Intricate Structure of the Analyticity Set for Solutions of a Class of Integral Equations

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March 13, 2019

[1]Partially supported by The Center for Nonlinear Analysis at Rutgers University.
[2]Partially supported by NSF Grants DMS-0701171, DMS-1201328, and by The Lefschetz Center for Dynamical Systems at Brown University.

Abstract

We consider a class of compact positive operators $L : X \to X$ given by $(Lx)(t) = \int_{\eta(t)}^{t} x(s)\, ds$, acting on the space X of continuous 2π-periodic functions x. Here η is continuous with $\eta(t) \leq t$ and $\eta(t+2\pi) = \eta(t)+2\pi$ for all $t \in \mathbf{R}$. We obtain necessary and sufficient conditions for the spectral radius of L to be positive, in which case a nonnegative eigensolution to the problem $\kappa x = Lx$ exists for some $\kappa > 0$ (equal to the spectral radius of L) by the Krein-Rutman Theorem. If additionally η is analytic, we study the set $\mathcal{A} \subseteq \mathbf{R}$ of points t at which x is analytic; in general $\mathcal{A}$ is a proper subset of $\mathbf{R}$, although x is C^∞ everywhere. Among other results, we obtain conditions under which the complement $\mathcal{N} = \mathbf{R} \setminus \mathcal{A}$ of $\mathcal{A}$ is a generalized Cantor set, namely, a nonempty closed set with empty interior and no isolated points. The proofs of this and of other such results depend strongly on the dynamical properties of the map $t \to \eta(t)$.

This paper is dedicated to the memory of Professor George R. Sell.

Key Words: Integral operator; spectral radius; delay-differential equation; variable delay; analytic solution; generalized Cantor set.

AMS Subject Classification (2010): Primary 26E05, 26E10, 37E10, 34K13, 47B65; Secondary 26A18, 34K99, 47G10.

1 Introduction

In this paper we study the equation

$$\kappa x(t) = \int_{\eta(t)}^{t} x(s)\, ds, \tag{1.1}$$

where $\eta : \mathbf{R} \to \mathbf{R}$ is a given continuous function which satisfies

$$\eta(t + 2\pi) = \eta(t) + 2\pi, \qquad \eta(t) \leq t, \tag{1.2}$$

for every $t \in \mathbf{R}$. Here $\kappa \in \mathbf{C}$ is an unknown eigenvalue which is to be found along with the eigenfunction x. We wish to find and study such eigenfunctions of period 2π, that is, lying in the space

$$X = \{x : \mathbf{R} \to \mathbf{C} \mid x \text{ is continuous and } x(t + 2\pi) = x(t) \text{ for all } t \in \mathbf{R}\}. \tag{1.3}$$

We endow X with the usual supremum norm $\|x\| = \sup_{t \in \mathbf{R}} |x(t)|$, thereby making it a Banach space.

Associated to equation (1.1) is the compact linear operator $L : X \to X$ given by

$$(Lx)(t) = \int_{\eta(t)}^{t} x(s)\, ds \tag{1.4}$$

for $x \in X$, and so (1.1) can be written as $Lx = \kappa x$. In light of the inequality in (1.2), one sees that L is a positive operator with respect to the cone

$$X^+ = \{x \in X \mid x(t) \geq 0 \text{ for all } t \in \mathbf{R}\}$$

of nonnegative functions in X; that is, if $x \in X^+$ then $Lx \in X^+$.

One point of interest is whether or not a nontrivial nonnegative solution $x \in X^+ \setminus \{0\}$ of (1.1) exists; we observe that necessarily $\kappa \geq 0$ for such a solution. Another point of interest, perhaps the main focus of this paper, concerns the regularity properties of solutions; specifically, if η is analytic and $x \in X$ is a solution to (1.1), is x analytic? We discuss this further below.

By the Krein-Rutman Theorem, if the spectral radius $r(L)$ of L is strictly positive, that is if $r(L) > 0$, then equation (1.1) has a solution $x \in X^+ \setminus \{0\}$ with $\kappa = r(L)$. On the other hand, if $r(L) = 0$ then the Krein-Rutman Theorem is silent, and a nontrivial nonnegative solution to equation (1.1) with $\kappa = 0$ may or may not exist.

Among other results, in this paper we obtain necessary and sufficient conditions on the function η for $r(L) > 0$ to hold; see Theorem A below.

To see examples of both existence of, and nonexistence of, an eigenfunction in $X^+ \setminus \{0\}$ when $r(L) = 0$, consider the case in which η is nondecreasing, that is, $\eta(t_1) \leq \eta(t_2)$ whenever $t_1 \leq t_2$, and where also $\eta(a) = a$ for some a. Then by Theorem A necessarily $r(L) = 0$ must hold as condition (2.1) fails. For such η denote $S = \{t \in \mathbf{R} \mid \eta(t) = t\}$. Then if $x \in X^+ \setminus \{0\}$ satisfies equation (1.1) with $\kappa = 0$, we claim that $x(t_0) = 0$ for every $t_0 \in \mathbf{R} \setminus S$. One sees this by taking $t = t_0 + \varepsilon$ with $\varepsilon > 0$ sufficiently small; then $t_0 \in (\eta(t), t)$ and $x(s) = 0$ for every $s \in (\eta(t), t)$, by equation (1.1) with $\kappa = 0$, and thus $x(t_0) = 0$. One concludes from this that if S has empty interior, that is if $\mathbf{R} \setminus S$ is dense, then there does not exist $x \in X^+ \setminus \{0\}$ satisfying (1.1) with $\kappa = 0$. (This in particular is the case if η is analytic with $\eta(t) \not\equiv t$.) On the other hand, suppose that S has nonempty interior, say $(p, q) \subseteq S$ where $p < q < p + 2\pi$. Then one easily checks that if $x \in X^+ \setminus \{0\}$ is such that $x(t) = 0$ for every $t \in [q, p + 2\pi]$ then x satisfies (1.1) with $\kappa = 0$. In this case $x(t) > 0$ is permitted for $t \in (p, q)$, and so solutions $x \in X^+ \setminus \{0\}$ to (1.1) with $\kappa = 0$ do exist; in fact, there is an infinite dimensional set of them.

If $\eta(t) < t$ for every $t \in \mathbf{R}$ then $r(L) > 0$ (again see Theorem A), and one easily sees from (1.1) that $x(t) > 0$ for every $t \in \mathbf{R}$, for every eigenfunction in $X^+ \setminus \{0\}$; or equivalently, $x \in \text{int}(X^+ \setminus \{0\})$. (Here and in what follows we let $\text{int}(S)$ denote the interior of a set S.) Indeed, if $x(t_0) > 0$ for some $t_0 \in \mathbf{R}$ then by (1.1) we have that $\kappa x(t) > 0$ for every $t \in [t_0, t_0 + \varepsilon]$, where $\varepsilon = \min_{t \in \mathbf{R}}(t - \eta(t)) > 0$, as $t_0 \in [\eta(t), t]$ for such t. After a finite number of iterations of this argument one concludes that $x(t) > 0$ for every $t \in \mathbf{R}$, as desired.

In the above case where $\eta(t) < t$ for every $t \in \mathbf{R}$ it is known that $x \in X^+ \setminus \{0\}$ with $\kappa = r(L) > 0$ is the unique solution of equation (1.1) in $(X^+ \setminus \{0\}) \times (0, \infty)$ up to scalar multiple. On the other hand, if $\eta(t) = t$ for some $t \in \mathbf{R}$ but still $r(L) > 0$, then it is not known whether or not the above solution $(x, \kappa) \in (X^+ \setminus \{0\}) \times (0, \infty)$ with $\kappa = r(L) > 0$ is unique (up to scalar multiple). It would be interesting to have an example of such a solution which is not unique in this sense; but so far this eludes us.

Concerning the existence of eigenfunctions of L which do not belong to $X^+ \setminus \{0\}$ (more precisely, for which no scalar multiple belongs to this space), we may consider the case in which $\eta(t) = t - r_0$ where $r_0 > 0$ is a constant. Then one easily checks that $x(t) = e^{int}$ is an eigenfunction for any $n \in \mathbf{Z} \setminus \{0\}$ with eigenvalue $\kappa_n = i(e^{-inr_0} - 1)/n$; this is in addition to the eigenfunction $x(t) = 1$ for $\kappa_0 = r_0$. By a standard spectral perturbation argument, these eigenvalues perturb continuously if r_0 is replaced by a continuous 2π-periodic function $r(t)$ which is near r_0. In particular, given any $n_0 \geq 1$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $|r(t) - r_0| \leq \delta$ for all t, then for every n with $|n| \leq n_0$ the operator L, with $\eta(t) = t - r(t)$, has an eigenvalue $\widetilde{\kappa}_n$ for which $|\widetilde{\kappa}_n - \kappa_n| \leq \varepsilon$.

Note finally that in the above example with $\eta(t) = t - r_0$, none of the eigenvalues are nonzero real numbers except for $\kappa_0 = r_0$. We do not know of any example of an operator L as above, but with $\eta(t) = t - r(t)$, which has more than one nonzero real eigenvalue; it would be interesting to find such an example, or to prove that none exists.

Our interest in equation (1.1) stems from our earlier studies of analyticity properties of solutions of delay-differential equation, and particularly solutions of such equations with variable delays. Early work in [13] showed that for a broad class of equations, including those of the form

$$\dot{x}(t) = f(x(t - r_1), x(t - r_2), \ldots, x(t - r_m))$$

where $f : \mathbf{R}^{mn} \to \mathbf{R}^n$ is analytic with $x \in \mathbf{R}^n$ and where $r_j \geq 0$ for $1 \leq j \leq m$ are given constants, any solution $x(t)$ which exists and is bounded as $t \to -\infty$ is analytic in t. The assumption that the delays r_j are constant is essential here; in general, if the r_j vary with time, for example as explicitly given functions $r_j = r_j(t)$ which are analytic in t, then while a solution $x(t)$ as above must be C^∞ in t, it need not be analytic. Such issues were explored in [8]. Quite generally, if a solution $x(t)$ exists for all $t \in \mathbf{R}$, then we may define sets $\mathcal{A}, \mathcal{N} \subseteq \mathbf{R}$ by

$$\mathcal{A} = \{t_0 \in \mathbf{R} \mid t \to x(t) \text{ is analytic for } |t - t_0| < \varepsilon, \text{ for some } \varepsilon > 0\}, \qquad \mathcal{N} = \mathbf{R} \setminus \mathcal{A}. \tag{1.5}$$

Certainly $\mathcal{A}$ is an open subset of $\mathbf{R}$, and $\mathcal{N}$ is closed. As was shown in [8], with f and each r_j analytic, it may occur that both $\mathcal{A} \neq \emptyset$ and $\mathcal{N} \neq \emptyset$ for the same solution x; in this case we speak of coexistence of analyticity and non-analyticity.

Even in the case of constant delays subtleties abound. In [9] the results of [13] are extended to a broader class of equations with constant delays, including nonautonomous systems. Nevertheless, for equations such as

$$\dot{x}(t) = \sin(t^q)x(t - 1) \qquad \text{and} \qquad \dot{x}(t) = \exp(it^q)x(t - 1)$$

where $q \geq 2$ is an integer, there exists a nontrivial solution $x(t)$ which is bounded as $t \to -\infty$ (see [10]); but it is unknown whether or not this solution is analytic for any t, although it is certainly C^∞ for all $t \in \mathbf{R}$.

It is worth mentioning that there are many classes of delay-differential equations, with variable delays, which arise quite naturally as models in the sciences. In [14] Kuang and Smith study a system of the form

$$\dot{x}(t) = -\nu x(t) + f(x(t - r)), \qquad \text{where} \qquad \int_{t-r}^{t} k(x(t), x(s))\, ds = 1. \tag{1.6}$$

Here the function k is positive-valued and the delay $r > 0$ is implicitly determined by the second equation in (1.6). Such systems occur in models of population growth, and they generalize the much-studied Mackey-Glass equation [11], for which the delay is constant. The delay r in (1.6) is a so-called state-dependent delay, in that it depends of the state of the system, namely on the history $x(s)$ for $s \leq t$. Alternatively, one could consider the same Mackey-Glass differential equation (1.6) but instead with the delay $r = r(t)$ given as an explicit function of t; such a delay could model, for example, external seasonal variations in the environment.

Another class of examples is given by Walther [15], who studies the problem of controlling a vehicle by means of the echo of a signal. Here the finite signal speed gives a variable time-delay which is dependent on the position of the vehicle.

In [2] Krisztin studies a class of models with analytic nonlinearities as in (1.6) in which the delay r is implicitly defined by a simpler integral condition

$$\int_{t-r}^{t} k(x(s))\, ds = 1,$$

in which the kernel function does not depend on $x(t)$. In this case he obtains analyticity of solutions bounded at $-\infty$, of course assuming all nonlinearities in the differential equation are analytic. (A version of this for multiple delays also holds.) It was observed later [8] that following an analytic change of the time variable, such a system can be transformed to a system with a constant delay to which the results of [13] apply.

For other classes of variable-delay equations, such as ones where the delay $r = r(x(t))$ is an explicit function of the present state $x(t)$, see for example the references in [4], [5], [6], and [7]. Also see [1] for a comprehensive survey of recent results on state-dependent delay-differential equations.

It is our belief that quite typically, solutions of delay-differential equations with analytic nonlinearities can fail to be analytic at some values of t, even for solutions which exist and are bounded as $t \to -\infty$. (In this respect the above examples of Krisztin are an exception to the expected behavior.) The examples studied in [8], for which coexistence of analyticity and non-analyticity was established for some solution x, had the form of equation (1.1) with η analytic. When differentiated, and assuming that $\kappa \neq 0$, one obtains a delay-differential equation with a variable delay, namely

$$\dot{x}(t) = \frac{1}{\kappa}\Big(x(t) - x(\eta(t))\dot{\eta}(t)\Big). \tag{1.7}$$

In the present paper we explore these issues in more detail. Here we take equation (1.1) as a tractable model example with a simple presentation, but which nonetheless exhibits a very rich

and intricate structure of the analyticity set $\mathcal{A}$ and its complement $\mathcal{N}$. In [8] it was shown that $\mathcal{A} \neq \emptyset$ and $\mathcal{N} \neq \emptyset$ could occur simultaneously for a particular solution of this equation, although the detailed structure of these sets was unclear. In the present paper we are able to give a complete characterization and description of the sets $\mathcal{A}$ and $\mathcal{N}$ for a broad class of equations. In particular, this characterization is intimately connected with the dynamics of the discrete map $t \to \eta(t)$, and shows in certain cases that $\mathcal{N}$ has a fractal-like structure similar to the Cantor set.

In particular, in Theorem B we give general conditions, with η analytic, under which $\mathcal{N}$ is uncountable and $\mathcal{A}$ has infinitely many connected components in any interval of length 2π. This structure is extended in Theorem C. In Theorem D we give additional conditions under which $\mathcal{N}$ is a generalized Cantor set, that is, a nonempty closed set with empty interior and with no isolated points; and in addition in Theorem D we precisely characterize the set $\mathcal{A}$ in terms of the dynamical properties of the map $\eta : \mathbf{R} \to \mathbf{R}$. Finally, in Theorem E we verify the conditions of Theorem D for an explicit and elementary class of examples.

Let us finally mention that we believe many of our results should extend naturally, and using basically the same techniques, to certain classes of nonlinear equations. We have in mind equations such as

$$\kappa x(t) = \int_{\eta(t)}^{t} f(s, x(s))\, ds \tag{1.8}$$

where as before $\eta : \mathbf{R} \to \mathbf{R}$ is analytic and satisfies (1.2), and where, for example, $f : \mathbf{R}^2 \to \mathbf{R}$ is analytic and satisfies

$$xf(s,x) > 0 \text{ for all } x \neq 0, \qquad f(s+2\pi, x) = f(s,x), \qquad \frac{\partial f(s,0)}{\partial x} > 0, \qquad \lim_{x \to \pm\infty} \frac{f(s,x)}{x} = 0,$$

for all s and x, with the final limit uniform in s. Then if $0 < \kappa < r(L)$, with L as in (1.4), but with $x(s)$ replaced by $(\partial f(s,0)/\partial x)x(s)$, there exists $x \in X^+ \setminus \{0\}$ satisfying (1.8); such a result follows from standard arguments involving degree theory for maps of cones. And in this situation it would be natural to expect analogs of Theorems B, C, and D to hold.

2 The Main Results

Here we state our main results, namely Theorems A, B, C, D, and E; the proofs of these results will be given in later sections. Throughout this section L is the linear operator given by (1.4), acting on the space X given by (1.3). We let $r(L)$ denote the spectral radius of L.

The first theorem gives a necessary and sufficient condition for $r(L)$ to be positive.

Theorem A. *Let $\eta : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy* (1.2) *for every $t \in \mathbf{R}$. Then $r(L) > 0$ if and only if*

$$\inf_{t \in [a,\infty)} \eta(t) < a \qquad \text{whenever } \eta(a) = a. \tag{2.1}$$

In particular, if $\eta(t) < t$ for every $t \in \mathbf{R}$, so that the condition (2.1) *is vacuously true, then $r(L) > 0$.*

As noted, we are interested in the case in which η is a real analytic function; and in particular, we are interested in the sets $\mathcal{A}, \mathcal{N} \subseteq \mathbf{R}$ defined in (1.5) for a given solution x of (1.1). In studying these sets, an important role is played by so-called Volterra intervals. We define the following.

