



Iterated linear maps on a cone and Denjoy–Wolff theorems

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Abstract

Let C be a closed cone with nonempty interior $\text{int}(C)$ in a finite dimensional Banach space X . We consider linear maps $f : X \rightarrow X$ such that $f(\text{int}(C)) \subset \text{int}(C)$ and f has no eigenvector in $\text{int}(C)$. For $q \in C^*$, with $q(x) > 0 \forall x \in C \setminus \{0\}$ we define $T(x) = \frac{f(x)}{q(f(x))}$ and $\Sigma_q = \{x \in C \mid q(x) = 1\}$. Let $\text{ri}(\Sigma_q)$ denote the relative interior of Σ_q . We are interested in the omega limit set $\omega(x; T)$ of $x \in \text{ri}(\Sigma_q)$ under T . We prove that the convex hull $\text{co}(\omega(x; T)) \subset \partial\Sigma_q$, and if C is polyhedral we also show that $\omega(x; T)$ is finite. Thus if C is polyhedral there is a face of C such that the orbit of any point in the interior of C under iterates of f approaches that face after scaling.

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1. Introduction

Let D be a bounded open convex set in a finite dimensional real Banach space X . D can be given a metric d called Hilbert’s metric with the aid of cross ratios (see [12]). Assume that $f : D \rightarrow D$ is nonexpansive with respect to d . If in addition D is strictly convex and f has no fixed point in D , then Beardon [4] showed that there exists a $z \in \partial D$ such that $\|f^k(x) - z\| \rightarrow 0$

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as $k \rightarrow \infty$ for all $x \in D$ (actually Beardon assumes that $d(f(x), f(y)) < d(x, y)$ but later work (see [12]) has shown that this assumption is not necessary). This result is an exact analogue for Hilbert's metric of the classical Denjoy–Wolff theorem for fixed point free analytic maps of the unit disc into itself. Further work in this direction can be found in [3,11,12].

Unfortunately, in almost all applications in analysis of which we are aware (see [5,13,15] for example) the set D is not strictly convex, and simple examples show that in this case the direct generalization of Beardon's theorem fails. Karlsson and Noskov [12] have considered the case in which D may not be strictly convex. If f has no fixed point in D , then for any $x \in D$, there must exist a point z in the omega limit set (see Section 2 for definitions) of x such that for any other point y in the omega limit set, the line segment connecting z and y lies in the boundary of D . This result raises the following question. For a Hilbert metric nonexpansive map $f : D \rightarrow D$, can there exist a point x which has an omega limit set which is not contained in a face of the boundary of $\text{cl}(D)$? Nussbaum has conjectured that the convex hull of the omega limit set $\omega(x; f)$ of any point $x \in D$ under f will be contained entirely in the boundary ∂D , at least for many important classes of functions from analysis (see [16] for further remarks). Karlsson has proposed a similar conjecture by e-mail communication in 2004.

Let C be a closed cone in X . An alternative formulation of Hilbert's metric can be given on the interior of C , denoted $\text{int}(C)$. In this formulation, we have a pseudo-metric d , called Hilbert's projective metric on $\text{int}(C)$, which is invariant under scaling of the vectors involved. Let $q : C \rightarrow \mathbb{R}$ be a continuous, homogeneous of degree one map with $q(x) > 0$ for all $x \in C \setminus \{0\}$ and define $\Sigma_q = \{x \in C \mid q(x) = 1\}$. Hilbert's projective metric restricted to the relative interior of Σ_q (defined to be $\text{ri}(\Sigma_q) = \Sigma_q \cap \text{int}(C)$) is in fact a metric, and if Σ_q is convex then this metric is the same as the Hilbert metric obtained using cross ratios. For the problems dealt with in this paper, we typically choose q to be an element of the dual cone C^* with the property that $q(x) > 0$ for all $x \in C \setminus \{0\}$. For finite dimensional cones, this poses no problems. However, such a linear functional may not exist when C is an infinite dimensional cone.

If $f : X \rightarrow X$ is homogeneous of degree one, leaves $\text{int}(C)$ invariant, and f preserves the ordering induced by the cone C on X , then it is known that f is nonexpansive in Hilbert's projective metric on $\text{int}(C)$ [15, Prop. 1.5]. This fact gives us an obvious place to look for examples of maps which are nonexpansive with respect to Hilbert's metric. In order to convert between the two approaches to Hilbert's metric, we define the map T by $T(x) := f(x)/q(f(x))$ which maps $\text{ri}(\Sigma_q) \rightarrow \text{ri}(\Sigma_q)$, and is nonexpansive with respect to Hilbert's metric. Note that T has a fixed point in $\text{ri}(\Sigma_q)$ if and only if f has an eigenvector in $\text{int}(C)$.

In a paper by Akian et al. [1], the authors consider polyhedral cones C and continuous maps $f : C \rightarrow C$ which are order-preserving and subhomogeneous (see Section 2 for definitions). If $x \in C$ and $\{f^k(x) \mid k \geq 0\}$ is bounded in norm it is proved that $\exists p = p(x)$ and $\xi = \xi(x)$ such that $\|f^{kp}(x) - \xi\| \rightarrow 0$ as $k \rightarrow \infty$. In particular the omega limit set $\omega(x; f)$ is finite. If in addition $f(\text{int}(C)) \subseteq \text{int}(C)$, $x \in \text{int}(C)$ and f has no fixed points in $\text{int}(C)$ the result of [1] is used in [16] to show that $\omega(x; f)$ lies in a single component of C (again see Section 2 for definitions) and $\text{co}(\omega(x; f)) \subseteq \partial C$. Thus this situation is reasonably well understood. However, if $\{f^k(x) \mid k \geq 0\}$ is unbounded in norm, the results of [1] give no information about $\omega(x; T)$, where T is the scaled version of f , $T(y) = f(y)/q(f(y))$.

In this paper we shall consider Hilbert metric nonexpansive maps arising from linear maps which are order-preserving and leave $\text{int}(C)$ invariant. The iterates of such maps need not be bounded even if they have spectral radius one. In Section 2 of this paper we review some relevant background material and prove some general results about order-preserving, homogeneous of degree one maps $f : C \rightarrow C$. In Section 3 we prove that for a Hilbert metric nonexpansive map

T arising from scaling a linear map $f : \text{int}(C) \rightarrow \text{int}(C)$ in the manner described above, either T has a fixed point in $\text{ri}(\Sigma_q)$ or the convex hull of the omega limit set of any point $x \in \text{ri}(\Sigma_q)$, $\text{co}(\omega(x; T))$, is contained in the boundary of Σ_q . In the case where Σ_q is polyhedral, we show that $\omega(x; T)$ is also finite.

