Generalizing the Krein-Rutman Theorem, Measures of Noncompactness and the Fixed Point Index

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Abstract

If $L : Y \to Y$ is a bounded linear map on a Banach space Y, the "radius of the essential spectrum" or "essential spectral radius" $\rho(L)$ of L is well-defined and there are well-known formulas for $\rho(L)$ in terms of measures of noncompactness. Now let $C \subset D$ be complete cones in a normed linear space $(X, \|\cdot\|)$ and $f : C \to C$ a continuous map which is homogeneous of degree one and preserves the partial ordering induced by D. We prove (see Section 2) that various obvious analogs of the formulas for the essential spectral radius for the case $f : C \to C$ have serious defects, even when f is linear on C. We propose (see equation (3.5)) a definition for $\rho_C(f)$, the "cone essential spectral radius of f," which avoids these difficulties. If $\widetilde{r}_C(f)$ denotes the (Bonsall) cone spectral radius of f, we conjecture (see Conjecture 4.1) that if $\rho_C(f) < \widetilde{r}_C(f)$, then there exists $u \in C \setminus \{0\}$ with $f(u) = ru$ where $r := r_C(f)u$. If f satisfies certain additional conditions (for example, if f is a compact perturbation of a map which is linear on C), we obtain the conclusion of the conjecture; but in general we observe (Remark 4.7) that the conjecture is intimately related to old and difficult conjectures in asymptotic fixed point theory. In Section 5 we briefly discuss extensions of generalized max-plus operators which were our original motivation and for which Conjecture 4.1 is already nontrivial.

Key Words: Nonlinear Krein-Rutman theorems; fixed points of cone maps; measures of noncompactness; essential spectral radius.

0 Introduction

Let $C \subset D$ be complete cones in a normed linear space $(X, \|\cdot\|)$ and suppose that $f : C \to C$ is continuous, homogeneous of degree one and preserves the partial ordering induced by D. In this framework there is a natural definition of what we shall call (see equation (1.5)) the "Bonsall cone spectral radius of f," denoted $\widetilde{r}_C(f)$. However, it is much less clear how one should define $\rho_C(f)$, the "cone essential spectral radius of f." If X is a Banach space and $L : X \to X$ is a bounded linear operator, there is a natural definition of the essential spectral radius $\rho(L)$ of L. With the aid of the Kuratowski measure of noncompactness, one can give a simple formula for $\rho(L)$; see [24]. Motivated by this simple formula, the authors (see [18] and [29]) have proposed what might seem a natural analog as a formula for $\rho_C(f)$. One goal of this paper is to give a variety of examples which show that, in general, the definition proposed in [18] and [29] for $\rho_C(f)$ has serious defects. We shall propose a new definition for $\rho_C(f)$ which avoids the problems of the earlier definitions.

If $f : C \to C$ is as before and $r := \widetilde{r}_C(f) > 0$, it is natural to ask whether f has an eigenvector $u \in C \setminus \{0\}$ with eigenvalue r, that is, satisfying $f(u) = ru$. If $\rho_C(f) < \widetilde{r}_C(f)$, where $\rho_C(f)$ is the new definition of the cone essential spectral radius of f, we conjecture that f has such an eigenvector. We prove this conjecture in a number of cases, for example, if f is a compact perturbation of a map $g : C \to C$ which is linear on C. In general, we argue that the conjecture is exactly analogous to a long-standing and apparently intractable problem in "asymptotic fixed point theory;" see Remark 4.7 below.

This paper is long, so an outline may be helpful. In an attempt to keep the paper self-contained, we list in Section 1 some standard facts about cones, and we recall the classical linear Krein-Rutman theorem [15] and generalization due to Bonsall [3]. We also recall some theorems concerning measures of noncompactness, including recently discovered results (see [20] and [21]) concerning existence of inequivalent measures of noncompactness; and we describe some nonlinear Krein-Rutman theorems for noncompact operators from [18] and [29]. In Section 2, we construct closed, total cones K and linear maps $L : K \to K$ for which the definition of cone essential spectral radius in [18] is seriously flawed. With the aid of results from [20] and [21], we also show that another plausible definition of cone essential spectral radius has serious defects for linear maps $L : K \to K$, where again K is a closed, total cone. In Section 3 we present our definition of $\rho_C(f)$; see Definition 3.2 and equation (3.5). We then derive a number of consequences which play a role in investigating the basic conjecture that

$\rho_C(f) < \widetilde{r}_C(f)$ implies that $f(u) = ru$ for some $u \in C \setminus \{0\}$ with $r := \widetilde{r}_C(f)$. In Section 4 we prove the basic conjecture if f satisfies a variety of additional assumptions; but we note that the general conjecture remains open. In Section 5 we consider a concrete class of maps $\mathcal{F}$ motivated by the general max-plus operators treated in [18], but the general conjecture again remains unsolved for the class $\mathcal{F}$; see Question E.

1 Background: Cones, Measures of Noncompactness and Theorems of Krein-Rutman Type

We begin by reminding the reader of some necessary background material.

If $(X, \|\cdot\|)$ is a normed linear space (or NLS) over $\mathbb{R}$ or $\mathbb{C}$, we shall call a subset C of X a "wedge" if C is convex and $tC \subset C$ for all $t \geq 0$, where $tC := \{tx \mid x \in C\}$. We shall call a wedge C a "cone" (with vertex at 0) if $C \cap (-C) = \{0\}$, where $-C := \{-x \mid x \in C\}$. If C is a cone (respectively, wedge) and C is a complete metric space in the metric induced by the norm on X, we shall call C a "complete cone" (respectively, "complete wedge"). If X is a Banach space, a wedge $C \subset X$ is complete if and only if it is closed.

A cone C in an NLS $(X, \|\cdot\|)$ induces a partial ordering $\leq_C$ on X by $x \leq_C y$ if and only if $y - x \in C$. If C is obvious, we shall write $\leq$ instead of $\leq_C$. Such a cone is called "normal" if there exists a constant M such that $\|x\| \leq M\|y\|$ whenever $0 \leq_C x \leq_C y$. If $(X, \|\cdot\|)$ is a Banach space and C is a complete, normal cone in X, it is known that there exists an equivalent norm $|||\cdot|||$ on X such that $|||x||| \leq |||y|||$ whenever $0 \leq_C x \leq_C y$. See [35] for more general results. If $(X, \|\cdot\|)$ is an NLS, one obtains the same result by taking the completion of X. A wedge C in an NLS $(X, \|\cdot\|)$ is called "reproducing" if $X = C - C := \{u - v \mid u, v \in C\}$, and C is called "total" if X equals the closure of $\{u - v \mid u, v \in C\}$. As will be illustrated in some later examples, it may easily happen in infinite dimensions that a complete cone in a Banach space is total but not reproducing. If C is a closed, reproducing cone in a real Banach space $(X, \|\cdot\|)$, one can define a norm $|||\cdot|||$ on X by

$$|||x||| := \inf\{\|u\| + \|v\| \mid x = u - v \text{ for some } u, v \in C\}, \tag{1.1}$$

and it is known (see [8], [13], [35]) that $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms on X and $|||x||| = \|x\|$ for all $x \in C$. More generally, if C is a closed cone in a Banach space $(X, \|\cdot\|)$ and if $Y := \{u - v \mid u, v \in C\}$, then Y is a real Banach space in the norm $|||\cdot|||$ defined by (1.1), and again $|||x||| = \|x\|$ for all $x \in C$.

If C is a closed cone in a Banach space X, we shall write

$$C^* := \{\phi \in X^* \mid \operatorname{Re}(\phi(x)) \geq 0 \text{ for all } x \in C\},$$

where X^* denotes the dual space of X and $\operatorname{Re}(\phi(x))$ denotes the real part of $\phi(x)$. In general C^* is a closed wedge; and if C is total, C^* is a closed cone. It is known (see [8], [35]) that C is normal if and only if C^* is reproducing and C is reproducing if and only if C^* is normal.

If C is a complete cone in an NLS $(X, \|\cdot\|)$, a map $f : C \to C$ will be called "homogeneous of degree one" or simply "homogeneous" if, for all $t \geq 0$ and for all $x \in C$,

$$f(tx) = tf(x).$$

A map $f : C \to C$ will be called C-linear if

$$f(ax + by) = af(x) + bf(y),$$

for all nonnegative scalars a and b and for all $x, y \in C$. As will be seen later, it may happen that a continuous, C-linear map $f : C \to C$, where C is a total cone in a Banach space X, does not have a continuous extension $F : X \to X$ as a linear map. A map $f : C \to C$ will be called C-order-preserving if, whenever $0 \leq_C x \leq_C y$,

$$f(x) \leq_C f(y).$$

It is sometimes the case that one has a complete cone C, a map $f : C \to C$ and a cone $C_1 \supset C$ such that for all $x, y \in C$ with $x \leq_{C_1} y$ one has $f(x) \leq_{C_1} f(y)$; and in this situation we shall say that f is "C_1-order-preserving." If $f : C \to C$ is C-linear, f is automatically C-order-preserving; but in general it may easily happen that $f : C \to C$ is continuous, homogeneous, C_1-order-preserving but not C-order-preserving. See the discussion of "reproduction-decimation operators" in [17].

If Y is a complex Banach space and $\Lambda : Y \to Y$ is a bounded, (complex) linear operator, $\sigma(\Lambda)$ will denote the spectrum of Λ, so $\sigma(\Lambda) := \{z \in \mathbb{C} \mid zI - \Lambda \text{ is not one-one and onto}\}$. If X is a real Banach space and $L : X \to X$ is a bounded, (real) linear map, one can form the complexification $\widetilde{X}$ of X and the complexification $\widetilde{L}$ of L, so $\widetilde{L} : X \to \widetilde{X}$ is a bounded, complex linear map. We define $\sigma(L) := \sigma(\widetilde{L})$. Recall that we have $\lim_{n\to\infty} \|L^n\|^{1/n} = \lim_{n\to\infty} \|\widetilde{L}^n\|^{1/n}$ and

$$r(L) := \sup\{|z| \mid z \in \sigma(L)\} = \lim_{n\to\infty} \|L^n\|^{1/n} = \inf_{n\geq 1} \|L^n\|^{1/n}. \tag{1.2}$$

We shall use the notation of equation (1.2). As usual, $r(L)$ is called the spectral radius of L.

If Y is a complex Banach space and $\Lambda : Y \to Y$ is a bounded, linear map, there are several inequivalent definitions of the essential spectrum $\mathrm{ess}(\Lambda)$ of Λ. For example, T. Kato [14] defines $\mathrm{ess}_1(\Lambda)$ by

$$\mathrm{ess}_1(\Lambda) := \{z \in \mathbb{C} \mid zI - \Lambda \text{ is not semi-Fredholm}\}.$$

F. Browder [5] gives a different definition, $\mathrm{ess}_2(\Lambda)$, while F. Wolf [37] defines $\mathrm{ess}_3(\Lambda)$ by

$$\mathrm{ess}_3(\Lambda) := \{z \in \mathbb{C} \mid zI - \Lambda \text{ is not Fredholm}\}.$$

If Y is infinite dimensional, it is known that the essential spectrum is nonempty. Furthermore, the quantity $\sup\{|z| \mid z \in \mathrm{ess}_j(\Lambda)\}$ is independent of $j = 1, 2, 3$. We shall write

$$\rho(\Lambda) := \sup\{|z| \mid z \in \mathrm{ess}_j(\Lambda)\} \tag{1.3}$$

and call $\rho(\Lambda)$ the "essential spectral radius of Λ." Note that $\rho(\Lambda) = 0$ if Λ^N is compact for some integer $N \geq 1$. If $\delta > 0$, there are at most finitely many elements $\lambda \in \sigma(\Lambda)$ with $|\lambda| \geq \rho(\Lambda) + \delta$ and each such λ is an eigenvalue of finite algebraic multiplicity. If X is a real Banach space and $L : X \to X$ is a bounded linear map, we consider the complexification $\widetilde{X}$ of X and the complexification $\widetilde{L} : \widetilde{X} \to \widetilde{X}$ of L, and we define $\rho(L) := \rho(\widetilde{L})$.

With these preliminaries we can describe the basic questions of interest in this paper. Suppose that C is a complete cone in an NLS $(X, \|\cdot\|)$. Suppose that $f : C \to C$ is continuous, homogeneous and either C-order-preserving or C_1-order-preserving for some complete cone $C_1 \supset C$. If $g : C \to C$ is continuous and homogeneous, define $\|g\|_C$ by

$$\|g\|_C := \sup\{\|g(x)\| \mid x \in C \text{ and } \|x\| \leq 1\}. \tag{1.4}$$

Define the "Bonsall cone spectral radius" $\widetilde{r}_C(f)$ by

$$\widetilde{r}_C(f) := \lim_{m \to \infty} \|f^m\|_C^{1/m} = \inf_{m \geq 1} \|f^m\|_C^{1/m}. \tag{1.5}$$

We note that (1.5) is well-defined even if f is not order-preserving; however, in this paper we generally shall assume that our functions are order-preserving.

Question A. Under what further conditions on f is it true that there exists $v \in C \setminus \{0\}$ with $f(v) = rv$, where $r := \widetilde{r}_C(f)$?

Question B. If $f : C \to C$ is continuous and C-linear, under what further conditions on f is it true that there exists $v \in C \setminus \{0\}$ with $f(v) = rv$, where $r := \widetilde{r}_C(f)$?

It is easy to see that some sort of compactness condition will be necessary to obtain the desired eigenvectors in Questions A and B. The hope is to find a condition which is optimal or close to optimal. If f in Question A or B is compact, the existing theory is satisfactory; but there are many interesting noncompact maps (see, for example, [18] and the linear maps in Sections 5 and 6 of [31]), and here the situation is much less satisfactory. Indeed, it is not generally recognized that the existing theory is inadequate even to handle Question B in the stated generality.

The historical starting point of our work here is the classical Krein-Rutman Theorem [15].

Theorem 1.1 (Krein and Rutman [15]). *Let C be a closed, total cone in a real Banach space X and $L : X \to X$ a bounded, compact linear map such that $LC \subset C$. If $r(L) > 0$ (see equation (1.2)), there exists $u \in C \setminus \{0\}$ with $Lu = ru$, where $r := r(L)$.*

It is interesting to note that the original Krein-Rutman paper [15] already has some discussion of eigenvectors of nonlinear maps $f : C \to C$; see Section 9 of [15].

F.F. Bonsall [3] has given a generalization of Theorem 1.1. If X is a real Banach space, C is a closed, total cone in X and $L : X \to X$ is a bounded, compact linear map, Bonsall proves that $\widetilde{r}_C(L) = r(L)$ (see equation (1.5)). However, if $L : X \to X$ is not compact, Bonsall gives a simple example of a bounded linear map $L : X \to X$ and a parameterized family of closed, total cones C_γ, for $\gamma > 0$, such that $LC_\gamma \subset C_\gamma$, with $L|C_\gamma$ compact, such that $\widetilde{r}_{C_\gamma}(L) = 2^{-\gamma}$ and $r(L) = 1$.

Theorem 1.2 (Bonsall [3]). *Let C be a complete cone in an NLS $(X, \|\cdot\|)$ and $L : C \to C$ a continuous, C-linear map. Assume that $L|C$ is compact and $\widetilde{r}_C(L) > 0$. Then there exists $u \in C \setminus \{0\}$ with $Lu = ru$, where $r := \widetilde{r}_C(L)$.*

Notice in Theorem 1.2 that even if L has a continuous linear extension $\widehat{L} : X \to X$, it is only assumed that $\widehat{L}|C$ is compact, not that $\widehat{L}$ is compact.

Many authors have given generalizations of Theorems 1.1 and 1.2; see, for example, [10], [18], [28], [29], [30], [31], [33], [34], [36] and the references in these papers.

To describe some of these theorems we need to recall the definition of a "measure of noncompactness" or MNC. If (X, d) is a metric space and S is a bounded subset of X, then K. Kuratowski [16] has defined $\alpha(S)$, the Kuratowski measure of noncompactness of S by

$$\alpha(S) := \inf\{\delta > 0 \mid S = \bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } \operatorname{diam}(S_i) \leq \delta, \text{ for } 1 \leq i \leq n < \infty\}.$$

Here $\operatorname{diam}(S_i) := \sup\{d(u, v) \mid u, v \in S_i\}$. If (X, d) is a complete metric space, one can easily verify that the Kuratowski MNC α satisfies the following properties:

(A1) $\alpha(S) = 0$ if and only if $\overline{S}$ is compact, for all bounded sets $S \subset X$;

(A2) $\alpha(S) \leq \alpha(T)$ for all bounded sets $S \subset T \subset X$;

(A3) $\alpha(S \cup \{x_0\}) = \alpha(S)$ for all bounded sets $S \subset X$ and all $x_0 \in X$; and

(A4) $\alpha(\overline{S}) = \alpha(S)$ for all bounded sets $S \subset X$.

Property (A1) explains the terminology "measure of noncompactness." Properties (A2)-(A4) are true for a general metric space (X, d).

If $(X, \|\cdot\|)$ is an NLS and S is a bounded subset of X, we shall denote by $\operatorname{co}(S)$ the convex hull of S, that is, the smallest convex set containing S. If T is also a bounded subset of X and λ is a scalar, we shall write

$$\lambda S := \{\lambda s \mid s \in S\}, \qquad S + T := \{s + t \mid s \in S \text{ and } t \in T\}.$$

More generally, if S_j for $1 \leq j \leq m$ are bounded subsets of X, we shall write

$$\sum_{j=1}^{m} S_j := \{\sum_{j=1}^{m} s_j \mid s_j \in S_j \text{ for } 1 \leq j \leq m\}.$$

G. Darbo [7] first observed that if $(X, \|\cdot\|)$ is an NLS with metric $d(x, y) := \|x - y\|$, then α satisfies the following additional properties:

(A5) $\alpha(\operatorname{co}(S)) = \alpha(S)$ for all bounded sets $S \subset X$;

(A6) $\alpha(S + T) \leq \alpha(S) + \alpha(T)$ for all bounded sets $S, T \subset X$; and

(A7) $\alpha(\lambda S) = |\lambda|\alpha(S)$ for all bounded sets $S \subset X$ and all scalars λ.