Definition. Let $\eta : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy $\eta(t) \leq t$ for every $t \in \mathbf{R}$. Suppose that $I = [a, b] \subseteq \mathbf{R}$ is a compact interval (possibly a single point). We say that I is a **Volterra interval** if $\eta(I) \subseteq I$, equivalently, if

$$a \leq \eta(t) \qquad \text{for every } t \in [a, b]. \tag{2.2}$$

We say that I is a **right-maximal Volterra interval** (abbreviated **RM-Volterra interval**) if it is a Volterra interval and if in addition

$$b = \sup\{t \in \mathbf{R} \mid a \leq t \text{ and } a \leq \eta(s) \text{ for every } s \in [a, t]\}, \tag{2.3}$$

that is, if b is maximal for the given a. We say that I is a **maximal Volterra interval** if it is an RM-Volterra interval which is not properly contained in another RM-Volterra interval.

The next proposition indicates the significance of Volterra intervals. The proposition that follows it establishes half of Theorem A, namely the necessity of (2.1) for $r(L) > 0$ to hold.

Proposition 2.1. *Let $\eta : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy* (1.2) *for every $t \in \mathbf{R}$. Suppose that $[a, b]$ is a Volterra interval. Also suppose that $Lx = \kappa x$, that is, equation* (1.1) *holds, for some $x \in X$ and $\kappa \neq 0$. Then*

$$x(t) = 0 \qquad \text{for every } t \in [a, b] \tag{2.4}$$

holds.

Proof. For every $t \in [a, b]$ we have that $a \leq \eta(t) \leq t$, and so from (1.1) we have that

$$|x(t)| = \left| \frac{1}{\kappa} \int_{\eta(t)}^{t} x(s)\, ds \right| \leq \frac{1}{|\kappa|} \int_{a}^{t} |x(s)|\, ds. \tag{2.5}$$

Thus (2.4) follows directly from Gronwall's inequality. ■

We remark that the terminology "Volterra interval" is motivated by the second integral in (2.5), which has the form of a Volterra operator.

The following result proves half of Theorem A, namely the necessity of (2.1) for $r(L) > 0$ to hold.

Corollary 2.2. *Let $\eta : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy (1.2) for every $t \in \mathbf{R}$, and suppose that (2.1) is false. Then $r(L) = 0$.*

Proof. Suppose to the contrary that $r(L) > 0$. Then by the Krein-Rutman Theorem there exists a solution $x \in X^+ \setminus \{0\}$ to (1.1) with $\kappa = r(L)$. Since (2.1) is false, there exists $a \in \mathbf{R}$ such that $a \leq \eta(t)$ for every $t \geq a$, and in particular for every $t \in [a, a + 2\pi]$. Thus $[a, a + 2\pi]$ is a Volterra interval, and so $x(t) = 0$ for every $t \in [a, a+2\pi]$ by Proposition 2.1. As x is 2π-periodic, it vanishes identically, a contradiction. Thus $r(L) = 0$. ■

It is perhaps instructive to consider the case of a one-parameter family of delays, with $\eta_\mu(t) = t - \mu r(t)$, where $r : \mathbf{R} \to [0, \infty)$ is 2π-periodic and $\mu > 0$ is a parameter. Note that $\eta_\mu(a) = a$ if and only if $r(a) = 0$; assume there exists some $a \in \mathbf{R}$ such that $r(a) = 0$. Also assume that r is C^1 and that all zeros of r are isolated. Then whenever $r(a) = 0$ we have that $r'(a) = 0$ and thus

$$\lim_{t \to a+} \frac{t - a}{r(t)} = \infty.$$

One sees from this that there exists $t_* = t_*(a) \in (a, a + 2\pi)$ with $r(t_*) \neq 0$ such that

$$\mu(a) = \frac{t_* - a}{r(t_*)} = \inf_{\substack{t \in (a, a+2\pi) \\ r(t) \neq 0}} \frac{t - a}{r(t)}, \tag{2.6}$$

where the first equality in (2.6) serves as the definition of $\mu(a)$; and further,

$$\frac{d}{dt}\left(\frac{t - a}{r(t)}\right)\bigg|_{t=t_*} = 0$$

for any such t_*. If we now define

$$\mu_* = \max_{\substack{a \in [0, 2\pi) \\ r(a) = 0}} \mu(a)$$

then one sees that condition (2.1) holds for $\eta = \eta_\mu$ if and only if $\mu > \mu_*$.

We mention a simple example for which μ_* can be calculated to high accuracy, and which is relevant to Theorem E. Take $r(t) = \pi(1 - \cos t)$, and so $r(0) = 0$ and $r(t) > 0$ for $t \in (0, 2\pi)$. One easily sees that

$$\mu_* = \mu(0) = \inf_{t\in(0,2\pi)} h_1(t), \qquad h_1(t) = \frac{t}{\pi(1-\cos t)}.$$

Using the identity $1 - \cos t = 2\sin^2(\frac{t}{2})$, one finds that

$$h_1'(t) = \frac{2\sin^2(t/2) - t\sin t}{4\pi\sin^4(t/2)} = \frac{\sin(t/2) - t\cos(t/2)}{2\pi\sin^3(t/2)}.$$

Writing $\tau = \frac{t}{2}$, one sees that $h_1'(t) = 0$ with $0 < t < 2\pi$ if and only if

$$\sin\tau - 2\tau\cos\tau = 0, \qquad \text{with } 0 < \tau < \pi.$$

Since $\sin\tau - 2\tau\cos\tau > 0$ for $\frac{\pi}{2} \le \tau < \pi$ it is enough to consider τ satisfying $0 < \tau < \frac{\pi}{2}$. Defining $h_2(\tau) = \tan\tau - 2\tau$, one observes that $h_2(0) = 0$ and $h_2'(0) < 0$, with $h_2''(\tau) > 0$ for $0 < \tau < \frac{\pi}{2}$, and also $\lim_{\tau\to\pi/2-} h_2(\tau) = \infty$, and thus there exists a unique quantity $\tau_* \in (0, \frac{\pi}{2})$ such that $h_2(\tau_*) = 0$. Now setting $t_* = 2\tau_*$, we have that $h_1'(t_*) = 0$ and

$$\mu_* = h_1(t_*) = \frac{\tau_*}{\pi\sin^2\tau_*}.$$

It is easy to check directly that $h_2(1.165) < 0$ and $h_2(1.166) > 0$, and thus $1.165 < \tau_* < 1.166$. Then, using Newton's method, one can refine this to $1.16556118 < \tau_* < 1.16556119$, and from that conclude that $0.43928360 < \mu_* < 0.43928361$.

The following lemma is not itself a "main result," but it provides context for the statements of Theorems B, C and D, in addition to being used in the proofs of these results; it will be proved at the end of this section. In this lemma and in what follows, let us denote

$$E(S) = \bigcup_{t\in S} [\eta(t), t], \tag{2.7}$$

for any set $S \subseteq \mathbf{R}$ provided that $\eta(t) \le t$ for all $t \in S$. Note that $S \subseteq E(S)$, and that if η is continuous and if S is an interval then so is $E(S)$, with S and $E(S)$ having the same right-hand endpoint.

Lemma 2.3. *Let $\eta : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy* (1.2) *for every $t \in \mathbf{R}$. Assume in addition that condition* (2.1) *holds. Then*

(a) *if* $[a,b]$ *is a Volterra interval then* $\eta(a)=a$*; if further* $[a,b]$ *is an RM-Volterra interval then* $\eta(b)=a$*; and*

(b) *if* $\eta(a)=a$ *for some* $a\in\mathbf{R}$ *then there exists a unique* $b\geq a$ *such that* $[a,b]$ *is an RM-Volterra interval; and further, every Volterra interval is contained in an RM-Volterra interval.*

Thus there exists a one-to-one correspondence between fixed points of η *and RM-Volterra intervals; and so if* η *is analytic then any finite interval contains only finitely many RM-Volterra intervals.*

We further have that

(c) *if* $I_1,I_2\subseteq\mathbf{R}$ *are RM-Volterra intervals, then either* $I_1\cap I_2=\emptyset$*, or* $I_1\subseteq I_2$*, or* $I_2\subseteq I_1$*;*

(d) *if* $I_1,I_2\subseteq\mathbf{R}$ *are maximal Volterra intervals, then either* $I_1\cap I_2=\emptyset$ *or* $I_1=I_2$*;*

(e) *every Volterra interval (and thus every RM-Volterra interval) is contained in a maximal Volterra interval, and every maximal Volterra interval has length strictly less than* 2π*; and*

(f) *if* $Lx=\kappa x$*, that is, equation* (1.1) *holds, for some* $x\in X\setminus\{0\}$ *and* $\kappa\neq 0$*, and if* $I\subseteq\mathbf{R}$ *is a maximal Volterra interval, then* $x(t)=0$ *for every* $t\in I$*; and further,* I *is maximal with respect to this property, that is, if* $x(t)=0$ *for every* $t\in J$ *for some interval* J *where* $I\subseteq J$*, then* $I=J$*.*

Finally,

(g) *for any* $b\in\mathbf{R}$*, and inductively for* $k\geq 1$*, let*

$$J_0(b)=\{b\},\qquad J_k(b)=E(J_{k-1}(b)),\qquad J_*(b)=\overline{\bigcup_{k\geq 0}J_k(b)};\tag{2.8}$$

then either $J_*(b)=[a,b]$ *for some* $a\leq b$ *or* $J_*(b)=(-\infty,b]$*; and* $J_*(b)=[a,b]$ *if and only if* b *is contained in a maximal Volterra interval; and further, there always exists* $b\in\mathbf{R}$ *such that* $J_*(b)=(-\infty,b]$*.*

Let us observe that by using part (e) of Lemma 2.3, a partial converse to part (f) can be proved. Assume that conditions (1.2) and (2.1) hold, and that $Lx=\kappa x$ for some $x\in X^+\setminus\{0\}$ and $\kappa\neq 0$. (Thus we are assuming an additional condition, that $x(t)\geq 0$ for all t, and so $\kappa>0$.) Suppose for some interval I that $x(t)=0$ for every $t\in I$ and that I is maximal with respect to this property;

necessarily I is compact. We claim that I is a maximal Volterra interval. To prove this note that for any $t \in I$ we have that

$$0 = \kappa x(t) = \int_{\eta(t)}^{t} x(s)\, ds,$$

hence $x(s) = 0$ identically on $[\eta(t), t]$; and this implies that $\eta(t) \in I$, due to the maximality of I. Thus $\eta(I) \subseteq I$, that is, I is a Volterra interval. By part (e), I is contained in a maximal Volterra interval J, and $x(t) = 0$ identically on J by Proposition 2.1. It follows that $I = J$, and so I is a maximal Volterra interval, as claimed.

We have the following result for equation (1.1), which gives a condition for coexistence of analyticity and non-analyticity, and which shows that the sets $\mathcal{A}$ and $\mathcal{N}$ can have a rather intricate structure. In this result and elsewhere, recall that a connected component of a topological space Y is a nonempty subset of Y which is connected and is maximal in the sense of set inclusion.

Theorem B. *Let $\eta : \mathbf{R} \to \mathbf{R}$ be real analytic and satisfy* (1.2) *for every $t \in \mathbf{R}$. Suppose that $[a, b]$ is a maximal Volterra interval. Also suppose that $Lx = \kappa x$, that is, equation* (1.1) *holds, for some $x \in X \setminus \{0\}$ and $\kappa \neq 0$. Then*

(a) $x(t) = 0$ for every $t \in [a, b]$, and thus $(a, b) \subseteq \mathcal{A}$; it is also the case that $a, b \in \mathcal{N}$;

(b) for every $\varepsilon > 0$ the interval $[a - \varepsilon, a]$ contains infinitely many connected components of $\mathcal{A}$ and uncountably many points of $\mathcal{N}$; and

(c) for every $\varepsilon > 0$ the interval $[b, b + \varepsilon]$ contains infinitely many connected components of $\mathcal{A}$ and uncountably many points of $\mathcal{N}$.

Here $\mathcal{A}$ and $\mathcal{N}$ are as in (1.5). *(Recall that $\mathcal{A}$ is an open set and $\mathcal{N}$ is a closed set.)*

In the setting of Theorem B the interval $[a, b]$ is a maximal interval on which the solution x vanishes identically. It would be of interest to have an asymptotic description of $x(t)$ for t in a neighborhood of this interval; for example one might ask how rapidly $x(t)$ approaches zero as $t \to a-$ or as $t \to b+$. Necessarily the rate of approach to zero would be more rapid than algebraic, as x is a C^∞ function. We note that η near a has the form $\eta(t) = t - K(t-a)^n + O(|t-a|^{n+1})$ for some $K > 0$ and $n \geq 2$ (in fact with n even, as $\eta(t) \leq t$). The theory of parabolic renormalization, and in particular the so-called Ecalle-Voronin modulus, applies here; it indicates that there is a continuum of distinct equivalence classes of such local maps, up to analytic conjugacy; see, for

example, [12]. While this theory might be useful here, we still expect a resolution of this question could be quite challenging.

Theorem C below extends the results of Theorem B. Following this is Theorem D, which describes a situation in which the set $\mathcal{N}$ is a generalized Cantor set, namely it is nonempty and closed, with empty interior, and has no isolated points.

Theorem C. *Let $\eta : \mathbf{R} \to \mathbf{R}$ be real analytic, let μ be a positive integer, and define*

$$\xi(t) = -\eta(-t) - 2\pi\mu.$$

Assume that

(a) η satisfies condition (1.2) *for every $t \in \mathbf{R}$;*

(b) ξ satisfies condition (1.2) *(with η replaced with ξ) for every $t \in \mathbf{R}$;*

(c) $\eta(a) = a$ for at least one $a \in \mathbf{R}$; and

(d) $\xi(v) = v$ for at least one $v \in \mathbf{R}$.

Then condition (2.1) *holds for η, and also for ξ; in particular, $r(L) > 0$. Suppose further that $Lx = \kappa x$, that is, equation* (1.1) *holds, for some $x \in X \setminus \{0\}$ and $\kappa \neq 0$. Then the conclusions of Theorem B hold for any maximal Volterra interval $[a, b]$ for η. Additionally, if $[v, w]$ is any maximal Volterra interval for ξ then $-w, -v \in \mathcal{N}$, and also conclusions (b) and (c) of Theorem B hold, but with a and b replaced, respectively, by $-w$ and $-v$. If further we have that*

(e) if $[v, w]$ is a maximal Volterra interval for the function ξ, then the strict inequalities $v < \xi(t) < t$ hold for every $t \in (v, w)$,

then $(-w, -v) \subseteq \mathcal{A}$ holds.

One sees in Theorem C that with condition (a) holding, condition (b) means simply that $\eta(t) \geq t - 2\pi\mu$ for every $t \in \mathbf{R}$, and condition (d) says that $\eta(-v) = -v - 2\pi\mu$ for some $v \in \mathbf{R}$. Also, we shall see that if $[v, w]$ is a maximal Volterra interval for ξ, then the solution x in the statement of Theorem C need not vanish identically on $[-w, -v]$; by contrast, x does vanish identically on $[a, b]$ if $[a, b]$ is a maximal Volterra interval for η. In particular, such $[-w, -v]$ is not an invariant interval for η while $[a, b]$ is. But do observe that $[-w, -v]$ is "mod 2π invariant," in the sense that if $t \in [-w, -v]$ then $\eta(t) + 2\pi\mu \in [-w, -v]$.

The proof of Theorem C will be given in Section 5, along with the proof of Theorem D. We need the following definition for Theorem D.

Definition. Suppose that $\eta : \mathbf{R} \to \mathbf{R}$ is C^1 and that $S \subseteq \mathbf{R}$. We say that η is **expansive on S** if whenever $t \in S$ is such that $\eta^k(t) \in S$ for every $k \geq 1$, then

$$\liminf_{k\to\infty} |\dot{\eta}^k(t)| > 1. \tag{2.9}$$

Here η^k denotes the k^{th} iterate of the function η, and $\dot{\eta}^k$ is the derivative of η^k.

Theorem D. *Assume all the conditions and notation in the statement of Theorem C; in particular, assume conditions (a) through (e) in the statement of that result. Let $[a_k, b_k]$ and $[v_k, w_k]$ be an enumeration (in any order) of all the maximal Volterra intervals for η and for ξ, respectively, where $k \in \mathbf{Z}$. Let*

$$\mathcal{B}_\eta = \bigcup_{k=-\infty}^{\infty} (a_k, b_k), \qquad \mathcal{B}_\xi = \bigcup_{k=-\infty}^{\infty} (-w_k, -v_k), \qquad \mathcal{B} = \mathcal{B}_\eta \cup \mathcal{B}_\xi, \qquad \mathcal{S} = \mathbf{R} \setminus \overline{\mathcal{B}}, \tag{2.10}$$

and suppose that η is expansive on the set $\mathcal{S}$. Then

(a) the set $\mathcal{A}$ is open and dense in $\mathbf{R}$, and the set $\mathcal{N}$ is nonempty and closed, with empty interior;

(b) defining

$$\mathcal{G}_* = \{t_0 \in \mathbf{R} \mid \text{there exists } n \geq 1 \text{ and } \varepsilon > 0 \text{ such that}$$

$$\eta^n(t_0) \in \partial\mathcal{B}_* \text{ and } \eta^n(t) \in \overline{\mathcal{B}} \text{ for every } t \text{ with } |t - t_0| \leq \varepsilon\}, \qquad * = \eta \text{ or } \xi, \tag{2.11}$$

$$\mathcal{G} \ = \mathcal{G}_\eta \cup \mathcal{G}_\xi,$$

we have that

$$\begin{aligned} \mathcal{A} \setminus \mathcal{G} &\subseteq \{t \in \mathbf{R} \mid \eta^n(t) \in \mathcal{B} \text{ for some } n \geq 0\}, \\ \mathcal{N} &\subseteq \{t \in \mathbf{R} \mid \eta^n(t) \notin \mathcal{B} \text{ for every } n \geq 0\}, \end{aligned} \tag{2.12}$$

and thus in the case that $\mathcal{G} = \emptyset$ both these two inclusions are equalities;

(c) the inclusion $\mathcal{G}_\eta \subseteq \mathcal{A}$ holds; and

(d) letting $\mathcal{I}$ denote the set of isolated points of $\mathcal{N}$, namely

$$\mathcal{I} = \{t \in \mathbf{R} \mid (t - \varepsilon, t + \varepsilon) \cap \mathcal{N} = \{t\} \text{ for some } \varepsilon > 0\},$$

we have that $\mathcal{I} \subseteq \mathcal{G}_\xi$; in particular, if $\mathcal{G}_\xi = \emptyset$, then $\mathcal{N}$ is a perfect set, namely a nonempty closed set with empty interior and with no isolated points.

