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Lower and upper bounds for ω-limit sets of nonexpansive maps*

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ABSTRACT

If D is a subset of $\mathbb{R}^n$ and $f : D \to D$ is an ℓ_1-norm nonexpansive map, then it is known that every bounded orbit of f approaches a periodic orbit. Moreover, the minimal period of each periodic point of f is bounded by $n!\,2^m$, where $m = 2^{n-1}$. In this paper we shall describe two different procedures to construct periodic orbits of ℓ_1-norm nonexpansive maps. These constructions yield that a lower bound for the largest possible minimal period of a periodic point of an ℓ_1-norm nonexpansive map is given by $3 \cdot 2^{n-1}$, $n \geq 3$. If $n \leq 5$, we shall also improve the upper bound for the largest possible minimal period.

1. INTRODUCTION

If D is a set and $f : D \to D$ is a map, then f^k will denote the k-fold composition of f with itself. A point $x \in D$ is called a *periodic point of f of minimal period p* if $f^p(x) = x$ and $f^j(x) \neq x$ for $1 \leq j < p$. We shall call a map $f : D \to V$, where D is a subset of a Banach space $(V, \|\cdot\|)$, *nonexpansive* (with respect to $\|\cdot\|$) if

$$\|f(x) - f(y)\| \leq \|x - y\| \quad \text{for all } x, y \in D.$$

As usual we define the ℓ_1-norm $\|\cdot\|_1$ on $\mathbb{R}^n$ by

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \text{where } x = (x_1, x_2, \ldots, x_n).$$

The metric induced by the ℓ_1-norm will be denoted by d_1. So $d_1(x, y) = \|x - y\|_1$.

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Let D be a closed subset of $\mathbb{R}^n$. If $f : D \to D$ is a nonexpansive map with respect to $\|\cdot\|_1$ and there exists a $x_0 \in D$ such that the sequence $(f^j(x_0))_j$ is bounded, then Akcoglu and Krengel [1] showed that for every $x \in D$, there exist a positive integer $p_x = p$ and a point $\xi_x = \xi \in D$ such that ξ is a periodic point of f of minimal period p and

$$(1) \qquad \lim_{k \to \infty} f^{kp}(x) = \xi.$$

Furthermore, the number p is bounded by $n!\,2^m$, where $m = 2^n$. The proof of (1) by Akcoglu and Krengel did not provide an upper bound for the integer p_x, $x \in D$, and the upper bound given here was established by Misiurewicz in [9].

It is known that property (1) actually holds for nonexpansive maps with respect to a given polyhedral norm, see Weller [18], Martus [8] and Nussbaum [10]. An important example of a polyhedral norm on $\mathbb{R}^n$, aside from the ℓ_1-norm, is the sup norm

$$\|x\|_\infty = \max\{|x_i| : 1 \leq i \leq n\}, \qquad \text{where } x = (x_1, x_2, \ldots, x_n).$$

In case of the sup norm, the second author conjectured that the optimal upper bound for the integer p_x equals 2^n. The conjecture has been proved in dimension $n = 1, 2$ and 3, see Lyons and Nussbaum [7].

In general, however, sharp bounds for the largest possible minimal period of nonexpansive maps with respect to a polyhedral norm are unknown. In this paper we shall improve the a priori bounds for the largest possible minimal period of general nonexpansive maps with respect to the ℓ_1-norm. Sharp bounds for the largest possible minimal period of an ℓ_1-norm nonexpansive map $f : D_f \to D_f$, $D_f \subset \mathbb{R}^n$ do seem difficult to obtain. One of the reasons is that the map f, in general, does not have an ℓ_1-norm nonexpansive extension $F : \mathbb{R}^n \to \mathbb{R}^n$ (a fact, however, which is true for nonexpansive maps with respect to the sup-norm). Therefore the problem depends nontrivially on the set D_f. For arbitrary sets D_f not much is known. In special cases, for example if $D_f = \mathbb{K}^n$, the positive cone in $\mathbb{R}^n$, much more is known and a complete characterization of the set of possible minimal periods has been obtained by Nussbaum, Scheutzow and Verduyn Lunel [11, 13, 14, 15].

The study of the behaviour of orbits of ℓ_1-norm nonexpansive maps naturally leads to a detailed analysis of the structure of ω-limit sets of nonexpansive maps. The main idea in the proof of (1) is to show that if $f : D_f \to D_f$ is ℓ_1-norm nonexpansive and there exists a $x_0 \in D$ such that the sequence $(f^j(x_0))_j$ is bounded, then there exists an a priori upper bound on the cardinality of ω-limit sets which only depends on the number of independent variables.

The organisation of this paper is as follows. In Section 2 we shall discuss the procedure to obtain a priori upper bounds on the cardinality of ω-limit of ℓ_1-norm nonexpansive maps, based on the approach introduced by Misiurewicz [9]. We shall give a sharper upper bound for the cardinality of ω-limit sets of ℓ_1-norm nonexpansive maps, if the dimension is less than six. In Section 3 we shall describe two different procedures to construct periodic orbits of ℓ_1-norm nonexpansive maps. These constructions yield that a lower bound for the largest possible minimal period of a periodic point of an ℓ_1-norm nonexpansive map is given by $3 \cdot 2^{n-1}$, $n \geq 3$. Finally, in Section 4 we shall discuss some consequences of our approach for sup norm nonexpansive maps.

2. UPPER BOUNDS ON THE CARDINALITY OF ω-LIMIT SETS

Let (X, d) be a complete metric space and D_f a closed subset of X. If $f : D_f \to D_f$ a map, then for each $x \in D_f$ the *ω-limit set*, $\omega(x) = \omega(x;f)$, is defined by

$$\omega(x) = \{y \in D_f \mid y = \lim_{i \to \infty} f^{k_i}(x) \text{ for some sequence of integers } k_i \to \infty\},$$

or, equivalently, $\omega(x) = \cap_{k \geq 1} cl(\cup_{j \geq k} f^j(x))$, where $cl(S)$ denotes the closure of the set S.

It is clear that $\omega(x)$ is closed and invariant under f, i.e., $f[\omega(x)] \subseteq \omega(x)$. Furthermore, if f is continuous and D_f is compact, then f maps $\omega(x)$ onto itself.

For nonexpansive maps ω-limit sets have additional properties (cf. [3]). In particular, f restricted to $\omega(x)$ is an isometry and $\omega(y) = \omega(x)$, for each $y \in \omega(x)$. From this last property it follows that, for each $y, z \in \omega(x)$ there exists a sequence of integers $k_i \to \infty$ such that

$$\lim_{i \to \infty} f^{k_i}(y) = z. \tag{2}$$

Since f is nonexpansive the set of iterates of f is equicontinuous. Therefore, if D_f is compact the Arzela-Ascoli Theorem implies that the sequence $(f^{k_i})_{i \geq 1}$ has a uniform convergent subsequence. If we let $F_{y,z}$ denote the pointwise limit, one can verify that the restriction of $F_{y,z}$ to $\omega(x)$ is an isometry of $\omega(x)$ onto itself and $F_{y,z}(y) = z$. Furthermore, since all the iterates of f commute we have that

$$F_{u,v} \circ F_{y,z} = F_{y,z} \circ F_{u,v} \quad \text{for all } u, v, y, z \in \omega(x).$$

This property motivates the following definition. A subset S of (X, d) *has a transitive and commutative family of isometries*, if there exists a commutative family Γ of isometries (with respect to d) of S onto itself, such that for each $y, z \in S$, there exists $F_{y,z} \in \Gamma$ with $F_{y,z}(y) = z$.

The key idea to obtain a priori bounds for omega limit sets is to analyse compact sets S that have a transitive and commutative family of isometries. First we need some preparations. Throughout the paper we shall work in the metric space $(\mathbb{R}^n, d_1)$ and therefore suppress the metric.