The case we are considering is really part of linear Perron–Frobenius theory on general finite dimensional cones. It is striking that the elementary questions we ask seem not to have been previously addressed.

2. Background material

In this section we collect some needed background material. By a closed cone C in a real Banach space $(X, \|\cdot\|)$ we shall mean a closed, convex set in X such that (a) $\lambda C \subseteq C$ for all $\lambda \geq 0$ and (b) $C \cap (-C) = \{0\}$. A closed, convex set C is called a *wedge* if (a) holds but (b) possibly fails. As usual, X^* will denote the dual of X ; and we shall always define $C^* \subseteq X^*$ by

$$C^* = \{\theta \in X^* | \theta(x) \geq 0 \text{ for all } x \in C\}. \tag{1}$$

C^* is always a wedge, and if the closed linear span of C equals X (in which case C is called *total*), C^* is a closed cone.

A closed cone C in a Banach space $(X, \|\cdot\|)$ induces a partial ordering \leq_C by $x \leq_C y$ if and only if $y - x \in C$. If the context is clear, we shall write \leq instead of \leq_C . If C is a closed cone in a Banach space X and Y , the linear span of C , is finite dimensional, we say that C is finite dimensional and we define $\text{dim}(C) = \text{dim}(Y)$. In this case it is easy to show that $\text{int}_Y(C)$, the interior of C as a subset of Y , is nonempty. In general, a closed cone C in a Banach space X is called *normal* if there exists a constant M such that $\|x\| \leq M\|y\|$ whenever $0 \leq x \leq y$. It is known [9,19] that every closed, finite dimensional cone is normal.

If C is a closed cone in a Banach space $(X, \|\cdot\|)$, $x \in C \setminus \{0\}$ and $y \in X$, we shall say that x *dominates* y (in the partial ordering from C) if there exists $\beta \in \mathbb{R}$ with $\beta x \geq y$. If $x, y \in C \setminus \{0\}$, we shall say that x and y are *comparable* (in the partial ordering from C) and we shall write $x \sim y$ if x dominates y and y dominates x . Comparability gives an equivalence relation on $C \setminus \{0\}$; and for $u \in C \setminus \{0\}$, we shall call $\{x \in C | x \sim u\}$ the *component of C containing u* and we shall write

$$C_u = \{x \in C \setminus \{0\} | x \sim u\}. \tag{2}$$

It is easy to see that C_u is convex and $\lambda C_u = C_u$ for all $\lambda > 0$. If $\text{int}(C) \neq \emptyset$ and $u \in \text{int}(C)$, it is clear that $C_u = \text{int}(C)$. If C is a closed, normal cone in $(X, \|\cdot\|)$, $u \in C \setminus \{0\}$ and X_u is a vector space defined by

$$X_u = \{x \in X | \exists a > 0 \text{ with } -au \leq x \leq au\}, \tag{3}$$

then it is well known see [9,19,21] and references to the literature on p. 494 of [7] that with a norm $\|x\|_u$ defined by

$$\|x\|_u = \inf\{a > 0 | -au \leq x \leq au\}. \tag{4}$$

$(X_u, \|\cdot\|_u)$ is a Banach space and C_u is the interior of $C \cap X_u$ in $(X_u, \|\cdot\|_u)$. If X is finite dimensional, X_u is the linear span of C_u , $\|\cdot\|_u$ and $\|\cdot\|$ are equivalent norms on X_u and C_u is the interior of $C \cap X_u$ in X_u .

If C is a closed cone in a Banach space $(X, \|\cdot\|)$, $x, y \in C \setminus \{0\}$ and x and y are comparable, we shall follow Bushell’s notation [5] and define

$$m(y/x; C) := \sup\{a \geq 0 | ax \leq y\} \tag{5}$$

and

$$M(y/x; C) := \inf\{b \geq 0 \mid y \leq bx\}. \tag{6}$$

If $x, y \in C \setminus \{0\}$ are comparable, $\alpha := m(y/x; C)$, $\beta := M(y/x; C)$, we define Hilbert’s projective metric $d(x, y; C)$ by

$$d(x, y; C) = \log(\beta/\alpha). \tag{7}$$

If it is clear what cone is referred to, we shall write $M(y/x)$, $m(y/x)$ and $d(x, y)$ instead of $M(y/x; C)$, $m(y/x; C)$ and $d(x, y; C)$. If $u \in C \setminus \{0\}$ and $x, y, z \in C_u$, then (a) $0 \leq d(x, y) = d(y, x)$, (b) $d(x, z) \leq d(x, y) + d(y, z)$ and (c) $d(x, y) = 0$ if and only if $y = tx$ for some $t > 0$. It follows that if $q : C_u \rightarrow (0, \infty)$ is a continuous map which is homogeneous of degree 1 (so $q(tx) = tq(x)$ for all $t > 0$ and all $x \in C_u$) and if S_q is defined by

$$S_q = \{x \in C_u \mid q(x) = 1\}, \tag{8}$$

then (S_q, d) is a metric space. If C is also normal, it is known that (S_q, d) is a complete metric space and the topology on (S_q, d) is the same as the topology induced by the norm on X : see Chapter 1 of [15] and p. 494 of [7] for references to the literature concerning this theorem. We shall be interested in maps $T : S_q \rightarrow S_q$ such that $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in S_q$, and we shall call such maps *nonexpansive* with respect to d .

A closed cone in a finite dimensional Banach space X is called *polyhedral* if there exist $\theta_1, \theta_2, \dots, \theta_N \in X^*$ such that

$$C = \{x \in X \mid \theta_j(x) \geq 0 \text{ for } 1 \leq j \leq N\}. \tag{9}$$

A *face* of a polyhedral cone C is a set $F = \{x \in C \mid \varphi(x) = 0\}$, where $\varphi \in C^*$. If $\dim(C) = n$ and $\dim(F) = n - 1$, F is called a *facet* of C . It is known (see [20, Section 8.4]) that if C is a closed polyhedral cone with nonempty interior in X and C has N facets, then there exist $\theta_i \in C^* \setminus \{0\}$, $1 \leq i \leq N$, such that Eq. (9) holds and such that each θ_i defines a facet of C .

