*Nonlinear Analysis, Theory, Methods & Applications*, Vol. 20, No. 7, pp. 855–870, 1993.
Printed in Great Britain.

0362-546X/93 $6.00 + .00


# A LIMIT SET TRICHOTOMY FOR SELF-MAPPINGS OF NORMAL CONES IN BANACH SPACES

ULRICH KRAUSE† and ROGER D. NUSSBAUM‡§

† Department of Mathematics, University of Bremen, 2800 Bremen, Germany; and
‡ Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, U.S.A.

(*Received* 15 *October* 1991; *received for publication* 2 *June* 1992)

*Key words and phrases*: Limit set trichotomy, iterated nonlinear positive operators, nonexpansive and contractive mappings, part metric.

## 1. INTRODUCTION

IN A RECENT paper Krause and Ranft [1] explored conditions on self-mappings $T$ of the positive orthant $\mathbb{R}^n_+$ in $\mathbb{R}^n$ which ensure that the following trichotomy holds.
Either:

(i) for every nonzero $x$ in $\mathbb{R}^n_+$ the orbit $\gamma^+(x) \equiv \bigcup_{j=0}^{\infty} T^j x$ is an unbounded set; or

(ii) for every $x \in \mathbb{R}^n_+$, $\lim_{j\to\infty} T^j x = 0$; or

(iii) there exists a unique fixed point $x_0$ of $T$ in the interior of $\mathbb{R}^n_+$ such that $\lim_{j\to\infty} T^j x = x_0$ for all nonzero $x \in \mathbb{R}^n_+$.

If we write $K^n$ and $\mathring{K}^n$ for $\mathbb{R}^n_+$ and the interior of $\mathbb{R}^n_+$ respectively, it has been shown in particular (see [1, corollary 1]), that the above trichotomy is valid for continuous maps $T: K^n \to K^n$ such that

$$T(\lambda x) > \lambda T(x) \text{ for all } 0 < \lambda < 1 \text{ and all } x > 0, \text{ and}$$
$$Tx > 0 \text{ for all } x \geq 0,\ x \neq 0, \text{ and } Tx \geq Ty \text{ for all } x, y \text{ with } x \geq y.$$

(Here, we use the notation $x > y$ and $x \geq y$ to mean, respectively, $x - y \in \mathring{K}^n$ and $x - y \in K^n$.) This result extends an earlier theorem of Smith [2] concerning "discrete dynamics of monotone, concave maps"; some interesting applications to differential equations can be found in [1, 2]. Another extension of Smith's theorem has been given by Takáč in [3].

In this paper we shall extend the above trichotomy in two directions. First, we shall allow general normal cones $K$ with nonempty interior in a Banach space. Second, we shall allow a class of maps which is considerably more general than classes allowed in earlier results, even for $K$, the positive orthant in $\mathbb{R}^n$. The key observation which we shall exploit centers about a metric $p$, called the *part metric* or *Thompson's metric* (see [1, 4–6] and the references given there; and Section 2 below) which is defined on the interior $\mathring{K}$ of a cone. The proper class of maps to study seems to be those maps $T: \mathring{K} \to \mathring{K}$ such that $T^m$, the $m$th iterate of $T$, satisfies

$$p(T^m x, T^m y) \leq p(x, y) \qquad \text{for all } x, y \in \mathring{K} \text{ or} \tag{1.1}$$

$$p(T^m x, T^m y) < p(x, y) \qquad \text{for all } x, y \in \mathring{K},\ x \neq y, \tag{1.2}$$

§ Partially supported by NSF DMS 89 030 18.

and some further mild assumptions. For example, if $K$ is a normal cone with nonempty interior $\mathring{K}$ in a Banach space and $T: K \to K$ is a compact, continuous map such that $T(K - \{0\}) \subset \mathring{K}$ and $T$ satisfies (1.2), then it is a special case of our later results that the trichotomy holds. If we only know that (1.1) holds, the situation is much more subtle. By using recent results from [10] (see, also, related work in [8, 11-16]), we obtain analogues of the trichotomy property for maps $T: \mathring{K} \to \mathring{K}$, where $T$ satisfies (1.1) and $K$ is a "polyhedral cone" in a finite-dimensional Banach space. (The cone $K^n$ in $\mathbb{R}^n$ is a simple example of a polyhedral cone; the general definition is given in Section 2.) In work in progress Nussbaum has shown that the above kinds of results can be used to generalize theorems of Hirsch [17, 18] Krasnosel'skii [19] and Smith [2] concerning concave, cooperative differential equations. See, also, Krause and Ranft [1]. In order to apply our results, we need to determine whether the map in question satisfies (1.1) or (1.2). Here we shall be content to study the case that $T: \mathring{K}^n \to \mathring{K}^n$ and $T$ is locally Lipschitz, so $T$ has a Fréchet derivative $T'x$ for almost all $x$ in $\mathring{K}^n$. We shall give simple conditions on $T'x$ which insure that $T$ satisfies (1.1) or (1.2).

## 2. METRIC PRELIMINARIES

Let $(X, d)$ be a complete metric space. A mapping $T: X \to X$ will be called *nonexpansive* (with respect to $d$) if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$, and $T$ will be called *contractive* if

$$d(Tx, Ty) < d(x, y) \qquad \text{for all } x, y \in X,\ x \neq y.$$

If $A$ is a bounded subset of $(X, d)$, Kuratowski [20] has defined the *measure of noncompactness* of $A$, $\alpha(A)$, by the formula

$$\alpha(A) = \inf\left\{r > 0: A = \bigcup_{i=1}^{n} A_i,\ n < \infty,\ \text{diameter } (A_i) \leq r\right\}.$$

Basic properties of the measure of noncompactness and references to the literature can be found in [21, Section 1]. Recall that a continuous map $T: X \to X$ is called *condensing* if

$$\alpha(T(A)) < \alpha(A) \qquad \text{for all bounded } A \subset X \qquad \text{with } \alpha(A) > 0.$$

Recall that $\alpha(A) = 0$ if and only if $\mathrm{cl}(A)$, the closure of $A$ in $(X, d)$, is compact; and from this one concludes that compact operators are condensing. Given $T: X \to X$, the *forward orbit* $\gamma^+(x; T)$ of $x \in X$ (with respect to $T$) is

$$\gamma^+(x; T) = \bigcup_{j=0}^{\infty} T^j x.$$

The *omega limit set* $\omega(x; T)$ of $x \in X$ (with respect to $T$) is

$$\omega(x; T) = \bigcap_{j \geq 1} \mathrm{cl}\left(\bigcup_{i \geq j} T^i x\right).$$