2.1. Preliminary results

A sequence $a^1, a^2, \ldots, a^m$ in $\mathbb{R}^n$ is called an *additive chain* with respect to the ℓ_1-metric, if

$$d_1(a^1, a^m) = \sum_{i=1}^{m-1} d_1(a^i, a^{i+1}).$$

A sequence $a^1, a^2, \dots, a^m$ in $\mathbb{R}^n$ is called *monotone*, if for each $j \in \{1, \dots, n\}$ either

$$a_j^1 \le a_j^2 \le \dots \le a_j^m \text{ or } a_j^1 \ge a_j^2 \ge \dots \ge a_j^m.$$

By definition, it follows that a sequence $a^1, a^2, \dots, a^m \in \mathbb{R}^n$ is monotone if and only if it is an additive chain. We will call the *length* of a sequence the number of distinct points in the sequence.

Definition 2.1. For each $a, b \in \mathbb{R}^n$ we define the set

$$U(a,b) = \{c \in \mathbb{R}^n \mid (a,b,c) \text{ is a monotone sequence}\}.$$

Moreover, we let $U^\circ(a,b)$ denote the interior of $U(a,b)$ with respect to the Euclidean norm.

The assertions in the following two lemmas are in essence contained in the work of Misiurewicz [9].

Lemma 2.1. *For each $a, b \in \mathbb{R}^n$ one has that*

$$U^\circ(a,b) = \{c \in U(a,b) \mid (a_j - b_j)(b_j - c_j) > 0 \ \text{ whenever } \ a_j - b_j \ne 0\}.$$

Proof. Suppose that $c \in U^\circ(a,b)$ and that there exists $j \in \{1, \dots, n\}$ such that $(a_j - b_j)(b_j - c_j) = 0$ and $a_j - b_j \ne 0$. For every $\epsilon > 0$ with $\epsilon \le |a_j - b_j|$ define the vector $\tilde{c} = \tilde{c}_\epsilon$ by

$$\tilde{c}_\epsilon = c + \operatorname{sgn}(a_j - b_j) \cdot \epsilon \cdot e^j,$$

where e^j denotes the j-th unit vector.

Since $(a_j - b_j)(b_j - c_j) = 0$ and $a_j - b_j \ne 0$, it follows that $b_j = c_j$. This implies that either

$$a_j \le \tilde{c}_j < b_j \quad \text{or} \quad b_j < \tilde{c}_j \le a_j.$$

Therefore $(a, b, \tilde{c})$ is not a monotone sequence, and hence $\tilde{c} \notin U(a,b)$.

By construction $\tilde{c}$ is an element of the Euclidean ball $B_\epsilon(c)$ around c with radius ϵ. So, we can conclude that for every ϵ sufficiently small $B_\epsilon(c)$ is not contained in $U(a,b)$. This, however, contradicts the fact that $c \in U^\circ(a,b)$, and therefore we have proved

$$U^\circ(a,b) \subseteq \{c \in U(a,b) \mid (a_j - b_j)(b_j - c_j) > 0 \text{ whenever } a_j - b_j \ne 0\}.$$

To show equality we consider $c \in U(a,b)$ with $(a_j - b_j)(b_j - c_j) > 0$ whenever $a_j - b_j \ne 0$. Select $\epsilon > 0$ such that $|b_j - c_j| > \epsilon$ whenever $a_j - b_j \ne 0$. If we take $\bar{c}$ in $B_\epsilon(c)$ arbitrary, then it is clear that for each $1 \le j \le n$ we have that $(a_j, b_j, \bar{c}_j)$ is a monotone sequence in $\mathbb{R}$. Therefore $(a, b, \bar{c})$ is a monotone sequence, and hence $\bar{c} \in U(a,b)$. This proves that $c \in U^\circ(a,b)$. □

Lemma 2.2. *If S is a compact set in $\mathbb{R}^n$ and S has a transitive and commutative family of isometries, then $U^\circ(a,b) \cap S = \emptyset$ for each $a, b \in S$ with $a \ne b$.*

Proof. Let S be a compact set in $\mathbb{R}^n$ and suppose there exists a commutative family Γ of isometries of S such that for each $y, z \in S$ there exists $F_{y,z} \in \Gamma$ with $F_{y,z}(y) = z$. We shall argue by contradiction.

So, assume that $a, b, c \in S$ such that $a \neq b$ and $c \in U^\circ(a,b)$. Since $a \neq b$ and $b \neq c$ we can take $\epsilon > 0$ such that $d_1(a,b) \geq \epsilon$ and $d_1(b,c) \geq \epsilon$. Define $\mathcal{F}$ to be the collection of monotone sequences in S, which start with (a,b,c) and are such that the d_1-distance between two consecutive elements is at least ϵ. Since S is a compact subset of $\mathbb{R}^n$ there exists an a priori bound on the length of the sequences in $\mathcal{F}$. Therefore there exists a unique maximal length of the sequences in $\mathcal{F}$, which will be denoted by r. Suppose that

$$x^1 = a,\ x^2 = b,\ x^3 = c,\ x^4,\ \ldots,\ x^r$$

is a sequence in $\mathcal{F}$ of maximal length r. For integers $1 \leq k, l \leq r$ we select an isometry $F_{k,l} \in \Gamma$ with $F_{k,l}(x^k) = x^l$. We define $x^{r+1} = F_{1,2}(x^r)$ and claim that the sequence

$$x^2 = b,\ x^3 = c,\ x^4,\ \ldots,\ x^r,\ x^{r+1} \tag{3}$$

is a monotone sequence in S with d_1-distance between two consecutive elements at least ϵ. (These facts are special cases of more general results in [7]. For sake of completeness, we provide the elementary proofs.) To prove the claim, we first verify that the distance between two consecutive elements is at least ϵ. By construction, it suffices to verify that $d_1(x^r, x^{r+1}) \geq \epsilon$. Since $x^r = F_{1,r}(x_1)$, it follows that

$$\begin{aligned} d_1(x^r, F_{1,2}(x^r)) &= d_1(F_{1,r}(x^1), F_{1,r}(F_{1,2}(x^1))) \\ &= d_1(F_{1,r}(x^1), F_{1,r}(x^2)) = d_1(x^1, x^2), \end{aligned}$$

so that

$$d_1(x^r, x^{r+1}) = d_1(x^1, x^2), \tag{4}$$

and this shows $d_1(x^r, x^{r+1}) \geq \epsilon$.

To prove the monotonicity of the sequence, it suffices to prove that the sequence is an additive chain. Using (4) we derive

$$\begin{aligned} d_1(x^2, x^{r+1}) = d_1(x^1, x^r) &= \sum_{i=1}^{r-1} d_1(x^i, x^{i+1}) \\ &= \Big(\sum_{i=2}^{r-1} d_1(x^i, x^{i+1})\Big) + d_1(x^1, x^2) \\ &= \Big(\sum_{i=2}^{r-1} d_1(x^i, x^{i+1})\Big) + d_1(x^r, x^{r+1}) = \sum_{i=2}^{r} d_1(x^i, x^{i+1}). \end{aligned}$$

This proves that the sequence (3) is monotone.

Since $c \in U^\circ(a,b)$ it follows from Lemma 2.1 that if $c_j = b_j$, then $a_j = b_j$. Furthermore, $\operatorname{sgn}(a_j - b_j) = \operatorname{sgn}(b_j - c_j)$ for all j with $a_j \neq b_j$. This implies that the extended sequence

$$x^1 = a,\ x^2 = b,\ x^3 = c,\ \ldots,\ x^r,\ x^{r+1}$$

also belongs to $\mathcal{F}$, which contradicts that r is maximal. Therefore the intersection of $U^\circ(a,b)$ with S is empty and the lemma follows. □

Motivated by Lemma 2.2 we make the following definition.