If C is a closed cone in a Banach space X , $\Gamma \subseteq C$ and D is a closed cone in a Banach space Y , a map $f : \Gamma \rightarrow D$ is called *order-preserving* (with respect to the partial orders on C and D) if $f(x) \leq f(y)$ whenever $x, y \in \Gamma$ and $x \leq y$. If $t\Gamma \subseteq \Gamma$ for all $t > 0$, f is called homogeneous of degree 1 if $f(tx) = tf(x)$ for all $x \in \Gamma$ and $t > 0$ and f is called subhomogeneous if $tf(x) \leq f(tx)$ for all $0 < t < 1$. Our main interest is in the case that $\Gamma = C$ or $\Gamma = C_u$ for some $u \in C \setminus \{0\}$ and $D = C$ or $D = [0, \infty)$. If $u \in C \setminus \{0\}$ and $q : C_u \rightarrow (0, \infty)$ is continuous and homogeneous of degree one, $S := \{x \in C_u \mid q(x) = 1\}$ and $f : C_u \rightarrow C_u$ is homogeneous of degree one and order-preserving, one can define $T : S \rightarrow S$ by $T(x) = f(x)/q(f(x))$, and it is easy to verify (see Chapter 1 of [15]) that T is nonexpansive with respect to d . Furthermore, if $f : C_u \rightarrow C$ is homogeneous of degree one, order-preserving and $f(x) \in C_u$ for some $x \in C_u$, then $f(y) \in C_u$ for all $y \in C_u$.

Let C be a closed cone in a finite dimensional Banach space X and $q : C \rightarrow \mathbb{R}$ be a continuous map which is homogeneous of degree one and satisfies $q(x) > 0$ for all $x \in C \setminus \{0\}$, a simple continuity and compactness argument implies that there are positive constants c_1 and c_2 with

$$c_1 \|x\| \leq q(x) \leq c_2 \|x\| \text{ for all } x \in C. \tag{10}$$

If C is a closed, finite dimensional cone, it is well known that there exists $\theta \in C^*$ and positive constants c_1 and c_2 with

$$c_1 \|x\| \leq \theta(x) \leq c_2 \|x\| \text{ for all } x \in C \tag{11}$$

and in this case we shall take $q = \theta \in C^*$, where θ satisfies Eq. (11). In infinite dimensions, there may not exist $\theta \in C^*$ which satisfies Eq. (11), and it may be necessary (see [15] or [16]) to take $q(x) = \|x\|$.

Lemma 1. *Let C be a closed cone in a finite dimensional Banach space $(X, \|\cdot\|)$ and $q : C \rightarrow [0, \infty)$ a continuous map which is homogeneous of degree one and satisfies $q(x) > 0$ for $x \in C \setminus \{0\}$. Define $\Sigma_q := \{x \in C | q(x) = 1\}$. Assume that $\langle x_k | k \geq 1 \rangle \subseteq \Sigma_q$ and $\langle y_k | k \geq 1 \rangle \subseteq \Sigma_q$ are sequences in Σ_q such that x_k and y_k are comparable for all $k \geq 1$ and $d(x_k, y_k) \leq R < \infty$ for all $k \geq 1$. If $\lim_{k \rightarrow \infty} \|x_k - \zeta\| = 0$ and $\lim_{k \rightarrow \infty} \|y_k - \eta\| = 0$, then ζ and η are comparable and $d(\zeta, \eta) \leq R$.*

Proof. There is a constant M such that whenever $0 \leq x \leq y$

$$q(x) \leq Mq(y). \tag{12}$$

If Eq. (12) were not satisfied there would exist points $\chi_k \leq \psi_k$ in $C \setminus \{0\}$ with $q(\psi_k)/q(\chi_k) \rightarrow 0$. By the homogeneity of q , we can assume that $q(\psi_k) = 1$. Using Eq. (10) we see that there are positive constants c_1 and c_2 with $c_1\|x\| \leq q(x) \leq c_2\|x\|$ for all $x \in C$. If we define $u_k = \chi_k/q(\chi_k)$ and $v_k = \psi_k/q(\chi_k)$, we conclude that $c_1\|u_k\| \leq 1 \leq c_2\|u_k\|$ and $\|v_k\| \rightarrow 0$. By taking a subsequence we can assume that $u_k \rightarrow u \in C \setminus \{0\}$, $v_k \rightarrow 0$ and $0 - u = -u \in C \setminus \{0\}$. Since $u \neq 0$, we have contradicted the definition of a cone. Thus Eq. (12) holds.

There exist $\alpha_k > 0$ and $\beta_k > 0$ with $\alpha_k x_k \leq y_k \leq \beta_k x_k$ and $\log(\beta_k/\alpha_k) \leq R$. Applying q and using Eq. (12) we see that

$$\alpha_k = \alpha_k q(x_k) \leq Mq(y_k) = M \quad \text{and} \quad 1 = q(y_k) \leq Mq(\beta_k x_k) = M\beta_k.$$

Since $\beta_k/\alpha_k \leq \exp(R)$ we deduce that

$$\beta_k \leq \alpha_k \exp(R) \leq M \exp(R) \quad \text{and} \quad \left(\frac{1}{\alpha_k}\right) \leq \left(\frac{1}{\beta_k}\right) \exp(R) \leq M \exp(R).$$

It follows that by taking a subsequence we can assume that $\alpha_k \rightarrow \alpha > 0$ and $\beta_k \rightarrow \beta$, and we deduce that $\alpha\zeta \leq \eta \leq \beta\zeta$, with $\log(\beta/\alpha) \leq R$. \square

If A is a subset of a Banach space $(X, \|\cdot\|)$, $\text{cl}(A)$ will denote the closure of A in the norm topology, and $\text{co}(A)$ will denote the convex hull of A . If $D \subseteq X$, $T : D \rightarrow D$ is a map and $x \in D$ we shall write

$$\gamma(x; T) = \{T^k(x) | k \geq 0\}. \tag{13}$$

We are interested in the “omega limit set of x under T ”, $\omega(x; T)$ defined by

$$\omega(x; T) = \bigcap_{m \geq 0} \text{cl}(\gamma(T^m(x); T)). \tag{14}$$

As is well known, $\xi \in \omega(x; T)$ if and only if there is a sequence of integers $k_i \uparrow \infty$ with $\|T^{k_i}(x) - \xi\| \rightarrow 0$ as $i \rightarrow \infty$.

