) and the conditions involving c, d, and γ in the statement of Theo-

rem 3.1. Consider any number $\eta \in J$. We claim that there exist unique $\zeta, \xi \in \boldsymbol{R}$ such that

$$(4.86) \qquad\qquad \eta = f(\zeta) = f(f(\xi)) \ ;$$

moreover, one has

$$(4.87) \qquad\qquad \zeta, \xi \in (-d, c) \cap I \ .$$

The existence of ζ and ξ satisfying (4.86) and (4.87) follows easily from the inclusion (4.85) and from property (2) in the statement of Theorem 3.1. What is not so obvious is the uniqueness of ζ and ξ, not only in $(-d, c) \cap I$, but in all of $\boldsymbol{R}$. Suppose, therefore, that $\eta = f(\zeta_0)$ for some $\zeta_0 \in \boldsymbol{R}$ with $\zeta_0 \neq \zeta$. Condition (2) in Theorem 3.1 implies that $\zeta_0 \in (-d, c)$. The monotonicity condition (1) in that theorem further implies that the function f is monotone decreasing on the interval between ζ and ζ_0, hence is constant on that interval because $f(\zeta) = f(\zeta_0)$. However, this cannot be, since $\zeta \in I$ implies $f'(\zeta) < 0$ by (4.84). This contradiction proves that ζ is unique. A similar argument proves that ξ also is unique.

Suppose now that the conclusion of the corollary is false. Then there exist sequences $\varepsilon_n \to 0^+$, $t_n \in [0, \bar{q}_n]$, and $x_n(t)$ such that $x_n(t)$ is a slowly oscillating periodic solution of equation $(3.1)_{\varepsilon_n}$ of minimal period $\bar{q}_n$, and such that

$$(4.88) \qquad\qquad x_n(t_n) \in J$$

and

$$(4.89) \qquad\qquad \lim_{n \to \infty} \varepsilon_n \dot{x}_n(t_n) = 0 \ .$$

Theorem 4.1 (with $c_0 < c$ and $d_0 < d$ positive constants such that $J \subseteq [-d_0, c_0]$) implies there exists a constant K_0 such that $t_n \in N(q_n, \bar{q}_n, K_0\varepsilon_n)$ for large n, where as usual q_n denotes the first zero of $x_n(t)$. Therefore, after taking a subsequence one may assume that either t_n/ε_n, $(q_n - t_n)/\varepsilon_n$, or $(\bar{q}_n - t_n)/\varepsilon_n$ is a bounded sequence. To be definite, assume that t_n/ε_n is bounded; in fact, assume without loss that

$$(4.90) \qquad\qquad \lim_{n \to \infty} \frac{t_n}{\varepsilon_n} = t_0$$

exists for some real number t_0.

Now let $(y(t), z(t), r)$ be as in Corollary 4.1. The definition of $y(t)$ and equations (4.88) and (4.89) imply that

$$(4.91) \qquad\qquad y(t_0) \in J$$

and

$$(4.92) \qquad\qquad \dot{y}(t_0) = 0 \ ;$$

thus the differential equation $(4.77)_r$ for $y(t)$ and equation (4.92) yield

$$(4.93) \qquad y(t_0) = f\big(z(t_0 - r)\big) \ .$$

The results (4.86) and (4.87) established above imply that

$$(4.94) \qquad z(t_0 - r) \in (-d, c) \cap I \ ,$$

and so f is differentiable at $z(t_0 - r)$ and satisfies

$$(4.95) \qquad f'\big(z(t_0 - r)\big) < 0 \ .$$

The existence of the derivative (4.95) in turn allows the differential equation $(4.77)_r$ to be differentiated at $t = t_0$ to give, using (4.92),

$$(4.96) \qquad \ddot{y}(t_0) = \dot{y}(t_0) - f'\big(z(t_0 - r)\big)\,\dot{z}(t_0 - r) = -f'\big(z(t_0 - r)\big)\,\dot{z}(t_0 - r) \ .$$

Now from (4.83), (4.85), and (4.91) one sees that $y(t)$ is monotone decreasing in a neighborhood of t_0. As the first derivative (4.92) vanishes at t_0, one sees that the second derivative $\ddot{y}(t_0)$ also equals zero. Thus the right-hand side of equation (4.96) equals zero, so one concludes using (4.95) that

$$(4.97) \qquad \dot{z}(t_0 - r) = 0 \ .$$

One may now switch the roles of y and z in the above argument and replace t_0 with $t_0 - r$, and obtain from (4.94) and (4.97) in a similar fashion the conclusions

$$(4.98) \qquad z(t_0 - r) = f\big(y(t_0 - 2r)\big) \ ,$$

$$(4.99) \qquad y(t_0 - 2r) \in (-d, c) \cap I \ ,$$

and

$$(4.100) \qquad \dot{y}(t_0 - 2r) = 0 \ .$$

Equations (4.93) and (4.98) yield

$$y(t_0) = f\big(f(y(t_0 - 2r))\big) \ ,$$

and this together with condition (3) in the statement of Theorem 3.1, and equations (4.91) and (4.99), imply that

$$(4.101) \qquad |y(t_0 - 2r)| \leqslant \gamma^{-1} |y(t_0)|$$

and that

$$y(t_0 - 2r) \in \gamma^{-1} J = \{\gamma^{-1} x: \; x \in J\} \,.$$

Because J is an interval containing the origin, and $\gamma > 1$, one has $\gamma^{-1} J \subseteq J$ and hence