Definition 2.2. A set S in $\mathbb{R}^n$ is called *ℓ_1-separated if $U^\circ(a,b) \cap S = \emptyset$ for each $a, b \in S$ with $a \neq b$.*

For $a, b \in \mathbb{R}^n$ we let $Q(a,b)$ denote the minimal closed box containing both a and b, with sides parallel to the axes, so

$$Q(a,b) = \{x \in \mathbb{R}^n \mid \min\{a_j, b_j\} \le x_j \le \max\{a_j, b_j\} \ \text{ for } \ 1 \le j \le n\}.$$

Theorem 2.1. *If $S \subset \mathbb{R}^n$ is ℓ_1-separated, then the following assertions hold*

(i) *The length of any monotone sequence contained in S, is bounded by $n+1$.*

(ii) *If S contains a monotone sequence of length $n+1$, say $a^1, a^2, \ldots, a^{n+1}$, then S is contained in the boundary of the box $Q(a^1, a^{n+1})$.*

Proof. Suppose S is an ℓ_1-separated set in $\mathbb{R}^n$ and $a^1, a^2, \ldots, a^m$ is a monotone sequence of length m in S. Define for $1 \le k < l \le m$ the set

$$I_{k,l} = \{j \in \{1, \ldots, n\} \mid a_j^k = a_j^l\}.$$

Since $a^1, a^2, , a^m$ is a monotone sequence, we obtain the following inclusions

$$(5) \qquad I_{1,2} \supseteq I_{1,3} \supseteq \ldots \supseteq I_{1,m}.$$

We shall show, by contradiction, that

$$I_{1,k} \neq I_{1,k+1} \quad \text{for} \quad 2 \le k \le m-1.$$

If $I_{1,k} = I_{1,k+1}$ for some $k \in \{2, \ldots, m-1\}$, then it follows that $I_{1,k} \subseteq I_{k,k+1}$, and therefore

$$(a_j^{k+1} - a_j^k)(a_j^k - a_j^1) > 0 \qquad \text{for } j \notin I_{k,k+1}.$$

By definition, this implies that $a^1 \in U^\circ(a^{k+1}, a^k)$, which contradicts the assumption that S is ℓ_1-separated. This shows that the inclusions in (5) are all strict inclusions. Since $|I_{1,2}| \le n-1$, the strict inclusions imply $m \le n+1$ and this proves (i).

To show (ii), we shall first prove by induction that for a monotone sequence $a^1, a^2, \ldots, a^{n+1}$ with length $n+1$ in an ℓ_1-separated set S in $\mathbb{R}^n$ and corresponding sets $I_{k,k+1}$, as defined above, the following equalities hold:

$$(6) \qquad |I_{k,k+1}| = n-1 \ \text{ for } \ 1 \le k \le n,$$

and

$$(7) \qquad \bigcup_{k=1}^{n} \{1, 2, \ldots, n\} \setminus I_{k,k+1} = \{1, 2, \ldots, n\}.$$

Since the equalities (6) and (7) are trivial for $n = 1$, it suffices to prove the induction step.

Assume that (6) and (7) hold for $n - 1$. Since S is ℓ_1-separated, we know that the points $a^2, a^3, \ldots, a^{n+1}$ are contained in the boundary $\partial U(a^1, a^2)$ of $U(a^1, a^2)$. Since the inclusions in (5) are strict it follows that $|I_{1,2}| = n - 1$. Therefore $\partial U(a^1, a^2)$ satisfies

$$\partial U(a^1, a^2) = \{x \in \mathbb{R}^n \mid x_{j_1} = a^2_{j_1}\},$$

where j_1 is the unique element in $\{1, 2, \ldots, n\} \setminus I_{1,2}$.

Consequently, the sequence $a^2, a^3, \ldots, a^{n+1}$ is a monotone sequence of length n in an $n - 1$ dimensional affine space in $\mathbb{R}^n$. Therefore, the induction hypothesis yields that $|I_{k,k+1}| = (n - 2) + 1 = n - 1$ for $2 \leq k \leq n$. This proves (6). Furthermore, it follows that

$$\bigcup_{k=2}^{n} \{1, 2, \ldots, n\} \setminus I_{k,k+1} = \{1, 2, \ldots, n\} \setminus \{j_1\}.$$

Since $j_1 \in \{1, 2, \ldots, n\} \setminus I_{1,2}$ we also obtain (7).

For $1 \leq k \leq n$, we define j_k to be the unique element in $\{1, 2, \ldots, n\} \setminus I_{k,k+1}$, and

$$V_k = \{x \in \mathbb{R}^n \mid \min\{a^k_{j_k}, a^{k+1}_{j_k}\} \leq x_{j_k} \leq \max\{a^k_{j_k}, a^{k+1}_{j_k}\}\}.$$

Observe that if $y \notin V_k$, then either $y \in U^\circ(a^k, a^{k+1})$ or $y \in U^\circ(a^{k+1}, a^k)$. Therefore it follows from the assumption that S is ℓ_1-separated that

$$S \subset \bigcap_{k=1}^{n} V_k.$$

Set $Q = \bigcap_{k=1}^{n} V_k$. From (7) it follows that Q is a closed box, with sides parallel to the axes, containing both a^1 and a^{n+1}. Since the sequence $a^1, a^2, \ldots, a^{n+1}$ is monotone, we conclude that $Q = Q(a^1, a^{n+1})$.

To complete the proof of (ii), it suffices to note that, if $y \in \mathbb{R}^n$ is contained in the interior of $Q(a^1, a^{n+1})$, then $a^{n+1} \in U^\circ(a^1, y)$. As S is ℓ_1-separated we conclude that S is contained in the boundary of $Q(a^1, a^{n+1})$. $\square$

2.2. Large sets have long monotone sequences

From combinatorial geometry, it is known that a set in $\mathbb{R}^n$ of large cardinality contains a long monotone sequence (see [2,4,5]). Moreover, given the dimension n one can give precise expressions for 'large' and 'long' in the previous statement. We will state the precise results and give references for proofs.

Hidden in a paper by Erdős and Szekeres [4], it is proved that every sequence of length $k^2 + 1$ in $\mathbb{R}$ contains a monotone subsequence of length $k + 1$. From the sequence:

$$k, k - 1, \ldots, 1, 2k, 2k - 1, \ldots, k + 1, \ldots, k^2, k^2 - 1, \ldots, (k - 1)k + 1,$$

it is clear that the number $k^2 + 1$ is the best possible bound. Several proofs for this result are known, see [5] and [9]. In unpublished work, N.G. de Bruijn showed the following generalization of the result by Erdős and Szekeres. For a proof of this theorem we refer to [2, Lemma 2.1].

Theorem 2.2. *Every sequence of vectors in $\mathbb{R}^n$ of length $k^{2^n}+1$ contains a monotone subsequence of length $k+1$. Furthermore, the length $k^{2^n}+1$ is the smallest length with this property.*

In particular, we have the following corollary.

Corollary 2.1. *If S is a subset of $\mathbb{R}^n$ with cardinality at least $k^{2^{n-1}}+1$, then S contains a monotone sequence of length $k+1$. Moreover, the number $k^{2^{n-1}}+1$ is the smallest cardinality with this property.*

Proof. For a given set $S \subset \mathbb{R}^n$, the elements can be labelled such that the resulting sequence is monotone in the first coordinate. Therefore, if we apply Theorem 2.2 with respect to the last $n-1$ coordinates the result follows. □

2.3. A priori upper bounds

There are several ways to proceed in order to obtain upper bounds for the cardinality of ℓ_1-separated sets. The first approach is based on the following idea. If S in $\mathbb{R}^n$ is a set of large cardinality, then either there exists a large subset B of S and a coordinate $i \in \{1,\ldots,n\}$ such that $x_i^k = x_i^l$ for all $x^k, x^l \in B$, or there exists a large subset C of S such that for each $x^k \neq x^l$ in C we have $x_i^k \neq x_i^l$ for all $i \in \{1,\ldots,n\}$. In the first case we can use a projection to reduce the dimension. In the second case the upper bound from Corollary 2.1 with $k=2$ can be applied. This approach was followed by Misiurewicz in [9] who showed that the size of an ℓ_1-separated sets is bounded by

$$(8) \qquad \tau_n := \sum_{k=1}^{n} \frac{n!}{k!} 2^{2^n - 2^{k-1}} < n!\, 2^{2^n},$$

In this section, we shall proceed a different way and start with an observation. A combination of Theorem 2.1 and Corollary 2.2 yields an upper bound for the cardinality of compact sets in $\mathbb{R}^n$ with a transitive and commutative family of isometries. We state the result as a lemma.

