



LATTICE ISOMORPHISMS AND ITERATES OF NONEXPANSIVE MAPS

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1. INTRODUCTION

If D is a subset of a Banach space $(E, \|\cdot\|)$ and $f: D \rightarrow E$ is a map, f is called nonexpansive (with respect to $\|\cdot\|$) if

$$\|f(x) - f(y)\| \leq \|x - y\| \quad \text{for all } x, y \in D.$$

If D is a compact subset of \mathbb{R}^n and the norm is the l_1 -norm $\|\cdot\|_1$ (so $\|z\|_1 = \sum_{i=1}^n |z_i|$ for $z \in \mathbb{R}^n$) and $f: D \rightarrow D$ is nonexpansive with respect to $\|\cdot\|_1$, Akcoglu and Krengel [1] have proved that for each $x \in D$ there exists a minimal positive integer $p_x = p$ such that

$$\lim_{k \rightarrow \infty} f^{kp}(x) = \eta, \quad \text{where } f^p(\eta) = \eta. \tag{1.1}$$

Recall that a norm $\|\cdot\|$ on a finite dimensional Banach space E is called polyhedral if $\{x \in E: \|x\| \leq 1\}$ is a polyhedron. Equivalently, a norm is polyhedral if there exist continuous linear functionals $\varphi_j \in E^*$, $1 \leq j \leq m$, such that

$$\|x\| = \max_{1 \leq j \leq m} |\varphi_j(x)| \quad \text{for } 1 \leq j \leq m. \tag{1.2}$$

It is easy to see that the l_1 norm and the sup norm $\|\cdot\|_\infty$ ($\|x\|_\infty = \max\{|x_i|: 1 \leq i \leq n\}$) on \mathbb{R}^n are polyhedral.

If E is a finite dimensional Banach space with a polyhedral norm $\|\cdot\|$, D is a compact subset of E and $f: D \rightarrow D$ is a nonexpansive map, Weller [2] has shown that for each $x \in D$, there again exists an integer p_x such that (1.1) holds.

The original arguments did not give upper bounds for the integer p_x , $x \in D$. However, subsequent work has proved (see [3-11]) that there exists an integer N , depending only on the integer m in equation (1.2), such that $p_x \leq N$ for all $x \in D$. In general, the problem of finding optimal upper bounds for the integers p_x appears to be a very difficult combinatorial-geometrical question: see [3-9, 12] for a more complete discussion.

The original motivation for the study of l_1 -nonexpansive maps in [1] was to understand nonlinear analogues of diffusion on finite state spaces. However, as discussed in [8], these results, for the case of the sup norm and other polyhedral norms, also have applications to certain cone mappings and, hence, to a variety of examples and applications in [12-14]. These ideas also apply to certain autonomous differential equations $x'(t) = f(x(t))$ (see [8]) and

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(in work in progress by the author) to certain differential equations $x'(t) = f(t, x(t))$, where $f(t, x)$ is periodic of period T in the t variable.

If K^n denotes the positive orthant in \mathbb{R}^n ,

$$K^n = \{x \in \mathbb{R}^n: x_i \geq 0 \text{ for } 1 \leq i \leq n\}, \tag{1.3}$$

K^n induces a partial ordering by

$$x \leq y \quad \text{if and only if } y - x \in K^n. \tag{1.4}$$

We shall also write

$$x < y \quad \text{if and only if } x \leq y \text{ and } x \neq y. \tag{1.5}$$

A map $f: D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called ‘‘monotonic’’ or ‘‘order-preserving’’ if $x \leq y$ ($x, y \in D$) implies that $f(x) \leq f(y)$. A norm $\|\cdot\|$ on \mathbb{R}^n is called ‘‘monotonic’’ (on K^n) if $0 \leq x \leq y$ implies that $\|x\| \leq \|y\|$. The norm will be called ‘‘strictly monotonic’’ on K^n if $0 \leq x < y$ implies that $\|x\| < \|y\|$. Note that the l_1 norm is strictly monotonic; the sup norm is monotonic but not strictly monotonic.

If $f: K^n \rightarrow K^n$ is an order-preserving map which is nonexpansive with respect to the l_1 norm and if $f(0) = 0$, Akcoglu and Krengel [1] have proved that the period p of any periodic point η of f satisfies

$$p \leq n! \tag{1.6}$$

(Recall that η is called a periodic point of f if $f^p(\eta) = \eta$; the minimal positive integer p such that $f^p(\eta) = \eta$ is the period of η .) Scheutzwow [10] has proved that if $f: K^n \rightarrow K^n$ is l_1 -nonexpansive and $f(0) = 0$, then the period p of any periodic point η of f satisfies

$$p \leq \text{lcm}(1, 2, \dots, n) \equiv L(n). \tag{1.7}$$

(In (1.7), $\text{lcm}(1, 2, \dots, n)$ denotes the least common multiple of $\{j: 1 \leq j \leq n\}$, and we shall write $L(n)$ for $\text{lcm}(1, 2, \dots, n)$. Generally, if S is a set of positive integers, $\text{lcm}(S)$ is the least common multiple of the integers in S .)

It is known (see [15]) that

$$L(n) \leq \exp(1.03883n) \quad \text{for all } n,$$

and $L(n)$ is asymptotically dominated by $\exp(n)$ (for n large), so the estimate in (1.7) is much better than that in (1.6). Nevertheless, $L(n)$ is not a sharp upper bound. In [9, p. 362] (and also in equation (3.1) in Section 3 of this paper) a function $\varphi(n)$ is defined. It is proved in [9] that $p \leq \varphi(n)$ for all n and $\varphi(n) < L(n)$ for $n > 2$. The function $\varphi(n)$ has been computed for $n \leq 24$ in [9]; and it has been proved that $\varphi(n)$ is an optimal bound for $n \leq 24$ in the sense that for each $n \leq 24$, there exists an l_1 -nonexpansive, order-preserving map $f: K^n \rightarrow K^n$ such that $f(0) = 0$ and f has a periodic point of period $\varphi(n)$. In Section 3 of this paper, we shall extend these results to the range $n \leq 32$ and discuss the function φ more fully.

Exact asymptotic formulas for $\varphi(n)$ are not known, but estimates in [9] yield

$$\varphi(n) \leq nL\left(\left\lceil \frac{n}{2} \right\rceil\right) \leq n \exp\left(1.03883 \left\lceil \frac{n}{2} \right\rceil\right) \quad \text{and} \quad \varphi(n) \leq 2^n \text{ for all } n,$$

where $[x]$ denotes the greatest integer $m \leq x$.

With these preliminaries, we can describe the main theorem of this paper.

THEOREM 1.1. Suppose that $\|\cdot\|$ is a strictly monotonic norm on $K^n \subset \mathbb{R}^n$. Assume that $f: K^n \rightarrow K^n$ is nonexpansive with respect to $\|\cdot\|$, f is order-preserving, and $f(0) = 0$. Then for every $x \in K^n$, there exists a minimal integer $p = p_x \leq \varphi(n)$ (where $\varphi(n)$ is the function defined in [9, p. 362] and in Section 3 of this paper) such that

$$\lim_{k \rightarrow \infty} f^{kp}(x) = \eta \quad \text{and} \quad f^p(\eta) = \eta. \tag{1.8}$$

Actually the tools used to prove theorem 1.1 provide strong information about the set of possible periods p of periodic points of maps f , for f as in theorem 1.1. We shall not pursue this observation systematically here, although we shall give a discussion of the case $n = 16$ at the end of Section 3.

The major difficulty in proving theorem 1.1 will be to prove that there is any finite integer p such that (1.8) holds. Of course, if the norm is polyhedral, this follows from our previous results, but the norm may even be Euclidean. Even if the norm is polyhedral and monotonic (for example, the sup norm), note that we do not obtain the estimate $p_x \leq \varphi(n)$ unless the norm is strictly monotonic.

In order to proceed further, we need to recall some notation and some results from [9, 10]. The vector space \mathbb{R}^n is a vector lattice, i.e. if $x, y \in \mathbb{R}^n$, we define $z = x \wedge y$, the minimum of x and y by

$$z_i = \min(x_i, y_i), \quad 1 \leq i \leq n.$$

Similarly, we can define $x \vee y$, the maximum of x and y . If S is a finite collection of vectors x^j in \mathbb{R}^n , $1 \leq j \leq k$, we define

$$z = \min\{x: x \in S\} \equiv \bigwedge_{j=1}^k x^j \quad \text{by} \quad z_i = \min\{x_i: x \in S\}.$$

If $A \subset \mathbb{R}^n$, we define V , “the lower semilattice generated by A ” to be the smallest closed set V such that $A \subset V$ and such that $x \wedge y \in V$ for all $x, y \in V$. If A is finite, V is finite. If V is a finite lower semilattice and if $T \subset V$ is a subset which has an upper bound in V (so there exists $u \in V$, $u \geq x$ for all $x \in T$), then we define

$$\max_V(T) = \min\{w \in V: w \geq x \text{ for all } x \in T\}.$$

An element $x \in V$ is called irreducible (with respect to V) if

$$x > \max_V\{z \in V: x > z\}.$$

V has a minimal element which is defined to be irreducible.

Definition 1.1. Suppose that $f: D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a map. We shall say that “ f has the lower semilattice extension property” if whenever $x \in D$ is a periodic point of period p , $A = A_x = \{f^j(x): 0 \leq j < p\}$ and V is the lower semilattice generated by A , $f|_A$ has an extension $\tilde{f}: V \rightarrow V$ such that $\tilde{f}(y \wedge z) = \tilde{f}(y) \wedge \tilde{f}(z)$ for all $y, z \in V$.

Scheutzwow [10] has shown that these concepts are directly relevant to our problem.

LEMMA 1.1. (See [10].) Let $f: K^n \rightarrow K^n$ be an l_1 -nonexpansive map such that $f(0) = 0$. If x is a periodic point of f of period p , $A = \{f^j(x): 0 \leq j < p\}$ and V is the lower semilattice generated by A , $f(V) \subset V$ and

$$f(y \wedge z) = f(y) \cup f(z) \quad \text{for all } y, z \in V.$$

Thus, if f is as in the lemma, f satisfies the lower semilattice extension property. In this notation, Scheutzwow has proved the following theorem.

THEOREM 1.2. (See [10].) Suppose that $f: D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a map which satisfies the lower semilattice extension property and that η is a periodic point of period p . Then it follows that p divides $L(n) = \text{lcm}(\{j: 1 \leq j \leq n\})$. If $A = \{f^j(\eta): 0 \leq j < p\}$, V is the lower semilattice generated by A and $y \in V$ is irreducible, then y is a periodic point of f of period $q_y \leq n$. If $\Sigma = \{y \in V: y \leq \eta \text{ and } y \text{ is irreducible}\}$ and q_y denotes the period of $y \in \Sigma$ and $S = \{q_y: y \in \Sigma\}$, then p divides $\text{lcm}(S)$.