(4.102) $$y(t_0 - 2r) \in J \,.$$

At this point note that we have concluded (4.100), (4.101), and (4.102) from (4.91) and (4.92) using only the properties of the function f, the solution $(y(t), z(t))$, and the sets I and J; the definition (4.90) of t_0 has not been used. Therefore, this entire argument may be repeated indefinitely; one concludes by doing this that for each integer $n \geqslant 1$

$$y(t_0 - 2nr) \in J \,, \quad \dot{y}(t_0 - 2nr) = 0 \quad \text{and} \quad |y(t_0 - 2nr)| \leqslant \gamma^{-1} |y(t_0 - 2(n-1)r)|$$

and hence

(4.103) $$|y(t_0 - 2nr)| \leqslant \gamma^{-n} |y(t_0)| \to 0 \quad \text{as } n \to \infty \,.$$

However, (4.103) contradicts (4.81). This completes the proof of the corollary.   □

The next two results describe the solutions of the transition layer equations obtained in Corollary 4.1 under the additional assumptions of monotonicity of f (Corollary 4.3) and hypothesis (H3) (Corollary 4.4).

COROLLARY 4.3. – *Assume that f satisfies* (H1) *and* (H2), *and that in addition f is monotone decreasing in* $[- B, A]$, *and satisfies* $|f(f(x))| > |x|$ *whenever* $x \in (- B, A)$ *and* $x \neq 0$. *Let $y(t)$, $z(t)$, and r be as in Corollary* 4.1. *Then $y(t)$ and $z(t)$ satisfy the monotonicity conditions*

(4.104) $$\dot{y}(t) \leqslant 0 \quad \text{and} \quad \dot{z}(t) \geqslant 0 \quad \text{for all } t$$

and possess the limits

(4.105) $$\lim_{t \to -\infty} \big(y(t), z(t)\big) = (A, - B) \quad \text{and} \quad \lim_{t \to \infty} \big(y(t), z(t)\big) = (- B, A) \,.$$

Thus $\big(y(t), z(t)\big)$ is a heteroclinic orbit joining the stationary points $(A, - B)$ and $(- B, A)$ of the transition layer equations $(4.77)_r$ *and* $(4.78)_r$.

PROOF. – Note that the hypotheses of Corollary 4.1 are satisfied if $c < A$ is any positive number and d is a positive number less than, but sufficiently near $- f(c)$.

Also note that $f(A) = -B$ and $f(-B) = A$; in particular, the number d can be chosen arbitrarily near B. By choosing c and d in this manner, one sees from the conclusions (4.80), (4.81) and (4.82) of Corollary 4.1 that (4.104) and (4.105) hold. ☐

COROLLARY 4.4. – *Assume that f satisfies* (H2) *and* (H3) (*so the hypotheses of Corollary* 4.1 *are satisfied for some* c, d, *and* γ). *Let* $y(t)$, $z(t)$, *and* r *be as in Corollary* 4.1. *Then, with* a *and* b *as in* (H3), *one has*

$$(4.106) \qquad \lim_{t \to -\infty} \big(y(t), z(t)\big) = (a, -b) \quad and \quad \lim_{t \to \infty} \big(y(t), z(t)\big) = (-b, a) \,.$$

Thus $(y(t), z(t))$ *is a heteroclinic orbit joining the stationary points* $(a, -b)$ *and* $(-b, a)$ *of the transition layer equations* $(4.77)_r$ *and* $(4.78)_r$.

PROOF. – Suppose one of the limits in (4.106) fails to hold; to be definite, suppose $\lim_{t \to -\infty} y(t) = a$ is false. Let $t_n \to -\infty$ be a sequence such that

$$(4.107) \qquad \lim_{n \to \infty} y(t_n) = a_0 \neq a$$

for some a_0, and such that the limits

$$\lim_{n \to \infty} y(t + t_n) = y_0(t) \quad and \quad \lim_{n \to \infty} z(t + t_n) = z_0(t) \,,$$

together with their first derivatives, exist uniformly on compact subsets of R. Using Ascoli's theorem one can show such a sequence exists; moreover, $y_0(t)$ and $z_0(t)$ are solutions of the transition layer equations $(4.77)_r$ and $(4.78)_r$ for all $t \in R$. Furthermore, the inequalities (4.80) and (4.81) hold for some positive c and d, so it follows that

$$c \leqslant y_0(t) \leqslant A \quad and \quad -B \leqslant z_0(t) \leqslant -d \quad for all \ t \in R \,.$$

Lemma 3.3 therefore implies that $y_0(0) = a$. However, (4.107) and the definition of $y_0(t)$ imply that $y_0(0) = a_0 \neq a$. This contradiction completes the proof. ☐

REMARK 4.4. – It is interesting that if one does not establish the connection between periodic solutions of $(3.1)_\varepsilon$ and solutions of the transition layer equations $(4.77)_r$ and $(4.78)_r$, then it is highly nontrivial even to prove the existence of $r > 0$ and solutions $y(t)$ and $z(t)$ satisfying the conclusions of Corollaries 4.1, 4.3, or 4.4.