Lemma 2.3. *If S is an ℓ_1-separated set in $\mathbb{R}^n$, then the number of elements in S is bounded by $(n+1)^{2^{n-1}}$.*

Proof. Since S is an ℓ_1-separated set in $\mathbb{R}^n$, it follows from Theorem 2.1 that the length of the longest monotone sequence in S is bounded by $n+1$. Therefore Corollary 2.1 implies that the cardinality of S is bounded by $(n+1)^m$, where $m = 2^{n-1}$. □

The second part of Theorem 2.1 gives additional information about the structure of ℓ_1-separated sets in $\mathbb{R}^n$ that contain a monotone sequence of length $n+1$. We shall consider this situation in detail for compact sets in $\mathbb{R}^n$ that have a transitive and commutative family of isometries.

Theorem 2.3. *Let S be a compact subset of $\mathbb{R}^n$ with a transitive and commutative family of isometries. If S contains a monotone sequence of length $n+1$, then the number of elements in S is bounded by 2^n.*

Proof. Let $a^1, a^2, \ldots, a^{n+1}$ be a monotone sequence of length $n+1$ in S. From Lemma 2.2 and Theorem 2.1, it follows that S is contained in the boundary of $Q(a^1, a^{n+1})$.

We claim that S is a subset of the set of vertices of the box $Q(a^1, a^{n+1})$. To prove the claim, suppose that $x \in S$ is an element of the boundary of the box $Q(a^1, a^{n+1})$, but not a vertex. Let $F: S \to S$ be an isometry in Γ that maps a^1 to x. Since F is an isometry, the sequence

$$x = F(a^1),\ F(a^2),\ \ldots,\ F(a^{n+1}),$$

is monotone and of length $n+1$. If we apply Theorem 2.1 to this sequence, we obtain that S is contained in the boundary of $Q(x, F(a^{n+1}))$.

On the other hand, the element x is (by assumption) not a vertex of $Q(a^1, a^{n+1})$, so that there exists a coordinate $j \in \{1, 2, \ldots, n\}$ such that either

$$a_j^1 < x_j < a_j^{n+1} \quad \text{or} \quad a_j^{n+1} < x_j < a_j^1.$$

This implies that a^1 or a^{n+1} is not contained in the boundary of $Q(x, F(a^{n+1}))$, which is a contradiction and this proves the claim.

From the claim it immediately follows that the number of elements in S is at most 2^n. □

Corollary 2.2. *If $n \geq 2$ and S is a compact set in $\mathbb{R}^n$ with a transitive and commutative family of isometries, then the number of elements of S is bounded by $n^{2^{n-1}}$.*

Proof. Suppose $n \geq 2$ and assume, to the contrary, that the cardinality of S is at least $n^m + 1$, where $m = 2^{n-1}$. From Corollary 2.1, it follows that S contains a monotone sequence of length $n+1$. Theorem 2.3 implies that the number of elements in S is bounded by 2^n. So, for $n \geq 2$, we obtain $2^n \geq n^m + 1$, where $m = 2^{n-1}$ and this is a contradiction. □

Corollary 2.2 improves the upper bound (14) by Misiurewicz in dimension less than six as can be seen from the following table.

n	new	old
2	4	20
3	81	976
4	65536	999680
5	152587890625	327575207936

We conclude this section with two remarks.

Remark 2.1. Let us consider the case $n = 3$ more closely. Suppose that S is a compact set in $\mathbb{R}^3$ with a transitive and commutative family of isometries. Furthermore, let r denote the length of longest monotone sequence in S. From Theorem 2.1 it follows that $r \leq 4$. Moreover, if $r = 4$, then Theorem 2.3 implies that $|S| \leq 8$. If $r = 2$, then it follows from Corollary 2.1 that $|S| \leq 16$. Thus, to present a better estimate for the cardinality of S in case $n = 3$, we have to analyse the following problem. Does there exist a compact set S in $\mathbb{R}^3$ with a transitive and commutative family of isometries such that $r = 3$ and $|S| > 16$?

Remark 2.2. Since compact sets S in $\mathbb{R}^n$ with a transitive and commutative family of isometries, are ℓ_1-separated, one could try to improve the upper bound in Corollary 2.2 by looking at ℓ_1-separated sets. However, one has to realize that there exists (by Corollary 2.1) a lower bound for the cardinality of ℓ_1-separated sets in $\mathbb{R}^n$ of 2^m, where $m = 2^{n-1}$.

3. TWO PROCEDURES TO CONSTRUCT LOWER BOUNDS

If $f : D_f \to D_f$ is an ℓ_1-norm nonexpansive map and D_f is compact subset of $\mathbb{R}^n$, then we have proved in the previous section that there exists an a priori upper bound on the cardinality of $\omega(x)$ which only depends on the dimension of the ambient space. Consequently, we can reformulate the problem of determining the set $\tilde{R}(n)$, which consists of possible minimal periods of periodic points of ℓ_1-norm nonexpansive maps $f : D_f \to D_f$ where D_f is a subset of $\mathbb{R}^n$, in the following way. Find the integers p for which there exists a sequence of distinct points $x^0, x^1, \ldots, x^{p-1}$ in $\mathbb{R}^n$ such that the map

$$F(x^i) = x^{i+1 \bmod p} \quad \textit{for } i = 0, \ldots, p-1$$

is an ℓ_1- norm isometry. To simplify the analysis we introduce the following definition.

Definition 3.1. *A finite sequence of distinct points $x^0, x^1, \ldots, x^{p-1}$ in a Banach space $(V, \|\cdot\|)$ is called a regular polygon of size p or simply a regular p-gon if*

$$\|x^{k+l} - x^k\| = \|x^l - x^0\| \quad \textit{for all } k, l = 0, \ldots, p-1.$$

Here the indices are counted modulo p.

Remark that a sequence $x^0, x^1, \ldots, x^{p-1}$ is a regular polygon in $(V, \|\cdot\|)$ if and only if the map $F(x^i) = x^{i+1 \bmod p}$ is an isometry. In this section we shall give two procedures to construct regular polygons in $\mathbb{R}^n$ with the ℓ_1-norm.

3.1. Doubling via the simplex

Before we can start with the first construction some definitions are required.

Let $\mathbb{K}^n = \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } 1 \leq i \leq n\}$ be the *positive cone in* $\mathbb{R}^n$, *and let* $\Delta_n = \{x \in \mathbb{K}^n \mid \sum_{i=1}^n x_i = 1\}$ *be the unit simplex in* $\mathbb{R}^n$.

Lemma 3.1. *If there exists a regular p-gon in Δ_n, then there exists a regular 2p-gon in Δ_{n+1}.*

Proof. Let the sequence $s^0, s^1, \ldots, s^{p-1}$ be a regular p-gon in Δ_n. Consider the sequence $t^0, t^1, \ldots, t^{2p-1}$ in $\mathbb{R}^{n+1}$ given by

$$t^i = \begin{cases} (s^{i/2}, 0) & \text{if } i \text{ is even} \\ (-s^{(i-1)/2}, 2) & \text{if } i \text{ is odd.} \end{cases}$$

We claim that $t^0, t^1, \ldots, t^{2p-1}$ is a regular $2p$-gon in $\mathbb{R}^{n+1}$. To prove the claim we have to show that

$$\|t^{m+l} - t^m\|_1 = \|t^l - t^0\|_1 \qquad \text{for each } m, l = 0, 1, \ldots, 2p-1.$$

So, take $m, l \in \{0, 1 \ldots, 2p-1\}$ arbitrary. If l is odd, then the following equalities hold

$$\begin{aligned} \|t^{m+l} - t^m\|_1 &= (\sum_{j=1}^n |t^{m+l}j - t_j^m|) + 2 \\ &= \|s^{[(m+l)/2]} + s^{[m/2]}\|_1 + 2 \\ &= \|s^{[(m+l)/2]}\|_1 + \|s^{[m/2]}\|_1 + 2 \\ &= 4. \end{aligned}$$

Here $[x]$ denotes the largest integer $m \leq x$.