We remark that for each $k \in \mathbf{Z}$ we have that

$$\eta([a_k, b_k]) \subseteq [a_k, b_k], \qquad \eta([-w_k, -v_k]) \subseteq [-w_k - 2\pi\mu, -v_k - 2\pi\mu], \qquad \text{hence} \qquad \eta(\overline{\mathcal{B}}) \subseteq \overline{\mathcal{B}},$$

and we know that the intervals $[a_k, b_k]$ are pairwise disjoint by (d) of Lemma 2.3, as are the intervals $[-w_k, -v_k]$. Additionally, it is the case that the intervals $[a_j, b_j]$ and $[-w_k, -v_k]$ are disjoint for any $j, k \in \mathbf{Z}$. To prove this, assume to the contrary that there exists $t \in [a_j, b_j] \cap [-w_k, -v_k]$ for some j and k. Then $\eta(t) \in [-w_k - 2\pi\mu, -v_k - 2\pi\mu]$, and one sees inductively that $\eta^n(t) \in [-w_k - 2\pi\mu n, -v_k - 2\pi\mu n]$ for every $n \geq 0$; thus $\lim_{n\to\infty} \eta^n(t) = -\infty$. On the other hand, $\eta^n(t) \in [a_j, b_j]$ for every such n as $\eta([a_j, b_j]) \subseteq [a_j, b_j]$, so $\eta^n(t)$ is a bounded sequence. With this contradiction we conclude that $[a_j, b_j] \cap [-w_k, -v_k] = \emptyset$. This also implies that

$$\begin{aligned} &\partial\mathcal{B}_\eta = \{a_k\}_{k=-\infty}^{\infty} \cup \{b_k\}_{k=-\infty}^{\infty}, \qquad \partial\mathcal{B}_\xi = \{-w_k\}_{k=-\infty}^{\infty} \cup \{-v_k\}_{k=-\infty}^{\infty}, \\ &\partial\mathcal{B} = \partial\mathcal{B}_\eta \cup \partial\mathcal{B}_\xi, \end{aligned} \tag{2.13}$$

and that the intervals $[a_k, b_k]$ and $[-w_k, -v_k]$ are the connected components in $\overline{\mathcal{B}}$.

One expects that "typically" the sets $\mathcal{G}_\eta$ and $\mathcal{G}_\xi$, and thus $\mathcal{G}$, are empty. More precisely, suppose that $t_0 \in \mathcal{G}_*$ where $* = \eta$ or ξ; say $* = \eta$ for definiteness. Let $n \geq 1$ be as in the definition (2.11) of $\mathcal{G}_\eta$. Define the set

$$\begin{aligned} \mathcal{M}_\eta = \{t \in \mathbf{R} \,|\, &\text{there exists an integer } m \geq 1 \text{ such that} \\ &\eta^{(k)}(t) = 0 \text{ for } 1 \leq k \leq 2m-1 \text{ but } \eta^{(2m)}(t) \neq 0\}, \end{aligned} \tag{2.14}$$

namely the points at which η has a local maximum or minimum. (Here $\eta^{(k)}$ denotes the k^{th} derivative of η; by contrast, as noted earlier, η^k denotes the k^{th} iterate of η.) One sees η^n has a local maximum or minimum at t_0, and from this it follows that η has a local maximum or minimum at $\eta^k(t_0)$ for some k with $0 \leq k \leq n-1$; that is, $t_* \in \mathcal{M}_\eta$ where $t_* = \eta^k(t_0)$. Further, $\eta^{n-k}(t_*) = \eta^n(t_0) \in \partial\mathcal{B}_\eta$. Thus a necessary condition for $\mathcal{G}_\eta$ to be nonempty is that $\eta^j(t_*) \in \partial\mathcal{B}_\eta$ for some $t_* \in \mathcal{M}_\eta$ and some $j \geq 1$. While such a condition is possible, one does not typically expect it due to the fact that both $\mathcal{M}_\eta$ and $\mathcal{G}_\eta$ are discrete sets, containing only finitely many points in any finite interval. Similar conclusions apply for $* = \xi$.

One also sees that if $\mathcal{G}_* \neq \emptyset$ then $\mathcal{G}_*$ is an infinite set. Indeed, for any $t_0 \in \mathcal{G}_*$ let $\nu(t_0) \geq 1$ be the smallest integer n such that the defining condition (2.11) for $\mathcal{G}_*$ holds for t_0. Also, for each $n \geq 1$ let $\mathcal{G}_{*n} = \{t_0 \in \mathcal{G}_* \,|\, \nu(t_0) = n\}$. Then $\mathcal{G}_*$ is the disjoint union of the sets $\mathcal{G}_{*n}$. One easily sees that $\mathcal{G}_{*(n+1)}$ is the inverse image of $\mathcal{G}_{*n}$ under the map $* = \eta$ or $* = \xi$. It follows that $\mathcal{G}_* \neq \emptyset$ if and only if $\mathcal{G}_{*n} \neq \emptyset$ for every $n \geq 1$, in which case $\mathcal{G}_*$ is an infinite set.

The next theorem provides an explicit and simple class of examples which satisfy the conditions of Theorem D.

Theorem E. *The function*

$$\eta(t) = t - \pi\mu(1 - \cos t),$$

where $\mu \geq 1$ is an integer, satisfies the hypotheses of Theorem D. In particular, there exists a quantity $b \in (0, \frac{\pi}{2})$ such that

$$[a_k, b_k] = [2\pi k, b + 2\pi k], \qquad [v_k, w_k] = [\pi + 2\pi k, b + \pi + 2\pi k], \qquad k \in \mathbf{Z}, \tag{2.15}$$

and therefore

$$\mathcal{S} = \Big(\bigcup_{k=-\infty}^{\infty} (-\pi + 2\pi k, 2\pi k) \Big) \cup \Big(\bigcup_{k=-\infty}^{\infty} (b + 2\pi k, -b + \pi + 2\pi k) \Big). \tag{2.16}$$

In addition $\mathcal{G} = \emptyset$, and so the inclusions in (2.12) are equalities and $\mathcal{N}$ is a perfect set.

The following lemma provides a practical criterion with which expansiveness of η, defined above, can be established. It will be used in the proof of Theorem E.

Lemma 2.4. *Suppose that $\eta : \mathbf{R} \to \mathbf{R}$ is C^1. Also suppose that $S \subseteq \mathbf{R}$ and that there exist $\alpha \in (0, 1)$ and $\beta > 0$ such that $\alpha\beta > 1$ and such that the following holds. If $t \in S$ is such that $\eta^k(t) \in S$ for every $k \geq 1$ then $|\dot\eta(t)| \geq \alpha$; and additionally, if $|\dot\eta(t)| \leq 1$ then there exists $n \geq 1$ such that*

$$|\dot\eta(\eta^k(t))| > 1 \qquad \text{for } 1 \leq k \leq n-1, \qquad \text{and} \qquad |\dot\eta(\eta^n(t))| \geq \beta. \tag{2.17}$$

Then η is expansive on S.

Proof. Let $S_* = \{t \in S \mid \eta^k(t) \in S \text{ for every } k \geq 1\}$. Fix any $t \in S_*$ and let $t_k = \eta^k(t)$ for $k \geq 0$ (and thus $t_0 = t$). Note here that $t_k \in S_*$ for every $k \geq 0$.

First suppose that there are infinitely many $k \geq 0$ for which $|\dot\eta(t_k)| \leq 1$; denote such k, in order, by $m_1 < m_2 < m_3 < \cdots$. Then by assumption (2.17), and as $t_{m_j} \in S_*$ for each $j \geq 1$, there exists n_j with $m_j < n_j < m_{j+1}$ such that

$$|\dot\eta(t_k)| \begin{cases} \geq \alpha, & \text{if } k = m_j \text{ for some } j \geq 1, \\ \geq \beta, & \text{if } k = n_j \text{ for some } j \geq 1, \\ > 1, & \text{otherwise.} \end{cases}$$

It follows that

$$|\dot\eta^{k+1}(t_0)| = \prod_{i=0}^{k} |\dot\eta(t_i)| \begin{cases} \geq (\alpha\beta)^j, & \text{for } n_j \leq k < m_{j+1} \text{ with } j \geq 1, \\ \\ \geq (\alpha\beta)^j \alpha, & \text{for } m_{j+1} \leq k < n_{j+1} \text{ with } j \geq 0, \end{cases}$$

and thus $\lim_{k\to\infty} |\dot\eta^k(t_0)| = \infty$ and so (2.9) holds.

In case there are only finitely many k for which $|\dot\eta(t_k)| \leq 1$, say $j_0 \geq 1$ of them, a slight modification of the above argument shows that $|\dot\eta^{k+1}(t_0)| \geq (\alpha\beta)^{j_0} > 1$ for all $k \geq n_{j_0}$. Finally, if $|\dot\eta(t_k)| \leq 1$ for no k, and so $|\dot\eta(t_k)| > 1$ for every k, then sees that $|\dot\eta^k(t_0)| > 1$ for every $k \geq 1$, and that $|\dot\eta^k(t_0)|$ is strictly increasing in k. In any case condition (2.9) holds, as desired. ■

We end this section with the proof of Lemma 2.3. Here and elsewhere we let $\text{len}(I)$ denote the length of an interval I.

Proof of Lemma 2.3. We prove the seven parts of this lemma in sequence.

(a) We have $a \leq \eta(t) \leq t$ for every $t \in [a, b]$ by (1.2) and (2.2), and thus $\eta(a) = a$. If further $[a, b]$ is an RM-Volterra interval then $\eta(b) = a$ follows immediately from (2.3).

(b) With $\eta(a) = a$, if $[a, b]$ is an RM-Volterra interval then b is uniquely determined by (2.3). Furthermore, for any a satisfying $\eta(a) = a$ the right-hand side of (2.3) is a finite quantity in light of condition (2.1); and moreover $[a, b]$ is a Volterra interval as (2.2) holds. Thus $[a, b]$ is an RM-Volterra interval. And further, if $[a, r]$ is a Volterra interval then b in (2.3) satisfies $b \geq r$; thus $[a, r]$ is contained in the RM-Volterra interval $[a, b]$.

(c) Suppose that $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are RM-Volterra intervals which are not disjoint, and that neither is a subset of the other. Without loss we have $a_1 < a_2 \leq b_1 < b_2$. But then $\eta(b_1) = a_1$ by (a) above, so $\eta(b_1) < a_2$. This now contradicts (2.2) for $[a, b] = [a_2, b_2]$ and $t = b_1 \in [a, b]$.

(d) This follows directly from (c) and from the definition of a maximal Volterra interval.

(e) Let us first note that if $J = [p, q]$ is an RM-Volterra interval (in particular, if it is a maximal Volterra interval) then $\text{len}(J) < 2\pi$. For if not, then $J' = [p + 2\pi, q + 2\pi]$ is also an RM-Volterra interval with $J \cap J' \neq \emptyset$, but neither J nor J' is contained in the other. This contradicts (c).

Now suppose $I = [a, r]$ is a Volterra interval, and consider

$$\mathcal{J} = \{J \subseteq \mathbf{R} \mid J \text{ is an RM-Volterra interval with } I \subseteq J\}.$$

Then $\mathcal{J}$ is a nonempty set by (b). (We remark that the intervals $J \in \mathcal{J}$ need not have a as their left-hand endpoint.) By (c), if $J_1, J_2 \in \mathcal{J}$ then either $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$, and so the set

$\mathcal{J}$ is totally ordered by inclusion. If $\mathcal{J}$ is a finite set then it contains a maximal element which is necessarily a maximal Volterra interval. If on the other hand $\mathcal{J}$ is an infinite set then there exists a sequence $J_k = [p_k, q_k] \in \mathcal{J}$ with $J_k \subseteq J_{k+1}$ for every $k \geq 1$, such that $\lim_{k\to\infty} \operatorname{len}(J_k) = \beta$, where $\beta = \sup_{J\in\mathcal{J}} \operatorname{len}(J) \leq 2\pi$. In this case $\lim_{k\to\infty} p_k = p_\infty$ and $\lim_{k\to\infty} q_k = q_\infty$ exist, and one sees that $J_\infty = [p_\infty, q_\infty]$ is a Volterra interval with $\operatorname{len}(J_\infty) = \beta$. In particular, J_∞ is contained in an RM-Volterra interval J_*. But from the definition of β necessarily $\beta = \operatorname{len}(J_\infty) \leq \operatorname{len}(J_*) \leq \beta$, hence $J_\infty = J_*$, and so J_∞ is an RM-Volterra interval. Again from the definition of β it follows that J_∞ is a maximal Volterra interval. This proves (e).

(f) By Proposition 2.1 we have that $x(t) = 0$ for every $t \in I$. Denoting $I = [a, b]$, let $[p, q] \supseteq [a, b]$ be the maximal interval containing $[a, b]$ on which x vanishes; that is,

$$\begin{aligned} p &= \inf\{t \in \mathbf{R} \mid t \leq a \text{ and } x(s) = 0 \text{ for every } s \in [t, a]\}, \\ q &= \sup\{t \in \mathbf{R} \mid t \geq b \text{ and } x(s) = 0 \text{ for every } s \in [b, t]\}. \end{aligned} \tag{2.18}$$

We wish to show that $[p, q] = [a, b]$. To begin, we shall show that

$$\eta(t) \geq p \qquad \text{for every } t \in [p, a]. \tag{2.19}$$

Let $r_1 = \inf_{t\in[p,a]} \eta(t)$. Certainly $r_1 \leq p$ as $\eta(p) \leq p$. Given any $c_1 \in [r_1, a]$, then since $\eta(a) = a$ there exists $t \in [p, a]$ such that $\eta(t) = c_1$. For this t we have that $x(t) = 0$ and thus

$$\kappa x(t) = 0 = \int_{\eta(t)}^{t} x(s)\, ds = \int_{c_1}^{p} x(s)\, ds + \int_{p}^{t} x(s)\, ds = \int_{c_1}^{p} x(s)\, ds.$$

As $c_1 \in [r_1, a]$ is arbitrary, the vanishing of the final integral above implies that $x(s) = 0$ for every $s \in [r_1, a]$. If $r_1 < p$ this violates the definition of p; thus necessarily $r_1 = p$ and thus (2.19) holds.

Now $\eta(t) \geq a$ for every $t \in [a, b]$, and thus by (2.19) $\eta(t) \geq p$ for every $t \in [p, b]$. This implies that $[p, b]$ is a Volterra interval and is thus contained in a maximal Volterra interval by (e) above. But by assumption $[a, b]$ is a maximal Volterra interval, and thus $[p, b] = [a, b]$ must hold. Thus $p = a$, as desired.

There remains to prove that $q = b$. Suppose to the contrary that $q > b$. Let $r_2 = \inf_{t\in[b,q]} \eta(t)$. Then in light of (2.3) necessarily $r_2 < a$. Now take any $c_2 \in [r_2, a]$. Then since $\eta(b) = a$, there exists $t \in [b, q]$ with $\eta(t) = c_2$; and as $x(t) = 0$ for this t, we have that

$$\kappa x(t) = 0 = \int_{\eta(t)}^{t} x(s)\, ds = \int_{c_2}^{a} x(s)\, ds + \int_{a}^{t} x(s)\, ds = \int_{c_2}^{a} x(s)\, ds.$$

Again, as $c_2 \in [r_2, a]$ is arbitrary, we have that $x(s) = 0$ for every $s \in [r_2, a]$. But from (2.18) this implies that $p \leq r_2 < a$, contradicting the above result that $p = a$. With this we have proved that $q = b$.

(g) As noted above, using (1.2), for each $k \geq 0$ the set $J_k(b)$ is a compact interval whose right-hand endpoint is b; and furthermore, these intervals are nested so that $J_{k-1}(b) \subseteq J_k(b)$ for $k \geq 1$. Thus either $J_*(b) = [a, b]$ for some $a \leq b$ or $J_*(b) = (-\infty, b]$.