Properties (A5)-(A7) have made α a very useful tool in functional analysis and fixed point theory. Indeed, Darbo's immediate motivation for establishing properties (A5)-(A7) was to use them to prove an elegant new fixed point theorem; see [7].

Notice that in a Banach space the properties (A1)-(A7) are *not* independent. For example, (A2), (A6) and (A7) imply (A4).

For general metric spaces (X, d), the Kuratowski MNC also satisfies the so-called "set-additivity property," namely

(A8) $\alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\}$ for all bounded sets $S, T \subset X$.

For our purposes here, (A8) will, for the most part, be irrelevant.

If W is a complete wedge in an NLS $(X, \|\cdot\|)$ then $\mathcal{B}(W)$ will denote the collection of all bounded subsets of W. If $(X, \|\cdot\|)$ is a Banach space then a map $\beta : \mathcal{B}(X) \to [0, \infty)$ will be called a "homogeneous measure of noncompactness" on X, or "homogeneous MNC" on X, if β satisfies properties (A1)-(A7) with β replacing α in the statements of these properties. If β also satisfies (A8) with β replacing α there, then β will be called a "homogeneous, set-additive MNC."

If W is a complete wedge in an NLS $(X, \|\cdot\|)$, a map $\beta : \mathcal{B}(W) \to [0, \infty)$ will be called "weakly homogeneous" if it satisfies the following property:

(A7w) $\beta(\lambda S) = \lambda\beta(S)$ for every $S \in \mathcal{B}(W)$ and every $\lambda \geq 0$.

A map $\beta : \mathcal{B}(W) \to [0, \infty)$ will be called a "weakly homogeneous MNC" on W if it satisfies (A7w) and also satisfies (A1)-(A6), with β replacing α and W replacing X in the statements of (A1)-(A6). If β also satisfies (A8), with β replacing α and W replacing X there, then β will be called a "weakly homogeneous, set-additive MNC" on W.

If β and γ are homogeneous MNC's on a Banach space $(X, \|\cdot\|)$, we say that β dominates γ if there exists a constant c such that, for all $S \in \mathcal{B}(X)$,

$$\gamma(S) \leq c\beta(S).$$

We say that β and γ are equivalent if β dominates γ and γ dominates β, that is, if there exist positive constants b and c such that, for all $S \in \mathcal{B}(X)$,

$$b\beta(S) \leq \gamma(S) \leq c\beta(S). \tag{1.6}$$

There are many examples known of MNC's, and in a given problem it may be important to work with an MNC which is natural for that problem. See, for example, [18] and [31]. For given equivalent homogeneous MNC's β and γ on an infinite dimensional Banach space X, considerable effort has been devoted (see [1] and [2] and the references there) to finding optimal constants b and c in equation (1.6). Curiously, it has only very recently been proven (see [20] and [21]) that for a wide variety of infinite dimensional classical Banach spaces X (in particular, for any infinite dimensional Hilbert space; for any infinite dimensional space $L^p(\Omega, \Sigma, \mu)$ where $1 \leq p \leq \infty$ and where (Ω, Σ, μ) is a general measure space; and for any infinite dimensional $C(K)$ where K is a compact Hausdorff space) that there exist uncountably many pairwise inequivalent homogeneous MNC's on X. The question of whether there exist inequivalent homogeneous MNC's on every infinite dimensional Banach space X remains open. We shall use later some special cases of results from [20] and [21].

If W is a complete wedge in an NLS $(X, \|\cdot\|)$ and β and γ are weakly homogeneous MNC's on W, the definitions of "β dominates γ" or "β and γ are equivalent" remain essentially the same and will not be repeated.

If W is a complete wedge in an NLS $(X, \|\cdot\|)$ and β is a weakly homogeneous MNC on W, and if $f : W \to W$ is a continuous map such that $f(tx) = tf(x)$ for all $x \in W$ and $t \geq 0$, then f maps bounded subsets of W to bounded subsets of W. In this case one defines

$$\beta(f) := \inf\{\lambda \geq 0 \mid \beta(f(S)) \leq \lambda\beta(S) \text{ for all } S \in \mathcal{B}(W)\}, \tag{1.7}$$

where we set $\beta(f) = \infty$ if the set in the right-hand side of (1.7) is empty. As is proved in [20], even if $W = X$ is a Banach space, $f : X \to X$ is a bounded linear map and β_X is a homogeneous, set-additive MNC on X, it may happen that $\beta_X(f^m) = \infty$ for all $m \geq 1$. In general, we follow notation in [20] and define $\beta^{\#}$ by

$$\beta^{\#}(f) := \limsup_{m\to\infty}(\beta(f^m))^{1/m}, \tag{1.8}$$

where $\beta^{\#}(f) = \infty$ is allowed. If $\beta(f) < \infty$, then for all $j \geq 1$ and $k \geq 1$, it is the case that $\beta(f^j)$, $\beta(f^k)$ and $\beta(f^{j+k})$ are finite and

$$\beta(f^{j+k}) \leq \beta(f^j)\beta(f^k).$$

A well-known calculus lemma now implies that, if $\beta(f) < \infty$, then

$$\beta^{\#}(f) = \lim_{m\to\infty}(\beta(f^m))^{1/m} = \inf_{m\geq 1}(\beta(f^m))^{1/m}. \tag{1.9}$$

If $W = C$ is a complete cone in an NLS $(X, \|\cdot\|)$, the quantity $\beta^{\#}(f)$ is defined in [18] on page 531 to be the "cone essential spectral radius of f." If β and γ are *equivalent*, weakly homogeneous MNC's on C, it is easy to see that $\beta^{\#}(f) = \gamma^{\#}(f)$; but in general we shall see that it may happen that $\beta^{\#}(f) \neq \gamma^{\#}(f)$. For this and other reasons we shall argue that $\beta^{\#}(f)$ is not an appropriate definition of the cone essential spectral radius of f.

If $(X, \|\cdot\|)$ is a real or complex Banach space, L is a bounded linear map and α denotes the Kuratowski MNC on X, it is shown in [24] that $\rho(L)$, the essential spectral radius of L, is given by

$$\rho(L) = \alpha^{\#}(L) = \lim_{m\to\infty} (\alpha(L^m))^{1/m} = \inf_{m\geq 1} (\alpha(L^m))^{1/m}, \tag{1.10}$$

and it follows easily that if β is any homogeneous MNC on X which is equivalent to α, then $\beta^{\#}(L) = \rho(L)$.

One might conjecture that $\beta^{\#}(L) = \rho(L)$ for any homogeneous MNC. However, if $Z := \ell^p(\mathbb{N}\times\mathbb{N})$, where $1 \leq p \leq \infty$ and $\Lambda : Z \to Z$ is defined by $\Lambda z = x$, where $x(i,j) = z(i+1,j)$ for $i, j \in \mathbb{N}$, then $\|\Lambda^m\| = 1$ for all $m \geq 1$, so $r(\Lambda) = 1$. Furthermore, it is proved in Theorem 8 of [20] that for each s with $1 < s \leq \infty$, there exists a homogeneous, set-additive MNC γ_s on Z such that $\gamma_s^{\#}(\Lambda) = s$. In particular, for $s = \infty$, it is the case that $\gamma_\infty(\Lambda^m) = \infty$ for all $m \geq 1$.

If X is a Banach space, β is a homogeneous MNC on X and $L : X \to X$ is a bounded linear operator, one can also define

$$\beta^*(L) := \limsup_{m\to\infty} (\beta(L^m B_1(0)))^{1/m},$$

where we write $B_r(x) := \{y \in X \mid \|y - x\| < r\}$. Notice that because $L^m B_1(0) \subset B_{\|L^m\|}(0) = \|L^m\| B_1(0)$, we have

$$\beta(L^m B_1(0)) \leq \|L^m\| \beta(B_1(0)),$$

so, as opposed to $\beta^{\#}(L)$, one has $\beta^*(L) \leq \lim_{m\to\infty} \|L^m\|^{1/m} = r(L)$. More generally, suppose that C is a complete cone in an NLS $(X, \|\cdot\|)$ and that $f : C \to C$ is continuous and homogeneous. If we define $V_r := \{x \in C \mid \|x\| \leq r\}$ and if β is a weakly homogeneous MNC on C, we can define

$$\beta^*(f) := \limsup_{m\to\infty} (\beta(f^m(V_1)))^{1/m}. \tag{1.11}$$

Notice that $f^m(V_1) \subset \|f^m\|_C V_1$, so

$$\beta(f^m(V_1)) \leq \|f^m\|_C \beta(V_1).$$

Using equations (1.5) and (1.11), one obtains, in contrast to $\beta^{\#}(f)$, that

$$\beta^*(f) \leq \widetilde{r}_C(f).$$

If X is a Banach space, $L : X \to X$ is a bounded linear operator and β is any homogeneous MNC on X, it is proved in [20] that

$$\beta^*(L) = \rho(L). \tag{1.12}$$

With these preliminaries we can describe some results from [29] which represent our proximate starting point; see also Proposition 6 on page 252 of [28] and Sections 2 and 3 of [18].

Theorem 1.3 (See Theorem 2.1 in [29]). *Let C be a complete cone in an NLS $(X, \|\cdot\|)$ and $f : C \to C$ a continuous, homogeneous map which is C-order-preserving. Assume that there exists a weakly homogeneous MNC β on C such that $\beta(f) < 1$. Assume also that there exists $u \in C$ such that $\{\|f^m(u)\| \mid m \geq 1\}$ is unbounded. Then there exists $\lambda \geq 1$ and $v \in C \setminus \{0\}$ with $f(v) = \lambda v$. If $f(x) \neq x$ for all $x \in C$ with $\|x\| = 1$ and if $V := \{x \in C \mid \|x\| < 1\}$, the fixed point index of $f : U \to C$ satisfies $i_C(f, V) = 0$.*

Theorem 2.1 is stated slightly less generally in [29], but the same proof applies. The relevant fixed point index is described in [6], [9] and [12], with generalizations in [19], [25], [27] and [30]. An examination of the proof in [29] shows that Theorem 1.3 remains true under the weaker assumption that f is C_1-order-preserving for some closed cone $C_1 \supset C$.

Theorem 1.4 (See Theorem 2.2 in [29]). *Let C be a closed cone in a Banach space X and β a weakly homogeneous MNC on C. Let $L : X \to X$ be a bounded linear map with $LC \subset C$. If $\beta^{\#}(L) < \widetilde{r}_C(L)$, where these quantities are defined by equations (1.8) and (1.5), then there exists $v \in C \setminus \{0\}$ with $Lv = rv$, where $r := \widetilde{r}_C(L)$.*

Theorem 1.4 is stated slightly less generally in [29], but the same proof applies. It is remarked in [29] that if C is reproducing, one can prove that $\alpha_C^{\#}(L) \leq \widetilde{r}_C(L)$, where α_C denotes the Kuratowski MNC on C. However, as we shall see, if C is a closed, total cone in X, it may easily happen that $\alpha_C^{\#}(L) > \widetilde{r}_C(L)$, in which case Theorem 1.4 provides no information.

Corollary 1.5 (See Corollary 2.2 in [29]). *Let X be a real Banach space and $L : X \to X$ a* *bounded linear map with $\rho(L) < r(L) := r$ (see equations (1.2), (1.3) and (1.10)). If C is a closed, total cone in X with $LC \subset C$, then $\widetilde{r}_C(L) = r(L)$ and there exists $v \in C \setminus \{0\}$ with $Lv = rv$. If C^* denotes the dual cone of C, there exists $\psi \in C^* \setminus \{0\}$ with $L^*\psi = r\psi$.*

If C in Corollary 1.5 is reproducing, part of Corollary 1.5 was obtained by a different argument by Edmunds, Potter and Stuart in [10].

2 The Cone Essential Spectral Radius: Counterexamples

If C is a complete cone in an NLS $(X, \|\cdot\|)$ and $f : C \to C$ is continuous, homogeneous and C-order-preserving, we want to give a "reasonable" definition of $\rho_C(f)$, the "cone essential spectral radius of f." We take the viewpoint that whatever definition is given, $\rho_C(f)$ should satisfy $\rho_C(f) \leq \widetilde{r}_C(f)$. Also, if a definition of $\rho_C(f)$ is given in terms of a weakly homogeneous MNC β on C, the number $\rho_C(f)$ should be independent of β; see equation (1.12).

Recall that if β is a weakly homogeneous MNC on a complete cone C and $f : C \to C$ is continuous, homogeneous and C-order-preserving, then $\rho_C(f; \beta)$, a cone essential spectral radius of f, possibly dependent on β, is defined in [18] by

$$\rho_C(f; \beta) := \beta^{\#}(f),$$

where $\beta^{\#}(f)$ is defined by equation (1.8). If α denotes the Kuratowski MNC on X and α_C its restriction to C, one might hope that $\alpha_C^{\#}(f) \leq \widetilde{r}_C(f)$. We shall show, in Theorem 2.8 below, that it may happen that $\alpha_C^{\#}(f) > \widetilde{r}_C(f)$ even if $f : C \to C$ is continuous and C-linear where C is a closed, total cone in a Banach space. To this end we introduce several spaces and maps which will be used below, as stated:

(B1) $(X, |\cdot|)$ is an infinite dimensional Banach space;

(B2) $Y := c_0(X)$ denotes the Banach space of sequences $y = \{x_j\}_{j\geq 1}$, with $x_j \in X$ for all $j \geq 1$ and $\lim_{j\to\infty} |x_j| = 0$, endowed with the norm

$$\|y\|_Y := \sup_{j\geq 1} |x_j| = \max_{j\geq 1} |x_j|;$$

(B3) $Z := \mathbb{R} \times Y$ is the Banach space of all pairs (t, y) with $t \in \mathbb{R}$ and $y \in Y$ endowed with the norm

$$\|(t, y)\|_Z := \max\{|t|, \|y\|_Y\},$$

where we view Y as a closed linear subspace of Z via the isometric embedding $j(y) := (0, y)$; and

(B4) we define bounded linear projections $Q : Z \to Z$ and $P_n, \widehat{P}_n : Z \to Z$, and $\pi_n : Z \to X$, for $n \geq 1$, by $Q(t, y) = (0, y)$ and $P_n(t, y) = \widehat{P}_n(0, y)$, where

$$\widehat{P}_n(t, y) = (t, \eta) \text{ and } \pi_j(t, \eta) = \begin{cases} \pi_j(t, y) & \text{for} \quad 1 \leq j \leq n, \\ 0 & \text{for} \quad j > n, \end{cases}$$

where $\pi_j(t, y) = x_j$ and $y := (x_1, x_2, \ldots, x_n, \ldots)$.

The setting given in (B1)-(B4), as well as in (B5) and (B6) below, will only be employed where explicitly stated. Since we identify Y with $\{0\} \times Y \subset Z$ via the isometry j, we may write $\pi_j y := \pi_j(0, y)$ for $y \in Y$.

Lemma 2.1. *Assuming (B1)-(B4), let α denote the Kuratowski MNC on Z and let γ denote the Kuratowski MNC on X. If S is a bounded subset of Z, we have that*

(a) $\alpha(S) = \alpha(QS)$; and

(b) $\alpha(P_n S) = \max\{\gamma(\pi_j S) \mid 1 \leq j \leq n\}$.

Proof. We let $\|Q\|$ and $\|\pi_j\|$ denote the usual operator norms of these maps.

If M is chosen so that $\|z\|_Z \leq M$ for all $z \in S$ and $K := \{(t, 0) \mid |t| \leq M\}$, then $S \subset QS + K$. Since K is compact, $\alpha(S) \leq \alpha(QS) + \alpha(K) = \alpha(QS)$. On the other hand, $\|Q\| \leq 1$, so $\alpha(QS) \leq \alpha(S)$, and we conclude that $\alpha(S) = \alpha(QS)$.

Because $\|\pi_j\| = 1$, and $\pi_j P_n S = \pi_j S$ for $1 \leq j \leq n$,

$$\gamma(\pi_j S) \leq \|\pi_j\| \alpha(P_n S) = \alpha(P_n S)$$

for such j. On the other hand, let $d := \max\{\gamma(\pi_j S) \mid 1 \leq j \leq n\}$ and select $\varepsilon > 0$. Then for $1 \leq j \leq n$ there exists an integer $n_j \geq 1$ and sets $S_{ij} \subset X$, for $1 \leq i \leq n_j$, such that $\pi_j S = \bigcup_{i=1}^{n} S_{ij}$ and $\operatorname{diam}(S_{ij}) \leq d + \varepsilon$ for $1 \leq i \leq n_j$. Let $\mathcal{I}_n$ denote the finite collection of all n-tuples of integers $I := (i_1, i_2, \ldots, i_n)$ with $1 \leq i_j \leq n_j$ for $1 \leq j \leq n$. Define $S_I \subset j(Z)$ by $(0, y) \in S_I$ if and only if $\pi_j y \in S_{i_j, j}$ for $1 \leq j \leq n$ and $\pi_j y = 0$ for $j > n$. By our construction, $P_n S \subset \bigcup_{I \in \mathcal{I}_n} S_I$ and

$\mathrm{diam}(S_I) \leq d + \varepsilon$. This shows that $\alpha(P_n S) \leq d + \varepsilon$, and since $\varepsilon > 0$ was arbitrary, the proof is complete. ■

We introduce some addititional notation:

(B5) $a = \{a_j\}_{j\geq 1}$ denotes a sequence of reals with $0 < a_j \leq 1$ for all $j \geq 1$ and $\lim_{j\to\infty} a_j = 0$; and

(B6) $C_a \subset Z$ is defined by

$$C_a := \{(t, y) \in Z \mid |\pi_j y| \leq a_j t \text{ for } j \geq 1\}.$$

Lemma 2.2. *Assume (B1)-(B6). Then C_a is a closed, normal and total cone in Z, but C_a is not reproducing.*

Proof. We leave to the reader the proof that C_a is a closed cone in Z. If $(s, u) \in C_a$, then $s \geq 0$ and $\|(s, u)\|_Z = s$ because $0 < a_j \leq 1$ for all j. If $(s, u), (t, v) \in C_a$ and $(s, u) \leq_{C_a} (t, v)$, it follows that $s \leq t$, so $\|(s, v)\|_Z = s \leq t = \|(t, u)\|_Z$, and thus C_a is normal.