On the other hand, if l is even, then we have that

$$\begin{aligned} \|t^{m+l} - t^m\|_1 &= \sum_{j=1}^n |t^{m+l}j - t_j^m| \\ &= \|s^{[(m+l)/2]} - s^{[m/2]}\|_1 \\ &= \|s^{[l/2]} - s^0\|_1 \\ &= \|t^l - t^0\|_1, \end{aligned}$$

where we have used the fact that $s^0, s^1, \ldots, s^{p-1}$ is a regular p-gon in the second last equality. This shows the claim.

To prove the lemma let $e \in \mathbb{R}^{n+1}$ be the vector with all coordinates equal to 1. Define the sequence $u^0, u^1, \ldots, u^{2p-1}$ by

$$u^j = t^j + e \qquad \text{for } 0 \leq j \leq 2p-1.$$

Observe that this sequence is again a regular $2p$-gon in $\mathbb{R}^{n+1}$. Since the sequence $s^0, \ldots, s^{p-1}$ is contained in Δ_n, it follows that $u^0, \ldots, u^{2p-1}$ is contained in $\mathbb{K}^{n+1}$ and for every $0 \leq j \leq 2p-1$

$$(9) \qquad \sum_{i=1}^{n+1} u_i^j = \sum_{i=1}^{n+1} t_i^j + \sum_{i=1}^{n+1} e_i = n + 2.$$

Now put $\alpha = (n+2)^{-1}$ and define the sequence $w^0, w^1, \dots, w^{2p-1}$ by

$$w^j = \alpha u^j \quad \text{for } 0 \le j \le 2p-1.$$

From equation (9) and the fact that the sequence $u^0, \dots, u^{2p-1}$ is a regular polygon in $\mathbb{K}^n$ it follows that $w^0, \dots, w^{2p-1}$ is a regular $2p$-gon in Δ_{n+1}. □

Theorem 3.1. *If there exists a regular p-gon in Δ_k, then for each $n \ge k$ there exists a regular polygon of size $p \cdot 2^{n-k+1}$ in $\mathbb{R}^n$.*

Proof. Suppose $n \ge k$ and let $s^0, s^1, \dots, s^{p-1}$ be a regular p-gon in Δ_k. We can apply Lemma 3.1 repeatedly until we obtain a regular polygon, say $v^0, v^1, \dots, v^{q-1}$, in Δ_n with $q = p \cdot 2^{n-k}$. Now we define the sequence $w^0, w^1, \dots, w^{2q-1}$ in $\mathbb{R}^n$ by

$$w^0 = v^0, w^1 = -v^0, w^2 = v^1, w^3 = -v^1, \dots, w^{2q-2} = v^{q-1}, w^{2q-1} = -v^{q-1}.$$

We claim that this is a regular polygon of size $p \cdot 2^{n-k+1}$ in $\mathbb{R}^n$.

Indeed take $m, l \in \{0, 1, \dots, 2r-1\}$ arbitrary, and consider

$$\|w^{m+l} - w^m\|_1.$$

If l is odd, then the following identities hold

$$\begin{aligned}\|w^{m+l} - w^m\|_1 &= \|v^{[(m+l)/2]} + v^{[m/2]}\|_1 \\ &= \|v^{[(m+l)/2]}\|_1 + \|v^{[m/2]}\|_1 \\ &= 2.\end{aligned}$$

If l is even, then we have that

$$\begin{aligned}\|w^{m+l} - w^m\|_1 &= \|v^{[(m+l)/2]} - v^{[m/2]}\|_1 \\ &= \|v^{[l/2]} - v^0\|_1 \\ &= \|w^l - w^0\|_1.\end{aligned}$$

This implies that $w^0, \dots, w^{2q-1}$ is a regular polygon of size $p \cdot 2^{n-k+1}$ □

To use Theorem 3.1 we have to search for regular polygons on the unit simplex. Let us start with a simple one.

Corollary 3.1. *There exists a regular 2^n-gon in $\mathbb{R}^n$ for $n \ge 1$.*

Proof. Remark that the sequence $x^0 = 1$ is a regular polygon in Δ_1. Thus Theorem 3.1 yields the result. □

For a candidate regular p-gon $x^0, x^1, \dots, x^{p-1}$ we have to verify that

$$\|x^{k+l} - x^k\|_1 = \|x^l - x^0\|_1 \quad \text{for} \quad 0 < k < p \quad \text{and} \quad 0 < l \le [p/2].$$

This can be done quickly by a computer. The major difficulty in finding regular polygons is caused by the fact that given a set of p points we need to find a suitable ordering on the elements. The following example in $\mathbb{R}^3$ was found by hand.

Example 3.1. The sequence $x^0, x^1, \ldots, x^5$ in $\mathbb{R}^3$, defined by

$$\begin{aligned} x^0 &= (0,1,2), & x^1 &= (0,2,1), \\ x^2 &= (1,2,0), & x^3 &= (2,1,0), \\ x^4 &= (2,0,1), & x^5 &= (1,0,2) \end{aligned}$$

is a regular polygon of size 6. Notice that we can rescale each element x^j by $1/3$ to obtain a regular polygon on the unit simplex in $\mathbb{R}^3$.

If we use Theorem 3.1 with respect to the polygon in Example 3.1, we obtain the following result.

Corollary 3.2. *There exists a regular polygon of size* $3 \cdot 2^{n-1}$ *in* $\mathbb{R}^n$ *for* $n \geq 3$.

Another interesting example occurs in dimension 5. This example was found using a computer.

Example 3.2. The sequence $y^0, y^1, \ldots, y^{19}$ in $\mathbb{R}^5$, defined by

$$\begin{aligned} y^0 &= (0,1,2,3,4), & y^1 &= (0,3,4,2,1), \\ y^2 &= (0,2,1,4,3), & y^3 &= (0,4,3,1,2), \\ y^4 &= (1,3,0,4,2), & y^5 &= (1,4,2,0,3), \\ y^6 &= (2,4,0,3,1), & y^7 &= (2,3,1,0,4), \\ y^8 &= (3,4,1,2,0), & y^9 &= (3,2,0,1,4), \\ y^{10} &= (4,3,2,1,0), & y^{11} &= (4,1,0,2,3), \\ y^{12} &= (4,2,3,0,1), & y^{13} &= (4,0,1,3,2), \\ y^{14} &= (3,1,4,0,2), & y^{15} &= (3,0,2,4,1), \\ y^{16} &= (2,0,4,1,3), & y^{17} &= (2,1,3,4,0), \\ y^{18} &= (1,0,3,2,4), & y^{19} &= (1,2,4,3,0) \end{aligned}$$

is a regular polygon. Remark that we can rescale each vector y^j by $1/10$ to obtain a regular polygon of size 20 on Δ_5.

The reader may wonder what happens in dimension 4. So far we do not know of any regular polygon of size bigger than 8 on Δ_4 except for size 12.

If we use Theorem 3.1 with respect to the polygon in Example 3.2, we obtain the following result.

Corollary 3.3. *There exists a regular polygon of size* $5 \cdot 2^{n-2}$ *in* $\mathbb{R}^n$ *for* $n \geq 5$.

Remark 3.1. We have seen in Corollary 3.2 that the regular 6-gon in Example 3.1 yields a regular polygon of size $3 \cdot 2^{n-1}$ in $\mathbb{R}^n$ for $n \geq 3$. This is the largest regular polygon we know so far. If we compare this result with the best known upper bound in Corollary 2.2, it follows that there exists a wide gap to bridge between the best known lower and upper bound.