The following theorem will be crucial for the main results of this paper. Note that assertion (a) is restatement of results which appear in Theorems 4.1 and 4.2 on p. 114 of [15]. For more details see [16].

Theorem 1. *Let C be a closed cone with nonempty interior, $\text{int}(C)$, in a finite dimensional Banach space $(X, \|\cdot\|)$. Let $q \in C^*$ be such that $q(x) > 0$ for all $x \in C \setminus \{0\}$. Define*

$\Sigma_q := \{x \in C \mid q(x) = 1\}$ and $\text{ri}(\Sigma_q) := \Sigma_q \cap \text{int}(C)$. Assume that $T : \text{ri}(\Sigma_q) \rightarrow \text{ri}(\Sigma_q)$ is nonexpansive with respect to Hilbert’s projective metric d and that T has no fixed points in $\text{ri}(\Sigma_q)$. Then the following hold:

- (a) For every $x \in \text{ri}(\Sigma_q)$, $\omega(x; T) \subseteq \partial C$.
- (b) If $x, y \in \text{ri}(\Sigma_q)$, then every element $\zeta \in \omega(x; T)$ is comparable to an element $\eta \in \omega(y; T)$.
- (c) If $x \in \text{ri}(\Sigma_q)$, every element $\xi \in \text{co}(\bigcup_{z \in \text{ri}(\Sigma_q)} \omega(z; T))$ is comparable to an element $\zeta \in \text{co}(\omega(x; T))$.
- (d) If there exists $x \in \text{ri}(\Sigma_q)$ and $\zeta \in \omega(x; T)$ such that T has a continuous extension $\bar{T} : \text{ri}(\Sigma_q) \cup \{\zeta\} \rightarrow \Sigma_q$ then ζ and $\bar{T}(\zeta)$ are comparable and $d(\zeta, \bar{T}(\zeta)) \leq d(x, Tx)$.

Proof. As stated above, a proof of assertion (a) can be found in Theorems 4.1 and 4.2 of [15] on p. 114.

Suppose that $x, y \in \text{ri}(\Sigma_q)$ and $\zeta \in \omega(x; T)$. By definition, there exists a sequence $k_i \rightarrow \infty$ with $\|T^{k_i}(x) - \zeta\| \rightarrow 0$. Because $\text{cl}(\gamma(y; T))$ is compact, by taking a further subsequence, which we label the same, we can assume that there exists $\eta \in C$ and $\|T^{k_i}(y) - \eta\| \rightarrow 0$. Because T is nonexpansive with respect to d , $d(T^{k_i}(x), T^{k_i}(y)) \leq d(x, y)$. Lemma 1 now implies that ζ is comparable to η and $d(\zeta, \eta) \leq d(x, y)$.

For $\zeta \in \omega(x; T)$ there exists a sequence $k_i \rightarrow \infty$ with $\|T^{k_i}(x) - \zeta\| \rightarrow 0$ as $i \rightarrow \infty$. If T has a continuous extension $\bar{T} : \text{ri}(\Sigma_q) \cup \{\zeta\} \rightarrow \Sigma_q$, then $\|T^{k_i}(T(x)) - \bar{T}(\zeta)\| \rightarrow 0$ as $i \rightarrow \infty$, so $\bar{T}(\zeta) \in \omega(x; T)$. By the argument given above ζ and $\bar{T}(\zeta)$ are comparable and $d(\zeta, \bar{T}(\zeta)) \leq d(x, Tx)$.

Suppose $\xi = \sum_{j=1}^N \lambda_j \eta_j$ where each $\lambda_j > 0$, $\sum_{j=1}^N \lambda_j = 1$, and each $\eta_j \in \omega(y_j; T)$ for some $y_j \in \text{ri}(\Sigma_q)$. We have proved that there exist $\zeta_j \in \omega(x; T)$ such that ζ_j is comparable to η_j for each $1 \leq j \leq N$. It follows that $\xi = \sum_{j=1}^N \lambda_j \zeta_j$ is comparable to $\sum_{j=1}^N \lambda_j \eta_j$. \square

Remark. We shall actually be considering a special case of Theorem 1: Suppose that $f : C \rightarrow C$ is continuous, order-preserving, homogeneous of degree one and $f : \text{int}(C) \rightarrow \text{int}(C)$. We know then that $T(x) = f(x)/q(f(x))$ defines a map of $\text{ri}(\Sigma_q)$ to $\text{ri}(\Sigma_q)$ which is nonexpansive with respect to d . If $G = \{x \in \Sigma_q \mid f(x) \neq 0\}$, it is clear from the definition of T that T extends to a continuous map $\bar{T} : G \rightarrow \Sigma_q$. Generally, the question of when a Hilbert metric nonexpansive map extends continuously to the boundary of its domain is critical in the problems we are studying. Although we will not use it in this paper, the following proposition implies that if $x \in \text{ri}(\Sigma_q)$ and $\zeta \in \omega(x; T)$ then $\zeta \in G$.

Proposition 1. Let C, X, q , and Σ_q be as in Theorem 1 and assume that $f : C \rightarrow C$ is continuous, order-preserving and homogeneous of degree one. Assume that $f(\text{int}(C)) \subseteq \text{int}(C)$ and define $T : \text{ri}(\Sigma_q) \rightarrow \text{ri}(\Sigma_q)$ by $T(x) = f(x)/q(f(x))$. If $x \in \text{ri}(\Sigma_q)$ and $\zeta \in \omega(x; T)$ then $f(\zeta) \neq 0$.

Proof. Choose $c > 0$ such that $cx \leq f(x)$. Since f is order-preserving, $cf^k(x) \leq f^{k+1}(x)$. So using Eq. (12) (which we may do by the arguments given in the proof of Theorem 1) we get that

$$cq(f^k(x)) \leq Mq(f^{k+1}(x))$$

and hence

$$\frac{q(f^{k+1}(x))}{q(f^k(x))} \geq \frac{c}{M} \quad \text{for all } k \geq 0.$$

This implies that

$$f\left(\frac{f^{k_i}(x)}{q(f^{k_i}(x))}\right) = f(T^{k_i}(x)) \rightarrow 0$$

for any subsequence $k_i \rightarrow \infty$. So if $\zeta \in \omega(x; T)$ then $f(\zeta) \neq 0$. \square

If C is a closed, two dimensional cone, all of our questions become trivial.

