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A basis theorem for a class of max-plus eigenproblems

John Mallet-Paret[a,*] and Roger D. Nussbaum[b]

[a] Division of Applied Mathematics, Brown University, Providence, RI 02912, USA
[b] Mathematics Department, Rutgers University, Piscataway, NJ 08854, USA

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Abstract

We study the max-plus equation

$$P + \psi(\xi) = \max_{\xi \leqslant s \leqslant M} (H(s) + \psi(\gamma(s))), \quad \xi \in [0, M], \tag{$*$}$$

where $H : [0, M] \to (-\infty, \infty)$ and $\gamma : [0, M] \to [0, M]$ are given functions. The function $\psi : [0, M] \to [-\infty, \infty)$ and the quantity P are unknown, and are, respectively, an eigenfunction and additive eigenvalue. Eigensolutions ψ are known to describe the asymptotics of certain solutions of singularly perturbed differential equations with state dependent time lags. Under general conditions we prove the existence of a finite set (a basis) of eigensolutions φ^i, for $1 \leqslant i \leqslant q$, with the same eigenvalue P, such that the general solution ψ to $(*)$ is given by

$$\psi(\xi) = (c^1 + \varphi^1(\xi)) \vee (c^2 + \varphi^2(\xi)) \vee \cdots \vee (c^q + \varphi^q(\xi)).$$

Here $c^i \in [-\infty, \infty)$ are arbitrary quantities and $\vee$ denotes the maximum operator. In many cases $q = 1$ so the solution ψ is unique up to an additive constant.

Keywords: Max-plus operator; Additive eigenvalue; Differential-delay equation

*Corresponding author.
E-mail addresses: jmp@cfm.brown.edu (J. Mallet-Paret), nussbaum@math.rutgers.edu (R.D. Nussbaum).

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1. Introduction

This paper is concerned with a class of so-called *max-plus equations*, specifically with equations of the form

$$P + \psi(\xi) = \max_{\xi \leqslant s \leqslant M} (H(s) + \psi(\gamma(s))), \quad \xi \in [0, M], \tag{1.1}$$

where H and γ are given functions satisfying appropriate conditions. We seek an unknown function ψ and an unknown quantity $P \in \mathbb{R}$, which we regard as an eigenfunction and an additive eigenvalue, respectively. Eq. (1.1) arises naturally in the study of the singularly perturbed differential-delay equation

$$\varepsilon\dot{x}(t) = f(x(t), x(t-r)), \quad r = r(x(t)),$$

with a state-dependent time lag r as indicated, as described in [16]. Indeed, the present paper is a companion to [16], playing an essential role in the analysis there. On the other hand, we believe the results on Eq. (1.1) contained herein are of sufficient independent interest to warrant a separate exposition, and to this end the present paper is completely self-contained.

We also mention here the paper [15] in which a more general class of max-plus eigenproblems of the form

$$P + \psi(\xi) = \max_{\alpha(\xi) \leqslant s \leqslant \beta(\xi)} (H(\xi, s) + \psi(s)) \tag{1.2}$$

are considered, although from the somewhat different perspective of topological fixed point theory. Here $\alpha, \beta : [0, M] \to [0, M]$ are given continuous functions satisfying $\alpha(s) \leqslant \beta(s)$ in $[0, M]$. (Note that Eq. (1.1) can be put in the form (1.2) by making the change of variables $\tilde{s} = \gamma(s)$.) Quite generally, one observes for Eq. (1.2) that adding a constant to a solution ψ yields another solution with the same P, just as multiplying the solution of a linear homogeneous equation by a constant yields another solution. Also, the maximum of two solutions of Eq. (1.2) with the same P is again a solution, just as the sum of two solutions of a homogeneous linear equation is a solution. Thus the max-plus eigenproblem (1.2) is analogous to the linear Fredholm equation

$$\lambda x(\xi) = \int_{\alpha(\xi)}^{\beta(\xi)} h(\xi, s)x(s)\, ds,$$

in which multiplication is replaced with addition, and addition (and integration) replaced with the max operator. Of course the same remarks apply to Eq. (1.1).

Eq. (1.2) has arisen in other contexts [3,4,12,17], at least when the functions α and β are constant. In this case it is known [3,17], that the right-hand side of (1.2) defines a compact operator, although this operator generally is not compact [15] for nonconstant α and β. The discrete version

$$p + z_i = \max_{j \in J(i)} (w_{ij} + z_j)$$

of (1.2) has arisen in many applications; see, for example, [1,5–7,13], and the extensive references therein. Here $W = (w_{ij})$ is a given $n \times n$ matrix, $z \in \mathbb{R}^n$ with $p \in \mathbb{R}$, and $J(i) \subseteq \{1, 2, \ldots, n\}$ is a nonempty subset for every $1 \leqslant i \leqslant n$. Max-plus operators arise quite generally in problems of optimal and stochastic control; see, for example, [8–11]. We mention the book [14] as a general reference for max-plus analysis.

The quantity $M > 0$ in Eq. (1.1), as it occurs in the delay equation problem [16], in fact plays the role of a parameter. That is, the given functions H and γ are defined on an interval $[0, C]$ for some fixed $C > 0$ and one considers the problem (1.1) for each $M \in (0, C]$. In the present paper we also take this point of view. In particular, we assume throughout, and without further comment, that

$$H : [0, C] \to \mathbb{R}, \quad \gamma : [0, C] \to [0, C]$$

are both continuous. Additionally, we assume that

$$\begin{aligned} &\gamma(\xi) \text{ is strictly increasing in } \xi \in [0, C] \text{ and} \\ &\gamma(\xi) < \xi \quad \text{for every } \xi \in (0, C). \end{aligned} \tag{1.3}$$

Observe that $\gamma(0) = 0$ must hold, although either $\gamma(C) = C$ or $\gamma(C) < C$ may occur.

Our main result is that under additional mild conditions on H and γ, for each M there exist a finite collection of nontrivial "basis" solutions φ^i, say for $1 \leqslant i \leqslant q$, from which the general solution of (1.1) for a particular P may be obtained. More precisely, ψ satisfies (1.1) if and only if

$$\psi(\xi) = (c^1 + \varphi^1(\xi)) \vee (c^2 + \varphi^2(\xi)) \vee \cdots \vee (c^q + \varphi^q(\xi)) \tag{1.4}$$

in $[0, M]$ for some constants $c^i \in [-\infty, \infty)$. We denote here $x \vee y = \max\{x, y\}$. It is easy to see that every such choice of c^i gives a solution to Eq. (1.1). Thus the thrust of our result is that (1.4) describes the general solution to (1.1).

The basis solutions φ^i and their number q in general depend on M. Remarkably, for any given M the eigenvalue P is the same for all nontrivial solutions to (1.1) and is given explicitly by

$$P = \max_{0 \leqslant \xi \leqslant M} H(\xi). \tag{1.5}$$

Clearly $P = P(M)$ depends on M, in general. Our results show that under mild conditions nontrivial solutions always exist with this P, that is, $q = q(M) \geqslant 1$. Further, a constructive and in some sense explicit description of the basis solutions φ^i can be given, and as well there is an explicit characterization of the quantity q.

Once the relation (1.5) has been established we may subtract $P = P(M)$ from both sides of Eq. (1.1) to obtain

$$\psi(\xi) = \max_{\xi \leqslant s \leqslant M} (A(s, M) + \psi(\gamma(s))), \tag{1.6}$$

where we denote

$$A(\xi, M) = H(\xi) - P(M). \tag{1.7}$$

Observe that the right-hand sides of (1.1) and (1.6) involve the values of ψ in the interval $[\gamma(\xi), \gamma(M)]$, which may contain ξ in its interior. This fact highlights the difficulty in obtaining solutions directly from (1.6) using any sort of evolutionary approach, that is, an approach which treats ξ as "time" and attempts to solve an initial value problem. On the other hand such an evolutionary approach is feasible for the equation

$$\psi(\xi) = \max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi)} (A(s, M) + \psi(\gamma(s))), \tag{1.8}$$

which is related to but different from (1.6). Here

$$\gamma_M^{-1}(\xi) = \begin{cases} \zeta \in [0, M] \text{ such that } \gamma(\zeta) = \xi & \text{if such } \zeta \text{ exists,} \\ M & \text{otherwise,} \end{cases}$$

which is the inverse of γ in the range of $[0, \gamma(M)]$, and M outside this range. Recalling (1.3), we see that $\gamma_M^{-1} : [0, M] \to [0, M]$ is well defined and continuous. Note that the right-hand side of (1.8) involves values of ψ only in the interval $[\gamma(\xi), \gamma(\gamma_M^{-1}(\xi))] \subseteq [\gamma(\xi), \xi]$, which lies to the left of ξ. (We caution the reader that $\gamma(\gamma_M^{-1}(\xi)) = \gamma(M) \neq \xi$ when $\xi > \gamma(M)$.) Our analysis will show that certain solutions of (1.8) also satisfy (1.6), and that solutions to (1.8) can be constructed by an evolutionary approach, in some sense solving an initial value problem. In particular, the basis solutions φ^i will be constructed in this fashion.

Our main result, the Basis Theorem, provides the representation (1.4) for the general solution of (1.1) and (1.6). It is stated in Section 2 and its proof given in Section 3. Theorem 2.3, whose statement precedes that of the Basis Theorem, is an essential component of this result as it gives a precise characterization of the canonically defined basis functions φ^i appearing in (1.4). The proof of Theorem 2.3 is also given in Section 3. All the results in these two sections are stated in terms of a given but arbitrary $M \in (0, C]$.