Suppose that $J_*(b) = [a, b]$. Taking any $t \in (a, b]$ we have that $t \in J_k(b)$ for some $k \geq 0$ and thus $\eta(t) \in J_{k+1}(b) \subseteq J_*(b)$. Thus $\eta((a, b]) \subseteq [a, b]$, which implies that $\eta([a, b]) \subseteq [a, b]$; that is, $J_*(b)$ is a Volterra interval. Thus by (e) above $J_*(b)$ is contained in a maximal Volterra interval and so b is contained in a maximal Volterra interval, as claimed.

Now suppose that $b \in [p, q]$ where $[p, q]$ is a maximal Volterra interval. As $\eta([p, q]) \subseteq [p, q]$, it follows that $J_k(b) \subseteq [p, q]$ for every $k \geq 0$, and so $J_*(b) \subseteq [p, q]$. Thus $J_*(b) = [a, b]$ for some $a \leq b$, again as claimed.

Finally, we need to show there exists some $b \in \mathbf{R}$ such that $J_*(b) = (-\infty, b]$, equivalently, that b is not contained in any maximal Volterra interval. This is trivial if there are no maximal Volterra intervals, so suppose there exists a maximal Volterra interval $[a_1, b_1]$. Necessarily $\inf_{t \in [b_1, b_1 + \varepsilon]} \eta(t) < a_1$ holds for any $\varepsilon > 0$, otherwise $[a_1, b_1 + \varepsilon]$ would be a Volterra interval, contradicting the maximality of $[a_1, b_1]$. Fix any $b > b_1$ so that $\eta(b) < a_1$. We claim that b is not contained in any maximal Volterra interval. Suppose to the contrary that b is contained in some maximal Volterra interval $[a_2, b_2]$; we seek a contradiction. Certainly $b_1 < b \leq b_2$. Also $\eta([a_2, b_2]) \subseteq [a_2, b_2]$ and so $a_2 \leq \eta(b) < a_1$. Thus $[a_1, b_1] \subseteq [a_2, b_2]$ with $[a_1, b_1] \neq [a_2, b_2]$ and this contradicts (d) above, completing the proof. ■

3 Is the Spectral Radius Positive? The Proof of Theorem A

In this section we prove Theorem A. Again, L is the linear operator given by (1.4) acting on the space X given by (1.3), and $r(L)$ denotes the spectral radius of L.

We first need the following lemmas.

Lemma 3.1. *Let $\eta : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy* (1.2) *for every $t \in \mathbf{R}$. Suppose there exist points $t_k \in \mathbf{R}$ for $0 \leq k \leq n$, for some $n \geq 1$, such that $t_k \in (\eta(t_{k-1}), t_{k-1})$ for $1 \leq k \leq n$, and such that also $t_n \equiv t_0 \pmod{2\pi}$. Then $r(L) > 0$.*

Proof. It is enough to show the existence of some $x \in X^+ \setminus \{0\}$ and some $\alpha > 0$ such that

$$(Lx)(t) \geq \alpha x(t) \qquad \text{for every } t \in \mathbf{R}, \tag{3.1}$$

that is, $Lx \geq \alpha x$ where $\geq$ denotes the partial ordering in X given by the cone X^+. For then we have $L^m x \geq \alpha^m x$ for every $m \geq 1$, and therefore $\|L^m\| \geq \alpha^m$ for the operator norm of L^m. Taking m^{th} roots and letting $m \to \infty$ gives $r(L) \geq \alpha > 0$, as desired. (We use here the fact that if $y \geq z \geq 0$ for $y, z \in X$, then $\|y\| \geq \|z\|$, which is a property of the cone X^+. The reader might compare this fact to [3, Lemma 2.2], which is a more general result valid for other cones.)

To establish (3.1) for some x and α as above, fix $\delta > 0$ and define a set $S \subseteq \mathbf{R}$ and a function $x \in X^+ \setminus \{0\}$ by

$$S = \{t_k + 2\pi j \,|\, 0 \leq k \leq n-1 \text{ and } j \in \mathbf{Z}\}, \qquad x(t) = \max\{\delta - \operatorname{dist}(t, S),\, 0\},$$

for $t \in \mathbf{R}$. Here $\operatorname{dist}(t, S)$ denotes the distance from a point t to the set S. It is clear that S is closed, and in fact is a discrete set, and is 2π-periodic (meaning $t \in S$ if and only if $t + 2\pi \in S$), and that $x \in X^+ \setminus \{0\}$. To choose δ, first let

$$\varepsilon = \min_{1 \leq k \leq n} \min\{t_k - \eta(t_{k-1}),\, t_{k-1} - t_k\}$$

which is a positive quantity, namely the minimum of the distances of the points t_k to the boundary of the intervals $(\eta(t_{k-1}), t_{k-1})$. Then from the continuity of η, there exists $\delta > 0$ such that if $|t - t_{k-1}| \leq \delta$ for some k with $1 \leq k \leq n$, then $|\eta(t) - \eta(t_{k-1})| \leq \varepsilon$. Additionally, we may assume that $\delta < \varepsilon$, and denoting $t_{k,j} = t_k + 2\pi j$, we observe that

$$t_{k,j} \in [\eta(t_{k-1,j}) + \varepsilon,\; t_{k-1,j} - \delta) \tag{3.2}$$

for $1 \leq k \leq n$ and $j \in \mathbf{Z}$.

Now suppose that $\operatorname{dist}(t, S) \leq \delta$. Then we have that $|t - t_{k-1,j}| \leq \delta$ for some k and j with $1 \leq k \leq n$ and $j \in \mathbf{Z}$. We also have that $|\eta(t) - \eta(t_{k-1,j})| \leq \varepsilon$, and therefore, using (3.2),

$$\eta(t) \leq \eta(t_{k-1,j}) + \varepsilon \leq t_{k,j} < t_{k-1,j} - \delta \leq t.$$

Thus

$$(Lx)(t) = \int_{\eta(t)}^{t} x(s)\, ds \geq \int_{\eta(t_{k-1,j})+\varepsilon}^{t_{k-1,j}-\delta} x(s)\, ds > 0,$$

where the positivity of the final integral above holds because $x(t_{k,j}) = \delta > 0$, with (3.2) holding. With this, we have established that $(Lx)(t) > 0$ for every $t \in Q$, where $Q = \{t \in \mathbf{R} |\, \operatorname{dist}(t, S) \leq \delta\}$.

As the set Q is closed and 2π-periodic, and the function Lx is 2π-periodic, it follows that there exists $\beta > 0$ such that $(Lx)(t) \geq \beta$ for every $t \in Q$.

We now establish (3.1) for every $t \in \mathbf{R}$, taking $\alpha = \beta\|x\|^{-1}$. We assume without loss that t is such that $x(t) > 0$, and thus $t \in Q$. Then

$$(Lx)(t) \geq \beta = \alpha\|x\| \geq \alpha x(t),$$

as desired. ■

For the next result recall the definition (2.7) of the set $E(S)$.

Lemma 3.2. *Let $\eta : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy $\eta(t) \leq t$ for every $t \in \mathbf{R}$. Suppose that J is a compact interval of positive length and that $p \in \mathrm{int}(E(J))$. Suppose also that $\eta(p) < p$. Then there exists $q \in \mathrm{int}(J)$ such that $\eta(q) < q$ and $p \in (\eta(q), q)$.*

Proof. Denote $J = [a, b]$ where $a < b$. Note that $p \in \mathrm{int}(E(J)) \subseteq (-\infty, b)$, and so $p < b$. First suppose that $p \geq a$. Then upon letting $q = p + \varepsilon$ for sufficiently small $\varepsilon > 0$, we see that $q \in (a, b)$, and that $\eta(q) < q$ and $p \in (\eta(q), q)$, since $\eta(p) < p$. Thus q is as desired.

Now suppose that $p < a$. Let $s \in [a, b]$ be such that $\eta(s) = \inf_{t \in [a,b]} \eta(t)$, and so $\eta(s)$ is the left-hand endpoint of the interval $E(J)$. Therefore $\eta(s) < p < a \leq s$, in particular because $p \in \mathrm{int}(E(J))$. Choose $q \in (a, b)$ and sufficiently near s; again we see that $\eta(q) < q$ and $p \in (\eta(q), q)$, as desired. ■

Proof of Theorem A. By Corollary 2.2, if (2.1) is false then $r(L) = 0$. Therefore suppose that (2.1) holds; we must prove that $r(L) > 0$. Recall the definitions (2.8) of $J_k(b)$ and $J_*(b)$ in Lemma 2.3; by part (g) of that lemma there exists $b \in \mathbf{R}$ such that $J_*(b) = (-\infty, b]$. Note that $\eta(b) < b$, for if $\eta(b) = b$ then we would have $J_*(b) = \{b\}$. Then $b - 2\pi \in J_*(b)$, hence $b - 2\pi \in \mathrm{int}(J_n(b))$ for some $n \geq 1$. Denote $t_0 = b$ and $t_n = b - 2\pi$. We claim there exist $t_k \in \mathbf{R}$, for $1 \leq k \leq n - 1$, such that

$$\begin{aligned} &\eta(t_k) < t_k \quad \text{and} \quad t_k \in \mathrm{int}(J_k(t_0)) \quad \text{for} \quad 1 \leq k \leq n, \\ &t_{k+1} \in (\eta(t_k), t_k) \quad \text{for} \quad 0 \leq k \leq n - 1. \end{aligned} \tag{3.3}$$

To prove (3.3) we begin by proving the first line but only for $k = n$. We have that $\eta(t_n) = \eta(t_0) - 2\pi < t_0 - 2\pi = t_n$, and certainly $t_n = t_0 - 2\pi \in \mathrm{int}(J_n(t_0))$ from above.

We next prove both lines of (3.3) for k in the range $1 \leq k \leq n - 1$, by induction, but in descending order beginning with $k = n - 1$. In making the inductive step only the first line of (3.3)

will be required, for $k+1$, in order to obtain both lines of (3.3) for k. Thus assume

$$\eta(t_{k+1}) < t_{k+1} \quad \text{and} \quad t_{k+1} \in \text{int}(J_{k+1}(t_0))$$

for some k with $1 \leq k \leq n-1$. As $J_{k+1}(t_0) = E(J_k(t_0))$, we have by Lemma 3.2 that there exists t_k satisfying the inequality and inclusions in both lines of (3.3); in particular $J = J_k(t_0)$, with $p = t_{k+1}$ and $q = t_k$, in the statement of the lemma. Note that the required condition $\text{len}(J_k(t_0)) > 0$ in Lemma 3.2 holds as $[\eta(t_0), t_0] = J_1(t_0) \subseteq J_k(t_0)$ and $\eta(t_0) < t_0$. This establishes (3.3) for all indicated values of k except the second line for $k = 0$, that is, $t_1 \in (\eta(t_0), t_0)$. However, this fact is already established by the first line of (3.3) for $k = 1$, namely $t_1 \in \text{int}(J_1(t_0)) = (\eta(t_0), t_0)$. Thus (3.3) is proved.

With the points t_k so constructed, we conclude immediately from Lemma 3.1 that $r(L) > 0$, as desired. ■

4 Intricate Structure of $\mathcal{A}$ and $\mathcal{N}$; the Proof of Theorem B

Throughout this section we will assume as standing hypotheses that η is continuous and satisfies (1.2) and (2.1); in particular, $r(L) > 0$ holds. Note that condition (2.1) is implicitly assumed in the statement of Theorem B, in that the existence of the eigenvalue $\kappa \neq 0$ implies that $r(L) > 0$, which is equivalent to (2.1).

We give several lemmas, followed by the proof of Theorem B.

Lemma 4.1. *Let $[a, b]$ be a maximal Volterra interval. Then given $\varepsilon > 0$ and $c \in \mathbf{R}$, there exists $n \geq 1$ such that*

$$\inf_{t \in [a-\varepsilon, a]} \eta^n(t) < c$$

holds.

Proof. Suppose the result is false. Then for some $\varepsilon > 0$ the quantity $r \leq a - \varepsilon$ defined by

$$r = \inf_{k \geq 1} \left(\inf_{t \in [a-\varepsilon, a]} \eta^k(t) \right)$$

is finite, that is, $r \neq -\infty$. As $\eta(a) = a$, it follows that if $s \in (r, a]$ then there exists $t \in [a - \varepsilon, a]$ and $k \geq 1$ such that $\eta^k(t) = s$. But then $r \leq \eta^{k+1}(t) = \eta(s) \leq s \leq a$ and so $\eta(s) \in [r, a]$. Thus $\eta((r, a]) \subseteq [r, a]$ which implies that $\eta([r, a]) \subseteq [r, a]$, and so $[r, a]$ is a Volterra interval. Therefore

by part (e) of Lemma 2.3 we have that $[r, a] \subseteq [p, q]$ where $[p, q]$ is a maximal Volterra interval. However $a \in [p, q] \cap [a, b] \neq \emptyset$ and $p \leq r < a$, which contradicts part (d) of Lemma 2.3. ■

As was shown in [8], if x satisfies an appropriate delay-differential equation then the sets $\mathcal{A}$ and $\mathcal{N}$ defined in (1.5) enjoy certain mapping properties. In the case of equation (1.7), which arises from (1.1) with $\kappa \neq 0$, one has the following; see [8, Corollary 3.5]. Assuming that $\eta : \mathbf{R} \to \mathbf{R}$ is analytic, and that $x : \mathbf{R} \to \mathbf{C}$ satisfies (1.7) on $\mathbf{R}$, recall the set $\mathcal{M}_\eta$ in (2.14), namely the set of points at which η has a local maximum or minimum. For simplicity denote $\mathcal{M} = \mathcal{M}_\eta$. Then

$$\eta(\mathcal{A} \setminus \mathcal{M}) \subseteq \mathcal{A}, \qquad \eta(\mathcal{N}) \subseteq \mathcal{N}. \tag{4.1}$$

Generally, if $\zeta : \mathbf{R} \to \mathbf{R}$ is analytic, we shall say that ζ has an extremum at $t \in \mathbf{R}$ if ζ has either a local maximum or a local minimum at t; in particular, $\mathcal{M}$ is the set of extrema of η.

Lemma 4.2. *Assume that η is analytic, and suppose for some $n \geq 1$ that the n^{th} iterate η^n does not have an extremum at t. Then η does not have an extremum at any of the points $\eta^k(t)$, for $0 \leq k \leq n-1$; that is,*

$$\eta^k(t) \notin \mathcal{M} \qquad \text{for } 0 \leq k \leq n-1 \tag{4.2}$$

holds.

Proof. First note that η^n is strictly monotone and thus one-to-one in a neighborhood of t. Suppose however that η has an extremum at $\eta^k(t)$ for some k with $0 \leq k \leq n-1$, and let m be the minimum such k. Then η is strictly monotone in a neighborhood of $\eta^j(t)$ for $0 \leq j \leq m-1$; thus η^m is strictly monotone in a neighborhood of t and so maps some neighborhood of t one-to-one onto a neighborhood of $\eta^m(t)$. On the other hand, η is not one-to-one on any neighborhood of $\eta^m(t)$, and so neither is η^{n-m}. It follows that the composition $\eta^n = \eta^{n-m} \circ \eta^m$ is not one-to-one on any neighborhood of t, a contradiction. ■

Lemma 4.3. *Assume that η is analytic. Also assume that $Lx = \kappa x$, that is, equation* (1.1) *holds, for some $x \in X \setminus \{0\}$ and $\kappa \neq 0$. Suppose that $c \in \operatorname{int}(\eta^n(I))$ for some interval $I \subseteq \mathbf{R}$ and some $n \geq 1$. Then if $c \in \mathcal{A}$ there exists $t \in I \cap \mathcal{A}$ such that $\eta^n(t) = c$; and if $c \in \mathcal{N}$ there exists $t \in I \cap \mathcal{N}$ such that $\eta^n(t) = c$.*

Proof. There exists $t \in I$ such that $\eta^n(t) = c$. If $c \in \mathcal{A}$ then also $t \in \mathcal{A}$ for any such t by (the contrapositive of) the second inclusion in (4.1).

Now suppose that $c \in \mathcal{N}$; in this case t must be chosen more carefully. There exist $p, q \in I$ such that $\eta^n(p) < c < \eta^n(q)$; let us assume that $p < q$, the argument when $p > q$ being similar. Now let

$$t = \sup\{s \in [p, q] \mid \eta^n(r) \leq c \text{ for every } r \in [p, s]\}.$$

Then $t \in (p, q) \subseteq I$ and $\eta^n(t) = c$. Further, $\eta^n - c$ changes sign at t and so η^n does not have an extremum at t; thus by Lemma 4.2 we have that (4.2) holds. We conclude from (the contrapositive of) the first inclusion in (4.1) that $\eta^k(t) \in \mathcal{N}$ for $0 \leq k \leq n-1$; in particular, $t \in \mathcal{N}$, as desired. ■

Lemma 4.4. *Let η, $[a, b]$, x, and κ be as in the statement of Theorem B, with $\mathcal{N}$ the set* (1.5) *associated to the solution x. Then given $\varepsilon > 0$ there exists $n \geq 1$ such that the following holds. Given any $c \in [a-2\pi(m+1)-\varepsilon, a-2\pi m] \cap \mathcal{N}$ for some integer m, there exists $t \in [a-2\pi m-\varepsilon, a-2\pi m] \cap \mathcal{N}$ such that $\eta^n(t) = c$.*

Proof. In proving the lemma now we may assume without loss that $m = 0$, due to the periodicity condition in (1.2); thus we have $c \in [a - 2\pi - \varepsilon, a] \cap \mathcal{N}$.