If $(\tau, y) \in Z$, then $\lim_{n\to\infty} \|(\tau, y) - \widehat{P}_n(\tau, y)\|_Z = 0$, so to prove that C_a is total, it suffices to prove that if $(\tau, y) \in Z$ and $n \geq 1$, then there exist $(s, u), (t, v) \in C_a$ with

$$(s, u) - (t, v) = \widehat{P}_n(\tau, y).$$

With (τ, y) and n fixed as above, select $s \geq \tau$ such that $a_j s \geq |\pi_j y|$ for $1 \leq j \leq n$ and define $t := s - \tau$. Then $\widehat{P}_n(s, y) \in C_a$, and $t \geq 0$ so $(t, 0) \in C_a$. Thus $\widehat{P}_n(s, y) - (t, 0) = \widehat{P}_n(\tau, y)$, and it follows that C_a is total.

To see that C_a is not reproducing, select $x_j \in X$ with $|x_j| = a_j^{1/2}$ for all $j \geq 1$ and define $y := (x_1, x_2, \ldots, x_n, \ldots) \in Y$ and $z := (0, y) \in Z$. Suppose that there exist $(s, u), (t, v) \in C_a$ with $(s, u) - (t, v) = (0, y)$. Denoting $u_j := \pi_j u$ and $v_j := \pi_j v$ for $j \geq 1$, then by the definition of C_a we have that $|u_j| \leq a_j s$ and $|v_j| \leq a_j t = a_j s$, so for all $j \geq 1$,

$$a_j^{1/2} = |x_j| = |u_j - v_j| \leq |u_j| + |v_j| \leq 2a_j s.$$

Since we assume that $\lim_{j\to\infty} a_j = 0$ and $a_j > 0$, the above inequality is impossible for large j. Thus C_a is not reproducing. ■

We also need to recall a result which was obtained independently by Furi and Vignoli in [11] and by Nussbaum in Section A of [25].

Lemma 2.3 (See [11] and Section A of [25]). *If $(X, \|\cdot\|)$ is an infinite dimensional Banach space, $V := \{x \in X \mid \|x\| \leq 1\}$ and α is the Kuratowski MNC on $(X, \|\cdot\|)$, then $\alpha(V) = 2$.*

For $(X, \|\cdot\|)$ an infinite dimensional Banach space, it follows from Lemma 2.3 that $\alpha(V_r(x_0)) = 2r$, where α is the Kuratowski MNC on X and $V_r(x_0) := \{x \mid \|x - x_0\| \leq r\}$.

Lemma 2.4. *Assume (B1)-(B6). If $\{(t_n, y_n)\}_{n\geq 1}$ is a sequence of vectors in C_a and also $(t, y) \in C_a$, then $\lim_{n\to\infty} \|(t_n, y_n) - (t, y)\|_Z = 0$ if and only if $\lim_{n\to\infty} t_n = t$ and $\lim_{n\to\infty} |\pi_i y_n - \pi_i y| = 0$ for all $i \geq 1$.*

Proof. Because π_i is a continuous linear map for $1 \leq i < \infty$, the implication in one direction is clear. To prove the implication in the other direction, select M such that $0 \leq t_n \leq M$ for all n, and so $0 \leq t \leq M$. It follows from the definition of C_a that

$$|\pi_i y_n|, |\pi_i y| \leq M a_i, \qquad \text{for } i \geq 1 \text{ and } n \geq 1. \tag{2.1}$$

Select $\varepsilon > 0$. Since $\lim_{i\to\infty} a_i = 0$, equation (2.1) implies that there exists an integer i_0 such that $|\pi_i y_n| < \varepsilon/2$ and $|\pi_i y| < \varepsilon/2$ for all $i > i_0$. Because $\lim_{n\to\infty} |\pi_i y_n - \pi_i y| = 0$ and $\lim_{n\to\infty} |t_n - t| = 0$, there exists an integer n_0 such that $|\pi_i y_n - \pi_i y| < \varepsilon$ and $|t_n - t| < \varepsilon$ for $1 \leq i \leq i_0$ and for all $n \geq n_0$. Combining these estimates we find that

$$\|(t_n, y_n) - (t, y)\|_Z < \varepsilon$$

for all $n \geq n_0$, which completes the proof. ■

With π_i, a and C_a as in (B4)-(B6), we define a C_a-linear map $L : C_a \to C_a$ by $L(t, y) = (t, \eta)$, where

$$\pi_i \eta := \left(\frac{a_i}{a_{i+1}}\right) \pi_{i+1} y, \tag{2.2}$$

for $i \geq 1$.

Lemma 2.5. *If $L : C_a \to C_a$ is defined as above, L is a continuous, C_a-linear map. If $m \geq 1$, and $L^m(t, y) = (t, \zeta)$,*

$$\pi_i \zeta = \left(\frac{a_i}{a_{i+m}}\right) \pi_{i+m} y. \tag{2.3}$$

For $m \geq 1$ we have (see equations (1.4) and (1.5)) that $\|L^m\|_{C_a} = 1$ and $\widetilde{r}_{C_a}(L) = 1$.

Proof. We leave to the reader the verification that $LC_a \subset C_a$, that equation (2.3) holds and that L is C_a-linear. The fact that $L : C_a \to C_a$ is continuous follows easily with the aid of Lemma 2.4.

Because $\|(t, y)\|_Z = t$ for all $(t, y) \in C_a$ and $L(t, y) = (t, \eta)$, we see that $\|L^m\|_{C_a} = 1$ for all m and thus $\widetilde{r}_{C_a}(L) = 1$. ■

Lemma 2.6. *Assume (B1)-(B6). Let β denote the restriction of the Kuratowski MNC α on Z to bounded subsets of C_a, so β is a weakly homogeneous MNC on C_a. If $L : C_a \to C_a$ is defined by equation (2.2), we have (see equation (1.7))*

$$\beta(L^m) = \sup_{i \geq 1} \left(\frac{a_i}{a_{i+m}} \right), \tag{2.4}$$

where we allow the possibility that $\beta(L^m) = \infty$.

Proof. Let $S_j := \{(1, y) \in C_a \mid \pi_i y = 0 \text{ for } i \neq j \text{ and } |\pi_j y| \leq a_j\}$. Since S_j is isometric to $\{x \in X \mid |x| \leq a_j\}$, Lemma 2.3 and the comment following Lemma 2.3 imply that $\beta(S_j) = 2a_j$. Lemma 2.5 implies that $L^m S_{j+m} = S_j$, and since $\beta(S_j)/\beta(S_{j+m}) = a_j/a_{j+m}$, we conclude that $\beta(L^m) \geq \sup_{j \geq 1}(a_j/a_{j+m})$. If $\sup_{j \geq 1}(a_j/a_{j+m}) = \infty$, we are done. If $\sup_{j \geq 1}(a_j/a_{j+m}) := M < \infty$, we can extend L^m to a bounded linear map Λ of Z to Z by $\Lambda(t, y) = (t, \zeta)$, where $\pi_j \zeta = (a_j/a_{j+m})\pi_{j+m} y$. Because we assume that $\lim_{j \to \infty} a_j = 0$ and $a_j > 0$ for all j, it must be the case that $M > 1$, and one can see that

$$\|\Lambda\| = \sup_{j \geq 1} \left(\frac{a_j}{a_{j+m}} \right) = M,$$

where $\|\Lambda\|$ denotes the usual operator norm. It follows that for all bounded $S \subset Z$,

$$\alpha(\Lambda S) \leq M\alpha(S),$$

so this same inequality must hold for all bounded $S \subset C_a$. This proves equation (2.4). ■

Remark 2.7. As was shown in Lemma 2.6, if $\sup_{j \geq 1}(a_j/a_{j+m}) < \infty$ then the continuous C_a-linear map $L^m : C_a \to C_a$ has a continuous linear extension $\Lambda : Z \to Z$ to all of Z. Conversely, if $\sup_{j \geq 1}(a_j/a_{j+m}) = \infty$ we claim that such an extension Λ of L^m does not not exist. If such Λ did exist, choose $e_j \in Y$ to satisfy $\pi_i e_j = 0$ for $i \neq j$ and $|\pi_j e_j| = 1$. Then $(1, a_j e_j) \in C_a$, and

$$\begin{aligned}\Lambda(0, a_{j+m} e_{j+m}) &= \Lambda(1,0) - \Lambda(1, a_{j+m} e_{j+m}) \\ &= L^m(1,0) - L^m(1, a_{j+m} e_{j+m}) = (1,0) - (1, \eta_j) = (0, -\eta_j),\end{aligned}$$

where $\|\eta_j\|_Y = a_j$. But then $\|\Lambda\| \geq \|\Lambda(0, e_{j+m})\|_Z = a_j / a_{j+m}$, so $\|\Lambda\| = \infty$, a contradiction.

Our next theorem shows that even if C is a closed total cone in a Banach space Z, with $f : C \to C$ a continuous, C-linear map and α_C the restriction of the Kuratowski MNC α on Z to the bounded subsets of C, it may still happen (see equations (1.5) and (1.9)) that

$$\alpha_C^{\#} = \lim_{m \to \infty} (\alpha_C(f^m))^{1/m} > \widetilde{r}_C(f). \tag{2.5}$$

Equation (2.5) suggests that the definition of cone essential spectral radius in [18], [29] has serious defects.

Theorem 2.8. *Assume (B1)-(B3). Then the Banach space $(Z, \|\cdot\|_Z)$ given there with Kuratowski MNC α has the following property: For each $\mu \in (1, \infty]$, there exists a closed, total cone $K_\mu \subset Z$ and a continuous, K_μ-linear map $L_\mu : K_\mu \to K_\mu$ such that (see equations (1.5), (1.8) and (1.9))*

$$\alpha_{K_\mu}^{\#}(L_\mu) = \lim_{m \to \infty} (\alpha_{K_\mu}(L_\mu^m))^{1/m} = \mu > \widetilde{r}_{K_\mu}(L_\mu) = 1.$$

Here α_{K_μ} is the weakly homogeneous MNC on K_μ obtained by restricting α to the bounded subsets of K_μ.

Proof. Assume additionally (B4)-(B6), where the sequence a in (B5) will depend on μ and will be given shortly, and where we take $K_\mu := C_a$ with C_a as in (B6). Also let $L_\mu : K_\mu \to K_\mu$ be the linear map L as defined in the sentence preceding Lemma 2.5. Then by Lemma 2.5, we have $\widetilde{r}_{K_\mu}(L_\mu) = 1$; and by Lemma 2.6,

$$\alpha_{K_\mu}(L_\mu^m) = \sup_{i \geq 1} \left(\frac{a_i}{a_{i+m}} \right).$$

If $1 < \mu < \infty$ we choose $a_i := \mu^{-i}$ for $i \geq 1$, and we find that $\alpha_{K_\mu}(L_\mu^m) = \mu^m$ and $\alpha_{K_\mu}^{\#}(L_\mu) = \mu$. If on the other hand $\mu = \infty$ we choose $a_i := i^{-i}$ for $i \geq 1$, and we find that $\alpha_{K_\mu}(L_\mu^m) = \infty$ for all m, so $\alpha_{K_\mu}^{\#}(L_\mu) = \infty$. ∎

Remark 2.9. In equation (1.8), we have defined $\beta^{\#}(f) = \limsup_{m \to \infty} (\beta(f^m))^{1/m}$, and we have noted that the limsup can be replaced by a limit (1.9) if $\beta(f) < \infty$. One might hope that the limsup can always be replaced by a limit, at least if one allows the value ∞ for the limit. However, this hope is false even for continuous, C-linear maps and total cones. To see this, we work in the setting of (B1)-(B6), as in Theorem 2.8 above. Select μ_1 and μ_2 with $\mu_1 > \mu_2 > 1$ and define $a_{2i-1} := \mu_1^{-i}$ and $a_{2i} := \mu_2^{-i}$ for $i \geq 1$, to give the sequence a as in (B5). Also let $C_a \subset Z$ be as in (B6), and let $L : C_a \to C_a$ be as defined in the sentence preceding Lemma 2.5. Letting α_{C_a} denote the weakly homogeneous MNC on C_a obtained by restricting the Kuratowski MNC α on Z to bounded sets of C_a, one easily checks that for $k \geq 1$,

$$\alpha_{C_a}(L^{2k}) = \sup_{j\geq 1}\left(\frac{a_j}{a_{j+2k}}\right) = \mu_1^k, \qquad \alpha_{C_a}(L^{2k-1}) = \sup_{j\geq 1}\left(\frac{a_j}{a_{j+2k-1}}\right) = \infty,$$

where Lemma 2.6 has been used. It follows that

$$\mu_1^{1/2} = \liminf_{m\to\infty} \alpha_{C_a}(L^m)^{1/m} < \limsup_{m\to\infty} \alpha_{C_a}(L^m)^{1/m} = \infty.$$

Remark 2.10. Even if Y is an infinite dimensional Banach space, $U : Y \to Y$ is a bounded linear map and β is a homogeneous MNC on Y, it may happen that $\beta(U^{2k-1}) = \infty$ for all $k \geq 1$ while $c^k \leq \beta(U^{2k}) \leq 1$ for all $k \geq 1$, where $0 < c < 1$. Thus the limsup in equation (1.8) is, in general, necessary. To see this, consider again the setting (B1)-(B6), with Y as in (B2). Let $P_n : Y \to Y$ be as in (B4), where here we identify Y with $\{0\} \times Y \subset Z$ via the isometry j. Thus $P_n y = \eta$ where $\pi_j \eta = \pi_j y$ for $1 \leq j \leq n$ and $\pi_j \eta = 0$ for $j > n$. Define $U : Y \to Y$ by $Uy = \eta$, where $\pi_1 \eta = 0$ and $\pi_j \eta = \pi_{j-1} y$ for $j \geq 2$, so U is just the right translation. Let α denote the Kuratowski MNC on Y and let

$$\mathcal{A}(Y) := \{S \in \mathcal{B}(Y) \mid \lim_{n\to\infty} \alpha((I - P_n)S) = 0\}.$$

Again select μ_1 and μ_2 with $\mu_1 > \mu_2 > 1$ and define $a_{2i-1} := \mu_1^{-i}$ and $a_{2i} := \mu_2^{-i}$ for $i \geq 1$, to give the sequence a as in (B5). Define a Banach space $(\widehat{Y}, \|\cdot\|_{\widehat{Y}})$ by

$$\begin{aligned}\widehat{Y} := \{y = (x_1, x_2, \ldots, x_n, \ldots) \mid x_j \in X \text{ for all } j \geq 1 \\ \text{and } \lim_{j\to\infty} a_j|x_j| = 0, \text{ with } \|y\|_{\widehat{Y}} := \sup_{j\geq 1} a_j|x_j|\},\end{aligned} \tag{2.6}$$

and let $\widehat{\alpha}$ denote the Kuratowski MNC on $\widehat{Y}$. Theorem 2.4 in [21] implies that we can define a homogeneous MNC β on Y by setting

$$\beta(S) := \inf\{\widehat{\alpha}(A) + \alpha(B) \mid S \subset A + B \text{ for some } A \in \mathcal{A}(Y) \text{ and } B \in \mathcal{B}(Y)\},$$

for any $S \in \mathcal{B}(Y)$. Further, it is the case that $\beta(S) = \widehat{\alpha}(S)$ for all $S \in \mathcal{A}(Y)$.

For $j \geq 1$ let

$$S_j := \{y \in Y \mid \pi_i y = 0 \text{ for } i \neq j \text{ and } |\pi_j y| \leq 1\}$$

so $U^m S_j = S_{j+m}$ and $S_j \in \mathcal{A}(Y)$ for all $j \geq 1$ and $m \geq 1$. It follows easily using Lemma 2.3 that $\beta(S_j) = \widehat{\alpha}(S_j) = 2a_j$ and thus $\beta(U^m S_j) = 2a_{m+j}$, and therefore

$$\beta(U^m) \geq \sup_{j \geq 1} \left(\frac{a_{m+j}}{a_j} \right).$$

Taking m odd, say $m := 2k-1$ where $k \geq 1$, we have for $j := 2i+1$ where $i \geq 0$, that

$$\beta(U^{2k-1}) \geq \frac{a_{2(k+i)}}{a_{2i+1}} = \frac{\mu_1^{i+1}}{\mu_2^{k+i}} \to \infty$$

as $i \to \infty$. Thus $\beta(U^{2k-1}) = \infty$. On the other hand, taking $m := 2k$ even, where $k \geq 1$, one can easily show that

$$\beta(U^{2k}) \geq \sup_{j \geq 1} \left(\frac{a_{2k+j}}{a_j} \right) = \mu_2^{-k},$$

Further, a calculation shows that U^{2k} extends to a continuous linear map of $\widehat{Y}$ to $\widehat{Y}$ with operator norm $\|U^{2k}\|_{\widehat{Y}} = \mu_2^{-k} \leq 1$ in this space. Since $\|U^{2k}\|_Y = 1$ for the norm in the space Y, it follows from the formula for $\beta(S)$ that $\beta(U^{2k}S) \leq \beta(S)$ for all $S \in \mathcal{B}(Y)$. Thus $c^k \leq \beta(U^{2k}) \leq 1$ for all $k \geq 1$, where $c := \mu_2^{-1}$, as claimed.

In view of equation (1.12), one might hope that the number $\beta^*(f)$ defined in equation (1.11) is independent of the weakly homogeneous MNC β on a cone C, at least if $f : C \to C$ is continuous and C-linear. However, this conjecture fails badly, as is described in the following theorem. Note (recall Lemma 2.2) that the cone $C_a \subset Z$ appearing there is in fact normal and total (although not reproducing).