REMARK 4.5. – Under the assumptions of Corollary 4.3 one sees the sense in which slowly oscillating periodic solutions $x(t)$ approach the square wave function

$$\text{sqw} \ (t) = \begin{cases} A & \text{if } 2n < t < 2n + 1 \\ -B & \text{if } 2n + 1 < t < 2n + 2 \end{cases}$$
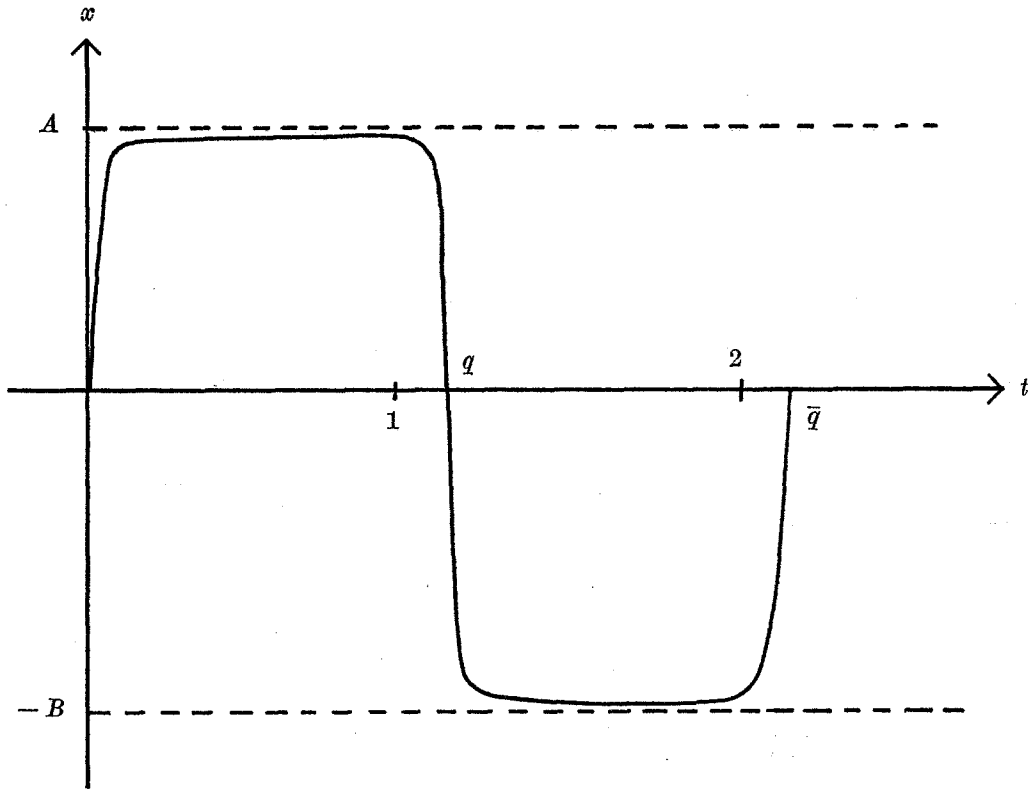
Fig. 15. A slowly oscillating periodic solution for small ε, with monotone f.

as $\varepsilon \to 0^+$. Indeed, $x(t)$ has the simple form shown in Figure 15. This follows from Corollary 4.3, and from the fact that for any positive $c < A$ and $d < B$ the solution $x(t)$ satisfies Property M between $-d$ and c.

If f satisfies (H2) and (H3) then the limits (4.106) of the transition layer solutions $(y(t), z(t))$ exist as $t \to \pm\infty$. However, the existence of these limits still gives no information about how slowly oscillating periodic solutions $x(t)$ behave away from the transition layers as $\varepsilon \to 0^+$. One cannot conclude from (4.83) that $x(t)$ is uniformly near

$$(4.108) \qquad \text{sqw}\,(t) = \begin{cases} a & \text{if } 2n < t < 2n + 1 \\ -b & \text{if } 2n + 1 < t < 2n + 2 \end{cases}$$

on compact subsets of $\mathbf{R} - \mathbf{Z}$. Nevertheless, this fact is true, as is shown in the following result.

THEOREM 4.2. – *Assume that f satisfies* (H2) *and* (H3), *and that a_1, a_2, b_1, and b_2 are positive numbers such that $a_1 < a < a_2$ and $b_1 < b < b_2$, where a and b are as in*

(H3). *Then there exists a number $K > 0$ such that if $x(t)$ is a slowly oscillating periodic solution of* $(3.1)_\varepsilon$ *for some positive* ε, *then*

(4.109) $$\{t \in [0, \bar{q}] : x(t) \notin (-b_2, -b_1) \cup (a_1, a_2)\} \subseteq N(q, \bar{q}, K\varepsilon)$$

where q and $\bar{q}$ are the first two zeros of $x(t)$.