It is well known that $\omega(x; T)$ is the set of $z \in X$ such that there exists a sequence $m_i \to \infty$ with

$$z = \lim_{i \to \infty} T^{m_i} x.$$

If $T$ is obvious, we shall write $\gamma^+(x)$ or $\omega(x)$ instead of $\gamma^+(x; T)$ and $\omega(x; T)$. Recall that if $(X, d)$ is a complete metric space and $T: X \to X$ is a map such that $T^m$ is condensing for some

$m$, then $\gamma^+(x)$ has compact closure in $X$ if an only if $\gamma^+(x)$ is bounded. This result is known, but we sketch the proof. If $\gamma^+(x)$ is not bounded (so $\mathrm{cl}(\gamma^+(x))$ is not bounded), it is easy to see that $\mathrm{cl}(\gamma^+(x))$ is not compact (the open covering by balls of radius $j$ about a specified point $x_0 \in X$ has no finite subcovering). Conversely, if $\gamma^+(x) = A$ is bounded,

$$\alpha(\mathrm{cl}(\gamma^+(x))) = \alpha(\gamma^+(x)) = \alpha\left(\bigcup_{j=m}^{\infty} T^j x\right) = \alpha(T^m(A)).$$

If $\alpha(A) > 0$, the assumption that $T^m$ is condensing implies that

$$\alpha(T^m(A)) < \alpha(A),$$

which would contradict the above equation. Thus $\alpha(A) = 0$ and $\mathrm{cl}(\gamma^+(x))$ is compact.

Our first lemma is related to work of Edelstein [22].

Lemma 2.1. Let $(X, d)$ be a complete metric space and $T$ a continuous self-mapping of $X$ such that $T^m$ is contractive for some integer $m \geq 1$. Assume that for every $x \in X$, $\mathrm{cl}(\gamma^+(x; T))$ is compact if $\gamma^+(x; T)$ is bounded. (Recall that this condition will be satisfied if $T^p$ is condensing for some $p \geq 1$.) If, for at least one point $\xi \in X$, $\omega(\xi; T^m)$ is nonempty (in particular if $\gamma^+(\xi; T)$ is bounded) then $T$ has a unique fixed point $x_0$ and

$$\lim_{k \to \infty} T^k x = x_0 \qquad \text{for all } x \in X. \tag{2.1}$$

*Proof.* In the case where $\gamma^+(\xi; T)$ is bounded, $\mathrm{cl}(\gamma^+(\xi; T))$ is compact by assumption and the subset $\gamma^+(\xi; T^m)$ contains a converging sequence, showing that $\omega(\xi; T^m) \neq \emptyset$. Hence let $\omega(\xi; T^m) \neq \emptyset$ for some $\xi \in X$. If we take $S = T^m$, then $\lim_{i \to \infty} S^{k_i}\xi = x_0 \in X$ for some sequence $k_i \to \infty$. Contractivity of $S$ implies that the sequence defined by $a_k = d(S^k\xi, S^{k+1}\xi)$ is decreasing and converges to some $a \geq 0$. It follows that

$$a = \lim_{i \to \infty} g_{k_i} = d(x_0, Sx_0),$$

and also

$$a = \lim_{i \to \infty} a_{k_i+1} = \lim_{i \to \infty} d(S(S^{k_i}\xi), S^2(S^{k_i}\xi)) = d(Sx_0, S^2x_0).$$

Hence $d(x_0, Sx_0) = d(Sx_0, S^2x_0)$, and $Sx_0 = x_0$ by contractivity of $S$. Therefore

$$S(Tx_0) = T^{m+1}x_0 = T(Sx_0) = Tx_0,$$

and we must have $Tx_0 = x_0$, since a contractive map has at most one fixed point.

It remains to establish (2.1). Because we can write

$$d(T^j x, x_0) = d(T^{km}(T^r x), x_0), \qquad 0 \leq r < m$$

it suffices to prove that for $S = T^m$

$$\lim_{k \to \infty} d(S^k u, x_0) = 0 \qquad \text{for all } u \in X.$$

We have that the sequence defined for fixed $u$ by

$$a_k = d(S^k u, x_0) = d(S^k u, S^k x_0)$$

is decreasing and converges to some $a \geq 0$. Because by contractivity of $S$

$$d(T^j u, x_0) = d(S^k(T^r u), x_0) \leq \max\{d(T^r u, x_0) : 0 \leq r < m\} \qquad \text{for all } j \geq 0,$$

$\gamma^+(u; T)$ is bounded and has compact closure. Since $\gamma^+(u; S) \subset \gamma^+(u; T)$ we can take a subsequence $S^{k_i}u$ such that $\lim_{i \to \infty} S^{k_i}u = v \in X$ and $d(v, x_0) = \lim_{i \to \infty} a_{k_i} = a$. If $a > 0$, we have that

$$d(Sv, x_0) = d(Sv, Sx_0) < d(v, x_0) = a,$$

and by contractivity we obtain

$$d(S^{k_i+1}u, x_0) < a \qquad \text{for } i \text{ large.}$$

This contradicts the choice of $a$, so $a = 0$. ■

*Remark* 2.1. By a result in [23], which is stated for Banach spaces but with a proof valid for complete metric spaces, if $S: X \to X$ is nonexpansive and $\omega(\xi; S)$ is nonempty for some $\xi \in X$, then $S \mid \omega(\xi; S)$ is an isometry of $\omega(\xi; S)$ onto itself. This result, if applied to the situation of lemma 2.1, provides an alternative way of deriving $Sx_0 = x_0$ from $x_0 \in \omega(\xi; S)$.

Our next lemma is a slight variant of lemma 2.1, but it has proved quite useful for the kinds of examples studied in [6, 7]. It follows easily from [6, lemma 2.3, p. 66].

LEMMA 2.2. Let $(X, d)$ be a connected metric space and $T$ a continuous self-mapping of $X$. Assume that $T$ has a fixed point $x_0$ and that there exists an open neighborhood $U$ of $x_0$ such that

$$\lim_{k \to \infty} T^k x = x_0 \qquad \text{for all } x \in U. \tag{2.2}$$

Assume further that there exists an integer $m$ such that $T^m$ is nonexpansive. Then it follows that

$$\lim_{k \to \infty} T^k x = x_0 \qquad \text{for all } x \in X.$$

*Proof.* By assumption, $S = T^m$ is nonexpansive and $\lim_{k \to \infty} S^k x = x_0$ for all $x \in U$. Lemma 2.3 in [6] implies that $\lim_{k \to \infty} S^k x = x_0$ for all $x \in X$, and one can see that this implies $\lim_{k \to \infty} T^k x = x_0$ for all $x \in X$. ■

*Remark* 2.2. Note that lemma 2.1 implies that (2.2) will be satisfied if there exists an open neighborhood $V$ of $x_0$ and an integer $m \geq 1$ such that $T^m \mid V$ is contractive and such that $\gamma^+(x; T)$ has compact closure for all $x \in V$.