3.2. Using the increment sequence

The regular polygons that are obtained by the procedure described in the proof of Theorem 3.1, are all contained in the boundary of an ℓ_1-norm sphere. This nice geometric property does not hold for every regular polygon, as can be seen from the following example. Consider the regular polygon $z^0, z^1, \ldots, z^7$ in $\mathbb{R}^3$ given by

$$\begin{aligned} z^0 &= (\ 1,\ 1, 1), & z^1 &= (0,\ 2, 2), \\ z^2 &= (\text{-}1,\ 1, 3), & z^3 &= (0,\ 0, 4), \\ z^4 &= (\ 1, \text{-}1, 3), & z^5 &= (0, \text{-}2, 2), \\ z^6 &= (\text{-}1, \text{-}1, 1), & z^7 &= (0,\ 0, 0). \end{aligned}$$

For this polygon, one can show that no $x \in \mathbb{R}^3$ exists such that

$$\|x - z^i\|_1 = \|x - z^j\|_1 \quad \text{for each} \quad 1 \leq i < j \leq 7.$$

This remark is related to the fact that an ℓ_1-norm nonexpansive map may not have an nonexpansive extension to the whole space. To be precise: any periodic orbit of an ℓ_1-norm nonexpansive map $f : \mathbb{R}^n \to \mathbb{R}^n$, is contained in the boundary of a sphere (see [12, page 187]), and therefore the isometry $F : \{z^0, \ldots, z^7\} \to \{z^0, \ldots, z^7\}$ given by

$$F(z^j) = z^{j+1 \bmod 8} \quad \text{for} \quad 1 \leq j \leq 7,$$

can not be extended in an ℓ_1-norm nonexpansive way to the whole of $\mathbb{R}^3$. Instead of the geometric argument, one can also use results by Scheutzow [16,17]. From his work it follows that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is an ℓ_1-norm nonexpansive map and $x \in \mathbb{R}^n$ is a periodic point of f of minimal period p, then $p \mid \text{lcm}(1, 2, \ldots, 2n)$. Since 8 does not divide $\text{lcm}(1, 2, \ldots, 6)$, it follows that F cannot be extended.

It turns out that the polygon $z^0, z^1, \ldots, z^7$ belongs to a family of regular 2^n-gons in $\mathbb{R}^n$ which are not contained on the boundary of an ℓ_1-norm sphere.

The *increment sequence* $y^0, y^1, \ldots, y^{p-1}$ of a polygon $x^0, x^1, \ldots, x^{p-1}$ is given by

$$y^j = x^j - x^{j-1} \quad \text{for} \quad 0 \leq j \leq p - 1,$$

where the indices are considered modulo p. We present a procedure to construct a regular polygon starting from an increment sequence. In order to simplify the construction, we introduce the following definition.

Definition 3.2. *A $p \times n$ matrix B, with successive rows $b^0, \ldots, b^{p-1}$ is called a regular block of size p in $(\mathbb{R}^n, \|\cdot\|)$ if* (*i*) $\|\sum_{j=0}^{p-1} b^j\| = 0$ and

(*ii*) $\|\sum_{j=k}^{k+l} b^{j \bmod p}\| = \|\sum_{j=0}^{l} b^j\| > 0$ for $0 \leq k \leq p-1$ and $0 \leq l < p-1$

It is a simple observation that if $a^0, a^1, \ldots, a^{p-1}$ is a regular p-gon, then the $p \times n$ block B with rows $b^j = a^j - a^{j-1}$ is a regular block of size p. However, the converse is also true. A regular polygon can be constructed from a regular block.

Lemma 3.2. *If B is a regular block of size p in $\mathbb{R}^n$, then the sequence a^0, $a^1, \ldots, a^{p-1}$ defined by $a^i = \sum_{j=0}^{i} b^j$, is a regular p-gon in $\mathbb{R}^n$.*

Proof. To show that the points in the sequence $a^0, a^1, \ldots, a^{p-1}$ are all distinct, we remark that for each $0 \leq l < k \leq p-1$ the following equalities hold

$$\|a^k - a^l\| = \|\sum_{j=0}^{k} b^j - \sum_{j=0}^{l} b^j\| = \|\sum_{j=l+1}^{k} b^j\| = \|\sum_{j=0}^{k-l-1} b^j\| > 0.$$

Now to show that the sequence $a^0, a^1, \ldots, a^{p-1}$ is a regular polygon we take $k, l \in \{0, \ldots, p-1\}$ arbitrary. The following equalities hold

$$\|a^{k+l} - a^k\| = \|\sum_{j=0}^{k+l} b^{j \bmod p} - \sum_{j=0}^{k} b^j\| = \|\sum_{j=k+1}^{k+l} b^{j \bmod p}\|$$
$$= \|\sum_{j=1}^{l} b^j\| = \|a^l - a^0\|.$$

Therefore we conclude that $a^0, a^1, \ldots, a^{p-1}$ is a regular polygon of size p in $\mathbb{R}^n$. □

The next step in the construction of the family of regular 2^n-gons in $\mathbb{R}^n$ is to define inductively, for $n \geq 1$, an $2^n \times n$ block B_n and to show that this block B_n is regular of size 2^n in $\mathbb{R}^n$.

Before we can state the definition of the blocks B_n, we need some more notation. If B is block with rows $b^0, b^1 \ldots, b^{p-1}$, then we let $\bar{B}$ denote the block with rows

$$\bar{b}^0 = b^{p-1}, \quad \bar{b}^1 = b^{p-2}, \ \ldots, \quad \bar{b}^{p-1} = b^0.$$

Remark that B is a regular block if and only if $\bar{B}$ is a regular block.

Definition 3.3. The $2^n \times n$ block B_n is inductively defined by

$$
B_1 = \begin{array}{|r|}\hline 1 \\ -1 \\ \hline\end{array}
\quad
B_2 = \begin{array}{|rr|}\hline 1 & 1 \\ -1 & 1 \\ -1 & -1 \\ 1 & -1 \\ \hline\end{array}
\quad
B_3 = \begin{array}{|rrr|}\hline 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & -1 \\ \hline\end{array}
\quad \cdots \quad
B_{n+1} = \begin{array}{|c|c|}\hline & 1 \\ B_n & \vdots \\ & 1 \\ \hline & -1 \\ B_n & \vdots \\ & -1 \\ \hline\end{array}
$$

Before we shall prove that B_n is a regular block, we shall prove a useful lemma. Let ϕ and ψ be permutations on $\{0, 1, \ldots, p-1\}$ and $\{1, 2, \ldots, n\}$, respectively. We define a transformation R_ϕ on a $p \times n$ block B by permuting the rows of B according to ϕ. Likewise we let T_ψ denote the transformation on B which permutes the columns of B according to ψ. Furthermore, for $1 \leq i \leq n$ we let S_i denote the transformation on B which changes the sign of each element in the i-th column of B. If T is a transformation on the block B, then we let $(T(B))^j$ denote the j-th row of the transformed block $T(B)$. We are now ready to prove the following lemma.

Lemma 3.3. *Let $n \geq 3$ and B_n be the block as defined in Definition 3.3. Suppose that the permutations $\phi = \phi_n$ and $\mu = \mu_n$ on $\{0, 1, \ldots, 2^n - 1\}$ are given by*

$$\phi(i) = i - 2^{n-2} \bmod 2^n \quad \textit{and} \quad \mu(i) = i + 2^{n-1} \bmod 2^n,$$

and that $\psi = \psi_n$ denotes the two-cycle $(n-1\ n)$ on $\{1, 2, \ldots, n\}$. If we define

$$\Gamma_n = S_{n-2} \circ S_n \circ T_\psi \circ R_\phi \quad \text{and} \quad \Lambda_n = S_{n-1} \circ S_n \circ R_\mu,$$

then we have that $\Gamma_n(B_n) = B_n$ and $\Lambda_n(B_n) = B_n$.