Theorem 2.11. *Fix $\mu > 1$ and assume (B1)-(B6), taking the sequence $a_j = \mu^{-j}$ for $j \geq 1$ in (B5). Define $U : Z \to Z$ by $U(t, y) = (t, \eta)$, where $\pi_1 \eta = 0$ and $\pi_j \eta = \mu^{-1} \pi_{j-1} y$ for $j \geq 2$. Then we have $UC_a \subset C_a$ and*

$$\alpha_{C_a}^*(U) = \lim_{m \to \infty} (\alpha(U^m V_1))^{1/m} = \mu^{-1},$$

where α denotes the Kuratowski MNC on Z and α_{C_a} its restriction to C_a, with $V_1 \subset Z$ given by

$$V_1 := \{z \in C_a \mid \|z\|_Z \leq 1\} = \{(t, y) \in C_a \mid 0 \leq t \leq 1\}.$$

Further, for each s with $0 \le s < 1$, there exists a homogeneous, set-additive MNC δ_s on Z with

$$\gamma_s^*(U) = \lim_{m\to\infty} (\delta_s(U^m V_1))^{1/m} = \mu^{-1} s, \tag{2.7}$$

where γ_s denotes the restriction of δ_s to C_a.

Proof. Let $\{\widehat{a}_j\}_{j\ge 1}$ be a decreasing sequence of positive reals with $\widehat{a}_1 \le 1$ and $\lim_{j\to\infty} \widehat{a}_j = 0$. Define a Banach space $(\widehat{Y}, \|\cdot\|_{\widehat{Y}})$ as in (2.6), but with $\widehat{a}_j$ in place of a_j. Let $\widehat{\alpha}$ denote the Kuratowski MNC on $\widehat{Z} := \mathbb{R} \oplus \widehat{Y}$, where the norm is defined by

$$\|(t, y)\|_{\widehat{Z}} := \max\{|t|, \|y\|_{\widehat{Y}}\},$$

and thus $\widehat{Y}$ is isometrically embedded in $\widehat{Z}$ by $y \to (0, y)$. For any bounded set $S \subset C_a$, one can easily check that $\lim_{n\to\infty} \alpha((I - P_n)S) = 0$. It now follows from Theorems 2.4 and 2.8 in [21] that there exists a homogeneous, set-additive MNC β on Z such that $\beta(S) = \widehat{\alpha}(S)$ for all bounded sets $S \subset Z$ such that $\lim_{n\to\infty} \alpha((I - P_n)S) = 0$. In particular, $\beta(S) = \widehat{\alpha}(S)$ for all bounded $S \subset C_a$.

An easy calculation shows that

$$U^m V_1 = \{(t, y) \in Z \mid \pi_j y = 0 \text{ for } 1 \le j \le m, \text{ and } |\pi_j y| \le \mu^{-j} t \text{ for } j > m\}.$$

Because $\operatorname{diam}((I - P_n)(U^m V_1))$ approaches zero as $n \to \infty$, Lemma 2.1 implies that

$$\alpha(U^m V_1) = \max\{\gamma(\pi_j U^m V_1) \mid j \ge 1\},$$

where γ denotes the Kuratowski MNC on X. Since, for $j > m$,

$$\pi_j U^m V_1 = \{x \in X \mid |x| \le \mu^{-j}\},$$

Lemma 2.3 implies that

$$\alpha(U^m V_1) = 2\mu^{-m-1}, \qquad \lim_{m\to\infty} (\alpha(U^m V_1))^{1/m} = \mu^{-1}.$$

Define a linear isometry $\Gamma : \widehat{Z} \to Z$ by setting $\Gamma(t, y) = (t, \eta)$, where $\pi_j \eta = \widehat{a}_j x_j$ and $y = (x_1, x_2, \ldots, x_n, \ldots) \in \widehat{Y}$. It follows that $\widehat{\alpha}(U^m V_1) = \alpha(\Gamma U^m V_1)$. But we have

$$\Gamma U^m V_1 = \{(t, \eta) \in Z \mid \pi_j \eta = 0 \text{ for } 1 \le j \le m, \text{ and } |\pi_j \eta| \le \widehat{a}_j \mu^{-j} \text{ for } j > m\},$$

so the same argument used above shows that

$$\widehat{\alpha}(U^m V_1) = \alpha(\Gamma U^m V_1) = \widehat{a}_{m+1}\mu^{-m-1}.$$

If $s := \lim_{k\to\infty} \widehat{a}_{k+1}^{1/k}$ exists, it follows that

$$\lim_{m\to\infty} (\widehat{\alpha}(U^m V_1))^{1/m} = \mu^{-1}s.$$

Thus given s satisfying $0 \le s < 1$, in order to obtain (2.7) it suffices to take $\widehat{a}_m = s^m$ if $s \neq 0$ and $\widehat{a}_m = m^{-m}$ if $s = 0$, where in either case we take $\delta_s := \beta$. ■

Notice that in Theorem 2.11, δ_s is a homogeneous, set-additive MNC on Z, not just a weakly homogeneous MNC on C, the bounded subsets of C. One might hope that a weakly homogeneous MNC β on C, where C is a closed, total cone in a (general) Banach space Z, necessarily has an extension to a homogeneous MNC $\widehat{\beta}$ on Z. The next theorem shows that this hope is false.

We note that in the following theorem, we do not specifically use the setting of (B1)-(B6). In particular, the space $(Z, \|\cdot\|_Z)$ and the cone C are not as in these conditions.

Theorem 2.12. *There exists a Banach space $(Z, \|\cdot\|_Z)$, a closed, total cone $C \subset Z$ and a weakly homogeneous MNC γ on C with the property that if β is any homogeneous MNC on Z, then the restriction of β to to C is not equivalent to γ. In particular, there does not exist an extension of γ as a homogeneous MNC on the Banach space $(Z, \|\cdot\|_Z)$.*

Proof. Let $(X, |\cdot|)$ be an infinite dimensional Banach space. For $\mu > 1$, let $(Y_\mu, \|\cdot\|_{Y_\mu})$ be the Banach space $(\widehat{Y}, \|\cdot\|_{\widehat{Y}})$ as in (2.6) with the choice $a_j = \mu^{-j}$ for $j \ge 1$. Also let $Z_\mu := \mathbb{R} \oplus Y_\mu$ with the norm $\|(t,y)\|_{Z_\mu} := \max\{|t|, \|y\|_{Y_\mu}\}$, and observe that Y_μ is isometrically embedded in Z_μ by $y \to (0, y)$. Let α_μ denote the Kuratowski MNC on Z_μ. Finally, let

$$C := \{(t,y) \in Z_\mu \mid |\pi_j y| \le t \text{ for } j \ge 1\},$$

where we denote $\pi_j y = x_j$ for $y = (x_1, x_2, \ldots, x_n, \ldots)$.

We leave to the reader the simple argument (compare Lemma 2.2) that C is a closed, total cone in Z_μ. Let us observe here that C, as a set, is in fact independent of μ. Denote by $d_\mu(u,v) = \|u - v\|_{Z_\mu}$ the metric induced on C from $(Z_\mu, \|\cdot\|_{Z_\mu})$. Then one can verify that if $\mu_1 > 0$ and $\mu_2 > 0$, it is the case that a subset $S \subset C$ is compact (respectively, closed or bounded) in the d_{μ_1}-metric if and only if it is compact (respectively, closed or bounded) in the d_{μ_2}-metric. In particular, it follows that α_{μ_2}, restricted to (C, d_{μ_1}), namely to C taken with the d_{μ_1} metric, is a weakly homogeneous MNC on (C, d_{μ_1}).

Now fix numbers μ_1 and μ_2 with $\mu_1 > \mu_2 > 1$. We claim that there does not exist a homogeneous MNC β on Z_{μ_1} such that the restriction of β to C is equivalent to α_{μ_2} restricted to C. (Here and below we are taking C with the d_{μ_1} metric.) If there existed such a homogeneous MNC β, then Proposition 2 in [20] would imply that there is a constant M such that $\beta(S) \leq M\alpha_{\mu_1}(S)$ for all $S \in \mathcal{B}(Z_{\mu_1})$. In particular this would be true for all $S \in \mathcal{B}(C)$, and there would exist a constant M' such that for all $S \in \mathcal{B}(C)$,

$$\alpha_{\mu_2}(S) \leq M'\alpha_{\mu_1}(S). \tag{2.8}$$

However, if $S_j := \{(1, y) \in C \mid \pi_i y = 0 \text{ for } i \neq j \text{ and } |\pi_j y| \leq 1\}$, it follows from Lemma 2.3 that $\alpha_{\mu_1}(S_j) = 2\mu_1^{-j}$ and $\alpha_{\mu_2}(S_j) = 2\mu_2^{-j}$. This contradicts (2.8), and so β does not have an extension as above. This proves the result with $Z := Z_{\mu_1}$ and $\gamma := \alpha_{\mu_2}$. ■

We have given a variety of examples of continuous C-linear maps $L : C \to C$, where C is a closed, total cone in a Banach space Z; and we have indicated why, in this generality, certain plausible definitions of the cone essential spectral radius of L are, in fact, inappropriate. We should note, however, that if C is reproducing and $L : C \to C$ is continuous and C-linear, it is clear how the cone essential spectral radius of L should be defined. For the sake of brevity we omit the proofs of Lemma 2.13 and Theorem 2.14.

Lemma 2.13. *Let $(Z, \|\cdot\|)$ be a real Banach space and $C \subset Z$ a closed, reproducing cone. Let α denote the Kuratowski MNC on Z and α_C the restriction of α to C. For each bounded set $R \subset Z$ define $\beta(R)$ by*

$$\beta(R) := \inf\{\alpha_C(S) + \alpha_C(T) \mid R \subset S + (-T), \text{ where } S, T \in \mathcal{B}(C)\}.$$

Then β is a homogeneous MNC on Z and β is equivalent to α.

Theorem 2.14. *Let $(Z, \|\cdot\|)$ be a real Banach space and $C \subset Z$ a closed, reproducing cone, and let $L : C \to C$ be a continuous, C-linear map. Then there exists a unique, continuous linear map $\widehat{L} : Z \to Z$ such that $\widehat{L}|C = L|C$. If $r(\widehat{L})$ denotes the spectral radius of $\widehat{L}$ and $\widetilde{r}_C(L)$ the Bonsall cone* *spectral radius of L (see equations (1.2) and (1.5)), then*

$$r(\widehat{L}) = \widetilde{r}_C(L).$$

If α denotes the Kuratowski MNC on Z and α_C the restriction of α to C, then (see equation (1.8)),

$$\alpha_C^{\#}(L) := \lim_{m\to\infty}(\alpha_C(L^m))^{1/m} = \rho(\widehat{L}),$$

where $\rho(\widehat{L})$ denotes the essential spectral radius of $\widehat{L}$.

Theorem 2.14 suggests that if $C \subset Z$ is a closed, reproducing cone and $L : C \to C$ is continuous and C-linear, then $\rho(\widehat{L})$ is the appropriate definition of the cone essential spectral radius of L.

3 A Definition for the Cone Essential Spectral Radius

Let C be a complete cone in a normed linear space $(X, \|\cdot\|)$ and $f : C \to C$ a continuous, homogeneous, C-order-preserving map. We shall propose here a possible reasonable definition of the cone essential spectral radius of f. The definition we shall give avoids the inadequacies described in Section 2.

Note that C is a metric space in the metric $d(x, y) := \|x - y\|$ inherited from $(X, \|\cdot\|)$, so if $U \subset C$ we shall say that U is "relatively open in C" if it is open as a subset of the metric space (C, d). Equivalently, $U \subset C$ is relatively open in C if there exists an open subset $O \subset X$ such that $O \cap C = U$. If $U \subset C$ is relatively open in C and $0 \in C$, we shall say that U is a "relatively open neighborhood of 0 in C." If $U \subset C$ is relatively open in C and $tU \subset U$ for $0 \le t \le 1$, we shall say that U is a "radial, relatively open neighborhood of 0 in C."

Now suppose that C is a complete cone in an NLS $(X, \|\cdot\|)$, that $g : C \to C$ is continuous and homogeneous, and that V is a bounded, relatively open neighborhood of 0 in C. Define $C_1(g; V)$ by

$$C_1(g; V) = g(V), \tag{3.1}$$

and for $k > 1$ define $C_k(g; V)$ inductively by

$$C_k(g; V) = g(V \cap C_{k-1}(g; V)). \tag{3.2}$$

A simple induction on k, which we leave to the reader, shows that

$$C_k(g; V) = \{g^k(x) \mid g^j(x) \in V \text{ for } 0 \le j < k\}. \tag{3.3}$$

If $f : C \to C$ is continuous and homogeneous and $\lambda > 0$, we shall always denote by f_λ the map

$$f_\lambda(x) := \lambda^{-1} f(x). \tag{3.4}$$

If V is a bounded and radial relatively open neighborhood of 0 in C and $0 < \lambda < \lambda_1$, we claim that, for $k \geq 1$

$$C_k(f_{\lambda_1}; V) \subset C_k(f_\lambda; V).$$

To see this, write $g = f_{\lambda_1}$, so $f_\lambda = (\lambda_1/\lambda)g$. If $y \in C_k(f_{\lambda_1}; V)$, then $y = g^k(x)$, where $g^j(x) \in V$ for $0 \leq j < k$. If $\xi := (\lambda/\lambda_1)^k x$, then $\xi \in V$, because V is radial. Also, $f_\lambda^j(\xi) = (\lambda/\lambda_1)^{k-j} g^j(x)$, so $f_\lambda^j(\xi) \in V$ for $0 \leq j < k$ and $f_\lambda^k(\xi) = g^k(x) \in C_k(f_\lambda; V)$, which proves the desired inclusion.

If (Y, d) is a complete metric space, let $\mathcal{Y}$ denote the collection of closed, bounded nonempty sets $A \subset Y$. For $A \in \mathcal{Y}$ and $y \in Y$, and $r > 0$, let $d(y, A) := \inf\{d(y, a) \mid a \in A\}$ and let $N_r(A) := \{y \in Y \mid d(y, A) < r\}$. If $A, B \in \mathcal{Y}$, define the Hausdorff metric D on $\mathcal{Y}$ by

$$D(A, B) := \inf\{r > 0 \mid A \subset N_r(B) \text{ and } B \subset N_r(A)\}.$$

It is known (see [23], pages 280-281) that $(\mathcal{Y}, D)$ is a complete metric space.

An old theorem of Kuratowski [16] gives conditions in terms of the Kuratowski MNC under which a decreasing sequence of closed, bounded, nonempty sets in a complete metric space (Y, d) converges in the Hausdorff metric to a nonempty, compact set. The same result holds for general measures of noncompactness; see [1], page 19, for a proof.

Proposition 3.1 (Compare Kuratowski [16]). *Let (Y, d) be a complete metric space and let $\mathcal{B}(Y)$ denote the bounded subsets of Y. Suppose that $\beta : \mathcal{B}(Y) \to [0, \infty)$ is a map which satisfies properties (A1)-(A4) in Section 1, with β replacing α in the statements of (A1)-(A4). Let A_k, for $k \geq 1$, be a decreasing sequence of closed, bounded, nonempty sets in Y and assume that $\lim_{k\to\infty} \beta(A_k) = 0$. Then $A_\infty := \bigcap_{k\geq 1} A_k$ is a compact, nonempty set and A_k converges to A_∞ in the Hausdorff metric.*

Let C be a complete cone in an NLS $(X, \|\cdot\|)$, let V be a bounded, relatively open neighborhood of 0 in C and let $f : C \to C$ be a continuous, homogeneous map.

Definition 3.2. The cone essential spectral radius of f, denoted $\rho_C(f)$, is defined by

$$\rho_C(f) := \inf\{\lambda > 0 \mid \lim_{k\to\infty} \alpha(C_k(f_\lambda, V)) = 0\}, \tag{3.5}$$

where α denotes the Kuratowski MNC on C and the notation is as in equations (3.1)-(3.4).

In Definition 3.2, it is easy to show, using Proposition 3.1, that $\rho_C(f)$ is the infimum of numbers $\lambda > 0$ such that $\overline{C_k(f_\lambda, V)}$ converges in the Hausdorff metric to a compact, nonempty set, so Definition 3.2 can be phrased without reference to MNC's. Also, it is the case that $\rho_C(f) \leq \alpha^*(f) \leq \widetilde{r}_C(f)$, as the reader can easily show. If X is a Banach space and β is any homogeneous MNC on X, one can see that

$$\rho_C(f) = \inf\{\lambda > 0 \mid \lim_{k\to\infty} \beta(C_k(f_\lambda; V)) = 0\}.$$

If γ is a weakly homogeneous MNC on C and if $\lim_{k\to\infty} \gamma(C_k(f_\lambda; V)) = 0$ for some $\lambda > 0$, then $\rho_C(f) \leq \lambda$.

Definition 3.2 ostensibly depends on V. However, if V and W are bounded, relatively open neighborhoods of 0 in C, there are positive constants a and b with $W \subset aV$ and $V \subset bW$. It is easy to check that $C_k(f_\lambda; W) \subset C_k(f_\lambda; aV) = aC_k(f_\lambda; V)$ and $C_k(f_\lambda; V) \subset bC_k(f_\lambda, W)$. It follows that $\alpha(C_k(f_\lambda; W)) \to 0$ if and only if $\alpha(C_k(f_\lambda, V)) = 0$, so Definition 3.2 is independent of the bounded, relatively open neighborhood of 0 which is used.

If V is a bounded and radial relatively open neighborhood of 0 in C and $0 < \lambda < \lambda_1$, we have already observed that $C_k(f_{\lambda_1}; V) \subset C_k(f_\lambda; V)$ for all $k \geq 1$. It follows easily that if $\lambda > \rho_C(f)$ and α denotes the Kuratowski MNC, then

$$\lim_{k\to\infty} \alpha(C_k(f_\lambda; V)) = 0.$$

This, in turn, implies that if $\lambda > \rho_C(f)$ and W is any bounded, relatively open neighborhood of 0 in C, then

$$\lim_{k\to\infty} \alpha(C_k(f_\lambda; W)) = 0.$$

If C is a complete cone in an NLS $(X, \|\cdot\|)$ and $f : C \to C$ is continuous and homogeneous, we have already defined the Bonsall cone spectral radius $\widetilde{r}_C(f)$ of f. For our purposes it will be useful to give a variant definition. If $x \in C$, define $\mu(x)$ by

$$\mu(x) := \limsup_{n\to\infty} \|f^n(x)\|^{1/n}.$$

We define $r_C(f)$, the "cone spectral radius of f," by

$$r_C(f) := \sup\{\mu(x) \mid x \in C\}.$$

Under the above hypotheses, the same argument used in Proposition 2.1 on page 525 of [18] shows that

$$r_C(f) \leq \widetilde{r}_C(f)$$

and, for all positive integers m,

$$r_C(f^m) = (r_C(f))^m, \qquad \widetilde{r}_C(f^m) = (\widetilde{r}_C(f))^m. \tag{3.6}$$

Without further restrictions on f, it may happen (see the remark on page 526 of [18]) that $r_C(f) = 0$ and $\widetilde{r}_C(f) = 1$.

