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Boundary layer phenomena for differential-delay equations with state-dependent time lags: III

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Abstract

We consider a class of singularly perturbed delay-differential equations of the form

$$\varepsilon\dot{x}(t) = f(x(t), x(t-r)),$$

where $r = r(x(t))$ is a state-dependent delay. We study the asymptotic shape, as $\varepsilon \to 0$, of slowly oscillating periodic solutions. In particular, we show that the limiting shape of such solutions can be explicitly described by the solution of a pair of so-called max-plus equations. We are able thereby to characterize both the regular parts of the solution graph and the internal transition layers arising from the singular perturbation structure.

Keywords: Delay-differential equation; State-dependent delay; Slowly oscillating periodic solution; Singular perturbation; Max-plus equation

1. Introduction

In this paper, which is a sequel to the papers [33,35], we study *slowly oscillating periodic solutions* (SOPSs) of the state-dependent delay differential equation

$$\varepsilon\dot{x}(t) = f(x(t), x(t-r)), \qquad r = r(x(t)). \tag{1.1}$$

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Here $x = x(t)$ is a scalar, f and r are given nonlinearities, and $\varepsilon > 0$ is a parameter. By an *SOPS* we mean a solution of (1.1) satisfying

$$\begin{cases} x(t) > 0 \text{ in } (0, \alpha_1), \qquad x(t) < 0 \text{ in } (\alpha_1, \alpha_2), \\ x(t + \alpha_2) = x(t) \quad \text{for every } t, \text{ where} \\ \alpha_1 > r(0), \qquad \alpha_2 - \alpha_1 > r(0), \end{cases} \tag{1.2}$$

for some quantities α_1 and α_2, where $r(0) > 0$ is assumed. Of course $x(0) = x(\alpha_1) = x(\alpha_2) = 0$ for such a solution. We note that by rescaling time we may always make the normalization $r(0) = 1$, which in fact we shall assume below.

Of particular interest is the singular perturbation case of small ε. The existence of such solutions for a range of parameter $0 < \varepsilon < \varepsilon_*$ for a general class of nonlinearities f and r was proved in [33,35]. Here $f(0,0) = 0$ and ε_* is a Hopf bifurcation point below which the origin is unstable. Theorem 1.1 below gives the existence results relevant to the present paper. In [35] a general theory of "limiting profiles" was developed in order to make a rigorous connection between Eq. (1.1) for small ε and the (implicit) difference equation

$$0 = f(\xi_n, \xi_{n-1}), \qquad \tau_{n-1} = \tau_n - r(\xi_n), \tag{1.3}$$

which is obtained formally in the limit $\varepsilon \to 0$ in (1.1). Our object in the present paper is to employ the machinery of [35] in order to obtain precise and explicit results on the asymptotic form of SOPSs as $\varepsilon \to 0$. Our approach will be to use this machinery to show how the limiting profiles can be expressed in terms of solutions of a system of so-called *max-plus equations*:

$$\begin{aligned} \psi_{2m}(\xi) &= \max_{-v \leqslant s \leqslant \xi} (r(s) + \psi_{2m-1}(g(s))), \\ \psi_{2m+1}(\xi) &= \max_{\xi \leqslant s \leqslant \mu} (r(s) + \psi_{2m}(g(s))). \end{aligned} \tag{1.4}$$

Indeed, the system (1.4) is at the heart of our analysis and is the centerpiece of Theorem A, our most general result.

Differential equations of the form (1.1), or more generally differential equations with variable delays, arise in many models of interest. Biology is a particularly rich source of models [1,5,6,24–27,29] as are the social sciences [7,28]. The equation $\dot{x}(t) = x(t - x(t))$ arises in models of crystal growth [38], and in fact the related equation $\dot{x}(t) = -ax(t - r(x(t)))$ was studied theoretically by Cooke [10] in 1967. Some of the earliest rigorous results on variable delay systems were given in the 1960s by Driver [14–17] and Driver and Norris [18], and in the 1970s by Alt [2,3], Nussbaum [39], and Winston [41,42].

Nevertheless, for state-dependent delay equations many basic dynamical issues are still undeveloped and unresolved. While the basic work in [20], and also in [13] and elsewhere, treats constant delay problems in great generality, many of the most fundamental results (invariant manifold theorems, Hopf and other bifurcations, linearized stability of equilibria and periodic solutions) have not been developed for the state-dependent case, at least not in any generality. However, some initial steps in this direction can be found in [4,8,11,12,19,21–23,30]. A very recent and significant contribution by Walther is found in [40].

For further work on state-dependent and variable delay problems see, for example, the articles in [9], the many references in [22], and as well [31–35].

Simple numerical experiments demonstrate a wealth of patterns, which spontaneously arise as limiting profiles, particularly in the singular perturbation case (1.1), and certainly for more general problems such as those with multiple delays [34]. (While we only consider the case (1.1) of a single delay in the present paper, extending our theory to systems with multiple delays as in [34] presents an intriguing challenge.) One observes that the limiting shapes of solutions of (1.1) as $\varepsilon \to 0$ seem very often to be unique, and very stable with respect to changes in initial conditions, so it is a fundamental problem to predict such limiting shapes from the differential equation. Indeed, one can think of the value $\varepsilon = 0$ as a "organizing center" about which a rich variety of dynamical phenomena might be sought.

Let us now fix our assumptions on the differential equation (1.1), although we shall make additional assumptions as necessary.

Standing assumptions. We make the following assumptions on f and r, to hold for the remainder of this paper unless noted otherwise. We assume there exist positive quantities C and D such that

$$f : I \times I \to \mathbb{R}, \qquad r : I \to [0, \infty), \qquad \text{where } I = [-D, C],$$

and where f and r are locally lipschitz. We assume that

$$r(0) = 1, \qquad r(\xi) > 0 \quad \text{for every } \xi \in (-D, C),$$

and that there exists a continuous function $g : I \to I$ such that

$$\begin{cases} \operatorname{sgn} f(\xi, \zeta) = \operatorname{sgn}(g(\xi) - \zeta) & \text{for every } (\xi, \zeta) \in I \times I, \\ g(0) = 0, \qquad g(\xi) \text{ is strictly decreasing in } \xi \in I, \\ g(C) = -D \text{ if } r(C) > 0, \\ g(-D) = C \text{ if } r(-D) > 0. \end{cases} \tag{1.5}$$

We further assume that

$$f \text{ is differentiable at } (0,0), \quad \text{with } D_2 f(0,0) < D_1 f(0,0) < 0, \tag{1.6}$$

where $D_1 f = \partial f / \partial \xi$ and $D_2 f = \partial f / \partial \zeta$ denote the partial derivatives of f. A final assumption is that

$$|g^2(\xi)| < |\xi| \text{ for every } \xi \in \operatorname{int}(I) \backslash \{0\}, \tag{1.7}$$

where g^2 denotes the second iterate of g, and "int" denotes the interior. We shall use (τ, ξ) to denote general coordinates in the plane $\mathbb{R}^2$, although we usually denote specific solutions of a differential equation by $x(t)$.

Observe from (1.5) that $f(\xi, g(\xi)) = 0$ for every $\xi \in I$. Also, $g'(0)$ exists with

$$g'(0) = -k^{-1}, \text{ where } k = D_2 f(0,0)/D_1 f(0,0) > 1. \tag{1.8}$$

If $r(C) = 0$ then $g(C) \geqslant -D$ since $g : I \to I$, with strict inequality a possibility. Similarly $g(-D) \leqslant C$ if $r(-D) = 0$. Let us denote

$$R = \max_{\xi \in I} r(\xi), \tag{1.9}$$

so we have that $R \geqslant r(0) = 1$.

With the function g we may now express the difference equation (1.3) as

$$(\tau_{n-1}, \xi_{n-1}) = \Phi(\tau_n, \xi_n),$$

where the map $\Phi : H_I \to H_I$ of the horizontal strip $H_I = \mathbb{R} \times I$ into itself is defined to be

$$\Phi(\tau, \xi) = (\tau - r(\xi),\ g(\xi)). \tag{1.10}$$

Although Φ need not map H_I onto itself, it is one-to-one in view of the strict monotonicity of g and the map Φ^{-1} is continuous in the range of Φ. Indeed, the range of Φ is the strip $\Phi(H_I) = H_J = \mathbb{R} \times J$ where

$$J = [g(C), g(-D)] \subseteq [-D, C],$$

and

$$\Phi^{-1}(\tau, \xi) = (\tau + r(g^{-1}(\xi)),\ g^{-1}(\xi))$$

for $(\tau, \xi) \in H_J$. Formally, the map Φ^{-1} describes the forward evolution of (τ_{n-1}, ξ_{n-1}) to (τ_n, ξ_n), and as such we shall refer to Φ as the *backdating map* and to Φ^{-1} as the *updating map*. Note that the τ-axis is invariant under Φ and Φ^{-1}. The fact that $-1 < g'(0) < 0$, which holds by (1.8), means that the τ-axis is unstable in the normal direction for the map Φ^{-1}.

Let us observe, as in [35], that with all the above conditions on f and r, the set I (or more precisely the set $C([-R, 0], I) \subseteq C([-R, 0])$ of continuous functions taking values in I) is positively invariant for Eq. (1.1). Indeed, this is a consequence of the inequalities

$$\begin{cases} f(C, \zeta) \leqslant 0 & \text{for every } \zeta \in I, \text{ if } r(C) > 0, \\ f(-D, \zeta) \geqslant 0 & \text{for every } \zeta \in I, \text{ if } r(-D) > 0, \\ f(C, C) < 0 & \text{and} \quad f(-D, -D) > 0, \quad \text{in any case,} \end{cases} \tag{1.11}$$

which imply that $\dot{x}(t_0) \leqslant 0$ whenever $x(t_0) = C$, and that $\dot{x}(t_0) \geqslant 0$ whenever $x(t_0) = -D$, for any solution of (1.1) for which $x(t) \in I$ for every $t \in [t_0 - R, t_0]$. The inequalities in (1.11) follow directly from the conditions (1.5). Also observe that as $|g'(0)| < 1$, the inequality in (1.7) holds for $\xi \neq 0$ near 0. If $g(-D) = C$ and $g(C) = -D$ then $g^2(\xi) = \xi$ at $\xi = -D, C$, that is, (1.7) becomes an equality there. In this case if the map g is defined outside the interval I in all of $\mathbb{R}$, then $\{-D, C\}$ is the period-two orbit of g which is nearest the origin. More generally, if g and r are defined in all of $\mathbb{R}$ then C is characterized as the smallest positive number for which either $g^2(C) = C$ or else $\min\{r(C), r(g(C))\} = 0$, with a similar characterization for $-D$. Condition (1.7) implies that if $\xi \in I$ is any periodic point of the map g, that is $g^n(\xi) = \xi$ for some $n > 0$, then $\xi \in \{0, -D, C\}$.

Under the above standing assumptions the results of [35] guarantee the existence of an SOPS for every sufficiently small $\varepsilon > 0$, with this solution taking values in the interval $[-D, C]$ and enjoying a sine-like monotonicity property. We have the following result.

Theorem 1.1 (see Mallet-Paret and Nussbaum [35, Theorems 4.6 and 4.15]). *Let* $0 < \varepsilon < \varepsilon_*$ *where*

$$\varepsilon_* = \frac{(B^2 - A^2)^{1/2}}{\arccos(-k^{-1})}, \qquad A = D_1 f(0,0), \qquad B = D_2 f(0,0),$$
$$\text{where } \frac{\pi}{2} < \arccos(-k^{-1}) < \pi,$$

with k as in (1.8). *Then Eq.* (1.1) *possesses a slowly oscillating periodic solution*, *namely a solution satisfying* (1.2) *for some* α_1 *and* α_2, *and in addition this solution satisfies*

$$-D \leqslant x(t) \leqslant C \tag{1.12}$$

for every t. Furthermore, *every such solution enjoys the following monotonicity property. One has that*

$$\dot{x}(t) \geqslant 0 \ \text{ in } [0, \beta_0] \cup [\beta_1, \alpha_2], \qquad \dot{x}(t) \leqslant 0 \ \text{ in } [\beta_0, \beta_1], \tag{1.13}$$

for some quantities β_0 *and* β_1 *satisfying* $0 < \beta_0 < \alpha_1 < \beta_1 < \alpha_2$.

The results of [35] in fact provide a continuum of SOPSs emanating from the Hopf bifurcation point $(x, \varepsilon) = (0, \varepsilon_*)$ and extending at least throughout the range $0 < \varepsilon < \varepsilon_*$. In general, one does not expect uniqueness of the solution for a given ε, although simple numerical simulations suggest that this may often be the case. In many cases the inequalities in (1.13) are strict away from the points β_0 and β_1, which are the locations of the maximum and minimum of x. In any case one always has that

$$\dot{x}(0) = \dot{x}(\alpha_2) > 0, \qquad \dot{x}(\alpha_1) < 0,$$

at the zeros of x, as one sees directly from the differential equation.

As noted, the main object of the present paper is to describe precisely and explicitly the asymptotic shape of the solutions given by Theorem 1.1 as $\varepsilon \to 0$. A central and deep result of [35], which is a first step in this direction, states that under appropriate conditions a family of SOPSs cannot tend uniformly to zero as $\varepsilon \to 0$. That is, a nontrivial limiting profile is approached. In particular the following holds.

Theorem 1.2 (see Mallet-Paret and Nussbaum [35, Theorem 5.1]). *Assume the function r satisfies*

$$\lim_{\xi \to 0} \frac{r(\xi) - 1}{\xi^n} = Q,$$

for some integer $n \geqslant 1$ *and some quantity* Q *satisfying*

$$Q \neq 0 \quad \textit{if } n \textit{ is odd}, \qquad Q > 0 \quad \textit{if } n \textit{ is even}.$$

Let x^k *be a sequence of slowly oscillating periodic solutions of Eq.* (1.1) *for some sequence of positive parameters* $\varepsilon^k \to 0$. *Then*

$$\liminf_{k \to 0} ||x^k|| > 0, \qquad ||x^k|| = \sup_{t \in \mathbb{R}} |x^k(t)|, \tag{1.14}$$

holds.

Upon making a change of variables $x \to -x$ in the differential equation one replaces Q with $-Q$ if n is odd, and thus the condition $Q \neq 0$ above is natural. For even n such a change of variables leaves Q unchanged, however. If n is even and $Q < 0$ then it is an open question whether the conclusion (1.14) must always hold, although simple numerical simulations strongly suggest that it is true. Note that if either

$$r'(0) \neq 0 \text{ or } r''(0) > 0$$

with r smooth enough then the above theorem applies, but this is not the case if $r'(0) = 0$ and $r''(0) < 0$.

We remark that the results of [35] are in fact a bit more general than those given in the theorem above. In particular, different quantities Q^- and Q^+ are allowed for the left and right limits provided they are suitably related, and the exponent n need not be an integer. An earlier result [31] also established (1.14) in the case of a constant delay $r(\xi) \equiv 1$, although the mechanisms and proofs in the two cases [31] and [35] are entirely different. Indeed, in both cases the proof of (1.14) is not at all trivial, and both require a considerable amount of effort which goes well beyond standard local arguments.

Another basic result is that the periods of SOPSs remain bounded as $\varepsilon \to 0$. Results in this direction are implied in some of the proofs of [33]. For definiteness we state a result in the following theorem in a form which will be useful to us. A self-contained and straightforward proof of this theorem will be given in Section 5.

Theorem 1.3. *Let* x^k *be a sequence of slowly oscillating periodic solutions of Eq.* (1.1) *for some sequence of positive parameters* $\varepsilon^k \to 0$, *and assume each solution satisfies the bounds* (1.12) *for every t. Then there exists* $P > 0$ *such that*

$$p^k \leqslant P$$

for every k, where p^k *is the minimal period of* x^k.

To give the flavor of some of our results consider the special case when

$$\begin{cases} r(\xi) \quad \text{is monotone increasing in } \xi \in [-D, C] \\ \qquad\quad \text{and is strictly increasing in } \xi \in [-D, 0], \\ r'(0) \ \text{exists and } r'(0) > 0, \\ h(\xi) \ \text{is monotone increasing in } \xi \in [0, C], \end{cases} \tag{1.15}$$

all hold, where $h : I \to \mathbb{R}$ is the function

$$h(\xi) = r(\xi) + r(g(\xi)). \tag{1.16}$$

Let x^k with positive parameters $\varepsilon^k \to 0$ be any sequence of SOPSs of Eq. (1.1) satisfying the bounds (1.12), such solutions existing by virtue of Theorem 1.1. Then it is a consequence of our results (Theorems A, B, and C in Section 3 below) that the sequence of graphs

$$\Gamma^k = \{(t, x^k(t)) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^2 \tag{1.17}$$

of our solutions approaches a limiting set $\Omega \subseteq \mathbb{R}^2$, which has the following properties. The set Ω is periodic in the horizontal direction, in particular

$$\Omega = \bigcup_{m=-\infty}^{\infty} \Theta_m, \qquad \Theta_m = \{(\tau + mp, \xi) \mid (\tau, \xi) \in \Theta_0\},$$

for some $p > 0$ and some $\Theta_0 \subseteq \mathbb{R}^2$. Here the period p is the quantity

$$p = h(C) = r(C) + r(-D) \tag{1.18}$$

and it also equals the limit $p^k \to p$ of the periods p^k of x^k (note that $g(C) = -D$ by (1.5), from the monotonicity of r, to give the second equality in (1.18)). The set Θ_0, which describes one period of Ω, is a union $\Theta_0 = \Theta_{\text{bot}} \cup \Theta_{\text{asc}} \cup \Theta_{\text{top}} \cup \Theta_{\text{dsc}}$ of four pieces given by

$$\Theta_{\text{bot}} = [-1,\ r(-D) - 1] \times \{-D\},$$

$$\Theta_{\text{asc}} = \{(\tau, \xi) \in \mathbb{R}^2 \mid \tau = r(\xi) - 1 \text{ and } \xi \in [-D, C]\},$$

$$\Theta_{\text{top}} = [r(C) - 1, r(C) + r(-D) - 1] \times \{C\},$$

$$\Theta_{\text{dsc}} = \{r(C) + r(-D) - 1\} \times [-D, C].$$

The sets Θ_{bot} and Θ_{top} are horizontal line segments each of length $r(-D)$ located at the levels $\xi = -D$ and $\xi = C$ in the vertical direction. The set Θ_{asc} is the "ascending" portion of Θ_0 which joins Θ_{bot} to Θ_{top}. If r is strictly increasing then in fact Θ_{asc} is a graph $\xi = r^{-1}(\tau + 1)$ and is the only part of Θ_0 which is neither a horizontal nor a vertical line segment. Finally Θ_{dsc} is the "descending" portion of Θ_0, which unlike Θ_{asc} is a vertical line segment. If it is the case that $r(-D) = 0$ then the line segments Θ_{bot} and Θ_{top} are each just single points.

In the notation of Theorem A, which is given in Section 3, we have in this example that

$$\psi_{2m}(\xi) = r(\xi) - 1 + mp, \qquad \psi_{2m+1}(\xi) = r(C) + r(-D) - 1 + mp, \tag{1.19}$$

for every integer m, with $\xi \in [-\nu, \mu] = [-D, C]$. The sets Θ_{asc} and Θ_{dsc} are just the graphs $\tau = \psi_0(\xi)$ and $\tau = \psi_1(\xi)$ for this range of ξ, while the graphs $\tau = \psi_{2m}(\xi)$ and $\tau = \psi_{2m+1}(\xi)$ are the horizontal translates of these sets in Θ_m. Also, $\Theta_{\text{bot}} = B_{-1} \times \{-\nu\}$ and $\Theta_{\text{top}} = B_0 \times \{\mu\}$ with $L_B = r(-D)$. As ψ_n is continuous at $\xi = 0$ each set $A_n = \{\psi_n(0)\}$ is a single point and $L_A = 0$. One readily checks that the functions ψ_n satisfy the system (1.4) of max-plus equations throughout $[-\nu, \mu]$. They also satisfy Eqs. (3.12) and (3.13) of Theorem B, which follow from (1.4) when r is monotone increasing. The quantity $\mu = C$ is uniquely determined by Theorem C, and one has that $-\nu = g(\mu)$. In the notation of that result it is the case that $\psi_*(\xi) \equiv 0$ identically in $[0, C]$ (it is easy to check this is a solution of (3.17), but its uniqueness is not immediately obvious), and so $F(M) = r(g(M))$. The assumptions on r ensure that $F(M) > 0$ for every $M \in (0, C)$, and so the unique solution to (3.19) guaranteed by the theorem is $\mu = C$.