PROOF. – Assume that the conclusion (4.109) of the theorem is false. Then there exist sequences $\varepsilon_n \to 0^+$, $K_n \to \infty$, $t_n \in [0, \bar{q}_n]$, and $x_n(t)$ such that $x_n(t)$ is a slowly oscillating periodic solution of equation $(3.1)_{\varepsilon_n}$ of minimal period $\bar{q}_n$, and such that

(4.110) $$x_n(t_n) \notin (-b_2, -b_1) \cup (a_1, a_2)$$

and

(4.111) $$t_n \notin N(q_n, \bar{q}_n, K_n \varepsilon_n) .$$

Here q_n denotes the first zero of $x_n(t)$. One may also assume, without loss, that $t_n \in [0, q_n]$ for each n. Clearly (4.111) now implies that one has

(4.112) $$\lim_{n \to \infty} \frac{t_n}{\varepsilon_n} = \lim_{n \to \infty} \frac{q_n - t_n}{\varepsilon_n} = \infty .$$

Now fix positive numbers c and d satisfying the conditions in the statement of Theorem 3.1 for some $\gamma > 1$. By Theorem 4.1 and (4.112) one has for each $t \in R$

(4.113) $$\liminf_{n \to \infty} x_n(t_n - \varepsilon_n t) \geqslant c ;$$

also, using the result $\frac{1}{2}\bar{q}_n = q_n + O(\varepsilon_n) = 1 + O(\varepsilon_n)$ of Theorem 3.2 one has

(4.114) $$\limsup_{n \to \infty} x_n(t_n + \frac{1}{2}\bar{q}_n - \varepsilon_n t) \leqslant -d .$$

Let $(y(t), z(t), r)$ be a transition layer solution associated with the sequence $(\varepsilon_n, x_n(t), t_n)$. Then the inequalities (4.113) and (4.114) and the definitions of $y(t)$ and $z(t)$ imply that

$$y(t) \geqslant c \quad \text{and} \quad z(t) \leqslant -d \quad \text{for all } t$$

and from (4.110) one obtains $y(0) \notin (a_1, a_2)$, so in particular

(4.115) $$y(0) \neq a ,$$

One also has the bounds $y(t) \leqslant A$ and $z(t) \geqslant - B$ from (ii) of Proposition 3.1. Now observe that the hypotheses of Lemma 3.3 are satisfied. One therefore concludes that

$$(4.116) \qquad\qquad y(t) = a \quad \text{for all } t;$$

however, (4.116) contradicts (4.115), completing the proof of the theorem. □

REMARK 4.6. – Corollary 4.4 and Theorem 4.2 give a very precise description of the way in which slowly oscillating periodic solutions $x(t)$ converge to the step function sqw (t) given by (4.108) as $\varepsilon \to 0^+$, if f satisfies (H2) and (H3). In contrast to the situation in Corollary 4.3 where the nonlinearity f and the solutions $y(t)$ and $z(t)$ were monotone functions, it is possible in Corollary 4.4 that either $y(t)$ or $z(t)$ is not a monotone function of t. Indeed, it is possible for the range of $y(t)$ or $z(t)$ to properly contain $[- b, a]$, as will be shown in Proposition 4.1. If this is so, then Corollary 4.4 and Theorem 4.2 imply that a peculiar type of «non-uniform» convergence of $x(t)$ to sqw (t), similar to the Gibbs phenomenon of classical Fourier series, must occur as $\varepsilon \to 0^+$.

To be specific, suppose that
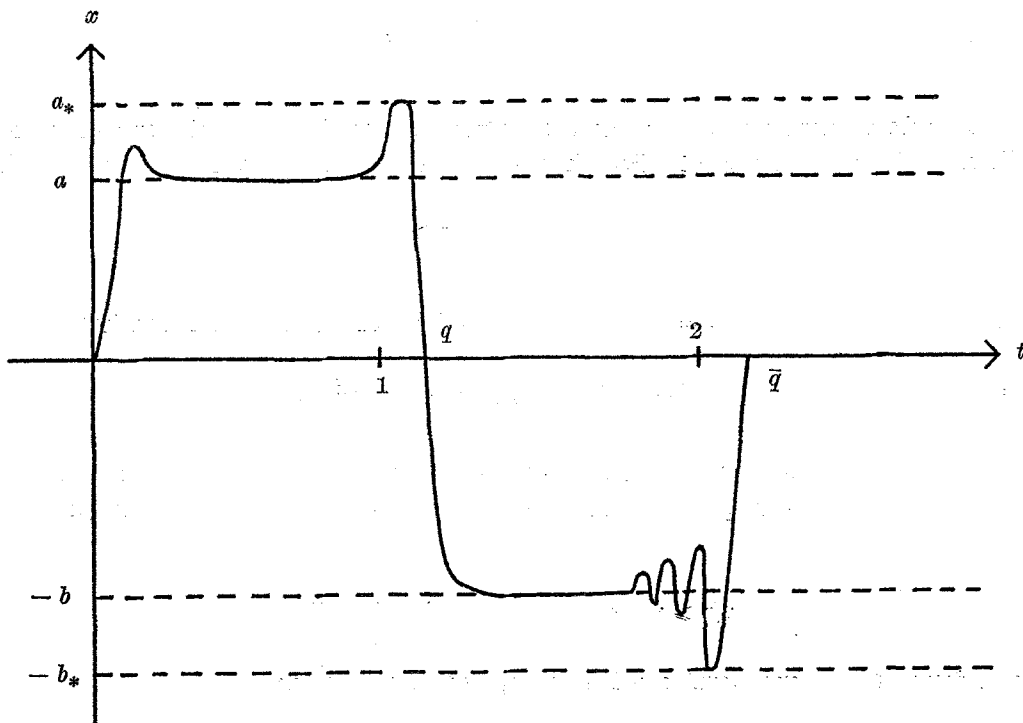
$$(4.117) \qquad\qquad \sup_t z(t) = a_* > a$$