Theorem 1.2 motivates the following definition.

Definition 1.2. Define $\alpha(n)$ to be the maximal positive integer p such that there exists a map $f: D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ which satisfies the lower semilattice extension property and has a periodic point of period p .

If $\varphi(n)$ is the function previously discussed it is proved in [9] that $\alpha(n) \leq \varphi(n)$ for all n and that $\alpha(n) = \varphi(n)$ for $1 \leq n \leq 24$. Our previous remarks show $p_x \leq \alpha(n) \leq \varphi(n)$ for p_x as in theorem 1.1. Furthermore, it is shown in [9] that there are constraints (see Section 3) on the set S in theorem 1.2.

2. LATTICES AND PERIODIC POINTS

In this section we shall exploit various ideas connected with homomorphisms of lattices in order to prove theorem 1.1. In the end we shall have to restrict attention to the cone $K^n \subset \mathbb{R}^n$, but many of our lemmas are true in much greater generality and may be of independent interest. Thus, we shall initially work in greater generality.

Recall that a cone C in a Banach space E is a closed, convex subset of E such that $tC \subset C$ for all $t \geq 0$ and $C \cap (-C) = \{0\}$. A cone induces a partial ordering by $x \leq y$ iff $y - x \in C$. We shall write $x < y$ if $x \leq y$ and $x \neq y$; and if C has nonempty interior, we shall write $x \ll y$ if $y - x \in \overset{\circ}{C}$. If $x, y \in E$ and there exists $z \in E$ such that $z \leq x$ and $z \leq y$ and $z \geq \zeta$ for every ζ such that $\zeta \leq x$ and $\zeta \leq y$, we shall write

$$z = x \wedge y.$$

If $x \wedge y$ exists for every $x, y \in C$, E is called a vector lattice (in the ordering induced by C). It is easy to see that there is a smallest element $w = x \vee y$ such that $w \geq x$ and $w \geq y$ and

$$x + y = (x \wedge y) + (x \vee y).$$

If E is a vector lattice in the ordering from a cone C and $A \subset E$, we define V , the lattice generated by A , to be the smallest closed set such that $A \subset V$ and such that for all $x, y \in V$, $x \wedge y \in V$ and $x \vee y \in V$. Similarly, one defines the lower semilattice generated by A as in Section 1.

As for the cone K^n , a norm $\|\cdot\|$ on E is called monotonic (in the ordering from C) if $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$; the norm is strictly monotonic if $0 \leq x \leq y$ implies $\|x\| < \|y\|$. A map $f: D \subset E \rightarrow E$ is called order-preserving if $x \leq y, (x, y \in D)$ implies $f(x) \leq f(y)$.

If D is a topological space and $f: D \rightarrow D$ a map and $z \in D$, we shall denote the forward orbit of z under f , $\{f^j(z): j \geq 0\}$, by $\gamma_+(z; f)$ or (if f is obvious) $\gamma_+(z)$.

PROPOSITION 2.1. Let C be a cone in a Banach space E and assume that E is a vector lattice in the ordering from C and that the map $(x, y) \rightarrow x \wedge y$ is continuous. Assume that the norm $\|\cdot\|$ on E is strictly monotonic. Let $T: C \rightarrow C$ be an order-preserving, nonexpansive map with $T(0) = 0$. Assume that, for each $z \in C$, $\gamma_+(z; T)$ has compact closure. Suppose that $x, y \in C$ and that there exists a sequence of positive integers $p_i \rightarrow \infty$ with

$$\lim_{i \rightarrow \infty} T^{p_i}(x) = x \quad \text{and} \quad \lim_{i \rightarrow \infty} T^{p_i}(y) = y. \tag{2.1}$$

Then it follows that for all $j \geq 1$

$$\|T^j(x)\| = \|x\| \quad \text{and} \quad \|T^j(y)\| = \|y\| \tag{2.2}$$

and

$$T(x \wedge y) = (Tx) \wedge (Ty) \quad \text{and} \quad T(x \vee y) = (Tx) \vee (Ty). \tag{2.3}$$

Proof. The order-preserving property of T^j implies that

$$T^j(x \wedge y) \leq T^j(x), \quad T^j(x \wedge y) \leq T^j(y) \quad \text{and} \quad T^j(x \wedge y) \leq (T^j x) \wedge (T^j y) \tag{2.4}$$

for all $j \geq 1$. We claim first that

$$\lim_{i \rightarrow \infty} \|T^{p_i}(x \wedge y) - (T^{p_i}x) \wedge (T^{p_i}y)\| = 0. \tag{2.5}$$

If not, then by taking a further subsequence $m_i \rightarrow \infty$ we can assume that for all $i \geq 1$

$$\|T^{m_i}(x \wedge y) - (T^{m_i}x) \wedge (T^{m_i}y)\| \geq \alpha > 0. \tag{2.6}$$

Because $T(0) = 0$ and T is nonexpansive, we see that for any $x \in C$, $\|T^j(x)\|$ is a decreasing sequence of nonnegative reals. Using this observation and (2.1) we obtain (2.2). Because we assume that $\text{cl}(\gamma_+(z))$ is compact for all z , by taking a further subsequence, which we also label m_i , we can assume that

$$\lim_{i \rightarrow \infty} T^{m_i}(x \wedge y) = \eta. \tag{2.7}$$

By taking the limit as $i \rightarrow \infty$ in (2.6) we obtain

$$\|\eta - x \wedge y\| \geq \alpha > 0. \tag{2.8}$$

We already know from (2.4) and (2.7) that

$$\eta < x \wedge y \quad \text{and} \quad x - (x \wedge y) < x - \eta. \tag{2.9}$$

The strict monotonicity of the norm now implies that

$$\|x - (x \wedge y)\| < \|x - \eta\|. \tag{2.10}$$

However, (2.10) gives a contradiction: by using the nonexpansiveness of T we obtain

$$\|x - \eta\| = \lim_{i \rightarrow \infty} \|T^{m_i}(x \wedge y) - T^{m_i}(x)\| \leq \|x - x \wedge y\|,$$

which contradicts (2.10). It follows that (2.5) is valid.

By using (2.5) we see that

$$\lim_{i \rightarrow \infty} \|T^{p_i}(x \wedge y) - x \wedge y\| = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \|T^{p_i}(x \wedge y)\| = \|x \wedge y\|.$$

Since $\|T^j(x \wedge y)\|$ is a decreasing sequence, we conclude that

$$\|T^j(x \wedge y)\| = \|x \wedge y\| \quad \text{for all } j \geq 1. \tag{2.11}$$

Applying (2.11) to $x' = Tx$ and $y' = Ty$, we also obtain

$$\|T^j(Tx \wedge Ty)\| = \|Tx \wedge Ty\| \quad \text{for all } j \geq 1. \tag{2.12}$$

It remains to prove that the inequality

$$T(x \wedge y) < Tx \wedge Ty \tag{2.13}$$

does not hold. If (2.13) holds we obtain from (2.12) and (2.11) and strict monotonicity of the norm that

$$\|T^{p_i}(x \wedge y)\| = \|T^{p_i-1}T(x \wedge y)\| = \|x \wedge y\| < \|Tx \wedge Ty\| = \|T^{p_i-1}(Tx \wedge Ty)\|. \tag{2.14}$$

However, (2.4) and (2.13) give

$$\|Tx \wedge Ty\| = \|T^{p_i-1}(Tx \wedge Ty)\| \leq \|T^{p_i}x \wedge T^{p_i}y\|; \tag{2.15}$$

and taking limits in the above equation yields

$$\|Tx \wedge Ty\| \leq \|x \wedge y\|,$$

which contradicts (2.14). Since we already know that

$$T(x \wedge y) \leq Tx \wedge Ty, \tag{2.16}$$

we must have equality in inequality (2.13).

The proof that

$$\|T(x \vee y)\| = \|Tx \vee Ty\|$$

is completely analogous and left to the reader. ■

Remark 2.1. Proposition 2.1 is false, even for linear maps, if the norm is not strictly monotonic. To see this, view elements of \mathbb{R}^3 as column vectors and define $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$T(x) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We have $T(K^3) \subset K^3$, T is order-preserving and T is nonexpansive with respect to the sup norm or l_∞ -norm. (Recall that an $n \times n$ matrix $A = (a_{ij})$ induces a nonexpansive map of \mathbb{R}^n into \mathbb{R}^n

with respect to the sup norm if and only if

$$\sum_{j=1}^n |a_{ij}| \leq 1 \quad \text{for } 1 \leq i \leq n.$$

On the other hand, if we define $z, w \in K^3$ by

$$z = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix},$$

it is easy to check that

$$T(z) = w \quad \text{and} \quad T(w) = z.$$

Thus, w and z are periodic points of T and $T^{2i}(w) = w$ and $T^{2i}(z) = z$ for $i \geq 1$. However, a calculation gives

$$T(w \wedge z) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \end{pmatrix} \neq Tw \wedge Tz \quad \text{and} \quad T(w \vee z) = \begin{pmatrix} 1 \\ 1 \\ \frac{3}{4} \end{pmatrix} \neq Tw \vee Tz.$$

Remark 2.2. As was noted in the proof of proposition 2.1, the facts that $T(0) = 0$ and T is nonexpansive imply that $\gamma_+(z; T)$ is bounded for all $z \in C$. Thus, if E is finite dimensional, $\text{cl}(\gamma_+(z))$ is automatically compact. In general, suppose that all hypotheses of proposition 2.1 hold except the assumption that $\text{cl}(\gamma_+(z; T))$ is compact for all $z \in C$. Instead suppose either that the norm is uniformly convex on C (so that if u_n and v_n are any sequences of points in C such that $\|u_n\| \rightarrow r, \|v_n\| \rightarrow r$ and $\|(u_n + v_n)/2\| \rightarrow r$, it follows that $\|u_n - v_n\| \rightarrow 0$) or that the norm is additive on C (so $\|u + v\| = \|u\| + \|v\|$ for all $u, v \in C$). Then if x and y are as in proposition 2.1 (so that (2.1) is satisfied), it follows that (2.2) and (2.3) are satisfied. Equation (2.2) follows immediately, as in the proof of proposition 2.3, so it remains to prove (2.3).