Perhaps, the simplest nontrivial example of Eq. (1.1) is

$$\varepsilon\dot{x}(t) = -x(t) - kx(t - r), \qquad r = 1 + cx(t), \tag{1.20}$$

where $k > 1$ and $c > 0$ are given constants. Although both the functions $f(\xi, \zeta) = -\xi - k\zeta$ and $r(\xi) = 1 + c\xi$ are linear, Eq. (1.20) is most certainly nonlinear. This equation is covered by our theory, and is a special case of the above example (1.15).

In particular, we see easily from (1.5), (1.16), and (1.18) that

$$g(\xi) = -k^{-1}\xi, \qquad h(\xi) = 2 + c(1 - k^{-1})\xi, \qquad [-D, C] = [-c^{-1}, c^{-1}k],$$
$$p = 1 + k.$$

The set Ω has the shape of a sawtooth wave, namely the graph of

$$\mathrm{saw}(\tau) = c^{-1}\tau \text{ for } \tau \in (-1, k), \qquad \mathrm{saw}(\tau + 1 + k) \equiv \mathrm{saw}(\tau),$$

with vertical lines $-c^{-1} \leqslant \xi \leqslant c^{-1}k$ at the discontinuities $\tau = k + n(1 + k)$ filled in. In the notation of Theorem A, the set Θ_{asc} is a line segment $\xi = c^{-1}\tau$ of slope c^{-1} in the (τ, ξ)-plane extending from $(-1, -c^{-1})$ to $(k, c^{-1}k)$ and the sets Θ_{bot} and Θ_{top} are single points, namely the endpoints $\{(-1, -c^{-1})\}$ and $\{(k, c^{-1}k)\}$ of Θ_{asc}. The set Θ_{dsc} is the vertical segment $\{k\} \times [-c^{-1}, c^{-1}k]$.

Also, it is not hard to check, with elementary numerical simulations, that (numerical) solutions to (1.20) converge rather quickly to the above sawtooth for reasonably small values of ε. Generally, beginning with any positive initial condition, the solution is observed after only three or four cycles to settle into a very stable sawtooth pattern.

Max-plus equations, such as the system (1.4) which occurs in the statement of Theorem A, and the eigenproblem (3.12) of Theorem B, are a central feature of our analysis. A number of relevant results on this topic have been developed independently, and in particular we mention [37] which should be viewed as a companion to the present paper. Some of the results of [37] are crucial to our analysis here, and we shall outline these in a later section.

This paper is organized as follows. Some basic notation and terminology will be established in Section 2. In Section 3 we state our main results, Theorems A, B, and C. In particular Theorem A is a general result about how limiting profiles of SOPSs of Eq. (1.1) can be described explicitly using solutions of the max-plus equations (1.4). Theorem B specializes this result to the case that $r(\xi)$ is monotone in ξ, and Theorem C specializes this further to the so-called quasimodal case (which in particular includes the conditions (1.15) above). In the quasimodal case it turns out that the limiting profile Ω of the sequence of SOPSs is uniquely, and in a sense explicitly, determined. In Section 3 we also outline the relevant results from the companion paper [37], and we prove Theorem C there. We do mention here that the proof of Theorem C relies on Theorems A and B, which are proved later in the paper.

A familiarity with some of the definitions and results in our earlier papers [33] and especially [35] is necessary to understand fully the statements of our theorems in Section 3. However, even the reader who is not familiar with this earlier work should acquire a general understanding of what we prove here from this section. A summary of the necessary material from [35] will be given in Sections 4 and 5; we present all the relevant concepts there so that the present paper is self-contained and can be read without reference to [33,35]. In particular, the theory of limiting profiles, from [35], will be summarized in Section 4. In Section 5 this theory will be specialized to SOPSs and further results from [35] will be presented.

The heart of our theory, the max-plus equations, is derived in Section 6, and this section culminates with the proof of Theorem A. In Section 7 we obtain both upper and lower bounds on the asymptotic period p. For a broad class of cases these bounds are equal and yield an explicit formula for p. With such a formula for p we are then able, in Section 8, to prove Theorem B.

2. Notation and terminology

By an *interval* we mean a nonempty connected subset of the real line $\mathbb{R}$, a single point in this sense being considered an interval. We denote the length of an interval J by $\ell(J)$.

If S_1 and S_2 are any two subsets of $\mathbb{R}$, we write

$$\begin{aligned} S_1 < S_2 \qquad & \text{if } a_1 < a_2 \text{ for every } a_1 \in S_1 \text{ and } a_2 \in S_2, \\ S_1 \leqslant S_2 \qquad & \text{if } a_1 \leqslant a_2 \text{ for every } a_1 \in S_1 \text{ and } a_2 \in S_2. \end{aligned}$$

In case one of the sets is a single point, say $S_1 = \{a\}$, we write $a < S_2$ in place of $\{a\} < S_2$, and so forth.

If $S \subseteq \mathbb{R}^N$ and $v \in \mathbb{R}^N$ we denote

$$S + v = \{a + v \mid a \in S\}, \qquad S - v = \{a - v \mid a \in S\}.$$

The minus sign $-$ will be reserved for the algebraic difference, as above. To denote the set-theoretic difference of two sets, we write

$$S_1 \backslash S_2 = \{a \in S_1 \mid a \notin S_2\}.$$

We let $\mathrm{int}(S)$ denote the interior of S.

If $S \subseteq \mathbb{R}$, then we define the vertical and horizontal strips, $V_S \subseteq \mathbb{R}^2$ and $H_S \subseteq \mathbb{R}^2$, respectively, as

$$V_S = S \times \mathbb{R}, \qquad H_S = \mathbb{R} \times S.$$

In case $S = \{a\}$ is a single point, for simplicity we shall write V_a and H_a in place of $V_{\{a\}}$ and $H_{\{a\}}$.

If $S \subseteq \mathbb{R}^2$, we say the set S is *monotone increasing* if

$$(\tau_1, \xi_1), \ (\tau_2, \xi_2) \in S, \text{ with } \tau_1 < \tau_2 \Rightarrow \xi_1 \leqslant \xi_2,$$

and *monotone decreasing* if instead we have $\xi_1 \geqslant \xi_2$ in the above implication. If $\varphi : J \to \mathbb{R}$ is a function defined in an interval J, we say φ is monotone increasing in J

in case the graph of φ is a monotone increasing subset of $\mathbb{R}^2$, that is,

$$\tau_1, \tau_2 \in J, \text{ with } \tau_1 < \tau_2 \Rightarrow \varphi(\tau_1) \leqslant \varphi(\tau_2). \tag{2.1}$$

We say φ is *strictly increasing* if instead we have a strict inequality $\varphi(\tau_1) < \varphi(\tau_2)$ in (2.1). Monotone and strictly decreasing functions are defined in the obvious way.

More generally, we may consider set-valued functions $\varphi : J \to 2^{\mathbb{R}}$, where $2^{\mathbb{R}}$ is the set of all subsets of $\mathbb{R}$. Such a function is said to be monotone increasing if (2.1) holds, and strictly increasing if again we have $\varphi(\tau_1) < \varphi(\tau_2)$ in (2.1), where we mean here the inequality of sets defined above. The obvious definitions of decreasing functions hold. We say that a set-valued function is *single-valued* in a subset $S \subseteq J$ if $\varphi(\tau)$ contains exactly one element for every $\tau \in S$.

Finally, we shall let $\vee$ denote the maximum operator, namely

$$a_1 \vee a_2 \vee \cdots \vee a_q = \max \{a_i\}_{i=1}^q$$

for real numbers a_i.

3. Statements of the main results

Our first main result gives a general description of the so-called limiting profile of a regular sequence of SOPSs of Eq. (1.1). The terms "limiting profile" and "regular sequence" will be defined precisely in Section 4. Roughly, a sequence x^k of SOPSs with $\varepsilon^k \to 0$ is regular if, in certain sufficiently large compact subsets of the plane, the graphs $\Gamma^k \subseteq \mathbb{R}^2$ of x^k as in (1.17) converge in the Hausdorff sense to a limiting set $\Omega \subseteq \mathbb{R}^2$ as $k \to \infty$. The set Ω is then called the limiting profile of the sequence x^k. It is the case that every sequence x^k of bounded solutions has a regular subsequence.

Note that the set $\text{graph}(\psi_n) \subseteq \mathbb{R}^2$ in this theorem is in fact what one would usually call the graph of ψ_n^{-1}, the coordinates τ and ξ having been switched. We trust that this slight irregularity in terminology will not be a problem.

Theorem A. *Let x^k with $\varepsilon^k \to 0$ be a regular sequence of slowly oscillating periodic solutions of Eq.* (1.1), *each solution satisfying the bounds* (1.12) *for every t. Let $\Omega \subseteq \mathbb{R}^2$ be the limiting profile of this sequence. Then either $x^k \to 0$ uniformly, in which case $\Omega = \mathbb{R} \times \{0\}$, or else there exist quantities $\mu > 0$ and $\nu > 0$ satisfying*

$$g([-\nu, \mu]) \subseteq [-\nu, \mu] \subseteq [-D, C] \tag{3.1}$$

and functions $\psi_n : [-\nu, \mu] \backslash \{0\} \to \mathbb{R}$ such that

$$\Omega = \left(\bigcup_{n=-\infty}^{\infty} \text{graph}(\psi_n) \right) \cup \left(\bigcup_{n=-\infty}^{\infty} B_n \times \{\lambda_n\} \right). \tag{3.2}$$

For every n we have in $[-\nu,\mu]\backslash\{0\}$ *that the function* ψ_n *is continuous with left- and right-hand limits at* 0, *the function* $(-1)^n\psi_n$ *is monotone increasing, and*

$$\psi_n(\xi) \leqslant \psi_{n+1}(\xi), \qquad \psi_{n+2}(\xi) = \psi_n(\xi) + p, \tag{3.3}$$

hold identically where $p \geqslant 2$ *is independent of n. We denote the set*

$$\mathrm{graph}(\psi_n) = \{(\tau,\xi) \mid \tau = \psi_n(\xi) \text{ and } \xi \in [-\nu,\mu]\backslash\{0\}\} \cup (A_n \times \{0\}) \tag{3.4}$$

and the compact intervals

$$A_n = \begin{cases} [\psi_n(0-), \psi_n(0+)], & n \text{ even}, \\ [\psi_n(0+), \psi_n(0-)], & n \text{ odd}, \end{cases} \qquad B_n = [\psi_n(\lambda_n), \psi_{n+1}(\lambda_n)], \tag{3.5}$$

where

$$\lambda_n = \begin{cases} \mu, & n \text{ even}, \\ -\nu, & n \text{ odd}. \end{cases} \tag{3.6}$$

The functions ψ_n *satisfy the systems of max-plus equations* (1.4) *with the first and second equations in* (1.4) *holding, respectively, for* ξ *in* $[-\nu,\delta_0]\backslash\{0\}$ *and for* ξ *in* $[-\delta_1,\mu]\backslash\{0\}$ *for some positive quantities* δ_0 *and* δ_1. *Also,*

$$0 \in A_0. \tag{3.7}$$

All the intervals A_n *have the same length* $\ell(A_n) = L_A$, *and all intervals* B_n *have the same length* $\ell(B_n) = L_B$. *If* $L_A = 0$ *then each* ψ_n *can be extended continuously to* $\xi = 0$, *with* (1.4) *and* (3.3) *holding at* $\xi = 0$ *and with* (3.7) *taking the form*

$$\psi_0(0) = 0. \tag{3.8}$$

Finally, if $p > 2$ *then* $L_A = 0$, *and if either* $g^2(\mu) \neq \mu$ *or* $g^2(-\nu) \neq -\nu$ *then* $L_B = 0$.

Remark. It is clear from Theorem A that the max-plus equations in (1.4) play a central role in determining the functions ψ_n and thus the limiting profile Ω. We shall see later in this section how the two equations in (1.4) can be combined with the aid of the periodicity relation in (3.3) to obtain a closed system. In particular this is done in Theorem B, where r is monotone. In the closed system the unknown quantity p, the period of Ω, plays the role of an additive eigenvalue.

Remark. In many cases we have in Theorem A that $\delta_0 = \mu$ and $\delta_1 = \nu$, and so both the max-plus equations (1.4) hold throughout the domain $[-\nu,\mu]\backslash\{0\}$ of the functions ψ_n. More generally, in Proposition 6.2 we shall provide a number of necessary conditions which the quantities δ_0 and δ_1 must satisfy, from which precise information about them may be deduced for specific systems.

Remark. The quantities μ, ν, and p are central players in our theory, and one sees that

$$\mu = \lim_{k\to\infty} \mu^k, \qquad \mu^k = \max_{t\in\mathbb{R}} x^k(t),$$
$$\nu = \lim_{k\to\infty} \nu^k, \qquad -\nu^k = \min_{t\in\mathbb{R}} x^k(t).$$

When the conclusion (1.14) of Theorem 1.2 holds then either $\mu > 0$ or $\nu > 0$ (in fact both inequalities must hold if one of them does, by Theorem A), while if (1.14) is false then $\mu = \nu = 0$. The set Ω is seen to be periodic in the horizontal direction with period p. More precisely we have that

$$\Omega = \Omega + (p, 0), \tag{3.9}$$

where this equation is interpreted as a translation by the vector $(p,0)\in\mathbb{R}^2$. The quantity p is in fact the limit of the periods $p^k > 2$ of the solutions x^k, and in particular $p \geqslant 2$ must hold.

Remark. The jump in the function ψ_n at $\xi = 0$ equals $\psi_n(0+) - \psi_n(0-) = (-1)^n L_A$, by (3.5). If $L_A > 0$ and so each ψ_n is discontinuous there, then strictly speaking one should write "sup" rather than "max" in the max-plus equations. However, we allow ourselves this slight abuse of notation.

Remark. If $\mu > 0$ and $\nu > 0$ then $g^2(\mu) = \mu$ if and only if $g^2(-\nu) = -\nu$, and if so then $\mu = C$ and $\nu = D$. This follows directly from (1.7).

Theorem A by itself does not generally provide sufficient information to completely determine the set Ω. For example, it does not give any direct indication of the values of μ, ν, or p, and it does not make claims about the set of all solutions of the system (1.4) of max-plus equations. Indeed, Theorem A leaves open the possibility of a trivial limit $x^k \to 0$ where Ω is the horizontal axis (although Theorem 1.2 often rules this out). In many cases we shall be able to determine μ, ν, and p through further arguments, but even without these arguments much information can be gleaned.

Assuming that Ω is not equal to the horizontal axis, we see that Ω consists of the sets graph(ψ_n) in sequential order, with horizontal line segments $B_n \times \{\lambda_n\}$ between the endpoints of graph(ψ_n) and graph(ψ_{n+1}). (Of course if $L_B = 0$ then these line segments are absent and graph(ψ_n) and graph(ψ_{n+1}) touch at their endpoints.) In addition to these horizontal line segments, if $L_A > 0$ then each set graph(ψ_n) contains the horizontal line segment $A_n \times \{0\}$.

If for some n the function ψ_n is constant in an interval $J \subseteq [-\nu, \mu]$ then Ω contains the vertical line segment $\{\tau\} \times J \subseteq \text{graph}(\psi_n)$, where $\tau = \psi_n(\xi)$ is the constant value in J. In this case Ω, which is the limit of the graphs Γ^k in (1.17), is not itself the graph of a function of τ. Such a vertical line segment typically arises as an internal transition layer due to the singular perturbation nature of Eq. (1.1). In the special case of (1.15) described in Section 1 we saw (1.19) that graph(ψ_{2m+1}) consisted entirely of such a vertical segment.

In view of the periodicity (3.3) of ψ_n in n, we are only dealing with two functions modulo additive constants, namely ψ_n for even n and ψ_n for odd n. Let us denote

$$E_0 = \psi_{2m}(\mu) - \psi_{2m}(-\nu), \qquad E_1 = \psi_{2m+1}(-\nu) - \psi_{2m+1}(\mu), \tag{3.10}$$

which are nonnegative quantities measuring the horizontal extent of the sets graph(ψ_n), where we note the monotonicity of ψ_n. Then the period p of Ω is easily seen to be

$$p = E_0 + E_1 + 2L_B, \tag{3.11}$$

corresponding to one ascending function ψ_{2m} and one descending function ψ_{2m+1}, together with two horizontal line segments $B_{2m} \times \{\mu\}$ and $B_{2m+1} \times \{-\nu\}$ at the top and bottom of Ω. Formula (3.11) links the quantities μ, ν, and p, at least implicitly, and is a step toward determining these quantities explicitly.

An important case occurs when the delay function r is monotone, as described in Section 1, and here much more detailed information can be obtained. In particular both the max-plus equations (1.4) hold throughout $[-\nu, \mu]\backslash\{0\}$ and one may easily combine them, along with the periodicity relation in (3.3), to obtain a single max-plus equation involving a function and a quantity p. It is enough here to consider only the case of monotone increasing r, as the case of monotone decreasing r reduces to this under the change of variables $x \to -x$ in the original equation (1.1).

Theorem B. *Assume that the function r is monotone increasing throughout the interval* $[-D, C]$. *Assume also that either* $\mu > 0$ *or* $\nu > 0$ (*which, if* $r'(0) > 0$, *is necessarily the case, by Theorem* 1.2). *Then both* $\mu > 0$ *and* $\nu > 0$, *and both equations in* (1.4) *hold throughout* $[-\nu, \mu]\backslash\{0\}$. *We also have that*

$$p + \psi_{2m+1}(\xi) = \max_{\xi \leqslant s \leqslant \mu} (h(s) + \psi_{2m+1}(g^2(s))) \tag{3.12}$$

in $[-\nu, \mu]\backslash\{0\}$, *or in* $[-\nu, \mu]$ *if* $L_A = 0$, *where the function h is given by* (1.16), *and that*

$$\psi_{2m}(\xi) = r(\xi) + \psi_{2m-1}(g(\xi)) \tag{3.13}$$

in $[-\nu, \mu]$. *Furthermore*

$$\psi_{2m+1}(\xi) \text{ is constant for } \xi \in [-\nu, 0), \tag{3.14}$$

and if $p > 2$ *then* $\psi_{2m+1}(\xi)$ *is constant in* $[-\nu, \delta]$ *for some* $\delta > 0$. *The period p of* Ω *is given by*

$$p = \max_{0 \leqslant \xi \leqslant \mu} h(\xi) = \max_{-\nu \leqslant \xi \leqslant \mu} h(\xi) \tag{3.15}$$

and if $r'(0)$ *exists with* $r'(0) > 0$ *then* $p > 2$, *and* $\mu > 0$ *and* $\nu > 0$. *Finally, we have that*

$$g(\mu) = -\nu \tag{3.16}$$

if it is the case that r is strictly increasing in $[-D, 0]$ *in addition to being monotone increasing in* $[-D, C]$.

Some of the conclusions of Theorem B are seen to follow directly from Theorem A and Theorem 1.2. In particular, one obtains Eq. (3.13) directly from the first equation in (1.4) as r is monotone increasing and both ψ_{2m-1} and g are monotone decreasing. Upon substituting Eq. (3.13) into the second equation in (1.4) and using periodicity (3.3) we obtain Eq. (3.12). What is notable and is not obvious about Eqs. (3.12) and (3.13) in Theorem B is that they hold throughout $[-\nu, \mu]\backslash\{0\}$, and as well at $\xi = 0$ if $L_A = 0$. That is, $\delta_0 = \mu$ and $\delta_1 = \nu$ in the notation of Theorem A.