Recall that a *cone* $K$ in a Banach space $(E, \|\cdot\|)$ is a closed, convex set such that $tK \subset K$ for all $t \geq 0$ and $K \cap (-K) = \{0\}$. A cone $K$ induces a *partial ordering* by $x \leq y$ iff $y - x \in K$. If $\mathring{K}$, the interior of $K$, is nonempty, we shall write $x < y$ iff $y - x \in \mathring{K}$ (note that sometimes the notation $x \ll y$ is used instead of $x < y$). $K - \{0\}$ denotes $K$ without 0.

A cone $K$ is called a *normal cone* if there exists a constant $M$ such that

$$0 \leq x \leq y \qquad \text{implies } \|x\| \leq M\|y\|.$$

If $K$ is normal, it is known (see [24]) that $E$ can be given an equivalent norm $|\cdot|$ such that

$$0 \le x \le y \qquad \text{implies } |x| \le |y|.$$

A mapping $T: K \to K$, $K$ a cone, we shall call *monotone* or *order-preserving* if

$$0 \le x \le y \qquad \text{implies } Tx \le Ty. \tag{2.3}$$

Thus, the above norm $|\cdot|$ is monotone.

One can define an equivalence relation on a cone $K$ by $x \sim y$ iff there exists $\alpha > 0$ such that $\alpha^{-1}x \le y \le \alpha x$. The equivalence classes under this equivalence relation are called the *parts* of $K$. If $K$ has a nonempty interior, $\mathring{K}$ is a part of $K$. If $J$ is any subset (possibly empty) of $\{j: 1 \le j \le n\}$ and $K_J^n$ denotes the set of vectors $x \in K^n$ such that $x_j = 0$ for all $j \in J$ and $x_j > 0$ for all $j \notin J$, then $K_J^n$ is a part of $K^n$. If $C$ is a part of a cone $K$ and $x, y \in C$, the *part metric* on $C$ is defined by

$$p(x, y) = \inf\{\log \alpha : \alpha^{-1}x \le y \le \alpha x\}.$$

$p$ can be extended to $K$ by setting $p(x, y) = \infty$ for $x$ and $y$ not lying in the same equivalence class.

It is easy to verify that $p$ is a metric on $C$, and it is proved [6, proposition 1.12, p. 34], that $(C, p)$ is metrically convex (see [6, pp. 24–37] for definitions and further references). Thompson [8] proved that $(C, p)$ is a complete metric space if $K$ is normal (see [4, 5, 9] for further results in this respect).

One verifies easily that for $K^n$ the part metric on the part $\mathring{K}^n = K_\phi^n$ becomes

$$p(x, y) = \max\{|\log y_j - \log x_j| : 1 \le j \le n\}.$$

Hence $\Phi(x) = \log x = (\log x_1, \dots, \log x_n)$ defines an isometry of $(\mathring{K}^n, p)$ onto $(\mathbb{R}^n, \|\cdot\|_\infty)$. For later work we shall need some simple relationships between the part metric and the distance defined by the norm on the cone. Related results are given in [6, Section 1]. See also [4, 5], where the relationship between the part metric and the topology on a locally convex vector space is considered.

LEMMA 2.3. (i) Let $K$ be a cone with nonempty interior in a Banach space $E$. If $x, y \in \mathring{K}$ and $r > 0$ is a number such that the closed norm ball of radius $r$ and center $x$ and $y$, respectively, is contained in $K$, then

$$p(x, y) \le \log\left(1 + \frac{\|x - y\|}{r}\right). \tag{2.4}$$

(ii) If $K$ is a normal cone in a Banach space $E$ and the norm $\|\cdot\|$ is monotone on $K$, then for $x, y \in K - \{0\}$,

$$\|x - y\| \le (2e^{p(x,y)} - e^{-p(x,y)} - 1)\min\{\|x\|, \|y\|\}. \tag{2.5}$$

*Proof.* (i) For $x = y$, (2.4) is trivial, so assume $x \ne y$. From the equation

$$\frac{r}{\|x - y\|}\left(\left(1 + \frac{\|x - y\|}{r}\right)x - y\right) = x + r\left(\frac{x - y}{\|x - y\|}\right) \in K,$$

it follows that

$$y \leq \left(1 + \frac{\|x - y\|}{r}\right)x \equiv \alpha x.$$

Interchanging the roles of $x$ and $y$ gives $x \leq \alpha y$, and we obtain (2.4) by using the definition of the part metric.

(ii) If $x$ is not equivalent to $y$, $p(x, y) = \infty$ and (2.5) is trivial. If $x \sim y$, write $\alpha = e^{p(x,y)}$, so $(1/\alpha)x \leq y \leq \alpha$. It follows that

$$-(\alpha - 1)x \leq x - y \leq \left(1 - \frac{1}{\alpha}\right)x \quad \text{and} \quad 0 \leq x - y + (\alpha - 1)x \leq \left(\alpha - \frac{1}{\alpha}\right)x.$$

Using the monotonicity of the norm, we obtain

$$\|x - y\| - (\alpha - 1)\|x\| \leq \|(x - y) + (\alpha - 1)x\| \leq \left(\alpha - \frac{1}{\alpha}\right)\|x\| \quad \text{or}$$

$$\|x - y\| \leq \left(2\alpha - \frac{1}{\alpha} - 1\right)\|x\|.$$

Interchanging the roles of $x$ and $y$ gives

$$\|x - y\| \leq \left(2\alpha - \frac{1}{\alpha} - 1\right)\|y\|,$$

and we obtain (2.5). ■

## 3. LIMIT SET BEHAVIOR FOR NONEXPANSIVE AND CONTRACTIVE MAPPINGS

The next lemma presents our first variant of a limit set trichotomy. It holds under rather weak assumptions, in that the cone is admitted to be infinite-dimensional, the self-mapping is only required to be nonexpansive and no assumptions about the mapping's behavior on the boundary of the cone are made.

LEMMA 3.1. Let $K$ be a normal cone with interior $\mathring{K} \neq \emptyset$ in a Banach space $(E, \|\cdot\|)$. Let $T$ be a continuous self-mapping of the space $(K, \|\cdot\|)$ with $T(\mathring{K}) \subset \mathring{K}$ and such that for some integer $r \geq 1$, $T^r$ is nonexpansive for the part metric on $\mathring{K}$. Then at least one of the following cases holds.