An examination of the proof of proposition 2.1 shows that the same argument is valid in our situation if we can prove that

$$\lim_{i \rightarrow \infty} \|(T^{p_i}x) \wedge (T^{p_i}y) - T^{p_i}(x \wedge y)\| = 0. \tag{2.17}$$

To see that (2.17) holds, first assume that the norm is uniformly convex. The order-preserving property of T gives

$$0 \leq T^{p_i}x - T^{p_i}x \wedge T^{p_i}y \leq T^{p_i}x - \left[\frac{T^{p_i}x \wedge T^{p_i}y + T^{p_i}(x \wedge y)}{2} \right] \leq T^{p_i}x - T^{p_i}(x \wedge y). \tag{2.18}$$

Because T is nonexpansive we find that

$$\|T^{p_i}(x) - T^{p_i}(x \wedge y)\| \leq \|x - (x \wedge y)\| = \rho. \tag{2.19}$$

We assume that $T^{p_i}x \rightarrow x$ and $T^{p_i}y \rightarrow y$, so the continuity of the lattice operation implies that

$$\lim_{i \rightarrow \infty} \|T^{p_i}x - (T^{p_i}x \wedge T^{p_i}y)\| = \|x - (x \wedge y)\| = \rho. \tag{2.20}$$

Combining equations (2.18)–(2.20) and using the monotonicity of the norm gives

$$\lim_{i \rightarrow \infty} \|T^{p_i}(x) - T^{p_i}(x \wedge y)\| = \rho = \lim_{i \rightarrow \infty} \left\| T^{p_i}(x) - \left[\frac{T^{p_i}x \wedge T^{p_i}y + T^{p_i}(x \wedge y)}{2} \right] \right\|. \tag{2.21}$$

Equations (2.19) and (2.21) and the uniform convexity of the norm on C give (2.17).

If the norm is additive, then we obtain from (2.18) that

$$\begin{aligned} \|T^{p_i}x - (T^{p_i}x \wedge T^{p_i}y)\| + \|T^{p_i}x \wedge T^{p_i}y - T^{p_i}(x \wedge y)\| &= \|T^{p_i}x - T^{p_i}(x \wedge y)\| \\ &\leq \|x - (x \wedge y)\|. \end{aligned} \tag{2.22}$$

By using (2.20) and (2.22), we immediately obtain (2.17).

Remark 2.3. If we apply proposition 2.1 to $C = K^n$ and if we know that (1.8) is satisfied for some $p \geq 1$ (as will be the case if the norm is polyhedral), then the results mentioned in Section 1 immediately give theorem 1.1 and the estimate $p \leq \varphi(n)$. Thus, the remainder of this section is devoted to proving that (1.8) is satisfied for some $p \geq 1$, even if the norm is not polyhedral.

Before proceeding further let us recall some basic facts about omega limit sets of non-expansive maps. If (M, d) is a complete metric space and $T: M \rightarrow M$ is a nonexpansive map (so $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in M$), define the omega limit set of a point $x \in M$ under T by

$$\omega(x; T) = \bigcap_{n \geq 1} \text{cl} \left(\bigcup_{j \geq n} T^j(x) \right),$$

where $\text{cl}(A)$ denotes the closure of a set A . If $\omega(x; T)$ is nonempty, it is known that $T|_{\omega(x; T)}$ is an isometry of $\omega(x; T)$ onto itself. Furthermore, it is known that $\omega(y; T) = \omega(x; T)$ for all $y \in \omega(x; T)$ and that given any two elements $y, z \in \omega(x; T)$, there exists a sequence $q_i \rightarrow \infty$ such that $T^{q_i}(y) \rightarrow z$. Finally, there exists a sequence $p_i \rightarrow \infty$ such that $T^{p_i}(y) \rightarrow y$ for all $y \in \omega(x; T)$. All of these results are proved in [16]; the proofs in [16] are given for the norm in a Banach space but apply equally well to a complete metric space.

PROPOSITION 2.2. Let C be a cone in a Banach space $(X, \|\cdot\|)$ and suppose that $\|\cdot\|$ is a strictly monotonic norm. Assume that X is a vector lattice in the partial ordering from C and that $(x, y) \rightarrow x \wedge y$ is continuous. Let $T: C \rightarrow C$ be an order-preserving map such that $T(0) = 0$ and T is nonexpansive with respect to $\|\cdot\|$. Assume that for each $y \in C$, $\gamma_+(y; T)$ has compact closure. Then for every $x \in C$, $\omega(x; T)$ is compact and nonempty. For any fixed $x_0 \in C$, there exists a sequence of integers $p_i \rightarrow \infty$ such that $T^{p_i}(z) \rightarrow z$ for all $z \in \omega(x_0; T)$. If S is defined by

$$S = \{z \in C : \lim_{i \rightarrow \infty} T^{p_i}(z) = z\},$$

S is a closed set such that $T(S) \subset S$ and $T|_S$ is an isometry of S onto S . If $y, z \in S$, it follows that $y \wedge z$ and $y \vee z$ are elements of S and T and $(T|_S)^{-1}$ preserve the lattice operations on S . If V is the lattice generated by $\omega(x_0; T)$, then $T(V) \subset V$ and $T|_V$ is an isometry of V onto V .

Proof. For $x \in C$, we have

$$\omega(x; T) = \bigcap_{n \geq 1} A_n(x), \quad A_n(x) \equiv \text{cl} \left(\bigcup_{j \geq n} T^j(x) \right),$$

so $\omega(x; T)$ is the intersection of a decreasing sequence of compact, nonempty sets, and, hence, compact and nonempty. (This is well known.) It follows by the remarks preceding proposition 2.2 that $T \upharpoonright \omega(x_0; T)$ is an isometry of $\omega(x_0; T)$ onto $\omega(x_0; T)$ and that there exists a sequence $p_i \rightarrow \infty$ such that $T^{p_i}(z) \rightarrow z$ for all $z \in \omega(x_0; T)$. Proposition 2.1 implies that if $y, z \in S$, then $y \wedge z \in S$ and $y \vee z \in S$, and T preserves the lattice operations on S . It is obvious that $T(S) \subset S$; and if $y, z \in S$ and

$$\|Ty - Tz\| < \|y - z\|,$$

we obtain a contradiction

$$\|y - z\| = \lim_{i \rightarrow \infty} \|T^{p_i-1}(Ty) - T^{p_i-1}(Tz)\| \leq \|Ty - Tz\| < \|y - z\|.$$

Thus, we see that $T \upharpoonright S$ is an isometry.

To see that S is closed, assume that $z_n \in S$ for $n \geq 1$ and $z_n \rightarrow z$. Given $\varepsilon > 0$, select N so that $\|z_N - z\| < \varepsilon/3$ and select i_0 so that

$$\|T^{p_i}(z_N) - z_N\| < \left(\frac{\varepsilon}{3}\right) \quad \text{for } i \geq i_0.$$

It follows (using the nonexpansiveness of T) that for $i \geq i_0$ we have

$$\|T^{p_i}(z_N) - z\| \leq \|T^{p_i}z - T^{p_i}z_N\| + \|T^{p_i}z_N - z_N\| + \|z_N - z\| < \varepsilon,$$

so $T^{p_i}z \rightarrow z$.

To see that T is onto S , observe that if $z \in S$,

$$z = \lim_{i \rightarrow \infty} T^{p_i}(z).$$

It follows that $T^{p_i}(z)$ is a Cauchy sequence, and because T is an isometry $T^{p_i-1}(z)$ is a Cauchy sequence. It is easy to see that $T^{p_i-1}(z) \in S$ for each $i \geq 1$, and since S is closed,

$$\lim_{i \rightarrow \infty} T^{p_i-1}(z) = y \in S, \quad \text{and} \quad Ty = z.$$

Notice that this proves that

$$T^{-1}(z) = \lim_{i \rightarrow \infty} T^{p_i-1}(z) \quad \text{for } z \in S,$$

so $(T \upharpoonright S)^{-1}$ is order-preserving and preserves the lattice operations.

It follows from what we have already proved that $V \subset S$ and $T(V)$ is a closed set. Writing $\omega = \omega(x_0; T)$, we know that $T(\omega) = \omega$, so $T(V) \supset \omega$. Proposition 2.1 implies that $T(V)$ is closed under the lattice operations, so the minimality of V gives

$$T(V) \supset V. \tag{2.23}$$

If $\Gamma = (T | S)^{-1}$ we also know (by using the formula $\Gamma(z) = \lim T^{p_i-1}(z)$ and previous arguments) that Γ is an isometry, $\Gamma(\omega) = \omega$ and Γ preserves the lattice operations on S , so

$$\Gamma(V) \supset V. \tag{2.24}$$

The inclusions (2.23) and (2.24) imply that

$$T(V) = \Gamma(V) = V. \quad \blacksquare$$

Before proceeding further we need a technical lemma which insures that the set V constructed in proposition 2.2 is compact. In the following lemma recall that a cone C is ‘‘normal’’ if there exists a constant M such that $\|x\| \leq M\|y\|$ for all x, y such that $0 \leq x \leq y$. It is well known that every cone in a finite dimensional Banach space is normal.

LEMMA 2.1. Let X be a Banach space and assume that $C \subset X$ is a normal cone and $\overset{\circ}{C}$, the interior of C , is nonempty. Assume that X is a vector lattice with respect to the partial ordering induced by C . If A is a compact subset of X , let V be the lattice generated by A (see the definition preceding proposition 2.1). Then V is compact.

Proof. If F is a finite subset of X , we first claim that there is a finite set G such that $G \supset F$ and G is closed under the lattice operations. To see this, first define

$$H = \left\{ x = \bigwedge_{\zeta \in T} \zeta : T \subset F \right\}.$$

It is clear that H is a finite set, $F \subset H$ and if $x, y \in H$, then $x \wedge y \in H$. Next define G by

$$G = \left\{ x = \bigvee_{\zeta \in T} \zeta : T \subset H \right\}.$$

Again it is clear that $H \subset G$, G is a finite set and if $x, y \in G$, then $x \vee y \in G$. However, we also claim that if $x, y \in G$, then $x \wedge y \in G$. Recall (see [17, p. 365]) that the lattice distributive laws hold in X

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z) \quad \text{and} \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z).$$

If $x, y \in G$, so there are sets $T_1 \subset H$ and $T_2 \subset H$ with

$$x = \bigvee_{\zeta_1 \in T_1} \zeta_1 \quad \text{and} \quad y = \bigvee_{\zeta_2 \in T_2} \zeta_2,$$

the lattice distributive laws give

$$x \wedge y = \bigvee_{\zeta_1 \in T_1, \zeta_2 \in T_2} (\zeta_1 \wedge \zeta_2).$$

Because $\zeta_1 \wedge \zeta_2 \in H$ for all $\zeta_1 \in T_1$ and $\zeta_2 \in T_2$, we see that $x \wedge y$ can be written

$$x \wedge y = \bigvee_{\zeta \in T} \zeta,$$

where T is a subset of H , so $x \wedge y \in G$.