Eq. (3.12) is a so-called *max-plus eigenproblem*, in which the quantity p is regarded as an additive eigenvalue. The difficulty presented by Eq. (3.12) is due to its nonlocal character. More precisely, the right-hand side of this equation depends on the values of ψ_{2m+1} in the interval $[g^2(\xi), g^2(\mu)]$, and this interval may contain ξ in its interior. Later in this section we describe results from the companion paper [37] which we use here to analyze this problem.

If $r'(0) > 0$ then one has $h'(0) > 0$ since $|g'(0)| < 1$, and with (3.15) this gives $p > 2$. Also $\mu > 0$ and $\nu > 0$ by Theorem 1.2 in this case. If $p > 2$ then $L_A = 0$ and each ψ_n is continuous throughout $[-\nu, \mu]$, by Theorem A.

If r is monotone increasing in $[-D, C]$ and is strictly increasing in $[-D, 0]$, and $r'(0) > 0$, then an examination of the statement of Theorem B shows that Ω is completely determined once μ and $\psi_{2m+1}(\xi)$ for $\xi \in [0, \mu]$ are known. Indeed, ν is given by (3.16), we have $\psi_{2m+1}(\xi) = \psi_{2m+1}(0)$ throughout $[-\nu, 0]$, and ψ_{2m} is given by (3.13). The functions ψ_{2m+1} for different m are related by (3.3), as are the functions ψ_{2m}. The horizontal segments $B_n \times \{\lambda_n\}$ in Ω are given by (3.5) and (3.6), and $A_n = \{\psi_n(0)\}$ is a single point as $p > 2$ implies $L_A = 0$, by Theorem A.

The next result shows that under an additional condition, namely that (h, g^2) is a so-called quasimodal pair, both μ and ψ_{2m+1} are uniquely determined and thus the limiting profile Ω is uniquely determined. Moreover, explicit characterizations of μ and ψ_{2m+1}, and thus of Ω, are given. We remark that the hypotheses of Theorem C include conditions (1.15) from the example given in Section 1. As described later, if h is monotone increasing in $[0, C]$ with $h'(0) > 0$ then the pair (h, g^2) is quasimodal.

Theorem C. *Assume that the function r is monotone increasing throughout the interval* $[-D, C]$ *and strictly increasing in* $[-D, 0]$. *Also assume that* $r'(0)$ *exists and* $r'(0) > 0$. *Finally, assume that the pair* (h, g^2) *is quasimodal and let* $\psi_* : [0, C] \to [-\infty, \infty)$ *denote the unique continuous solution to the max-plus equation*

$$\begin{aligned} P_* + \psi_*(\xi) &= \max_{\xi \leqslant s \leqslant C} (h(s) + \psi_*(g^2(s))), \quad \xi \in [0, C], \\ P_* &= \max_{0 \leqslant \xi \leqslant C} h(\xi), \end{aligned} \tag{3.17}$$

normalized so that $\psi_*(0) = 0$. *Define a function* $F : [0, C] \to [-\infty, \infty)$ *by*

$$F(M) = \psi_*(g^2(M)) + r(g(M)). \tag{3.18}$$

Then the unique solution of the problem

$$F(M) \in \mathscr{G}(M), \qquad \textit{where } \mathscr{G}(M) = \begin{cases} \{0\}, & M \in (0, C), \\ [0, \infty), & M = C, \end{cases} \tag{3.19}$$

is the quantity $M = \mu$ *in Theorems A and B, and it is the case that*

$$F(\mu) = L_B. \tag{3.20}$$

Additionally, one has that

$$\psi_{2m+1}(\xi) - \psi_{2m+1}(0) = \begin{cases} \psi_*(\xi), & \xi \in [0, g^2(\mu)], \\ \max\limits_{\xi \leqslant s \leqslant \mu} (h(s) - p + \psi_*(g^2(s))), & \xi \in [g^2(\mu), \mu], \end{cases} \tag{3.21}$$

where

$$\psi_{2m+1}(0) = (m+1)p - 1, \tag{3.22}$$

with p *in* (3.21) *the same quantity as in* (3.15).

It is necessary to make precise certain notions in the statement of Theorem C and also to justify some of the incidental claims in the statement. This, along with the proof of Theorem C, relies on an understanding of the max-plus eigenproblem (3.12). Quite generally max-plus eigenproblems are analogous to linear Fredholm equations of the form

$$\lambda x(\xi) = \int_{\alpha(\xi)}^{\beta(\xi)} h(\xi, s) x(s)\, ds,$$

in which multiplication is replaced with addition, and addition (and integration) replaced with maximization. A detailed study of a general class of max-plus eigenproblems is found in [36]. In the companion paper [37] we obtained a representation of the general solution of a class of max-plus eigenproblems including (3.12). As we make essential use of the results of [37], we provide here a brief exposition of them.

In [37] problems of the form

$$P + \psi(\xi) = \max_{\xi \leqslant s \leqslant M} (H(s) + \psi(\gamma(s))), \qquad \xi \in [0, M], \tag{3.23}$$

modeled on Eq. (3.12), were considered. Here P, M, H, and γ correspond to p, μ, h, and g^2, respectively. It was assumed in [37] that

$$H : [0, C] \to \mathbb{R}, \qquad \gamma : [0, C] \to [0, C]$$

are given continuous functions where

$$\gamma(\xi) \text{ is strictly increasing in } \xi \in [0, C], \text{ and}$$
$$\gamma(\xi) < \xi \text{ for every } \xi \in (0, C),$$

and where the quantity $M \in (0, C]$ is treated as a parameter. Note that necessarily $\gamma(0) = 0$, but that either $\gamma(C) = C$ or $\gamma(C) < C$ may hold. Solutions $\psi : [0, M] \to [-\infty, \infty)$ are permitted to take the value $-\infty$; however, the value $+\infty$ is not allowed (in this spirit the constant function $\psi(\xi) \equiv -\infty$ is to be regarded as the trivial solution). Also, ψ is required to be continuous, where $[-\infty, \infty)$ is endowed with the standard topology in which the sets $[-\infty, \xi)$ form a neighborhood basis for $-\infty$. The additive eigenvalue $P \in \mathbb{R}$ is required to be finite. It was shown [37, Proposition 2.1] that the only point at which a nontrivial solution could fail to be finite, namely $\psi(\xi) = -\infty$, was at $\xi = M = C = \gamma(C)$. This necessary condition is also sufficient for $\psi(\xi) = -\infty$ if $H(C) < \max_{0 \leqslant s \leqslant C} H(s)$.

The main result of [37], the *Basis Theorem*, is that under additional mild conditions, for every M there exists a finite collection $\{\varphi^i\}_{i=1}^{q}$ of canonically defined solutions to (3.23), such that the general solution of (3.23) has the form

$$\psi(\xi) = (c^1 + \varphi^1(\xi)) \vee (c^2 + \varphi^2(\xi)) \vee \cdots (c^q + \varphi^q(\xi)), \tag{3.24}$$

where $c^i \in [-\infty, \infty)$ are any quantities. Also, the additive eigenvalue P is the same quantity

$$P = \max_{0 \leqslant \xi \leqslant M} H(\xi) \tag{3.25}$$

for every nontrivial solution, although it depends on M. It is easily seen that if φ^i are solutions to (3.23) then so is the right-hand side of (3.24) for any c^i. Indeed, formula (3.24) is the analog of a linear combination of solutions, wherein multiplication and addition are replaced by addition and maximization, respectively. In general the quantity q and the basis solutions φ^i depend on M just as does P.

The precise conditions for the Basis Theorem to hold for a given M are encoded in the set

$$Z(M) = \{\zeta \in [0, M] \mid H(\zeta) = P(M), \text{ and } H(\xi) < P(M)$$
$$\text{whenever } \gamma(\zeta) < \gamma(\xi) \leqslant \zeta \text{ and } \xi \in [0, M]\},$$

where $P(M)$ is the quantity (3.25). If $Z(M)$ is a finite set, and if also $0 \notin Z(M)$, then the above conclusion of the Basis Theorem holds and the quantity $q = q(M)$ equals the cardinality of $Z(M)$.

Note that $Z(M) \neq \phi$ for every $M \in (0, C]$ as $Z(M)$ contains the rightmost maximum of H in $[0, M]$. Also note that if $H'(0) > 0$ then $0 \notin Z(M)$ for every M, as $H(0) < P(M)$. This corresponds to $h'(0) > 0$ in (3.12), which holds if $r'(0) > 0$. It was noted in [37] that among C^2 smooth functions H for which $H'(0) > 0$ it is generically the case that $Z(M)$ is a finite set for every γ, and in fact that $1 \leqslant q(M) \leqslant 2$ for every M with $q(M) = 2$ holding only for finitely many M.

A particularly interesting case occurs when $q(M) = 1$, that is $Z(M)$ is a singleton, for every M. In this direction we present the following definition which was introduced in [37].

Definition. The pair (H, γ) is said to be *quasimodal* if for every $M \in (0, C]$ the set $Z(M)$ contains exactly one element, and also $0 \notin Z(M)$.

As noted in [37], if the function H is monotone increasing in $[0, M_0]$ for some $M_0 \in (0, C]$, and $H(\xi) < H(M_0)$ for every $\xi \in (M_0, C]$, and also $H(\xi) > H(0)$ for every $\xi \in (0, M_0]$, then (H, γ) is quasimodal for any γ. This includes the case in which H is monotone increasing throughout $[0, C]$, where here $M_0 = C$, and also the so-called unimodal case in which H is monotone increasing to the left of a maximum at $M_0 \in (0, C)$ and then monotone decreasing to the right of M_0, where in both cases we need $H(\xi) > H(0)$ for ξ near 0. However, there exist quasimodal pairs (H, γ) for which H is neither monotone nor unimodal. Let us remark also that if (H, γ) is quasimodal then the unique element $\zeta(M) \in Z(M)$ need not depend continuously on M.

In the quasimodal case Eq. (3.23) has, for every M, a unique nontrivial solution up to an additive constant, by the Basis Theorem. Let us denote this solution by $\varphi(\xi, M)$, which we may assume normalized so that $\varphi(0, M) = 0$. Then it was shown in [37, Theorem 4.1] that these solutions are related by

$$\varphi(\xi, M) = \begin{cases} \varphi(\xi, C), & \xi \in [0, \gamma(M)], \\ \max\limits_{\xi \leqslant s \leqslant M} (H(s) - P(M) + \varphi(\gamma(s), C)) \leqslant \varphi(\xi, C), & \xi \in [\gamma(M), M], \end{cases} \tag{3.26}$$

as M varies. Also, in the unimodal case it follows from an observation in [37] that

$$\varphi(\xi, M) = \sum_{n=0}^{\infty} A_*(\gamma^n(\xi)) \quad \text{for every } \xi \in [0, M],$$

where

$$A_*(\xi) = \begin{cases} 0, & \xi \in [0, M_0], \\ H(\xi) - H(M_0), & \xi \in [M_0, C]. \end{cases}$$

Here $M_0 \in (0, C]$ is the location of the rightmost maximum of H.

Following the above discussion it is now quite easy to see how Theorem C follows from Theorems A and B and from the results of [37].

Proof of Theorem C. The existence and uniqueness of $\psi_* = \varphi(\cdot, C)$ in the statement of Theorem C follows from the Basis Theorem of [37] described above, using the assumption that (h, g^2) is quasimodal. The above discussion also gives Eq. (3.21), which is just Eq. (3.26) rewritten in this case. Formula (3.22) for $\psi_{2m+1}(0)$ follows directly from the periodicity condition in (3.3), from Eq. (3.8) in Theorem A, and from (3.13) in Theorem B. Note that as $r'(0) > 0$ we have that $p > 2$ by Theorem B, hence $L_A = 0$ by Theorem A, which is needed for (3.8).

Consider now the quantities E_0 and E_1 given by (3.10). Let us eliminate both ψ_{2m} and $-v$ from these formulas by making the substitutions (3.13) and (3.16) and using again periodicity (3.3). With a short calculation we obtain

$$E_0 = r(\mu) + \psi_{2m+1}(g(\mu)) - r(g(\mu)) - \psi_{2m+1}(g^2(\mu)),$$

$$\begin{aligned} E_1 &= \psi_{2m+1}(g(\mu)) - \psi_{2m+1}(\mu) \\ &= \psi_{2m+1}(g(\mu)) + p - h(\mu) - \psi_{2m+1}(g^2(\mu)), \end{aligned}$$

where in the case of E_1 we have made an additional substitution $\psi_{2m+1}(\mu) = -p + h(\mu) + \psi_{2m+1}(g^2(\mu))$, which is just Eq. (3.12) at $\xi = \mu$. We obtain further, using the fact that $\psi_{2m+1}(g(\mu)) = \psi_{2m+1}(0)$ by Theorem B, and using the formula (1.16) for h, that

$$\begin{aligned} E_0 + E_1 &= 2\psi_{2m+1}(0) - 2\psi_{2m+1}(g^2(\mu)) + r(\mu) - r(g(\mu)) + p - h(\mu) \\ &= 2\psi_{2m+1}(0) - 2\psi_{2m+1}(g^2(\mu)) - 2r(g(\mu)) + p \\ &= -2F(\mu) + p, \end{aligned} \tag{3.27}$$

where $F(\mu)$ is as in (3.18) with (3.21) used here. Upon substituting (3.27) into Eq. (3.11) we obtain Eq. (3.20). If $\mu < C$ then $g^2(\mu) < \mu$ and so $L_B = 0$ by Theorem A, while if $\mu = C$ then $L_B \geqslant 0$. In either case $L_B \in \mathcal{G}(\mu)$ and so (3.19) holds.

Finally, we observe that the function F is strictly decreasing, in particular because r is strictly increasing in $[-D, 0]$ and ψ_* is monotone decreasing. As $F(0) = 1$ it follows that the problem (3.19) has a unique solution, which is $M = \mu$. □

Before closing this section let us return to the general context of Theorem A. Just as was done in Theorem B one may combine the two equations in (1.4) to form a max-plus eigenproblem with an unknown parameter p. Upon substituting the first equation of (1.4) into the second we obtain

$$\begin{aligned} \psi_{2m+1}(\xi) &= \max_{\xi \leqslant s \leqslant \mu} \left(r(s) + \max_{-v \leqslant t \leqslant g(s)} (r(t) + \psi_{2m-1}(g(t))) \right) \\ &= \max_{\xi \leqslant s \leqslant \mu} \left(r(s) + \max_{g^2(s) \leqslant \tau \leqslant g(-v)} (r(g^{-1}(\tau)) + \psi_{2m-1}(\tau)) \right) \\ &= \max_{\xi \leqslant s \leqslant \mu} \left(r(s) + \max_{g^2(s) \leqslant \tau \leqslant \mu} (r(g^{-1}(\tau)) + \psi_{2m-1}(\tau)) \right) \\ &= \max_{g^2(\xi) \leqslant \tau \leqslant \mu} \left(\left(\max_{\xi \leqslant s \leqslant g^{-2}(\tau)} r(s) \right) + r(g^{-1}(\tau)) + \psi_{2m-1}(\tau) \right). \end{aligned}$$

Here we adopt the convention that $g^{-1}(\tau) = \mu$ if $-v \leqslant \tau < g(\mu)$, and that $g^{-1}(\tau) = -v$ if $g(-v) < \tau \leqslant \mu$, with a similar convention for $g^{-2}(\tau)$. In particular this is used in the third line of the above formula, where we also use the monotonicity of ψ_{2m-1}. Now using (3.3) the above formula becomes

$$p + \psi_{2m+1}(\xi) = \max_{g^2(\xi) \leqslant \tau \leqslant g(-v)} (h_1(\xi, \tau) + \psi_{2m+1}(\tau)), \tag{3.28}$$

where

$$h_1(\xi, \tau) = \left(\max_{\xi \leqslant s \leqslant g^{-2}(\tau)} r(s) \right) + r(g^{-1}(\tau)). \tag{3.29}$$

Note that (3.28) reduces to the max-plus eigenproblem (3.12) of Theorem B when r is monotone increasing. One obtains analogously the equation

$$p + \psi_{2m}(\xi) = \max_{g(\mu) \leqslant \tau \leqslant g^2(\xi)} (h_0(\xi, \tau) + \psi_{2m}(\tau)), \tag{3.30}$$

where

$$h_0(\xi, \tau) = \left(\max_{g^{-2}(\tau) \leqslant s \leqslant \xi} r(s) \right) + r(g^{-1}(\tau)) \tag{3.31}$$

for the even-indexed functions. The range of ξ for which (3.28) is valid depends on the quantities δ_0 and δ_1 in Theorem A. Generally, (3.28) holds if both $-v \leqslant g(\xi) \leqslant \delta_0$ and $-\delta_1 \leqslant \xi \leqslant \mu$ hold, that is, provided

$$\max\{g^{-1}(\delta_0), -\delta_1\} \leqslant \xi \leqslant \mu, \tag{3.32}$$

with $\xi \neq 0$ unless $L_A = 0$. In any case, as $g^{-1}(\delta_0)$ and $-\delta_1$ are both negative Eq. (3.28) is valid at least for $\xi \in (0, \mu]$. Similar remarks apply to Eq. (3.30).

4. The limiting profile Ω

In this section we recall the basic elements of the theory of "limiting profiles" developed in [35]. This theory was developed specifically to analyze problems of the form (1.1), in particular to make rigorous the connection between the differential equation (1.1) and the relation (1.3). It will be the main tool we use to analyze solutions of (1.1) and prove the results of Section 3.

We consider a sequence $x^k : \mathbb{R} \to \mathbb{R}$ of solutions of Eq. (1.1), with positive parameter values $\varepsilon = \varepsilon^k \to 0$, each of these solutions satisfying the bounds (1.12) in $\mathbb{R}$ with C and D independent of k. Here and for the remainder of this section we do not specifically assume the standing assumptions on f and r given in Section 1, as our purpose here is to describe the general machinery of [35] needed to solve our problem. In this section we assume only that $f : I \times I \to \mathbb{R}$ and $r : I \to \mathbb{R}$ are continuous, with $I=[-D,C]$, and that the solutions x^k satisfy (1.1) and (1.12) in $\mathbb{R}$ (so no assumption of periodicity or slow oscillation of these solutions is made).

We assume the sequence x^k is a so-called *regular sequence* [35]. That is, the graphs Γ^k in (1.17) converge to a limiting set $\Omega\subseteq\mathbb{R}^2$ in the following sense: There exists a nested sequence of compact sets

$$K^1\subseteq K^2\subseteq\cdots\subseteq\mathbb{R}^2 \quad \text{with} \quad \bigcup_{j=1}^{\infty}\operatorname{int}(K^j)=\mathbb{R}^2$$

such that for each j the limit

$$\lim_{k\to\infty}\Gamma^k\cap K^j=G^j$$

exists in the Hausdorff topology of compact sets, and for which

$$\Omega=\bigcup_{j=1}^{\infty}G^j.$$

There is no loss in making this assumption, since as shown in [35], every uniformly bounded sequence of solutions x^k possesses a regular subsequence. For a regular sequence x^k the limiting set Ω can be characterized as

$$\begin{aligned}\Omega&=\{(\tau,\xi)\in\mathbb{R}^2\mid \text{there exist } t^{k'}\to\tau \text{ with } x^{k'}(t^{k'})\to\xi, \text{ for some subsequence } k'\to\infty\}\\&=\{(\tau,\xi)\in\mathbb{R}^2\mid \text{there exist } t^{k}\to\tau \text{ with } x^{k}(t^{k})\to\xi\},\end{aligned}\tag{4.1}$$

where the equality of the two sets above follows from the regularity of the sequence. The set Ω is called the *limiting profile* of the sequence x^k. Typically Ω is not itself a graph as it can contain vertical line segments, although it inherits certain connectedness properties from the graphs Γ^k as described below.