Proposition 2. *Let C be a closed, two dimensional cone in a two dimensional Banach space X and let $q : C \rightarrow [0, \infty)$ be a continuous map which is homogeneous of degree one and satisfies $q(x) > 0$ for all $x \in C \setminus \{0\}$. Let $\Sigma_q := \{x \in C | q(x) = 1\}$ and $\text{ri}(\Sigma_q) := \text{int}(C) \cap \Sigma_q$. Assume that $T : \text{ri}(\Sigma_q) \rightarrow \text{ri}(\Sigma_q)$ is a continuous map which has no fixed points in $\text{ri}(\Sigma_q)$. There exist points $v_1, v_2 \in \partial C \cap \Sigma_q$ such that $\Sigma_q = \text{ri}(\Sigma_q) \cup \{v_1, v_2\}$ and either (a) $T^k(x) \rightarrow v_1$ as $k \rightarrow \infty$ for all $x \in \text{ri}(\Sigma_q)$ or (b) $T^k(x) \rightarrow v_2$ as $k \rightarrow \infty$ for all $x \in \text{ri}(\Sigma_q)$.*

Proof. It is easy to verify [6] that there exist linearly independent vectors $v_1, v_2 \in X$ such that $C = \{\lambda_1 v_1 + \lambda_2 v_2 | \lambda_1 \text{ and } \lambda_2 \geq 0\}$ and $\text{int}(C) = \{\lambda_1 v_1 + \lambda_2 v_2 | \lambda_1 \text{ and } \lambda_2 > 0\}$. By homogeneity of q , we can assume that $q(v_1) = q(v_2) = 1$. There exists a homeomorphism $\Phi : [0, 1] \rightarrow \Sigma_q$ defined by $\Phi(t) = \frac{(1-t)v_1 + tv_2}{q((1-t)v_1 + tv_2)}$, and $\Phi((0, 1)) = \text{ri}(\Sigma_q)$. If we define $\widehat{T} : (0, 1) \rightarrow (0, 1)$ by

$$\widehat{T} = (\Phi^{-1} \circ T \circ \Phi)(t),$$

it is easy to see that $\widehat{T}^k(t) \rightarrow 0$ for all $t \in (0, 1)$ if and only if $T^k(x) \rightarrow v_1$ for all $x \in \text{ri}(\Sigma_q)$. Similarly $\widehat{T}^k(t) \rightarrow 1$ for all $t \in (0, 1)$ if and only if $T^k(x) \rightarrow v_2$ for all $x \in \text{ri}(\Sigma_q)$. Because T has no fixed points in $\text{ri}(\Sigma_q)$, \widehat{T}^k has no fixed points in $(0,1)$. The intermediate value theorem implies that either (a) $\widehat{T}(t) < t$ for all $t \in (0, 1)$ or (b) $\widehat{T}(t) > t$ for all $t \in (0, 1)$. In case (a) we deduce that for $0 < t < 1$, $(\widehat{T}^k(t) | k \geq 0)$ is a strictly decreasing sequence of real numbers in $(0, 1)$. It follows that there exists $\tau \in [0, 1)$ so that $\lim_{k \rightarrow \infty} |\widehat{T}^k(t) - \tau| = 0$. Thus $\widehat{T}(\tau) = \tau$, and if $\tau > 0$, \widehat{T} has a fixed point in $(0,1)$ which is a contradiction. Thus, in case (a), $\lim_{k \rightarrow \infty} T^k(x) = v_1$ for all $x \in \text{ri}(\Sigma_q)$. An analogous argument shows that in case (b) $\lim_{k \rightarrow \infty} T^k(x) = v_2$ for all $x \in \text{ri}(\Sigma_q)$. \square

3. Hilbert metric nonexpansive maps from linear maps

For any closed cone C with nonempty interior in a finite dimensional Banach space $(X, \|\cdot\|)$, we may choose a linear functional q in the dual cone C^* with the property that $q(x) > 0$ for all $x \in C \setminus \{0\}$. We define $\Sigma_q := \{x \in C | q(x) = 1\}$ and $\text{ri}(\Sigma_q) := \Sigma_q \cap \text{int}(C)$. As mentioned before, if $f : C \rightarrow C$ is continuous, order-preserving, homogeneous of degree one and $f(\text{int}(C)) \subseteq \text{int}(C)$, then the map $T : \text{ri}(\Sigma_q) \rightarrow \text{ri}(\Sigma_q)$ defined by $T(x) = f(x)/q(f(x))$ is a nonexpansive map in the Hilbert metric on $\text{ri}(\Sigma_q)$ (see Chapter 1 of [15]). In particular, if f is a linear, it will give rise to a map $T : \text{ri}(\Sigma_q) \rightarrow \text{ri}(\Sigma_q)$ which is nonexpansive in Hilbert’s metric. In this section we will collect some results relating to the omega limit sets of points $x \in \text{ri}(\Sigma_q)$ under a map $T(x) = f(x)/q(f(x))$ where f is linear.

Theorem 2. *Let C be any closed polyhedral cone with nonempty interior in a finite dimensional Banach space X . Suppose that $f : C \rightarrow C$ is a linear map such that $f(\text{int}(C)) \subseteq \text{int}(C)$. Let $q \in C^*$ be such that $q(x) > 0$ for all $x \in C \setminus \{0\}$, and define Σ_q and $\text{ri}(\Sigma_q)$ as before. Let*

$T(x) = f(x)/q(f(x))$. If T has no fixed points in $\text{ri}(\Sigma_q)$, then for any $x \in \text{ri}(\Sigma_q)$, $\omega(x; T)$ is a finite subset of $C_u \cap \Sigma_q$ for some $u \in \partial C$, and C_u does not depend on x .

Before we can prove this theorem, we need the following lemma. For completeness we give a proof, although Bit-Shun Tam has pointed out that this lemma can be obtained as a corollary of Theorem 3.6 in [8]. For comparison, see Theorem 9.1 of [17].