If $B_r(x)$ denotes a closed ball of radius r and center x , and if $u_1 \in \overset{\circ}{K}$, then there exists $r > 0$ such that

$$u_1 + B_r(0) \subset K.$$

This immediately implies that, if $u = r^{-1}u_1$,

$$-u \leq z \leq u \quad \text{for all } z \text{ with } \|z\| \leq 1.$$

As usual, if $x \leq y$ we write

$$[x, y] = \{z: x \leq z \leq y\}.$$

By using the normality of K it is not hard to see that there exists a constant M such that

$$[-u, u] \subset B_M(0).$$

Since A is compact, given $\varepsilon > 0$, there exist $x_j \in A$, $1 \leq j \leq n$, such that

$$A \subset \bigcup_{j=1}^n B_{\varepsilon M^{-1}}(x_j).$$

It follows that (for $\varepsilon_1 = \varepsilon M^{-1}$)

$$A \subset \bigcup_{j=1}^n [x_j - \varepsilon_1 u, x_j + \varepsilon_1 u].$$

If F denotes the finite set $\{x_j: 1 \leq j \leq n\}$, we know that there is a finite set $G \supset F$ such that G is closed under the lattice operations. We claim that

$$V \subset \bigcup_{y \in G} [y - \varepsilon_1 u, y + \varepsilon_1 u] = V_1. \tag{2.25}$$

We know that V_1 is closed, and $V_1 \supset A$, so to prove (2.25), it suffices to prove that V_1 is closed under the lattice operations. However, if $w_1, w_2 \in V_1$, there exist $y, z \in G$ such that

$$y - \varepsilon_1 u \leq w_1 \leq y + \varepsilon_1 u \quad \text{and} \quad z - \varepsilon_1 u \leq w_2 \leq z + \varepsilon_1 u.$$

It follows easily that

$$w_1 \wedge w_2 \in [y \wedge z - \varepsilon_1 u, y \wedge z + \varepsilon_1 u] \quad \text{and} \quad w_1 \vee w_2 \in [y \vee z - \varepsilon_1 u, y \vee z + \varepsilon_1 u].$$

Since $y \wedge z \in G$ and $y \vee z \in G$, this proves that $w_1 \wedge w_2 \in V_1$ and $w_1 \vee w_2 \in V_1$.

Since we know that

$$V \subset V_1 \subset \bigcup_{y \in G} B_\varepsilon(y),$$

and G is finite, we conclude that V is totally bounded, hence compact. ■

Remark 2.4. Lemma 2.1 can be generalized slightly, but perhaps it is more interesting to note that lemma 2.1 fails badly if C has empty interior and one works in $L^p[0, 1]$ for $1 \leq p < \infty$. Specifically, we can construct a compact set $A \subset L^p$ such that A is bounded in L^∞ and such that V , the lattice generated by A , is not compact. To construct such an A , for each $n \geq 1$ and for $0 \leq j \leq 2^n - 1$ let $\Delta_{j,n} = [j2^{-n}, (j+1)2^{-n}]$. Let $f_{j,n}$ denote the characteristic function of

the interval $\Delta_{j,n}$ and define

$$A = \{0\} \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{j=0}^{2^n-1} \{f_{j,n}\} \right) \subset L^p[0, 1], \quad 1 \leq p \leq \infty.$$

To prove that A is compact in $L^p[0, 1]$, we have to prove that if $g_k, k \geq 1$, is any sequence of points in A , g_k has a convergent subsequence. If g_k has a subsequence g_{k_i} such that $g_{k_i} = g_{k_j}$ for all $i, j \geq 1$, we are done. Otherwise, given any integer N , there exists k_N such that

$$g_k \in \bigcup_{n=N}^{\infty} \bigcup_{j=0}^{2^n-1} \{f_{j,n}\}$$

for all $k \geq k_N$. This implies that

$$\|g_k\|_{L^p}^p \leq \frac{1}{2^N} \quad \text{for } k \geq k_N,$$

so $g_k \rightarrow 0$ in L^p .

If T is any subset of $\{j: 0 \leq j \leq 2^n - 1\}$ and g is given by

$$g = \bigvee_{j \in T} f_{j,n},$$

g is the characteristic function of $\bigcup_{j \in T} \Delta_{j,n}$. In particular, if we define $h_n(x)$ by

$$h_n(x) = \begin{cases} 1, & \text{if } j2^{-n} \leq x \leq (j+1)2^{-n} \text{ and } j \text{ is even, } 0 \leq j \leq 2^n - 1, \\ 0, & \text{otherwise,} \end{cases}$$

we see that $h_n \in V$ for $n \geq 1$. If $1 \leq m < n$, it is not hard to see that

$$\|h_n - h_m\|_{L^p}^p = \frac{1}{2} \quad \text{for } p < \infty \text{ and } \|h_n - h_m\|_{\infty} = 1.$$

This proves that V is not totally bounded and, hence, not compact.

We shall also need a lemma which gives conditions under which it makes sense to talk about $\text{sup}(A)$, where A is a subset of a Banach space X with a partial ordering induced by a cone $C \subset X$. Of course, if B is given by

$$B = \{z \mid z \geq a \text{ for all } a \in A\}, \tag{2.26}$$

by $\text{sup}(A)$ we mean an element $\zeta \in B$ such that $\zeta \leq z$ for all $x \in B$, if such an element ζ exists.

LEMMA 2.2. Let X be a Banach space with a normal cone C and assume that X is a vector lattice with respect to the partial ordering induced by C . Let A be a nonempty subset of X . Assume either: (a) X is reflective and B (defined as in (2.26)) is nonempty; or (b) A is compact and $\overset{\circ}{C}$, the interior of C , is nonempty. Then $\text{sup}(A)$ is defined.

Proof. First assume hypothesis (a). Fix $x_0 \in B$ and define $B_0 = \{x \in B: x \leq x_0\}$. If $a \in A$, we have that

$$a \leq x \leq x_0 \quad \text{for all } x \in B_0,$$

so the normality of C implies that B_0 is bounded. It is also easy to see that B_0 is closed and convex, so reflexivity implies that B_0 is compact in the weak topology. If $x \in B$ and

$B_x = \{y \in B: y \leq x\}$, the same argument shows that B_x is compact in the weak topology. It follows that in order to prove

$$\bigcap_{x \in B_0} B_x \neq \emptyset,$$

it suffices to prove that if $F = \{x_i: 1 \leq i \leq k\}$ is any finite collection of points in B_0

$$\bigcap_{x \in F} B_x \neq \emptyset. \tag{2.27}$$

However, we have

$$\bigwedge_{i=1}^k x_i \in \bigcap_{x \in F} B_x,$$

so (2.27) is valid. If $x \in B$, we have that $x \wedge x_0 \in B_0$ and

$$B_x \supset B_{x \wedge x_0}.$$

Using this we see that

$$\bigcap_{x \in B} B_x = \bigcap_{x \in B_0} B_x \neq \emptyset.$$

It is not hard to see that $\bigcap_{x \in B} B_x$ contains only one element and

$$\bigcap_{x \in B} B_x = \sup(A).$$

The second part of the lemma follows easily from lemma 2.1. If V is the lattice generated by A , lemma 2.1 implies that V is compact. If $u \in \overset{\circ}{C}$ and m is a positive integer, the same argument used in lemma 2.1 shows that there are points $a_{im} \in A$, $1 \leq i \leq N(m)$, such that

$$A \subset \bigcup_{i=1}^{N(m)} [a_{im} - m^{-1}u, a_{im} + m^{-1}u].$$

If we define $y_m \in V$ by

$$y_m = \bigvee_{i=1}^{N(m)} a_{im},$$

it is clear that

$$a \leq y_m + \left(\frac{1}{m}u\right) \quad \text{for all } a \in A. \tag{2.28}$$

Since V is compact, we can assume, by taking a subsequence, that $y_m \rightarrow y \in V$; and we have by taking the limit in (2.28) that

$$a \leq y \quad \text{for all } a \in A.$$

If $z \geq a$ for all $a \in A$, it is clear that $z \geq y_m$ for all m , so $z \geq y$ and $y = \sup(A)$.

Obviously, the same argument also gives the existence of $\inf(A)$ in this case. ■

Remark 2.5. Suppose that C is a normal cone with nonempty interior in a Banach space X and assume that X is a vector lattice. Suppose that A_k , $k \geq 1$, is a decreasing sequence of compact nonempty sets and write

$$A = \bigcap_{k \geq 1} A_k.$$

As is well known, given any open neighborhood U of A , there exists an integer $k(U)$ such that $A_k \subset U$ for all $k \geq k(U)$. For any fixed $\delta > 0$ and $u \in \overset{\circ}{C}$ let

$$U = \{w \in X: w - x + \delta u \in \overset{\circ}{C} \text{ and } -w + x + \delta u \in \overset{\circ}{C} \text{ for some } x \in A\}.$$

Lemma 2.2. implies that $\sup(A_k) = s_k$ exists and $\sup(A) = s$ exists and obviously $s_k \geq s$ for all k . However, if $k \geq k(U)$, we have $w \leq s + \delta u$ for all $w \in A_k$ and $s_k \leq s + \delta u$. The normality of C now implies that $\lim_{k \rightarrow \infty} s_k = s$.

Suppose now that assumptions and notation are as in lemma 2.1. We need to generalize the idea (defined in Section 1) of an irreducible element of V . If V is as in lemma 2.1, $\varepsilon > 0$ and $x \in V$, we shall say that “ x is ε -irreducible” (with respect to V) if

$$\zeta = \sup\{z \in V \mid z < x, \|x - z\| \geq \varepsilon\} < x,$$

so $\zeta \leq x$ and $\zeta \neq x$. Lemma 2.2 implies that ζ is well defined and $\zeta \in V$ (assuming that $\{z \in V \mid z < x, \|x - z\| \geq \varepsilon\}$ is nonempty). If there does not exist $z \in V$ such that $z < x$ and $\|x - z\| \geq \varepsilon$, we shall still say that x is ε -irreducible.

LEMMA 2.3. Let notation and assumptions be as in proposition 2.2 and suppose in addition that C is normal with nonempty interior. If $x \in V$ is ε -irreducible for some $\varepsilon > 0$, then $T(x)$ is also ε -irreducible.