Either:

(i) for every $x \in K$, $\gamma^+(x; T)$ does not have compact closure in the norm topology;

(ii) for every $x \in \mathring{K}$, $\lim_{k \to \infty} \|T^k x\| = 0$;

(iii) there exists some $x \in K$ such that for every $s \geq 1$, $\omega(x; T^s)$, the omega limit set in the norm topology, contains a point of $K - \{0\}$.

If case (i) holds, but not case (iii), for $s = 1$, and if $T$ maps norm-bounded subsets of $\mathring{K}$ to sets which have compact closure in $K$ for the norm topology, then

$$\lim_{k \to \infty} \|T^k x\| = \infty \quad \text{for all } x \in \mathring{K}. \tag{3.1}$$

*Proof.* Since $K$ is normal, we can assume the norm is monotone on $K$. Assume that neither (i) nor (ii) hold, then there exists $x_0 \in \mathring{K}$ such that $\gamma^+(x_0; T)$ has compact closure. Furthermore, for $s \geq 1$ there exists $\xi \in \mathring{K}$, $\delta > 0$ and a sequence $k_i \to \infty$ such that

$$\delta \leq \|T^{sk_i}\xi\| \qquad \text{for all } i.$$

Otherwise,

$$\lim_{k\to\infty} \|T^{sk}y\| = 0 \qquad \text{for all } y \in \mathring{K},$$

and hence

$$\lim_{k\to\infty} \|T^{sk}(T^j x)\| = 0 \qquad \text{for all } x \in \mathring{K}, \text{ all } 0 \leq j < s,$$

which contradicts the assumption that (ii) does not hold.

If we define $\alpha$ by

$$\alpha = \max\{p(T^j x_0, T^j \xi) : 0 \leq j < r\}$$

then because $T^r$ is nonexpansive for the part metric $p$, we find that

$$p(T^j x_0, T^j \xi) \leq \alpha \qquad \text{for all } j \geq 0.$$

Lemma 2.3 now implies that

$$\|T^j x_0 - T^j \xi\| \leq (2e^{\alpha} - e^{-\alpha} - 1)\|T^j x_0\| \qquad \text{for all } j \geq 0.$$

Taking the construction of $\xi$ into account we obtain, for some $\varepsilon > 0$,

$$\varepsilon \leq \|T^{sk_i} x_0\| \qquad \text{for all } i.$$

Because we know that $\gamma^+(x_0; T)$, and hence also $\gamma^+(x_0; T^s)$, has compact closure, we may, by taking a subsequence, assume as well that there exists $y_s \in K$, $\|y_s\| \geq \varepsilon$, such that

$$\lim_{i\to\infty} \|T^{sk_i} x_0 - y_s\| = 0.$$

Thus, case (iii) holds.

It remains to prove (3.1) under the additional assumptions made. If not, there exists $x_0 \in \mathring{K}$, a real number $M > 0$ and an increasing sequence $k_i \to \infty$ such that

$$\|T^{k_i} x_0\| \leq M \qquad \text{for all } i \geq 0.$$

From (i) it follows that $\gamma^+(x_0; T)$ is unbounded. Otherwise, $T(\gamma^+(x_0; T))$ has compact closure and because of

$$\gamma^+(x_0; T) \subset \{x_0\} \cup T(\gamma^+(x_0; T))$$

$\gamma^+(x_0; T)$ has compact closure which contradicts (i). Because $\gamma^+(x; T)$ is unbounded, there exists for each $k_i$ *a* smallest integer $j_i > k_i$ such that $\|T^{j_i} x_0\| > M$. We obtain that

$$\|T^{j_i - 1} x_0\| \leq M \qquad \text{and} \qquad T^{j_i} x_0 = T(T^{j_i - 1} x_0).$$

It follows by the additional assumption for $T$ that (by taking a further subsequence) we can assume that $\lim_{i\to\infty} T^{j_i} x_0$ exists in $K - \{0\}$. This contradicts the assumption that (iii) does not hold for $s = 1$. ∎

In the next theorem we continue to work in a general Banach space, but we obtain a stronger limit set trichotomy by strengthening the assumptions on the self-mapping.

THEOREM 3.1. Let $K$ be a normal cone with nonempty interior in a Banach space $(E, \|\cdot\|)$. Let $T$ be a continuous self-mapping of the space $(K, \|\cdot\|)$ with $T(\mathring{K}) \subset \mathring{K}$. Assume further that for some integer $m \geq 1$, $T^m$ is contractive for the part metric on $\mathring{K}$ and that $T^m(K - \{0\}) \subset \mathring{K}$. For all bounded orbits $\gamma^+(x; T)$, $x \in K$, assume that the closure is compact in the norm topology. Then the following trichotomy holds.
Either:

(i) $\gamma^+(x; T)$ is unbounded in norm for all $x \in K - \{0\}$; or

(ii) $\lim_{k \to \infty} \|T^k x\| = 0$ for all $x \in K$; or

(iii) there exists $x_0 \in \mathring{K}$ such that $Tx_0 = x_0$ and $\lim_{k \to \infty} \|T^k x - x_0\| = 0$ for all $x \in K - \{0\}$.

If case (i) holds and if $T$ is compact in the norm topology, then in fact we have

$$\lim_{k \to \infty} \|T^k x\| = \infty \qquad \text{for all } x \in K - \{0\}. \tag{3.1}$$

*Proof.* Under our given assumptions, we can apply lemma 3.1 with $r = s = m$. It follows easily that cases (i) and (ii) of theorem 3.1 are implied by the corresponding cases of lemma 3.1. Thus, if we assume that cases (i) and (ii) of theorem 3.1 do not hold, then case (iii) of lemma 3.1 holds, and there exists $\xi \in \mathring{K}$, $\eta \in K - \{0\}$ and a sequence $k_i \to \infty$ such that, writing $S = T^m$,

$$\lim_{i \to \infty} \|S^{k_i}\xi - \eta\| = 0.$$

Since $S(K - \{0\}) \subset \mathring{K}$, we have $x_0 = S\eta \in \mathring{K}$ and

$$\lim_{i \to \infty} \|S^{k_i+1}\xi - x_0\| = 0.$$

It is known, or one can easily derive from lemma 2.3, that the norm topology on $\mathring{K}$ and the topology induced by the part metric $p$ on $\mathring{K}$ are the same. (Recall, we can assume the norm is monotone on $K$.) Thus we conclude that