Fig. 16. A slowly oscillating periodic solution displaying the Gibbs Phenomenon.

for some number a_*, and that one considers a sequence $x_n(t)$ of slowly oscillating periodic solutions of $(3.1)_{\varepsilon_n}$ where $\varepsilon_n \to 0^+$ and $z(t)$ is given by the limit (4.76) as in Corollary 4.1. If f is assumed to satisfy (H2) and (H3), then by Theorem 4.2 $x_n(t)$ converges uniformly to a on compact subsets of $(0, 1)$, and uniformly to $-b$ on compact subsets of $(1, 2)$. However, near the transition point $t = 1$, $x(t)$ will overshoot the value a by an amount which does *not* become small as $\varepsilon_n \to 0$. More precisely, one has from (4.117) and the definition of $z(t)$ that if $0 < \delta < 1$, then

$$(4.118) \qquad \lim_{n \to \infty} \max_{|t-1| \leqslant \delta} x_n(t) = a_* > a .$$

In the same fashion, if one has

$$\inf_t y(t) = -b_*$$

for some positive number $b_* > b$, then

$$(4.119) \qquad \lim_{n \to \infty} \min_{|t| \leqslant \delta} x_n(t) = -b_* < -b$$

for each $\delta < 1$.

Figure 16 illustrates this phenomenon for a typical solution, for small ε. The following result gives a sufficient condition on f for this Gibbs phenomenon to occur.

PROPOSITION 4.1. – *Assume that f satisfies* (H2) *and* (H3), *and that there exists a positive number* $a_0 < a$ *such that*

$$(4.120) \qquad a_0 \leqslant x < a \Rightarrow f(x) < -b .$$

Let $y(t)$, $z(t)$, and r be as in Corollary 4.1. Then either

$$(4.121) \qquad \inf_t y(t) < -b$$

or else

$$(4.122) \qquad \sup_t z(t) > a ,$$

so that either the Gibbs phenomenon (4.118) *or* (4.119) *described in Remark 4.6 must occur. The same conclusion holds if in place of* (4.120) *one assumes*

$$(4.123) \qquad -b < x \leqslant -b_0 \Rightarrow f(x) > a$$

for some positive number $b_0 < b$.

PROOF. – Assuming condition (4.120) let us suppose that both (4.121) and (4.122) are false. By Corollary 4.4 one has $\lim\limits_{t \to \infty} z(t) = a$, so there exists a number T such that

$$a_0 \leqslant z(t) \leqslant a \quad \text{if } t \geqslant T \quad \text{and} \quad z(T) = a_0 .$$

From (4.77)$_r$ and (4.120), and the fact that $y(t) \geqslant - b$ for all t, it follows that

$$(4.124) \qquad \dot{y}(t) \geqslant y(t) + b \geqslant - b + b = 0 \quad \text{if } t \geqslant T + r$$

with the strict inequality

$$(4.125) \qquad \dot{y}(T + r) > 0$$

because $f(z(T)) = f(a_0) < b$. From (4.124), from the fact that (4.121) is assumed false, and from the fact that $\lim\limits_{t \to \infty} y(t) = - b$, one concludes that

$$(4.126) \qquad y(t) = - b \quad \text{if } t \geqslant T + r .$$

However, (4.126) contradicts (4.125).

If (4.123) is assumed in place of (4.120) then the proof is similar.   □

If the function f is odd, then the results of this section, in particular Corollaries 4.1, 4.3, and 4.4, have analogs in which « slowly oscillating periodic solution » is replaced with « S-solution ». Recall that an S-solution is a slowly oscillating periodic solution satisfying the symmetry condition (1.42), and that the existence of S-solutions was proved in Theorem 1.2. In this situation one easily sees from (1.42) and the oddness of f that the system of transition layer equations (4.77)$_r$ and (4.78)$_r$ reduces to the single equation

$$(4.127)_r \qquad \dot{y}(t) = y(t) + f(y(t - r)) ,$$

because the symmetry of the S-solution $x(t)$ yields the relation

$$(4.128) \qquad z(t) = - y(t) \quad \text{for all } t .$$

One therefore obtains the following result.

COROLLARY 4.5. – *Assume, in addition to the hypotheses of either Corollary 4.1, 4.3, or 4.4, that the function f is odd (and hence $A = B$, $a = b$, and without loss $c = d$). Then the analog of the respective Corollary 4.1, 4.3, or 4.4 holds but with « S-solution » replacing « slowly oscillating periodic solution », with $z(t)$ given by (4.128) and with the single transition layer equation (4.127)$_r$ replacing the system (4.77)$_r$ and (4.78)$_r$.*

In particular, the solution $y(t)$ of equation (4.127)$_r$ obtained satisfies

$$-A \leqslant y(t) \leqslant A \quad \text{for all } t \quad \text{and} \quad -c \leqslant y(t) \leqslant c \Rightarrow \dot{y}(t) \leqslant 0 \,.$$

In Corollaries 4.3 and 4.4 $y(t)$ is a heteroclinic orbit of (4.127)$_r$, as it satisfies

$$\lim_{t \to \pm \infty} y(t) = \mp A$$

in Corollary 4.3, and

$$\lim_{t \to \pm \infty} y(t) = \mp a$$

in Corollary 4.4.