Section 4 is devoted to understanding how the basis solutions vary with M, which we regard here as a parameter. This approach is very much motivated by the role our theory plays in the companion paper [16]. In this section we study the situation in which for every $M \in (0, C]$ there is a unique basis element, that is, $q(M) = 1$. In this case the function H (or more precisely the pair (H, γ)) is said to be quasimodal. The quasimodal case includes both the case in which H is monotone increasing in $[0, C]$, and also the case in which H is monotone increasing in $[0, M_0]$ to a maximum at some $M_0 \in (0, C)$ and then satisfies $H(\xi) < H(M_0)$ to the right of M_0, where in both cases we need $H(\xi) > H(0)$ for ξ near 0. Theorem 4.1, the main result of Section 4, describes how the unique basis solution $\varphi^1(\cdot, M)$ for each $M \in (0, C]$ is related to the basis solution $\varphi^1(\cdot, C)$ at $M = C$.

Throughout this paper we say a function f is *strictly increasing* if $f(\xi_1) < f(\xi_2)$ whenever $\xi_1 < \xi_2$. If the monotonicity is not strict, and so $f(\xi_1) \leqslant f(\xi_2)$ whenever $\xi_1 < \xi_2$, then we say that f is *monotone increasing*. The terms *strictly decreasing* and *monotone decreasing* are defined correspondingly.

2. The basis theorem

We shall allow solutions of Eqs. (1.1), (1.6), and (1.8) to take the value $-\infty$ (but not $+\infty$) as well as taking values in $\mathbb{R}$, that is

$$\psi : [0, M] \to [-\infty, \infty).$$

(Note however that the kernel functions H and A and the additive eigenvalue P are always finite.) We consider only continuous solutions ψ, where by continuity we mean that $[-\infty, \infty)$ is endowed with the standard topology in which the sets $[-\infty, \xi)$ form a neighborhood basis for $-\infty$. By the *trivial solution* to any of these equations we mean the solution $\psi(\xi) = -\infty$ identically in $[0, M]$, all other solutions being termed *nontrivial*. We note that every solution ψ to (1.1) and (1.6) is monotone decreasing in $[0, M]$, and in particular $\psi(\xi) \leqslant \psi(\gamma(\xi))$ in $[0, M]$ by (1.3).

Let us now define $P(M)$ to be the quantity P given by Eq. (1.5) (the fact that the additive eigenvalue in (1.1) necessarily equals this quantity for any nontrivial solution will be established later, in Proposition 2.2). Next let the function A be defined by (1.7). Then clearly $A(\xi, M) \leqslant 0$ in $[0, M]$, with $A(\xi_0, M) = 0$ for at least one $\xi_0 \in [0, M]$ which in general depends on M. A central role in our analysis will be played by the set $Z(M) \subseteq [0, M]$ defined as

$$Z(M) = \{\zeta \in [0, M] \mid A(\zeta, M) = 0 \quad \text{and} \quad A(\xi, M) < 0 \quad \text{for every } \xi \in (\zeta, \gamma_M^{-1}(\zeta)]\},$$

which we observe is nonempty as it contains the right-most zero of $A(\cdot, M)$. Indeed, we shall see that the cardinality of $Z(M)$ equals the number q of basis solutions and that there is a canonical one-to-one correspondence between elements of $Z(M)$ and basis solutions. This correspondence is described precisely in Theorem 2.3 below.

Note that $0 \in Z(M)$ if and only if $A(0, M) = 0$ and that $M \in Z(M)$ if and only if $A(M, M) = 0$. Also, it is easy to see using (1.3) that the only possible cluster points of the set $Z(M)$ are 0 and M and that in any case $Z(M)$ is closed. Furthermore, M cannot be a cluster point of $Z(M)$ if $\gamma(M) < M$, which is always the case if $M < C$. The condition

$$H(0) < \max_{0 \leqslant \xi \leqslant \varepsilon} H(\xi) \text{ for every small } \varepsilon > 0, \tag{2.1}$$

which holds in particular if $H'(0) > 0$, is necessary and sufficient for $0 \notin Z(M)$, that is $A(0, M) < 0$, for every $M \in (0, C]$. This condition thus precludes, for every M, the clustering of $Z(M)$ at 0. It also ensures that $Z(M)$ is a finite set for every $M < C$.

It is not hard to give conditions which preclude clustering of $Z(M)$ at M when $M = C$. As noted, one such condition is that $\gamma(C) < C$. Another condition is that

$$H(\xi) < \max_{0 \leqslant s \leqslant C} H(s) \text{ for every } \xi \in [C - \varepsilon, C) \tag{2.2}$$

for some $\varepsilon > 0$, as this ensures that either $A(C, C) < 0$ or else that $\xi = C$ is an isolated zero of $A(\cdot, C)$. Thus if either $\gamma(C) < C$ or condition (2.2) holds, and if also (2.1) holds, then the finiteness of $Z(M)$ for every $M \in (0, C]$ is assured.

Clustering at either $\xi = 0$ or at $\xi = C$ can also be precluded by assuming a nondegenerate derivative, $H^{(k)}(\xi) \neq 0$ for some $k \geqslant 1$, at this point, where H is smooth enough.

Let us now state two key propositions which provide fundamental properties of solutions. We then state Theorem 2.3 which provides the existence of the basis solutions. In particular, Theorem 2.3 associates to each nonzero element of $Z(M)$ a solution of Eq. (1.6) in a canonical fashion. It also provides a link between Eqs. (1.6) and (1.8). Lastly, we state our main result, the Basis Theorem, which gives a representation of the general solution of (1.6), and hence of (1.1), in terms of basis solutions.

Proposition 2.1. *If ψ is a nontrivial solution of* (1.1) *for some $P \in \mathbb{R}$ then $\psi(\xi) > -\infty$ for every $\xi \in [0, M)$. If $\gamma(M) < M$, which in particular is the case if $M < C$, then $\psi(M) > -\infty$ also holds. On the other hand if $\gamma(M) = M = C$ and if also $H(C) \neq P$ (necessarily $H(C) \leqslant P$ with $P = P(C)$ as in* (1.5)*), then $\psi(C) = -\infty$.*

Proposition 2.2. *If ψ is a nontrivial solution of* (1.1) *then P is given by Eq.* (1.5).

Theorem 2.3. *Fix $M \in (0, C]$ and take any $\zeta \in Z(M)$. Assume that $\zeta > 0$. Then there exists a solution $\varphi : [0, M] \to [-\infty, \infty)$ to Eq.* (1.6) *such that*

$$\begin{aligned} &\varphi(\xi) = 0 \quad \textit{for every } \xi \in [0, \zeta], \textit{ and} \\ &\varphi(\xi) \textit{ satisfies Eq. (1.8)} \quad \textit{for every } \xi \in [\zeta, M]. \end{aligned} \tag{2.3}$$

Moreover, the two conditions (2.3) *uniquely characterize φ from among all continuous functions $\psi : [0, M] \to [-\infty, \infty)$ which are monotone decreasing. It is furthermore the case that if $\zeta < M$ then*

$$\varphi(\xi) < 0 \quad \textit{for every } \xi \in (\zeta, M]. \tag{2.4}$$

Also, for every $\xi_0 \in [\zeta, M)$ we have that

$$\varphi(\xi) = \max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi_0)} (A(s, M) + \varphi(\gamma(s))) \quad \textit{for every } \xi \in [\xi_0, \xi_0 + \varepsilon] \tag{2.5}$$

for some $\varepsilon = \varepsilon(\xi_0) > 0$. *Finally, if* $\tilde{\zeta} \in Z(M)$ *with* $\tilde{\zeta} > \zeta$ *and* $\tilde{\varphi}$ *is the solution of* (1.6), (2.3) *with* $\tilde{\zeta}$ *in place of* ζ, *then*

$$\varphi(\xi) = \varphi(\tilde{\zeta}) + \tilde{\varphi}(\xi) \quad \text{for every } \xi \in [\gamma(\tilde{\zeta}), M]. \tag{2.6}$$

In particular $\varphi(\xi) = \varphi(\tilde{\zeta})$ *is constant in* $[\gamma(\tilde{\zeta}), \tilde{\zeta}]$.

Main Theorem (The Basis Theorem). *Fix* $M \in (0, C]$ *and assume that* $Z(M)$ *is a finite set with* $0 \notin Z(M)$, *say* $Z(M) = \{\zeta^i\}_{i=1}^q$ *with* $q \geq 1$ *elements. Let* $\varphi^i : [0, M] \to [-\infty, \infty)$ *denote the solution associated to* ζ^i *by Theorem* 2.3, *namely the solution to Eq.* (1.6) *satisfying* (2.3) *with* $\zeta = \zeta^i$. *Then* $\psi : [0, M] \to [-\infty, \infty)$ *is a solution to* (1.6) *if and only if* ψ *is given by* (1.4) *for some constants* $c^i \in [-\infty, \infty)$.

It is clear from the first line in (2.3) and from (2.4) that different points of $Z(M)$ yield different solutions in Theorem 2.3. Thus the q basis solutions φ^i in the statement of the Basis Theorem are distinct. In general these solutions, and their number q, depend on M. Also note that for any $1 \leq j \leq q$ in the Basis Theorem we may take $c^i = -\infty$ for every $i \neq j$ and $c^j = 0$. This recovers the basis solution $\psi = \varphi^j$ in (1.4). Taking $c^i = -\infty$ for every i yields the trivial solution.

Remark. Our earlier comment that a constructive description of the basis solutions can be given is embodied in Eq. (2.5) and the surrounding claim. This formula allows for a direct construction of the solution φ by moving ξ forward from ζ to M by steps of (albeit variable) size ε.