$$\lim_{i \to \infty} p(S^{k_i+1}\xi, x_0) = 0.$$

This shows that $\omega(\xi; T^m)$, taken in the metric space $(\mathring{K}, p)$, is nonempty. We want to apply lemma 2.1 to $(X, d) = (\mathring{K}, p)$. By using lemma 2.3 one can see that if $A \subset \mathring{K}$ is bounded in $(\mathring{K}, p)$, then $A$ is bounded in the norm and the norm closure of $A$ is contained in $\mathring{K}$; and if $A \subset \mathring{K}$ and $A$ is compact in $(\mathring{K}, \|\cdot\|)$, then $A$ is compact in $(\mathring{K}, p)$. It follows that if $x \in \mathring{K}$ and $\gamma^+(x; T)$ is bounded in $(\mathring{K}, p)$, then $\gamma^+(x; T)$ bounded in the norm, has a compact norm closure contained in $\mathring{K}$ and therefore has compact closure in $(K, p)$. Thus all assumptions of lemma 2.1 are satisfied. If we recall that $T^m(K - \{0\}) \subset \mathring{K}$ and switch to the norm topology, (2.1) yields case (iii) of theorem 3.1.

Note that in theorem 3.1, cases (i)–(iii) are mutually exclusive. Therefore, in theorem 3.1, the statement regarding (3.1) is implied by the corresponding statement of lemma 3.1. ■

*Remark* 3.1. In [22] Edelstein has given an example of an affine linear map $T: l_2 \to l_2$ such that $T$ is an isometry with respect to the $l_2$-norm and such that $\gamma^+(0; T)$ is unbounded and

$\omega(0; T) \neq \emptyset$. Thus the fact that $\gamma^+(x; T)$ is unbounded and $T$ is nonexpansive by no means insures, in general, that (3.1) is satisfied.

Simple sufficient conditions for nonexpansiveness of a mapping $T: \mathring{K} \to \mathring{K}$, for any cone $K$ in a Banach space $E$ with $\mathring{K} \neq \emptyset$, are given by the following properties of monotonicity and subhomogeneity (sublinearity), respectively:

$$\begin{aligned} &\text{(a)} \qquad x, y \in \mathring{K} \text{ and } x \leq y \text{ imply } Tx \leq Ty; \\ &\text{(b)} \qquad x \in \mathring{K} \text{ and } 0 < t < 1 \text{ imply } tTx \leq T(tx). \end{aligned} \tag{3.2}$$

It is not hard to see that (a) and (b) imply that $T$ is nonexpansive on $\mathring{K}$ with respect to the part metric and that, moreover, $T$ is contractive if in addition (b) holds with strict inequality. A similar statement holds with respect to properties obtained from (a) and (b) by order-reversal, i.e. that $Tx \geq Ty$ for $x, y \in \mathring{K}$, $x \leq y$ and that $t^{-1}Tx \geq T(tx)$ for $x \in \mathring{K}$, $0 < t < 1$, respectively (cf. the argument given in corollary 2.2 of [6] with respect to Hilbert's projective metric). With $K$ instead of $\mathring{K}$, mappings $T$ satisfying (a) as above and (b) with strict inequality are considered in [1]. Mappings $T$ satisfying (b) as above and (a) with strict inequality, i.e. $x, y \in K$, $x \leq y$, $x \neq y$ imply $Tx < Ty$, are considered in [3]. There [3, theorem 1] a convergence result for the iterates of $T$ is obtained which is related to theorem 3.1 in some way, but employs rather different assumptions. In [5] so-called ascending operators are considered, which satisfy a stronger condition than (3.2) in that $tx \leq y$ implies $\varphi(t)Tx \leq Ty$ for some continuous self-mapping $\varphi$ of the unit interval with $t < \varphi(t)$ for all $0 < t < 1$. There [5, theorem 4] a convergence result for the iterates of an ascending operator is obtained which is also related to theorem 3.1, without assuming, however, that bounded orbits have compact closure. From the above discussion it follows in particular that any bounded linear map $L$ on a Banach space containing a normal cone $K$ with $\mathring{K} \neq \emptyset$ such that $L(K) \subset K$ and $L(\mathring{K}) \subset \mathring{K}$ is nonexpansive with respect to the part metric. Hence lemma 3.1 applies to such a mapping $L$. It can be seen from simple examples that all three cases in the lemma can occur. (For linear mappings see also remark 3.2 below.) Neither property (a) nor property (b), however, is necessary for a mapping to be nonexpansive for the part metric, as can be seen from simple examples.

We now want to go more deeply into the case in which the map $T^m$ is only nonexpansive with respect to the part metric $p$. Since the situation here is far more subtle than in the case of contractivity, we now suppose that the underlying Banach space $E$ is finite-dimensional. Assume that $K$ is a cone in $E$ with $\mathring{K} \neq \emptyset$, that $T: \mathring{K} \to \mathring{K}$ is nonexpansive for $p$ and that $T$ has no fixed point in $\mathring{K}$. Then it is proved [6, theorem 4.4, p. 117] that for every $x \in \mathring{K}$, $T^k x$ approaches $\partial K$ in the sense that, given any closed bounded set $C \subset \mathring{K}$ and any $x \in \mathring{K}$, there exists an integer $m = m(x, C)$ such that $T^k x \notin C$ for $k > m$. It follows that $\omega(x; T)$, the omega limit set in $(\mathring{K}, p)$, is empty for all $x \in \mathring{K}$. (Of course, the omega limit set $\tilde{\omega}(x; T)$ in $(K, \|\cdot\|)$ may very well be nonempty for $x \in \mathring{K}$.)

If $K$ is a cone in a finite-dimensional Banach space $E$, $K$ is called *polyhedral*, if there exist continuous linear functionals $\varphi_j \in E^*$, $1 \leq j \leq m$, such that

$$K = \{x \in E: \varphi_j(x) \geq 0 \text{ for } 1 \leq j \leq m\}. \tag{3.3}$$

Obviously $K^n \subset \mathbb{R}^n$ is polyhedral with $m = n$. If $K$ is a polyhedral cone with nonempty interior $\mathring{K}$, it follows, as for $\mathring{K}^n$ in Section 2 (see [10] for a proof), that there is an isometry $\psi: (\mathring{K}, p) \to (\mathbb{R}^m, \|\cdot\|_\infty)$ of $(\mathring{K}, p)$ into $\mathbb{R}^m$ with the sup norm $\|\cdot\|_\infty$, where $m$ is as in (3.3). If $D$

is a compact subset of $\mathring{K}$ and $T: D \to D$ is nonexpansive with respect to the part metric $p$, then it is proved in [10] that for each $x \in D$, there exists a minimal integer $\nu = \nu(x)$ such that

$$\lim_{k \to \infty} T^{k\nu}x = \xi, \qquad \text{where } T^{\nu}\xi = \xi.$$

It is also proved that there exists an integer $N = N(m)$ such that $\nu(x) \leq N(m)$ for all $x \in D$. An upper bound for $N(m)$ is given in [10] and a different argument in [13] implies a somewhat better result, namely,

$$N(m) \leq 2^m(m!).$$

However, this estimate is far from the best possible. It is known (see [10, 14]) that simple examples force $N(m) \geq 2^m$, and it is conjectured in [10] that $N(m) = 2^m$. Unpublished joint work of Lyons and Nussbaum has proved the conjecture for $m = 1, 2$ and 3. The case $m = 3$ already seems nontrivial; the result for $m = 2$ has also been obtained in [12].