Our next theorem, although stated for a complete cone in an NLS instead of a closed cone in a Banach space, follows by the same arguments used in Theorems 2.2 and 2.3 of [18] and the remark on page 528 of [18].

Theorem 3.3 (Compare Section 2 of [18]). *Let C be a complete cone in an NLS $(X, \|\cdot\|)$ and $f : C \to C$ a continuous, homogeneous map. Then $r_C(f) = \widetilde{r}_C(f)$ if any one of the following additional conditions holds:*

(a) f is C-linear;

(b) there exists $m \geq 1$ such that f^m is compact; or

(c) there exists a complete, normal cone D with $C \subset D$ such that $f : C \to C$ is D-order-preserving, that is, f preserves the partial ordering $\leq_D$.

In the statement of Theorem 3.3, recall that a continuous map $g : C \to C$ is called compact if $\overline{g(V)}$ is compact for every bounded set $V \subset C$.

The next theorem gives another condition under which $r_C(f) = \widetilde{r}_C(f)$.

Theorem 3.4. *Let C and D be complete cones in an NLS $(X, \|\cdot\|)$ and assume that $C \subset D$. Let $f : C \to C$ be continuous, homogeneous and D-order-preserving. Assume that $\rho_C(f) < \widetilde{r}_C(f)$. Then it follows that $r_C(f) = \widetilde{r}_C(f)$.*

Proof. Since we know that $r_C(f) \leq \widetilde{r}_C(f)$, assume, by way of contradiction, that $r_C(f) < \widetilde{r}_C(f)$. Select λ satisfying both $r_C(f) < \lambda < \widetilde{r}_C(f)$ and also $\rho_C(f) < \lambda$. In the first part of the proof of

Theorem 2.2 in [18], a Baire category argument is used to prove that there exist $a > 0$ and $x_0 \in C$ with

$$\sup\{\|f_\lambda^k(y)\| \mid k \geq 0 \text{ and } y \in C, \text{ with } \|y - x_0\| \leq a\} < \infty.$$

It follows that

$$\sup\{\|f_\lambda^k(x_0 + z)\| \mid k \geq 0 \text{ and } z \in C, \text{ with } \|z\| \leq a\} < \infty.$$

Since $\lambda < \widetilde{r}_C(f)$ we have $\lim_{n\to\infty} \|f_\lambda^n\|_C = \infty$, so there exists an increasing sequence $n_i \to \infty$ of integers with $\|f_\lambda^{n_i}\|_C > \|f_\lambda^j\|_C$ for $0 \leq j < n_i$. It follows that there exists $w_i \in C$ with $\|w_i\| = a$ and

$$\|f_\lambda^{n_i}(w_i)\| > a\|f_\lambda^j\|_C, \qquad \text{for } 0 \leq j < n_i. \tag{3.7}$$

Because f_λ is D-order-preserving, we see that

$$f_\lambda^{n_i}(w_i) \leq f_\lambda^{n_i}(x_0 + w_i),$$

where $\leq$ denotes the partial ordering on X induced by D. For notational convenience, we define $R_i := \|f_\lambda^{n_i}(w_i)\| \to \infty$ and

$$u_i := R_i^{-1} f_\lambda^{n_i}(w_i), \qquad z_i := R_i^{-1} f_\lambda^{n_i}(x_0 + w_i).$$

Our construction insures that $\lim_{i\to\infty} z_i = 0$ and $\|u_i\| = 1$, with

$$z_i - u_i \in D. \tag{3.8}$$

Define $S := \{u_i \,|\, i \geq 1\}$. If we can prove that S has compact closure, then by taking a subsequence, which we also label u_i, we can assume $u_i \to u \in C \subset D$ and $\|u\| = 1$. On the other hand, by taking limits in equation (3.8), we obtain $-u \in D$; and since $u \in D$ and $-u \in D$, we have a contradiction.

Thus it suffices to prove that $\alpha(S) = 0$, where α denotes the Kuratowski MNC. If we write $\Gamma_k := \{u_i \mid i > k\}$, we see that $\alpha(S) = \alpha(\Gamma_k)$. We claim that

$$\Gamma_k \subset C_{n_{k+1}}(f_\lambda; V_1), \tag{3.9}$$

where $V_1 := \{x \in C \mid \|x\| < 1\}$. To see this, note by (3.7) that if $i > k$, the norm of $R_i^{-1} f_\lambda^j(w_i)$ is strictly less than $+1$ for $0 \leq j < n_i$. For $i > k$, this implies that

$$R_i^{-1} f_\lambda^j(w_i) \in C_j(f_\lambda; V_1) \cap V_1$$

for $0 < j < n_i$, so

$$R_i^{-1} f_\lambda^{n_i}(w_i) \in C_{n_i}(f_\lambda; V_1) \subset C_{n_{k+1}}(f_\lambda, V_1),$$

which proves equation (3.9). Because $\rho_C(f) < \lambda$, we see that $\lim_{k\to\infty} \alpha(C_{n_k}(f_\lambda; V_1)) = 0$; and we conclude that

$$\alpha(S) = \lim_{k\to\infty} \alpha(\Gamma_k) \leq \lim_{k\to\infty} \alpha(C_{n_k}(f_\lambda; V_1)) = 0,$$

so $\alpha(S) = 0$. ■

There are naturally occurring examples of maps $f : C \to C$ which are D-order-preserving but not necessarily C-order-preserving. See, for instance, [17] and the "renormalization operators" which occur in discussing diffusion on fractals.

Question C. If C is a complete cone in an NLS $(X, \|\cdot\|)$ and $f : C \to C$ is continuous, homogeneous and C-order-preserving, does it follow that $r_C(f) = \widetilde{r}_C(f)$?

The assumption that $\rho_C(f) < \widetilde{r}_C(f)$ is a compactness assumption, so the following compactness result concerning eigenvectors of f is unsurprising.

Theorem 3.5. *Let the hypotheses and notation be as in the statement of Theorem 3.4. If* $\rho_C(f) < \lambda < \widetilde{r}_C(f)$, *define*

$$T := \{x \in C \mid \|x\| = 1 \text{ and } f(x) = tx \text{ for some } t \geq \lambda\}.$$

Then T is compact (possibly empty).

Proof. If $t \in T$, it is easy to see that $\mu(x) = t$, so $t \leq r_C(f) = \widetilde{r}_C(f)$. Take λ_1 with $\rho_C(f) < \lambda_1 < \lambda$. If $x \in T$ with $f(x) = tx$ and $n \geq 1$, set $\varepsilon := (\lambda_1/t)^n$. Then one has that

$$f_{\lambda_1}^n(\varepsilon x) = x.$$

Since $t \geq \lambda > \lambda_1$, one has $f_{\lambda_1}^j(\varepsilon x) = (\lambda_1/t)^{n-j} x \in V_1 := \{y \in C \mid \|y\| < 1\}$ for $0 \leq j < n$. This shows that $x \in C_n(f_{\lambda_1}; V_1)$ and thus $T \subset C_n(f_{\lambda_1}; V_1)$. If α denotes the Kuratowski MNC, it follows that

$$\alpha(T) \leq \alpha(C_n(f_{\lambda_1}; V_1))$$

for all $n \geq 1$. Since $\rho_C(f) < \lambda_1$, we conclude that $\alpha(T) = 0$; and since T is clearly closed, T is compact. ■

Corollary 3.6. *Let hypotheses and notation be as in Theorem 3.4. Assume that* $s_k \to r_C(f)$ *and* $f(x_k) = s_k x_k$*, where* $x_k \in C$ *and* $\|x_k\| = 1$*. Then there exists a sequence of integers* $k_i \to \infty$ *with* $x_{k_i} \to x$ *and* $f(x) = rx$*, where* $r := r_C(f)$*.*

Proof. By Theorem 3.5, the set $\{x_k \mid k \geq 1\}$ has compact closure, so there is a sequence $k_i \to \infty$ with $x_{k_i} \to x$; and the corollary follows from the continuity of f. ■

If $\rho(L)$ denotes the essential spectral radius of a bounded linear operator L on a Banach space X, it is well-known (use equation (1.10)) that $\rho(L^m) = \rho(L)^m$. An analogous result is true for continuous, homogeneous cone mappings.

Theorem 3.7. *Let* C *be a complete cone in an NLS* $(X, \|\cdot\|)$ *and* $f : C \to C$ *continuous and homogeneous. Then*

$$\rho_C(f^m) = (\rho_C(f))^m$$

for every positive integer m*.*

Proof. Fix m. For notational convenience we write $g = f^m$; and if $\lambda > 0$, we shall write $\sigma := \lambda^m$ so $\lambda = \sigma^{1/m}$. In the notation of equation (3.4), we have $g_\sigma = f_\lambda^m$. Let V be a bounded, relatively open neighborhood of 0 in C and, for $\lambda > 0$, let $W = W_\lambda$ be a bounded, relatively open neighborhood of 0 in C such that $f_\lambda^j(W) \subset V$ for $0 \leq j < m$. Let $\lambda > \rho_C(f)$. Then equation (3.3) shows that

$$C_k(g_\sigma; W) = \{g_\sigma^k(x) \mid g_\sigma^j(x) \in W \text{ for } 0 \leq j < k\}.$$

Because $g_\sigma^j(x) = f_\lambda^{jm}(x)$ and $f_\lambda^p(W) \subset V$ for $0 \leq p < m$, we see that if $g_\sigma^j(x) \in W$ for $0 \leq j < k$, then $f_\lambda^s(x) \in V$ for $0 \leq s < km$, so $f_\lambda^{km}(x) \in C_{km}(f_\lambda; V)$. It follows that

$$C_k(g_\sigma; W) \subset C_{km}(f_\lambda; V). \tag{3.10}$$

Since we assume that $\lambda > \rho_C(f)$, equation (3.10) implies (denoting the Kuratowski MNC by α) that

$$\lim_{k\to\infty} \alpha(C_{km}(f_\lambda; V)) = 0, \qquad \text{hence } \lim_{k\to\infty} \alpha(C_k(g_\sigma; W)) = 0,$$

so $\sigma = \lambda^m > \rho_C(g)$. Letting λ approach $\rho_C(f)$, we conclude that

$$(\rho_C(f))^m \geq \rho_C(g) = \rho_C(f^m). \tag{3.11}$$

Using equation (3.3) again yields

$$C_k(g_\sigma; V) = \{g_\sigma^k(x) = f_\lambda^{km}(x) \mid f_\lambda^{kj}(x) \in V \text{ for } 0 \leq j < m\},$$

and it follows that

$$C_k(g_\sigma; V) \supset C_{km}(f_\lambda; V).$$

If $\sigma > \rho_C(g)$ then $\lim_{k\to\infty} \alpha(C_k(g_\sigma; V)) = 0$, so we conclude that

$$\lim_{k\to\infty} \alpha(C_{km}(f_\lambda; V)) = \lim_{n\to\infty} \alpha(C_n(f_\lambda; V)) = 0$$

and hence that $\lambda = \sigma^{1/m} > \rho_C(f)$ so $\sigma > (\rho_C(f))^m$. Letting σ approach $\rho_C(f)$ yields that

$$\rho_C(f^m) \geq (\rho_C(f))^m, \tag{3.12}$$

and combining equations (3.11) and (3.12) completes the proof. ∎

4 Positive Eigenvectors of Homogeneous, Order-Preserving Noncompact Operators

The starting point of this section is the following conjecture.

Conjecture 4.1. Let C and D be complete cones with $C \subset D$ in an NLS $(X, \|\cdot\|)$. Let $f : C \to C$ be continuous, homogeneous, and D-order-preserving. Assume that $\rho_C(f) < \widetilde{r}_C(f)$. Then there exists $x \in C$ with $\|x\| = 1$ satisfying $f(x) = rx$, where $r := \widetilde{r}_C(f)$.

If f in Conjecture 4.1 is nonlinear and noncompact, we are very far from proving the conjecture. However, we know of no counterexample.

Lemma 4.2. *Let C, D, X and f be as in Conjecture 4.1. Select λ satisfying $\rho_C(f) < \lambda < \widetilde{r}_C(f)$ and define $f_\lambda(x) := \lambda^{-1} f(x)$. Then there exists $u \in C$ with $\limsup_{k\to\infty} \|f_\lambda^k(u)\| = \infty$. Further, if $s > 0$ then the equation $f_\lambda(x) + su = x$ has no solution in C.*

Proof. By Theorem 3.4 we have $r_C(f) = \widetilde{r}_C(f)$, so the definition of $r_C(f)$ implies the existence of u with the stated property.

Assume by way of contradiction that $s > 0$ and that $f_\lambda(x) + su = x$, where $x \in C$. If $\leq$ denotes the partial ordering on X induced by D, then $su \leq x$. We claim that $sf_\lambda^n(u) \leq x$ for all $n \geq 0$. Assume, by mathematical induction, that $sf_\lambda^n(u) \leq x$ for some $n \geq 0$. Because f_λ is D-order-preserving,

$$sf_\lambda^{n+1}(u) + su \leq f_\lambda(x) + su = x,$$

which proves that $sf_\lambda^{n+1}(u) \leq x$. This establishes the claim. Because of the assumption on u in the statement of the lemma, there exists a strictly increasing sequence of integers n_i, for $i \geq 1$, such that

$$\|f_\lambda^j(u)\| < \|f_\lambda^{n_i}(u)\|, \qquad \text{for } 0 \leq j < n_i.$$

Define $v_i := R_i^{-1} f_\lambda^{n_i}(u)$ where $R_i := \|f_\lambda^{n_i}(u)\|$, and let $S := \{v_i \mid i \geq 1\}$. Because $sf_\lambda^j(u) \leq x$ for all $j \geq 1$, we have that

$$R_i^{-1}x - sv_i \in D, \qquad \text{for } i \geq 1. \tag{4.1}$$

If $\overline{S}$ is compact, we can assume by taking a further subsequence that $v_i \to v \in C$ where $\|v\| = 1$. Then taking the limit as $i \to \infty$ in equation (4.1), we then obtain that $-v \in D$. But since $v \in C \subset D$ and D is a cone, this is a contradiction.

Thus it suffices to prove that $\overline{S}$ is compact, or, equivalently, that $\alpha(S) = 0$ where α is the Kuratowski MNC. We argue as in Theorem 3.4. If we let $V := \{x \in C \mid \|x\| < 1\}$, then by our definition of n_i

$$R_i^{-1} f_\lambda^j(u) \in V, \qquad \text{for } 0 \leq j < n_i.$$

Since $n_i \geq i$, we have $v_i \in C_k(f_\lambda; V)$ for all $i \geq k$ and it follows that

$$\alpha(S) = \alpha(\{v_i \mid i \geq k\}) \leq \alpha(C_k(f_\lambda; V)).$$

Since $\alpha(C_k(f_\lambda; V)) \to 0$ as $k \to \infty$, we have $\alpha(S) = 0$, as desired. ■

Our approach to Conjecture 4.1 will be through the "fixed point index." We refer the reader to [6], [9], [12], [19], [25] and [30] for descriptions of the classical fixed point index and some of its generalizations. If we could define a "reasonable" fixed point index for maps f as in Conjecture 4.1, then we could prove Conjecture 4.1. The problem is that no such generalization of the fixed point index is known; and even if f is C-linear, there are some technical difficulties.

For purposes of describing situations in which a reasonable fixed point index is defined, it will be useful to establish some notation. For the remainder of this section the following hypotheses and notation will generally be assumed:

(C1) $C \subset D$ are complete cones in an NLS $(X, \|\cdot\|)$ and $f : C \to C$ is continuous, homogeneous and D-order-preserving.

If C is a complete cone in an NLS $(X, \|\cdot\|)$ and V is a bounded, relatively open neighborhood of 0 in C, and if $u \in C \setminus \{0\}$ and $g : C \to C$ is continuous and homogeneous, we define $G_m := G_m(g; V, u)$ for $m \geq 0$ inductively by

$$G_0(g; V, u) := \{tu \mid 0 \leq t \leq 1\} := S_u \tag{4.2}$$

and

$$G_m(g; V, u) := \text{co}(S_u + g(V \cap G_{m-1}(g; V, u))), \qquad \text{for } m \geq 1. \tag{4.3}$$

Recall that $\text{co}(T)$ denotes the convex hull of a set $T \subset X$. In general, if S is a bounded subset of C, we define $K_n := K_n(g; V, S)$ for $n \geq 1$ inductively by

$$K_1(g; V, S) := \text{co}(S + g(V)) \tag{4.4}$$

and

$$K_n(g; V, S) := \text{co}(S + g(V \cap K_{n-1}(g; V, S)), \qquad \text{for } n \geq 2. \tag{4.5}$$

Sets like K_n have been used extensively in [25], and sets like G_m have been used by H. Mönch; see Theorem 2.1 in [22]. It is easy to see that $S_u \subset G_m$ for all m, that $G_m \subset G_{m+1}$ for all m, and that $S_u + g(V \cap G_m) \subset G_{m+1}$ for all m. Similarly, we have that $S \subset K_n$ for all n, that $K_n \supset K_{n+1}$ for all n, and that $g(V \cap K_n) + S \subset K_{n+1}$ for all n.