Lemma 2. *If $A \neq 0$ is a nonnegative $n \times n$ matrix, with spectral radius $r(A) = 1$, then there is a positive integer p and a nonnegative integer m such that $\lim_{k \rightarrow \infty} \frac{A^{kp}}{k^m} = L$ exists, and L is a nonnegative matrix not equal to zero.*

Proof. If A is irreducible and the spectral radius $r(A) = 1$, then it is well known that every eigenvalue λ of A with $|\lambda| = r(A) = 1$ is a root of unity [10, Cor 8.4.10]. In fact this is true for any nonnegative matrix with spectral radius $r(A) = 1$. To see this, it suffices to note that A is conjugate via a permutation matrix to its Perron normal form, which is a block upper triangular matrix with irreducible blocks along the diagonal ([14, p. 142]). Therefore, there is a p such that the only eigenvalue of A^p with absolute value 1 is 1. Thus the Jordan form $J = S^{-1}A^pS$ of A^p will consist of Jordan blocks corresponding to the eigenvalue 1, and Jordan blocks corresponding to eigenvalues with modulus strictly less than 1.

If $J_q(1)$ is a $q \times q$ Jordan block of J corresponding to the eigenvalue 1, then $J_q(1) = I_q + N_q$ where I_q is the $q \times q$ identity matrix, and N_q is the nilpotent matrix with entries $v_{ij} = 1$ if $j - i = 1$ and $v_{ij} = 0$ otherwise. The binomial theorem gives

$$J_q^k(1) = I_q + \sum_{t=1}^k \binom{k}{t} N_q^t$$

and N_q^t is the matrix with entries $v_{ij}^{(t)} = 1$ if $j - i = t$; $v_{ij}^{(t)} = 0$ otherwise. In particular, if $t > q - 1$, $N_q^t = 0$. Thus

$$\lim_{k \rightarrow \infty} \frac{J_q^k(1)}{k^{(q-1)}} = \lim_{k \rightarrow \infty} \sum_{t=1}^{(q-1)} \frac{1}{k^{(q-1)}} \binom{k}{t} N_q^t = \frac{1}{(q-1)!} N_q^{(q-1)}.$$

Thus if q is the size of the largest Jordan block in J corresponding to the eigenvalue 1, it follows that if J_i is any Jordan block of J which either corresponds to the eigenvalue 1, but has size less than q , or corresponds to an eigenvalue with modulus less than 1, we have

$$\lim_{k \rightarrow \infty} \frac{J_i^k}{k^{(q-1)}} = 0.$$

This implies that

$$\lim_{k \rightarrow \infty} J^k = J^\infty$$

exists and is not zero. Thus,

$$\lim_{k \rightarrow \infty} \frac{A^{kp}}{k^{(q-1)}} = \lim_{k \rightarrow \infty} S \left(\frac{J^k}{k^{(q-1)}} \right) S^{-1} = SJ^\infty S^{-1} = L$$

exists. Then since each A^{kp} is a nonnegative matrix, we conclude that L must also be nonnegative, and since J^∞ has a strictly positive norm, $L \neq 0$. \square

Using this lemma, and the results of the previous section, we can now prove Theorem 2.

Proof (of Theorem 2). Since C is a polyhedral cone and q is a linear functional, the set Σ_q is a convex compact polyhedral set, and thus Σ_q has a finite number of extreme points. Suppose that $\zeta_1, \zeta_2, \dots, \zeta_N$ are the extreme points of Σ_q . If $x \in \text{ri}(\Sigma_q)$, there exist $\lambda_1, \lambda_2, \dots, \lambda_N \geq 0$ such that $x = \lambda_1\zeta_1 + \lambda_2\zeta_2 + \dots + \lambda_N\zeta_N$. In fact, we may assume that each $\lambda_i > 0$ for $1 \leq i \leq N$. This is because x is in the interior of Σ_q and can therefore be written as a convex combination of the point $z = \frac{1}{N} \sum_{i=1}^N \lambda_i$ and some other point in Σ_q on the ray from z passing through x . Furthermore, since f is linear, $f(x) = \lambda_1 f(\zeta_1) + \lambda_2 f(\zeta_2) + \dots + \lambda_N f(\zeta_N)$. Since $f : C \rightarrow C$ and $\text{span}\{\zeta_1, \zeta_2, \dots, \zeta_N\}$ contains C , note that $f(\zeta_j) = \sum_{i=1}^N a_{ij}\zeta_i$ with each $a_{ij} \geq 0$. Let A be the $N \times N$ matrix defined by $A = [a_{ij}]$. Let $\phi : \mathbb{R}^N \rightarrow X$ be the map given by $\phi(u_1, u_2, \dots, u_N) = u_1\zeta_1 + u_2\zeta_2 + \dots + u_N\zeta_N$. Using the linearity of ϕ we can see that if $v = u_1\zeta_1 + u_2\zeta_2 + \dots + u_N\zeta_N$ then $f(v) = \phi(Au)$ where $u = (u_1, u_2, \dots, u_N) \in \mathbb{R}^N$. Since $v = \phi(u)$ implies $f(v) = \phi(Au)$ it follows that $f^k(v) = \phi(A^k u)$ whenever $v = \phi(u)$. In particular, for our $x \in \text{ri}(\Sigma_q)$, $f^k(x) = \phi(A^k y)$ where $y = (\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathbb{R}^n$. By Lemma 2, there exists a p and an m such that $\frac{A^{kp}}{k^m \rho(A)^{kp}} \rightarrow L \neq 0$ as $k \rightarrow \infty$. Since ϕ is continuous, $\frac{f^{kp}(x)}{k^m \rho(A)^{kp}} = \phi\left(\frac{(A^{kp}y)}{k^m \rho(A)^{kp}}\right) \rightarrow \phi(Ly)$ as $k \rightarrow \infty$, and since all entries of y are strictly positive $Ly \neq 0$. It then follows that $T^{kp}(x) \rightarrow \zeta = \phi(Ly)/q(\phi(Ly))$. Since T has no fixed points in the interior and $x \in \text{ri}(\Sigma_q)$, Theorem 1 tells us that $\zeta \in \partial C \cap \Sigma_q$. Because $T^i(x) \in \text{ri}(\Sigma_q)$ for $0 < i \leq p - 1$, the same argument shows that the sequence $\langle T^{kp+i}(x) | k \geq 0 \rangle$ converges as $k \rightarrow \infty$ for all i , $0 < i \leq p - 1$. Because $T^{kp+i}(x)$ and $T^{kp}(x)$ are comparable for $k \geq 0$, with $d(T^{kp+i}(x), T^{kp}(x)) \leq d(T^i(x), x)$, Lemma 1 implies that $\lim_{k \rightarrow \infty} T^{kp}(x)$ and $\lim_{k \rightarrow \infty} T^{kp+i}(x)$ are comparable for $0 < i \leq p - 1$. Also since $\lim_{k \rightarrow \infty} T^{kp}(x) = \lim_{k \rightarrow \infty} T^{kp+p}(x) = \zeta$, it follows from Theorem 1 that the omega limit set $\omega(x; T)$ consists of at most p points in $C_\zeta \cap \Sigma_q$. It also follows from Theorem 1 that for any other $y \in \text{ri}(\Sigma_q)$, $\omega(y; T) \subseteq C_\zeta$. \square