As $[a, b]$ is a maximal Volterra interval, by Lemma 4.1 there exists $n \geq 1$ such that

$$\inf_{t \in [a-\varepsilon, a]} \eta^n(t) < a - 2\pi - \varepsilon. \tag{4.3}$$

If $c = a$ then we may take $t = a$, as $\eta(a) = a \in \mathcal{N}$ by part (a) of Lemma 2.3. (Note that in this case we are assuming that $c \in \mathcal{N}$ and thus $a \in \mathcal{N}$.) Suppose therefore that $c \neq a$ and let $I = [a - \varepsilon, a]$. Then by (4.3) we have that $c \in [a - 2\pi - \varepsilon, a) \subseteq \operatorname{int}(\eta^n(I))$, and so from Lemma 4.3 there exists $t \in I \cap \mathcal{N}$ such that $\eta^n(t) = c$, as desired. ■

Proof of Theorem B. We prove the three parts of this theorem in sequence.

(a) The fact that $\kappa \neq 0$ implies that $r(L) > 0$, and so (2.1) holds and thus Lemma 2.3 applies. By part (f) of that result we have that $x(t) = 0$ for every $t \in [a, b]$, and so $(a, b) \subseteq \mathcal{A}$. Further by part (f), the solution x does not vanish identically on $[a - \varepsilon, a]$ or on $[b, b + \varepsilon]$, for any $\varepsilon > 0$; it follows that x is not analytic at a or b, that is, $a, b \in \mathcal{N}$. This completes the proof of part (a) of the theorem.

(b) We first show that the set $[a - \varepsilon, a] \cap \mathcal{N}$ is uncountable for every $\varepsilon > 0$; without loss we may assume that $\varepsilon < 2\pi$. With ε fixed, let $n \geq 1$ be as in Lemma 4.4. Also denote

$$I_m = [a - 2\pi m - \varepsilon, a - 2\pi m]$$

for any integer m, and note that these intervals are pairwise disjoint. Now let $\rho_j \in \{0,1\}$ for $j \geq 1$ be any sequence of zeros and ones, and define integers m_j for $j \geq 0$ by

$$m_0 = 0, \qquad m_j = \sum_{i=1}^{j} \rho_i. \tag{4.4}$$

We shall construct a point $t_* \in [a-\varepsilon, a] \cap \mathcal{N}$, such that

$$\eta^{nj}(t_*) \in I_{m_j} \qquad \text{for } j \geq 0. \tag{4.5}$$

One easily sees that different sequences of zeros and ones yield different points; indeed, if $\{\rho_j\}_{j=1}^{\infty}$ and $\{\widetilde{\rho}_j\}_{j=1}^{\infty}$ are such sequences, say with $\rho_{j_0} \neq \widetilde{\rho}_{j_0}$ for some $j_0 \geq 1$, and if t_* and $\widetilde{t}_*$ are the corresponding points in $[a-\varepsilon, a] \cap \mathcal{N}$, then by (4.4) either $m_{j_0-1} \neq \widetilde{m}_{j_0-1}$ or $m_{j_0} \neq \widetilde{m}_{j_0}$ must hold (with the obvious notation for $\widetilde{m}_j$). Thus either $\eta^{n(j_0-1)}(t_*) \neq \eta^{n(j_0-1)}(\widetilde{t}_*)$ or else $\eta^{nj_0}(t_*) \neq \eta^{nj_0}(\widetilde{t}_*)$ by (4.5); in any case, $t_* \neq \widetilde{t}_*$. As there are uncountably many sequences $\{\rho_j\}_{j=1}^{\infty}$, it follows that the set $[a-\varepsilon, a] \cap \mathcal{N}$ is uncountable.

We shall construct t_* by a sequence of approximations, each using only a finite number of the ρ_j. For every $k \geq 0$ we construct points $t_{k,j}$ such that

$$t_{k,j} \in I_{m_j} \cap \mathcal{N} \quad \text{for } 0 \leq j \leq k,$$

$$t_{k,k} = a - 2\pi m_k, \qquad t_{k,j} = \eta^n(t_{k,j-1}) \quad \text{for } 1 \leq j \leq k.$$

Indeed, this is done easily using Lemma 4.4, inducting on j in descending order beginning with $j = k$. First, from above $a \in \mathcal{N}$, and thus $t_{k,k} = a - 2\pi m_k \in I_{m_k} \cap \mathcal{N}$, to begin the induction. Assuming that $t_{k,j} \in I_{m_j} \cap \mathcal{N}$ holds for some j with $1 \leq j \leq k$, and noting that either $m_j = m_{j-1}+1$ or $m_j = m_{j-1}$, we have that

$$\begin{aligned} t_{k,j} \in I_{m_j} \cap \mathcal{N} &\subseteq (I_{m_{j-1}+1} \cup I_{m_{j-1}}) \cap \mathcal{N} \\ &\subseteq [a - 2\pi(m_{j-1}+1) - \varepsilon, a - 2\pi m_{j-1}] \cap \mathcal{N}. \end{aligned}$$

It follows by Lemma 4.4 that there exists $t_{k,j-1} \in I_{m_{j-1}} \cap \mathcal{N}$ with $\eta^n(t_{k,j-1}) = t_{k,j}$, as desired; and this completes the induction. Now with $t_{k,0} \in [a-\varepsilon, a] \cap \mathcal{N}$ constructed, there exists a subsequence $k' \to \infty$ such that $t_{k',0}$ converges, and we set $t_* = \lim_{k' \to \infty} t_{k',0}$. Then $t_* \in \mathcal{N}$ as $\mathcal{N}$ is closed. Also,

$$\eta^{nj}(t_*) = \lim_{k' \to \infty} \eta^{nj}(t_{k',0}) = \lim_{k' \to \infty} t_{k',j} \in I_{m_j}$$

for every $j \geq 0$, since $\eta^{nj}(t_{k',0}) = t_{k',j} \in I_{m_j}$ for $k' \geq j$. With this, the proof that $[a-\varepsilon, a] \cap \mathcal{N}$ is uncountable is complete.

We next show that $[a-\varepsilon, a]$ contains infinitely many connected components of $\mathcal{A}$, for every $\varepsilon > 0$. From above $[a-\varepsilon, a) \cap \mathcal{N} \neq \emptyset$ for every $\varepsilon > 0$, so it is enough to show that $[a-\varepsilon, a) \cap \mathcal{A} \neq \emptyset$ for every $\varepsilon > 0$. By Lemma 4.1 let $n \geq 1$ be such that $\inf_{t \in [a-\varepsilon, a]} \eta^n(t) < a - 2\pi$; thus there exists $t_0 \in [a-\varepsilon, a)$ such that $\eta^n(t_0) \in (a-2\pi, b-2\pi)$. (Note that $t_0 \neq a$ as $\eta(a) = a$.) But $[a-2\pi, b-2\pi]$ is a maximal Volterra interval, and so $(a-2\pi, b-2\pi) \subseteq \mathcal{A}$, and thus $\eta^n(t_0) \in \mathcal{A}$. It follows that $t_0 \in \mathcal{A}$ by (the contrapositive of) the second inclusion in (4.1), and we conclude that $[a-\varepsilon, a) \cap \mathcal{A} \neq \emptyset$ holds, as desired.

(c) To complete the proof of the theorem we consider the interval $[b, b+\varepsilon]$ for small $\varepsilon > 0$. We recall that $\eta(b) = a$ and we note that $\dot{\eta}(t) < 0$ for every $t > b$ sufficiently near b, as the interval $[a, b]$ is an RM-Volterra interval. Therefore there exist positive quantities ε_1 and ε_2 such that $\eta : [b, b+\varepsilon_1] \to [a-\varepsilon_2, a]$ is a surjective homeomorphism. Furthermore, we may choose ε_1 so that η is monotone in a neighborhood of $[b, b+\varepsilon_1]$, and therefore $[b, b+\varepsilon_1] \cap \mathcal{M} = \emptyset$. It follows from (4.1) that $\eta([b, b+\varepsilon_1] \cap \mathcal{A}) = [a-\varepsilon_2, a] \cap \mathcal{A}$ and $\eta([b, b+\varepsilon_1] \cap \mathcal{N}) = [a-\varepsilon_2, a] \cap \mathcal{N}$, and thus for any positive $\varepsilon \leq \varepsilon_1$ part (c) of Theorem B follows from part (b) above. ■

5 Cantor Set Structure of $\mathcal{N}$; the Proofs of Theorems C and D

We require several lemmas before first proving Theorem C.

Lemma 5.1. *If $n \geq 1$ we have that*

$$|\sin n\theta| \leq n|\sin\theta| \qquad \text{for all } \theta \in \mathbf{R}, \tag{5.1}$$

and thus

$$\sin n\theta \leq n \sin\theta \qquad \text{for } 0 \leq \theta \leq \frac{\pi}{3(n-1)} \tag{5.2}$$

holds for $n \geq 2$.

Proof. We prove (5.1) for $n \geq 1$ by inducting on n. The result is trivial for $n = 1$, so assume that (5.1) holds for some $n \geq 1$. Then

$$|\sin((n+1)\theta)| = |\sin n\theta \cos\theta + \cos n\theta \sin\theta| \leq |\sin n\theta| + |\sin\theta| \leq (n+1)|\sin\theta|,$$

as desired. This establishes (5.1), and (5.2) follows immediately. ■

Lemma 5.2. *Assume that $U \subseteq \mathbf{C}$ is a neighborhood of a point $v \in \mathbf{R}$, and that $\xi : U \to \mathbf{C}$ is real analytic (that is, analytic, with $\xi(t) \in \mathbf{R}$ for every $t \in U \cap \mathbf{R}$). Further assume that*

$$\xi(t) = t - K(t-v)^n + O(|t-v|^{n+1}) \tag{5.3}$$

in a neighborhood of the point v, for some $K > 0$ and integer $n \geq 2$. Then if $\varepsilon > 0$ is sufficiently small, the region $D \subseteq \mathbf{C}$ defined as

$$D = \{v + re^{i\theta} \in \mathbf{C} \,|\, 0 < r < \varepsilon \text{ and } |\theta| < \alpha\}, \qquad \alpha = \frac{\pi}{3(n-1)}, \tag{5.4}$$

satisfies $\xi(\overline{D} \setminus \{v\}) \subseteq D$.

Proof. Henceforth, by taking ε sufficiently small, we assume the bound

$$|\xi(t) - t + K(t-v)^n| \leq M|t-v|^{n+1}$$

on the remainder term in (5.3) for every $t \in \overline{D}$, for some M. We also assume that ε is small enough that the inequalities

$$\begin{aligned} &K^2\varepsilon^{n-1} + 2M\varepsilon < K, \qquad && K\varepsilon^{n-1} + M\varepsilon^n < \cos\alpha, \\ &Kn\varepsilon^{n-1} \leq \cos\alpha, && 2M\varepsilon \leq K\sin(\pi/3), \end{aligned} \tag{5.5}$$

all hold. Then taking $t = v + re^{i\theta} \in \overline{D} \setminus \{v\}$, and so $0 < r \leq \varepsilon$ and $|\theta| \leq \alpha$, we have that

$$\begin{aligned} |\xi(t) - v| &\leq |re^{i\theta} - Kr^n e^{in\theta}| + Mr^{n+1} = r\Big(|1 - Kr^{n-1}e^{i(n-1)\theta}| + Mr^n\Big) \\ &= r\bigg(\Big(1 - 2Kr^{n-1}\cos((n-1)\theta) + K^2r^{2n-2}\Big)^{1/2} + Mr^n\bigg) \\ &\leq r\bigg(\Big(1 - Kr^{n-1}\cos((n-1)\theta) + \frac{K^2r^{2n-2}}{2}\Big) + Mr^n\bigg) \\ &\leq r\bigg(\Big(1 - \frac{Kr^{n-1}}{2} + \frac{K^2r^{2n-2}}{2}\Big) + Mr^n\bigg) < r \leq \varepsilon, \end{aligned} \tag{5.6}$$

provided that $K^2r^{2n-2} + 2Mr^n < Kr^{n-1}$, equivalently, $K^2r^{n-1} + 2Mr < K$, which holds due to the first inequality in (5.5). (Note that the second inequality in (5.6) follows from the fact that $(1+\sigma)^{1/2} \leq 1 + \frac{1}{2}\sigma$ for any $\sigma \geq -1$.)

We next note that

$$\mathrm{Re}(\xi(t)-v) \geq r\cos\theta - Kr^n\cos n\theta - Mr^{n+1} \geq r\cos\alpha - Kr^n - Mr^{n+1} > 0, \tag{5.7}$$

where the final inequality in (5.7) follows from the second inequality in (5.5). In particular, $\xi(t) \neq v$ and so

$$0 < |\xi(t)-v| < \varepsilon \tag{5.8}$$

from (5.6). Now write $\xi(t)-v = |\xi(t)-v|e^{i\varphi}$, where we may assume $|\varphi| < \frac{\pi}{2}$ by (5.7). For the remainder of the proof we assume without loss that $0 \leq \theta \leq \alpha$; the case in which $-\alpha \leq \theta \leq 0$ is similar and is omitted. Then

$$|\,\mathrm{Im}(\xi(t)-v)| \leq |r\sin\theta - Kr^n\sin n\theta| + Mr^{n+1} = r\sin\theta - Kr^n\sin n\theta + Mr^{n+1} \tag{5.9}$$

provided $Kr^{n-1}\sin n\theta \leq \sin\theta$; but this holds by Lemma 5.1 and from the third inequality in (5.5) since

$$Kr^{n-1}\sin n\theta \leq K\varepsilon^{n-1}\sin n\theta \leq Kn\varepsilon^{n-1}\sin\theta \leq \cos\alpha\sin\theta \leq \sin\theta.$$

Therefore, from (5.7) and (5.9),

$$|\tan\varphi| = \frac{|\sin\varphi|}{\cos\varphi} = \frac{|\,\mathrm{Im}(\xi(t)-v)|}{\mathrm{Re}(\xi(t)-v)} \leq \frac{\sin\theta - Kr^{n-1}\sin n\theta + Mr^n}{\cos\theta - Kr^{n-1}\cos n\theta - Mr^n}. \tag{5.10}$$

We claim that the final term in (5.10) is strictly less than $\tan\alpha$, that is,

$$\frac{\sin\theta - Kr^{n-1}\sin n\theta + Mr^n}{\cos\theta - Kr^{n-1}\cos n\theta - Mr^n} < \frac{\sin\alpha}{\cos\alpha}. \tag{5.11}$$

Upon cross-multiplying, we see that proving (5.11) is equivalent to proving that

$$g(\theta) = \sin\alpha\Big(\cos\theta - Kr^{n-1}\cos n\theta - Mr^n\Big) - \cos\alpha\Big(\sin\theta - Kr^{n-1}\sin n\theta + Mr^n\Big) > 0, \tag{5.12}$$

where the above formula serves as the definition of $g(\theta)$, for a given fixed r. We have that

$$\begin{aligned} g'(\theta) &= \sin\alpha\Big(-\sin\theta + Knr^{n-1}\sin n\theta\Big) - \cos\alpha\Big(\cos\theta - Knr^{n-1}\cos n\theta\Big) \\ &= -\cos(\alpha-\theta) + Knr^{n-1}\cos(\alpha - n\theta) \leq -\cos\alpha + Kn\varepsilon^{n-1} \leq 0 \end{aligned} \tag{5.13}$$

for every θ satisfying $0 \leq \theta \leq \alpha$, again from the third inequality in (5.5). Also,

$$g(\alpha) = r^{n-1}\Big(K\sin((n-1)\alpha) - Mr(\sin\alpha + \cos\alpha)\Big) > r^{n-1}\Big(K\sin(\pi/3) - 2M\varepsilon\Big) \geq 0 \tag{5.14}$$

by the fourth inequality in (5.5). Thus from (5.13) and (5.14) it follows that (5.12) holds. Therefore, from (5.10) and (5.11) we have that $|\varphi| < \alpha$, and with (5.8) we have that $\xi(t) \in D$, as desired. ■

Let us remark that with somewhat more effort, one can prove the following sharper version of Lemma 5.2. We omit the proof as it is not needed here.

Lemma 5.2′. *Let the assumptions and notation be as in Lemma 5.2 and let* $\beta = \pi/(2(n-1))$. *Then there exists a continuous even function* $\rho : [-\beta, \beta] \to \mathbf{R}$ *which is decreasing on* $[0, \beta]$, *positive on* $(-\beta, \beta)$, *and with* $\rho(\pm\beta) = 0$, *such that if*

$$E = \{v + re^{i\theta} \in \mathbf{C} \mid 0 < r < \rho(\theta) \text{ and } |\theta| < \beta\},$$

then $\xi(\overline{E} \setminus \{v\}) \subseteq E$. *If* $z = v + re^{i\theta} \in \overline{E} \setminus \{v\}$ *and we denote* $\xi(z) = v + \widetilde{r}e^{i\widetilde{\theta}}$, *where* $|\theta|, |\widetilde{\theta}| < \beta$, *then* $0 < \widetilde{r} < r$; *and also* $0 < \widetilde{\theta} < \theta$ *if* $\theta > 0$, *and* $\theta < \widetilde{\theta} < 0$ *if* $\theta < 0$, *and* $\widetilde{\theta} = 0$ *if* $\theta = 0$.