The limiting profile Ω for a sequence of SOPSs is the central object of study, and for a nontrivial class of equations we shall show that it is uniquely and explicitly determined by the nonlinearities f and r. In particular, the uniqueness of Ω will imply, a posteriori, that any sequence x^k of SOPSs satisfying (1.1) and (1.12) with $\varepsilon^k\to0$ is already regular, and so converges to Ω.

We recall from [35] the main properties of Ω and the features of the theory developed therein. Let us denote in general

$$\Omega(S)=\Omega\cap V_S,$$

the intersection of Ω with the vertical strip over the set $S\subseteq\mathbb{R}$. In case $S=(\tau_1,\tau_2)$ or $S=[\tau_1,\tau_2]$ is an interval we write simply $\Omega(\tau_1,\tau_2)$ or $\Omega[\tau_1,\tau_2]$, and we write $\Omega(\tau)$ if $S=\{\tau\}$ is a singleton. Then if J is an interval the set $\Omega(J)$ is nonempty and connected. The connectedness in particular is a consequence of the fact that x^k is a regular sequence. Of course Ω is a closed set and is contained in the horizontal strip

H_I, by (1.12), so $\Omega(J)$ is closed (compact) if J is closed (compact). If $J = \{\tau\}$ is a point, then $\Omega(J)$ is simply a vertical compact interval (or point) in the plane. We denote the set

$$\overline{\underline{x}}(\tau) = \{\xi \in \mathbb{R} \mid (\tau, \xi) \in \Omega\}$$

for every τ, and so $\Omega(\tau) = \{\tau\} \times \overline{\underline{x}}(\tau)$. We also define $\underline{x}(\tau)$ and $\bar{x}(\tau)$ to be the endpoints of the compact interval $\overline{\underline{x}}(\tau)$, that is,

$$\overline{\underline{x}}(\tau) = [\underline{x}(\tau), \bar{x}(\tau)].$$

The functions $\underline{x}$ and $\bar{x}$ are lower and upper semi-continuous, respectively; this is an immediate consequence of the closedness of Ω.

Now we describe the precise relation between the set Ω and the difference equation (1.3) which was proved in [35] and which is at the heart of our theory. We begin with a fundamental decomposition

$$\Omega = \Omega^+ \cup \Omega^- \cup \Omega^*$$

of the limiting profile into three disjoint subsets. The sets Ω^+ and Ω^- correspond to the transition layers (classically known as the inner solution in singular perturbation theory), while Ω^* corresponds to the regular part of the solution, that is, the outer solution. The precise definitions of these sets are

$$\begin{aligned} \Omega^{\pm} = \Big\{ (\tau, \xi) \Big| \liminf_{k \to \infty} \ (\pm \varepsilon^k \dot{x}^k(t^k)) > 0 \quad &\text{for every sequence } t^k \to \tau \\ &\text{with } x^k(t^k) \to \xi \Big\}, \\ \Omega^* = \Omega \backslash (\Omega^+ \cup \Omega^-). \end{aligned} \tag{4.2}$$

Clearly, these three sets are disjoint. Note particularly that a full sequence, not just a subsequence, is required in the definitions of $\Omega^{\pm}$. The sets Ω^+ and Ω^- are locally like vertical lines in the following sense. If $(\tau, \xi) \in \Omega^{\pm}$ for some choice $+$ or $-$ of $\pm$, then there exists a neighborhood $U \subseteq \mathbb{R}^2$ of (τ, ξ) such that

$$\Omega \cap U = \Omega^{\pm} \cap U = V_\tau \cap U. \tag{4.3}$$

That is, in a neighborhood of (τ, ξ) the set $\Omega^{\pm}$ is simply the vertical line through τ, and with no other points of the larger set Ω present. Thus Ω^+ and Ω^- each are relatively open subsets of the closed set Ω, and hence Ω^* is closed. Also, from these facts it follows that both inequalities $\underline{x}(\tau) < \xi < \bar{x}(\tau)$ hold if $(\tau, \xi) \in \Omega^+ \cup \Omega^-$. Thus for every $\tau \in \mathbb{R}$ we have that

$$(\tau, \underline{x}(\tau)), \ \ (\tau, \bar{x}(\tau)) \in \Omega^* \tag{4.4}$$

for the endpoints of $\Omega(\tau)$.

It is also the case that the vertical lines which comprise Ω^+ are "upward" lines in the following sense. If $(\tau, \xi) \in \Omega^+$, then there are exactly two connected components Ω^L and Ω^R of $\Omega \backslash \{(\tau, \xi)\}$, and they are given by

$$\begin{aligned} \Omega^L &= \Omega(-\infty, \tau) \cup (\{\tau\} \times [\underline{x}(\tau), \xi)), \\ \Omega^R &= \Omega(\tau, \infty) \cup (\{\tau\} \times (\xi, \bar{x}(\tau)]). \end{aligned} \tag{4.5}$$

On the other hand, if $(\tau, \xi) \in \Omega^-$, then

$$\begin{aligned} \Omega^L &= \Omega(-\infty, \tau) \cup (\{\tau\} \times (\xi, \bar{x}(\tau)]), \\ \Omega^R &= \Omega(\tau, \infty) \cup (\{\tau\} \times [\underline{x}(\tau), \xi)) \end{aligned} \tag{4.6}$$

are the corresponding connected components, that is, the vertical lines of Ω^- go "downward".

One intuitively thinks of the set Ω^* as corresponding to those points at which one is justified in setting $\varepsilon = 0$ in the differential equation (1.1), thereby obtaining the relation

$$0 = f(x(t), x(t-r)), \qquad r = r(x(t)),$$

between the current part $(t, x(t))$ and the history part $(t-r, x(t-r))$ of the solution. At points of $\Omega^\pm$ the idea is that one instead obtains an inequality

$$\pm f(x(t), x(t-r)) > 0.$$

The precise results, proved in [35], are as follows. For every $(\tau, \xi) \in \Omega^*$, there exists $(\tilde{\tau}, \tilde{\xi}) \in \Omega$ such that

$$f(\xi, \tilde{\xi}) = 0, \qquad \tilde{\tau} = \tau - r(\xi). \tag{4.7}$$

For every $(\tau, \xi) \in \Omega^\pm$, for some choice $+$ or $-$ of $\pm$, there exists $(\tilde{\tau}, \tilde{\xi}) \in \Omega$ such that

$$\pm f(\xi, \tilde{\xi}) > 0, \qquad \tilde{\tau} = \tau - r(\xi). \tag{4.8}$$

Note that if, as in Section 1, we have that $\operatorname{sgn} f(x,y) = \operatorname{sgn}(g(x) - y)$ for $(x,y) \in I \times I$, for some function $g : I \to I$ and if $\Phi : H_I \to H_I$ is the backdating map given in (1.10), then the statements above can be reformulated as follows. We have first that

$$\Phi(\Omega^*) \subseteq \Omega, \tag{4.9}$$

which expresses Eq. (4.7). The relation (4.8), in which the inequality is equivalent to $\pm(g(\xi) - \tilde{\xi}) > 0$, is expressed by the fact that

$$\begin{aligned} &\text{if } (\tau, \xi) \in \Omega^\pm \text{ and } (\bar{\tau}, \bar{\xi}) = \Phi(\tau, \xi), \text{ then there exists} \\ &(\tilde{\tau}, \tilde{\xi}) \in \Omega \text{ with } \bar{\tau} = \tilde{\tau} \text{ and } \pm(\bar{\xi} - \tilde{\xi}) > 0. \end{aligned} \tag{4.10}$$

Roughly, (4.10) says that under the map Φ every point in the image $\Phi(\Omega^+)$ of Ω^+ lies above some point of Ω, and every point of $\Phi(\Omega^-)$ lies below some point of Ω. In making the heuristic connection of this to the differential equation (1.1), one should identify (τ, ξ) with $(t, x(t))$ and $(\tilde{\tau}, \tilde{\xi})$ with $(t-r, x(t-r))$ where $r = r(x(t))$. Properties (4.9) and (4.10) will be used extensively in the rest of this paper, and we shall often refer to them as the *mapping properties* of Φ.

5. The limiting profile for slowly oscillating periodic solutions

Following the standing assumptions of Section 1, we have by Theorem 1.1 that for every ε in some range $0<\varepsilon<\varepsilon_*$ Eq. (1.1) possesses an SOPS satisfying the bounds (1.12) in $\mathbb{R}$. In order to apply the theory of limiting profiles described in the previous section we take a sequence $\varepsilon^k \to 0$ of positive parameters and a corresponding sequence x^k of SOPSs satisfying (1.12) which form a regular sequence. We assume for the remainder of this section that such a sequence has been fixed and we let $\Omega \subseteq \mathbb{R}^2$ denote the associated limiting profile.

The focus in this section is on obtaining properties of Ω which follow directly from the properties of slow oscillation and associated monotonicity properties of the solutions. Each solution x^k has a sine-like shape in the sense that it satisfies (1.2) and (1.13) for some quantities $0<\beta_0^k<\alpha_1^k<\beta_1^k<\alpha_2^k$. From the periodicity of x^k we may define α_n^k and β_n^k for every integer n by requiring that

$$\alpha_{n+2}^k = \alpha_n^k + p^k, \qquad \beta_{n+2}^k = \beta_n^k + p^k, \text{ where}$$
$$p^k = \alpha_2^k \text{ is the minimal period of } x^k.$$

Thus

$$\alpha_n^k < \beta_n^k < \alpha_{n+1}^k, \qquad \alpha_0^k = 0. \tag{5.1}$$

Each solution x^k is positive in the interval $(\alpha_{2m}^k, \alpha_{2m+1}^k)$ and is negative in $(\alpha_{2m+1}^k, \alpha_{2m+2}^k)$, and is monotone increasing in $[\beta_{2m-1}^k, \beta_{2m}^k]$ and monotone decreasing in $[\beta_{2m}^k, \beta_{2m+1}^k]$. The maximum and minimum μ^k and $-\nu^k$ of x^k are given, respectively, by

$$\mu^k = x^k(\beta_{2m}^k), \qquad -\nu^k = x^k(\beta_{2m+1}^k),$$

for every m.

Observe that $0 = \varepsilon^k \dot{x}^k(\beta_n^k) = f(x^k(\beta_n^k), x^k(\eta^k(\beta_n^k)))$, which implies by (1.5) that

$$x^k(\eta^k(\beta_n^k)) = g(x^k(\beta_n^k)) = \begin{cases} g(\mu^k), & n \text{ even}, \\ g(-\nu^k), & n \text{ odd}, \end{cases} \tag{5.2}$$

and hence that

$$(-1)^{n+1}x^k(\eta^k(\beta_n^k)))>0. \tag{5.3}$$

Here $\eta^k:\mathbb{R}\to\mathbb{R}$ is the history function associated to x^k, defined as

$$\eta^k(t)=t-r(x^k(t)). \tag{5.4}$$

We next observe that

$$\alpha_n^k-R<\beta_n^k-R\leqslant\eta^k(\beta_n^k)<\alpha_n^k \tag{5.5}$$

with R as in (1.9), where the final inequality in (5.5) follows from fact (5.3) that $x^k(\eta^k(\beta_n^k))$ and $x^k(\beta_n^k)$ have opposite signs. Let us also note from (5.2) that both $g(\mu^k)$ and $g(-\nu^k)$ belong to the range $[-\nu^k,\mu^k]$ of x^k, and so

$$\begin{aligned}&g([-\nu^k,\mu^k])\subseteq[-\nu^k,\mu^k],\\&\text{hence } g^2(\mu^k)\leqslant g(-\nu^k) \text{ and } g(\mu^k)\leqslant g^2(-\nu^k),\end{aligned} \tag{5.6}$$

from the monotonicity of g.

At this point it is convenient to prove Theorem 1.3, which establishes an upper bound on the periods of our solutions.

Proof of Theorem 1.3. We shall obtain explicit upper bounds

$$\alpha_1^k-\alpha_0^k=\alpha_1^k\leqslant P_0,\qquad \alpha_2^k-\alpha_1^k\leqslant P_1, \tag{5.7}$$

which together give the bound $p^k\leqslant P_0+P_1$ on the period. Here P_0 and P_1 will be independent of k. We shall only obtain the first bound in (5.7) as the derivation of the second is similar.

Let $K>0$ be such that

$$f(\xi,\zeta)\leqslant-K(\xi+\zeta),\qquad g^2(\xi)\geqslant K\xi, \tag{5.8}$$

hold for every $\xi,\zeta\in[0,C]$. Such K exists by virtue of conditions (1.5), (1.6), and (1.8) on f and g. We shall obtain the bound

$$\alpha_1^k-\alpha_0^k\leqslant 2R+\frac{\varepsilon^k}{K^2} \tag{5.9}$$

with R as in (1.9). Assume that $\alpha_1^k-\alpha_0^k>2R$ otherwise we are done. We consider t in the interval $[\alpha_0^k+R,\ \alpha_1^k]$, which by (5.5) lies to the right of β_0^k, and so $\dot{x}^k(t)\leqslant 0$ for such t. Thus in the smaller interval $[\alpha_0^k+R,\ \alpha_1^k-R]$ we have that

$$x^k(t)\geqslant x^k(\alpha_1^k-R)\geqslant x^k(\eta^k(\beta_1^k))=g(-\nu^k)\geqslant g^2(\mu^k)\geqslant K\mu^k \tag{5.10}$$

again by (5.5), and by (5.8), (5.2), and (5.6). We also have there that $x^k(\eta^k(t)) \geqslant 0$, and hence

$$\varepsilon^k \dot{x}^k(t) = f(x^k(t), x^k(\eta^k(t))) \leqslant -Kx^k(t) \leqslant -K^2\mu^k \tag{5.11}$$

by (5.8) and (5.10). In light of the estimate (5.11) in $[\alpha_0^k + R, \alpha_1^k - R]$ we have the upper bound

$$(\alpha_1^k - R) - (\alpha_0^k + R) \leqslant \frac{x^k(\alpha_0^k + R) - x^k(\alpha_1^k - R)}{(\varepsilon^k)^{-1} K^2 \mu^k} \leqslant \frac{\varepsilon^k x^k(\alpha_0^k + R)}{K^2 \mu^k} \leqslant \frac{\varepsilon^k}{K^2}$$

for the length of this interval, which gives (5.9) as desired. □

It will be convenient to assume (and we do without loss) the existence of the limits

$$\alpha_n^k \to \alpha_n, \qquad \beta_n^k \to \beta_n, \qquad p^k \to p,$$

for every integer n, as $k \to \infty$ (in fact the limit of the sequence p^k must exist as x^k is a regular sequence). Of course these limits are finite by Theorem 1.3, and it follows that

$$\begin{aligned} &\alpha_n \leqslant \beta_n \leqslant \alpha_{n+1}, \qquad \alpha_{n+1} - \alpha_n \geqslant 1, \\ &\alpha_{n+2} = \alpha_n + p, \qquad \beta_{n+2} = \beta_n + p, \end{aligned} \tag{5.12}$$

for every n, and that

$$\alpha_0 = 0, \qquad \alpha_2 = p \geqslant 2. \tag{5.13}$$

It is certainly possible for some of the inequalities in (5.12) to be equalities, and in fact this is often the case. We also observe the existence of the limits

$$\mu^k \to \mu, \qquad v^k \to v, \tag{5.14}$$

following from the regularity of the sequence x^k, and we recall Theorem 1.2 which gives conditions under which $\mu > 0$ and $v > 0$ is assured. Let us note that $-D \leqslant -v \leqslant 0 \leqslant \mu \leqslant C$, that $\Omega \subseteq H_{[-v,\mu]}$, and that

$$\phi \neq \Omega \cap H_{-v} \subseteq \Omega^*, \qquad \phi \neq \Omega \cap H_{\mu} \subseteq \Omega^*, \tag{5.15}$$

where the inclusions in (5.15) follow from (4.4). Upon taking the limits in (5.6) we obtain the claim (3.1) of Theorem A. Let us note here that (3.1) implies that

$$\mu = 0 \;\Leftrightarrow\; v = 0. \tag{5.16}$$

Indeed, if $\mu > 0$ then $-v \leqslant g(\mu) < 0$, hence $v > 0$, and similarly for the converse. Thus either the limiting profile Ω contains both points above and below the horizontal axis, or else $x^k \to 0$ uniformly as $k \to \infty$ and $\Omega = \mathbb{R} \times \{0\}$ is trivial.

The limiting profile Ω inherits properties corresponding to the periodicity and piecewise monotonicity of the solutions x^k. Certainly Ω has period p in the horizontal direction, that is, (3.9) holds. Now define sets $A_n, B_n \subseteq \mathbb{R}$ by

$$A_n = \{\tau \in \mathbb{R} \mid \text{there exist } t^{k'} \in [\beta_{n-1}^{k'}, \beta_n^{k'}] \text{ with } t^{k'} \to \tau \text{ and } x^{k'}(t^{k'}) \to 0 \\ \text{for some subsequence } k' \to \infty\}, \tag{5.17}$$

$$B_n = \{\tau \in \mathbb{R} \mid \text{there exist } t^{k'} \in [\alpha_n^{k'}, \alpha_{n+1}^{k'}] \text{ with } t^{k'} \to \tau \text{ and } x^{k'}(t^{k'}) \to \lambda_n \\ \text{for some subsequence } k' \to \infty\},$$

where λ_n is as in (3.6). It is immediate from the characterization (4.1) of Ω that A_n and B_n are nonempty compact sets and that

$$\begin{cases} (\tau, 0) \in \Omega \iff \tau \in \bigcup_{n=-\infty}^{\infty} A_n, \\ (\tau, \mu) \in \Omega \iff \tau \in \bigcup_{m=-\infty}^{\infty} B_{2m}, \\ (\tau, -\nu) \in \Omega \iff \tau \in \bigcup_{m=-\infty}^{\infty} B_{2m+1}. \end{cases} \tag{5.18}$$

In addition

$$\begin{array}{lll} \alpha_n \in A_n \subseteq [\beta_{n-1}, \beta_n], & A_n \times \{0\} \subseteq \Omega, & A_{n+2} = A_n + p, \\ \beta_n \in B_n \subseteq [\alpha_n, \alpha_{n+1}], & B_n \times \{\lambda_n\} \subseteq \Omega^*, & B_{n+2} = B_n + p, \end{array} \tag{5.19}$$

hold for every n, from the ordering (5.1) and from (5.15).

Remark. We could have chosen a slightly simpler definition of the sets A_n and B_n by taking full sequences, $(t^k, x^k(t^k)) \to (\tau, 0)$ or (τ, λ_n), rather than by taking subsequences. The sets $\tilde{A}_n$ and $\tilde{B}_n$ thereby obtained would be subsets of A_n and B_n. However, in light of the equality (4.1) it is easy to see that such sets can only differ at most at their endpoints, specifically, $A_n \backslash \tilde{A}_n \subseteq \{\beta_{n-1}, \beta_n\}$ and $B_n \backslash \tilde{B}_n \subseteq \{\alpha_n, \alpha_{n+1}\}$. We choose the definitions (5.17) as it is clear from them that these sets are closed.

It is in fact the case that the sets A_n and B_n are intervals, as the following result shows. In addition, each portion of the set Ω which projects to the interior of one of these intervals on the τ-axis is simply a line segment. The proof of these facts is not completely trivial: It not only uses the monotonicity properties of the solutions, but also the fact that x^k is a regular sequence.