Remark. As $Z(M)$ is closed it contains a leftmost (minimum) point, namely $\zeta_* \in Z(M)$ satisfying $\zeta_* \leq \zeta$ for every $\zeta \in Z(M)$. Denote by φ_* the solution given by Theorem 2.3 corresponding to the point ζ_*, where we assume that $\zeta_* > 0$. Then the solution corresponding to any other point $\zeta \in Z(M)$ for which $\zeta < M$ can be given in terms of φ_* by the formula

$$\varphi(\xi) = \begin{cases} 0, & \xi \in [0, \zeta], \\ \varphi_*(\xi) - \varphi_*(\zeta), & \xi \in [\zeta, M], \end{cases}$$

from (2.6). (If $\zeta = M$ this formula is still valid, namely $\varphi(\xi) = 0$ identically in $[0, M]$, although it requires proper interpretation in the event that $\varphi_*(M) = -\infty$.) In the Basis Theorem, where $Z(M)$ is a finite set, the basis solutions φ^i and hence also the general solution of Eq. (1.1) or (1.6) can thus be obtained from φ_* alone (we have $\varphi_* = \varphi^1$ in the notation of the proof of this theorem). One can thereby regard φ^1 as a "generating solution" for this equation.

Remark. An explicit formula for φ in Theorem 2.3 can be given in the case that $\zeta = \zeta^*$ is the rightmost (maximum) point of $Z(M)$ if the function H is monotone decreasing in $[\zeta^*, M]$. Let us assume that $\zeta^* < M$ otherwise the desired solution is just the zero function (of course $\zeta^* > 0$ by assumption).

Denoting $\varphi = \varphi^*$ for this solution and defining

$$A^*(\xi) = \begin{cases} 0, & \xi \in [0, \zeta^*], \\ H(\xi) - H(\zeta^*), & \xi \in [\zeta^*, M], \end{cases} \tag{2.7}$$

we claim that

$$\varphi^*(\xi) = \sum_{n=0}^{\infty} A^*(\gamma^n(\xi)) \quad \text{for every } \xi \in [0, M]. \tag{2.8}$$

To establish this claim note first that as $\zeta^* \in Z(M)$ we have $A(\zeta^*, M) = 0$ and hence $A(\xi, M) = H(\xi) - H(\zeta^*)$ in $[0, M]$. Thus $A^*(\xi) = A(\xi, M)$ in $[\zeta^*, M]$. Also, A^* is monotone decreasing in $[0, M]$ and $A^*(\xi) < 0$ in $(\zeta^*, M]$ otherwise ζ^* would not be the rightmost point of $Z(M)$. As $A^*(\xi) = 0$ in an interval to the right of the origin, we see that the sum (2.8) has only finitely many nonzero terms unless both $\xi = M = C$ and $\gamma(C) = C$ hold, in which case $\varphi^*(C) = -\infty$. In any case φ^* is a continuous monotone decreasing function and it vanishes in $[0, \zeta^*]$. Thus by the uniqueness claim in Theorem 2.3 it suffices to verify that φ^* satisfies Eq. (1.8) in $[\zeta^*, M]$. In this range the maximum in (1.8) is achieved at $s = \xi$, so there (1.8) for the function φ^* becomes

$$\varphi^*(\xi) = A^*(\xi) + \varphi^*(\gamma(\xi)). \tag{2.9}$$

One now easily verifies (2.9) using formula (2.8).

Remark. The condition $\zeta > 0$ in Theorem 2.3 is necessary as the following example shows. Suppose that H is monotone decreasing with $H(\xi) < H(0)$ throughout $(0, M]$. Then $Z(M) = \{0\}$. Arguing as in the above remark, one sees that a monotone decreasing solution of Eq. (1.8) as in Theorem 2.3 with $\zeta = 0$, or in fact any nontrivial solution of Eq. (1.6) in $[0, M]$, would satisfy Eq. (2.9) in that interval with (2.7) and $\zeta^* = 0$. Such a solution, normalized to vanish at $\xi = 0$, would then again be given by (2.8). However for certain H and γ sum (2.8) diverges to $-\infty$ for every $\xi \in (0, M)$ and so no such solution exists. For example, if $\gamma(\xi) \geqslant \xi - K\xi^2$ holds in a neighborhood of $\xi = 0$ for some $K > 0$ then the iterates $\xi_n = \gamma^n(\xi)$ of every $\xi = \xi_0 \in (0, M)$ satisfy $\liminf_{n\to\infty} n\xi_n > 0$ (see [15, Proposition 4.20] for this estimate, where a related example is given). If also $H'(0) < 0$ then one sees further that $\limsup_{n\to\infty} nA^*(\gamma^n(\xi)) < 0$, and so (2.8) diverges.

Remark. The generic situation is exemplified by the class of functions $\mathscr{G} \subseteq C_+^2[0, C]$ defined as follows. First let $C_+^2[0, C] = \{H \in C^2[0, C] \mid H'(0) > 0\}$, which is an open subset of $C^2[0, C]$. We consider only $H \in C^2_+[0, C]$ in order to ensure that $0 \notin Z(M)$, as in the statement of the Basis Theorem. Now let $\mathscr{G}$ denote the set of all $H \in C^2_+[0, C]$ which enjoy the following two properties: (1) whenever $H'(\xi) = 0$ for some $0 < \xi \leqslant C$ then $H''(\xi) \neq 0$; and (2) whenever $H'(\xi_i) = 0$ and $H''(\xi_i) < 0$ for $0 < \xi_1 < \xi_2 < C$ then $H(\xi_1) \neq H(\xi_2)$. Then $\mathscr{G} \subseteq C^2_+[0, C]$ is an open dense subset. We claim that if $H \in \mathscr{G}$ then $q(M) = 1$ for all but finitely many $M \in (0, C]$, for which $q(M) = 2$. This claim follows directly from the following straightforward observations. First, if $H \in \mathscr{G}$ then for every $M \in (0, C]$ the function H achieves its maximum in $[0, M]$ at most twice in this interval, never at $\xi = 0$, and if twice then one of the points where the maximum is achieved is M. Second, there are only finitely many M for which the maximum of H in $[0, M]$ is achieved twice in that interval.

Remark. As stated at the beginning of this section, we do not consider solutions ψ which take the value $+\infty$. This is a genuine restriction as the following example shows. Let $H(\xi) = 0$ and $\gamma(\xi) = \xi/2$ identically in $[0, M] = [0, 1]$. Then for every $P > 0$ one has that $\psi(\xi) = -K \log \xi$ satisfies (1.1) where $K = P/\log 2$ and where $\psi(0) = \infty$. Thus ψ is not a solution as considered above. Certainly, the existence of such ψ is not covered by either Proposition 2.2 or by the Basis Theorem. Of course, one can give a parallel theory in which one seeks solutions $\tilde{\psi} : [0, M] \to (-\infty, \infty]$ to the equation

$$\tilde{P} + \tilde{\psi}(\xi) = \min_{\xi \leqslant s \leqslant M} (\tilde{H}(s) + \tilde{\psi}(\gamma(s))), \quad \xi \in [0, M], \tag{2.10}$$

where here $\tilde{\psi}(\xi) = -\psi(\xi)$, and also $\tilde{P} = -P$ and $\tilde{H}(\xi) = -H(\xi)$. One obtains analogs of Theorem 2.3 and the Basis Theorem for Eq. (2.10).

Remark. The requirement that solutions ψ be continuous is significant. If a function ψ satisfying (1.1) possessed a discontinuity (necessarily a jump of size $\psi(\xi_0+) - \psi(\xi_0-) = -\kappa < 0$, by monotonicity) at some $\xi_0 \in (0, M)$ then Eq. (1.1) implies that ψ would possess a jump of size at least $-\kappa$, that is of size $-\tilde{\kappa} \leqslant -\kappa$, at $\xi_1 = \gamma(\xi_0)$. Thus at each iterate $\xi_n = \gamma^n(\xi_0)$ we would have $\psi(\xi_n+) - \psi(\xi_n-) \leqslant -\kappa$. But this would force $\psi(\xi) \to \infty$ as $\xi \to 0$ and so $\psi(0) = \infty$.

Remark. Eqs. (1.1) and (2.10), and in particular the example above with $H(\xi) = 0$ and $\gamma(\xi) = \xi/2$, are closely related to the theory of linear *Perron–Frobenius operators*. To see a simple case of this connection define $\theta(\xi) = e^{-\psi(\xi)}$, and also $\lambda = e^{-P}$ and $G(\xi) = e^{-H(\xi)}$, and observe that (1.1) becomes

$$\lambda\theta(\xi) = \min_{\xi \leqslant s \leqslant M} G(s)\theta(\gamma(s)).$$

Note that θ is monotone increasing, and that if G is also monotone increasing then the above equation becomes

$$\lambda\theta = \mathscr{F}(\theta), \quad \mathscr{F}(\theta)(\xi) = G(\xi)\theta(\gamma(\xi)), \tag{2.11}$$

where the second equation in (2.11) defines the transformation $\mathscr{F}$, which is an example of a Perron–Frobenius operator. The special case $G(\xi) = 1$ and $\gamma(\xi) = \xi/2$ treated above was studied long ago by Bonsall [2]. For further remarks and references to Perron–Frobenius operators see [18], and particularly Remark 3.7 and Section 5 in that paper.

We close this section with the proofs of Propositions 2.1 and 2.2. We leave the proofs of Theorem 2.3 and the Basis Theorem to the next section.