Lemma 4.3. *If C, V, u and g are as above, and $S := \{tu \mid 0 \leq t \leq 1\}$, then $G_m \subset K_n$ for all $m \geq 0$ and $n \geq 1$, where G_m and K_n are as in (4.2)-(4.5).*

Proof. Since $S \subset K_n$ for all $n \geq 1$, we have that $G_0 = S \subset \bigcap_{n \geq 1} K_n$. Now assume that $G_m \subset \bigcap_{n \geq 1} K_n$ for some $m \geq 0$. Then $G_m \subset K_n$ for all $n \geq 1$, so

$$G_{m+1} = \text{co}(S + g(V \cap G_m)) \subset \text{co}(S + g(V \cap K_n)) = K_{n+1}$$

for all $n \geq 1$, and since $K_n \supset K_{n+1}$ for all n, we have that $G_{m+1} \subset \bigcap_{n \geq 1} K_n$. By mathematical induction, the lemma follows. ■

It will be convenient to define $G_\infty := G_\infty(g; V, u)$ and $K_\infty := K_\infty(g; V, S)$ by

$$G_\infty(g; V, u) = \bigcup_{m \geq 0} G_m(g; V, u)$$

and

$$K_\infty(g; V, S) = \bigcap_{n \geq 0} K_n(g; V, S).$$

The reader can verify that if $0 \leq t \leq 1$ then

$$g(x) + tu \in \overline{G_\infty(g; V, u)}$$

for $x \in V \cap \overline{G_\infty(g; V, u)}$, and that

$$g(x) + tu \in \overline{K_\infty(g; V, u)}$$

for $x \in V \cap \overline{K_\infty(g; V, u)}$, where the horizontal bar as usual denotes the closure of a set.

With this notation we can state a hypothesis which ensures the existence of eigenvectors for maps $f : C \to C$ as in (C1):

(C2) Assume that (C1) is satisfied and that $\rho_C(f) < \widetilde{r}_C(f) := r$, where $\rho_C(f)$ and $\widetilde{r}_C(f)$ are given by equations (3.5) and (1.5), respectively. For $\lambda > 0$ define $f_\lambda : C \to C$ by $f_\lambda(x) := \lambda^{-1} f(x)$. Assume that there exist a bounded, relatively open neighborhood V of 0 in C, a sequence $\{\lambda_k\}_{k \geq 1}$ with $\rho_C(f) < \lambda_k < r$ for all k and $\lim_{k \to \infty} \lambda_k = r$, and a sequence of vectors $u_k \in C$ for $k \geq 1$ such that $\limsup_{j \to \infty} \|f^j_{\lambda_k}(u_k)\| = \infty$ and $\overline{G_\infty(f_{\lambda_k}; V, u_k)}$ is compact for $k \geq 1$.

Theorem 4.4. *Assume that (C2) is satisfied. Then there exists $v \in C \setminus \{0\}$ with $f(v) = rv$.*

Proof. By Corollary 3.6 it suffices to prove that f_{λ_k} has an eigenvector in C with eigenvalue $s_k \geq 1$, for each $k \geq 1$. Note in particular that this means $\lambda_k s_k$ is an eigenvalue of f, and thus must satisfy $\lambda_k \leq \lambda_k s_k \leq r$. For notational convenience we define $D_k := \overline{G_\infty(f_{\lambda_k}; V, u_k)}$, so D_k is compact and convex. By our previous remarks, if $x \in V \cap D_k$ and $0 \leq t \leq 1$, then $f_{\lambda_k}(x) + tu_k \in D_k$, so the same is true if $x \in \overline{V \cap D_k}$. If $f_{\lambda_k}(x) = x$ for some $x \in \overline{V \cap D_k} \setminus (V \cap D_k)$, then we have the desired eigenvector, so we assume that $f_{\lambda_k}(x) \neq x$ for all $x \in \overline{V \cap D_k} \setminus (V \cap D_k)$. Lemma 4.2 implies that $f_{\lambda_k}(x) + tu_k \neq x$ for all $x \in \overline{V \cap D_k}$ and $0 < t \leq 1$. Because D_k is compact and convex, the fixed point index is defined for continuous functions $h : \overline{V \cap D_k} \to D_k$ with $h(x) \neq x$ for all $x \in \overline{V \cap D_k} \setminus (V \cap D_k)$. It follows by considering the homotopy $f_{\lambda_k}(x) + tu_k$, for $0 \le t \le 1$, and using the properties of the fixed point index, that

$$i_{D_k}(f_{\lambda_k}, V \cap D_k) = 0.$$

On the other hand, suppose that $f_{\lambda_k}(x) \neq sx$ for $s \ge 1$ and $x \in \overline{V \cap D_k} \setminus (V \cap D_k)$. Then it follows, by considering the homotopy $tf_{\lambda_k}(x)$ for $0 \le t \le 1$, that

$$i_{D_k}(f_{\lambda_k}, V \cap D_k) = 1,$$

a contradiction. Thus f_{λ_k} has an eigenvector with eigenvalue $s_k \ge 1$, and we are done. ■

Corollary 4.5. *Assume that (C2) is satisfied, but replace the assumption that $\overline{G_\infty(f_{\lambda_k}; V, u_k)}$ is compact for $k \ge 1$ by the assumption that $\overline{K_\infty(f_{\lambda_k}; V, S_k)}$ is compact for $k \ge 1$, where $S_k := \{tu_k \mid 0 \le t \le 1\}$. Then there exists $v \in C \setminus \{0\}$ with $f(v) = rv$.*

Proof. If $\overline{K_\infty(f_{\lambda_k}; V, S_k)}$ is compact, then Lemma 4.3 implies that $\overline{G_\infty(f_{\lambda_k}; V, u_k)}$ is compact, and thus Corollary 4.5 follows from Theorem 4.4. ■

Our next corollary gives the main results of Section 2 of [29]; see also Section 3 of [18] and Proposition 6 on page 525 of [28].

Corollary 4.6 (See Section 2 of [29]). *Assume that (C1) holds. Also assume that $r := \widetilde{r}_C(f) > 0$ and that there exist μ with $0 < \mu < r$, a weakly homogeneous MNC β on C and a quantity $k < 1$, such that*

$$\beta(f_\mu(S)) \le k\beta(S) \tag{4.6}$$

for all bounded sets $S \subset C$. Then $\rho_C(f) \le k\mu$, and f has an eigenvector in C with eigenvalue r.

Proof. If V is a bounded, relatively open neighborhood of 0 in C, it is clear that $C_n(f_\mu; V) \subset f_\mu^n(V)$, so

$$\beta(C_n(f_\mu; V)) \le k^n \beta(V) \to 0,$$

and $\rho_C(f) \le k\mu$. Note that equation (4.6) remains true if f_μ is replaced by f_λ and $\lambda \ge \mu$. By Theorem 3.4 we have that $r_C(f) = \widetilde{r}_C(f)$, so by Lemma 4.2 there exists a sequence $\{\lambda_j\}_{j \ge 1}$, with $\mu < \lambda_j < r$ and $\lim_{j \to \infty} \lambda_j = r$, and $u_j \in C$ for $j \ge 1$ with $\limsup_{n \to \infty} \|f^n_{\lambda_j}(u_j)\| = \infty$. Let $V := \{x \in C \mid \|x\| < 1\}$

and $S_j := \{tu_j \mid 0 \leq t \leq 1\}$. By Corollary 4.5, it suffices to prove that, for $j \geq 1$,

$$\lim_{n\to\infty} \beta(K_n(f_{\lambda_j}; V, S_j) = 0,$$

as that implies that $\overline{K_\infty(f_{\lambda_j}; V, S_{u_j})}$ is compact. Fix j, write $g := f_{\lambda_j}$ and $S := S_j$, and let $K_n := K_n(g; V, S)$. We have that

$$\beta(K_1) = \beta(\text{co}(S + g(V))) = \beta(S + g(V)) = \beta(g(V)) \leq k\beta(V).$$

Assume, by induction, that $\beta(K_n) \leq k^n\beta(V)$. Then we have

$$\beta(K_{n+1}) = \beta(\text{co}(S + g(V \cap K_n))) = \beta(S + g(V \cap K_n)) = \beta(g(V \cap K_n)) \leq k\beta(K_n).$$

Since $\beta(K_n) \leq k^n\beta(V)$, this completes the inductive step and proves the corollary. ■

Remark 4.7. Assume all the hypotheses of Corollary 4.6 hold, except that in place of (4.6) assume that for some integer $p \geq 1$ we have

$$\beta(f_\mu^p(S)) \leq k\beta(S)$$

for all bounded sets $S \subset C$. (Here f_μ^p denotes the p^{th} iterate of the map f_μ.) Then Corollary 4.6 implies (since $f_\mu^p = \mu^{-p} f^p$) that $\rho_C(f^p) \leq k\mu^p$; and Theorem 3.7 implies that $\rho_C(f) < \mu$. Since (see equation (3.6) and [18]) we have $\widetilde{r}_C(f^p) = (\widetilde{r}_C(f))^p$, Corollary 4.6 implies that there exists $x \in C \setminus \{0\}$ with $f^p(x) = r^p x$.

However, Conjecture 4.1 suggests that there exists $u \in C \setminus \{0\}$ with $f(u) = ru$, which is *not* known. The discrepancy here is closely analogous to an old and apparently intractable conjecture in "asymptotic fixed point theory." If G is a closed, bounded convex set in a Banach space and $f : G \to G$ is a continuous map such that f^p is compact for some integer $p \geq 2$, then it has long been conjectured that f has a fixed point. Although a variety of partial results are known (see [19], [26] and [27]), the general conjecture remains open. The difficulties in proving this conjecture are analogous to the difficulties in studying Conjecture 4.1.

We shall now consider the case in which our map $f : C \to C$ is a compact perturbation of a C-linear map. In this case, as we shall see, Conjecture 4.1 is essentially true. We collect some relevant assumptions in the following hypothesis:

(C3) $C \subset D$ are complete cones in an NLS $(X, \|\cdot\|)$. The map $g : C \to C$ is continuous, C-linear and D-order-preserving, and the map $h : C \to C$ is continuous, compact, homogeneous and D-order-preserving.

Lemma 4.8. *Assume that (C3) holds. Define $f(x) := g(x) + h(x)$ for $x \in C$. Then with $h_j : C \to C$ defined by the equation $f^j(x) = g^j(x) + h_j(x)$ for $j \geq 1$, it is the case that h_j is continuous, compact, homogeneous and D-order-preserving for $j \geq 1$. If additionally D is normal, then $\rho_C(f) \leq \rho_C(g)$ and $\widetilde{r}_C(g) \leq \widetilde{r}_C(f)$.*

Proof. We prove the first claim, concerning the map h_j, by mathematical induction. This claim is true for $j = 1$ by (C3), so assume for some $j \geq 1$ that the map h_j is continuous, compact, homogeneous and D-order-preserving. Because g is C-linear, it follows that

$$f^{j+1}(x) = g^{j+1}(x) + g(h_j(x)) + h_j(g^j(x) + h_j(x)) = g^{j+1}(x) + h_{j+1}(x).$$

The composition of two continuous, homogeneous maps from C to C, with one of the maps being compact, is necessarily compact, so h_{j+1} is a sum of compact maps and is continuous, compact and homogeneous. The composition of D-order-preserving maps from C to C is D-order-preserving, and thus h_{j+1} satisfies the required properties. The proves the first claim of the lemma.

Now assuming D is normal, we can assume that the norm $\|\cdot\|$ on X satisfies $\|u\| \leq \|v\|$ whenever $0 \leq u \leq v$, where $\leq$ denotes the partial ordering induced on X by the cone D. Let $V := \{x \in C \mid \|x\| < 1\}$. By definition, we have, recalling equations (3.1)-(3.3) and defining $f_\lambda(x) := \lambda^{-1} f(x) := g_\lambda(x) + h_\lambda(x)$ for $\lambda > 0$,

$$f_\lambda(V) = C_1(f_\lambda; V) \subset g_\lambda(V) + h_\lambda(V) = C_1(g_\lambda; V) + S_1,$$

where $S_1 := h_\lambda(V)$ and $\overline{S}_1$ is compact. We claim that for every $k \geq 1$

$$C_k(f_\lambda; V) \subset C_k(g_\lambda; V) + S_k, \tag{4.7}$$

where $S_k \subset C$ and $\overline{S}_k$ is compact. Assume, using mathematical induction, that (4.7) is true for some $k \geq 1$. If $y \in C_{k+1}(f_\lambda; V)$ we know that $y = f_\lambda^{k+1}(x)$ where $f_\lambda^j(x) \in V$ for $0 \leq j \leq k$. By the first part of the lemma

$$f_\lambda^j(x) = g_\lambda^j(x) + h_{\lambda,j}(x) \in V$$

for $0 \le j \le k$; and using the normality of D we conclude that for such j

$$\|g_\lambda^j(x)\| \le \|f_\lambda^j(x)\| < 1.$$

It follows that $g_\lambda^{k+1}(x) \in C_{k+1}(g_\lambda; V)$ and

$$f_\lambda^{k+1}(x) = g_\lambda^{k+1}(x) + h_{\lambda,k+1}(x).$$

If we define $S_{k+1} := h_{\lambda,k+1}(V)$, we conclude that

$$C_{k+1}(f_\lambda; V) \subset C_{k+1}(g_\lambda; V) + S_{k+1},$$

where $S_{k+1} \subset C$ and $\overline{S}_{k+1}$ is compact.

If $\lambda > \rho_C(g)$ and α denotes the Kuratowski MNC on X, it follows from equation (4.7) that

$$\lim_{k\to\infty} \alpha(C_{k+1}(f_\lambda; V)) = 0,$$

which implies that $\lambda > \rho_C(f)$ and thus $\rho_C(f) \le \rho_C(g)$.

Because C is normal, the fact that $f^j(x) = g^j(x) + h_j(x)$, where $h_j(x) \in C$, implies that $\|f^j(x)\| \ge \|g^j(x)\|$ for all $x \in C$ and $j \ge 1$. This, in turn, implies that $\|g^j\|_C \le \|f^j\|_C$ for $j \ge 1$ and $\widetilde{r}_C(g) \le \widetilde{r}_C(f)$. ■

Theorem 4.9. *Assume that (C3) holds and define $f(x) := g(x) + h(x)$ for $x \in C$. Also assume that D is normal and that either*

(a) $\rho_C(g) < \widetilde{r}_C(g)$; or

(b) $\widetilde{r}_C(g) < \widetilde{r}_C(f)$.

Then we have that

$$\rho_C(f) \le \rho_C(g), \qquad r_C(f) = \widetilde{r}_C(f) \ge \widetilde{r}_C(g), \qquad \widetilde{r}_C(f) > \rho_C(f)$$

and that f has an eigenvector in C with eigenvalue equal to $r_C(f)$.

Proof. By Lemma 4.8 we have $\rho_C(f) \le \rho_C(g)$ and $\widetilde{r}_C(f) \ge \widetilde{r}_C(g)$. In either case (a) or case (b) of the theorem it follows that $\widetilde{r}_C(f) > \rho_C(f)$, and Theorem 3.3 or Theorem 3.4 implies that $r_C(f) = \widetilde{r}_C(f)$. Select a sequence of positive reals $\{\lambda_k\}_{k\ge1}$ with $\rho_C(g) < \lambda_k < r_C(f)$ for all k in case (a), or with

$\widetilde{r}_C(g) < \lambda_k < r_C(f)$ for all k in case (b), and which also satisfies $\lim_{k\to\infty} \lambda_k = r_C(f)$ in either case. By definition of $r_C(f)$, there exist $u_k \in C$ such that $\|u_k\| = 1$ for $k \geq 1$ and $\limsup_{j\to\infty} \|f^j_{\lambda_k}(u_k)\| = \infty$. Since D is normal, we may assume that $\|u\| \leq \|v\|$ whenever $0 \leq u \leq v$, where $\leq$ denotes the partial ordering induced on X by the cone C; and we define $V := \{x \in C \,|\, \|x\| < 1\}$ and $S_k := \{tu_k \,|\, 0 \leq t \leq 1\}$ for $k \geq 1$.

By Corollary 4.5, it suffices to prove that $\overline{K_\infty(f_{\lambda_k}; V, S_k)}$ is compact for $k \geq 1$. If α denotes the Kuratowski MNC on X, it suffices (see equation (4.5)) to prove that

$$\lim_{n\to\infty} \alpha(K_n(f_{\lambda_k}; V, S_k)) = 0. \tag{4.8}$$

Now fix $k \geq 1$. Because $\lambda_k > \rho_C(g)$ in case (a) or case (b), we know that

$$\lim_{n\to\infty} \alpha(C_n(g_{\lambda_k}; V)) = 0,$$

so to prove equation (4.8) it suffices to prove that, for $n \geq 1$,

$$\alpha(K_n(f_{\lambda_k}; V, S_k)) \leq \alpha(C_n(g_{\lambda_k}; V)). \tag{4.9}$$

Equation (4.9) will hold if we prove that, for $n \geq 1$,

$$K_n(f_{\lambda_k}; V, S_k) \subset C_n(g_{\lambda_k}; V) + T_n, \tag{4.10}$$

where T_n is a convex subset of C and $\overline{T}_n$ is compact. We shall prove this by mathematical induction. Define $T_1 := \text{co}(h_{\lambda_k}(V) + S_k)$. Then $T_1 \subset C$ is convex and $\overline{T}_1$ is compact because $\overline{h_{\lambda_k}(V)}$ and S_k are compact. Also,

$$f_{\lambda_k}(V) + S_k \subset g_{\lambda_k}(V) + \text{co}(h_{\lambda_k}(V) + S_k) = g_{\lambda_k}(V) + T_1.$$

Because $g_{\lambda_k}(V)$ and T_1 are convex, so is $g_{\lambda_k}(V) + T_1$ and

$$K_1(f_{\lambda_k}; V, S_k) = \text{co}(f_{\lambda_k}(V) + S_k) \subset g_{\lambda_k}(V) + T_1 = C_1(g_{\lambda_k}; V) + T_1.$$

This establishes equation (4.10) for $n = 1$. Now assume that (4.10) holds for some $n \geq 1$, with $T_n \subset C$ convex and $\overline{T}_n$ compact. If $y \in K_n(f_{\lambda_k}; V, S_k) \cap V$, it follows that $\|y\| < 1$ and $y = u + v$ there $u \in C_n(g_{\lambda_k}; V)$ and $v \in T_n$. Since C is normal, $\|u\| \leq \|y\| < 1$, so $u \in C_n(g_{\lambda_k}; V) \cap V$ and $g_{\lambda_k}(u) \in C_{n+1}(g_{\lambda_k}; V)$. It follows that

$$f_{\lambda_k}(y) = g_{\lambda_k}(y) + h_{\lambda_k}(y) = g_{\lambda_k}(u) + g_{\lambda_k}(v) + h_{\lambda_k}(y) \subset C_{n+1}(g_{\lambda_k}; V) + g_{\lambda_k}(T_n) + h_{\lambda_k}(V).$$

This implies that

$$S_k + f_{\lambda_k}(K_n(f_{\lambda_k}; V, S_k) \cap V) \subset C_{n+1}(g_{\lambda_k}; V) + g_{\lambda_k}(T_n) + \mathrm{co}(h_{\lambda_k}(V) + S_k).$$

If we define

$$T_{n+1} := g_{\lambda_k}(T_n) + \mathrm{co}(h_{\lambda_k}(V) + S_k),$$

then $T_{n+1} \subset C$ is convex and $\overline{T}_{n+1}$ is compact, where we have used C-linearity and continuity of g_{λ_k} and compactness of h_{λ_k}. Since $C_{n+1}(g_{\lambda_k}; V)$ is convex, so is $C_{n+1}(g_{\lambda_k}; V) + T_{n+1}$, and (4.10) holds with $n+1$ in place of n. This completes the proof. ■

The argument in Theorem 4.9 uses the normality of D. The following variant theorem does not require that D be normal but imposes a stronger condition on g.