Lemma 5.1. *The compact sets A_n and B_n defined in* (5.17) *are intervals, and moreover*

$$\Omega(\mathrm{int}(A_n)) = \mathrm{int}(A_n) \times \{0\}, \qquad \Omega(\mathrm{int}(B_n)) = \mathrm{int}(B_n) \times \{\lambda_n\}. \tag{5.20}$$

Denoting

$$A_n = [\alpha_n^-, \alpha_n^+], \qquad B_n = [\beta_n^-, \beta_n^+], \tag{5.21}$$

define intervals

$$P_n = [\beta_{n-1}^+, \beta_n^-], \qquad Q_n = [\beta_{n-1}^-, \beta_n^+] = B_{n-1} \cup P_n \cup B_n. \tag{5.22}$$

Then if $\mu > 0$ and $\nu > 0$ hold we have that

$$B_{n-1} \leqslant A_n \leqslant B_n, \qquad A_n \subseteq P_n \subseteq Q_n, \tag{5.23}$$

for every n.

Observe immediately that $A_{n-1} \leqslant A_n$ and $B_{n-1} \leqslant B_n$ hold, from the leftmost inclusions in (5.19). Thus P_n and Q_n in the above result are well defined with

$$B_{n-1} \leqslant P_n \leqslant B_n. \tag{5.24}$$

Indeed, the intervals B_{n-1} and P_n abut, having exactly one point in common, as do P_n and B_n. Note here the inequalities $\alpha_n^- \leqslant \alpha_n \leqslant \alpha_n^+$ and $\beta_n^- \leqslant \beta_n \leqslant \beta_n^+$ which follow from (5.19) and (5.21). Finally note that the requirements $\mu > 0$ and $\nu > 0$ are necessary for (5.23) to hold. If $\mu = \nu = 0$ then $A_n = [\beta_{n-1}, \beta_n]$ and $B_n = [\alpha_n, \alpha_{n+1}]$ for every n, and so (5.23) is impossible.

Proof of Lemma 5.1. First take any $\tau_1, \tau_2 \in A_n$ with $\tau_1 < \tau_2$, and let $t_1^{k'} \in [\beta_{n-1}^{k'}, \beta_n^{k'}]$ with $t_1^{k'} \to \tau_1$ and $x^{k'}(t_1^{k'}) \to 0$, and similarly $t_2^{k''} \in [\beta_{n-1}^{k''}, \beta_n^{k''}]$ with $t_2^{k''} \to \tau_2$ and $x^{k''}(t_2^{k''}) \to 0$, be subsequences as in the definition (5.17) for $\tau = \tau_1, \tau_2$. Fix any $\tau_0 \in (\tau_1, \tau_2)$ and let $k''' \to \infty$ be a third subsequence, with $t_0^{k'''} \to \tau_0$ and $x^{k'''}(t_0^{k'''}) \to \xi$ for some ξ. We shall prove that $\xi = 0$. Note this implies that $\tau_0 \in A_n$ hence A_n is an interval, and as well establishes the first equation of (5.20).

We may regard

$$(t_0^{k'''}, x^{k'''}(t_0^{k'''})) \to (\tau_0, \xi) \tag{5.25}$$

as a subsequence of points on the graphs $\Gamma^{k'''}$ converging to $(\tau_0, \xi) \in \Omega$. From the fact that x^k is a regular sequence, and in particular from the equality of the two sets in (4.1), the subsequence (5.25) can be extended to a full sequence

$$(t_0^k, x^k(t_0^k)) \to (\tau_0, \xi). \tag{5.26}$$

The full sequence (5.26) may now be compared with the first two subsequences on either side. For definiteness assume that n is even, and so x^k is monotone increasing in the interval $[\beta_{n-1}^k, \beta_n^k]$. As $t_1^{k'} < t_0^{k'}$ and $t_0^{k''} < t_2^{k''}$ for large k' and k'', respectively, we have that

$$x^{k'}(t_0^{k'}) \geqslant x^{k'}(t_1^{k'}) \to 0, \qquad x^{k''}(t_0^{k''}) \leqslant x^{k''}(t_2^{k''}) \to 0.$$

It follows that $x^k(t_0^k) \to 0$, hence $\xi = 0$ as claimed.

The proof that B_n also is an interval and that the second equation of (5.20) holds follows similar lines, but with enough differences that we present it. As before, we begin with sequences $(t_1^{k'}, x^{k'}(t_1^{k'})) \to (\tau_1, \lambda_n)$ and $(t_2^{k''}, x^{k''}(t_2^{k''})) \to (\tau_2, \lambda_n)$, and $(t_0^{k'''}, x^{k'''}(t_0^{k'''})) \to (\tau_0, \xi)$, where $\tau_1 < \tau_0 < \tau_2$. We have $t_1^{k'} \in [\alpha_n^{k'}, \alpha_{n+1}^{k'}]$ and similarly with $t_2^{k''}$ and $t_0^{k'''}$, and we must prove that $\xi = \lambda_n$. Again assume n is even, and so $\lambda_n = \mu$. Now extend the two subsequences for τ_1 and τ_2 (as opposed to the one for τ_0 as before) to full sequences $(t_i^k, x^k(t_i^k)) \to (\tau_i, \mu)$ for $i = 1, 2$. Observe that for these extended sequences the points t_i^k belong to the intervals $(\alpha_n^k, \alpha_{n+1}^k)$, at least for large k, as $x^k(t_i^k) > 0$ and because the adjacent intervals $(\alpha_{n-1}^k, \alpha_n^k)$ and $(\alpha_{n+1}^k, \alpha_{n+2}^k)$ where x^k is negative each has length greater than 1. Therefore, since x^k in $[\alpha_n^k, \alpha_{n+1}^k]$ consists of a monotone increasing part followed by a monotone decreasing part, we have that

$$x^{k'''}(t_0^{k'''}) \geqslant \min\{x^{k'''}(t_1^{k'''}), x^{k'''}(t_2^{k'''})\} \to \mu$$

at the point $t_0^{k'''}$ between $t_1^{k'''}$ and $t_2^{k'''}$. This implies that $x^{k'''}(t_0^{k'''}) \to \mu$, that is, $\xi = \mu$ as desired.

The proof of (5.23) follows a similar construction, so for simplicity we only sketch the proof that $A_n \leqslant B_n$. Assume to the contrary that there exist $\tau_1 \in B_n$ and $\tau_2 \in A_n$ with $\tau_1 < \tau_2$, let

$$(t_1^{k'}, x^{k'}(t_1^{k'})) \to (\tau_1, \lambda_n), \qquad (t_2^{k''}, x^{k''}(t_2^{k''})) \to (\tau_2, 0), \tag{5.27}$$

much as before, and extend the subsequence converging to (τ_1, λ_n) to a full sequence. Then $t_1^k \in (\alpha_n^k, \alpha_{n+1}^k)$ for the extended sequence, and as well $t_2^{k''} \in [\beta_{n-1}^{k''}, \beta_n^{k''}]$, which by (5.1) forces $t_1^{k''}, t_2^{k''} \in (\alpha_n^{k''}, \beta_n^{k''}]$. But this is incompatible with the limits (5.27) in the light of the monotonicity of x^k in $[\alpha_n^k, \beta_n^k]$ and the fact that $\lambda_n \neq 0$, which holds because $\mu > 0$ and $\nu > 0$. With this we have a contradiction. □

While the above result describes portions of Ω which are horizontal line segments, the more difficult matter is to describe the rest of Ω. Let us decompose Ω into monotone pieces by setting

$$\Omega_n = \{(\tau, \xi) \in \mathbb{R} \mid \text{there exist } t^{k'} \in [\beta_{n-1}^{k'}, \beta_n^{k'}] \text{ with } t^{k'} \to \tau \text{ and } x^{k'}(t^{k'}) \to \xi \text{ for some subsequence } k' \to \infty\}, \tag{5.28}$$

for each n. Then Ω_n is compact, Ω is the union of the Ω_n,

$$(\beta_{n-1}, \lambda_{n-1}),\ (\beta_n, \lambda_n) \in \Omega_n \tag{5.29}$$

holds, and using (5.22) we see that

$$\Omega(\mathrm{int}(P_n)) \subseteq \Omega(\beta_{n-1}, \beta_n) \subseteq \Omega_n \subseteq \Omega[\beta_{n-1}, \beta_n] \subseteq \Omega(Q_n). \tag{5.30}$$

The set Ω_n also has the expected monotonicity property.

Lemma 5.2. *The set $\Omega_n \subseteq \mathbb{R}^2$ is monotone increasing if n is even, and is monotone decreasing if n is odd.*

Proof. The proof of this result is in the same spirit as that of Lemma 5.1, and so will be omitted. We do however note that the regularity of the sequence x^k is again used. □

It will be very useful to express the ascending and descending parts of Ω as graphs of functions $\tau = \psi_n(\xi)$ parameterized by the vertical coordinate ξ. Indeed, as one sees from Theorem A such functions will play a key role in determining the limiting profile Ω. To this end we define for every n a set-valued function

$$\psi_n : [-\nu, \mu] \to 2^{P_n},$$

where 2^{P_n} is the set of all subsets of P_n, by letting

$$\psi_n(\xi) = \{\tau \in P_n \mid (\tau, \xi) \in \Omega_n\} \text{ for every } \xi \in [-\nu, \mu]. \tag{5.31}$$

Clearly $\psi_n(\xi)$ is a compact set for every ξ. Also define the set

$$\mathrm{graph}(\psi_n) = \{(\tau, \xi) \in \mathbb{R}^2 \mid \xi \in [-\nu, \mu] \text{ and } \tau \in \psi_n(\xi)\}, \tag{5.32}$$

or equivalently

$$\mathrm{graph}(\psi_n) = \Omega_n \cap V_{P_n}, \tag{5.33}$$

which also is a compact. Notice that while Ω_n extends horizontally from β_{n-1} to β_n, in the definition of $\psi_n(\xi)$ we only take $\tau \in P_n$, that is, Ω_n is truncated at the left and right so that τ lies between β_{n-1}^+ and β_n^-. (This truncation is done simply for technical reasons.) One can check that the only parts of Ω_n so removed are horizontal line segments at the levels λ_{n-1} and λ_n, and indeed this is shown in Proposition 5.3 below. Also, observe that $A_n \times \{0\} \subseteq \Omega_n$ from the definitions of these sets. It is clear that

$$\Omega(\mathrm{int}(P_n)) \subseteq \mathrm{graph}(\psi_n) \subseteq \Omega(P_n), \tag{5.34}$$

and from Lemma 5.2 that

$$\operatorname{graph}(\psi_n) \text{ is a monotone } \begin{cases} \text{increasing set for } n \text{ even,} \\ \text{decreasing set for } n \text{ odd.} \end{cases} \tag{5.35}$$

Also, the ordering and periodicity claims (3.3) of Theorem A about the functions ψ_n hold, although we have not yet established the single-valuedness of these set-valued functions.

The following result makes precise the relation between the graphs of the functions ψ_n, the horizontal line segments over the intervals B_n, and the set Ω. In particular, the basic description (3.2) of Ω is established, albeit with the set $\operatorname{graph}(\psi_n)$ given by (5.32) rather than by (3.4). (One needs still to reconcile these two formulas.)

Proposition 5.3. *We have that*

$$\Omega_n \subseteq \operatorname{graph}(\psi_n) \cup (B_{n-1} \times \{\lambda_{n-1}\}) \cup (B_n \times \{\lambda_n\}) \tag{5.36}$$

for every n, and thus Ω is given by (3.2). *We also have that*

$$\Omega^+ \cap \operatorname{graph}(\psi_{2m+1}) = \phi, \qquad \Omega^- \cap \operatorname{graph}(\psi_{2m}) = \phi, \tag{5.37}$$

for every m and that the inclusions

$$\Omega^+ \subseteq \bigcup_{m=-\infty}^{\infty} \operatorname{graph}(\psi_{2m}), \qquad \Omega^- \subseteq \bigcup_{m=-\infty}^{\infty} \operatorname{graph}(\psi_{2m+1}), \tag{5.38}$$

hold.

Before proving Proposition 5.3 we comment on some of the more subtle points which need to be considered. The reader may have noted the somewhat technically detailed and pedantic nature of the proof of Lemma 5.1. Although the sinusoidal shape of the solutions x^k is very suggestive of the shape of the limiting profile Ω, proper care must be taken. In particular, it is worth keeping in mind several somewhat pathological possibilities for the limiting profile Ω. While these pathologies can generally be ruled out, they must be considered as possibilities, and implicitly taken into account in our proofs.

In one scenario the sequence x^k converges to 0 uniformly on compact subsets of $\mathbb{R}\backslash p\mathbb{Z}$, that is, everywhere except near the integer multiples np of the limiting period. The minimum and maximum of x^k, say near $t = 0$, occur at $\beta^k_{-1} < 0$ and $\beta^k_0 > 0$, with both $\beta^k_{-1} \to 0$ and $\beta^k_0 \to 0$ as $k \to \infty$. Thus the graph of x^k has a narrow downward trough immediately to the left of $t = 0$, followed by a narrow upward peak immediately to the right of $t = 0$. In the limit the trough and peak become a vertical line segment (a "spike" from $-v$ to μ), and we have

$$\Omega = (\mathbb{R} \times \{0\}) \cup \bigcup_{n=-\infty}^{\infty} (\{np\} \times [-v, \mu]),$$

$$A_{2m} = B_{2m-1} = B_{2m} = \{mp\}, \qquad A_{2m+1} = [mp, (m+1)p],$$

$$\psi_{2m}(\xi) = \{mp\}, \qquad \psi_{2m+1}(\xi) = \begin{cases} \{(m+1)p\}, & \xi \in [-v, 0), \\ A_{2m+1}, & \xi = 0, \\ \{mp\}, & \xi \in (0, \mu]. \end{cases} \tag{5.39}$$

Note, incidentally, that in this case we have $\Omega = \Omega^*$ and $\Omega^{\pm} = \phi$. The spikes $\{mp\} \times [-v, \mu]$ go neither upward nor downward in the sense of (4.5) and (4.6), and so belong to Ω^*, and not to $\Omega^{\pm}$.

In a slightly different scenario the order of the peak and trough is reversed: One has $\beta_0^k < \alpha_1^k < \beta_1^k$, with $\beta_0^k \to \alpha_1$ and $\beta_1^k \to \alpha_1$, so the peak is followed immediately by the trough near $t = \alpha_1$. The limiting profile obtained is the same as in (5.39) but horizontally translated by an amount α_1, although now

$$A_{2m} = [(m-1)p + \alpha_1,\ mp + \alpha_1], \qquad A_{2m+1} = B_{2m} = B_{2m+1} = \{mp + \alpha_1\},$$

$$\psi_{2m}(\xi) = \begin{cases} \{(m-1)p + \alpha_1\}, & \xi \in [-v, 0), \\ A_{2m}, & \xi = 0, \\ \{mp + \alpha_1\}, & \xi \in (0, \mu], \end{cases} \qquad \psi_{2m+1}(\xi) = \{mp + \alpha_1\},$$

in contrast to (5.39).

One can also conceive of a profile Ω in which the minimum and maximum spikes occur at different locations on the τ-axis.

Although Ω is obtained as the limit of the graphs of piecewise monotone functions, we see that Ω itself need not be the graph of a function. In addition to the possibility of spikes as above, Ω might (locally) take the form of a graph $\xi = \varphi(\tau)$ of a monotone function with jump discontinuities. The possibility that such φ has countably infinitely many jumps in a finite interval, and these jumps occur for a dense set of τ, cannot yet be excluded. For example, φ could resemble the inverse to the classical Cantor function.

We caution the reader to keep in mind the sort of pathologies described above throughout our analysis.

We need the following lemma before giving the proof of Proposition 5.3.

Lemma 5.4. *We have that*

$$(\beta_{n-1}^+, \lambda_{n-1}) \in \Omega_n, \qquad (\beta_n^-, \lambda_n) \in \Omega_n. \tag{5.40}$$

In addition, we have the implications

$$\tau < \beta_{n-1}^{+} \Rightarrow \xi = \lambda_{n-1}, \qquad \tau > \beta_n^{-} \Rightarrow \xi = \lambda_n, \tag{5.41}$$

for $(\tau, \xi) \in \Omega_n$.

Proof. We prove the second formula in (5.40), the proof of the first being similar. If $\beta_n^{-} = \beta_n$ then we are done by (5.29), so assume that $\beta_n^{-} < \beta_n$. We have that

$$(\beta_n^{-}, \beta_n) \times \{\lambda_n\} \subseteq \mathrm{int}(B_n) \times \{\lambda_n\} = \Omega(\mathrm{int}(B_n)) \subseteq \Omega$$

by (5.20), hence

$$(\beta_n^{-}, \beta_n) \times \{\lambda_n\} \subseteq \Omega(\beta_{n-1}, \beta_n) \subseteq \Omega_n$$

by (5.30). Thus $[\beta_n^{-}, \beta_n] \times \{\lambda_n\} \subseteq \Omega_n$ as Ω_n is closed, so (5.40) holds as desired.

We again prove only the second implication in (5.41). If n is even then $(\beta_n^{-}, \mu) \in \Omega_n$ by (5.40) and the set Ω_n is monotone increasing by Lemma 5.2. This forces $\xi = \mu$ for any point $(\tau, \xi) \in \Omega_n$ with $\tau > \beta_n^{-}$, as desired. The proof for odd n is similar. □

Proof of Proposition 5.3. Take any $(\tau, \xi) \in \Omega_n$. If $\tau \in P_n$ then $(\tau, \xi) \in \mathrm{graph}(\psi_n)$ by (5.33). If $\tau \notin P_n$ then either $\tau < \beta_{n-1}^{+}$ or $\tau > \beta_n^{-}$ and thus either $(\tau, \xi) \in B_{n-1} \times \{\lambda_{n-1}\}$ or $(\tau, \xi) \in B_n \times \{\lambda_n\}$ by (5.41) of Lemma 5.4. This establishes the inclusion (5.36). It also establishes (3.2), as Ω is the union of the Ω_n.

To prove the first equation in (5.37) take any $(\tau, \xi) \in \mathrm{graph}(\psi_{2m+1})$. Then $(\tau, \xi) \in \Omega_{2m+1}$ by (5.33), so one has that $(t^{k'}, x^{k'}(t^{k'})) \to (\tau, \xi)$ for a subsequence as in the definition (5.28) of Ω_{2m+1}. As $t^{k'} \in [\beta_{2m}^{k'}, \beta_{2m+1}^{k'}]$ one has that $\dot{x}^{k'}(t^{k'}) \leqslant 0$. By (4.1) this subsequence may be extended to a full sequence $(t^k, x^k(t^k)) \to (\tau, \xi)$; however, one sees the condition in the definition (4.2) of Ω^{+} is violated. Thus $(\tau, \xi) \notin \Omega^{+}$, as desired. The proof of the second equation in (5.37) is similar.

To prove (5.38) take any $(\tau, \xi) \in \Omega^{+}$. Then $\xi \in (-\nu, \mu)$ by (5.15) and so $(\tau, \xi) \in \mathrm{graph}(\psi_n)$ for some n, by (3.2), where necessarily n is even by (5.37). This proves the first conclusion in (5.38) and the second is proved similarly. □

The next result describes the relation between the intervals A_n and B_n, and the function ψ_n.

Lemma 5.5. *Assume that* $\mu > 0$ *and* $\nu > 0$. *Then we have that*

$$\psi_n(0) = A_n, \tag{5.42}$$

and also that

$$\begin{aligned} \psi_{2m}(-\nu) &= \{\beta_{2m-1}^{+}\}, \qquad \psi_{2m}(\mu) = \{\beta_{2m}^{-}\}, \\ \psi_{2m+1}(-\nu) &= \{\beta_{2m+1}^{-}\}, \qquad \psi_{2m+1}(\mu) = \{\beta_{2m}^{+}\}, \end{aligned} \tag{5.43}$$

or equivalently that

$$\psi_n(\lambda_{n-1}) = \{\beta^+_{n-1}\}, \qquad \psi_n(\lambda_n) = \{\beta^-_n\}, \tag{5.44}$$

for every n and m.