Proof of Proposition 2.1. If $\psi(\xi) = -\infty$ for some ξ then (1.1) implies that $\psi(\gamma(\xi)) = -\infty$, and hence that $\psi(\gamma^n(\xi)) = -\infty$ for every $n \geqslant 1$. If either $\gamma(\xi) < \xi$ or $\xi = 0$ then $\gamma^n(\xi) \to 0$ as $n \to \infty$, so $\psi(0) = -\infty$. This, with the monotonicity of ψ, forces ψ to be the trivial solution, a contradiction. Thus $\gamma(\xi) = \xi > 0$ and so $\xi = M$, to establish the first two sentences of the proposition. To prove the final claim set $\xi = M = C$ in Eq. (1.1) to obtain

$$P + \psi(C) = H(C) + \psi(C).$$

The inequality $H(C) \neq P$ now forces $\psi(C) = -\infty$. □

Proof of Proposition 2.2. For any $\xi \in [0, M)$ we have, taking $s = \xi$ in (1.1), that

$$P + \psi(\xi) \geqslant H(\xi) + \psi(\gamma(\xi)) \geqslant H(\xi) + \psi(\xi).$$

As $\psi(\xi) > -\infty$ by Proposition 2.1 it follows that $P \geqslant H(\xi)$ and so

$$P \geqslant \max_{0 \leqslant \xi \leqslant M} H(\xi). \tag{2.12}$$

Now let $\xi_0 \in [0, M]$ be any point at which H achieves its maximum in that interval. Then again from (1.1) with $\xi = 0$ and from the monotonicity of ψ

$$P + \psi(0) \leqslant H(\xi_0) + \max_{0 \leqslant s \leqslant M} \psi(\gamma(s)) = H(\xi_0) + \psi(0).$$

Canceling $\psi(0) > -\infty$ gives the opposite inequality to (2.12). This establishes (1.5). □

3. The proofs of Theorem 2.3 and the Basis Theorem

We maintain the notation and conventions of the previous section. We begin with a technical lemma, which is followed by a result which establishes a uniqueness property for monotone solutions of Eq. (1.8).

Lemma 3.1. *Let* $\alpha, \beta : [\xi_1, \xi_2] \to \mathbb{R}$ *and* $Q : [\alpha(\xi_1), \beta(\xi_2)] \to [-\infty, \infty)$ *be continuous, with both* α *and* β *monotone increasing and* $\alpha(\xi) \leqslant \beta(\xi)$ *holding in* $[\xi_1, \xi_2]$. *Let*

$$\varphi(\xi) = \max_{\alpha(\xi) \leqslant s \leqslant \beta(\xi)} Q(s). \tag{3.1}$$

Suppose $\xi_0 \in [\xi_1, \xi_2)$ *is such that*

$$\varphi(\xi) \leqslant \varphi(\xi_0) \tag{3.2}$$

for every $\xi \in [\xi_0, \xi_0 + \delta]$ *for some* $\delta > 0$, *and that* $\alpha(\xi_0) < \beta(\xi_0)$. *Then the following two properties hold. First, there exists* $\varepsilon > 0$ *such that*

$$\varphi(\xi) = \max_{\alpha(\xi) \leqslant s \leqslant \beta(\xi_0)} Q(s) \tag{3.3}$$

for every $\xi \in [\xi_0, \xi_0 + \varepsilon]$. *Second, if additionally inequality* (3.2) *is strict for every* $\xi \in (\xi_0, \xi_0 + \delta]$ *then*

$$Q(s) < \varphi(\xi_0) = Q(\alpha(\xi_0)) \tag{3.4}$$

for every $s \in (\alpha(\xi_0), \beta(\xi_0)]$.

Proof. We have from (3.1) that

$$Q(\beta(\xi_0)) \leqslant \varphi(\xi_0) \tag{3.5}$$

and to prove (3.3) we consider two cases depending on whether or not this inequality is strict. Assume first that (3.5) is an equality and let $0 < \varepsilon \leqslant \delta$ be small enough that $\alpha(\xi) \leqslant \beta(\xi_0)$ for every $\xi \in [\xi_0, \xi_0 + \varepsilon] \subseteq [\xi_1, \xi_2]$. Then for such ξ

$$\varphi(\xi_0) = Q(\beta(\xi_0)) \leqslant \max_{\alpha(\xi) \leqslant s \leqslant \beta(\xi_0)} Q(s) \leqslant \max_{\alpha(\xi) \leqslant s \leqslant \beta(\xi)} Q(s) = \varphi(\xi) \leqslant \varphi(\xi_0), \tag{3.6}$$

where we have used (3.2). The inequalities in (3.6) are all equalities and so we have (3.3). Assume now that inequality (3.5) is strict and choose ε so that both

$$\max_{\beta(\xi_0) \leqslant s \leqslant \beta(\xi)} Q(s) < \varphi(\xi) \tag{3.7}$$

and $\alpha(\xi) \leqslant \beta(\xi_0)$ hold for every $\xi \in [\xi_0, \xi_0 + \varepsilon] \subseteq [\xi_1, \xi_2]$. For such ξ let us write (3.1) as

$$\varphi(\xi) = \left(\max_{\alpha(\xi) \leqslant s \leqslant \beta(\xi_0)} Q(s) \right) \vee \left(\max_{\beta(\xi_0) \leqslant s \leqslant \beta(\xi)} Q(s) \right). \tag{3.8}$$

It follows from (3.7) that the first of the two terms in the right-hand side of (3.8) achieves the maximum there, again giving (3.3).

We now prove (3.4). Fixing any $s \in (\alpha(\xi_0), \beta(\xi_0)]$, we have that $s \in [\alpha(\xi), \beta(\xi)]$ for every $\xi > \xi_0$ sufficiently near ξ_0, and for such ξ we have from (3.1) that $\varphi(\xi) \geqslant Q(s)$. Combining this with (3.2), which is assumed strict, gives us the strict inequality in

(3.4). The equality in (3.4) now holds because the maximum in (3.1), when $\xi = \xi_0$, is achieved only at the point $\alpha(\xi_0)$. $\square$

Proposition 3.2. *Let* $\psi, \tilde{\psi} : [\gamma(\xi_1), \xi_2] \to [-\infty, \infty)$ *be continuous in* $[\gamma(\xi_1), \xi_2]$, *where* $[\xi_1, \xi_2] \subseteq [0, M]$ *and* $\xi_1 > 0$. *Also suppose that* $\psi(\xi) = \tilde{\psi}(\xi)$ *for every* $\xi \in [\gamma(\xi_1), \xi_1]$, *and that both* ψ *and* $\tilde{\psi}$ *satisfy Eq.* (1.8) *and are monotone decreasing in* $[\xi_1, \xi_2]$. *Then* $\psi(\xi) = \tilde{\psi}(\xi)$ *for every* $\xi \in [\gamma(\xi_1), \xi_2]$.

Proof. Let $\xi_0 = \sup\{\xi \in [\xi_1, \xi_2] \mid \psi(\xi) = \tilde{\psi}(\xi)\}$. If $\xi_0 = \xi_2$ we are done, so suppose that $\xi_0 \in [\xi_1, \xi_2)$. We use Lemma 3.1, taking $\alpha(\xi) = \xi$ and $\beta(\xi) = \gamma_M^{-1}(\xi)$, with $Q(s) = A(s, M) + \psi(\gamma(s))$. Thus the function φ in (3.1) satisfies $\varphi(\xi) = \psi(\xi)$ in $[\xi_1, \xi_2]$. Noting that $\alpha(\xi_0) = \xi_0 < \gamma_M^{-1}(\xi_0) = \beta(\xi_0)$ as $\xi_0 \in (0, M)$, we have the first conclusion (3.3) of Lemma 3.1, namely that

$$\psi(\xi) = \max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi_0)} (A(s, M) + \psi(\gamma(s))) \tag{3.9}$$

for every $\xi \in [\xi_0, \xi_0 + \varepsilon]$ for some $\varepsilon > 0$. In a similar fashion we obtain again (3.9) in such an interval but with $\tilde{\psi}$ replacing ψ. However, the right-hand sides of both equations involve only the values of ψ and $\tilde{\psi}$ to the left of ξ_0, where they agree, so it follows that $\psi(\xi) = \tilde{\psi}(\xi)$ throughout $[\xi_0, \xi_0 + \varepsilon]$. But this contradicts the definition of ξ_0. $\square$

The next result establishes a basic property of solutions of Eq. (1.6). It is followed by the proof of Theorem 2.3.

Proposition 3.3. *Let* ψ *be a solution of* (1.6) *in* $[0, M]$. *Suppose that* $A(\zeta, M) = 0$ *for some* $\zeta \in [0, M]$. *Then* $\psi(\gamma(\zeta)) = \psi(\zeta)$ *and hence* ψ *is constant in* $[\gamma(\zeta), \zeta]$.

Proof. With ζ as in the statement of the proposition we have that

$$\psi(\gamma(\zeta)) \leqslant \max_{\zeta \leqslant s \leqslant M} (A(s, M) + \psi(\gamma(s))) = \psi(\zeta) \leqslant \psi(\gamma(\zeta)),$$

where the monotonicity of ψ gives the final inequality above. This proves the result. $\square$

Proof of Theorem 2.3. With ζ as in the statement of the theorem consider the set

$$\begin{aligned} \mathscr{P} = \{(\psi, \zeta_0) \mid \zeta_0 \in [\zeta, M] \quad &\text{and} \quad \psi : [0, \zeta_0] \to [-\infty, \infty) \text{ is continuous} \\ &\text{and monotone decreasing, with } \psi(\xi) = 0 \quad \text{for every } \xi \in [0, \zeta] \\ &\text{and with Eq. (1.8) holding in } [\zeta, \zeta_0]\}. \end{aligned}$$

We seek a unique element of $\mathscr{P}$ for which $\zeta_0 = M$, as this will provide a solution to the problem (2.3). We begin by observing that $\mathscr{P}$ is nonempty as it contains the pair (ψ, ζ_0) with ψ the zero function in $[0, \zeta]$ and $\zeta_0 = \zeta$. This is the case because $A(\zeta, M) = 0$ and so (1.8) holds at $\xi = \zeta$.