Proof. We know from proposition 2.2 that T is an order-preserving isometry of V onto V . If $\Gamma = (T \mid V)^{-1}$, it was shown in the proof of proposition 2.2 that Γ is also an order-preserving isometry of V onto V .

If A is any compact subset of V , lemma 2.2 implies $\sup(A)$ exists. More generally, if A is any subset of V , one can easily see that $\sup(\bar{A}) = \sup(A)$; and because V is compact (see lemma 2.1), $\sup(A)$ exists for any $A \subset V$.

We next claim that if A is any subset of V ,

$$T(\sup(A)) = \sup(T(A)).$$

To see this, let $z_1 = \sup(A)$ and $z_2 = \sup(T(A))$. Because T is order-preserving, we have

$$T(z_1) \geq T(a) \quad \text{for all } a \in A,$$

so $T(z_1) \geq z_2$. Because $\Gamma(T(A)) = A$ and Γ is order-preserving the same argument shows that

$$\Gamma(z_2) \geq z_1.$$

If equality does not hold in the above inequality, we obtain

$$z_2 = T(\Gamma(z_2)) > T(z_1) \geq z_2,$$

a contradiction. Thus, we must have $T(z_1) = z_2$.

For $\xi \in V$ and $\varepsilon > 0$, define a set $A_\varepsilon(\xi)$ by

$$A_\varepsilon(\xi) = \{y \in V: y \leq \xi \text{ and } \|y - \xi\| \geq \varepsilon\}. \tag{2.29}$$

Because T and Γ are order-preserving isometries we have

$$T(A_\varepsilon(x)) \subset A_\varepsilon(Tx) \quad \text{and} \quad \Gamma(A_\varepsilon(Tx)) \subset A_\varepsilon(x). \tag{2.30}$$

Because Γ and T are one-one and $(\Gamma T)(A_\varepsilon(x)) = A_\varepsilon(x)$, we conclude that both of the inclusions in (2.30) must be equalities. It follows that

$$T(\sup(A_\varepsilon(x))) = \sup(A_\varepsilon(Tx)),$$

which shows that x is ε -irreducible if and only if Tx is ε -irreducible. ■

If C is a normal cone with nonempty interior in a Banach space X and X is a vector lattice in the partial ordering induced by C , then it is not hard to see that $(x, y) \rightarrow (x \wedge y)$ is continuous. Thus, the hypotheses of lemma 2.3 are slightly redundant.

We also need an ‘‘epsilonized’’ version of the idea of the height of a point x in a lattice V (see [10]). If X is a vector lattice and a normed linear space and $V \subset X$ is closed under the lattice operations and $x \in V$, we define the ‘‘ ε -height of x ’’ (with respect to V), $h_\varepsilon(x)$, by

$$h_\varepsilon(x) = \sup\{k \mid \exists e_i \in V, 0 \leq i \leq k, \text{ such that } e_k = x \text{ and } e_i < e_{i+1} \text{ for } 0 \leq i < k \\ \text{and } \|e_{i+1} - e_i\| \geq \varepsilon \text{ for } 0 \leq i < k\}. \tag{2.31}$$

If there is no element $w \in V$ such that $w < x$ and $\|w - x\| \geq \varepsilon$, we define $h_\varepsilon(x) = 0$. If the norm on X is monotonic and e_j is as in (2.31) we have

$$\|e_p - e_j\| \geq \|e_p - e_{p-1}\| \geq \varepsilon \quad \text{for } 0 \leq j < p \leq k. \tag{2.32}$$

If, in addition, we assume that V is compact, there exists for each $\varepsilon > 0$ an integer $N = N(\varepsilon)$ such that V is contained in the union of N open balls of radius $\varepsilon/2$ and centers in V . By using (2.32) and the pigeon-hole principle, it is easy to see that

$$h_\varepsilon(x) < N(\varepsilon) < \infty.$$

Our next lemma is a straightforward generalization of the corresponding result for irreducible elements in finite semilattices.

LEMMA 2.4. Let C be a normal cone with nonempty interior in a Banach space X and suppose that the norm of X is monotonic. Assume that X is a vector lattice with respect to the partial ordering induced by C . Let V be a compact subset of X such that V is closed under the lattice operations. If $\varepsilon > 0$ and $x \in V$, then $x = \sup\{z \in V: z \leq x \text{ and } z \text{ is } \varepsilon\text{-irreducible with respect to } V\}$.

Proof. For $w \in V$, let $h_\varepsilon(w)$ denote the ε -height of w with respect to V . We shall prove the lemma by induction on $h_\varepsilon(x)$. If $h_\varepsilon(x) = 0$, x is ε -irreducible by definition and the lemma is true. For an integer $k > 0$, assume that the lemma is true for all $x \in V$ such that $h_\varepsilon(x) < k$. If x is not ε -irreducible,

$$x = \sup\{z \in V \mid z \leq x \text{ and } \|z - x\| \geq \varepsilon\} = \sup(S). \tag{2.33}$$

If $z \in S$, it is easy to see that $h_\varepsilon(z) < k$: if $h_\varepsilon(z) \geq k$, we would obtain $h_\varepsilon(x) \geq k + 1$. By induction we have (for $z \in S$)

$$z = \sup\{w \in V \mid w \leq z, w \text{ is } \varepsilon\text{-irreducible}\}. \tag{2.34}$$

We obtain from (2.33) and (2.34) that

$$x = \sup\{w \in V \mid w \leq x, \|w - x\| \geq \varepsilon \text{ and } w \text{ is } \varepsilon\text{-irreducible}\},$$

which completes the proof. ■

Remark 2.6. In general it is necessary in lemma 2.4 that $\varepsilon > 0$. If $\varepsilon = 0$, 0-irreducibility is the same as irreducibility. However, lemma 2.4 may fail for $\varepsilon = 0$: to see this take $X = \mathbb{R}$ and $V = [0, 1]$ and note that 0 is the only irreducible element of V .

We prove our next lemma in somewhat greater generality than we shall actually need. No real simplification is gained by considering a less general case. In the statement of the following lemma, we shall call a partial ordering on a metric space (ω, d) “closed” if whenever $x_k \leq y_k$ for all $k \geq 1$ and $x_k \rightarrow x$ and $y_k \rightarrow y$, it follows that $x \leq y$.

LEMMA 2.5. Let (M, d) be a complete metric space and $T: M \rightarrow M$ a nonexpansive map with respect to d . Assume that $\xi \in M$ and $x_0 \in M$ are such that $\omega(\xi; T)$ is compact and nonempty and $Tx_0 = x_0$. Then for all $x, y \in \omega(\xi; T)$

$$d(x, x_0) = d(y, x_0).$$

If, in addition, there is a closed partial ordering defined on $\omega(\xi; T)$ such that

$$d(x, y) \leq d(x, z) \quad \text{and} \quad d(y, z) \leq d(x, z) \tag{2.35}$$

whenever $x, y, z \in \omega(\xi; T)$ and $x \leq y \leq z$ and if T preserves the partial ordering on $\omega(\xi; T)$, then there do not exist elements $x, y \in \omega(\xi; T)$ such that $x \leq y$ and $x \neq y$.

Proof. If $x \in \omega(\xi; T)$, we know that there exists a sequence $p_i \rightarrow \infty$ such that $T^{p_i}(x) \rightarrow x$. It follows that

$$d(x, x_0) = \lim_{i \rightarrow \infty} d(T^{p_i}x, x_0) = \lim_{i \rightarrow \infty} d(T^{p_i}x, T^{p_i}x_0). \tag{2.36}$$

On the other hand, T is nonexpansive, so $d(T^m x, T^m x_0)$ is a decreasing sequence of reals. Using this fact and (2.36) we see that

$$d(T^m x, x_0) = d(x, x_0) \quad \text{for all } m \geq 1. \tag{2.37}$$

For any $y \in \omega(\xi; T)$, there exists a sequence $q_i \rightarrow \infty$ such that $T^{q_i}(x) \rightarrow y$, so

$$d(y, x_0) = \lim_{i \rightarrow \infty} d(T^{q_i}(x), x_0) = d(x, x_0).$$

To prove the second part of the lemma, assume, to the contrary, that there exist $x, y \in \omega(\xi; T)$ with $x < y$. We know that there exists a sequence $q_i \rightarrow \infty$ such that $T^{q_i}(x) \rightarrow y$. Since $T|_\omega$ is an isometry of ω onto ω , $T^{q_i}|_\omega$ is also an isometry of ω onto ω . We assume that ω is compact, so the Ascoli–Arzela theorem implies that there exists a map $S: \omega \rightarrow \omega$ and a subsequence of q_i (which we label the same) so that $T^{q_i}|_\omega$ approaches S uniformly on ω . We have

$$d(S\xi, S\eta) = \lim_{i \rightarrow \infty} d(T^{q_i}\xi, T^{q_i}\eta) = d(\xi, \eta) \quad \text{for all } \xi, \eta \in \omega,$$

so S is an isometry. To see that S is onto, choose $z \in \omega$ and $x_i \in \omega$ such that

$$T^{q_i}(x_i) = z.$$

Since T^{q_i} approaches S uniformly on ω we have that

$$\lim_{i \rightarrow \infty} S(x_i) = \lim_{i \rightarrow \infty} z_i = z,$$

so z_i is a Cauchy sequence. It follows, because S is an isometry, that x_i is a Cauchy sequence and

$$\lim_{i \rightarrow \infty} x_i = \xi \in \omega \quad \text{with } S(\xi) = z.$$

Notice that S is also order-preserving, because T^{q_i} is order-preserving and the partial ordering is closed.

We now have, using the order-preserving property of S , that

$$x < Sx = y \quad \text{and} \quad S^j x < S^m x \quad \text{for } 0 \leq j < m. \quad (2.38)$$

By using (2.35) and (2.38) and the fact that S is an isometry,

$$0 < d(x, Sx) = d(S^j x, S^{j+1} x) \leq d(S^j x, S^m x) \quad \text{for } 0 \leq j < m. \quad (2.39)$$

On the other hand, (2.39) implies that the sequence $S^j(x)$ can have no convergent subsequence, which contradicts the assumption that ω is compact. ■

Remark 2.7. An examination of the proof of lemma 2.5 shows that it suffices to replace (2.35) by the assumption that there exists a constant M such that

$$d(x, y) \leq Md(x, z) \quad \text{and} \quad d(y, z) \leq Md(x, z)$$

whenever $x, y, z \in \omega(\xi; T)$ and $x \leq y \leq z$. One can also see that the final assertion of lemma 2.5 holds without the assumption that T has a fixed point x_0 .

We shall also need a technical lemma. We shall only use lemma 2.6 in finite dimension Banach spaces, but we shall prove it in greater generality.