Proof. From the definitions (5.17) and (5.28) of A_n and Ω_n we have that $A_n = \{\tau \in \mathbb{R} \mid (\tau, 0) \in \Omega_n\}$. As $A_n \subseteq P_n$ by (5.23) and the fact that $\mu > 0$ and $\nu > 0$, the claim (5.42) follows from the definition (5.31) of $\psi_n(\xi)$.

Now consider (5.43). We prove only that $\psi_{2m}(\mu) = \{\beta^-_{2m}\}$ as the proofs of the other three equations there are similar. We have $(\beta^-_{2m}, \mu) \in \Omega_{2m}$ by Lemma 5.4, hence $\beta^-_{2m} \in \psi_{2m}(\mu)$, so consider any point $\tau \in \psi_{2m}(\mu)$ with $\tau \neq \beta^-_{2m}$. As $\tau \in P_{2m}$ we must have that $\tau < \beta^-_{2m}$. Thus $\tau < B_{2m}$, and by (5.18) we have $\tau \in B_{2k}$ for some k with $k < m$. Thus $\tau \leqslant \beta^+_{2k} \leqslant \beta^+_{2m-2}$. But now consider the point $\alpha_{2m} \in A_{2m} = \psi_{2m}(0)$. From (5.23) and (5.12)

$$\tau \leqslant \beta^+_{2m-2} \leqslant \alpha_{2m-1} \leqslant \alpha_{2m} - 1 < \alpha_{2m}. \tag{5.45}$$

The strict inequality (5.45) contradicts the fact that $\psi_{2m}(0) \leqslant \psi_{2m}(\mu)$, which follows from the monotonicity property (5.35). Thus we have (5.43).

Eqs. (5.44) are merely restatements of (5.43). This completes the proof. □

Lemma 5.6. *The set $\psi_n(\xi)$ is a nonempty compact interval (or point) for every $\xi \in [-\nu, \mu]$.*

Proof. As noted earlier $\psi_n(\xi)$ is a compact set, so we must show it is nonempty and connected. This result is already established for $\xi = \mu, -\nu$ in Lemma 5.5 when $\mu > 0$ and $\nu > 0$ (if $\mu = \nu = 0$ one can check directly that $\psi_n(0) = \{\alpha_n\}$), so it is enough to consider $\xi \in (-\nu, \mu)$. Fix such ξ. Then $\xi \in (-\nu^k, \mu^k)$ for large k, and so $x^k(t^k) = \xi$ for some $t^k \in [\beta^k_{n-1}, \beta^k_n]$. Upon taking the limit of a subsequence $t^{k'} \to \tau$ we have $(\tau, \xi) \in \Omega_n$. By (5.36) of Proposition 5.3 and because $\xi \neq \mu, -\nu$, we have that $(\tau, \xi) \in \mathrm{graph}(\psi_n)$, that is $\tau \in \psi_n(\xi)$, and so $\psi_n(\xi) \neq \phi$.

To prove connectedness of $\psi_n(\xi)$ let $\tau_1, \tau_2 \in \psi_n(\xi)$ with $\tau_1 < \tau_2$, for some $\xi \in (-\nu, \mu)$, and take any $\tau_0 \in (\tau_1, \tau_2)$. Also take any $\xi_0 \in \overline{x}(\tau_0) \neq \phi$. Then $\tau_0 \in \mathrm{int}(P_n)$, and so $\tau_0 \notin P_k$ for every $k \neq n$, and $\tau_0 \notin B_k$ for every $k \in \mathbb{Z}$ by (5.24). Also, $(\tau_0, \xi) \in \Omega_n$ by (5.30), and so $(\tau_0, \xi) \in \mathrm{graph}(\psi_n)$ by (5.36) of Proposition 5.3. By considering the three points (τ_i, ξ), for $i = 1, 2$, and (τ_0, ξ_0), it follows from the monotonicity of $\mathrm{graph}(\psi_n)$ that $\xi_0 = \xi$, that is, $\tau_0 \in \psi_n(\xi)$. Thus $\psi_n(\xi)$ is connected. □

Lemma 5.7. *Suppose $(\tau_0, \xi_0) \in \Omega^{\pm}$ for some choice of sign $\pm$. Then $\xi_0 \in (-\nu, \mu)$, and there exists a unique integer n such that $\tau_0 \in \psi_n(\xi_0)$. Moreover, $\psi_n(\xi) = \psi_n(\xi_0) = \{\tau_0\}$ is constant and single-valued for all ξ in some neighborhood of ξ_0 and $(-1)^n = \pm 1$.*

Proof. For definiteness suppose $(\tau_0, \xi_0) \in \Omega^+$. Then $\xi_0 \in (-\nu, \mu)$ by (5.15). By (4.3) we have $(\tau_0, \xi) \in \Omega^+$ for every ξ near ξ_0, and these are the only points of Ω near (τ_0, ξ_0).

Thus (5.38) of Proposition 5.3 implies that $\tau_0 \in \psi_{2m}(\xi)$ for each such ξ, for some $m = m(\xi)$. As τ_0 is an isolated point of $\psi_{2m}(\xi)$ we have that $\psi_{2m}(\xi) = \{\tau_0\}$ for this m, since $\psi_{2m}(\xi)$ is a connected set by Lemma 5.6. From the second equation in (3.3) we see also that this $m(\xi)$ is uniquely determined for each ξ.

We claim the function $m(\xi)$ is constant in a neighborhood of $\xi = \xi_0$. Certainly $m(\xi)$ is bounded, so if it is not locally constant there exists a sequence $\xi_k \to \xi_0$ and integers m_0, m_1, with $m_0 \neq m_1$, such that $m(\xi_k) = m_1$, while $m(\xi_0) = m_0$. But then $(\tau_0, \xi_k) \in \mathrm{graph}(\psi_{2m_1})$, hence in the limit $(\tau_0, \xi_0) \in \mathrm{graph}(\psi_{2m_1})$ as $\mathrm{graph}(\psi_{2m_1})$ is closed. Thus $\tau_0 \in \psi_{2m_1}(\xi_0)$, and so $m(\xi_0) = m_1$. This contradicts $m(\xi_0) = m_0$ and proves that $m(\xi)$ is constant in ξ.

The uniqueness of the integer n for which $\tau_0 \in \psi_n(\xi_0)$ among even integers has been noted. If n is odd then $\tau_0 \notin \psi_n(\xi_0)$ by (5.37) of Proposition 5.3. This shows uniqueness of $n = 2m$ among all integers. $\square$

6. The max-plus equations and the proof of Theorem A

Our object in this section is to show that the functions ψ_n, which describe the ascending and descending portions of the limiting profile Ω, satisfy the system of max-plus equations (1.4) of Theorem A. In particular we shall show that these functions are single-valued and continuous except for a possible jump discontinuity at $\xi = 0$. We shall also give a more precise characterization of the quantities δ_0 and δ_1 in that theorem which describe the range over which the max-plus equations are valid.

The standing assumptions on f and r in Section 1 continue to hold in this section, and we shall make liberal use of the results of the previous section. Also, we shall take

$$\mu > 0 \text{ and } \nu > 0 \tag{6.1}$$

as an additional standing assumption throughout this section so as to avoid trivialities. While our techniques are elementary, there is enough detail that we again caution the reader to keep in mind the potential pathologies that were described earlier.

The following are two principal results of this section. In particular, Proposition 6.2 establishes the max-plus equations (1.4) of Theorem A and characterizes the ranges of ξ where they hold.

Proposition 6.1. *The set-valued function ψ_n is single-valued and continuous (considered as a real-valued function) in $[-\nu, \mu] \backslash \{0\}$. The left- and right-hand limits $\psi_n(0-)$ and $\psi_n(0+)$ of this function at $\xi = 0$ are the endpoints of the interval A_n, that is, the formula for A_n in (3.5) holds.*

Proposition 6.2. *There exists* $\delta_0 > 0$ *such that the first max-plus equation in* (1.4) *holds for every* $\xi \in [-\nu, \delta_0]\backslash\{0\}$, *for every m. This equation also holds at* $\xi = 0$, *except that the left-hand side* $\psi_{2m}(0)$ *is replaced with the right-hand endpoint* $\psi_{2m}(0+) = \alpha_{2m}^{+}$ *of the interval* $\psi_{2m}(0) = A_{2m}$ *in case this interval has positive length. In any case the quantity* δ_0 *can be chosen so that either* $\delta_0 = \mu$, *or else that* $0 < \delta_0 < \mu$ *and*

$$r(\delta_0) + \psi_{2m-1}(g(\delta_0)) < \max_{-\nu \leqslant s \leqslant \delta_0} (r(s) + \psi_{2m-1}(g(s))) = \psi_{2m}(\delta_0) = r(\delta_0) + \psi_{2m}(g(\delta_0)), \tag{6.2}$$

with also

$$\psi_{2m}(\delta_0) < \psi_{2m}(\xi) \quad \textit{and} \quad r(\delta_0) < r(\xi) \quad \textit{for every } \xi \in (\delta_0, \mu]. \tag{6.3}$$

The second max-plus equation in (1.4) *holds for* $\xi \in [-\delta_1, \mu]\backslash\{0\}$ *for some* $\delta_1 > 0$, *where here either* $\delta_1 = \nu$, *or else* $0 < \delta_1 < \nu$ *and*

$$r(-\delta_1) + \psi_{2m}(g(-\delta_1)) < \max_{-\delta_1 \leqslant s \leqslant \mu} (r(s) + \psi_{2m}(g(s))) = \psi_{2m+1}(-\delta_1) = r(-\delta_1) + \psi_{2m+1}(g(-\delta_1))$$

with also

$$\psi_{2m+1}(-\delta_1) < \psi_{2m+1}(\xi) \textit{ and } r(-\delta_1) < r(\xi) \textit{ for every } \xi \in [-\nu, -\delta_1),$$

and with a similar interpretation as above at $\xi = 0$.

Remark. Once Proposition 6.1 is established we may allow an abuse of notation in which we write $\psi_n(\xi) = \tau$ rather than $\psi_n(\xi) = \{\tau\}$ when $\psi_n(\xi)$ is a singleton set. That is, we regard $\psi_n(\xi)$ as a real number.

Remark. We observe from (3.2) as established in Proposition 5.3 that if $\xi \in (-\nu, \mu)\backslash\{0\}$, then $(\tau, \xi) \in \Omega$ if and only if $\psi_n(\xi) = \tau$ for some n.

To illustrate how Proposition 6.2 can be used, and in particular to see the significance of conditions (6.2) and (6.3) involving δ_0 and the corresponding conditions involving δ_1, we present the following two results. Note that Corollary 6.3 guarantees that both equations in (1.4) hold throughout the full interval $[-\nu, \mu]$ if r is both monotone increasing in $[-\nu, 0]$ and monotone decreasing in $[0, \mu]$, as for example with $r(\xi) = 1 - c\xi^2$ where $c > 0$. The same conclusion holds by Corollary 6.4 if r is monotone throughout $[-\nu, \mu]$.

Corollary 6.3. *If r is monotone decreasing in* $[0, \mu]$ *then* $\delta_0 = \mu$, *while if r is monotone increasing in* $[-\nu, 0]$ *then* $\delta_1 = \nu$.

Proof. If $\delta_0 < \mu$ then r cannot be monotone decreasing in $[0, \mu]$ from the second inequality in (6.3). The claim about δ_1 is proved similarly. □

Corollary 6.4. *If r is monotone increasing throughout $[-\nu, \mu]$ then both $\delta_0 = \mu$ and $\delta_1 = \nu$. In addition Eq.* (3.13) *holds for every $\xi \in [-\nu, \mu]$.*

If r is monotone decreasing throughout $[-\nu, \mu]$ the corresponding result holds.

Proof. The first equation in (1.4), which holds throughout $[-\nu, \delta_0]$, takes the form (3.13) in that interval as both r and the composition of ψ_{2m-1} with g are monotone increasing functions. Thus the strict inequality in (6.2) is impossible, which by Proposition 6.2 implies that $\delta_0 = \mu$. One has that $\delta_1 = \nu$ by Corollary 6.3.

The result when r is monotone decreasing is proved similarly. $\square$

We now proceed with the proofs of our results. We begin with an analysis of so-called plateaus, which are horizontal line segments in Ω.

Definition. A *plateau* in Ω is a subset $J \times \{\xi\} \subseteq \Omega$, where J is an interval of positive length and $\xi \in \mathbb{R}$. We call ξ the *level* of the plateau.

Remark. If $J \times \{\xi\}$ is a plateau in Ω then $J \times \{\xi\} \subseteq \Omega^*$, as $\Omega^+ \cup \Omega^-$ consists locally of vertical line segments. Also, the horizontal translates $(J + np) \times \{\xi\}$ of a plateau, modulo the period p of Ω, are plateaus. If it is the case that $\xi \in (-\nu, \mu)$ then

$$J \subseteq \bigcup_{n=-\infty}^{\infty} \psi_n(\xi) \subseteq \bigcup_{-\infty}^{\infty} P_n$$

by (3.2). In this case there exists a subinterval $J_* \subseteq J$ of positive length such that $J_* \subseteq \operatorname{int}(P_m)$ for some m, and thus $J_* \subseteq \psi_m(\xi)$ from (5.34).

Lemma 6.5. *Suppose that Ω contains a plateau $J \times \{\xi\}$. Then $(J - r(\xi)) \times \{g(\xi)\}$ is also a plateau. Moreover, ξ is a periodic point of g, that is, ξ is a fixed point of some iterate of g, and hence $\xi \in \{0, -D, C\}$.*

Proof. Since $J \times \{\xi\} \subseteq \Omega^*$ we have from the mapping properties that

$$\Phi(J \times \{\xi\}) = (J - r(\xi)) \times \{g(\xi)\} \subseteq \Omega,$$

and this proves the first claim.

Now suppose $J \times \{\xi\}$ is a plateau but ξ is not a periodic point of g. Then from (1.7) we obtain a sequence $J_n \times \{\xi_n\}$ of plateaus with distinct levels ξ_n, where $J_n = J_{n-1} - r(\xi_{n-1})$ and $\xi_n = g(\xi_{n-1})$ for $n \geqslant 1$, with $J_0 = J$ and $\xi_0 = \xi$. Since all intervals J_n have the same length, two of them have an overlap, modulo the period p, of positive length. Thus there exist $\xi_{n_1} < \xi_{n_2}$ and an interval J_* of positive length such that $J_* \times \{\xi_{n_i}\}$ are plateaus for $i = 1, 2$. We may also assume that $\xi_{n_i} \in (-\nu, \mu)$ for both levels. The interval J_* need not be the maximal one for which either set $J_* \times \{\xi_{n_i}\}$ is a plateau, and indeed, from the above remark we may choose J_* so that $J_* \subseteq \operatorname{int}(P_m)$ hence $J_* \subseteq \psi_m(\xi_{n_i})$ for $i = 1, 2$, for some m. But then neither

$\psi_m(\xi_{n_1}) \leqslant \psi_m(\xi_{n_2})$ nor $\psi_m(\xi_{n_2}) \leqslant \psi_m(\xi_{n_1})$ holds, contradicting the fact that the function ψ_m is monotone. □

We next prove one of the results stated above.

Proof of Proposition 6.1. By Lemma 5.6 the set $\psi_n(\xi)$ is nonempty and connected, so if $\psi_n(\xi)$ were not single-valued for some ξ then $\psi_n(\xi) \times \{\xi\} \subseteq \Omega$ would be a plateau. Necessarily $\xi \in \{0, -D, C\}$ by Lemma 6.5, hence $\xi \in \{0, -\nu, \mu\}$. But $\psi_n(-\nu)$ and $\psi_n(\mu)$ are single-valued by Lemma 5.5, and so $\xi = 0$.

It follows directly from the fact that graph(ψ_n) is a closed set that ψ_n, considered as a real-valued function, is continuous in $[-\nu, \mu] \backslash \{0\}$. Finally, the closedness and monotonicity of the set graph(ψ_n), along with (5.42) of Lemma 5.5, imply the left- and right-hand limits at $\xi = 0$ are the appropriate endpoints of A_n. □

The following result is an important component in the derivation of the max-plus equations (1.4). Let us note that if ψ_n is single-valued at $\xi = 0$ then at that point the inequalities (6.4) can easily be obtained by taking the limit $\xi \to 0$, and the strict inequalities (6.6) trivially hold. On the other hand, if $\psi_n(0) = A_n$ is an interval of positive length then the left-hand inequality in (6.4) will be shown in fact to be an equality. This fact, and others, are established later in Lemma 6.9 and Proposition 6.10. Note finally that the claim involving (6.5) is made even for $\xi_0 = 0$, where the inequalities are interpreted as between sets.

Before proving Proposition 6.2 we need to establish several preliminary results.

Lemma 6.6. *We have that*

$$\psi_{n-1}(g(\xi)) \leqslant \psi_n(\xi) - r(\xi) \leqslant \psi_n(g(\xi)) \quad \textit{for every } \xi \in [-\nu, \mu] \backslash \{0\}, \tag{6.4}$$

for every n. If for some $\xi = \xi_0 \in (-\nu, \mu)$ *both inequalities in* (6.4) *are strict, so*

$$\psi_{n-1}(g(\xi_0)) < \psi_n(\xi_0) - r(\xi_0) < \psi_n(g(\xi_0)), \tag{6.5}$$

then $(\psi_n(\xi_0), \xi_0) \in \Omega^{\pm}$ *with* $(-1)^n = \pm 1$ *and* $\psi_n(\xi) = \psi_n(\xi_0)$ *for every* ξ *near* ξ_0. *Finally, we have the strict inequalities*

$$\begin{aligned} &\psi_{2m}(\xi) - r(\xi) < \psi_{2m}(g(\xi)) \quad \textit{for every } \xi \in (-\nu, 0), \\ &\psi_{2m+1}(\xi) - r(\xi) < \psi_{2m+1}(g(\xi)) \quad \textit{for every } \xi \in (0, \mu) \end{aligned} \tag{6.6}$$

for every m.

Proof. For every k consider for each n the set

$$\mathscr{R}_n^k = \{(\tau, \xi) \in \mathbb{R}^2 \mid (-1)^{n+1}(\xi - x^k(\tau)) \geqslant 0, \\ \text{where } \beta_{n-1}^k \leqslant \tau \leqslant \beta_{n+1}^k \text{ and } -\nu^k < \xi < \mu^k\},$$

and also the two sets

$$\mathscr{R}^k_\pm = \{(\tau, \xi) \in \mathbb{R}^2 \mid \pm(\xi - x^k(\tau)) \geqslant 0 \text{ and } -\nu^k < \xi < \mu^k\},$$

bounded by portions of the graph of the function x^k and by the horizontal lines $\xi = \mu^k$ and $\xi = -\nu^k$. Observe that

$$\mathscr{R}^k_+ = \bigcup_{m=-\infty}^{\infty} \mathscr{R}^k_{2m+1}, \qquad \mathscr{R}^k_- = \bigcup_{m=-\infty}^{\infty} \mathscr{R}^k_{2m},$$

and that the connected components of $\mathscr{R}^k_+$ and $\mathscr{R}^k_-$ are precisely the sets $\mathscr{R}^k_n$ for odd and even n, respectively, in view of the monotonicity properties of x^k.

Fix n and consider the solution x^k for t in the set

$$S^k_n = \{t \in (\beta^k_{n-1}, \beta^k_n) \mid -\nu^k < x^k(t) < \mu^k\},$$

which is an open interval containing α^k_n. For such t we have that

$$-\nu^k < g(x^k(t)) < \mu^k \tag{6.7}$$

by (5.6). Assuming for definiteness that n is even and letting η^k denote the history function as in (5.4), we have further that $0 \leqslant \varepsilon^k \dot{x}^k(t) = f(x^k(t), x^k(\eta^k(t)))$ and so from (1.5) we have that

$$g(x^k(t)) - x^k(\eta^k(t)) \geqslant 0. \tag{6.8}$$

(Note here that inequality (6.8) is strict if $\dot{x}^k(t) \neq 0$.) We conclude from (6.7) and (6.8) that $(\eta^k(t), g(x^k(t))) \in \mathscr{R}^k_+$, and so the point $(\eta^k(t), g(x^k(t)))$ lies in one of the connected components $\mathscr{R}^k_{2m+1}$ of $\mathscr{R}^k_+$ for every $t \in S^k_n$. Taking $t = \alpha^k_n$ gives $(\eta^k(\alpha^k_n), g(x^k(\alpha^k_n))) = (\eta^k(\alpha^k_n), 0) \in \mathscr{R}^k_{n-1}$ since $\beta^k_{n-2} < \alpha^k_{n-1} < \eta^k(\alpha^k_n) = \alpha^k_n - 1 < \alpha^k_n < \beta^k_n$. Therefore

$$(\eta^k(t), g(x^k(t))) \in \mathscr{R}^k_{n-1} \text{ for every} t \in S^k_n \tag{6.9}$$

for every n and k.