Next suppose we have two elements $(\psi, \zeta_0), (\tilde{\psi}, \tilde{\zeta}_0) \in \mathscr{P}$ with $\zeta_0 \leqslant \tilde{\zeta}_0$. Let $\hat{\psi} = \tilde{\psi}|_{[0,\zeta_0]}$ denote the restriction of $\tilde{\psi}$ to $[0, \zeta_0]$. Then $(\hat{\psi}, \zeta_0) \in \mathscr{P}$, and in particular $\hat{\psi}$ satisfies Eq. (1.8) in $[\zeta, \zeta_0]$, where in making this observation we use the fact that the right-hand side of (1.8) depends only on values of the solution to the left of ξ. Thus $\hat{\psi} = \psi$ by the uniqueness result Proposition 3.2, wherein we take $\xi_1 = \zeta$ and $\xi_2 = \zeta_0$, and so $\tilde{\psi}$ is an extension of ψ from $[0, \zeta_0]$ to the larger interval $[0, \tilde{\zeta}_0]$. We conclude that the set $\mathscr{P}$ is totally ordered with respect to the order given by extension of a function, and that for every ζ_0 there is at most one element $(\psi, \zeta_0) \in \mathscr{P}$.

Now let

$$\xi_0 = \sup\{\zeta_0 \in [\zeta, M] \mid \text{there exists } (\psi, \zeta_0) \in \mathscr{P}\}.$$

We claim there exists an element $(\psi, \xi_0) \in \mathscr{P}$, that is, there exists a maximal element of $\mathscr{P}$. From the total ordering we have immediately the existence of some $\psi : [0, \xi_0) \to [-\infty, \infty)$ with $\psi(\xi) = 0$ in $[0, \zeta]$, with (1.8) holding in $[\zeta, \xi_0)$, and with ψ continuous and monotone decreasing. Indeed, ψ is just the common extension of all the elements of $\mathscr{P}$ with $\zeta_0 < \xi_0$. Now set $\psi(\xi_0) = \lim_{\xi \to \xi_0 -} \psi(\xi)$, thereby extending ψ continuously to the closed interval $[0, \xi_0]$. By continuity ψ satisfies Eq. (1.8) at $\xi = \xi_0$ and hence throughout $[\zeta, \xi_0]$, and so $(\psi, \xi_0) \in \mathscr{P}$, as desired.

Let us denote by (φ, ξ_0) the maximal element of $\mathscr{P}$ obtained above. We now claim that $\xi_0 = M$. Assume to the contrary that $\xi_0 < M$ and define $\tilde{\varphi} : [0, \gamma_M^{-1}(\xi_0)] \to [-\infty, \infty)$ by

$$\tilde{\varphi}(\xi) = \begin{cases} \varphi(\xi), & \xi \in [0, \xi_0], \\ \max\limits_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi_0)} (A(s, M) + \varphi(\gamma(s))), & \xi \in [\xi_0, \gamma_M^{-1}(\xi_0)], \end{cases} \tag{3.10}$$

observing that $\tilde{\varphi}$ is continuous and monotone decreasing in $[0, \gamma_M^{-1}(\xi_0)]$. Next extend φ to the right of ξ_0 by setting

$$\varphi(\xi) = \max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi)} (A(s, M) + \tilde{\varphi}(\gamma(s))) \quad \text{for every } \xi \in [\xi_0, \gamma_M^{-1}(\xi_0)]. \tag{3.11}$$

Note the different formulas for the upper limits of the maxima in (3.10) and (3.11). In particular note that the right-hand side of (3.10) depends only on values of φ to the left of ξ_0, while the right-hand side of (3.11) depends only on values of $\tilde{\varphi}$ to the left of $\gamma_M^{-1}(\xi_0)$. Also observe that the extension (3.11) of φ is continuous at ξ_0 since at that point Eq. (3.11) reduces to (1.8).

We now consider two cases, in each case seeking a contradiction. First, if $\xi_0 \geqslant \gamma(M)$ then in fact the two upper limits in the maxima in (3.10) and (3.11) are the same, $\gamma_M^{-1}(\xi_0) = \gamma_M^{-1}(\xi) = M$. Also, both maxima are taken over a region where φ

and $\tilde{\varphi}$ agree, lying to the left of $\gamma(M)$, so it follows that $\varphi(\xi) = \tilde{\varphi}(\xi)$ everywhere in $[0, M]$. One also sees by replacing $\tilde{\varphi}$ with φ in (3.11) that the extension of φ given there is monotone decreasing and satisfies Eq. (1.8) in $[\xi_0, M]$ and hence throughout $[\zeta, M]$. Thus $(\varphi, M) \in \mathscr{P}$ and this implies that $\xi_0 = M$, a contradiction.

For the second case we assume that $\xi_0 < \gamma(M)$. Here we wish to apply Lemma 3.1 to Eq. (3.11), taking $\alpha(\xi) = \xi$ and $\beta(\xi) = \gamma_M^{-1}(\xi)$, and $Q(s) = A(s, M) + \tilde{\varphi}(\gamma(s))$. Note first that Eq. (3.11) in fact holds for every $\xi \in [\zeta, \gamma_M^{-1}(\xi_0)]$, as φ and $\tilde{\varphi}$ agree in $[0, \xi_0]$ and φ satisfies (1.8) in $[\zeta, \xi_0]$. We therefore take $\xi_1 = \zeta$ and $\xi_2 = \gamma_M^{-1}(\xi_0)$ in Lemma 3.1. Also note that $\zeta \leqslant \xi_0 < \gamma_M^{-1}(\xi_0)$, in particular because $\zeta > 0$. Thus all that remains to be checked for the lemma to apply is inequality (3.2) in some interval to the right of ξ_0. To this end we observe that

$$\max_{\gamma_M^{-1}(\xi_0) \leqslant s \leqslant \gamma_M^{-1}(\xi)} (A(s, M) + \tilde{\varphi}(\gamma(s))) \leqslant \max_{\gamma_M^{-1}(\xi_0) \leqslant s \leqslant \gamma_M^{-1}(\xi)} \tilde{\varphi}(\gamma(s)) = \tilde{\varphi}(\xi_0) = \varphi(\xi_0) \tag{3.12}$$

for $\xi \in [\xi_0, \gamma_M^{-1}(\xi_0)]$, as $A(s, M) \leqslant 0$ and $\tilde{\varphi}$ is monotone decreasing. Note that in the penultimate equality in (3.12) we have used the fact that $\gamma(\gamma_M^{-1}(\xi_0)) = \xi_0$, which holds because $\xi_0 < \gamma(M)$. From (3.11) and (3.12) we have for such ξ that

$$\begin{aligned}
\varphi(\xi) &= \max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi)} (A(s, M) + \tilde{\varphi}(\gamma(s))) \leqslant \max_{\xi_0 \leqslant s \leqslant \gamma_M^{-1}(\xi)} (A(s, M) + \tilde{\varphi}(\gamma(s))) \\
&= \left(\max_{\xi_0 \leqslant s \leqslant \gamma_M^{-1}(\xi_0)} (A(s, M) + \tilde{\varphi}(\gamma(s))) \right) \vee \left(\max_{\gamma_M^{-1}(\xi_0) \leqslant s \leqslant \gamma_M^{-1}(\xi)} (A(s, M) + \tilde{\varphi}(\gamma(s))) \right) \\
&\leqslant \varphi(\xi_0) \vee \varphi(\xi_0) = \varphi(\xi_0),
\end{aligned}$$

to give (3.2) as desired. We now conclude from Lemma 3.1 that (3.3) holds for $\xi \in [\xi_0, \xi_0 + \varepsilon]$ for some ε. However, the right-hand side of (3.3) for such ξ is the same as the right-hand side of (3.10), in particular because we are taking a maximum over a range where φ and $\tilde{\varphi}$ agree. We thus have that $\varphi(\xi) = \tilde{\varphi}(\xi)$ in $[\xi_0, \xi_0 + \varepsilon]$. Now having shown that φ and $\tilde{\varphi}$ agree in this interval, we may replace $\tilde{\varphi}$ with φ in (3.11) for this range of ξ and conclude that φ satisfies Eq. (1.8) there. We conclude that φ satisfies (1.8) in $[\zeta, \xi_0 + \varepsilon]$ and so $(\varphi, \xi_0 + \varepsilon) \in \mathscr{P}$, which contradicts the definition of ξ_0.

Our original assumption that $\xi_0 < M$ is therefore false, and thus $\xi_0 = M$ and we have that φ satisfies (2.3), as desired. Also, formula (2.5) with $\varepsilon = \varepsilon(\xi_0)$ holds by Lemma 3.1, and we have seen it play a central role in the above construction of φ.

We next show that φ constructed above satisfies Eq. (1.6) in $[0, M]$. Taking any $\xi_0 \in [\zeta, M]$, let us first show that φ satisfies (1.6) at this point. With $\psi = \varphi$ in Eq. (1.8), we maximize both sides of this equation over the range $\xi \in [\xi_0, M]$. For the left-hand side of (1.8) we obtain $\varphi(\xi_0)$ for the maximum as φ is monotone decreasing. For the right-hand side of (1.8) we obtain the maximum of $A(s, M) + \varphi(\gamma(s))$ over the union of the intervals $[\xi, \gamma_M^{-1}(\xi)]$ for such ξ, namely over the interval $[\xi_0, M]$. However, this gives precisely Eq. (1.6) at the point ξ_0, as desired.