LEMMA 2.6. Let C be a normal cone with nonempty interior in a Banach space X and assume that X is a vector lattice with respect to the partial ordering from C . Let $V \subset X$ be a compact set and let $T: V \rightarrow V$ be a nonexpansive, order-preserving map of V to V . Given $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that if $x, y \in \omega(\xi; T)$ for some $\xi \in V$ and $\|x - y\| \geq \delta$, then

$$\|x - (x \wedge y)\| \geq \varepsilon(\delta) \quad \text{and} \quad \|y - (x \wedge y)\| \geq \varepsilon(\delta).$$

Proof. We leave to the reader the exercise of proving (under the assumptions on C) that if $x_k \rightarrow x$ and $y_k \rightarrow y$ are any two convergent sequences in X , then $x_k \wedge y_k \rightarrow x \wedge y$ and $x_k \vee y_k \rightarrow x \vee y$.

We assume lemma 2.6 is false and try to obtain a contradiction. If the lemma is false, there exists two sequences $x_k \in \omega(\xi_k; T)$ and $y_k \in \omega(\xi_k; T)$ ($\xi_k \in V$) such that $\|x_k - y_k\| \geq \delta$ and

$$\|x_k - (x_k \wedge y_k)\| \rightarrow 0.$$

Because $\omega(\xi_k; T) = \omega(x_k; T)$, we can assume that $\xi_k = x_k$. Because V is compact, we can assume that $x_k \rightarrow x$ and $y_k \rightarrow y$. We claim that $x, y \in \omega(x; T)$. If we can prove this we will be done: lemma 2.5 and remark 2.7 imply that x and y are not comparable in the partial ordering. However, we have that

$$\lim_{k \rightarrow \infty} \|x_k - (x_k \wedge y_k)\| = \|x - (x \wedge y)\| = 0,$$

which implies that $x \leq y$.

Thus, it suffices to prove that $x, y \in \omega(x; T)$. The properties of omega limit sets imply that for each $k \geq 1$ there exist sequences of integers p_{ik} and q_{ik} such that $p_{ik} \rightarrow \infty$ as $i \rightarrow \infty$ and

$$\lim_{i \rightarrow \infty} T^{p_{ik}}(x_k) = x_k \quad \text{and} \quad \lim_{i \rightarrow \infty} T^{q_{ik}}(x_k) = y_k.$$

Given an integer $j \geq 1$, choose x_k and y_k such that

$$\|x_k - x\| < 2^{-j-2} \quad \text{and} \quad \|y_k - y\| < 2^{-j-2}.$$

Choose an integer i such that $p_{ik} \geq j$, $q_{ik} \geq j$ and $\|T^{p_{ik}}(x_k) - x_k\| < 2^{-j-1}$ and $\|T^{q_{ik}}(x_k) - y_k\| < 2^{-j-1}$. For this choice of i and k , define $p_j = p_{ik}$ and $q_j = q_{ik}$. By the triangle inequality we obtain

$$\begin{aligned} \|T^{p_j}(x) - x\| &\leq \|x - x_k\| - \|x_k - T^{p_j}(x_k)\| + \|T^{p_j}(x_k) - T^{p_j}(x)\| \\ &< 2^{-j-2} + 2^{-j-1} + 2^{-j-2} = 2^{-j}. \end{aligned}$$

It follows that $T^{p_j}(x) \rightarrow x$ and $x \in \omega(x; T)$. An analogous argument shows that $T^{q_j}(x) \rightarrow y$ and $y \in \omega(x; T)$. ■

It may be worthwhile to indicate a simpler variant of lemma 2.6 which would also be adequate for our applications.

LEMMA 2.6'. Let X be a Banach space which is also a vector lattice and assume that the norm on X is strictly monotonic. Assume that the map $(x, y) \rightarrow (x \wedge y)$ is continuous. Let V be a compact subset of X . Given $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that if $x, y \in V$, $\|x\| = \|y\|$ and $\|x - y\| \geq \delta$, then

$$\|x - (x \wedge y)\| \geq \varepsilon(\delta) \quad \text{and} \quad \|y - (x \wedge y)\| \geq \varepsilon(\delta).$$

Proof. Suppose the lemma is false. Then there are sequences $x_k \in V$, $y_k \in V$, such that

$$\|x_k\| = \|y_k\|, \quad \|x_k - y_k\| \geq \delta \quad \text{and} \quad \lim_{k \rightarrow \infty} \|x_k - (x_k \wedge y_k)\| = 0.$$

Because V is compact, we can take a subsequence and assume that $x_k \rightarrow x$ and $y_k \rightarrow y$ and

$$\|x\| = \|y\|, \quad \|x - y\| \geq \delta \quad \text{and} \quad \|x - (x \wedge y)\| = \lim_{k \rightarrow \infty} \|x_k - (x_k \wedge y_k)\| = 0.$$

It follows that $x < y$ and $\|x\| = \|y\|$, which contradicts the strict monotonicity of the norm. ■

For the remainder of this section we shall confine ourselves to finite dimensions and to $C = K^n$.

LEMMA 2.7. Let $T: K^n \subset \mathbb{R}^n \rightarrow K^n$ be an order-preserving map which is nonexpansive with respect to a strictly monotonic norm $\|\cdot\|$. Assume that $T(0) = 0$. For a given $\xi \in K^n$, let V be the lattice generated by $\omega(\xi; T)$. (Thus, our previous results imply that V is compact, T is an isometry of V onto V , and T and $\Gamma = (T|V)^{-1}$ preserve the lattice operations.)

For a given $\delta > 0$, let $\varepsilon(\delta) = \varepsilon > 0$ be as in lemma 2.6. If $x \in V$ is $\varepsilon(\delta)$ -irreducible, then there exists p such that $1 \leq p \leq n$ and

$$\|T^p(x) - x\| < \delta.$$

Proof. There exists a sequence $p_i \rightarrow \infty$ such that $T^{p_i}(y) \rightarrow y$ for all $y \in \omega(\xi; T)$; and it follows easily that $T^{p_i}(z) \rightarrow z$ for all $z \in V$. Thus, if $z \in V$, $z \in \omega(z; T)$.

If $\varepsilon > 0$ and $y \in V$ is ε -irreducible, define $I_\varepsilon(y) = \{i \mid z_i < y_i\}$, where

$$z = \sup\{w \in V: w \leq y \text{ and } \|w - y\| \geq \varepsilon\}. \tag{2.40}$$

Now suppose that $x \in V$ is ε -irreducible, $\varepsilon = \varepsilon(\delta)$. We know (lemma 2.3) that $T^j(x)$ is ε -irreducible for all $j \geq 0$. If $\|x - T^j(x)\| \geq \delta$ for some $j > 0$, we claim that

$$I_\varepsilon(x) \cap I_\varepsilon(T^j x) = \emptyset.$$

To prove this, we suppose not, so for some i , $1 \leq i \leq n$, we have

$$i \in I_\varepsilon(x) \cap I_\varepsilon(T^j x).$$

Lemma 2.6 (or lemma 2.6') implies that

$$\|x - (x \wedge T^j x)\| \geq \varepsilon(\delta) = \varepsilon \quad \text{and} \quad \|T^j x - (x \wedge T^j x)\| \geq \varepsilon.$$

If we define z and ζ by

$$z = \sup\{w \in V \mid w \leq x \text{ and } \|w - x\| \geq \varepsilon\} \quad \text{and} \quad \zeta = \sup\{w \in V \mid w \leq T^j x \text{ and } \|w - x\| \geq \varepsilon\},$$

we have that

$$(x \wedge T^j x)_i \leq z_i < x_i \quad \text{and} \quad (x \wedge T^j x)_i \leq \zeta_i < (T^j x)_i, \tag{2.41}$$

where a subscript i denotes the i th component of a vector.

Equation (2.41) gives a contradiction.

For $1 \leq i \leq n$ let $P_i = \{m: 0 \leq m \leq n \text{ and } i \in I_\varepsilon(T^m x)\}$. If $m_1, m_2 \in P_i$ and $m_1 < m_2$, our remarks above show that

$$\|T^{m_2} x - T^{m_1} x\| = \|T^{m_2 - m_1} x - x\| < \delta.$$

However, because there are $n + 1$ iterates $T^m(x)$, $0 \leq m \leq n$ and n sets P_i , $1 \leq i \leq n$, the pigeon-hole principle implies that there exists i , $1 \leq i \leq n$, which contains at least two elements. This completes the proof. ■

We now restate and prove our main theorem.

THEOREM 2.1. Let $K^n = \{x \in \mathbb{R}^n: x_i \geq 0 \text{ for } 1 \leq i \leq n\}$ and let $T: K^n \rightarrow K^n$ be a map which is nonexpansive with respect to a strictly monotonic norm on \mathbb{R}^n . Assume that $T(0) = 0$ and T is order-preserving. If $\xi \in K^n$, there exists a minimal positive integer $p_\xi = p \leq \alpha(n) \leq \varphi(n)$ (where

$\alpha(n)$ is defined in definition 1.2 and $\varphi(n)$ is defined in [9, p. 362] and in Section 3 below) such that

$$\lim_{k \rightarrow \infty} T^{kp}(\xi) = \eta \quad \text{and} \quad T^p(\eta) = \eta.$$

Proof. Let V be the lattice generated by $\omega(\xi; T)$.