Lemma 5.3. *Assume for some compact interval* $[v, w]$ *that* $\xi : [v, w] \to [v, w]$ *is analytic. Further assume that*

$$v < \xi(t) < t \qquad \text{for every } t \in (v, w). \tag{5.15}$$

Let $\kappa, \gamma \in \mathbf{C}$ *with* $\kappa \neq 0$, *and consider the equation*

$$\kappa y(t) = -\int_{\xi(t)}^{t} y(s)\, ds + \gamma. \tag{5.16}$$

Then equation (5.16) *has a unique solution* $y \in C([v, w], \mathbf{C})$. *Further, this solution is analytic in the interior* (v, w).

Proof. The proof that there is at most one solution in $C([v, w], \mathbf{C})$ is essentially the same argument as given in the proof of Proposition 2.1. We do not repeat it.

Let us first construct a continuous solution of (5.16) on the interval $[v, v + \varepsilon]$, for sufficiently small ε, and which is analytic in $(v, v + \varepsilon]$. To this end we require a set $D \subseteq \mathbf{C}$ in the complex plane such that

(a) $D \subseteq \mathbf{C}$ is open, bounded, and convex, with $\operatorname{diam}(D) < |\kappa|$;

(b) $v \in \overline{D}$ and $(v, v + \varepsilon) \subseteq D$; and

(c) ξ is analytic in a neighborhood of $\overline{D}$, and $\xi(\overline{D} \setminus \{v\}) \subseteq D$.

(Here diam(S) denotes the diameter of a set S.) Note that $0 \le \dot{\xi}(v) \le 1$ by (5.15). If $\dot{\xi}(v) < 1$ then we take $D = \{t \in \mathbf{C} \,|\, |t - v| < \varepsilon\}$ for sufficiently small ε, and observe that conditions (a), (b), and (c) above hold as ξ is a contraction in $\overline{D}$. If on the other hand $\dot{\xi}(v) = 1$, then by (5.15) the Taylor series of ξ about v must have the form (5.3) for some $K > 0$ and $n \ge 2$. In this case we take D to be the set (5.4), as given by Lemma 5.2, for sufficiently small ε. Again, one sees that conditions (a), (b), and (c) hold.

With D so chosen, define a Banach space

$$Z = \{y : \overline{D} \to \mathbf{C} \,|\, y \text{ is continuous in } \overline{D}, \text{ and analytic in } D\},$$

taking the supremum norm $\|y\| = \sup_{t \in \overline{D}} |y(t)|$. Then by the properties of D, and in particular (c) above, we may define a bounded linear operator $\Lambda : Z \to Z$ by

$$(\Lambda y)(t) = \frac{1}{\kappa} \int_{\xi(t)}^{t} y(s)\, ds$$

for $t \in \overline{D}$. Then equation (5.16) can be written as

$$y = -\Lambda y + \frac{\gamma}{\kappa}. \tag{5.17}$$

One sees from property (a) that $\|\Lambda\| \le |\kappa|^{-1} \operatorname{diam}(D) < 1$ for the operator norm, and it follows that equation (5.17), and thus (5.16), has a unique solution in Z. This solution y is certainly analytic in D, by the definition of Z. In fact y is analytic in $\overline{D} \setminus \{v\}$, and thus analytic in a neighborhood of this set. Indeed, taking $t_0 \in \overline{D} \setminus \{v\}$, one observes that y is analytic in a neighborhood of t_0 as follows. From equation (5.16) one has that

$$\dot{y}(t) = \frac{1}{\kappa}\Big(- y(t) + y(\xi(t))\dot{\xi}(t) \Big) \tag{5.18}$$

for t near t_0; and ξ is assumed analytic there, with $\xi(t_0) \in D$. Thus $t \to y(\xi(t))\dot{\xi}(t)$ is analytic in a neighborhood of t_0, and so y is also analytic in such a neighborhood by (5.18). In particular, this establishes analyticity of the solution $y \in Z$ in a neighborhood of the half-open interval $(v, v + \varepsilon]$.

To complete the proof, we must extend the solution y from the interval $[v, v + \varepsilon]$ to all of $[v, w]$, and show that this extension is analytic in (v, w). Let

$$r = \sup\{t \in [v + \varepsilon, w] \,|\, \text{the solution } y \in Z$$

$$\text{of (5.17) has an analytic extension to } (v, t)\}.$$

Then there exists an analytic extension z of y to (v, r), and further $z : [v, r) \to \mathbf{C}$ satisfies (5.16) and thus (5.18) on $[v, r)$. Then from (5.16) and using (5.15)

$$|z(t)| \leq \frac{1}{|\kappa|} \int_v^t |z(s)|\, ds + |\gamma|$$

for $t \in [v, r)$ and so

$$|z(t)| \leq |\gamma| e^{(t-v)/|\kappa|} \leq |\gamma| e^{(r-v)/|\kappa|}$$

by Gronwall's inequality. Thus by (5.18)

$$|\dot{z}(t)| \leq \left|\frac{\gamma}{\kappa}\right| \left(1 + \sup_{s \in [v,r]} |\dot{\xi}(s)|\right) e^{(r-v)/|\kappa|},$$

and from this bound it follows that $\lim_{t \to r} z(t)$ exists and is finite. Thus the extension z of y is continuous in $[v, r]$ and analytic in (v, r). It remains to show that $r = w$.

Suppose that $r < w$. Then $v < \xi(r) < r$ by (5.15), so $z(\xi(t))$ is analytic for t in a neighborhood of r. It follows by solving the differential equation (5.18) that z can be extended analytically to a neighborhood of r, and thus to $[r, r + \delta)$ for some $\delta > 0$. But this contradicts the definition of r, and so $r = w$, as desired. ■

We now prove Theorem C.

Proof of Theorem C. To begin, assume that conditions (a) through (d) in the statement of the theorem hold. We first show that condition (2.1) holds for η. Take any $a \in \mathbf{R}$ satisfying $\eta(a) = a$. Then by conditions (b) and (d) there exists $v \in (-a - 2\pi, -a]$ such that $\xi(v) = v$, equivalently, $\eta(-v) = -v - 2\pi\mu$. Thus $-v \in [a, a + 2\pi)$ and so $\eta(-v) < a$, verifying (2.1). The proof that (2.1) also holds for ξ is similar, but instead we use (a) and (c). We omit the details.

We note that with (2.1) holding for both η and ξ, the conclusions of Lemma 2.3 hold for these functions.

Also, the conclusions of Theorem B hold for any maximal Volterra interval $[a, b]$ for η, as the hypotheses of Theorem B are fulfilled.

Now let $[v, w]$ be a maximal Volterra interval for ξ, and $x \in X \setminus \{0\}$ and $\kappa \neq 0$ as in the statement of the theorem. We wish to show that (b) and (c) of Theorem B hold with a and b replaced, respectively, with $-w$ and $-v$, and also that $-w, -v \in \mathcal{N}$. We begin by establishing (c) of Theorem B but with $-v$ in place of b. We shall first show that the set $[-v, -v + \varepsilon] \cap \mathcal{N}$ is uncountable for any $\varepsilon > 0$; note that this implies that $-v$ is a cluster point of $\mathcal{N}$ and thus $-v \in \mathcal{N}$

as $\mathcal{N}$ is closed. Then we shall show that $[-v, -v+\varepsilon] \cap \mathcal{A} \neq \emptyset$ and thus $(-v, -v+\varepsilon] \cap \mathcal{A} \neq \emptyset$ for any $\varepsilon > 0$; it follows directly from this and the fact that $(-v, -v+\varepsilon] \cap \mathcal{N} \neq \emptyset$ for every $\varepsilon > 0$ that $\mathcal{A}$ has infinitely many connected components in $(-v, -v+\varepsilon]$. Thus fix $\varepsilon > 0$ and let $[a, b]$ be a maximal Volterra interval for η such that $-a < v$. Then by Lemma 4.1 applied to ξ, there exists $n \geq 1$ such that

$$\inf_{t \in [v-\varepsilon, v]} \xi^n(t) < -a, \qquad \text{equivalently,} \qquad \sup_{t \in [-v, -v+\varepsilon]} \eta^n(t) > a - 2\pi\mu n, \tag{5.19}$$

where we use the easily verified fact that $\eta^n(t) = -\xi^n(-t) - 2\pi\mu n$. Note also that $\xi^n(t) \leq v$ for every $t \leq v$, and as $\xi^n(v) = v$ we have that

$$\inf_{t \in [-v, -v+\varepsilon]} \eta^n(t) = -v - 2\pi\mu n < a - 2\pi\mu n. \tag{5.20}$$

Thus by (5.19) and (5.20),

$$[a - \delta - 2\pi\mu n, a - 2\pi\mu n] \subseteq \operatorname{int}(\eta^n(I)), \qquad \text{where } I = [-v, -v+\varepsilon],$$

for some $\delta > 0$. By Lemma 4.3, for every $c \in [a - \delta - 2\pi\mu n, a - 2\pi\mu n] \cap \mathcal{N}$ there exists $t \in I \cap \mathcal{N}$ such that $\eta^n(t) = c$. There are uncountably many such c, by part (b) of Theorem B applied to the maximal Volterra interval $[a - 2\pi\mu n, b - 2\pi\mu n]$; thus there are uncountably many such t, and so $I \cap \mathcal{N}$ is an uncountable set. Further, the set $[a - \delta - 2\pi\mu n, a - 2\pi\mu n) \cap \mathcal{A}$ is nonempty and again by Lemma 4.3 so is $I \cap \mathcal{A}$; and this is as desired.

To prove part (b) of Theorem B holds, but with $-w$ in place of a, we argue as in the proof of part (c) of Theorem B. In particular, as $[v, w]$ is a maximal Volterra interval for ξ, we have that ξ is a homeomorphism from $[w, w+\varepsilon_1]$ onto $[v-\varepsilon_2, v]$ for some positive ε_1 and ε_2, and that moreover ξ is monotone in a neighborhood of $[w, w+\varepsilon_1]$. It follows that η is a homeomorphism from $[-w-\varepsilon_1, -w]$ onto $[-v - 2\pi\mu, -v - 2\pi\mu + \varepsilon_2]$ and that $[-w - \varepsilon_1, -w] \cap \mathcal{M} = \emptyset$. Thus $\eta([-w - \varepsilon_1, -w] \cap \mathcal{A}) = [-v - 2\pi\mu, -v - 2\pi\mu + \varepsilon_2] \cap \mathcal{A}$ and $\eta([-w - \varepsilon_1, -w] \cap \mathcal{N}) = [-v - 2\pi\mu, -v - 2\pi\mu + \varepsilon_2] \cap \mathcal{N}$ by (4.1). Therefore (b) of Theorem B, but with $-w$ in place of a, follows from (c) of Theorem B, but with $-v$ in place of b, as proved in the paragraph above.

Now assume that condition (e) in the statement of the theorem holds, in addition to conditions (a) through (d). Still with $[v, w]$ a maximal Volterra interval for ξ, and $x \in X \setminus \{0\}$ and $\kappa \neq 0$ as above, let $y(t) = x(-t)$. One sees from equation (1.1) that

$$\kappa y(t) = -\int_{\xi(t)}^{t} y(s)\, ds + \gamma, \qquad \gamma = \mu \int_0^{2\pi} y(s)\, ds,$$

holds; indeed, one has that

$$\begin{aligned}\kappa y(t) &= \kappa x(-t) = \int_{\eta(-t)}^{-t} x(s)\,ds = \int_{\eta(-t)}^{-t} y(-s)\,ds \\ &= -\int_{-\eta(-t)}^{t} y(s)\,ds = -\int_{\xi(t)+2\pi\mu}^{t} y(s)\,ds \\ &= -\int_{\xi(t)}^{t} y(s)\,ds - \int_{\xi(t)+2\pi\mu}^{\xi(t)} y(s)\,ds = -\int_{\xi(t)}^{t} y(s)\,ds + \mu\int_{0}^{2\pi} y(s)\,ds,\end{aligned}$$

using the fact that y is 2π-periodic. It follows now from Lemma 5.3 that y is analytic in (v, w), that is, x is analytic in $(-w, -v)$ and so $(-w, -v) \subseteq \mathcal{A}$. Note in particular the strict inequalities (5.15) hold by virtue of condition (e). ■

If $[v, w]$ is a maximal Volterra interval for ξ as in Theorem C, then the nature of the solution x near the endpoints $-w$ and $-v$ is far from clear. In particular, even though these points belong to the set $\mathcal{N}$, it is not ruled out that $x(t)$ for $t \in [-w, -v]$ might still have an analytic extension in a full neighborhood of $-w$ or of $-v$. (Such an extension of course would be different from the solution $x \in X$ at hand.) Even if no such extension exists, it would be of interest to understand the analytic continuation and corresponding Riemann surface of x beyond the interval $(-w, -v)$.

We require several more lemmas before proving Theorem D.

Lemma 5.4. *Assume the conditions (a) through (d) in the statement of Theorem C. Also let the sets $\mathcal{B}$ and $\mathcal{S}$ be as in* (2.10) *in the statement of Theorem D. Suppose that I is an interval of positive length satisfying $I \cap \mathcal{B} = \emptyset$ and that either*

$$I \subseteq \mathcal{A} \qquad \text{or} \qquad I \subseteq \mathcal{N}. \tag{5.21}$$

Then $\overline{I} \cap \overline{\mathcal{B}} = \emptyset$, equivalently, $\overline{I} \subseteq \mathcal{S}$.

Proof. Suppose there exists some $t \in \overline{I} \cap \overline{\mathcal{B}}$. Then $t \in \partial I \cap \partial\mathcal{B}$, and in particular $t = a_k$, b_k, $-w_k$, or $-v_k$, for some $k \in \mathbf{Z}$, by (2.13). For definiteness assume that $t = a_k$, the other cases being handled similarly. Necessarily t is the right-hand endpoint of I (as I and $\mathcal{B}$ are disjoint) and so $[a_k - \varepsilon, a_k) \subseteq I$ for some $\varepsilon > 0$. But both $[a_k - \varepsilon, a_k) \cap \mathcal{A} \neq \emptyset$ and $[a_k - \varepsilon, a_k) \cap \mathcal{N} \neq \emptyset$ hold by Theorem C, and thus both $I \cap \mathcal{A} \neq \emptyset$ and $I \cap \mathcal{N} \neq \emptyset$ hold. This contradicts (5.21), completing the proof. ■

Lemma 5.5. *Assume the conditions (a) through (e) in the statement of Theorem C. Also let the sets $\mathcal{B}$ and $\mathcal{S}$ be as in* (2.10) *in the statement of Theorem D. Suppose that I is an interval of positive length with $I \subseteq \mathcal{A}$. Then*

$$\text{int}(\eta^k(I)) \subseteq \mathcal{A} \text{ for every } k \geq 0 \tag{5.22}$$

and either

$$\begin{gathered} \eta^k(I) \subseteq \overline{\mathcal{B}} \text{ for all large } k, \text{ or} \\ \overline{\eta^k(I)} \subseteq \mathcal{S} \text{ for every } k \geq 0 \end{gathered} \tag{5.23}$$

holds. If instead of $I \subseteq \mathcal{A}$ we have that $I \subseteq \mathcal{N}$, then

$$\overline{\eta^k(I)} \subseteq \mathcal{S} \cap \mathcal{N} \text{ for every } k \geq 0 \tag{5.24}$$

must hold.

Proof. Before proceeding let us recall that $\eta(\overline{\mathcal{B}}) \subseteq \overline{\mathcal{B}}$; see in particular the remarks following the statement of Theorem C. Also note that $\mathcal{B} \subseteq \mathcal{A}$ and $\partial\mathcal{B} \subseteq \mathcal{N}$, following from Theorem C.

First consider the case that $I \subseteq \mathcal{A}$. To show that (5.22) holds, assume to the contrary that there exists $c \in \text{int}(\eta^n(I)) \cap \mathcal{N}$ for some $n \geq 0$. Then by Lemma 4.3 there exists $t \in I \cap \mathcal{N}$ with $\eta^n(t) = c$, contradicting $I \subseteq \mathcal{A}$.

To prove (5.23), first observe that if $\text{int}(\eta^n(I)) \cap \mathcal{B} \neq \emptyset$ for some $n \geq 0$, then $\text{int}(\eta^n(I)) \subseteq \mathcal{B}$ must hold; for if not, then $\text{int}(\eta^n(I))$ would have nonempty intersection with $\partial\mathcal{B} \subseteq \mathcal{N}$, contradicting (5.22). Further, if this is the case then $\eta^n(I) \subseteq \overline{\mathcal{B}}$ and we have that $\eta^k(I) = \eta^{k-n}(\eta^n(I)) \subseteq \eta^{k-n}(\overline{\mathcal{B}}) \subseteq \overline{\mathcal{B}}$ for every $k \geq n$, to give the first conclusion in (5.23).

If on the other hand $\text{int}(\eta^k(I)) \cap \mathcal{B} = \emptyset$ for every $k \geq 0$, then $\overline{\eta^k(I)} = \overline{\text{int}(\eta^k(I))} \subseteq \mathcal{S}$ holds by Lemma 5.4, to give the second conclusion in (5.23).

Now consider the case that $I \subseteq \mathcal{N}$. Then $\eta^k(I) \subseteq \mathcal{N}$ and hence $\overline{\eta^k(I)} \subseteq \mathcal{N}$, for every $k \geq 0$, by the second inclusion in (4.1) and as $\mathcal{N}$ is closed. Also $\eta^k(I) \cap \mathcal{B} = \emptyset$ as $\mathcal{B} \subseteq \mathcal{A}$, and so $\overline{\eta^k(I)} \subseteq \mathcal{S}$ by Lemma 5.4. This gives (5.24). ■

For the next result recall the definition of an expansive map on a set, given in Section 2.