To show that (1.6) also holds in $[0, \zeta]$ we have for every ξ in that interval that

$$\begin{aligned}
0 = \varphi(\xi) = \varphi(\zeta) &= \max_{\zeta \leqslant s \leqslant M} (A(s, M) + \varphi(\gamma(s))) \leqslant \max_{\xi \leqslant s \leqslant M} (A(s, M) + \varphi(\gamma(s))) \\
&= \left(\max_{\xi \leqslant s \leqslant \zeta} (A(s, M) + \varphi(\gamma(s)))\right) \vee \left(\max_{\zeta \leqslant s \leqslant M} (A(s, M) + \varphi(\gamma(s)))\right) \\
&\leqslant \left(\max_{\xi \leqslant s \leqslant \zeta} \varphi(\gamma(s))\right) \vee \varphi(\zeta) = \varphi(\gamma(\xi)) \vee \varphi(\zeta) = 0 \vee 0 = 0. \qquad (3.13)
\end{aligned}$$

All the inequalities in (3.13) are equalities, to give (1.6). Note that we have used in (3.13) the fact that (1.6) holds at the point ζ, which we established in the paragraph above.

We now prove (2.4), and by monotonicity it is enough to establish this inequality in $(\zeta, \zeta + \varepsilon]$ for some ε. Taking $\xi_0 = \zeta$ in (2.5) and ε as in the surrounding statement, we have for every $\xi \in (\zeta, \zeta + \varepsilon]$ that

$$\varphi(\xi) = \max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\zeta)} (A(s, M) + \varphi(\gamma(s))) = \max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\zeta)} A(s, M) < 0. \qquad (3.14)$$

In particular $\varphi(\gamma(s)) = 0$ throughout the range in (3.14) as $\gamma(s) \leqslant \zeta$, and the strict inequality in (3.14) holds by virtue of the fact that $\zeta \in Z(M)$.

Finally, let us establish (2.6). With $\tilde{\zeta}$ as given, we have that $A(\tilde{\zeta}, M) = 0$ as $\tilde{\zeta} \in Z(M)$ and so both φ and $\tilde{\varphi}$ are constant in $[\gamma(\tilde{\zeta}), \tilde{\zeta}]$ by Proposition 3.3. This gives the equation in (2.6) in that interval, as $\tilde{\varphi}(\tilde{\zeta}) = 0$. Denoting $\hat{\varphi}(\xi) = \varphi(\xi) - \varphi(\tilde{\zeta})$, we have that both $\hat{\varphi}$ and $\tilde{\varphi}$ satisfy Eq. (1.8) in $[\tilde{\zeta}, M]$ and agree in $[\gamma(\tilde{\zeta}), \tilde{\zeta}]$ and so Proposition 3.2 implies that they agree throughout $[\gamma(\tilde{\zeta}), M]$. Thus (2.6) holds as stated. $\square$

The next result shows that solutions of Eq. (1.6) also satisfy Eq. (1.8) for certain ranges of ξ. This will allow us to make the connection between general solutions of (1.6) and the basis solutions, which satisfy (1.8) in certain intervals.

Proposition 3.4. *Let ψ satisfy Eq.* (1.6) *in* $[0, M]$ *and set*

$$U = \{\xi \in [0, M] \mid \psi(\xi) > \psi(\gamma_M^{-1}(\xi))\}. \qquad (3.15)$$

Then $U \subseteq (0, M)$ is an open set and Eq. (1.8) *holds at every $\xi \in \bar{U}$. If $(\xi_1, \xi_2) \subseteq U$ is a maximal connected component of U then $\xi_1 = \gamma(\zeta)$ for some $\zeta \in Z(M)$ which satisfies*

$$\text{either } \zeta \in (\xi_1, \xi_2) \text{ or else } \zeta = 0 \qquad (3.16)$$

and also

$$\psi(\xi) < \psi(\zeta) \quad \text{for every } \xi \in (\zeta, M]. \qquad (3.17)$$

If we have an interval $[\xi_1, \xi_2] \subseteq [0, M] \backslash U$ *in the complement of* U *then* $\psi(\xi) = \psi(\xi_1)$ *is constant for* $\xi \in [\xi_1, \gamma_M^{-1}(\xi_2)]$.

Proof. Without loss ψ is a nontrivial solution. Clearly U is a relatively open subset of $[0, M]$, and as $\gamma_M^{-1}(0) = 0$ and $\gamma_M^{-1}(M) = M$ we have that $0, M \notin U$. Thus U is open. Now fix any $\xi \in U$. Then breaking the interval $[\xi, M]$ into two pieces at the point $\gamma_M^{-1}(\xi)$, we may write Eq. (1.6) as

$$\begin{aligned} \psi(\xi) &= \left(\max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi)} (A(s, M) + \psi(\gamma(s))) \right) \vee \left(\max_{\gamma_M^{-1}(\xi) \leqslant s \leqslant M} (A(s, M) + \psi(\gamma(s))) \right) \\ &= \left(\max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi)} (A(s, M) + \psi(\gamma(s))) \right) \vee \psi(\gamma_M^{-1}(\xi)). \end{aligned} \tag{3.18}$$

As $\psi(\xi) > \psi(\gamma_M^{-1}(\xi))$ holds we have that the first of the two terms to the right of the final equality in (3.18) achieves the maximum, which gives Eq. (1.8). With (1.8) thus holding in U, it follows by continuity that (1.8) holds throughout $\bar{U}$.

Let $(\xi_1, \xi_2) \subseteq U$ be a maximal connected component of U. Suppose first that $\xi_1 > 0$, hence $\xi_1 < \gamma_M^{-1}(\xi_1)$, and so $\xi \in (\xi_1, \gamma_M^{-1}(\xi_1))$ for every $\xi > \xi_1$ sufficiently near ξ_1. For such ξ we have that

$$\psi(\xi_1) = \psi(\xi) = \psi(\gamma_M^{-1}(\xi_1)) > \psi(\gamma_M^{-1}(\xi)) \tag{3.19}$$

as $\xi_1 \notin U$ but $\xi \in U$, where the fact that ψ is monotone decreasing is used. In fact, monotonicity implies that (3.19) holds for every $\xi \in (\xi_1, \gamma_M^{-1}(\xi_1)]$ so it follows that $(\xi_1, \gamma_M^{-1}(\xi_1)] \subseteq U$, hence $(\xi_1, \gamma_M^{-1}(\xi_1)] \subseteq (\xi_1, \xi_2)$ and so

$$\gamma_M^{-1}(\xi_1) \in (\xi_1, \xi_2). \tag{3.20}$$

Also observe from (3.19) that $\gamma_M^{-1}(\xi_1) < \gamma_M^{-1}(\xi) \leqslant M$ and thus $\xi_1 = \gamma(\zeta)$ for some $\zeta \in (0, M)$. Thus the inclusion in (3.16) holds, which is just (3.20). Writing $\tilde{\xi} = \gamma_M^{-1}(\xi)$ in (3.19) gives $\psi(\tilde{\xi}) < \psi(\zeta)$ for every $\tilde{\xi} > \zeta$ near ζ, and hence for every $\tilde{\xi} \in (\zeta, M]$ by monotonicity. This directly gives (3.17).

We now show that $\zeta \in Z(M)$. We apply Lemma 3.1 at the point $\xi_0 = \zeta$ with ξ_1 and ξ_2 as above, and taking $\alpha(\xi) = \xi$ and $\beta(\xi) = \gamma_M^{-1}(\xi)$ with $Q(s) = A(s, M) + \psi(\gamma(s))$. With these choices we have $\varphi(\xi) = \psi(\xi)$ for the function φ given by (3.1) for every $\xi \in [\xi_1, \xi_2] \subseteq \bar{U}$, as Eq. (1.8) holds in that interval. Note that $\alpha(\xi_0) < \beta(\xi_0)$ as $\xi_0 \in (\xi_1, \xi_2) \subseteq (0, M)$. Also, the inequality in (3.17) gives inequality (3.2), which is strict. Thus the second conclusion (3.4) of Lemma 3.1 holds and so

$$A(s, M) + \psi(\gamma(s)) < \psi(\xi_0) = A(\xi_0, M) + \psi(\gamma(\xi_0)) \tag{3.21}$$

for every $s \in (\xi_0, \gamma_M^{-1}(\xi_0)]$. Now $\xi_1 = \gamma(\xi_0) < \gamma(s) \leqslant \xi_0 = \gamma_M^{-1}(\xi_1)$ for such s, and so $\psi(\gamma(\xi_0)) = \psi(\gamma(s)) = \psi(\xi_0)$ in light of the equalities in (3.19) and again from the monotonicity of ψ. Subtracting this quantity from (3.21) gives

$$A(s, M) < 0 = A(\xi_0, M)$$

for every $s \in (\xi_0, \gamma_M^{-1}(\xi_0)]$, and we have from this that $\zeta = \xi_0 \in Z(M)$, as desired.

Now suppose that $(\xi_1, \xi_2) \subseteq U$ is a maximal connected component of U with $\xi_1 = 0$. Then taking $\zeta = 0$ we have that $\psi(\xi) < \psi(\gamma(\xi)) \leqslant 0$ for every $\xi > 0$ near 0, as $\gamma(\xi) \in U$, and this gives (3.17). Setting $\xi = 0$ in Eq. (1.8), which holds there as $0 \in \bar{U}$, gives $A(0, M) = 0$ which directly implies that $\zeta = 0 \in Z(M)$.