Remark. The number p in the statement of Theorem 2 will always be the order of a cyclic subgroup of the symmetric group on N elements, where N is the number of extreme points of Σ_q , see [14].

Remark. Theorem 2 implies that the omega limit set under T of any point on the interior of Σ_q is finite. This result is not true in general for cones C which are not polyhedral. In fact, if $C \subseteq \mathbb{R}^4$ is the cone defined by

$$C = \{x \in \mathbb{R}^4 | x_1 \geq 0, x_2^2 + x_3^2 \leq x_1^2, \text{ and } 0 \leq x_4 \leq x_1\}$$

and $f : C \rightarrow C$ is the linear map defined by $f(x) = Ax$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & -b & a & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

where $a^2 + b^2 = 1$, and $\tan^{-1}(b/a)/2\pi$ is irrational. Then for $x = \left(1, \frac{1}{2}, 0, \frac{1}{2}\right) \in \text{int}(C)$, the omega limit set $\omega(x; f)$ contains infinitely many points in ∂C . If we define $\Sigma = \{x \in C | x_1 = 1\}$, and $T : \Sigma \rightarrow \Sigma$ by $T(x) = f(x)/(f(x)_1)$, then $\omega(x; T)$ will not be finite for $x = \left(1, \frac{1}{2}, 0, \frac{1}{2}\right)$.

For an arbitrary closed cone C with nonempty interior in a finite dimensional Banach space X , and a linear map $f : C \rightarrow C$, with $f(\text{int}(C)) \subseteq \text{int}(C)$, it will not always be the case that the omega limit sets of points in $\text{ri}(\Sigma_q)$ under the action of the Hilbert metric nonexpansive map T arising from f are finite. We can, however, prove a weaker result, which tells us that if T has no fixed points on the interior of Σ_q , then the convex hull of the omega limit set $\omega(x; T)$ will lie in the boundary of Σ_q . The proof of this result relies on the following observation.

Lemma 3. *For an affine map $f(x) = Lx + b$, where L is linear and order-preserving with respect to C , $b \in C$, and with q and Σ_q as in Theorem 2, let $T(x) = f(x)/q(f(x))$. Such a map T is a convexity-preserving map defined on the set $G = \{x \in \Sigma_q \mid f(x) \neq 0\}$. That is, if $x \in G$ is a convex combination of z_1, z_2, \dots, z_k in G , then $T(x)$ will be a convex combination of $T(z_1), T(z_2), \dots, T(z_k)$.*

Proof. For $x = \lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_k z_k$, with $\lambda_i > 0$ and $\sum_i \lambda_i = 1$

$$\begin{aligned} T(x) &= \frac{f(x)}{q(f(x))} = \frac{Lx + b}{q(Lx + b)} \\ &= \frac{\lambda_1 Lz_1 + \dots + \lambda_k Lz_k + b}{\lambda_1 q(Lz_1) + \dots + \lambda_k q(Lz_k) + q(b)} \\ &= \frac{\lambda_1(Lz_1 + b) + \dots + \lambda_k(Lz_k + b)}{\lambda_1 q(Lz_1 + b) + \dots + \lambda_k q(Lz_k + b)} \\ &= \frac{\lambda_1 q(Lz_1 + b)T(z_1) + \dots + \lambda_k q(Lz_k + b)T(z_k)}{\lambda_1 q(Lz_1 + b) + \dots + \lambda_k q(Lz_k + b)} \\ &= \mu_1 T(z_1) + \mu_2 T(z_2) + \dots + \mu_k T(z_k) \end{aligned}$$

with

$$\mu_i = \frac{\lambda_i q(Lz_i + b)}{\lambda_1 q(Lz_1 + b) + \dots + \lambda_k q(Lz_k + b)}. \quad \square$$

Remark. For a characterization of convexity-preserving maps on a subset of a vector space see [2].

Theorem 3. *Let C be a closed cone with nonempty interior in a finite dimensional Banach space X . Let $f : X \rightarrow X$ be a linear map such that $f(\text{int}(C)) \subseteq \text{int}(C)$. Let $q \in C^*$ be such that $q(x) > 0$ for all $x \in C \setminus \{0\}$ and define $\Sigma_q = \{x \in C \mid q(x) = 1\}$ and $\text{ri}(\Sigma_q) = \Sigma_q \cap \text{int}(C)$. If the function T defined on Σ_q by $T(x) = f(x)/q(f(x))$ has no fixed point in $\text{ri}(\Sigma_q)$, then for any $x \in \text{ri}(\Sigma_q)$, $\text{co}(\omega(x; T)) \subseteq \partial \Sigma_q := \Sigma_q \setminus \text{ri}(\Sigma_q)$.*

Proof. We argue by induction on $n = \dim(X)$. If $n = 1$, $\Sigma_q = \text{ri}(\Sigma_q)$ and T has a fixed point in $\text{ri}(\Sigma_q)$, so the theorem is vacuously true. If $\dim(X) = 2$, Proposition 2 proves that the theorem holds. Assume, for some $n > 2$, that the theorem is true for $\dim(X) < n$. Let X be a Banach space with $\dim(X) = n$ and let C and f be as in the theorem. If f is not one-to-one, $Y = \{f(x) \mid x \in X\}$ satisfies $\dim(Y) < n$, $f(Y) \subseteq Y$ and $f(\text{int}(C)) \subseteq \text{int}(C) \cap Y$. By our inductive hypothesis applied to the cone $C \cap Y$ in Y , $\text{co}(\omega(y; T)) \subseteq \partial_Y(\text{int}(C) \cap Y) \subseteq \partial C$ for all $y \in \text{int}(C) \cap Y \cap \Sigma_q$; and since $T(x) \in \text{int}(C) \cap Y \cap \Sigma_q$ for all $x \in \text{ri}(\Sigma_q)$, $\text{co}(\omega(x; T)) \subseteq \partial \Sigma_q$ for all $x \in \text{ri}(\Sigma_q)$.