Appendix: Results on linear autonomous differential-delay equations.

In this appendix we obtain some results on the characteristic equation of the linear differential-delay equation

$$(A.1) \qquad \dot{x}(t) = -\lambda x(t) - \lambda k x(t-1)$$

and on the characteristic equation of the linear system

$$(A.2) \qquad \dot{y}(t) = y(t) + kz(t-r) \,, \quad \dot{z}(t) = z(t) + ky(t-r) \,.$$

We assume throughout that $x(t)$, $y(t)$, and $z(t)$ are scalars, that λ, k, and r are real parameters, and that $r \geqslant 0$. Many of the results on the characteristic equation of (A.1) are stated in [46]; for completeness, however, we provide proofs here.

We shall also summarize in this appendix some results on the general linear system

$$(A.3) \qquad \dot{W}(t) = L(W_t) \,.$$

Here $W(t)$ is an n-vector, $W_t \in C = C([-r, 0], \mathbf{R}^n)$ denotes the translate

$$W_t(s) = W(t+s) \quad \text{for } -r \leqslant s \leqslant 0$$

where $r \geqslant 0$ is fixed, and L is a given bounded linear transformation

$$L \colon C \to \mathbf{R}^n \,.$$

Our setting for equation (A.3) is identical to that of HALE [25, Chapter 7], and we shall assume that the reader is familiar with the results there. See also BELLMAN and COOKE [1] and EL'SGOL'TS and NORKIN [16].

Recall that the characteristic equation of the system (A.3) is given by

$$\Delta(\zeta) = 0$$

where Δ is the entire function defined by

$$\Delta(\zeta) = \det\left[\zeta I - L(\exp[\zeta \cdot])\right]$$

for $\zeta \in C$. $\Big($Here $L(\exp[\zeta \cdot])$ is the $n \times n$ matrix obtained from the vector $L(\eta_j \exp[\zeta \cdot])$, where $\eta_1, \ldots, \eta_n$ is the standard basis in $\mathbf{R}^n$. Also, one complexifies the space C and the transformation L in the usual fashion.$\Big)$ There exists a solution of (A.3) of the form $W(t) = w \exp[\zeta t]$ for some nonzero vector $w \in C^n$ if and only if $\Delta(\zeta) = 0$; in general if ζ is a zero of the function Δ of multiplicity d, then there exists a d-dimensional linear space of solutions of the form $W(t) = w(t) \exp[\zeta t]$ where $w(t)$ is a polynomial taking values in C^n. In addition, the degree of the polynomial $w(t)$ for any such solution is strictly less than d. These facts generally follow from the theory presented in [25].

If $W(t)$ is a solution of (A.3) for all $t \in \mathbf{R}$, and in addition is bounded on $(-\infty, 0]$, then using Theorem 7.4.1 of [25] one can easily show

$$(A.4) \qquad\qquad W(t) = \sum_{j=1}^{p} w_j(t) \exp[\zeta_j t]$$

where each term $w_j(t) \exp[\zeta_j t]$ is an exponential-polynomial solution as described above (in particular $\Delta(\zeta_j) = 0$), and for each j one has that

$$\mathrm{Re}\,\zeta_j \geqslant 0, \quad \text{and} \quad w_j(t) \quad \text{is a constant if} \quad \mathrm{Re}\,\zeta_j = 0.$$

That is, $W(t)$ is a finite sum of exponential-polynomial solutions each of which is bounded as $t \to -\infty$. If in addition one has

$$\sup_{t \in \mathbf{R}} |W(t)| < \infty$$

then it is an exercise to show that each of the exponential-polynomial terms in (A.4) is also bounded as $t \to \infty$, that is, one has

$$W(t) = \sum_{j=1}^{p} w_j \exp[i v_j t]$$

for constant vectors w_j and roots $\zeta_j = i v_j$ of the function Δ with zero real part. This is the case in particular if $W(t)$ is a periodic solution.

One sees that the characteristic equation of the differential equation (A.1) is

$$(A.5) \qquad\qquad \zeta + \lambda + \lambda k \exp[-\zeta] = 0$$

and that of the system (A.2) is

$$\text{(A.6)} \qquad \begin{vmatrix} \zeta - 1 & - k \exp\left[-\zeta\right] \\ - k \exp\left[-\zeta\right] & \zeta - 1 \end{vmatrix} = 0 \, .$$

Note that the determinant in (A.6) factors to give

$$\Delta_+(\zeta)\Delta_-(\zeta) = 0$$

where the functions $\Delta_\pm$ are given by

$$\text{(A.7)} \qquad \Delta_\pm(\zeta) = \zeta - 1 \pm k \exp\left[-\zeta\right] \, .$$

We now prove some results about solutions of the characteristic equations (A.5) and (A.6).

PROPOSITION A.1. - *If $\lambda > 0$ and $-1 < k \leqslant 1$, then each solution of equation* (A.5) *satisfies* $\operatorname{Re}\zeta < 0$.