If $\delta > 0$ and $\varepsilon = \varepsilon(\delta)$ is as in lemma 2.6 and $N = \text{lcm}(1, 2, \dots, n)$, we claim that for any ε -irreducible element $x \in V$ one has

$$\|T^N(x) - x\| \leq N\delta.$$

To see this, note that lemma 2.7 implies that

$$\|T^m x - x\| \leq \delta \quad \text{for some } m, 1 \leq m \leq n.$$

We can write $N = mp_1$, p_1 an integer. Because

$$\|T^{m(j+1)}x - T^{mj}x\| \leq \delta \quad \text{for } 0 \leq j < p_1,$$

the triangle inequality implies

$$\|T^N x - x\| \leq p_1 \delta \leq N\delta.$$

If x now denotes any fixed element of V and p_1 is as above, define a compact set A_δ ($\delta > 0$) by

$$A_\delta = \{y \in V: y \leq x \text{ and } \|T^N y - y\| \leq N\delta\}.$$

By our remarks above, A_δ contains all $\varepsilon(\delta)$ -irreducible elements $y \in V$ with $y \leq x$. Thus, lemma 2.4 implies that

$$x = \sup(A_\delta).$$

Select a sequence $\delta_j \downarrow 0$, so A_{δ_j} is a decreasing sequence of compact, nonempty sets. If we define A_0 by

$$A_0 = \bigcap_{j=1}^{\infty} A_{\delta_j},$$

it follows from remark 2.5 that

$$\sup(A_0) = \lim_{j \rightarrow \infty} \sup(A_{\delta_j}) = x.$$

On the other hand, one can easily see that

$$A_0 = \{y \in V \mid y \leq x \text{ and } T^N(y) = y\}.$$

Because both T^N and Γ^N are order-preserving, where $\Gamma = (T \upharpoonright V)^{-1}$, one can see that

$$T^N(\sup(A_0)) = \sup(T^N(A_0)).$$

Thus, we have that

$$T^N(x) = T^N(\sup(A_0)) = \sup(T^N(A_0)).$$

The definition of A_0 shows that $T^N(A_0) = A_0$, so

$$\sup(T^N(A_0)) = \sup(A_0) = x.$$

Thus, we have proved that for $N = L(n)$ and all $x \in V$

$$T^N(x) = x.$$

If we take $x \in \omega(\xi; T)$, we see that $\omega(\xi; T)$ is the orbit of the periodic point x and the period p_x of x divides N . If V_0 is the semilattice generated by $\omega(\xi; T)$, $T(V_0) \subset V_0$ and T preserves the lattice operation on V_0 , so by definition 1.2 we have $p_x \leq \alpha(n)$. ■

Remark 2.8. In the l_1 -norm case, Scheutzow [10] has observed that one need not assume that T is order-preserving. However, Scheutzow strongly uses the Akcoglu–Krengel result [1] that $\omega(\xi; T)$ is a finite set for every $\xi \in K^n$ if $T: K^n \rightarrow K^n$, $T(0) = 0$ and T is l_1 -nonexpansive. It is interesting that a slight variant of our proof of theorem 2.1 gives a different proof that $\omega(\xi; T)$ is finite when $T: K^n \rightarrow K^n$, $T(0) = 0$ and T is l_1 -nonexpansive. For a given $\xi \in K^n$, define a set M by

$$M = \{z: \exists x \in \omega(\xi; T) \text{ such that } 0 \leq z \leq x\}.$$

A generalization of arguments in [9, 10] shows that T is integral-preserving (so $\sum_{i=1}^n (Tz)_i = \sum_{i=1}^n z_i$ for $z \in M$) and order-preserving on M . There exists a sequence of integers $p_i \rightarrow \infty$ such that $T^{p_i}(x) \rightarrow x$ for all $x \in \omega(\xi; T)$, and we define M_0 by

$$M_0 = \{z \in M: \lim_{i \rightarrow \infty} T^{p_i}(z) = z\}.$$

One can prove that if $x_1, x_2 \in M_0$, then $x_1 \wedge x_2 \in M_0$ and $T(x_1 \wedge x_2) = T(x_1) \wedge T(x_2)$. Similarly if $x_1, x_2 \in M_0$ and $x_1 \vee x_2 \in M$, then $x_1 \vee x_2 \in M_0$ and $T(x_1 \vee x_2) = T(x_1) \vee T(x_2)$. Also one can show that $T|_{M_0}$ is an order-preserving isometry of M_0 onto M_0 and $\Gamma = (T|_{M_0})^{-1}$ is also an order-preserving isometry of M_0 onto M_0 . By defining ε -irreducibility, the ε -height of $x \in M_0$, etc., with respect to M_0 instead of V , one can mimic the proof of theorem 2.1 to prove that $\omega(\xi; T)$ is finite and $|\omega(\xi; T)| \leq \alpha(n)$.

Remark 2.9. Theorem 2.1 does not apply to an interesting case, namely the sup norm on \mathbb{R}^n . Is the conclusion of theorem 2.1 still true for the sup norm? If not, what sort of estimates can one obtain on $|\omega(\xi; T)|$ when $T: K^n \rightarrow K^n$ is order-preserving, nonexpansive with respect to the sup norm and $T(0) = 0$?

Remark 2.10. If the norm is the Euclidean norm in \mathbb{R}^n , $n \geq 3$, and all assumptions of theorem 2.1 are satisfied except that T is order-preserving, it is easy to construct examples for which $\omega(\xi; T)$ is infinite for almost all $\xi \in K^n$. See [18, p. 224].

3. COMPUTING $\alpha(n)$ AND $\varphi(n)$

In this section we want to prove more information about the functions $\alpha(n)$ and $\varphi(n)$ and extend a table of values of $\alpha(n)$ and $\varphi(n)$ which was given in [9].

Recall that a lower semilattice $W \subset \mathbb{R}^n$ is a set such that for all $x, y \in W$, $x \wedge y \in W$.

Definition 3.1. $Q_0(n)$ is the set of integers $p \geq 1$ such that there exists a lower semilattice $W \subset \mathbb{R}^n$ and a map $T: W \rightarrow W$ such that $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in W$ and T has a periodic point $\xi \in W$ of period p .

It is clear (see definition 1.2) that

$$\alpha(n) = \sup\{p: p \in Q_0(n)\}.$$

Definition 3.2. $Q_1(n)$ is the set of integers $p \geq 1$ such that there exists a map $T: K^n \rightarrow K^n$ such that $T(0) = 0$, T is l_1 -nonexpansive and T has a periodic point $\xi \in K^n$ of period p .

The results of Section 1 show that $Q_1(n) \subset Q_0(n)$.

If $T: D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, T is called “integral-preserving” if

$$\sum_{i=1}^n (Tx)_i = \sum_{i=1}^n x_i \quad \text{for all } x \in D.$$

If $D = \mathbb{R}^n$ or K^n and T is integral-preserving, it is known [19] that T is l_1 -nonexpansive if and only if T is order-preserving. We shall denote by u the vector in \mathbb{R}^n all of whose components equal one.

Definition 3.3. A map $T: K^n \rightarrow K^n$ will be said to satisfy H3.1 if T is integral-preserving and order-preserving and $T(cu) = cu$ for all $c \geq 0$.

By our remarks, if T satisfies H3.1, T is l_1 -nonexpansive.

Definition 3.4. $P(n)$ is the set of integer $p \geq 1$ such that there exists a map $T: K^n \rightarrow K^n$ which satisfies H3.1 and has a periodic point $\xi \in K^n$ of period p .

It is easy to see that

$$P(n) \subset Q_1(n) \subset Q_0(n) \quad \text{and} \quad P(n) \subset P(n + 1)$$

and

$$Q_j(n) \subset Q_j(n + 1) \quad \text{for } j = 0, 1.$$

Also, if T is in the class of maps allowed for definitions 3.1, 3.2 or 3.4, respectively, and r is a positive integer, then T^r is also allowed for definitions 3.1, 3.2 or 3.4, respectively. Thus, if $p \in Q_0(n)$, $Q_1(n)$ or $P(n)$, respectively, and $p = mr$, then $m \in Q_0(n)$, $Q_1(n)$ or $P(n)$, respectively. If Σ denotes $Q_0(n)$, $Q_1(n)$ or $P(n)$, we can put a partial ordering on Σ by $p_1 \prec p_2$ if p_1 is a factor of p_2 . In this ordering, Σ has maximal elements, and Σ comprises precisely all divisors of its maximal elements.

It is clear that the order p of any permutation on n letters is an element of $P(n)$, so $n \in P(n)$. More generally, it is proved in Section 3 of [9] that if $n = mr$ and $p_1, p_2, \dots, p_r \in P(m)$, then

$$r \text{ lcm}(\{p_j: 1 \leq j \leq r\}) \in P(n).$$

Furthermore, if $n_1 + n_2 = n$ and $p_j \in P(n_j)$ for $j = 1, 2$, it is shown that $p = \text{lcm}(p_1, p_2) \in P(n)$.

It is proved in [9] that if $p \in Q_0(n)$, there is a set $S \subset \{j: 1 \leq j \leq n\}$ such that p divides $\text{lcm}(S)$. However, not every subset S of $\{j: 1 \leq j \leq n\}$ is possible. It is proved in [9] that there are constraints on S . To indicate some of these constraints, recall that if T is a set of positive integers, $\text{gcd}(T)$ denotes the greatest common divisor of the elements of T . Recall also the following conditions from [9] concerning nonempty subsets S of $\{j: 1 \leq j \leq n\}$.

Condition A. S does not contain a subset Q such that (1) $\gcd(i, j) = 1$ for all $i, j \in Q$ with $i \neq j$ and (2) $\sum_{i \in Q} i > n$.

Condition B. S does not contain disjoint subsets Q and R which satisfy the following properties:

- (1) $\gcd(i, j) = 1$ for all $i, j \in Q$ with $i \neq j$;
- (2) $\gcd(i, k) = 1$ for all $i \in Q$ and $k \in R$;
- (3) R has $r + 1$ elements, $r \geq 1$, $i > r$ for all $i \in R$, and $\gcd(i, j)$ divides r for all $i, j \in R$ with $i \neq j$;
- (4) $i + j > n - (\sum_{k \in Q} k)$ for all $i, j \in R$ with $i \neq j$.

The possibility that Q is the empty set in condition B is allowed. In that case conditions (1) and (2) in condition B are vacuous and $\sum_{k \in Q} k = 0$. It is also not hard to see that if S satisfies condition B (with $r = 1$), then S satisfies condition A, but we prefer to state condition A separately.

Condition C. S does not contain disjoint subsets Q and R with the following properties:

- (1) $\gcd(i, j) = 1$ for all $i, j \in Q$ such that $i \neq j$;
- (2) $\gcd(i, k) = 1$ for all $i \in Q$ and $k \in R$;
- (3) R has $m + 1$ elements where $m = \rho^2 - \rho + 1$ and $\rho \geq 2$ is an integer and $\gcd(i, j)$ divides $r = \rho^2$ for all $i, j \in R$ with $i \neq j$;
- (4) there exists $\gamma \in R$ such that $\gcd(\gamma, j)$ divides ρ for all $j \in R$, $j \neq \gamma$, and $\gamma > \rho$ and $j > \rho^2$ for $j \neq \gamma$;
- (5) $i + j > n - (\sum_{k \in Q} k)$ for all $i, j \in R$ with $i \neq j$.

Condition D. S does not contain a set R with the following properties:

- (1) $R = \{p_j \mid 1 \leq j \leq m + r - 1\}$, where $m \geq 2$, $r \geq 2$, and $p_i \neq p_j$ for $1 \leq i < j \leq m + r - 1$;
- (2) $\gcd(p_i, p_j)$ divides r for $1 \leq i < j \leq m + r - 1$ and $p_i > r$ for $1 \leq i \leq m + r - 1$;
- (3) $\sum_{j=1}^m p_j > n$ and $p_j + p_k > n$ for all $j \neq k$ such that $1 \leq j \leq m + r - 1$ and $m < k \leq m + r - 1$.