Finally, suppose that $[\xi_1, \xi_2] \subseteq [0, M] \backslash U$. For every $\xi \in [\xi_1, \xi_2]$ we have that $\psi(\xi) = \psi(\gamma_M^{-1}(\xi))$ and hence that ψ is constant in $[\xi, \gamma_M^{-1}(\xi)]$ by monotonicity. As $\xi < \gamma_M^{-1}(\xi)$, at least for $\xi \neq \xi_1, \xi_2$, we see that ψ is constant in the union of these intervals, namely in the set $[\xi_1, \gamma_M^{-1}(\xi_2)]$. This is as claimed. $\square$

Corollary 3.5. *Let ψ be a nontrivial solution of* (1.6) *in* $[0, M]$. *Suppose that* $\psi(\xi_0) = \psi(\gamma_M^{-1}(\xi_0))$ *for some* $\xi_0 \in [0, \gamma(M)]$. *Then either there exists* $\zeta \in Z(M) \cap [\xi_0, M]$ *such that*

$$\psi(\xi) \begin{cases} = \psi(\xi_0), & \xi \in [\xi_0, \zeta], \\ < \psi(\xi_0), & \xi \in (\zeta, M], \end{cases} \tag{3.22}$$

or else $\psi(\xi_0) = -\infty$ *in which case* $\xi_0 = M = C$ *and* $\gamma(C) = C$.

Proof. Let U be as in (3.15) and note that $\xi_0 \notin U$. Suppose first that $U \cap [\xi_0, M] \neq \phi$ and observe that the point $\xi_1 = \inf(U \cap [\xi_0, M])$ is either the left-hand endpoint of a maximal connected component $(\xi_1, \xi_2) \subseteq U$ of U of else the decreasing limit of such points. By Proposition 3.4 it is the case that $\xi_1 = \gamma(\zeta)$ for some $\zeta \in \overline{Z(M)} = Z(M)$. Also, as $[\xi_0, \xi_1] \subseteq [0, M] \backslash U$ we have by this result that ψ is constant in $[\xi_0, \gamma_M^{-1}(\xi_1)] = [\xi_0, \zeta]$ to give the equality in (3.22). Finally, the inequality in (3.17) implies the inequality in (3.22).

Now suppose that $U \cap [\xi_0, M] = \phi$. Again by Proposition 3.4 we have that ψ is constant in $[\xi_0, \gamma_M^{-1}(M)] = [\xi_0, M]$, with a value which we may assume to be finite otherwise $\xi_0 = M = C$ and $\gamma(C) = C$ by Proposition 2.1. Also by assumption $\gamma(M) \in [\xi_0, M]$, and so $\psi(M) = \psi(\gamma(M))$. This equation implies, upon setting $\xi = M$ in Eq. (1.6), that $A(M, M) = 0$, from which it follows immediately that $M \in Z(M)$. Thus (3.22) holds with $\zeta = M$, as desired. $\square$

Remark. If $A(\zeta, M) = 0$ with $\zeta > 0$ but $\zeta \notin Z(M)$ then the construction in the proof of Theorem 2.3 still yields a solution φ of (1.6) which satisfies (2.3). However, (2.4) can no longer be true by Corollary 3.5. In fact if we let $\tilde{\zeta} \in Z(M)$ with $\tilde{\zeta} > \zeta$ denote the first point of $Z(M)$ to the right of ζ and let $\tilde{\varphi}$ denote the solution of (1.6), (2.3) with $\tilde{\zeta}$ in place of ζ, then in fact it is the case that the solutions φ and $\tilde{\varphi}$ are the same. Indeed, $\varphi(\xi) = 0$ holds for $\xi \in [0, \zeta]$ and hence at least for $\xi \in [0, \tilde{\zeta}]$ by Corollary 3.5, where we take $\xi_0 = 0$ in that result. Of course $\tilde{\varphi}(\xi) = 0$ in $[0, \tilde{\zeta}]$ as well. Both φ and $\tilde{\varphi}$ are monotone decreasing and satisfy Eq. (1.8) in $[\tilde{\zeta}, M]$, so these functions are identical by the uniqueness claim of Theorem 2.3.

With the above results we are ready to prove our main theorem.

Proof of the Basis Theorem. With φ^i as in the statement of the theorem, as noted earlier we have that for any choice of $c^i \in [-\infty, \infty)$ the right-hand side of (1.4) is a solution of (1.6). Thus we must show that every solution ψ of (1.6) has the form (1.4) for some c^i. We label the q elements of $Z(M)$ corresponding to the order

$$\zeta^1 < \zeta^2 < \cdots < \zeta^q$$

and we let ψ be any nontrivial solution of (1.6). By Corollary 3.5 with $\xi_0 = 0$ this solution satisfies (3.22) for $\zeta = \zeta^j \in Z(M)$, for some unique $1 \leqslant j \leqslant q$. Let us call the integer j the *index* of the solution ψ.

Our proof that every nontrivial solution of Eq. (1.6) has the form (1.4) will proceed by reverse induction on the index, that is, we assume the result holds for solutions with indices k satisfying $j+1 \leqslant k \leqslant q$ and we prove it for solutions with index j. The case $j = q$ begins the induction. The reader will observe that our argument below applies both to the case where $1 \leqslant j < q$, for which the induction hypothesis is assumed, and also to the initial case $j = q$.

We keep our solution ψ of index j fixed from now on, where $1 \leqslant j \leqslant q$, and with the induction assumption holding if $j < q$. Also, without loss we may assume that $\psi(0) = 0$, as adding a constant to a solution preserves the form of representation (1.4).

Consider the set U in (3.15) associated to ψ. If $U = \phi$ then $\psi(\xi) = 0$ identically in $[0, M]$ from the final statement of Proposition 3.4, and so $M \in Z(M)$ by Corollary 3.5 where we take $\xi_0 = 0$. In this case $M = \zeta^q$, the rightmost element of $Z(M)$, so $j = q$. Observing in this case that $\varphi^q(\xi) = 0$ identically in $[0, M]$, we thus have that $\psi(\xi) = \varphi^q(\xi)$ in $[0, M]$. This gives (1.4) with $c^i = -\infty$ for $1 \leqslant i \leqslant q-1$ and $c^q = 0$.

Now suppose that $U \neq \phi$. Then by Proposition 3.4 there is a leftmost maximal connected component (ξ_1, ξ_2) of U where $\xi_1 = \gamma(\zeta)$ for some $\zeta \in Z(M)$ satisfying $\zeta \in (\xi_1, \xi_2)$ as in (3.16). It is clear from the definition of U and from Proposition 3.4 that $\zeta = \zeta^j$ with j the index of ψ as above. Also, ψ satisfies Eq. (1.8) throughout $[\xi_1, \xi_2]$, again by Proposition 3.4. Note additionally that ψ satisfies Eq. (1.8) in the interval $[\gamma(M), M]$ since in this range $\gamma_M^{-1}(\xi) = M$ and so Eqs. (1.6) and (1.8) have the same form. Thus if $\xi_2 \geqslant \gamma(M)$ then ψ satisfies Eq. (1.8) throughout $[\zeta^j, M] \subseteq [\xi_1, \xi_2] \cup [\gamma(M), M]$. As ψ vanishes identically in $[0, \zeta^j]$ it therefore satisfies the conditions (2.3) with $\zeta = \zeta^j$ which characterize the basis solution φ^j. In this case $\psi(\xi) = \varphi^j(\xi)$ in $[0, M]$, and (1.4) holds with $c^i = -\infty$ for $1 \leqslant i \leqslant q$ for which $i \neq j$ and with $c^j = 0$.

There remains to consider the case when $U \neq \phi$ and $\xi_2 < \gamma(M)$. As $\xi_2 \notin U$ we have that $\psi(\xi_2) = \psi(\gamma_M^{-1}(\xi_2))$ and so we have (3.22) of Corollary 3.5 for some $\zeta \in Z(M)$, where we take $\xi_0 = \xi_2$ in that result. Denoting by $\zeta = \zeta^{\tilde{j}}$ the quantity in that result, we have that ψ is constant throughout $[\xi_2, \zeta^{\tilde{j}}]$ but not in $[\xi_2, \zeta^{\tilde{j}} + \varepsilon]$ for any $\varepsilon > 0$, so necessarily $\zeta^{\tilde{j}} \geqslant \gamma_M^{-1}(\xi_2)$, and therefore

$$\xi_1 < \gamma_M^{-1}(\xi_1) = \zeta^j < \xi_2 < \gamma_M^{-1}(\xi_2) \leqslant \zeta^{\tilde{j}}. \tag{3.23}$$

Thus $j+1 \leqslant \tilde{j} \leqslant q$. Let us define a function $\tilde{\psi} : [0, M] \to [-\infty, \infty)$ by

$$\tilde{\psi}(\xi) = \begin{cases} \psi(\xi_2), & \xi \in [0, \xi_2], \\ \psi(\xi), & \xi \in [\xi_2, M]. \end{cases} \tag{3.24}$$

Then $\tilde{\psi}(\xi) = \psi(\xi_2)$ is constant throughout $[0, \zeta^{\tilde{j}}]$ and this is the maximal interval containing 0 for which this is true. We claim that $\tilde{\psi}$ satisfies Eq. (1.6) throughout $[0, M]$ and also that

$$\psi(\xi) = \varphi^j(\xi) \vee \tilde{\psi}(\xi) \quad \text{for every } \xi \in [0, M]. \tag{3.25}$$

If we prove these two claims we are done. In particular, the solution $\tilde{\psi}$ has index $\tilde{j} > j$ so by the induction hypothesis it has the form (1.4), but with coefficients which we may denote by $\tilde{c}^i$. With (3.25) this gives the desired form (1.4) for ψ, with coefficients $c^i = \tilde{c}^i$ for $i \neq j$ and $c^j = 0 \vee \tilde{c}^j$.