Theorem 4.10. *Assume that (C3) holds and define $f(x) := g(x) + h(x)$ for $x \in C$. Let $V := \{x \in C \mid \|x\| < 1\}$, and assume there exists a weakly homogeneous MNC β on C and a quantity λ satisfying $0 < \lambda < \tilde{r}_C(g)$ such that*

$$\lim_{j\to\infty} \beta((g_\lambda^j(V)) = 0.$$

Then it follows that $\rho_C(f) \le \lambda$ and $\tilde{r}_C(f) = r_C(f) \ge \tilde{r}_C(g)$. Also, there exists $v \in C$ with $\|v\| = 1$ and $f(v) = rv$ where $r := \tilde{r}_C(f)$.

Proof. We first prove the theorem in the case $h(x) \equiv 0$. For $t \ge \lambda$, our hypothesis implies that $\lim_{j\to\infty} \beta(g_t^j(V)) = 0$. It is easy to see that

$$C_n(g_t; V) \subset g_t^n(V),$$

so $\lim_{n\to\infty} \beta(C_n(g_t; V)) = 0$ for $t \ge \lambda$, which implies that $\rho_C(g) \le \lambda$. By Theorem 3.4, it follows that $r_C(g) = \tilde{r}_C(g)$, so there exists a sequence $\{t_k\}_{k\ge 1}$ with $\lambda < t_k < r_C(g)$ and $\lim_{k\to\infty} t_k = r_C(g)$ and a sequence of vectors $u_k \in C$ with $\|u_k\| = 1$ and

$$\limsup_{j\to\infty} \|g_{t_k}^j(u_k)\| = \infty.$$

By Corollary 4.5, if $S_k := \{su_k \mid 0 \le s \le 1\}$ and $\overline{K_\infty(g_{t_k}; V, S_k)}$ is compact, then there exists $v \in C$ with $\|v\| = 1$ and $g(v) = r_C(g)v$. Thus it suffices to prove that if $u \in C$ with $\|u\| = 1$, and $S := \{su \mid 0 \le s \le 1\}$ and $t > \lambda$, then

$$\lim_{n\to\infty} \beta(K_n(g_t; V, S)) = 0.$$

The latter equation will hold if we prove that for each $n \geq 1$, there exists a convex set $T_n \subset C$ with $\overline{T}_n$ compact such that

$$K_n(g_t; V, S) \subset g_t^n(V) + T_n. \tag{4.11}$$

If $n = 1$ we have

$$K_1(g_t; V, S) = \text{co}(g_t(V) + S) = g_t(V) + S,$$

which proves equation (4.11) for $n = 1$. Arguing by mathematical induction, assume that equation (4.11) holds for some $n \geq 1$. Then we have

$$\begin{aligned} K_{n+1}(g_t; V, S) &= \text{co}(g_t(K_n(g_t; V, S) \cap V) + S) \\ &\subset \text{co}(g_t(g_t^n(V) + T_n) + S) \subset g_t^{n+1}(V) + g_t(T_n) + S. \end{aligned}$$

If we define $T_{n+1} := g_t(T_n) + S$, this completes the inductive step. It follows that there exists $v \in C$ with $\|v\| = 1$ such that

$$g(v) = r_C(g)v. \tag{4.12}$$

The reason for establishing (4.12) is to prove that $r_C(f) \geq r_C(g)$, which is trivially true with D is normal. Letting $\leq$ denote the partial ordering induced on X by D, then because f is D-order-preserving, we obtain from equation (4.12) that, for $k \geq 1$,

$$v \leq (r_C(g))^{-k} f^k(v). \tag{4.13}$$

If $r_C(f) < r_C(g)$, then (4.13) implies, by letting $k \to \infty$, that $-v \in D$, which contradicts the fact that D is a cone. It follows that $r_C(f) \geq r_C(g)$.

Another straightforward induction argument, which we leave to the reader, shows that for each $n \geq 1$,

$$C_n(f_t; V) \subset g_t^n(V) + \Gamma_n,$$

where $\Gamma_n \subset C$ is convex and $\overline{\Gamma}_n$ is compact. It follows that $\rho_C(f) \leq \lambda < r_C(f)$, so $\tilde{r}_C(f) = r_C(f)$.

Select a sequence $\{t_k\}_{k \geq 1}$ with $\lambda \leq t_k < r_C(f)$ and $t_k \to r_C(f)$, and for each k select $u_k \in C$ with $\|u_k\| = 1$ with

$$\limsup_{j \to \infty} \|f_{t_k}^j(u_k)\| = \infty.$$

If $S_k := \{su_k \mid 0 \leq s \leq 1\}$, Corollary 4.5 implies that to complete the proof it suffices to prove that $\overline{K_\infty(f_{t_k}; V, S_k)}$ is compact for every $k \geq 1$. As in Theorem 4.9, if $\lambda \leq t < r_C(f)$, and $u \in C$ and

$S := \{su \mid 0 \le s \le 1\}$, it suffices to prove that

$$\lim_{n\to\infty} \beta(K_n(f_t; V, S)) = 0,$$

and the latter equation will hold if, for each $n \ge 1$, there exists a convex set $\Gamma_n \subset C$ with $\overline{\Gamma}_n$ compact such that

$$K_n(f_t; V, S) \subset g_t^n(V) + \Gamma_n. \tag{4.14}$$

If we define $\Gamma_1 := \text{co}(h_t(V) + S)$, we clearly have

$$K_1(f_t; V, S) \subset g_t(V) + \Gamma_1,$$

and $\Gamma_1 \subset C$ is convex with $\overline{\Gamma}_1$ compact. If we argue by induction and assume that equation (4.14) is satisfied, the reader can verify that

$$K_{n+1}(f_t; V, S) \subset g_t^{n+1}(V) + \Gamma_{n+1},$$

where

$$\Gamma_{n+1} := g_t(\Gamma_n) + \text{co}(h_t(V)) + S,$$

and that $\Gamma_{n+1} \subset C$ is convex and $\overline{\Gamma}_{n+1}$ is compact. This completes the proof. ■

Aside from the assumption that D is normal, Theorem 4.9 is essentially the best possible result concerning positive eigenvalues and eigenvectors of a continuous C-linear map $g : C \to C$.

Question D. Is Theorem 4.9 true without the assumption that D is normal?

5 A Class of Examples: Max-Type Operators

We shall briefly discuss in this concluding section some new results concerning concrete classes of operators for which Conjecture 4.1 remains unresolved. The operators we consider generalize max-type operators treated in Section 4 of [18]. In a limiting case, our own operators become so-called linear "Perron-Frobenius operators," which arise in a variety of applications. See, for example, Sections 5 and 6 of [30] and [32].

Throughout this section (M, d) will always denote a compact metric space M with metric d, and $\mathcal{M}$ will always denote the collection of closed, nonempty subsets of M. If D_d denotes the Hausdorff metric on $\mathcal{M}$, recall that $(\mathcal{M}, D_d)$ is also a compact metric space. A map $J : M \to \mathcal{M}$ will be called Lipschitzian with Lipschitz constant L if

$$D_d(J(s), J(t)) \leq Ld(s,t) \tag{5.1}$$

for all $s, t \in M$. As usual, $\mathrm{Lip}(J)$ will denote the infimum of numbers L for which equation (5.1) is satisfied for all $s, t \in M$. If $J : M \to \mathcal{M}$ is continuous, we shall define a map $\widehat{J} : \mathcal{M} \to \mathcal{M}$ by

$$\widehat{J}(A) := \bigcup_{s \in A} J(s).$$

We leave to the reader the exercise of proving that $\widehat{J}(A)$ is compact and nonempty for $A \in \mathcal{M}$ and that $\widehat{J}$ is continuous as a map from $(\mathcal{M}, D_d)$ to itself. The reader can also verify that if $J : M \to \mathcal{M}$ is Lipschitz with Lipschitz constant L, then $\widehat{J} : \mathcal{M} \to \mathcal{M}$ is also Lipschitz with Lipschitz constant L, and thus $\mathrm{Lip}(J) = \mathrm{Lip}(\widehat{J})$. Also, if $\Phi_j : \mathcal{M} \to \mathcal{M}$ are Lipschitz maps for $j = 1, 2$, then so is the composition $\Phi_2 \circ \Phi_1$ and one has

$$\mathrm{Lip}(\Phi_2 \circ \Phi_1) \leq \mathrm{Lip}(\Phi_1)\,\mathrm{Lip}(\Phi_2).$$

If $J : M \to \mathcal{M}$ is continuous, we shall usually abuse notation and write $J : \mathcal{M} \to \mathcal{M}$ instead of $\widehat{J}$, and we shall also let $J^n : \mathcal{M} \to \mathcal{M}$ denote the composition of J with itself n times, for $n \geq 1$. Note that $\mathrm{Lip}(J^n) \leq (\mathrm{Lip}(J))^n$. Let us also define the set $\mathcal{S}(J) \subset M \times M$ by

$$\mathcal{S}(J) := \{(s,t) \in M \times M \mid t \in J(s) \text{ and } s \in M\},$$

so $\mathcal{S}(J)$ is a compact subset of $M \times M$.

We shall denote by $C(M)$ the (real) Banach space of real-valued, continuous functions $x : M \to \mathbb{R}$ with the usual norm

$$\|x\| := \max_{s \in M} |x(s)|.$$

If $0 < \delta \leq 1$, then $C^\delta(M)$ will denote the (real) Banach space of real-valued, Hölder continuous functions $x : M \to \mathbb{R}$ with Hölder exponent δ and norm $\|x\|_\delta$ given by

$$\|x\|_\delta := \max_{s \in M} |x(s)| + \sup_{\substack{s,t \in M \\ s \neq t}} \left(\frac{|x(s) - x(t)|}{d(s,t)^\delta} \right).$$

We also let

$$C_+(M) := \{x \in C(M) \mid x(s) \geq 0 \text{ for all } s \in M\}, \qquad C_+^\delta(M) := C^\delta(M) \cap C_+(M),$$

so $C_+(M)$ and $C_+^\delta(M)$ are closed cones in $C(M)$ and $C^\delta(M)$, respectively. For notational convenience, if $\delta = 0$ we shall write $C^0(M) := C(M)$ and $C_+^0(M) := C_+(M)$.

If $S \subset M \times M$ is a closed set and if $a : S \to \mathbb{R}$ is continuous, we shall say that a is Hölder continuous with Hölder exponent δ, for $0 \leq \delta \leq 1$, if there exists a constant $C \geq 0$ such that

$$|a(s_1, t_1) - a(s_2, t_2)| \leq C\Big(d(s_1, s_2) + d(t_1, t_2)\Big)^\delta$$

whenever $(s_j, t_j) \in S$ for $j = 1, 2$ are distinct points. Note that for $\delta = 0$, any such continuous function is automatically Hölder continuous with Hölder exponenent 0. If additionally the function a satisfies $a(s,t) > 0$ for all $(s,t) \in M \times M$, then one easily checks that the function $\log a(s,t)$ is Hölder continuous with Hölder exponent δ if and only if there exists $C \geq 0$ such that

$$a(s_1, t_1) \leq \exp\left(C\Big(d(s_1, s_2) + d(t_1, t_2)\Big)^\delta\right) a(s_2, t_2)$$

whenever $(s_j, t_j) \in S$ for $j = 1, 2$ are distinct points.

With these preliminaries we can describe some continuous, homogeneous, order-preserving maps of interest. For $1 \leq i \leq N$, assume that $J_i, \widetilde{J}_i : M \to \mathcal{M}$ are Lipschitz, and also assume that $a_i, \widetilde{a}_i : \mathcal{S}(J_i) \to [0, \infty)$ are nonnegative and Hölder continuous with Hölder exponent δ, where δ is independent of the map a_i or $\widetilde{a}_i$ and satisfies $0 \leq \delta \leq 1$. Also define maps $F_i, F, \widetilde{F}_i, \widetilde{F} : C^\delta(M) \to C^\delta(M)$, for $1 \leq i \leq N$, by

$$(F_i(x))(s) := \max_{t \in J_i(s)} a_i(s,t)x(t), \qquad (F(x))(s) := \sum_{i=1}^N (F_i(x))(s), \tag{5.2}$$

and

$$(\widetilde{F}_i(x))(s) := \min_{t \in \widetilde{J}_i(s)} \widetilde{a}_i(s,t)x(t), \qquad (\widetilde{F}(x))(s) := \sum_{i=1}^N (\widetilde{F}_i(x))(s).$$

Under the above assumptions, if $x \in C^\delta(M)$ one can prove that $F(x), \widetilde{F}(x) \in C^\delta(M)$, and that further, both maps $F, \widetilde{F} : C^\delta(M) \to C^\delta(M)$ are continuous. Additionally, $F(C_+^\delta(M)) \subset C_+^\delta(M)$ and $\widetilde{F}(C_+^\delta(M)) \subset C_+^\delta(M)$, and the restrictions $F|C_+^\delta(M)$ and $\widetilde{F}|C_+^\delta(M)$ are homogeneous and preserve the partial ordering induced by $C_+^\delta(M)$. We omit the proofs. If follows that one can consider the smallest class $\mathcal{F}$ of functions $\Phi : C^\delta(M) \to C^\delta(M)$ containing all maps $F, \widetilde{F} : C^\delta(M) \to C^\delta(M)$, and which is closed under the operations of composition, addition, maximum, and minimum; that is, if $\Phi_1, \Phi_2 \in \mathcal{F}$ then all the maps $\Phi_2 \circ \Phi_1$, $\Phi_1 + \Phi_2$, $\Phi_1 \vee \Phi_2$ and $\Phi_1 \wedge \Phi_2$ belong to $\mathcal{F}$. (Here, as usual, $(\Phi_1 \vee \Phi_2)(x) := \max\{\Phi_1(x), \Phi_2(x)\}$ and $(\Phi_1 \wedge \Phi_2)(x) := \min\{\Phi_1(x), \Phi_2(x)\}$.) It follows that if $\Phi \in \mathcal{F}$, then Φ is homogeneous and preserves the partial ordering induced by $C^{\delta}_{+}(M)$, and in particular, $\Phi(C^{\delta}_{+}(M)) \subset C^{\delta}_{+}(M)$.

Question E. For a fixed δ with $0 \le \delta \le 1$, let $\mathcal{F}$ be the collection of functions $\Phi : C^{\delta}(M) \to C^{\delta}(M)$ described above. Suppose that $K \subset C^{\delta}_{+}(M)$ is a closed cone. Is Conjecture 4.1 true for all $\Phi \in \mathcal{F}$ for which $\Phi(K) \subset K$? In other words, if $\Phi \in \mathcal{F}$ and $\Phi(K) \subset K$ and $\rho_K(\Phi) < r := \widetilde{r}_K(\Phi)$, does there exist $u \in K \setminus \{0\}$ with $\Phi(u) = ru$?

If $J_i(s)$ is a single point for each $s \in M$, say $J_i(s) = \{\theta_i(s)\}$, then the function F in equation (5.2) becomes a linear "Perron-Frobenius operator," and we have

$$(F(x))(s) = \sum_{i=1}^{N} a_i(s, \theta_i(s)) x(\theta_i(s)). \tag{5.3}$$

This linear case is already non-trivial; see Sections 5 and 6 of [31].

If the functions $a_i(s, \theta_i(s))$ in equation (5.3) are Hölder continuous on M and strictly positive (as opposed to nonnegative), and if $\mathrm{Lip}(\theta_i) < 1$ for $1 \le i \le N$, a relatively simple argument (see Sections 5 and 6 of [31]) shows that F has a strictly positive eigenvector which is Hölder continuous. We wish to show that a similar observation applies to the map F in equation (5.2).

We shall make the following assumptions:

(D1) $J_i : (M, d) \to (\mathcal{M}, D_d)$ is Lipschitz with Lipschitz constant $\kappa < 1$ for $1 \le i \le N$, so

$$D_d(J_i(s), J_i(t)) \le \kappa d(s, t)$$

for all $s, t \in M$; and

(D2) $a_i : \mathcal{S}(J_i) \to \mathbb{R}$ is a strictly positive continuous function. Also, there exists δ satisfying $0 < \delta \le 1$, and a constant $C > 0$ such that

$$a_i(s_1, t_1) \le \exp\left(C \left(d(s_1, s_2) + d(t_1, t_2) \right)^{\delta} \right) a_i(s_2, t_2) \tag{5.4}$$

for all $(s_1, t_1), (s_2, t_2) \in \mathcal{S}(J_i)$ and $1 \le i \le N$.