PROOF. - If there were a solution $\zeta = \mu + i\nu$ with $\mu \geqslant 0$ and $\nu \in \boldsymbol{R}$, then taking the norms of both sides of $\zeta + \lambda = -\lambda k \exp\left[-\zeta\right]$ yields

$$\text{(A.8)} \qquad \lambda \leqslant \sqrt{(\mu + \lambda)^2 + \nu^2} = |\zeta + \lambda| = |\lambda k \exp\left[-\zeta\right]| = \lambda|k|\exp\left[-\mu\right] \leqslant \lambda \, .$$

Thus the inequalities in (A.8) are actually equalities, and this is seen to imply $\zeta = 0$. However, $\zeta = 0$ is not a solution of (A.5) if $-1 < k \leqslant 1$. □

PROPOSITION A.2. - *Assume that $k > 1$. Then there exists a sequence*

$$\ldots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots$$

such that if $\lambda \neq 0$, then equation (A.5) *has a solution with real part zero if and only if $\lambda = \lambda_m$ for some m. If $0 < \lambda < \lambda_0$, then all solutions ζ of* (A.5) *satisfy* $\operatorname{Re}\zeta < 0$. *If $\lambda = \lambda_m$, then* (A.5) *has a pair of complex conjugate roots*

$$\zeta = \pm i\nu_m \neq 0 \, ;$$

these are simple roots, and there are no other roots with zero real part. For λ near λ_m there is a unique pair $\zeta_m(\lambda), \overline{\zeta_m(\lambda)}$ of complex conjugate roots near $\pm i\nu_m$. The function $\zeta_m(\lambda)$ depends analytically on λ, satisfies $\zeta_m(\lambda_m) = i\nu_m$, and its derivative satisfies

$$\text{(A.9)} \qquad \operatorname{sgn}(\lambda_m)\operatorname{Re}\zeta_m'(\lambda_m) > 0 \, .$$

The values of v_m and λ_m are given by

(A.10)
$$v_m = v_0 + 2\pi m$$

and

(A.11)
$$\lambda_m = \frac{v_0 + 2\pi m}{\sqrt{k^2 - 1}}$$

where v_0 is the unique solution of

(A.12)
$$\cos v_0 = \frac{-1}{k}, \quad v_0 \in \left(\frac{\pi}{2}, \pi\right).$$

PROOF. – Assume that $\lambda \neq 0$ and that $\zeta = iv$ is a solution of (A.5) for some real number v, where we assume without loss that

(A.13)
$$\lambda v \geqslant 0 .$$

Taking the real and imaginary parts of (A.5) yields

(A.14)
$$\lambda(1 + k \cos v) = 0$$

and

(A.15)
$$v = \lambda k \sin v$$

respectively. Equations (A.13) and (A.15) imply that $\sin v \geqslant 0$, and so (A.14) holds if and only if $v = v_m$ for some integer m, where v_m is given by (A.10) and (A.12). Equation (A.15) and the fact that

(A.16)
$$\sin v_m = \sqrt{1 - \frac{1}{k^2}}$$

then imply that $\lambda = \lambda_m$, where λ_m is as in (A.11).

Now consider the left-hand side of (A.5) as an analytic function

$$\Delta(\zeta, \lambda) = \zeta + \lambda + \lambda k \exp[-\zeta]$$

of both ζ and λ. One has $\Delta(iv_m, \lambda_m) = 0$, and an easy calculation using (A.12) and (A.16) reveals that

$$\frac{\partial \Delta}{\partial \zeta}(iv_m, \lambda_m) = 1 - \lambda_m k \exp[-iv_m] = 1 + \lambda_m + i\lambda_m \sqrt{k^2 - 1} \neq 0 .$$

Therefore iv_m is a simple root of the characteristic equation, and by the implicit function theorem there is an analytic function $\zeta_m(\lambda)$, defined for λ near λ_m and

satisfying $\zeta_m(\lambda_m) = i\nu_m$ and

(A.17) $\qquad\qquad \Delta(\zeta_m(\lambda), \lambda) = 0 \quad$ for all λ near λ_m .

Of course one also has $\Delta(\overline{\zeta_m(\lambda)}, \lambda) = 0$ for (real) λ near λ_m. Implicit differentiation of (A.17) yields

$$\zeta_m'(\lambda_m) = -\frac{\partial\Delta}{\partial\lambda}(i\nu_m, \lambda_m)\frac{\partial\Delta}{\partial\zeta}(i\nu_m, \lambda_m)^{-1} = i\sqrt{k^2-1}\,(1 + \lambda_m + i\lambda_m\sqrt{k^2-1})^{-1}$$

and one sees from this that $\operatorname{Re}\zeta_m'(\lambda_m)$ is non-zero and has the same sign as λ_m, thereby proving (A.9).