Proof. To prove the lemma we simply follow the transformations on the block. If $n \geq 3$, we see that

$B_n \quad \longrightarrow \quad R_\phi(B_n) \quad \longrightarrow \quad T_\psi(R_\phi(B_n))$

B_n:

	1	1
B_{n-2}	⋮	⋮
	1	1
	−1	1
$\bar{B}_{n-2}$	⋮	⋮
	−1	1
	−1	−1
B_{n-2}	⋮	⋮
	−1	−1
	1	−1
$\bar{B}_{n-2}$	⋮	⋮
	1	−1

$R_\phi(B_n)$:

	−1	1
$\bar{B}_{n-2}$	⋮	⋮
	−1	1
	−1	−1
B_{n-2}	⋮	⋮
	−1	−1
	1	−1
$\bar{B}_{n-2}$	⋮	⋮
	1	−1
	1	1
B_{n-2}	⋮	⋮
	1	1

$T_\psi(R_\phi(B_n))$:

	1	−1
$\bar{B}_{n-2}$	⋮	⋮
	1	−1
	−1	−1
B_{n-2}	⋮	⋮
	−1	−1
	−1	1
$\bar{B}_{n-2}$	⋮	⋮
	−1	1
	1	1
B_{n-2}	⋮	⋮
	1	1

Remark that $S_k(\bar{B}_k) = B_k$ and $S_k(B_k) = \bar{B}_k$ for each $k \geq 1$. Therefore, if we apply $S_{n-2} \circ S_n$ on $T_\psi(R_\phi(B_n))$ we obtain $\Gamma_n(B_n) = B_n$ for $n \geq 3$. To derive the other identity remark that

$B_n \quad \longrightarrow \quad R_\mu(B_n) \quad \longrightarrow \quad S_n(R_\mu(B_n))$

B_n:

	1
B_{n-1}	⋮
	1
	−1
$\bar{B}_{n-1}$	⋮
	−1

$R_\mu(B_n)$:

	−1
$\bar{B}_{n-1}$	⋮
	−1
	1
B_{n-1}	⋮
	1

$S_n(R_\mu(B_n))$:

	1
$\bar{B}_{n-1}$	⋮
	1
	−1
B_{n-1}	⋮
	−1

Since $S_{n-1}(\bar{B}_{n-1}) = B_{n-1}$ and $S_{n-1}(B_{n-1}) = \bar{B}_{n-1}$, we obtain that $\Lambda_n(B_n) = B_n$, $n \geq 3$. This completes the proof of the lemma. □

We are now ready to prove the main result of this section.

Theorem 3.2. *The block B_n is a regular block of size 2^n in $\mathbb{R}^n$ for $n \geq 1$.*

Proof. The first property of Definition 3.2 follows from the fact that each vertex of the n dimensional unit cube appears exactly once as a row of the block B_n.

To prove the second property of Definition 3.2, it suffices to show that

$$(10) \qquad \|\sum_{i=k}^{k+l} b^{i \bmod 2^n}\|_1 = \|\sum_{i=0}^{l} b^i\|_1 > 0,$$

for $0 \le k < 2^n$ and $0 \le l < 2^{n-1}$.

We shall first prove the equality in (10) by induction. A direct computation shows that equality holds for the blocks B_1 and B_2. Now assume that $n \ge 3$ and that equality in (32) holds for all blocks B_m with $1 \le m < n$. To prove the induction step we have to show equality for B_n.

For $q = 0, 1, 2, 3$ define the set of integers A_q by

$$A_q = \{j + q \cdot 2^{n-2} \mid 0 \le j < 2^{n-2}\}.$$

According to the definition of B_n, see the first figure in the proof of Lemma 3.3, we distinguish 6 cases:

1. $k \in A_0 \cup A_1$ and $k + l \in A_0 \cup A_1$,
2. $k \in A_2 \cup A_3$ and $k + l \in A_2 \cup A_3$,
3. $k \in A_0$ and $k + l \in A_2$,
4. $k \in A_1$ and $k + l \in A_3$,
5. $k \in A_1$ and $k + l \in A_2$,
6. $k \in A_2 \cup A_3$ and $k + l \in A_0 \cup A_1$.

In Case 1 and 2 the equality in (10) follows from a direct application of the induction hypothesis on the block B_{n-1}.

In Case 3, we have

$$\|\sum_{j=k}^{k+l} b^j\|_1 = \sum_{i=1}^{n-2} |\sum_{j=k}^{k+l} b_i^j| + \sum_{i=n-1}^{n} |\sum_{j=k}^{k+l} b_i^j|.$$

Note that since $0 \le l < 2^{n-1}$:

$$\sum_{i=n-1}^{n} |\sum_{j=0}^{l} b_i^j| = 2^{n-1}$$

and hence it suffices to show that

$$\sum_{i=n-1}^{n} |\sum_{j=k}^{k+l} b_i^j| = 2^{n-1} \quad \text{and} \quad \sum_{i=1}^{n-2} |\sum_{j=k}^{k+l} b_i^j| = \sum_{i=1}^{n-2} |\sum_{j=0}^{l} b_i^j|. \tag{11}$$

To prove the first equality remark that

$$\sum_{i=n-1}^{n} |\sum_{j=k}^{k+l} b_i^j| = |2^{n-2} - k - 2^{n-2} - k - l + 2^{n-1} - 1|$$
$$+ |2^{n-2} - k + 2^{n-2} - k - l + 2^{n-1} - 1|$$
$$= |2^{n-1} - 1 - 2k - l| + |2^n - 1 - 2k - l| = 2^{n-1}.$$

where we have used the fact that $2^{n-1} \le k + l < 3 \cdot 2^{n-2}$ and $0 \le k < 2^{n-2}$.

To verify the second identity in (11) we shall use in each step the induction hypothesis on B_{n-2}.

$$
\begin{aligned}
\sum_{i=1}^{n-2} |\sum_{j=k}^{k+l} b_i^j| &= \sum_{i=1}^{n-2} | \sum_{j=k}^{2^{n-2}-1} b_i^j + \sum_{j=2^{n-1}}^{k+l} b_i^j| \\
&= \sum_{i=1}^{n-2} | \sum_{j=0}^{2^{n-2}+l-2^{n-1}} b_i^j| \\
&= \sum_{i=1}^{n-2} | \sum_{j=2^{n-2}}^{l} b_i^j| \\
&= \sum_{i=1}^{n-2} | \sum_{j=0}^{2^{n-2}-1} b_i^j + \sum_{j=2^{n-2}}^{l} b_i^j| \\
&= \sum_{i=1}^{n-2} |\sum_{j=0}^{l} b_i^j|.
\end{aligned}
$$

This prove equality in (10) in Case 3.

To prove equality in Case 4–6, we let the permutations ϕ, ψ and μ, and the transformations Γ_n and Λ_n be as in Lemma 3.3. We remark that

$$
(12) \qquad \|\sum_{j=k}^{k+l} b^j\|_1 = \|\sum_{j=k-2^{n-2}}^{k+l-2^{n-2}} (\Gamma_n((B_n))^j\|_1 = \|\sum_{j=k-2^{n-2}}^{k+l-2^{n-2}} b^j\|_1.
$$

Note that if $k \in A_1$ and $k+l \in A_3$, then $k - 2^{n-2} \in A_0$ and $k+l-2^{n-2} \in A_2$. Therefore, by using (12) we can conclude from Case 3 that equality in (10) holds also in Case 4. Likewise, equality in Case 5 follows from Case 1.

In Case 6 we use the other transformation and the following identities

$$
(13) \qquad \|\sum_{j=k}^{k+l} b^{j \bmod 2^n}\|_1 = \|\sum_{j=k+2^{n-1}}^{k+l+2^{n-1}} (\Lambda_n(B_n))^j\|_1 = \|\sum_{j=k+2^{n-1}}^{k+l+2^{n-1}} b^j\|_1.
$$

If we look at the last sum in (13), and assume k and l to be as in Case 6, then we can conclude from Case 3,4 or 5 that equality in (10) holds.