Lemma 5.6. *Suppose that $\eta : \mathbf{R} \to \mathbf{R}$ is C^1 and is expansive on a set $S \subseteq \mathbf{R}$. Suppose also that $I \subseteq S$ is a finite interval of positive length and that $\eta^k(I) \subseteq S$ for every $k \geq 1$. Then the strict inequality*

$$\text{len}(\eta^m(I)) > \text{len}(I) \tag{5.25}$$

holds for some $m \geq 1$. Further, the map η^k is strictly monotone (either increasing or decreasing) on I for every $k \geq 1$.

Proof. Fix any $t \in I$. Then for any $k \geq 1$ we have that $\dot{\eta}^k(t) = \prod_{j=0}^{k-1} \dot{\eta}(\eta^j(t))$, with $\dot{\eta}^k(t) \neq 0$ for all large k by (2.9). Thus $\dot{\eta}(\eta^j(t)) \neq 0$ for every $j \geq 0$, hence $\dot{\eta}^k(t) \neq 0$ for every $k \geq 1$. It follows now that η^k is strictly monotone on I for every $k \geq 1$. Letting $p < q$ denote the endpoints of I, we thus have that

$$\mathrm{len}(I) = q - p < \int_p^q \liminf_{k\to\infty} |\dot{\eta}^k(t)|\, dt \leq \liminf_{k\to\infty} \int_p^q |\dot{\eta}^k(t)|\, dt = \liminf_{k\to\infty} \mathrm{len}(\eta^k(I)),$$

using (2.9) in the first inequality, Fatou's Lemma in the second inequality, and the monotonicity of η^k in the final equality. The desired conclusion (5.25) now follows, in fact for all large m. ■

Proof of Theorem D. We prove the four parts of this theorem in sequence.

(a) We begin with the observation that if $P \subseteq \mathbf{R}$ is any set which is periodic, with $\mathrm{int}(P) \neq \emptyset$ and $P \neq \mathbf{R}$, then for all sufficiently small $\varepsilon > 0$ the set of lengths

$$\begin{aligned} \Gamma_\varepsilon = \{\sigma \in [\varepsilon, \infty) \,|\, \sigma = \mathrm{len}(I) \text{ for some } I \subseteq P \\ \text{where } I \text{ is a connected component of } P\} \end{aligned} \tag{5.26}$$

is a finite nonempty set. This holds because such connected components are finite intervals (or points) which are pairwise disjoint, with at least one of them of positive length.

Now we wish to prove that $\mathrm{int}(\mathcal{N}) = \emptyset$, so suppose to the contrary that $\mathrm{int}(\mathcal{N}) \neq \emptyset$. We also know from Theorem C that $\mathcal{A} \neq \emptyset$, and so $\mathcal{N} \neq \mathbf{R}$, and thus we may take $P = \mathcal{N}$ in the above remark. In particular, there exists a connected component $I \subseteq \mathcal{N}$ of $\mathcal{N}$ which maximizes the length of all such connected components; such exists because the set Γ_ε above is finite. Then $\eta^k(I) \subseteq \mathcal{S} \cap \mathcal{N}$ for every $k \geq 0$ by Lemma 5.5; and further, by Lemma 5.6 the inequality (5.25) holds for some $m \geq 1$. But then $\eta^m(I)$ is a connected subset of $\mathcal{N}$ of length greater than the maximum possible. With this contradiction we conclude that $\mathrm{int}(\mathcal{N}) = \emptyset$, and thus also that $\mathcal{A}$ is dense.

(b) Let us now prove (2.12). First note that the second inclusion in (2.12) follows immediately from the second inclusion in (4.1) and the fact that $\mathcal{B} \subseteq \mathcal{A}$; thus we only need establish the first inclusion in (2.12). To this end take any $t_0 \in \mathcal{A} \setminus \mathcal{G}$, denote $t_k = \eta^k(t_0)$ for $k \geq 1$, and assume that $t_k \notin \mathcal{B}$ for every $k \geq 0$. We seek a contradiction. We shall first show that $t_k \in \mathcal{A} \setminus \mathcal{G}$ for every $k \geq 0$, and then letting I_k denote the connected component of $\mathcal{A}$ containing t_k, we shall show there

exists $m \geq 1$ such that

$$\text{len}(I_m) > \text{len}(I_0).$$

If this is shown, it follows inductively that there exist $0 = m_0 < m_1 < m_2 < \cdots$ such that $\text{len}(I_{m_j}) < \text{len}(I_{m_{j+1}})$ for every $j \geq 0$. (Here m above equals m_1.) This, however, contradicts the fact that for any small $\varepsilon > 0$ the set Γ_ε in (5.26) with $P = \mathcal{A}$ is finite. (Note that $\text{int}(\mathcal{A}) \neq \emptyset$, and that $\mathcal{A} \neq \mathbf{R}$ as $\mathcal{N} \neq \emptyset$, again by Theorem C.)

To begin, observe that if $t \notin \mathcal{G}$, then $\eta(t) \notin \mathcal{G}$; indeed, this follows immediately from the definition of $\mathcal{G}$. Thus with $t_0 \in \mathcal{A} \setminus \mathcal{G}$ given we have that $t_k \notin \mathcal{G}$ for every $k \geq 0$.

Now let $I \subseteq \mathcal{A}$ denote the connected component of $\mathcal{A}$ containing t_0. Then by Lemma 5.5 one of the two possibilities in (5.23) holds. We eliminate the first possibility by assuming that $\eta^k(I) \subseteq \overline{\mathcal{B}}$ for all large k. Then $t_k \in \eta^k(I) \subseteq \overline{\mathcal{B}}$ for such k, and as $t_k \notin \mathcal{B}$ is assumed we have that $\eta^k(t_0) = t_k \in \partial\mathcal{B} = \partial\mathcal{B}_\eta \cup \partial\mathcal{B}_\xi$. Further, we have $t \in I$ for all t sufficiently near t_0 as I is open, and so $\eta^k(t) \in \overline{\mathcal{B}}$ for such t. This implies that $t_0 \in \mathcal{G}$, a contradiction

Thus the second possibility in (5.23) holds, and so $\eta^k(I) \subseteq \mathcal{S}$ for every $k \geq 0$. By Lemma 5.6 and the expansiveness of η on $\mathcal{S}$, the map η^k is strictly monotone on I and so $t_k \in \eta^k(I) = \text{int}(\eta^k(I)) \subseteq \mathcal{A}$, where (5.22) of Lemma 5.5 is used. Thus $t_k \in \mathcal{A} \setminus \mathcal{G}$, as claimed. Also by Lemma 5.6, there exists $m \geq 1$ such that (5.25) holds, and thus

$$\text{len}(I_m) \geq \text{len}(\eta^m(I)) > \text{len}(I) = \text{len}(I_0),$$

as desired. With this (2.12) is established.

(c) We next prove that $\mathcal{G}_\eta \subseteq \mathcal{A}$. Fix $t_0 \in \mathcal{G}_\eta$ and let $n \geq 1$ be as in the definition (2.11) of $\mathcal{G}_\eta$. Further, let k, uniquely determined, be such that $\eta^n(t_0) \in [a_k, b_k]$, and so $[a_k, b_k]$ is the maximal Volterra interval of η to which $\eta^n(t)$ belongs for all t near t_0. Of course $\eta^n(t_0) \in \{a_k, b_k\}$ and also $x(t) = 0$ identically in $[a_k, b_k]$. Denote $Q = [a_k, b_k]$. Also denote $t_j = \eta^j(t_0)$ for $1 \leq j \leq n$, and define $y_j(t)$, for t in a neighborhood of t_j, inductively (beginning with $j = n$ and descending to $j = 0$) by the differential equations

$$\begin{aligned} &y_n(t) = 0 \text{ identically,} \\ &\dot{y}_j(t) = \frac{1}{\kappa}\Big(y_j(t) - y_{j+1}(\eta(t))\dot{\eta}(t)\Big), \qquad y_j(t_j) = x(t_j), \qquad 0 \leq j \leq n-1. \end{aligned} \tag{5.27}$$

(Compare this with the differential equation (1.7) which $x(t)$ satisfies.) Observe that $y_j(t)$ is analytic in t, for each j.

We claim the following holds for $0 \le j \le n$: If η^j maps $[t_{n-j}, t_{n-j} + \varepsilon]$ into Q for some $\varepsilon > 0$ then $x(t) = y_{n-j}(t)$ identically for $t \in [t_{n-j}, t_{n-j} + \varepsilon']$ for some $\varepsilon' > 0$; and the corresponding claim also holds for $[t_{n-j} - \varepsilon, t_{n-j}]$ in place of $[t_{n-j}, t_{n-j} + \varepsilon]$. Note that this claim, for $j = n$, implies the desired result that $t_0 \in \mathcal{A}$. Indeed, η^n maps both $[t_0, t_0 + \varepsilon]$ and $[t_0 - \varepsilon, t_0]$ into Q and thus $x(t) = y_0(t)$ identically in $[t_0 - \varepsilon', t_0 + \varepsilon']$, with $y_0(t)$ analytic in t.

We prove the above claim by inducting on j. The result is trivial for $j = 0$. Namely, if η^0 (the identity map) maps $[t_n, t_n + \varepsilon]$ into Q then simply $[t_n, t_n + \varepsilon] \subseteq Q$ and $x(t) = y_n(t) = 0$ identically in $[t_n, t_n + \varepsilon]$; and similarly for $[t_n - \varepsilon, t_n]$. Suppose now that the claim holds for some j with $0 \le j \le n - 1$; we wish to establish it for $j + 1$. Thus assume that η^{j+1} maps I into Q where either $I = [t_{n-j-1}, t_{n-j-1} + \varepsilon]$ or $I = [t_{n-j-1} - \varepsilon, t_{n-j-1}]$, for some $\varepsilon > 0$. Also, by possibly reducing ε we have that $\eta(I) = J$ where J is an interval either of the form $J = [t_{n-j}, t_{n-j} + \delta]$ or $J = [t_{n-j} - \delta, t_{n-j}]$; and thus η^j maps J into Q. By the induction hypothesis it follows that $x(t) = y_{n-j}(t)$ identically in an interval J' of the form either $J' = [t_{n-j}, t_{n-j} + \delta']$ or $J' = [t_{n-j} - \delta', t_{n-j}]$, and without loss $J' \subseteq J$. Therefore $x(\eta(t)) = y_{n-j}(\eta(t))$ identically in an interval $I' \subseteq I$ of the form either $I' = [t_{n-j-1}, t_{n-j-1} + \varepsilon']$ or $I' = [t_{n-j-1} - \varepsilon', t_{n-j-1}]$; and it follows immediately from the differential equations (1.7) and (5.27) that $x(t) = y_{n-j-1}(t)$ identically in I', which is as desired. With this $\mathcal{G}_\eta \subseteq \mathcal{A}$ is proved.

(d) Here we prove that $\mathcal{I} \subseteq \mathcal{G}_\xi$. We make an initial observation, to be used later, that $\eta(\partial\mathcal{B}) \subseteq \partial\mathcal{B}$, and that if $t_0 \in \partial\mathcal{B}$ then η is strictly monotone in a neighborhood of t_0.

Define a set

$$\mathcal{H} = \{t_0 \in \mathbf{R} \mid \text{there exists } n \ge 0 \text{ and } \varepsilon > 0 \text{ such that } \eta^n(t) \in \overline{\mathcal{B}} \text{ for every } t \text{ with } |t - t_0| \le \varepsilon\}. \tag{5.28}$$

It is clear that $\mathcal{H}$ is open. Also,

$$\mathcal{A} \setminus \mathcal{G} \subseteq \mathcal{H}, \qquad \mathcal{G} \subseteq \mathcal{H}, \qquad \text{hence} \qquad \mathcal{A} \subseteq \mathcal{H};$$

indeed, the first inclusion above follows from (2.12) proved in part (b) above and by the openness of $\mathcal{B}$, and the second inclusion above follows immediately from the definition of $\mathcal{G}$. Also define $\nu(t_0) \ge 0$, for $t_0 \in \mathcal{H}$, to be the minimum integer $n \ge 0$ such that the condition in the definition (5.28) holds for some $\varepsilon > 0$. One sees immediately that for any $t_0 \in \mathcal{H}$, the inequality $\nu(t) \le \nu(t_0)$ holds for all t in some neighborhood of t_0; that is, the function ν is upper semicontinuous on $\mathcal{H}$. Now let $I \subseteq \mathcal{H}$ be any connected component of $\mathcal{H}$. Our goal is to prove that there is some iterate of η which maps all of I into $\overline{\mathcal{B}}$. In particular, let

$$m = \min_{t \in I} \nu(t), \qquad O = \{t \in I \mid \nu(t) = m\};$$

we shall show that $O = I$. By the upper semicontinuity of ν one sees that O is a nonempty open subset of I, and so the conclusion $O = I$ holds if we show that O is a relatively closed subset of I. Assume to the contrary that O is not relatively closed in I. Then there exists a sequence $t_k \in O$ for $k \geq 1$ such that the limit $t_* = \lim_{k\to\infty} t_k$ exists and satisfies $t_* \in I$, but that $t_* \notin O$. Certainly $\eta^m(t_*) \in \overline{\mathcal{B}}$ holds as $\eta^m(t_k) \in \overline{\mathcal{B}}$ for each k. As $t_* \notin O$, we have that $\nu(t_*) > m$ and so there does not exist $\varepsilon > 0$ such that $\eta^m(t) \in \overline{\mathcal{B}}$ for all t satisfying $|t - t_*| \leq \varepsilon$. This in particular implies that $\eta^m(t_*) \notin \mathcal{B}$, as $\mathcal{B}$ is open, and thus $\eta^m(t_*) \in \partial\mathcal{B}$. Denote $c = \eta^m(t_*)$. From the Taylor series of η^m about t_*, one sees that there exists $\varepsilon > 0$ such that for all $t \in (t_*, t_* + \varepsilon]$ either $\eta^m(t) > c$ or $\eta^m(t) < c$, and that η^m is strictly monotone in $(t_*, t_* + \varepsilon]$. A similar conclusion applies to the interval $[t_* - \varepsilon, t_*)$. Observing that there exists points, namely t_k, arbitrarily near t_* which are in $\mathcal{B}$, and also that there exits points arbitrarily near t_* which are not in $\mathcal{B}$ (because $t_* \notin O$), it follows that η^m must be strictly monotone in a neighborhood of t_*; in other words $\eta^m(t) > c$ on one side of t_* and $\eta^m(t) < c$ on the other side (for $t \neq t_*$ near t_*). But from our initial observation above, we see that all iterates $\eta^j(t_*)$ for $j \geq m$ lie in $\partial\mathcal{B}$ and also that η^j is strictly monotone in some neighborhood (possibly depending on j) of t_*. It follows that for each $j \geq m$, there does not exist $\varepsilon > 0$ such that $\eta^k(t) \in \overline{\mathcal{B}}$ holds for all t satisfying $|t - t_*| \leq \varepsilon$; and thus $t_* \notin \mathcal{H}$. This contradicts our assumption that $t_* \in I \subseteq \mathcal{H}$.

At this point we have shown that if I is any connected subset of $\mathcal{H}$ (in particular if I is a connected subset of $\mathcal{A}$), then there exists $m \geq 0$ such that $\eta^m(I) \subseteq \overline{\mathcal{B}}$. Now suppose that $t_0 \in \mathcal{I}$. Then there exists $\varepsilon > 0$ such that $[t_0 - \varepsilon, t_0) \cup (t_0, t_0 + \varepsilon] \subseteq \mathcal{A}$ but where $t_0 \in \mathcal{N}$. We conclude from above that there exist some $m \geq 0$ such that $\eta^m([t_0 - \varepsilon, t_0)) \cup \eta^m((t_0, t_0 + \varepsilon]) \subseteq \overline{\mathcal{B}}$, and thus $\eta^m([t_0 - \varepsilon, t_0 + \varepsilon]) \subseteq \overline{\mathcal{B}}$. However $\eta^m(t_0) \in \mathcal{N}$ by the second inclusion in (4.1) and so $\eta^m(t_0) \notin \mathcal{B}$. One concludes that $\eta^m(t_0) \in \partial\mathcal{B}$, and this immediately implies that $t_0 \in \mathcal{G}$. Further, $t_0 \in \mathcal{G}_\eta$ is impossible, as that would imply $t_0 \in \mathcal{A}$ from part (c) above. Thus $t_0 \in \mathcal{G}_\xi$ holds, as desired. With this the proof is complete. ■

6 An Explicit Example; the Proof of Theorem E

Throughout this section η denotes the function in the statement of Theorem E, and ξ denotes the function in the statement of Theorem C, namely,

$$\eta(t) = t - \pi\mu(1 - \cos t), \qquad \xi(t) = t - \pi\mu(1 + \cos t),$$

which one sees after a short calculation to obtain ξ. Here $\mu \geq 1$ is a fixed integer.

The following lemma verifies some of the conditions of Theorem C for η.