It remains to prove that $\operatorname{Re}\zeta < 0$ for each solution of (A.5) when $0 < \lambda < \lambda_0$. For this range of λ any solution ζ of (A.5) with $\operatorname{Re}\zeta \geqslant 0$ also satisfies

$$|\zeta| = |\lambda + \lambda k \exp[-\zeta]| \leqslant \lambda(1 + k) \quad \text{and} \quad \operatorname{Re}\zeta > 0$$

and so lies in the open half-disc

$$H(\lambda_0, k) = \{\zeta \in C\colon |\zeta| < \lambda_0(1 + k) \text{ and } \operatorname{Re}\zeta > 0\} .$$

In particular, no solution lies on the boundary $\partial H(\lambda_0, k)$ so by Rouché's theorem the number of solutions inside $H(\lambda_0, k)$ is independent of λ in the range $0 < \lambda < \lambda_0$. Now consider λ near zero. If $\lambda = 0$, then $\zeta = 0$ is the only solution of equation (A.5) in the closed half-disc $\overline{H(\lambda_0, k)}$; moreover, this solution is a simple root and lies on the boundary $\partial H(\lambda_0, k)$. For λ near λ_0 an application of the implicit function theorem, as above, shows that this root continues as an analytic function $\zeta(\lambda)$ and satisfies

(A.18) $\qquad\qquad\qquad \zeta'(0) = -1 - k < 0 .$

In particular, (A.18) implies that $\zeta(\lambda) \notin H(\lambda_0, k)$ for λ positive and sufficiently small. Therefore, a simple continuity argument proves that $H(\lambda_0, k)$ contains no roots of the equation (A.5) for small $\lambda > 0$. As noted above, all roots of equation (A.5) in the closed right-hand plane lie in $H(\lambda_0, k)$ for $0 < \lambda < \lambda_0$, and the number of such roots is constant for this range of λ. One concludes from this that if $0 < \lambda < \lambda_0$, then equation (A.5) has no solutions satisfying $\operatorname{Re}\zeta \geqslant 0$. $\square$

The next proposition concerns the characteristic equation of the system (A.2).

PROPOSITION A.3. – *Assume that $k > 1$. Then equation* (A.6) *has at most two solutions (counting multiplicity) with real part zero. Such solutions, if they exist, are complex conjugates $\zeta = \pm i\nu$ where $\nu > 0$.*

PROOF. - Assume that $\zeta = i\nu$ is a solution of (A.6) for some real number ν; without loss one can assume $\nu \geqslant 0$. Writing equation (A.6) as

(A.19) $(\zeta - 1)^2 = k^2 \exp[-2\zeta r]$

and taking the norms of each side at the solution $\zeta = i\nu$ yields $1 + \nu^2 = k^2$, and hence $\nu = \sqrt{k^2 - 1}$ is uniquely determined and strictly positive. It remains then to show that if $\zeta = i\nu$ is a root of equation (A.6), then it is a simple root. That is, one must show the derivative

(A.20) $2(\zeta - 1) + 2rk^2 \exp[-2\zeta r]$

of the left-hand side of (A.6) is not zero if $\zeta = i\nu$ satisfies equation (A.19). This is easily done by using (A.19) to replace $k^2 \exp[-2\zeta r]$ with $(\zeta - 1)^2$ in (A.20), then removing the factor $2(\zeta - 1) \neq 0$. What remains is the quantity

$$1 + r(\zeta - 1) = 1 - r + ir\sqrt{k^2 - 1} \neq 0 .$$

This completes the proof.   □

The last result concerns the function Δ_+ in equation (A.7).

PROPOSITION A.4. - *Assume that* $k > 0$ *and* $r > 0$. *Then the equation*

(A.21) $\zeta - 1 + k \exp[-\zeta r] = 0$

has a solution satisfying

(A.22) $0 \leqslant \operatorname{Im} \zeta < \dfrac{\pi}{r} .$

PROOF. - Fix quantities k_1 and k_2 satisfying $0 < k_1 < 1 < k_2$, and consider k as a parameter in the closed interval $k_1 \leqslant k \leqslant k_2$, while $r > 0$ is fixed. Let $R(K, r)$ denote the rectangle

$$R(K, r) = \left\{ \zeta \in C : |\operatorname{Re} \zeta| < K \text{ and } |\operatorname{Im} \zeta| < \frac{\pi}{r} \right\}$$

where the constant K is chosen so that

(A.23) $K \geqslant 1 + k_2$

and

(A.24) $k_1 \exp[Kr] \geqslant 1 + K + \dfrac{\pi}{r} .$

It is sufficient to prove that $R(K, r)$ contains a root of (A.21), for either such a root or its complex conjugate will satisfy (A.22).

One first observes that if $k_1 \leqslant k \leqslant k_2$, then the equation (A.21) has no solution on the boundary $\partial R(K, r)$ of the above rectangle. First, if $|\text{Im } \zeta| = \pi/r$, then $\exp[-\zeta r]$ is real, and this implies (by taking the imaginary part of the left-hand side) that equation (A.21) cannot hold. Next, if $\text{Re } \zeta = K$ for a root of (A.21), then one has

$$K - 1 \leqslant |\zeta - 1| = |k \exp[-\zeta r]| < k_2$$

which contradicts (A.23). Finally, if ζ is a root of (A.21) satisfying $\text{Re } \zeta = -K$ and $|\text{Im } \zeta| < \pi/r$, then

$$k_1 \exp[Kr] \leqslant k \exp[Kr] = |k \exp[-\zeta r]| = |\zeta - 1| < 1 + K + \frac{\pi}{r}$$

contradicting (A.24).

Rouché's theorem therefore implies that the number of solutions of equation (A.21) in $R(K, r)$ is independent of k in the range $k_1 \leqslant k \leqslant k_2$. Setting $k = 1$, one observes that $\zeta = 0$ is indeed a solution in $R(K, r)$, hence $R(K, r)$ contains a solution of (A.21) for each k in this range. As k_1 and k_2 are arbitrary, this completes the proof. □

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