To complete the proof of the theorem remark that

$$
\|\sum_{j=0}^{l} b^j\|_1 > 0 \qquad \text{for } 0 \le l \le 2^{n-1} - 1,
$$

since the last coordinate in each b^j in the sum is equal to 1. □

Applying Lemma 3.2 to the regular block B_3 yields the regular 8-gon $z^0, \ldots, z^7$ given at the beginning of this subsection. As another example we give the regular polygon of size 16 that arises from the regular block B_4. The regular 16-gon $x^0, x^1, \ldots, x^{15}$ in $\mathbb{R}^4$ is given by

$$
\begin{aligned}
x^0 &= (\ 1,\ 1,\ 1,\ 1), & x^1 &= (\ 0,\ 2,\ 2,\ 2),\\
x^2 &= (-1,\ 1,\ 3,\ 3), & x^3 &= (\ 0,\ 0,\ 4,\ 4),\\
x^4 &= (\ 1,-1,\ 3,\ 5), & x^5 &= (\ 0,-2,\ 2,\ 6),\\
x^6 &= (-1,-1,\ 1,\ 7), & x^7 &= (\ 0,\ 0,\ 0,\ 8),\\
x^8 &= (\ 1,\ 1,-1,\ 7), & x^9 &= (\ 0,\ 2,-2,\ 6),\\
x^{10} &= (-1,\ 1,-3,\ 5), & x^{11} &= (\ 0,\ 0,-4,\ 4),\\
x^{12} &= (\ 1,-1,-3,\ 3), & x^{13} &= (\ 0,-2,-2,\ 2),\\
x^{14} &= (-1,-1,-1,\ 1), & x^{15} &= (\ 0,\ 0,\ 0,\ 0).
\end{aligned}
$$

To end this section we note that the question, whether for a given p there exists a regular p-gon in $\mathbb{R}^n$, can be answered in finite time (see Lemmens [6]). The main idea is to show that it suffices to look for all regular p-gons in a finite subset of $\mathbb{Z}^n$. So far, however, no upper bound for the finite subset of $\mathbb{Z}^n$ is known that would allow us to do an exhaustive search for regular p-gons in reasonable time.

4. RELATIONS WITH THE SUP NORM

There exists a linear isometric embedding of the $(\mathbb{R}^n, d_1)$ into $(\mathbb{R}^m, d_\infty)$, where $m = 2^{n-1}$. In fact, take the set $v^1, \dots, v^{2^{n-1}}$ of vertices of the unit cube in $\mathbb{R}^{n-1}$ and define for each $1 \leq i \leq 2^{n-1}$ the linear functional $\theta_i : \mathbb{R}^n \to \mathbb{R}$ by $\theta_i(x) = x \cdot (1, v^i)$. One can show that the map $h : (\mathbb{R}^n, d_1) \to (\mathbb{R}^m, d_\infty)$, where $m = 2^{n-1}$, defined by

$$h(x) = (\theta_1(x), \theta_2(x), \dots, \theta_{2^{n-1}}(x)) \quad \text{for each} \quad x \in \mathbb{R}^n,$$

is an isometric embedding. Thus the problem of finding an upper bound on the size of a regular polygon in $(\mathbb{R}^n, d_1)$ is related the problem of determining the maximum size of a regular polygon in $\mathbb{R}^m$, where $m = 2^{n-1}$, under the sup norm. An upper bound on the size of a regular polygon in $\mathbb{R}^{2^{n-1}}$ under the sup norm implies an upper bound for the largest regular polygon in $\mathbb{R}^n$ under the ℓ_1-norm. In particular, if the 2^n-conjecture is true, then $2^{2^{n-1}}$ would be an upper bound for the ℓ_1-norm case. Of course, a further reduction is likely because $(\mathbb{R}^n, d_1)$ is an n-dimensional subspace of $(\mathbb{R}^m, d_\infty)$, where $m = 2^{n-1}$. However, the tempting conjecture that the correct upper bound would be 2^n fails, since we can conclude from Corollary 3.2 and 3.3 that there exist regular polygons in an n-dimensional, $n \geq 3$, linear subspace of $\mathbb{R}^{2^{n-1}}$ under the sup norm, that have a size bigger then $3 \times 2^{n-1}$.

Finally, we remark that in the proof of the 2^n-conjecture in dimension 3, see Lyons and Nussbaum [7], the notion of an additive chain plays an important role. Moreover, it is shown that the length of an sup norm additive chain in a compact set in $\mathbb{R}^n$ with a transitive and commutative family of sup norm isometries is bounded by $n + 1$ ([7, Theorem 2.1]). Furthermore, if the set contains an additive chain of length $n + 1$, then it is shown in unpublished work ([7, Remark 2.2]) that its cardinality is bounded by $2n$. These results are similar to the assertions in Theorem 2.1 and 2.3.

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REFERENCES

1. Akcoglu, M.A. and U. Krengel – Nonlinear models of diffusion on a finite space. Probab. Theory Related Fields **76**, 411–420 (1987).
2. Alon, N., Z. Füredi and M. Katchalski – Separating pairs of points by standard boxes. European J. Combin. **6**, 205–210 (1985).
3. Dafermos, C.M. and M. Slemrod – Asymptotic behaviour of nonlinear contraction semigroups. J. Funct. Anal. **13**, 97–106 (1973).
4. Erdős, P. and G. Szekeres – A combinatorial problem in geometry. Compositio Math. **2**, 463–470 (1935).
5. Kruskal, J.B. – Monotonic subsequences. Proc. Amer. Math. Soc. **4**, 264–274 (1953).
6. Lemmens, B. – Integral rigid sets and periods of nonexpansive maps. Indag. Mathem. N.S. **10**, 437–447 (1999).
7. Lyons, R. and R.D. Nussbaum – On transitive and commutative finite groups of isometries, pp. 189–228 in Fixed Point Theory and Applications, (K.-K. Tan, ed.), World Scientific, Singapore, 1992.
8. Martus, P. – Asymptotic properties of nonstationary operator sequences in the nonlinear case, Ph.D. dissertation, Friedrich-Alexander Univ., Erlangen-Nürnberg, 1989 (in German).
9. Misiurewicz, M. – Rigid sets in finite dimensional l_1-spaces, Report, Mathematica Göttingensis Schriftenreihe des Sonderforschungsbereichs Geometrie und Analysis, Heft 45, 1987.
10. Nussbaum, R.D. – Omega limit sets of nonexpansive maps: finiteness and cardinality estimates. Differential Integral Equations **3**, 523–540 (1990).
11. Nussbaum, R.D. – Estimates of the periods of periodic points of nonexpansive operators. Israel J. Math. **76**, 345–380 (1991).
12. Nussbaum, R.D. – A nonlinear generalization of Perron-Frobenius theory and periodic points of nonexpansive maps, pp. 183–198 in Recent Developments in Optimization Theory and Nonlinear Analysis, (Y. Censor and S. Reich, ed.), Contemporary Mathematics, vol. 204, American Math. Society, Providence, R.I., 1997.
13. Nussbaum, R.D. and M. Scheutzow – Admissible arrays and a nonlinear generalization of Perron-Frobenius theory. J. London Math. Soc. **2**, 526–544 (1998).
14. Nussbaum, R.D., M. Scheutzow and S.M. Verduyn Lunel – Periodic points of nonexpansive maps and nonlinear generalizations of the Perron-Frobenius theory. Selecta Math. (N.S.) **4**, 1–41 (1998).
15. Nussbaum, R.D. and S.M. Verduyn Lunel – Generalizations of the Perron-Frobenius theorem for nonlinear maps, Memoirs of the American Mathematical Society **138**, number 659, 1–98 (1999).
16. Scheutzow, M. – Periods of nonexpansive operators on finite l_1-spaces. European J. Combin. **9**, 73–78(1988).
17. Scheutzow, M. – Corrections to periods of nonexpansive operators on finite l_1-spaces. European J. Combin. **12**, 183 (1991).
18. Weller, D. – Hilbert's metric, part metric and self mappings of a cone, Ph.D. dissertation, Univ. of Bremen, Germany, 1987.

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