Lemma 6.1. *Conditions (a) through (e) in the statement of Theorem C hold. In addition, there exists a unique quantity b satisfying*

$$b \in (0, \pi/2) \qquad \text{and} \qquad \eta(b) = 0. \tag{6.1}$$

Furthermore, for this quantity b we have that

$$\eta(t) > 0 \text{ for every } t \in (0, b), \qquad \eta(t) < 0 \text{ for every } t \in (b, \pi/2]. \tag{6.2}$$

Also, all the maximal Volterra intervals $[a_k, b_k]$ for η and the maximal Volterra intervals $[v_k, w_k]$ for ξ may be enumerated as in (2.15)*, and thus the set $\mathcal{S}$ is as given in* (2.16)*.*

Proof. The verification of conditions (a) and (b) is trivial; we omit this. Also, $\eta(a) = a$ for $a = 0$ and $\xi(v) = v$ for $v = \pi$, which establishes (c) and (d). We verify condition (e) below.

The existence of a quantity b satisfying (6.1) follows from the fact that $\eta(0) = 0$ and $\dot\eta(0) > 0$, while $\eta(\frac{\pi}{2}) < 0$. The uniqueness of such b follows from the fact that $\ddot\eta(t) < 0$ for all $t \in (0, \frac{\pi}{2})$, and this also implies that (6.2) holds; and further, it implies that (2.2) and (2.3) hold with $a = 0$ and thus $[0, b]$ is an RM-Volterra interval for η. From the periodicity condition in (1.2) we have that $[2\pi k, b+2\pi k]$ is also an RM-Volterra interval for η, for every $k \in \mathbf{Z}$. There are no other RM-Volterra intervals for η, due to parts (a) and (b) of Lemma 2.3 and the fact that the points $2\pi k$ are the only fixed points of η. Thus the intervals $[a_k, b_k]$ in (2.15) account for all the RM-Volterra intervals for η, and as there are no others, they are maximal Volterra intervals.

The proof that the intervals $[v_k, w_k]$ in (2.15) is an enumeration of all the maximal Volterra intervals for ξ follows from the identity $\xi(t) = \eta(t-\pi)+\pi$, which implies that the maximal Volterra intervals for ξ are just those for η but translated by π; we omit the details.

One now sees that the set $\mathcal{S}$ given by (2.16) is as in the statement of Theorem C.

Finally, condition (e) in the statement of Theorem C follows easily from the strict inequality $\ddot\xi(t) < 0$ which holds for every $t \in [v_k, w_k]$ and every $k \in \mathbf{Z}$. ■

In verifying the expansiveness of η on the set $\mathcal{S}$, the following lemmas will be used. For notational simplicity let us denote

$$\mathcal{S}_k^+ = (-\pi + 2\pi k, 2\pi k), \qquad \mathcal{S}_k^- = (b + 2\pi k, -b + \pi + 2\pi k),$$

for $k \in \mathbf{Z}$. These intervals are the various connected components of the set $\mathcal{S}$. In the following lemma each set $\mathcal{S}_k^-$ will be divided into three disjoint parts, denoted $\mathcal{S}_k^{\mathrm{L}}$, $\mathcal{S}_k^{\mathrm{C}}$, and $\mathcal{S}_k^{\mathrm{R}}$.

Before proceeding let us note the elementary inequalities

$$\cos t > 1 - \frac{t^2}{2}, \qquad \sin t > t - \frac{t^3}{6}, \qquad \cos t < 1 - \frac{t^2}{2} + \frac{t^4}{24},$$

which are valid for every $t > 0$. These inequalities are easily obtained by successive integration, beginning with the inequality $\sin t < t$. We shall use them repeatedly in what follows.

Lemma 6.2. *We have that*

$$\dot\eta(t) > 1 \qquad \textit{for every } t \in \mathcal{S}_k^+.$$

Also, we have that $\frac{1}{\mu} \in (b, \frac{\pi}{2})$*, and if we denote*

$$\begin{aligned} &\mathcal{S}_k^{\mathrm{L}} = (b + 2\pi k, 1/\mu + 2\pi k), \qquad \mathcal{S}_k^{\mathrm{R}} = (-1/\mu + \pi + 2\pi k, -b + \pi + 2\pi k), \\ &\mathcal{S}_k^{\mathrm{C}} = [1/\mu + 2\pi k, -1/\mu + \pi + 2\pi k], \end{aligned} \tag{6.3}$$

then

$$\begin{aligned} &\dot\eta(t) < -\frac{23}{27} \qquad \textit{for every } t \in \mathcal{S}_k^{\mathrm{L}} \cup \mathcal{S}_k^{\mathrm{R}}, \\ &\dot\eta(t) < -1 \qquad \textit{for every } t \in \mathcal{S}_k^{\mathrm{C}}, \end{aligned} \tag{6.4}$$

both hold.

Proof. Due to the periodicity condition in (1.2), without loss we may take $k = 0$. Note that $\dot\eta(t) = 1 - \pi\mu \sin t$; thus for $t \in \mathcal{S}_0^+ = (-\pi, 0)$ we have that $\dot\eta(t) > 1$.

In obtaining the inequalities (6.4) for $t \in \mathcal{S}_0^- = (b, \pi - b)$, it is enough to consider only t in the left half of this interval, namely $t \in (b, \frac{\pi}{2}]$; this is due to the symmetry relation $\dot\eta(t) = \dot\eta(\pi - t)$. To begin we note that

$$\begin{aligned} \eta(1/\mu) &= \frac{1}{\mu} - \pi\mu(1 - \cos(1/\mu)) \\ &< \frac{1}{\mu} - \pi\mu\left(\frac{1}{2\mu^2} - \frac{1}{24\mu^4}\right) \le \frac{1}{\mu} - \pi\mu\left(\frac{1}{2\mu^2} - \frac{1}{24\mu^2}\right) = \left(1 - \frac{11\pi}{24}\right)\frac{1}{\mu} < 0, \end{aligned}$$

and so it follows from Lemma 6.1 that $\frac{1}{\mu} \in (b, \frac{\pi}{2})$ and that the three intervals (6.3) are well-defined. (Note that they are disjoint and their union is $\mathcal{S}_k^-$.)

We next observe that

$$\eta(2/(\pi\mu)) = \frac{2}{\pi\mu} - \pi\mu(1-\cos(2/(\pi\mu))) > \frac{2}{\pi\mu} - \frac{\pi\mu}{2}\left(\frac{2}{\pi\mu}\right)^2 = 0,$$

and therefore $\frac{2}{\pi\mu} \in (0,b)$, again by Lemma 6.1. Thus for every $t \in \mathcal{S}_0^{\mathrm{L}} = (b, \frac{1}{\mu})$ we have that

$$\begin{aligned}\dot\eta(t) &= 1 - \pi\mu\sin t < 1 - \pi\mu\sin(2/(\pi\mu)) \\ &< 1 - \pi\mu\left(\frac{2}{\pi\mu} - \frac{1}{6}\left(\frac{2}{\pi\mu}\right)^3\right) = -1 + \frac{4}{3\pi^2\mu^2} \le -1 + \frac{4}{3\pi^2} < -\frac{23}{27},\end{aligned}$$

to give the first inequality in (6.4).

Now take any $t \in \mathcal{S}_0^{\mathrm{C}} \cap (b, \frac{\pi}{2}] = [\frac{1}{\mu}, \frac{\pi}{2}]$. We have that

$$\begin{aligned}\dot\eta(t) &= 1 - \pi\mu\sin t \le 1 - \pi\mu\sin(1/\mu) \\ &< 1 - \pi\mu\left(\frac{1}{\mu} - \frac{1}{6\mu^3}\right) = 1 - \pi + \frac{\pi}{6\mu^2} \le 1 - \frac{5\pi}{6} < -1,\end{aligned}$$

to give the second inequality in (6.4). This completes the proof. ■

Lemma 6.3. *We have that*

$$\eta(t) \in (-\pi/2 + 2\pi k, 2\pi k) \qquad \textit{for every } t \in \mathcal{S}_k^{\mathrm{L}}$$

holds.

Proof. Again without loss $k = 0$. As $\dot\eta(t) < 0$ throughout $\mathcal{S}_0^{\mathrm{L}} = (b, \frac{1}{\mu})$ by (6.4) in Lemma 6.2, we have for any such t that

$$\eta(t) > \eta(1/\mu) = \frac{1}{\mu} - \pi\mu(1-\cos(1/\mu)) > \frac{1}{\mu} - \frac{\pi\mu}{2}\left(\frac{1}{\mu^2}\right) > -\frac{\pi}{2\mu} \ge -\frac{\pi}{2},$$

as desired. Also, $\eta(t) < 0$ for such t by (6.2) in Lemma 6.1, again as desired. ■

Lemma 6.4. *There exists a unique quantity q satisfying*

$$-q \in (-\pi/2, 0) \qquad \textit{and} \qquad \eta(-q) = -\frac{\pi}{2}. \tag{6.5}$$

Furthermore,

$$\dot\eta(t) > \frac{41}{18} \qquad \textit{for every } t \in [-\pi/2, -q] \tag{6.6}$$

holds.

Proof. The existence of q satisfying (6.5) follows from the fact that $\eta(-\frac{\pi}{2}) < -\frac{\pi}{2} < \eta(0) = 0$; and the uniqueness of such q holds because $\dot\eta(t) > 0$ throughout $(-\frac{\pi}{2}, 0)$ as is easily seen. Now define a quantity p given by

$$p = \frac{-1 + (1+\pi^2\mu)^{1/2}}{\pi\mu}, \qquad \text{and so} \qquad \left(\frac{\pi\mu}{2}\right)p^2 + p - \frac{\pi}{2} = 0, \tag{6.7}$$

as one sees by a simple calculation. Certainly $p > 0$, and $p < \frac{\pi}{2}$ follows from the second equation in (6.7), and thus $-p \in (-\frac{\pi}{2}, 0)$. Further, we have that

$$\eta(-p) = -p - \pi\mu(1 - \cos p) > -p - \left(\frac{\pi\mu}{2}\right)p^2 = -\frac{\pi}{2} = \eta(-q),$$

again using the second equation in (6.7); and again as $\dot\eta(t) > 0$ in $(-\frac{\pi}{2}, 0)$ it follows that $-p > -q$. Thus if $t \in [-\frac{\pi}{2}, -q]$ as in the statement of the lemma, we have that $t < -p$ and so

$$\dot\eta(t) = 1 - \pi\mu\sin t = 1 + \pi\mu\sin|t| > 1 + \pi\mu\sin p > 1 + \pi\mu\left(p - \frac{p^3}{6}\right).$$

Thus in order to obtain the conclusion (6.6) of the lemma, it suffices to prove that

$$1 + \pi\mu\left(p - \frac{p^3}{6}\right) \geq \frac{41}{18}.$$

To this end, we first note that

$$1 - \frac{p^2}{6} = 1 - \frac{1}{6\mu} + \left(\frac{1}{3\pi\mu}\right)p$$

where the second equation in (6.7) is used to substitute for p^2. Multiplying by p and making another such substitution gives

$$p - \frac{p^3}{6} = \left(1 - \frac{1}{6\mu}\right)p + \frac{1}{3\pi\mu}\left(\frac{1}{\mu} - \left(\frac{2}{\pi\mu}\right)p\right) = \frac{1}{3\pi\mu^2} + \left(1 - \frac{1}{6\mu} - \frac{2}{3\pi^2\mu^2}\right)p.$$

Now substituting the first equation in (6.7) for p above, and rearranging terms, gives

$$\begin{aligned} 1 + \pi\mu\left(p - \frac{p^3}{6}\right) &= \frac{1}{2\mu} + \frac{2}{3\pi^2\mu^2} + \left(1 - \frac{1}{6\mu} - \frac{2}{3\pi^2\mu^2}\right)(1+\pi^2\mu)^{1/2} \\ &> \left(1 - \frac{1}{6\mu} - \frac{2}{3\pi^2\mu^2}\right)(1+\pi^2\mu)^{1/2} \\ &\geq \left(1 - \frac{1}{6} - \frac{2}{3\pi^2}\right)(1+\pi^2)^{1/2} > \left(\frac{5}{6} - \frac{2}{27}\right) \times 3 = \frac{41}{18}. \end{aligned}$$

With this, the proof is complete. ■

Proof of Theorem E. We shall use Lemma 2.4 to verify the expansiveness of η on $\mathcal{S}$. Let $\alpha = \frac{23}{27}$ and $\beta = \frac{41}{18}$, and observe that $\alpha\beta = \frac{943}{486} > 1$.

First note that if $t \in \mathcal{S}$ then $|\dot\eta(t)| \geq \alpha$, by Lemma 6.2. Suppose now that $t \in \mathcal{S}$ is such that $\eta^k(t) \in \mathcal{S}$ for every $k \geq 1$, and also suppose that $|\dot\eta(t)| \leq 1$. Again by Lemma 6.2, we have that $t \in \mathcal{S}_j^{\mathrm{L}} \cup \mathcal{S}_j^{\mathrm{R}}$ for some $j \in \mathbf{Z}$; without loss assume that $j = 0$. Consider first the case that $t \in \mathcal{S}_0^{\mathrm{L}} = (b, \frac{1}{\mu})$. Then $\eta(t) \in (-\frac{\pi}{2}, 0) \subseteq \mathcal{S}_0^+$ by Lemma 6.3. With $-q$ as in the statement of Lemma 6.4, let $n = \min\{k \geq 1 \mid \eta^k(t) \notin (-q, 0)\}$. One sees that n is well-defined, for if not, then $\eta^k(t) \in (-q, 0)$ for all $k \geq 1$; but then $\eta^k(t)$ would be a nonincreasing sequence in $(-q, 0)$ which would converge to a fixed point of η in $[-q, 0)$. However, no such fixed point exists. Now if $n \geq 2$ then $\eta^n(t) \leq \eta^{n-1}(t) \in (-q, 0)$ and $\eta^n(t) \notin (-q, 0)$, and thus $\eta^n(t) \leq -q$. Also, $\eta^{n-1}(t) > -q$ hence $\eta^n(t) > \eta(-q) = -\frac{\pi}{2}$ as η is strictly increasing in $\mathcal{S}_0^+$. Thus $\eta^n(t) \in (-\frac{\pi}{2}, -q]$; and from the definition above of n this conclusion also holds if $n = 1$. Thus in any case $\dot\eta(\eta^n(t)) > \beta$ by (6.6) of Lemma 6.4. And also, $\dot\eta(\eta^k(t)) > 1$ for $1 \leq k \leq n-1$ by Lemma 6.2, as $\eta^k(t) \in (-q, 0) \subseteq \mathcal{S}_0^+$ for such k. This is as required by Lemma 2.4.

Now consider the case that $t \in \mathcal{S}_0^{\mathrm{R}} = (\pi - \frac{1}{\mu}, \pi - b)$. First note that for arbitrary $t \in \mathbf{R}$ we have that $t \in \mathcal{S}$ if and only if $\zeta(t) \in \mathcal{S}$, where we denote $\zeta(t) = \pi - t$. Also, $\zeta \circ \eta \circ \zeta^{-1}(t) = \pi - \eta(\pi - t) = \eta(t) + 2\pi\mu$, and so $\zeta \circ \eta^k \circ \zeta^{-1}(t) = \eta^k(t) + 2\pi\mu k$ for every $k \geq 1$. Thus

$$\eta^k(\pi - t) = \pi - \eta^k(t) - 2\pi\mu k, \qquad \text{hence} \qquad \dot\eta^k(\pi - t) = \dot\eta^k(t), \tag{6.8}$$

identically in t for all such k. Now fix any $t \in \mathcal{S}_0^{\mathrm{R}}$ such that $\eta^k(t) \in \mathcal{S}$ for all $k \geq 0$. Let $\widetilde{t} = \pi - t$ and note that $\widetilde{t} \in \mathcal{S}_0^{\mathrm{L}}$, and also, from the first equation in (6.8), that $\eta^k(\widetilde{t}) \in \mathcal{S}$ for all $k \geq 0$. If follows now immediately from the second equation in (6.8), and using the results in the paragraph above, that if $|\dot\eta(t)| \leq 1$ then the conditions (2.17) of Lemma 2.4 hold for t, for some $n \geq 1$, as desired. We omit the details. This now establishes the expansiveness of η on $\mathcal{S}$.

To complete the proof of the theorem we show that $\mathcal{G} = \emptyset$. To begin, denote $p = \arcsin(\frac{1}{\pi\mu})$ and observe that $\dot\eta(t) = 0$ if and only if either $t = p + 2\pi k$ or $t = \pi - p + 2\pi k$, for some $k \in \mathbf{Z}$. Further, $\ddot\eta(t) \neq 0$ at each such point, and so these points precisely constitute the set $\mathcal{M}$. Note also that $t \in \mathcal{B}$ for each such t; in particular, $p \in (0, b)$ as it is the maximum of η in this interval, and $\pi - p \in (\pi - b, \pi)$. And also note that $\eta(\mathcal{B}) \subseteq \mathcal{B}$ (that is, not merely $\subseteq \overline{\mathcal{B}}$). Now suppose there exists some $t_0 \in \mathcal{G}$. Then there exists $n \geq 1$ such that $\eta^n(t_0) \in \partial\mathcal{B}$ with $\eta^n(t) \in \mathcal{B}$ for all t sufficiently near t_0. Necessarily t_0 is either a local maximum or a local minimum of η^n, and it follows that

$t_* = \eta^k(t_0)$ is either a local maximum of a local minimum for η, for some k with $0 \le k \le n-1$; that is, $t_* \in \mathcal{M}$. Thus t_* and all its iterates belong to $\mathcal{B}$, in particular $\eta^n(t_0) = \eta^{n-k}(t_*) \in \mathcal{B}$. However, this contradicts $\eta^n(t_0) \in \partial\mathcal{B}$ and completes the proof. ■

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