With the aid of the above theorems we can now easily give the following limit set dichotomy.

THEOREM 3.2. Let $K$ be a polyhedral cone with nonempty interior in a finite-dimensional Banach space $E$. Assume that $T: \mathring{K} \to \mathring{K}$ is a continuous map and that there is an integer $r \geq 1$ such that $T^r$ is nonexpansive with respect to the part metric $p$. Then exactly one of the following two possibilities holds:

(i) $T^r$ has a fixed point in $\mathring{K}$. For every $x \in \mathring{K}$, there exists a minimal integer $\nu = \nu(x) \geq 1$ such that

$$\lim_{k \to \infty} T^{k\nu}x = \xi \in \mathring{K} \qquad \text{and} \qquad T^{\nu}\xi = \xi.$$

Furthermore, if $K$ is defined by $m$ continuous linear functionals as in (3.3), then

$$\nu = \nu(x) \leq r2^m(m!) \qquad \text{for all } x \in \mathring{K} \text{ and}$$

$$\nu = \nu(x) \leq r2^m \qquad \text{for all } x \in \mathring{K} \text{ if } m = 1, 2 \text{ and } 3;$$

(ii) $T^r$ does not have a fixed point in $\mathring{K}$. For every $x \in \mathring{K}$ and every closed, bounded set $C \subset \mathring{K}$ (in the norm topology) there exists an integer $n(x, C)$ such that $T^k x \notin C$ for all $k > n(x, C)$.

*Proof.* By assumption, the map $T^r$ is nonexpansive with respect to the part metric $p$, so theorem 3.2 follows in a straightforward way from the previously cited results (see theorem 4.4 in [6, 10]). Details are left to the reader. ■

*Remark* 3.2. If $A$ is a nonnegative matrix which has no zero rows, then for the induced linear mapping $Lx = A \cdot x$ (considering elements of $\mathbb{R}^n$ as column vectors) it holds that $L(K^n) \subset K^n$ and $L(\mathring{K}^n) \subset \mathring{K}^n$. It has been pointed out, that such a bounded linear map $L$, which possesses the properties of monotonicity (a) and subhomogeneity (b), is nonexpansive for the part metric on $\mathring{K}$. Thus mappings induced by permutation matrices provide an illustration of case (i) of theorem 3.2. If $K = K^2 \subset \mathbb{R}^2$ and $Tx = (x_1, \frac{1}{2}x_2)$ for $x = (x_1, x_2) \in \mathring{K}$, we obtain an illustration of case (ii): for every $x \in \mathring{K}$, $\lim_{k \to \infty} T^k x = (x_1, 0) \notin \mathring{K}$. A slightly more sophisticated example for $K = K^3 \subset \mathbb{R}^3$ is provided by $Tx = (x_2, x_1, \frac{1}{2}x_3)$, $x = (x_1, x_2, x_3) \in K$. For $x \in \mathring{K}$ one sees that

$$\lim_{k \to \infty} T^{2k}x = (x_1, x_2, 0) \qquad \text{and} \qquad \lim_{k \to \infty} T^{2k+1}x = (x_2, x_1, 0).$$

In these examples it is not true that, for some $m \geq 0$, $T^m(K - \{0\}) \subset \mathring{K}$. For a more natural example which arises in the study of so-called *DAD-theorems* see [7, Section 4]. As these examples suggest, the assumption in theorem 3.1 that $T^m(K - \{0\}) \subset \mathring{K}$ is not always a reasonable one. For polyhedral cones, theorem 3.2 provides a more flexible tool.

Our next corollary follows immediately from theorem 3.2.

COROLLARY 3.1. Let assumptions and notation be as in theorem 3.2. In addition assume that if $T^k x = x$ for any $k \geq 1$ and any $x \in \mathring{K}$, then $Tx = x$. Then either

(i) $T$ has a fixed point in $K$ and for every $x \in \mathring{K}$,

$$\lim_{k \to \infty} T^k x = \xi \text{ exists} \qquad \text{and} \qquad T\xi = \xi, \quad \text{or}$$

(ii) $T$ has no fixed point in $\mathring{K}$ and for every $x \in \mathring{K}$ and every closed, bounded set $C \subset \mathring{K}$ there exists an integer $n = n(x, C)$ such that

$$T^k x \notin C \qquad \text{for } k > n(x, C).$$

Our next theorem presents a limit set trichotomy in a situation where the self-mapping is only known to be nonexpansive for the part metric but the cone is polyhedral in finite dimensions.

THEOREM 3.3. Let notation and assumptions be as in theorem 3.2. In addition assume that $T$ extends continuously to a map of $K$ to $K$ and that $T^r(K - \{0\}) \subset \mathring{K}$ for some $r \geq 1$. Then the following trichotomy holds.
Either:

(i) $\lim_{k \to \infty} \|T^k x\| = \infty$ for all $x \in K - \{0\}$; or

(ii) $\lim_{k \to \infty} \|T^k x\| = 0$ for all $x \in K$; or

(iii) $T^r$ has a fixed point in $\mathring{K}$ and the properties of case (i) of theorem 3.2 are satisfied.