Definition 3.5. If $S \subset \{j : 1 \leq j \leq n\}$ we say that S is “admissible for n ” if $S = \{1\}$ or if $1 \notin S$ and S satisfies conditions A–D.

It is proved in [9] that if $p \in Q_0(n)$, then there exists a set $S \subset \{j : 1 \leq j \leq n\}$ which is admissible for n and is such that p is a divisor of $\text{lcm}(S)$. Motivated by this fact, an ad hoc function $\varphi(n)$ is defined by

$$\varphi(n) = \sup\{\text{lcm}(S) : S \text{ is admissible for } n\}. \tag{3.1}$$

If we define functions $\beta(n)$ and $\gamma(n)$ by

$$\beta(n) = \sup\{p \mid p \in Q_1(n)\} \quad \text{and} \quad \gamma(n) = \sup\{p \mid p \in P(n)\}, \tag{3.2}$$

we have that

$$\gamma(n) \leq \beta(n) \leq \alpha(n) \leq \varphi(n). \tag{3.3}$$

If a set S is admissible for n , it is admissible for $n + 1$, so $\varphi(n)$ is a monotonic increasing function, and clearly $\gamma(n)$, $\alpha(n)$ and $\beta(n)$ are monotonic increasing functions.

In [9] it is proved that $\gamma(n) = \varphi(n)$ for $1 \leq n \leq 24$ and the values of $\gamma(n)$ are tabulated for $n \leq 24$. Here we want to extend this table for $n \leq 32$ and prove $\gamma(n) = \varphi(n)$ for $n \leq 32$. By using the previously mentioned properties of $P(n)$, it is relatively easy to find lower bounds for $\gamma(n)$.

LEMMA 3.1. For $25 \leq n \leq 32$ we have the following lower bounds for $\gamma(n)$

$$\gamma(25) \geq 2640 = 16 \times 3 \times 5 \times 11,$$

$$\gamma(26) \geq 3120 = 16 \times 3 \times 5 \times 13,$$

$$\gamma(27) \geq 5040 = 16 \times 9 \times 5 \times 7,$$

$$\gamma(n) \geq 9240 = 8 \times 3 \times 5 \times 7 \times 11 \quad \text{for } 28 \leq n \leq 30,$$

$$\gamma(31) \geq 18480 = 16 \times 3 \times 5 \times 7 \times 11$$

and

$$\gamma(32) \geq 36960 = 32 \times 3 \times 5 \times 7 \times 11.$$

Proof. It has been proved in [9, theorem 3.4], that $2640 \in P(24)$, so we certainly have $2640 \in P(25)$ and $\gamma(25) \geq 2640$.

There are permutation maps of K^3 to K^3 which have periodic points of period 3 and 2, respectively. Thus, $2, 3 \in P(3)$ and our previous remarks imply $2 \operatorname{lcm}(2, 3) = 12 \in P(6)$. There is a permutation map of K^6 to itself which has a periodic point of period 5, so $5 \in P(6)$. It follows that $120 = 2 \operatorname{lcm}(5, 12) \in P(12)$, so $120 \in P(13)$. There is a permutation map of K^{13} with a periodic point of period 13, so $13 \in P(13)$. It follows that $3120 = 2 \operatorname{lcm}(120, 13) \in P(26)$.

There are permutation maps of K^5 of order 5 and 4, respectively, so $2 \operatorname{lcm}(4, 5) = 40 \in P(10)$. There is also a permutation map of K^{10} of order 9, so $9 \in P(10)$. It follows that $2 \operatorname{lcm}(9, 40) = 720 \in P(20)$. There is a permutation map of order 7 of K^7 , so $7 \in P(7)$. It follows that

$$5040 = \operatorname{lcm}(7, 720) \in P(27).$$

It is proved in remark 3.2 of [9] that $9240 \in P(28)$, so $\gamma(n) \geq 9240$ for $28 \leq n \leq 30$.

We have already remarked that $2640 \in P(24)$, and by considering a permutation map we have that $7 \in P(7)$. Because $7 + 24 = 31$, we conclude that $\operatorname{lcm}(7, 2640) = 18480 \in P(31)$.

There are permutation maps of K^4 of orders 4 and 3, respectively, so $2 \operatorname{lcm}(4, 3) = 24 \in P(8)$. There is also a permutation map of order 7 of K^8 , so $7 \in P(8)$ and $336 = 2 \operatorname{lcm}(7, 24) \in P(16)$. There is a permutation map of K^{16} of order $55 = 5 \times 11$, so $55 \in P(16)$. It follows that $2 \operatorname{lcm}(336, 55) = 36960 \in P(32)$. ■

We claim that $\gamma(n)$ actually equals the lower bounds for it given in lemma 3.1 for $25 \leq n \leq 32$ and that $\gamma(n) = \varphi(n)$ for $25 \leq n \leq 32$. The method of proof is to show $\varphi(n) \leq \gamma(n)$ for $25 \leq n \leq 32$. The problem is that the computation of $\varphi(n)$ for values of n like $n = 30$ or 31 is highly nontrivial and involves a very laborious and lengthy case-by-case analysis in which conditions A and B are primarily used but conditions C and D also play important roles. Reasons of length and aesthetics preclude a full proof, but notes are available from the author for those who wish to see details. Here we shall be satisfied to give the outlines of the proof for $28 \leq n \leq 32$.

First, consider the case $28 \leq n \leq 30$. Because $\gamma(28) \geq 9240$, it suffices to assume that $T \subset \{j: 1 \leq j \leq 30\}$ is admissible for $n = 30$ and that $\text{lcm}(T) \geq 9240$ and try to prove that necessarily $\text{lcm}(T) \leq 9240$. First, one proves for such a T that 29, 27, 25, 23, 19 and 17 are not factors of $\text{lcm}(T)$. It is only necessary to use conditions A and B to obtain this result. Next one proves (this is more difficult) that 13 is not a factor of $\text{lcm}(T)$; here conditions A-D are all needed.

Finally, one first proves that 16 is not a factor of $\text{lcm}(T)$ and then that 9 is not a factor of $\text{lcm}(T)$; again, conditions A-D are all needed. Using these results one finds that

$$\text{lcm}(T) \leq 8 \times 3 \times 5 \times 7 \times 11 = 9240 \quad \text{and} \quad \varphi(30) \leq 9240.$$

If T is admissible for $n(n = 31 \text{ or } 32)$ and $\text{lcm}(T) \geq 18480$ (for $n = 31$) or $\text{lcm}(T) \geq 36960$ (for $n = 32$), one proves fairly easily that 31, 29, 27, 25, 23, 19 and 17 are not factors of $\text{lcm}(T)$. With much more effort one proves that 13 is not a factor of $\text{lcm}(T)$; conditions A-D are all needed. If $n = 31$ or $n = 32$, one proves that 9 is not a factor of $\text{lcm}(T)$, so $\text{lcm}(T) \leq 16 \times 3 \times 5 \times 7 \times 11 = 18480$ and $\varphi(31) \leq 18480$ for $n = 31$ and $\text{lcm}(T) \leq 32 \times 3 \times 5 \times 7 \times 11 = 36960$ for $n = 32$.

Similar arguments apply for $25 \leq n \leq 27$ and one obtains the following theorem.

THEOREM 3.1. For $25 \leq n \leq 32$, $\gamma(n) = \beta(n) = \alpha(n) = \varphi(n)$. Also, one has that $\gamma(25) = 2640 = 16 \times 3 \times 5 \times 11$, $\gamma(26) = 3120 = 16 \times 3 \times 5 \times 13$, $\gamma(27) = 16 \times 9 \times 5 \times 7 = 5040$, $\gamma(n) = 8 \times 3 \times 5 \times 7 \times 11 = 9240$ for $28 \leq n \leq 30$, $\gamma(31) = 16 \times 3 \times 5 \times 7 \times 11 = 18480$, and $\gamma(32) = 32 \times 3 \times 5 \times 7 \times 11 = 36960$.

With the aid of theorem 3.1 we can extend the table of values of $\gamma(n)$ in [9].

For $1 \leq n \leq 32$, $\alpha(n) = \beta(n) = \gamma(n) = \varphi(n)$ (see Table 1).

We should emphasize that there are constraints on the set S in theorem 1.2 other than conditions A-D, and it is by no means clear that $\alpha(n)$, $\beta(n)$ or $\gamma(n)$ behaves asymptotically like $\varphi(n)$ for large n . Nevertheless, for values of $n \leq 32$, one can obtain remarkably precise results concerning $P(n)$, $Q_1(n)$ and $Q_0(n)$. As an illustration, we mention the following easy result: define numbers $\alpha_1 = 16 \times 3 \times 7$, $\alpha_2 = 16 \times 3 \times 5$, $\alpha_3 = 3 \times 13$, $\alpha_4 = 2 \times 13$, $\alpha_5 = 2 \times 3 \times 11$, $\alpha_6 = 5 \times 11$, $\alpha_7 = 4 \times 11$, $\alpha_8 = 4 \times 9 \times 5$, $\alpha_9 = 2 \times 9 \times 7$ and $\alpha_{10} = 4 \times 5 \times 7$. The elements

Table 1. Values of $\varphi(n)$ for $1 \leq n \leq 32$

n	$\varphi(n)$	n	$\varphi(n)$
1	1	17	420
2	2	18	420
3	3	19	840
4	4	20	1680
5	6	21	1680
6	12	22	1680
7	12	23	1680
8	24	24	2640
9	24	25	2640
10	60	26	3120
11	60	27	5040
12	120	28	9240
13	120	29	9240
14	168	30	9240
15	180	31	18480
16	336	32	36960

of $P(16) \cup \{\alpha_9\}$ and the elements of $Q_0(16) \cup \{\alpha_9\}$ are precisely the numbers which are the divisors of α_j for some j , $1 \leq j \leq 10$. It follows that, aside from the troublesome α_9 , we know all elements of $P(16)$ and

$$P(16) \cup \{\alpha_9\} = Q_1(16) \cup \{\alpha_9\} = Q_0(16) \cup \{\alpha_9\}.$$

The difficulty with α_9 arises from the fact that

$$\alpha_9 = \text{lcm}(6, 7, 9).$$

The set $T = \{6, 7, 9\}$ is admissible for $n = 16$, but we do not know whether $\alpha_9 \in P(16)$ or $\alpha_9 \in Q_0(16)$. However, since $\beta_9 = 9 \times 7 \in P(16)$, the problem is not too serious.

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