Now fix $\xi \in (-\nu, \mu) \backslash \{0\}$, keeping n as before, in particular with n even. Assume also that ξ is simultaneously a regular value of all the functions x^k, that is, $\dot{x}^k(t) \neq 0$ for every t and k such that $x^k(t) = \xi$. By Sard's theorem almost every ξ satisfies this property. Then $\xi \in (-\nu^k, \mu^k)$ for all large k so there exists $\tau^k \in (\beta^k_{n-1}, \beta^k_n)$ such that $x^k(\tau^k) = \xi$. Then $\tau^k \in S^k_n$, so the inclusion (6.9) at $t = \tau^k$ implies that

$$g(\xi) - x^k(\eta^k(\tau^k)) > 0, \qquad \beta^k_{n-2} \leqslant \eta^k(\tau^k) \leqslant \beta^k_n, \tag{6.10}$$

with the strict inequality in (6.10) holding because $\dot{x}^k(\tau^k) \neq 0$. There also exist $\sigma^{k,1}, \sigma^{k,2} \in (\beta^k_{n-2}, \beta^k_n)$ such that

$$x^k(\sigma^{k,1}) = x^k(\sigma^{k,2}) = g(\xi), \qquad \beta^k_{n-2} < \sigma^{k,1} < \beta^k_{n-1} < \sigma^{k,2} < \beta^k_n,$$

and so

$$x^k(\sigma^{k,i}) > x^k(\eta^k(\tau^k)) \tag{6.11}$$

for $i = 1, 2$ from the first inequality in (6.10). It now follows from the strict inequality (6.11), using the fact that x^k is monotone decreasing, respectively monotone increasing, in $(\beta^k_{n-2}, \beta^k_{n-1})$, respectively $(\beta^k_{n-1}, \beta^k_n)$, that

$$\sigma^{k,1} < \eta^k(\tau^k) = \tau^k - r(\xi) < \sigma^{k,2}. \tag{6.12}$$

Now pass to a subsequence $k' \to \infty$ and take limits $\tau^{k'} \to \tau^*$ and $\sigma^{k',i} \to \sigma^{*,i}$. Then $(\tau^{k'}, x^{k'}(\tau^{k'})) \to (\tau^*, \xi) \in \Omega_n$ from the definition (5.28) of Ω_n, and so $\psi_n(\xi) = \tau^*$ from a remark above. In a similar fashion $\sigma^{k',1} \to \psi_{n-1}(g(\xi))$ and $\sigma^{k',2} \to \psi_n(g(\xi))$. Taking these limits in (6.12) yields the desired inequalities (6.4), at least for almost every ξ in $(-v, \mu)\backslash\{0\}$. Continuity in ξ of the functions in (6.4) now yields the inequalities throughout $[-v, \mu]\backslash\{0\}$.

Suppose the strict inequalities (6.5) hold at some $\xi_0 \in (-v, \mu)$. Then taking any $\tau_0 \in \psi_n(\xi_0)$ we have $(\tau_0, \xi_0) \in \Omega$ and

$$\Phi(\tau_0, \xi_0) = (\tau_0 - r(\xi_0),\ g(\xi_0)) \in (\psi_n(\xi_0) - r(\xi_0)) \times \{g(\xi_0)\}.$$

Thus $\Phi(\tau_0, \xi_0) \notin \Omega$ by (3.2) and because $g(\xi_0) \neq \mu, -v$. Therefore $(\tau_0, \xi_0) \notin \Omega^*$ from the mapping properties, and so $(\tau_0, \xi_0) \in \Omega^\pm$. Lemma 5.7 now implies that $(-1)^n = \pm 1$ and that $\psi_n(\xi) = \psi_n(\xi_0)$ for ξ near ξ_0.

To prove the first inequality in (6.6) we see for $\xi \in (-v, 0)$ that $\xi < 0 < g(\xi)$ and so $\psi_{2m}(\xi) \leqslant \psi_{2m}(g(\xi))$. As $r(\xi) > 0$ here the desired inequality holds. The second inequality in (6.6) is proved similarly. □

The following result along with the description (3.2) of Ω is needed to determine how much space lies between the sets graph(ψ_n) and graph(ψ_{n+1}).

Lemma 6.7. *All intervals B_n have the same length, say $\ell(B_n) = \beta^+_n - \beta^-_n = L_B$. Let $b_n \geqslant 0$ be defined by*

$$b_n = \beta^+_n - \beta^+_{n-1} = \beta^-_n - \beta^-_{n-1},$$

that is, $B_{n-1} = B_n - b_n$. Then

$$b_n \geqslant r(\lambda_n), \qquad p = b_n + b_{n+1} \tag{6.13}$$

hold for every n. If in addition $L_B > 0$ then

$$b_n = r(\lambda_n) \tag{6.14}$$

for every n and also

$$\mu = C \quad and \quad \nu = D, \qquad g(C) = -D \quad and \quad g(-D) = C, \tag{6.15}$$

both hold.

Proof. From the periodicity property $B_{n+2} = B_n + p$ in (5.19) all even-indexed intervals have the same length $\ell(B_{2m}) = L_0$, as do all odd-indexed intervals $\ell(B_{2m+1}) = L_1$. Assume first that $\max\{L_0, L_1\} > 0$, and without loss that $L_0 > 0$ and $L_0 \geqslant L_1$. Then each $B_{2m} \times \{\mu\}$ is a plateau, which by Lemma 6.5 implies that $\mu = C$, and $g(C) = -D$ and $g(-D) = C$. As $g(\mu) \geqslant -\nu$ by (5.6) we thus have $\nu = D$, giving (6.15). The ordering $B_{n-1} \leqslant B_n$ of the intervals together with $L_0 > 0$ forces a strict separation $B_{2m-1} < B_{2m+1}$ of the odd-indexed intervals, as B_{2m} lies between. By Lemma 6.5 the set $(B_{2m} - r(\mu)) \times \{g(\mu)\} = (B_{2m} - r(\mu)) \times \{-\nu\}$ is also a plateau and so

$$B_{2m} - r(\mu) \subseteq B_k \tag{6.16}$$

for some odd $k < 2m$, by (5.18). Note from (6.16) that $L_0 \leqslant L_1$, hence $L_0 = L_1$. Upon setting $\xi = \mu$ with $n = 2m$ in the left-hand inequality of (6.4), we have using Lemma 5.5 that

$$\beta^+_{2m-3} < \beta^-_{2m-1} = \psi_{2m-1}(-\nu) = \psi_{2m-1}(g(\mu)) \leqslant \psi_{2m}(\mu) - r(\mu) = \beta^-_{2m} - r(\mu).$$

Thus $B_{2m-3} < B_{2m} - r(\mu)$, which forces $k = 2m - 1$ in (6.16). In fact (6.16) is an equality as $L_0 = L_1$, and this establishes (6.14) for $n = 2m$. One now obtains these formulas for odd n by a symmetric argument.

Now suppose that $L_0 = L_1 = 0$, so $B_n = \{\beta_n\}$ with $\beta_n = \beta^{\pm}_n$. Setting $\xi = \mu$ with $n = 2m + 1$ in the left-hand inequality of (6.4) gives

$$\beta_{2m-1} = \psi_{2m}(-\nu) \leqslant \psi_{2m}(g(\mu)) \leqslant \psi_{2m+1}(\mu) - r(\mu) = \beta_{2m} - r(\mu) \tag{6.17}$$

where in general $-\nu \leqslant g(\mu)$ and the monotonicity of ψ_{2m} is used. The inequality $b_{2m} \geqslant r(\mu) = r(\lambda_{2m})$ as in (6.13) follows immediately from (6.17) and one similarly obtains the result for odd subscripts.

All that remains is to establish the formula for p in (6.13) under either case $L_B = 0$ or $L_B > 0$. But this follows directly from the periodicity property (5.19) and the definition of b_n. □

The next result establishes the first max-plus equation in (1.4) at the endpoint $\xi = -\nu$, and the second equation at $\xi = \mu$. These will serve as starting points in the derivation of these equations for general ξ.

Lemma 6.8. *We have that*

$$\psi_{n-1}(g(\lambda_{n-1})) = \psi_n(\lambda_{n-1}) - r(\lambda_{n-1}) \tag{6.18}$$

for every n.

Proof. If $L_B > 0$ then by (5.44) of Lemma 5.5 and from Lemma 6.7

$$\begin{aligned}\psi_{n-1}(g(\lambda_{n-1})) &= \psi_{n-1}(\lambda_{n-2}) = \beta^+_{n-2} = \beta^+_{n-1} - b_{n-1}\\ &= \beta^+_{n-1} - r(\lambda_{n-1}) = \psi_n(\lambda_{n-1}) - r(\lambda_{n-1}),\end{aligned}$$

proving (6.18). If $L_B = 0$ then we have

$$\psi_{n-1}(\xi) - r(\xi) \leqslant \psi_{n-1}(g(\xi)) \leqslant \psi_n(\xi) - r(\xi)$$

for $\xi \in [-v, \mu] \backslash \{0\}$ by (6.4) of Lemma 6.6. Letting $\xi = \lambda_{n-1}$ and using the fact that $\psi_{n-1}(\lambda_{n-1}) = \beta^-_{n-1} = \beta^+_{n-1} = \psi_n(\lambda_{n-1})$, which holds by (5.44) of Lemma 5.5 and because $L_B = 0$, gives (6.18). □

At this point it is not difficult to establish the inequality

$$\psi_{2m}(\xi) \geqslant \max_{-v \leqslant s \leqslant \xi} (r(s) + \psi_{2m-1}(g(s))) \tag{6.19}$$

for $\xi \in [-v, 0)$, and the analogous inequality corresponding to the second equation in (1.4), using Lemma 6.6 and the monotonicity of ψ_n. As ψ_{2m} is monotone increasing we have that

$$\psi_{2m}(\xi) \geqslant \psi_{2m}(s) \geqslant r(s) + \psi_{2m-1}(g(s)) \tag{6.20}$$

for any $-v \leqslant s \leqslant \xi < 0$, from which (6.19) follows directly. To prove equality in (6.19) over the range of ξ given in Proposition 6.2 we need an additional argument which is based on the second claim of Lemma 6.6.

Before proving Proposition 6.2, however, we must first deal with some technical issues that arise from the possible jump discontinuity in ψ_n at $\xi = 0$, or equivalently, from the possibility that the interval A_n has positive length, in which case $A_n \times \{0\}$ is a plateau. In this direction we have the following result, which in spirit is not unlike Lemma 6.7. It is followed by Proposition 6.10, which extends the inequalities (6.4) to $\xi = 0$.

Lemma 6.9. *All intervals A_n have the same length, say $\ell(A_n) = \alpha^+_n - \alpha^-_n = L_A$, and moreover $L_A \leqslant 1$. Let $a_n \geqslant 0$ be defined by*

$$a_n = \alpha^+_n - \alpha^+_{n-1} = \alpha^-_n - \alpha^-_{n-1},$$

that is, $A_{n-1} = A_n - a_n$. *Then*

$$a_n \geqslant 1, \qquad p = a_n + a_{n+1}, \tag{6.21}$$

hold for every n. If in addition $L_A > 0$ *then*

$$a_n = 1 \tag{6.22}$$

for every n, and so $p = 2$.

Proof. From the periodicity property $A_{n+2} = A_n + p$ in (5.19) all even-indexed intervals have the same length $\ell(A_{2m}) = L_0$, as do all odd-indexed intervals $\ell(A_{2m+1}) = L_1$. Also, we have the inequalities

$$\alpha_{n-1}^+ \leqslant \alpha_n^+ - 1 \leqslant \alpha_n^-, \qquad \alpha_{n-1}^- \leqslant \alpha_n^- - 1 \leqslant \alpha_n^+, \tag{6.23}$$

which follow by taking the limits $\xi \to 0$ from the left and right in (6.4) using Proposition 6.1. The second inequality in the first display of (6.23) gives $\ell(A_n) = \alpha_n^+ - \alpha_n^- \leqslant 1$ for every n, and so $\max\{L_0, L_1\} \leqslant 1$. If it is the case that $L_0 = L_1$ then all intervals A_n have the same length $L_A \leqslant 1$, and moreover one has $a_n \geqslant 1$ directly from (6.23) while the second formula in (6.21) follows from (5.19). In particular, if $L_0 = L_1 = 0$ then we are done.

Thus we assume for the remainder of the proof that $\max\{L_0, L_1\} > 0$. We must show that $L_0 = L_1$ and also that (6.22) holds. First assume that $\min\{L_0, L_1\} < 1$. We claim in this case that there is a strict separation

$$A_{n-1} < A_n \tag{6.24}$$

between adjacent intervals. To prove this suppose first that $L_0 < 1$. Then using (6.23) with $n = 2m$ and $2m + 1$ gives

$$\alpha_{2m-1}^+ < \alpha_{2m-1}^+ + 1 - L_0 = \alpha_{2m-1}^+ + 1 - (\alpha_{2m}^+ - \alpha_{2m}^-) \leqslant \alpha_{2m}^-,$$

$$\alpha_{2m}^+ < \alpha_{2m}^+ + 1 - L_0 = \alpha_{2m}^- + 1 \leqslant \alpha_{2m+1}^-,$$

and so $A_{2m-1} < A_{2m} < A_{2m+1}$ which implies (6.24). If instead $L_1 < 1$ then one argues similarly.

Still assuming that $\min\{L_0, L_1\} < 1$, we have that $\max\{L_0, L_1\} > 0$ and so without loss $L_0 > 0$ and $L_0 \geqslant L_1$. Then $A_{2m} \times \{0\}$ is a plateau hence so is $\Phi(A_{2m} \times \{0\}) = (A_{2m} - 1) \times \{0\}$, thus

$$A_{2m} - 1 \subseteq A_k \tag{6.25}$$

for some $k < 2m$ in the light of the strict separation (6.24). As $\alpha_{2m-2}^+ < \alpha_{2m-1}^- \leqslant \alpha_{2m}^- - 1$ by (6.23) and (6.24), we have that $A_{2m-2} < A_{2m} - 1$ so necessarily $k = 2m - 1$ in (6.25). Thus $L_0 \leqslant L_1$, hence $L_0 = L_1$, and so (6.25) is an equality and we have

$a_{2m} = 1$. Thus proves all intervals A_n have the same length, and now a symmetric argument shows that $a_n = 1$ also for odd n.

There remains to prove (6.22) when $L_0 = L_1 = 1$. In this case we still have $A_{n-1} \leqslant A_n$, although the gap condition (6.24) can fail and the intervals abut. As before each $A_n \times \{0\}$ is a plateau and hence so is $(A_n - 1) \times \{0\}$, but now

$$A_n - 1 \subseteq \bigcup_{k=-\infty}^{n-1} A_k.$$

But from this inclusion one sees directly that $A_n - 1 = A_{n-1}$ must hold, giving (6.22). □

Proposition 6.10. *If $L_A = 0$ then the inequalities in* (6.4) *both hold at $\xi = 0$, while if $L_A > 0$ then*

$$\psi_{n-1}(0) = \psi_n(0) - 1 \leqslant \psi_n(0) \tag{6.26}$$

holds.

Proof. If $L_A = 0$ then ψ_n and ψ_{n-1} are single valued and continuous at $\xi = 0$, so the result holds by taking limits $\xi \to 0$ from the left and right and (6.4).

If $L_A > 0$ then $\psi_n(0) = A_n$ and $\psi_{n-1}(0) = A_{n-1}$. We have $A_{n-1} = A_n - 1$ by Lemma 6.9, so the equality in (6.26) holds. The right-hand inequality $A_n - 1 \leqslant A_n$ holds because $\ell(A_n) \leqslant 1$. □

We now prove the second main result of this section.

Proof of Proposition 6.2. Without loss we consider even $n = 2m$ and establish the first equation in (1.4) in $[-v, \delta_0]$, for some $\delta_0 > 0$ as in the statement of the proposition. We assume for simplicity of exposition that $L_A = 0$, and so the functions ψ_{n-1} and ψ_n are single-valued and continuous throughout $[-v, \mu]$, including at $\xi = 0$. In case $L_A > 0$ these functions have jumps at $\xi = 0$ and our proof must be appropriately modified.

From the fact that ψ_{2m} is monotone increasing and from the first inequality in (6.4), where we also recall Proposition 6.10, we have that (6.20) holds whenever $-v \leqslant s \leqslant \xi \leqslant \mu$. Thus (6.19) holds for every $\xi \in [-v, \mu]$. Let $\delta_0 \in [-v, \mu]$ denote the last point to the right of $-v$ such that (6.19) is an equality throughout $[-v, \delta_0]$, that is,

$$\begin{aligned} \delta_0 = \sup\{\delta \in [-v, \mu] \mid & \text{ the inequality (6.19) is an equality} \\ & \text{for every } \xi \in [-v, \delta]\}. \end{aligned} \tag{6.27}$$

Noting that (6.19) is an equality at $\xi = -v$ by Lemma 6.8, we see that δ_0 is well-defined. If $\delta_0 = \mu$ then we are done, so assume for the remainder of this proof that

$\delta_0 < \mu$. Then (6.19) is an equality:

$$\psi_{2m}(\delta_0) = \max_{-v \leqslant s \leqslant \delta_0} (r(s) + \psi_{2m-1}(g(s))) \tag{6.28}$$

at $\xi = \delta_0$, and so the first equality in (6.2) holds. Although we do not yet know that $\delta_0 > 0$, we observe that this fact will follow immediately once the second equality in (6.2) is established, in the light of (6.6) which also holds at $\xi = 0$.

Let us now establish the strict inequality

$$r(\delta_0) + \psi_{2m-1}(g(\delta_0)) < r(\delta_0) + \psi_{2m}(g(\delta_0)) \tag{6.29}$$

of the leftmost and rightmost terms of (6.2). Indeed, if (6.29) fails then we have the equality $\psi_{2m-1}(g(\delta_0)) = \psi_{2m}(g(\delta_0))$, and therefore

$$\psi_{2m-1}(g(\xi)) = \psi_{2m}(g(\xi)) \text{ for every } \xi \in [\delta_0, \mu],$$

from the monotonicity (5.35) of ψ_{2m-1} and ψ_{2m} and from the inequality in (3.3). But in this case the inequalities

$$\begin{aligned} r(\xi) + \psi_{2m-1}(g(\xi)) &\leqslant \max_{-v \leqslant s \leqslant \xi} (r(s) + \psi_{2m-1}(g(s))) \\ &\leqslant \psi_{2m}(\xi) \leqslant r(\xi) + \psi_{2m}(g(\xi)), \end{aligned}$$

which follow from (6.4) and (6.19), are equalities throughout $[\delta_0, \mu]$ and this contradicts the definition of δ_0. Thus (6.29) holds.