*Proof.* The theorem follows easily from theorem 3.2 and lemma 3.1. Assuming that $T^r$ has no fixed point in $\mathring{K}$ we show cases (i) and (ii) above. Because of $T^r(K - \{0\}) \subset \mathring{K}$ it suffices to show these cases for all $x \in \mathring{K}$. If $\xi \in \omega(x; T)$ (where the limit set is taken in $(K, \|\cdot\|)$), then by case (ii) of theorem 3.2 and $T^r(K - \{0\}) \subset \mathring{K}$, $\xi$ cannot be different from zero. Therefore in lemma 3.1 case (iii) is impossible. Since the map $T$ is automatically compact, lemma 3.1 implies cases (i), (ii) of theorem 3.3. ■

## 4. CONDITIONS WHICH INSURE NONEXPANSIVENESS OR CONTRACTIVENESS

The results of Section 3 are only useful if one has verifiable conditions which insure that a map $T: K \to K$ is nonexpansive or contractive with respect to the part metric $p$. Some such conditions have been given already by (3.2). If $S: \mathring{K} \to \mathring{K}$ and $T: \mathring{K} \to \mathring{K}$ are both nonexpansive with respect to $p$, it is not hard to check that $T \circ S$ and $T + S$ are both contractive if $T$ is contractive and $S$ is nonexpansive. Thus one obtains further examples of nonexpansive and contractive maps.

If one restricts attention to $K^n \subset \mathbb{R}^n$, more structure is available. If $x$ and $y$ are vectors in $\mathbb{R}^n$, we can take $x \wedge y \in \mathbb{R}^n$, the minimum of $x$ and $y$:

$$x \wedge y = z, \qquad z_i = x_i \wedge y_i = \min(x_i, y_i), \qquad 1 \leq i \leq n.$$

Similarly, one can define $x \vee y \in \mathbb{R}^n$, the maximum of $x$ and $y$. If $K = K^n$ and $T: \mathring{K} \to \mathring{K}$ and $S: \mathring{K} \to \mathring{K}$ are nonexpansive with respect to $p$ (contractive with respect to $p$), then one can easily see that

$$f(x) = T(x) \wedge S(x) \qquad \text{and} \qquad g(x) = T(x) \vee S(x)$$

are both nonexpansive (contractive) with respect to $p$. Interesting examples (see [6, p. 131; 10, remarks 8, 11]) can be obtained by using this observation.

In this section we wish to give necessary and sufficient conditions which insure that a locally Lipschitz map $f: \mathring{K}^n \to \mathring{K}^n$ is nonexpansive or contractive with respect to the part metric. We remind the reader that any locally Lipschitzian map $g$ from an open subset $D$ of $\mathbb{R}^n$ to $\mathbb{R}^n$ is Fréchet differentiable almost everywhere. Conditions involving properties of differentiability have been used in [1, 2] to obtain order-theoretic properties of the mapping $T$ (as, e.g. (3.2)). With the criteria given below, one can easily see that the theorems of Section 3 generalize results in [1, 2].

LEMMA 4.1. Let $D$ be an open, convex subset of $\mathbb{R}^n$ and $g: D \to \mathbb{R}^n$ a locally Lipschitzian map such that

$$\mu(x) = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} \left| \frac{\partial g_i}{\partial x_j}(x) \right| \right) \leq c \tag{4.1}$$

for almost all $x \in D$. Then for all $x, y \in D$ we have

$$\|g(x) - g(y)\|_\infty \leq c\|x - y\|_\infty, \tag{4.2}$$

where $\|\cdot\|_\infty$ denotes the sup norm. If $g$ is $C^1$ and strict inequality holds in (4.1) for all $x$, then strict inequality holds in (4.2) for all $x, y$ with $x \neq y$.

*Proof.* If we consider elements of $\mathbb{R}^n$ as column vectors, an $n \times n$ matrix $A = (a_{ij})$ induces a linear map $L: \mathbb{R}^n \to \mathbb{R}^n$ by

$$L(x) = Ax.$$

It is well known that the norm of $L$, $\|L\|_\infty$, as a map from $(\mathbb{R}^n, \|\cdot\|_\infty)$ into itself is given by

$$\|L\|_\infty = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} |a_{ij}| \right).$$

If $g: D \to \mathbb{R}^n$ is $C^1$ and $x, y \in D$ and we write $x_t = (1 - t)x + ty$ for $0 \leq t \leq 1$, we have

$$g(y) - g(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} g(x_t)\,\mathrm{d}t = \int_0^1 g'(x_t)(y - x)\,\mathrm{d}t.$$

This gives (using (4.1))

$$\|g(y) - g(x)\|_\infty \leq \int_0^1 \|g'(x_t)\|_\infty \|y - x\|_\infty\,\mathrm{d}t \leq c\|y - x\|_\infty. \tag{4.3}$$

If $x \neq y$ and strict inequality holds in (4.1) for all $x$, we obtain strict inequality in (4.3).

It remains to consider the case that $g$ is only locally Lipschitz. To prove (4.2) for a given $x, y \in D$ we can assume that $x = 0$. Let $V = \{z \in \mathbb{R}^n : y \cdot z = 0\}$, where $y \cdot z$ denotes the usual scalar product on $\mathbb{R}^n$. For $\delta > 0$, let $U_\delta = \{z \in V : \|z\| < \delta\}$ and choose $\delta > 0$ so small that $\{z + ty : z \in U_\delta, 0 \leq t \leq 1\} = O_\delta$ is contained in $D$. We know that for almost all points $w \in O_\delta$, $g'(w)$ exists. Since $O_\delta$ is diffeomorphic to $\{(z, t) : z \in U_\delta, 0 \leq t \leq 1\} = W_\delta$, it follows that for almost all $(z, t) \in W_\delta$, $g'(z + ty)$ exists. An application of Fubini's theorem now implies that for almost all $z \in U_\delta$, $g'(z + ty)$ exists for almost all $t \in [0, 1]$. Select a sequence $z^j \in V$ such that $z^j \to 0$ and $g'(z^j + ty)$ exists for almost all $t \in [0, 1]$. Because $g$ is Lipschitz, the map $t \to g'(z^j + ty)$ is absolutely continuous and

$$g(z^j + y) - g(z^j) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} g(z^j + ty)\,\mathrm{d}t = \int_0^1 g'(z^j + ty)(y)\,\mathrm{d}t.$$

It follows as in the $C^1$ case that

$$\|g(z^j + y) - g(z^j)\|_\infty \leq c\|y\|_\infty.$$

Taking the limit as $j \to \infty$ gives the desired result. ■

If $x, y \in \mathring{K}^n$ and $0 < t < 1$, we use the notation $x^{1-t}y^t$ to denote the vector whose $i$th component, $1 \leq i \leq n$, is $x_i^{1-t}y_i^t$. If $G$ is a subset of $K^n$, we say that $G$ is *logarithmically convex* if, whenever $x, y \in G$, it follows that $x^{1-t}y^t \in G$ for $0 < t < 1$.