Thus we may restrict attention to the case that f is one-to-one. In this case, the map T is defined continuously on Σ_q . Let us suppose by way of contradiction that there is a point $x \in \text{ri}(\Sigma_q)$ such that $\text{co}(\omega(x; T)) \cap \text{ri}(\Sigma_q)$ is nonempty. Let $y \in \text{co}(\omega(x; T)) \cap \text{ri}(\Sigma_q)$. Thus y is a convex combination of points in $\omega(x; T)$. By Carathéodory’s theorem (see [18]), we may assume that y is a convex combination of at most n points $z_1, z_2, \dots, z_n \in \omega(x; T)$.

Consider $\omega(y; T)$. By Theorem 1(c), we know that $\text{co}(\omega(y; T))$ has a nonempty intersection with $\text{ri}(\Sigma_q)$. At the same time, we claim that: $\omega(y; T) \subseteq \text{co}(U)$ where $U = \omega(z_1; T) \cup \omega(z_2; T) \cup \dots \cup \omega(z_n; T)$. Note that T extends continuously to any point on $\partial\Sigma_q$ since f is one-to-one. Therefore the omega limit sets $\omega(z_i; T)$ are well defined. To prove the claim, note that for each $k \geq 0$, $T^k(y) = \lambda_1^{(k)} T^k(z_1) + \dots + \lambda_n^{(k)} T^k(z_n)$, with each $\lambda_i^{(k)} \geq 0$, and $\sum_i \lambda_i^{(k)} = 1$, by Lemma 3. Taking a subsequence, $\langle k_j \rangle$, we can arrange for $1 \leq i \leq n$ that $T^{k_j}(z_i) \rightarrow z'_i$ as $j \rightarrow \infty$ and simultaneously $\lambda_i^{(k_j)} \rightarrow \lambda'_i$ with each $\lambda'_i \geq 0$ and $\sum_i \lambda'_i = 1$. Thus, each point $z' \in \omega(y; T)$ is a convex combination $\sum_{i=1}^n \lambda'_i z'_i$ with each $z'_i \in \omega(z_i; T)$, proving the claim.

Let y^1 be a point in $\text{co}(\omega(y; T)) \cap \text{ri}(\Sigma_q)$, and choose $z_1^1, z_2^1, \dots, z_n^1 \in U$ such that $y^1 \in \text{co}(\{z_1^1, z_2^1, \dots, z_n^1\})$. Note that each point $z_j^1 \in \omega(z_i; T)$ for some i . Furthermore, since $z_i \in \partial\Sigma_q$, we know that z_i must lie in a component C_i of C which has dimension at most $n - 1$. Theorem 1(d) implies that $T(\text{cl}(C_i) \cap \Sigma_q) \subseteq \text{cl}(C_i) \cap \Sigma_q$. If C_i contains a fixed point of T then since T is nonexpansive in the Hilbert metric on C_i , every T -orbit in $C_i \cap \Sigma_q$ must remain within a bounded Hilbert metric distance of that fixed point. On the other hand, if C_i does not contain a fixed point, then applying Theorem 1(a) to $T|_{C_i \cap \Sigma_q}$ implies that z_j^1 is contained in a component of C on the boundary of $\text{cl}(C_i)$. Such a component would have to have dimension strictly less than $n - 1$.

Repeat this process to obtain a sequence of points $y^1, y^2, \dots, y^{n-2} \in \text{ri}(\Sigma_q)$ with the property that each $y^i \in \text{co}(\omega(y^{i-1}; T))$ and more importantly $y^i \in \text{co}(\{z_1^i, z_2^i, \dots, z_n^i\})$ where each z_j^i is contained in a component of C with dimension less than $n - i$ or is contained in a component of C on which T has a fixed point. This means that y^{n-2} is a point in $\text{ri}(\Sigma_q)$ which is a convex combination of points $z_1^{n-2}, z_2^{n-2}, \dots, z_n^{n-2}$ which all lie in components of C containing fixed points of T . For each $1 \leq i \leq n$, let p_i be a fixed point in the component which contains z_i^{n-2} . Suppose that $y^{n-2} = \lambda_1 z_1^{n-2} + \lambda_2 z_2^{n-2} + \dots + \lambda_n z_n^{n-2}$ and let $\zeta = \lambda_1 p_1 + \lambda_2 p_2 + \dots + \lambda_n p_n$. Observe that ζ is comparable to y^{n-2} since $z_i^{n-2} \sim p_i$ for all $1 \leq i \leq n$. Thus $\zeta \in \text{ri}(\Sigma_q)$. Now, since $T(p_i) = p_i$ iff $f(p_i) = r_i p_i$ we have

$$T^k(\zeta) = \frac{f^k(\zeta)}{q(f^k(\zeta))} = \frac{\sum \lambda_i r_i^k p_i}{\sum \lambda_i r_i^k q(p_i)}.$$

If $r = \max r_i$ and $J = \{i | r_i = r\}$ then the reader can verify that as $k \rightarrow \infty$

$$T^k(\zeta) \rightarrow \frac{\sum_{i \in J} \lambda_i p_i}{\sum_{i \in J} \lambda_i q(p_i)}$$

which is a single point in Σ_q . Since there are no bounded orbits in $\text{ri}(\Sigma_q)$, this limit point must be on the boundary $\partial\Sigma_q$. However, if $\omega(\zeta; T)$ is a single point, then $\text{co}(\omega(\zeta; T)) \subseteq \partial\Sigma_q$, a contradiction by Theorem 1(c). \square

Remark. Theorem 3 also applies to maps $T(x) = f(x)/q(f(x))$ where $f : X \rightarrow X$ is an affine map, so long as $f(\text{int}(C)) \subseteq \text{int}(C)$. The proof of Lemma 2.1 in [15] shows that such a map T is nonexpansive in Hilbert’s metric on $\text{ri}(\Sigma_q)$.

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