We therefore wish to prove the rightmost equality in (6.2). Assuming it is false, so that $\psi_{2m}(\delta_0) < r(\delta_0) + \psi_{2m}(g(\delta_0))$, one has by continuity that for any $\gamma_1 \in (\delta_0, \mu]$ sufficiently near δ_0,

$$\psi_{2m}(\xi) < r(\xi) + \psi_{2m}(g(\xi)) \text{ for every } \xi \in [\delta_0, \gamma_1]. \tag{6.30}$$

From the definition (6.27) of δ_0 one may choose such γ_1 so that inequality (6.19) is strict at $\xi = \gamma_1$, and so

$$\begin{aligned} r(\xi) + \psi_{2m-1}(g(\xi)) \leqslant \max_{-v \leqslant s \leqslant \xi} (r(s) + \psi_{2m-1}(g(s))) < \psi_{2m}(\gamma_1) \\ \text{for every } \xi \in [-v, \gamma_1]. \end{aligned} \tag{6.31}$$

Now let

$$\gamma_2 = \inf\{\gamma \in [\delta_0, \gamma_1] \mid \psi_{2m}(\xi) = \psi_{2m}(\gamma_1) \text{ for every } \xi \in [\gamma, \gamma_1]\}. \tag{6.32}$$

Necessarily $\psi_{2m}(\gamma_2) = \psi_{2m}(\gamma_1)$, and so (6.30) and (6.31) imply that both inequalities in (6.4), with $n = 2m$, are strict at $\xi = \gamma_2$, that is, (6.5) holds there. Thus by Lemma 6.6 we have $\psi_{2m}(\xi) = \psi_{2m}(\gamma_2)$ for every ξ near γ_2, which contradicts the definition (6.32) of γ_2 if $\gamma_2 > \delta_0$. Thus $\gamma_2 = \delta_0$. But this is impossible, as with (6.31) it implies that

$$\max_{-\nu \leqslant s \leqslant \delta_0} (r(s) + \psi_{2m-1}(g(s))) < \psi_{2m}(\gamma_1) = \psi_{2m}(\delta_0),$$

contradicting (6.28). This establishes (6.2).

To prove the first inequality in (6.3) we assume that $\delta_0 < \mu$ and note that the maximum taken in (6.2) does not occur at the endpoint $s = \delta_0$, in view of the strict inequality. Thus the maximum in the right-hand side of the inequality in (6.19) is constant for ξ in a neighborhood of δ_0. On the other hand, from the definition (6.27) of δ_0 and from the monotonicity of ψ_{2m} we conclude that the inequality in (6.19) is strict for every $\xi \in (\delta_0, \gamma_3)$, for some $\gamma_3 > \delta_0$, and so $\psi_{2m}(\xi) > \psi_{2m}(\delta_0)$ for every such ξ. We conclude from this, again using the monotonicity of ψ_{2m}, that the first inequality in (6.3) holds throughout $(\delta_0, \mu]$.

To prove the second inequality in (6.3) we note for every $\xi \in (\delta_0, \mu]$ that

$$r(\delta_0) + \psi_{2m}(g(\delta_0)) = \psi_{2m}(\delta_0) < \psi_{2m}(\xi) \leqslant r(\xi) + \psi_{2m}(g(\xi)) \leqslant r(\xi) + \psi_{2m}(g(\delta_0)),$$

where in addition to the first inequality in (6.3) we have used (6.2) and (6.4), and as well the monotonicity of ψ_{2m}. □

We are now in a position to prove our first main theorem.

Proof of Theorem A. All the results claimed in this theorem have already been established in this and earlier sections, and only need be put in context.

Assuming it is not the case that $x^k \to 0$ uniformly, we have the limits μ and ν in (5.14), at least one of which is nonzero. The inclusion in (3.1) follows by taking the corresponding limit in (5.6), and as noted (5.16) both $\mu > 0$ and $\nu > 0$.

Proposition 5.3 establishes formula (3.2) with sets graph(ψ_n) of the form (5.32), and with the sets B_n. The continuity and single-valuedness of the set-valued functions ψ_n in $[-\nu, \mu]\backslash\{0\}$ follows from Lemma 5.6 and Proposition 6.1, with the latter result providing the limits of $\psi_n(\xi)$ as $\xi \to 0$ and the formula for the intervals A_n in (3.5). These facts together with (5.42) of Lemma 5.5 show the formulas (3.4) and (5.32) for the set graph(ψ_n) are equivalent. The monotonicity of ψ_n is noted in (5.35), and the ordering and periodicity properties (3.3) of these functions are also noted. Formula (3.5) for the sets B_n, which are intervals, is given in Lemma 5.5, with Lemma 5.1. Formula (3.7), which by (5.42) is equivalent to (3.8) when A_0 has length zero, holds as $0 = \alpha_0 \in A_0$ where (5.13) is used. The claims about the lengths L_A and L_B of the intervals A_n and B_n follow from Lemmas 6.9 and 6.7, respectively, and the continuity of ψ_n at $\xi = 0$ when $L_A = 0$ follows from (3.5).

Finally, Proposition 6.2 establishes the max-plus equations (1.4), and their extension to $\xi = 0$ when $L_A = 0$ follows by continuity. □

7. Bounds on and exact values of p

Here we use the max-plus equations to derive upper and lower bounds for the additive eigenvalue p. Remarkably, for a large class of nonlinearities these upper and lower bounds coincide and can be given in a simple and explicit form. We also see how p can be interpreted as the spectral radius of a nonlinear operator in the spirit of [36]. This analysis leads to a dynamical systems problem which involves the iteration of a set-valued map.

Recall the function h given by Eq. (1.16). Also define a function $\tilde{h}$ by

$$\tilde{h}(\xi) = r(\xi) + r(g^{-1}(\xi))$$

and note that $h(0) = \tilde{h}(0) = 2$.

We continue to assume (6.1) holds in addition to our standing assumptions.

Proposition 7.1. *We have that*

$$p \geqslant \max_{-v \leqslant \xi \leqslant \mu} h(\xi). \tag{7.1}$$

If r possesses a nonzero derivative $r'(0) \neq 0$ at the origin then $p > 2$ and hence $L_A = 0$.

Proof. From the first inequality in (6.4), we have that

$$\psi_n(\xi) - \psi_{n-1}(g(\xi)) \geqslant r(\xi), \tag{7.2}$$

and by replacing n with $n-1$ and ξ with $g(\xi)$ in (7.2) we have

$$\psi_{n-1}(g(\xi)) - \psi_{n-2}(g^2(\xi)) \geqslant r(g(\xi)) \tag{7.3}$$

for every $\xi \in [-v, \mu] \backslash \{0\}$. Combining (7.2) and (7.3) and using the periodicity condition in (3.3) gives

$$\psi_n(\xi) - \psi_n(g^2(\xi)) \geqslant r(\xi) + r(g(\xi)) - p = h(\xi) - p. \tag{7.4}$$

If n is even then the left-hand side of (7.4) is nonpositive for every $\xi \in [-v, 0)$ from the monotonicity of ψ_n, while the same holds for every $\xi \in (0, \mu]$ if n is odd. In any case we conclude that $h(\xi) - p \leqslant 0$ for every $\xi \in [-v, \mu] \backslash \{0\}$, hence for every $\xi \in [-v, \mu]$, and so (7.1) holds.

If $r'(0)$ exists and is nonzero then $h'(0) = (1 - k^{-1})r'(0)$ also exists and is nonzero by (1.8). Thus $h(\xi) > h(0) = 2$ for some ξ near 0 and this gives $p > 2$ from (7.1). Thus $L_A = 0$ by Lemma 6.9. □

In the following result recall our conventions, given near the end of Section 3, on g^{-1} and g^{-2}.

Proposition 7.2. *We have that*

$$p \leqslant \max_{-\nu \leqslant t \leqslant 0} \left(\left(\max_{0 \leqslant s \leqslant g^{-1}(t)} r(s) \right) + r(t) \right) = \max_{0 \leqslant s \leqslant \mu} \left(\left(\max_{-\nu \leqslant t \leqslant g(s)} r(t) \right) + r(s) \right) \tag{7.5}$$

and also that

$$p \leqslant \max_{0 \leqslant t \leqslant \mu} \left(\left(\max_{g^{-1}(t) \leqslant s \leqslant 0} r(s) \right) + r(t) \right) = \max_{-\nu \leqslant s \leqslant 0} \left(\left(\max_{g(s) \leqslant t \leqslant \mu} r(t) \right) + r(s) \right) \tag{7.6}$$

both hold.

Proof. Noting that Eq. (3.28), which was obtained from the max-plus equations (1.4), holds at least for $\xi \in (0, \mu]$, we replace $\psi_{2m+1}(\tau)$ with $\psi_{2m+1}(g^2(\xi))$ in that formula to obtain

$$p + \psi_{2m+1}(\xi) - \psi_{2m+1}(g^2(\xi)) \leqslant \max_{g^2(\xi) \leqslant \tau \leqslant g(-\nu)} h_1(\xi, \tau),$$

where the monotonicity of ψ_{2m+1} justifies this replacement. Upon letting $\xi \to 0$ we obtain

$$\begin{aligned} p &\leqslant \max_{0 \leqslant \tau \leqslant g(-\nu)} h_1(0, \tau) = \max_{-\nu \leqslant t \leqslant 0} h_1(0, g(t)) \\ &= \max_{-\nu \leqslant t \leqslant 0} \left(\left(\max_{0 \leqslant s \leqslant g^{-1}(t)} r(s) \right) + r(t) \right) = \max_{0 \leqslant s \leqslant \mu} \left(\left(\max_{-\nu \leqslant t \leqslant g(s)} r(t) \right) + r(s) \right), \end{aligned}$$

to give (7.5), where we have used the definition (3.29) of the function h_1 and where the final equality above comes about by switching the order in which the maxima are taken. The proof of (7.6) is similar. □

Proposition 7.3. *If r is monotone increasing in $[-\nu, 0]$ then the formula* (3.15) *for p holds. The analogous result holds if r is monotone decreasing in* $[0, \mu]$.

Proof. From (7.5) and the monotonicity assumption on r in $[-\nu, 0]$ we have that

$$p \leqslant \max_{0 \leqslant s \leqslant \mu} (r(g(s)) + r(s)) = \max_{0 \leqslant s \leqslant \mu} h(s).$$

This inequality together with (7.1) yields the result. The proof for r decreasing in $[0, \mu]$ is similar. □

Proposition 7.4. *If r is monotone decreasing in $[-\nu, 0]$ then*

$$p = \max_{0 \leqslant \xi \leqslant \mu} \tilde{h}(\xi) = \max_{-\nu \leqslant \xi \leqslant \mu} \tilde{h}(\xi). \tag{7.7}$$

The analogous result holds if r is monotone increasing in $[0, \mu]$.

Proof. From (7.6) and the monotonicity assumption on r in $[-\nu, 0]$ we have that

$$p \leqslant \max_{0 \leqslant t \leqslant \mu} (r(g^{-1}(t)) + r(t)) = \max_{0 \leqslant t \leqslant \mu} \tilde{h}(t). \tag{7.8}$$

From Eq. (3.30) we have for every $\xi \in [-\nu, 0)$ that

$$\begin{aligned} p + \psi_{2m}(\xi) &= \max_{g(\mu) \leqslant \tau \leqslant g^2(\xi)} (h_0(\xi, \tau) + \psi_{2m}(\tau)) = \max_{g(\xi) \leqslant t \leqslant \mu} (h_0(\xi, g(t)) + \psi_{2m}(g(t))) \\ &\geqslant \max_{g(\xi) \leqslant t \leqslant g^{-1}(\xi)} (h_0(\xi, g(t)) + \psi_{2m}(g(t))) \\ &\geqslant \left(\max_{g(\xi) \leqslant t \leqslant g^{-1}(\xi)} h_0(\xi, g(t)) \right) + \psi_{2m}(\xi), \end{aligned}$$

where the monotonicity of ψ_{2m} has been used. Therefore from the formula (3.31) for h_0

$$p \geqslant \max_{g(\xi) \leqslant t \leqslant g^{-1}(\xi)} \left(\left(\max_{g^{-1}(t) \leqslant s \leqslant \xi} r(s) \right) + r(t) \right) = \max_{g(\xi) \leqslant t \leqslant g^{-1}(\xi)} (r(g^{-1}(t)) + r(t)),$$

where again the monotonicity of r has been used. Noting that the union of the intervals $[g(\xi), g^{-1}(\xi)]$ for $\xi \in [-\nu, 0)$ equals $(0, \mu]$, we maximize the above expression over such ξ to give

$$p \geqslant \max_{0 \leqslant t \leqslant \mu} (r(g^{-1}(t)) + r(t)) = \max_{0 \leqslant t \leqslant \mu} \tilde{h}(t),$$

which with (7.8) yields the first equality in (7.7).

To prove the second equality in (7.7) take any $t \in [-\nu, 0)$ and let $s = g^{-1}(t)$, so $s \in (0, \mu]$. Then $t \geqslant g^{-2}(t)$ and so $r(t) \leqslant r(g^{-2}(t)) = r(g^{-1}(s))$, hence $\tilde{h}(t) \leqslant \tilde{h}(s)$. This implies the desired inequality.

The proofs for r increasing in $[0, \mu]$ are similar. $\square$

Let us now interpret the additive eigenvalue p as the spectral radius of a nonlinear operator. We refer to [36], in which some of these ideas are more fully and systematically developed. Denoting

$$J(\xi) = [g^2(\xi), g(-\nu)],$$

we have the inequality

$$\psi_{2m+1}(\xi) \geqslant -p + h_1(\xi, \tau) + \psi_{2m+1}(\tau) \tag{7.9}$$

for every $\tau \in J(\xi)$ from Eq. (3.28), at least for ξ in the range (3.32) where this equation is valid. Let us restrict $\xi \in [0, \mu]$, noting that every such ξ lies in the range (3.32) and satisfies $J(\xi) \subseteq [0, \mu]$. Then we may substitute (7.9) into itself repeatedly to obtain

$$\psi_{2m+1}(\xi_0) \geqslant -np + \sum_{i=1}^{n} h_1(\xi_{i-1}, \xi_i) + \psi_{2m+1}(\xi_n) \tag{7.10}$$

for any so-called *admissible sequence* $(\xi_0, \xi_1, \xi_2, \ldots, \xi_n)$, namely a sequence satisfying $\xi_0 \in [0, \mu]$ and $\xi_i \in J(\xi_{i-1})$ for every $1 \leqslant i \leqslant n$. (Heuristically one might think of such sequences dynamically as orbits obtained by iterating the set-valued map J.) As noted in [36], for every $\xi_0 \in [0, \mu]$ and $n \geqslant 1$ there exists an admissible sequence for which (7.10) is an equality, and so

$$np + \psi_{2m+1}(\xi_0) = \max_{\substack{\xi_i \in J(\xi_{i-1}) \\ 1 \leqslant i \leqslant n}} \left(\sum_{i=1}^{n} h_1(\xi_{i-1}, \xi_i) + \psi_{2m+1}(\xi_n) \right). \tag{7.11}$$

Upon dividing (7.11) by n and taking the limit $n \to \infty$ we obtain

$$p = \lim_{n \to \infty} \left(\max_{\substack{\xi_i \in J(\xi_{i-1}) \\ 1 \leqslant i \leqslant n}} \left(\frac{1}{n} \sum_{i=1}^{n} h_1(\xi_{i-1}, \xi_i) \right) \right), \tag{7.12}$$

valid for any $\xi_0 \in [0, \mu]$. Note also that for any fixed $n \geqslant 1$ the maximum on the right-hand side of (7.12) differs from p by an amount of order $O(n^{-1})$. As in [36], Eq. (7.12) can be viewed as an additive and nonlinear analog of the well-known formula $r(T) = \lim_{n \to \infty} ||T||^{1/n}$ for the spectral radius of a linear operator T.

8. The proof of Theorem B

This section is devoted to the proof of our second main theorem.

Proof of Theorem B. The fact that at least one of the quantities μ and v is positive means that the sequence x^k does not tend uniformly to zero, and so Theorem A implies that both $\mu > 0$ and $v > 0$. Corollary 6.4 with Proposition 6.2 implies that both the max-plus equations (1.4) of Theorem A hold throughout $[-v, \mu] \backslash \{0\}$, and also at $\xi = 0$ if $L_A = 0$. Corollary 6.4 also gives Eq. (3.13), which when substituted into the second equation of (1.4) and periodicity (3.3) used yields Eq. (3.12). Note that (3.13) holds at $\xi = 0$ even when $L_A > 0$, by (6.26) of Proposition 6.10. Proposition 7.3 establishes the formula (3.15) for p. If $r'(0) > 0$ then $p > 2$, and also $\mu > 0$ and $v > 0$, as in a discussion following the statement of Theorem B.

To complete the proof of Theorem B there remains to prove the constancy property (3.14), and also Eq. (3.16) under the strict monotonicity condition.

Let us first prove (3.14). Taking any $\xi \in (-v, 0)$ we claim that

$$\psi_{2m+1}(g^2(\xi)) \leqslant \psi_{2m+1}(\xi) \leqslant \max_{\xi \leqslant s \leqslant \mu} \psi_{2m+1}(g^2(s)) = \psi_{2m+1}(g^2(\xi)), \tag{8.1}$$

so that all terms in (8.1) are actually equal. To establish (8.1) we note that $\xi \leqslant g^2(\xi)$ together with the monotonicity of ψ_{2m+1} gives the first inequality there. The final equality also holds by monotonicity. The second inequality in (8.1) is a consequence of (3.12) and the fact that $h(s) \leqslant p$ for every s in that formula, which follows from (3.15). We therefore conclude from (8.1) that $\psi_{2m+1}(\xi) = \psi_{2m+1}(g^{2n}(\xi))$ for every n, and taking the limit $n \to \infty$ gives $\psi_{2m+1}(\xi) = \psi_{2m+1}(0-)$. This shows constancy in $(-v, 0)$ and hence in $[-v, 0)$ by continuity, to give (3.14).

If $p > 2$ then $L_A = 0$ by Theorem A, and so $\psi_{2m+1}(\xi)$ is continuous at $\xi = 0$. We observe further that $h(0) = 2$ and so the maximum on the right-hand side of Eq. (3.12) does not occur at $s = \xi$ when $\xi = 0$. This implies that the right-hand side of (3.12) is constant as ξ varies in a neighborhood of $\xi = 0$, and thus throughout $[-v, \delta]$ for some $\delta > 0$, as claimed.

Now let us prove (3.16), assuming that r is strictly increasing in $[-D, 0]$. If $L_B > 0$ then (3.16) holds by (6.15) of Lemma 6.7. Therefore we assume that $L_B = 0$, and we know that $g(\mu) \geqslant -v$ by (3.1). We have that

$$r(\mu) + \psi_{2m-1}(g(\mu)) = \psi_{2m}(\mu) = \beta^-_{2m} = \beta^+_{2m} = \psi_{2m+1}(\mu) = r(\mu) + \psi_{2m}(g(\mu))$$

by (3.13), by (5.43) of Lemma 5.5, and by Lemma 6.8. Thus $\psi_{2m-1}(g(\mu)) = \psi_{2m}(g(\mu))$. It follows from this, from the ordering (3.3), and from the fact that ψ_{2m-1} is monotone decreasing and ψ_{2m} is monotone increasing, that

$$\psi_{2m-1}(\xi) = \psi_{2m}(\xi) = \kappa \quad \text{for every } \xi \in [-v, g(\mu)] \tag{8.2}$$

for some constant κ. Now taking the left inequality in (6.4) of Lemma 6.6 for $n = 2m$, and also the right inequality in (6.4) but for $n = 2m - 1$, gives

$$\psi_{2m-1}(g(\xi)) \leqslant \psi_{2m}(\xi) - r(\xi) = \kappa - r(\xi) = \psi_{2m-1}(\xi) - r(\xi) \leqslant \psi_{2m-1}(g(\xi))$$

in the range (8.2), and hence

$$r(\xi) + \psi_{2m-1}(g(\xi)) = \kappa. \tag{8.3}$$

But the composition of ψ_{2m-1} and g is monotone increasing, and r is strictly increasing in $[-v, g(\mu)] \subseteq [-D, 0]$, and so (8.3) forces $g(\mu) = -v$ to hold. $\square$

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