THEOREM 4.1. Let $G$ be an open, logarithmically convex subset of $\mathring{K}^n \subset \mathbb{R}^n$ and $T: G \to \mathring{K}^n$ a locally Lipschitzian map. If

$$\sum_{j=1}^n u_j \cdot \left|\frac{\partial T_i}{\partial u_j}(u)\right| \leq cT_i(u) \tag{4.4}$$

for almost all $u \in G$ and for $1 \leq i \leq n$, then

$$p(Tu, Tv) \leq cp(u, v) \qquad \text{for all } u, v \in G, \tag{4.5}$$

where $p$ denotes the part metric. If $T$ is $C^1$ and the strict inequality holds in (4.4) for all $u \in G$, then the strict inequality holds in (4.5) for all $u \neq v$.

*Proof.* As noted in Section 2, $\Phi(x) = \log x$ is an isometry of $(\mathring{K}^n, p)$ onto $(\mathbb{R}^n, \|\cdot\|)$ and $\Phi^{-1}(y) = \mathrm{e}^y$. (Here $\log x$ and $\mathrm{e}^y$ are interpreted coordinate-wise.) It follows that if we define

$$g(y) = \log(T(\mathrm{e}^y)), \qquad g: \log(G) = \hat{G} \to \mathbb{R}^n,$$

theorem 4.1 will follow if we can prove that

$$\|g(x) - g(y)\|_\infty \leq c\|x - y\|_\infty \qquad \text{for all } x, y \in \hat{G} \tag{4.6}$$

with strict inequality holding under appropriate further assumptions. By lemma 4.1, it suffices to prove that

$$\max_{1 \leq i \leq n}\left(\sum_{j=1}^n \left|\frac{\partial g_i}{\partial y_j}(y)\right|\right) \leq c \qquad \text{for almost all } y \in \hat{G}. \tag{4.7}$$

However, writing $u = e^y$, a calculation gives

$$\frac{\partial g_i}{\partial y_j}(y) = \left(\frac{u_j}{T_i u}\right)\left(\frac{\partial T_i}{\partial u_j}(u)\right), \tag{4.8}$$

so (4.7) follows from (4.4). The case of strict inequality also follows immediately. ∎

*Remark* 4.1. If $T: G \to \mathring{K}^n$ is order-preserving and locally Lipschitzian on an open set $G \subset \mathring{K}^n$ (so $(\partial T_i/\partial u_j)(u) \geq 0$ almost everywhere on $G$), (4.4) is satisfied for $1 \leq i \leq n$ and almost all $u \in G$ iff

$$T'(u)(u) \leq cTu \qquad \text{for almost all } u \in G. \tag{4.9}$$

If, for almost all $u \in G$, there exists $\delta = \delta_u > 0$ such that

$$T(tu) \geq tTu \qquad \text{for } 1 - \delta_u \leq t \leq 1, \tag{4.10}$$

(4.9) is satisfied almost everywhere for $c = 1$. To see this, note that if $T'u$ exists and $1 - \delta_u \leq t < 1$ we have

$$\lim_{t \to 1^-} \frac{Tu - T(tu)}{1 - t} = T'(u)(u) \leq \lim_{t \to 1^-} \frac{Tu - tTu}{1 - t} = Tu.$$

(See also lemma 1 in [1].)

If $A = (a_{ij})$ is an $n \times n$ matrix, we shall write $|A|$ for the $n \times n$ matrix whose $i, j$ entry is $|a_{ij}|$. It is known (see [25, Chapter 1]) that the spectral radius of $A$ is less than or equal to the spectral radius of $|A|$. Also, (4.4) is equivalent to

$$|T'(u)| < cTu \qquad \text{for almost all } u \in G.$$

An $n \times n$ matrix $B$ with nonnegative entries is called *primitive* if there exists an integer $m \geq 1$ such that $B^m$ has all positive entries. If $B$ is primitive, $Bu \leq u$ for some $u \in K - \{0\}$ and $Bu \neq u$, it is known [25, Chapter 1] that the spectral radius of $B$, $r(B)$, is less than one.

COROLLARY 4.1. Let $T: \mathring{K}^n \to \mathring{K}^n$ be a locally Lipschitzian map. Assume that for almost all $u \in \mathring{K}^n$ and for $1 \leq i \leq n$,

$$\sum_{j=1}^{n} u_j \cdot \left|\frac{\partial T_i}{\partial u_j}(u)\right| \leq T_i u. \tag{4.11}$$

Then $T$ is nonexpansive with respect to the part metric $p$. Furthermore, exactly one of the following two possibilities holds:

(i) $T$ has a fixed point in $\mathring{K}^n$. For every $x \in \mathring{K}^n$ there exists a minimal integer $\nu = \nu(x)$ such that

$$\lim_{k \to \infty} T^{k\nu}x = \xi \text{ exists and } T^\nu \xi = \xi.$$

Moreover, $\nu = \nu(x)$ satisfies

$$\nu(x) \leq 2^n n! \text{ for all } n \qquad \text{and} \qquad \nu(x) \leq 2^n \text{ for } 1 \leq n \leq 3.$$

(ii) $T$ has no fixed point in $K^n$. For every closed bounded set $C \subset K^n$ and every $x \in \mathring{K}^n$, there exists $n(x, C)$ such that $T^k x \notin C$ for $k > n(x, C)$.

*Proof.* This is immediate from theorems 3.2 and 4.1. ■

COROLLARY 4.2. Let notation and assumptions be as in corollary 4.1. In addition, assume that if $Tu = u$ for some $u \in \mathring{K}^n$, then $T$ is $C^1$ near $u$, $|T'u|$ is primitive and the strict inequality holds in (4.11) for some $i$, $1 \leq i \leq n$. Then $T$ has at most one fixed point in $\mathring{K}^n$, and if $T$ has a fixed point $u \in \mathring{K}^n$,

$$\lim_{k \to \infty} T^k x = u \qquad \text{for all } x \in K^n.$$

*Proof.* If $T$ has a fixed point $u \in \mathring{K}^n$, we know that

$$|T'u|u \leq u, \tag{4.12}$$

and inequality does not hold in (4.12). It follows from our earlier remark 4.1 that

$$r(T'u) \leq r(|T'u|) < 1,$$

where $r(A)$ denotes the spectral radius of a matrix $A$. Thus there is an open neighborhood $U$ of $u$ such that

$$\lim_{k \to \infty} T^k x = u \qquad \text{for all } x \in U.$$

Since $T$ is nonexpansive with respect to $p$, lemma 2.2 implies $\lim_{k \to \infty} T^k x = u$ for all $x \in \mathring{K}^n$. ■

Corollary 4.2 is closely related to theorem 3.2 of [6].

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