For a given constant $C_0 > 0$ and δ with $0 < \delta \le 1$, we define a closed cone $K(C_0, \delta) \subset C_{+}(M) \subset C(M)$ by

$$K(C_0, \delta) := \{u \in C_{+}(M) \mid u(t_1) \le \exp(C_0 d(t_1, t_2)^{\delta}) u(t_2) \text{ for all } t_1, t_2 \in M\}. \tag{5.5}$$

An easy argument (see Lemma 5.4 in [31]) shows, using the norm from $C(M)$, that the closed unit ball $\{u \in K(C_0, \delta) \mid \|u\| \leq 1\}$ is compact in $C(M)$.

The essential observation is contained in the following lemma.

Lemma 5.1. *Assume (D1) and (D2) hold and let F be defined by equation (5.2). Then there exists a constant C_0 such that $F(x) \in K(C_0, \delta)$ for all $x \in K(C_0, \delta)$.*

Proof. If $F_i(x)$ is defined by equation (5.2), it suffices to show that for all sufficiently large $C_0 > 0$ it is the case that $F_i(x) \in K(C_0, \delta)$ whenever $x \in K(C_0, \delta)$, for $1 \leq i \leq N$. Let i be fixed. For C, κ, and δ as in (D1) and (D2), select C_0 so that

$$\frac{C(1+\kappa)^\delta}{1-\kappa^\delta} \leq C_0.$$

Given $s, \widetilde{s} \in M$ and $x \in K(C_0, \delta)$, select $s_1 \in J_i(s)$ such that $a_i(s, s_1)x(s_1) = (F_i(x))(s)$. By (D1) there exists $\widetilde{s}_1 \in J(\widetilde{s})$ with $d(s_1, \widetilde{s}_1) \leq \kappa d(s, \widetilde{s})$. By (D2) we know that

$$a_i(s, s_1) \leq \exp\left(C\Big(d(s, \widetilde{s}) + d(s_1, \widetilde{s}_1)\Big)^\delta\right) a_i(\widetilde{s}, \widetilde{s}_1) \leq \exp\left(C(1+\kappa)^\delta d(s, \widetilde{s})^\delta\right) a_i(\widetilde{s}, \widetilde{s}_1).$$

Because $x \in K(C_0, \delta)$ we have that

$$x(s_1) \leq \exp\left(C_0 d(s_1, \widetilde{s}_1)^\delta\right) x(\widetilde{s}_1) \leq \exp\left(C_0 \kappa^\delta d(s, \widetilde{s})^\delta\right) x(\widetilde{s}_1).$$

Combining these two inequalities, we see that

$$a_i(s, s_1)x(s_1) = (F_i(x))(s) \leq \exp\left(\left(C(1+\kappa)^\delta + C_0\kappa^\delta\right) d(s, \widetilde{s})^\delta\right) a_i(\widetilde{s}, \widetilde{s}_1)x(\widetilde{s}_1).$$

Now $a_i(\widetilde{s}, \widetilde{s}_1)x(\widetilde{s}_1) \leq (F_i(x))(\widetilde{s})$ by definition of F_i, and our choice of C_0 shows that $C(1+\kappa)^\delta + C_0\kappa^\delta \leq C_0$, so

$$(F_i(x))(s) \leq \exp\left(C_0 d(s, \widetilde{s})^\delta\right) (F_i(x))(\widetilde{s})$$

and thus $F_i(x) \in K(C_0, \delta)$. ■

Using Corollary 4.6, we obtain the following result.

Lemma 5.2. *Let $K := K(C_0, \delta)$ be defined by equation (5.5), where $C_0 > 0$ and $0 < \delta \leq 1$, and write $K_1 := C_+(M)$. Assume that $\Phi : K \to K$ is continuous, homogeneous and K_1-order-preserving. If*

$r := r_K(\Phi) > 0$, there exists $u \in K \setminus \{0\}$ with $\Phi(u) = ru$. If further Φ has an extension $\Phi_1 : K_1 \to K_1$ which is continuous, homogeneous and K_1-order-preserving, then $r_{K_1}(\Phi_1) = r_K(\Phi)$.

Proof. Because $\{u \in K \mid \|u\| \leq 1\}$ is compact, Φ is compact, so Corollary 4.6 implies there exists $u \in K \setminus \{0\}$ with $\Phi(u) = ru$. Because $u(t) > 0$ for all $t \in M$, the fact that $r_{K_1}(\Phi_1) = r_K(\Phi)$ follows easily. ■

Theorem 5.3. *Assume that (D1) and (D2) hold and take $C_0 \geq C(1+\kappa)^\delta(1-\kappa^\delta)^{-1}$. If $K := K(C_0, \delta)$ is given by equation (5.5) and $K_1 := C_+(M)$, and if $F : C(M) \to C(M)$ is defined by equation (5.2), then F is continuous, $F(K) \subset K$ and $F(K_1) \subset K_1$, and $F|K_1$ is homogeneous and K_1-order-preserving. Denoting $\Phi := F|K$ and $\Phi_1 := F|K_1$, we have that $r := r_K(\Phi) = r_{K_1}(\Phi_1) > 0$ and there exists $u \in K \setminus \{0\}$ with $F(u) = ru$.*

Proof. The facts that $F(K_1) \subset K_1$ and that Φ_1 is K_1-order-preserving are obvious. If $e(t) \equiv 1$ for all $t \in M$, it is easy to see that $F(e) \geq_{K_1} \eta e$ for some $\eta > 0$, so $r_K(\Phi) \geq \eta$. The remainder of the theorem follows directly from Lemmas 5.1 and 5.2. ■

If $N = 1$ in equation (5.2) there is a much sharper result than Theorem 5.3. We collect assumptions in the following hypotheses:

(D3) $J : (M, d) \to (\mathcal{M}, D_d)$ is Lipschitz with Lipschitz constant Q. There exists an integer $n \geq 1$ and a constant κ with $0 < \kappa < 1$ such that the iterate J^n is Lipschitz with Lipschitz constant κ; and

(D4) $a : \mathcal{S}(J) \to \mathbb{R}$ is a strictly positive continuous function. Also, there exists δ satisfying $0 < \delta \leq 1$, and a constant C such that a satisfies equation (5.4) with a replacing a_i there.

Under assumptions (D3) and (D4) we define $F : C(M) \to C(M)$ by

$$(F(x))(s) = \max_{t \in J(s)} a(s,t)x(t), \tag{5.6}$$

as in equation (5.2).

Theorem 5.4. *Assume (D3) and (D4) hold and let $F : C(M) \to C(M)$ be given by equation (5.6). Then there exists a constant C_0 such that, for n as in (D3), we have $F^n(K(C_0, \delta)) \subset K(C_0, \delta)$ where*

$K(C_0, \delta)$ is as in equation (5.5). Further, there exists $u \in K(C_0, \delta) \setminus \{0\}$ such that $F(u) = ru$ where $r := r_{K_1}(F) > 0$ and $K_1 := C_+(M)$.

Proof. If e is the function identically equal to $+1$, there exists $\eta > 0$ such that $F(e) \geq \eta e$ in the partial ordering from K_1. Our previous remarks show that $F : K_1 \to K_1$ is continuous, homogeneous and K_1-order-preserving. It follows that $r_K(F) = r_{K_1}(F)$, with $r_K(F^n) = (r_K(F))^n > 0$ and $r_{K_1}(F^n) = (r_{K_1}(F))^n > 0$.

We use the notation of (D3) and (D4). We claim that for C_0 sufficiently large, $F^n(K(C_0, \delta)) \subset K(C_0, \delta)$. Take $v \in K(C_0, \delta)$, where C_0 will be chosen later. By increasing Q in (D3), we can assume that $Q \geq \kappa$. By our previous remarks, $\mathrm{Lip}(J^i) \leq Q^i$ for $i \geq 1$ and by (D3) we have $\mathrm{Lip}(J^n) \leq \kappa < 1$. Take $s, \widetilde{s} \in M$ and $x \in K(C_0, \delta)$. Be relabelling, we can assume that $(F^n(x))(\widetilde{s}) \leq (F^n(x))(s)$. One can see, for $s_0 := s$, that

$$(F^n(x))(s_0) = \max\{\left(\prod_{i=1}^{n} a(s_{i-1}, s_i)\right)x(s_n) \mid s_i \in J(s_{i-1}) \text{ for } 1 \leq i \leq n\},$$

so there exist s_i for $1 \leq i \leq n$ such that $s_i \in J(s_{i-1})$ and

$$(F^n(x))(s_0) = \left(\prod_{i=1}^{n} a(s_{i-1}, s_i)\right)x(s_n).$$

Take $\widetilde{s}_0 := \widetilde{s}$ and choose $\widetilde{s}_i \in J(\widetilde{s}_{i-1})$, for $1 \leq i \leq n$, to be a point in $J(\widetilde{s}_{i-1})$ closest to s_i, that is, $d(\widetilde{s}_i, s_i) \leq d(\widehat{s}_i, s_i)$ for every $\widehat{s}_i \in J(\widetilde{s}_{i-1})$. (Such a point $\widetilde{s}_i$ exists, but may not be unique.) By our construction, $s_i \in J^i(s)$ and $\widetilde{s}_i \in J^i(\widetilde{s})$ so

$$d(s_i, \widetilde{s}_i) \leq D_d(J^i(s), J^i(\widetilde{s})) \leq Q^i d(s, \widetilde{s}), \qquad \text{for } 1 \leq i \leq n,$$

and

$$d(s_n, \widetilde{s}_n) \leq D_d(J^n(s), J^n(\widetilde{s})) \leq \kappa d(s, \widetilde{s}).$$

Using the above inequalities in conjunction with (D4) gives, for $1 \leq i \leq n$,

$$\begin{aligned} a(s_{i-1}, s_i) &\leq \exp\left(C\Big(d(s_{i-1}, \widetilde{s}_{i-1}) + d(s_i, \widetilde{s}_i)\Big)^{\delta}\right)a(\widetilde{s}_{i-1}, \widetilde{s}_i) \\ &\leq \exp\left(CQ^{(i-1)\delta}(1+Q)^{\delta} d(s, \widetilde{s})^{\delta}\right)a(\widetilde{s}_{i-1}, \widetilde{s}_i). \end{aligned}$$

Because we assume that $x \in K(C_0, \delta)$,

$$x(s_n) \le \exp\left(C_0 d(s_n, \widetilde{s}_n)^\delta\right) x(\widetilde{s}_n) \le \exp\left(C_0 \kappa^\delta d(s, \widetilde{s})^\delta\right) x(\widetilde{s}_n).$$

Combining these inequalities gives

$$\left(\prod_{i=1}^{n} a(s_{i-1}, s_i)\right) x(s_n)$$
$$\le \exp\left(\left(\sum_{i=1}^{n} C(1+Q)^\delta Q^{(i-1)\delta} + C_0\kappa^\delta\right) d(s,\widetilde{s})^\delta\right)\left(\prod_{i=1}^{n} a(\widetilde{s}_{i-1}, \widetilde{s}_i)\right) x(\widetilde{s}_n).$$

It follows that if C_0 is chosen so that

$$C(1+Q)^\delta \left(\sum_{j=0}^{n-1} Q^{j\delta}\right)(1-\kappa^\delta)^{-1} \le C_0, \tag{5.7}$$

then for $x \in K(C_0, \delta)$

$$(F^n(x))(s) \le \exp\left(C_0 d(s, \widetilde{s})^\delta\right)\left(\prod_{i=1}^{n} a(\widetilde{s}_{i-1}, \widetilde{s}_i)\right) x(\widetilde{s}_n) \le \exp\left(C_0 d(s,\widetilde{s})^\delta\right)(F^n(x))(\widetilde{s}),$$

so $F^n(K(C_0, \delta)) \subset K(C_0, \delta)$ if equation (5.7) is satisfied. If we now apply Lemma 5.2 to $\Phi := F^n$, we see that there exists $v \in K(C_0, \delta) \setminus \{0\}$ with $F^n(v) = r^n v$, where $r := r_{K_1}(F) = r_K(F) > 0$.

We leave to the reader the exercise of proving that if $x, y \in K(C_0, \delta)$, then $x \vee y \in K(C_0, \delta)$ for the maximum of these two functions. The reader can also verify that, for F as in equation (5.4), we have $F(x \vee y) = F(x) \vee F(y)$. If $v \in K(C_0, \delta)$ is as above and we define $w_i := r^{-i} F^i(v)$ for $0 < i < n$ and $w_0 := v$, it follows from these observations that $w := w_0 \vee w_1 \vee \cdots \vee w_{n-1} \in K(C_0, \delta)$ and $F(w) = rw$. ■

In [18] the authors studied the operator R in equation (5.6) for the special case $M = [c, d]$ and for $J(s) = [\alpha(s), \beta(s)]$, where $\alpha, \beta : [c, d] \to [c, d]$ are continuous maps. Here we shall make the following assumptions:

(D5) $[c, d]$ is a compact interval and $\alpha, \beta : [c, d] \to [c, d]$ are Lipschitz maps such that $\alpha(s) \le \beta(s)$ for all $s \in [c, d]$. The maps α and β have unique fixed points s_* and t_*, respectively. There exist $\delta > 0$ and $k < 1$ such that $\alpha_* := \alpha|[s_* - \delta, s_* + \delta] \cap [c, d]$ and $\beta_* := \beta|[t_* - \delta, t_* + \delta] \cap [c, d]$ satisfy $\mathrm{Lip}(\alpha_*) \le k$ and $\mathrm{Lip}(\beta_*) \le k$.

(D6) With α and β as in (H5), define $J(s) := [\alpha(s), \beta(s)]$ and $\mathcal{S}(J) := \{(s,t) | c \leq s \leq d \text{ and } t \in J(s)\}$. Assume that $a : \mathcal{S}(J) \to (0, \infty)$ is a strictly positive Hölder continuous function with H older exponent $\delta > 0$.

As noted previously, (D6) implies that there is a constant C such that equation (5.4) is satisfied, with a replacing a_i in equation (5.4).

Lemma 5.5. *Assume that (D5) holds and let J be defined as in (D6). Assume also that α and β are nondecreasing in $[c,d]$. If $k < k_1 < 1$, then there exists an integer n such that $\mathrm{Lip}(J^n) \leq k_1$.*

Proof. We follow the notation of (D5). Because $\mathrm{Lip}(\alpha_*) \leq k < 1$, we have $\alpha(s) > s$ for $s \in [s_* - \delta, s_*) \cap [c,d]$ and $\alpha(s) < s$ for $s \in (s_*, s_* + \delta] \cap [c,d]$. As $\alpha(s) \neq s$ for all $s \neq s_*$, it further follows from continuity that $\alpha(s) > s$ for $s \in [c, s_*)$ and $\alpha(s) < s$ for $s \in (s_*, d]$. Because α is nondecreasing, $\alpha(s_*) = s_* \leq \alpha(s) < s$ for $s \in (s_*, d]$, and upon iterating we find for such s that $s_* \leq \alpha^{j+1}(s) \leq \alpha^j(s)$ for $j \geq 0$, where α^j denotes the j^{th} iterate of α. Thus $\lim_{j\to\infty} \alpha^j(s) := \sigma_*$ exists with $\alpha(\sigma_*) = \sigma_*$, and therefore $\sigma_* = s_*$ by the uniqueness of the fixed point. The analogous argument for $s \in [c, s_*)$ shows that $\alpha^j(s) \leq \alpha^{j+1} \leq s_*$ for $j \geq 0$ and $\lim_{j\to\infty} \alpha^j(s) = s_*$. Similarly, $\lim_{j\to\infty} \beta^j(s) = t_*$ for all $s \in [c,d]$.

Because α and β are nondecreasing and continuous, one can see that $J^j(s) = [\alpha^j(s), \beta^j(s)]$, and because $\alpha^j(c) \leq \alpha^j(s) \leq \alpha^j(d)$ for all $j \geq 1$, there exists n_1 such that $\alpha^j(s) \in [s_* - \delta, s_* + \delta] \cap [c,d]$ and $\beta^j(s) \in [t_* - \delta, t_* + \delta] \cap [c,d]$ for all $j \geq n_1$ and $s \in [c,d]$. Moreover, because α^{n_1} and β^{n_1} are Lipschitz with, say, $\mathrm{Lip}(\alpha), \mathrm{Lip}(\beta) \leq Q_0$ for some Q_0, and because $\mathrm{Lip}(\alpha_*), \mathrm{Lip}(\beta_*) \leq k < 1$, we have for all $s, t \in [c,d]$ and $j \geq 0$ that

$$|\alpha^{n_1+j}(s) - \alpha^{n_1+j}(t)| \leq Q_0 k^j |s-t|, \qquad |\beta^{n_1+j}(s) - \beta^{n_1+j}(t)| \leq Q_0 k^j |s-t|.$$

It follows that there exists an integer j_1 such that $Q_0 k^{j_1} \leq k_1$ for, so letting $n = n_1 + j_1$ it follows that

$$|\alpha^n(s) - \alpha^n(t)| \leq k_1 |s-t|, \qquad |\beta^n(s) - \beta^n(t)| \leq k_1 |s-t|.$$

Thus

$$D(J^n(s), J^n(t)) \leq k_1 |s-t|$$

where D denotes the Hausdorff metric, as desired. ■

Theorem 5.6. *Assume that (D5) and (D6) hold and that α and β are nondecreasing. For $M := [c,d]$,* *let* $F : C(M) \to C(M)$ *be defined by equation (5.6), and let* $K_1 := C_+(M)$ *and* $r := r_{K_1}(F) > 0$ *Then there exist* $C_0 > 0$ *and* $u \in K(C_0, \delta) \setminus \{0\}$ *(see equation (5.5)) with* $F(u) = ru$.

Proof. With the aid of Lemma 5.5, the result follows directly from Theorem 5.4. ■

Theorem 5.6 directly generalizes Theorem 1.1 in [18]. It also generalizes, in a number of ways, results in Section 4 of [18], although it demands slightly greater regularity of the functions α, β and a then is usually assumed in [18].

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