Let us first show that $\tilde{\psi}$ is a solution of (1.6) in $[0, M]$. Clearly (1.6) holds for $\tilde{\psi}$ in the interval $[\gamma_M^{-1}(\xi_2), M]$, as in this range the right-hand side of (1.6) involves only values of $\tilde{\psi}$ to the right of ξ_2, where $\tilde{\psi}$ and ψ agree (the fact that $\gamma(\gamma_M^{-1}(\xi_2)) = \xi_2$, which holds because $\xi_2 < \gamma(M)$, is used here). If $\xi \in [0, \gamma_M^{-1}(\xi_2)]$ then

$$\begin{aligned} \tilde{\psi}(\xi) = \tilde{\psi}(\gamma_M^{-1}(\xi_2)) &= \max_{\gamma_M^{-1}(\xi_2) \leqslant s \leqslant M} (A(s, M) + \tilde{\psi}(\gamma(s))) \leqslant \max_{\xi \leqslant s \leqslant M} (A(s, M) + \tilde{\psi}(\gamma(s))) \\ &= \left(\max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi_2)} (A(s, M) + \tilde{\psi}(\gamma(s))) \right) \vee \left(\max_{\gamma_M^{-1}(\xi_2) \leqslant s \leqslant M} (A(s, M) + \tilde{\psi}(\gamma(s))) \right) \\ &\leqslant \left(\max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi_2)} \tilde{\psi}(\gamma(s)) \right) \vee \tilde{\psi}(\gamma_M^{-1}(\xi_2)) = \tilde{\psi}(\gamma(\xi)) \vee \tilde{\psi}(\gamma_M^{-1}(\xi_2)) = \tilde{\psi}(\gamma_M^{-1}(\xi_2)), \end{aligned} \tag{3.26}$$

where the constancy of $\tilde{\psi}$ in $[0, \zeta^{\tilde{j}}]$ is used and we note (3.23). All inequalities in (3.26) are thus equalities, so (1.6) also holds for $\tilde{\psi}$ in $[0, \gamma_M^{-1}(\xi_2)]$ and therefore throughout $[0, M]$.

We now prove (3.25). We have that $\psi(\xi) = \varphi^j(\xi) = 0$ in $[0, \zeta^j]$. Also ψ satisfies (1.8) in $[\zeta^j, \xi_2] \subseteq [\xi_1, \xi_2] \subseteq \bar{U}$, as does φ^j by definition. Thus $\psi(\xi) = \varphi^j(\xi)$ in $[0, \xi_2]$ by the uniqueness result Proposition 3.2, and since also $\tilde{\psi}(\xi) = \psi(\xi_2) \leqslant \psi(\xi)$ there we obtain Eq. (3.25) in that interval. To obtain (3.25) in $[\xi_2, M]$ it is enough, by (3.24), to prove that $\varphi^j(\xi) \leqslant \tilde{\psi}(\xi)$ there. This inequality holds throughout $[\xi_2, \zeta^{\tilde{j}}]$ since for ξ

in this interval we have that $\varphi^j(\xi) \leqslant \varphi^j(\xi_2) = \psi(\xi_2) = \tilde{\psi}(\xi)$, where we use, respectively, the monotonicity of φ^j, an equality noted immediately above, and the fact that $\tilde{\psi}(\xi) = \psi(\xi_2)$ is constant in $[0, \zeta^{\tilde{j}}]$, noted earlier in this proof. Let $\xi_* = \sup\{\xi \in [\zeta^{\tilde{j}}, M] \mid \varphi^j(\xi) \leqslant \tilde{\psi}(\xi)\}$ and suppose that $\xi_* < M$. We seek a contradiction. Applying Theorem 2.3 to φ^j, in particular using (2.5) with ξ_* in place of ξ_0 in that equation, we have that

$$
\begin{aligned}
\varphi^j(\xi) &= \max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi_*)} (A(s, M) + \varphi^j(\gamma(s))) \\
&\leqslant \max_{\xi \leqslant s \leqslant \gamma_M^{-1}(\xi_*)} (A(s, M) + \tilde{\psi}(\gamma(s))) \leqslant \max_{\xi \leqslant s \leqslant M} (A(s, M) + \tilde{\psi}(\gamma(s))) = \tilde{\psi}(\xi) \quad (3.27)
\end{aligned}
$$

for every $\xi \in [\xi_*, \xi_* + \varepsilon]$. In particular, the first inequality in (3.27) follows from the fact that $\varphi^j(\zeta) \leqslant \tilde{\psi}(\zeta)$ in the interval

$$[\gamma(\xi), \gamma(\gamma_M^{-1}(\xi_*))] \subseteq [\gamma(\xi_*), \xi_*] \subseteq [\xi_2, \xi_*], \qquad (3.28)$$

where the second inclusion in (3.28) holds because $\xi_2 = \gamma(\gamma_M^{-1}(\xi_2)) \leqslant \gamma(\zeta^{\tilde{j}}) \leqslant \gamma(\xi_*)$, as $\xi_2 < \gamma(M)$ and $\gamma_M^{-1}(\xi_2) \leqslant \zeta^{\tilde{j}}$ by (3.23). But now the inequalities in (3.27) contradict the definition of ξ_*, and completes the proof. □

4. Varying the parameter M

Our object in this section is to understand how the solutions of Eq. (1.1) vary with $M \in (0, C]$. In particular, from the way that Eq. (1.1) arises in [16] it is very natural to consider the case in which it has a unique solution for every M, that is where $q = q(M) = 1$ for every M in the statement of the Basis Theorem. This motivates the following definition.

Definition. The pair (H, γ) is said to be *quasimodal* if the set $Z(M)$ is a singleton $Z(M) = \{\zeta(M)\}$, with $\zeta(M) > 0$, for every $M \in (0, C]$.

If H is monotone increasing throughout $[0, C]$ with $H(\xi) > H(0)$ in $(0, C]$, then for any γ the pair (H, γ) is quasimodal with $Z(M) = \{M\}$ and $P(M) = H(M)$ for every $M \in (0, C]$. Also, a function is sometimes called *unimodal* if it is monotone increasing to the left of a maximum and monotone decreasing to the right. One easily sees that if H is unimodal with $H(\xi) > H(0)$ for every ξ near 0, then (H, γ) is quasimodal for any γ. Indeed, if $M_0 \in (0, C)$ denotes the location of the rightmost maximum of a unimodal function H then

$$Z(M) = \begin{cases} \{M\}, & M \in (0, M_0], \\ \{M_0\}, & M \in [M_0, C], \end{cases} \quad P(M) = \begin{cases} H(M), & M \in (0, M_0], \\ H(M_0), & M \in [M_0, C]. \end{cases} \tag{4.1}$$

More generally, if H is monotone increasing in $[0, M_0]$ with $H(\xi) > H(0)$ for ξ near 0, and satisfies $H(\xi) < H(M_0)$ in $(M_0, C]$, for some $M_0 \in (0, C]$, then (H, γ) is quasimodal with (4.1) holding.

Remark. If (H, γ) is quasimodal then the unique element $\zeta(M)$ of $Z(M)$ need not vary continuously with M. For example, suppose that H achieves its maximum in $[0, C]$ at exactly two points $M_1 < M_2$, with $\gamma(M_2) < M_1$ and with H monotone increasing in $[0, M_1]$. Then (H, γ) is quasimodal. One first sees that $Z(M) = \{M\}$ for $M \in (0, M_1]$ and that $Z(M) = \{M_1\}$ for $M \in [M_1, M_2)$. However, $Z(M) = \{M_2\}$ for $M \in [M_2, C]$ and so $\zeta(M)$ undergoes a jump at $M = M_2$.

We see from the above remark that the function H in a quasimodal pair (H, γ) need not have a unique maximum. Although it is somewhat awkward to write down a succinct set of necessary and sufficient conditions for (H, γ) to be quasimodal, it is straightforward to check whether a given (H, γ) satisfies the definition of a quasimodal pair.

A principal result about Eq. (1.1), or (1.6), when (H, γ) is quasimodal, is the following theorem. It describes the solutions for various M in terms of the canonical solution when $M = C$.

Theorem 4.1. *Assume the pair (H, γ) is quasimodal. For every $M \in (0, C]$ let $\varphi(\cdot, M)$ denote the solution to* (1.6) *given by Theorem* 2.3 *corresponding to the unique element $\zeta(M)$ of $Z(M)$. Then*

$$\varphi(\xi, M) = \begin{cases} \varphi(\xi, C), & \xi \in [0, \gamma(M)], \\ \max\limits_{\xi \leqslant s \leqslant M} (A(s, M) + \varphi(\gamma(s), C)) \leqslant \varphi(\xi, C), & \xi \in [\gamma(M), M], \end{cases} \tag{4.2}$$

holds.

The following results are needed before we prove Theorem 4.1.

Proposition 4.2. *Assume that (H, γ) is quasimodal and suppose for some $M_0 \in (0, C]$ that $\zeta(M_0) \leqslant \gamma(M_0)$. Then $P(M) = P(M_0)$ and $\zeta(M) = \zeta(M_0)$ for every $M \in [\zeta(M_0), C]$.*

Corollary 4.3. *Assume that (H, γ) is quasimodal. Then if $\zeta(M_0) \leqslant \gamma(M_0)$ for some $M_0 \in (0, C]$ we have that $\zeta(M) < \gamma(M)$ for every $M \in (M_0, C]$. If on the other hand $\zeta(M_0) \geqslant \gamma(M_0)$ for some $M_0 \in (0, C]$ then $\zeta(M) > \gamma(M)$ for every $M \in (0, M_0)$.*

Proof of Proposition 4.2. We first observe that

$$\gamma_M^{-1}(\zeta(M_0)) \leqslant \gamma_{M_0}^{-1}(\zeta(M_0)) \quad \text{for every } M \in [\zeta(M_0), C]. \tag{4.3}$$

Certainly (4.3) is immediate when $M \leqslant M_0$. On the other hand when $M > M_0$ then the inequality $\zeta(M_0) \leqslant \gamma(M_0)$ ensures that in fact $\gamma_M^{-1}(\zeta(M_0)) = \gamma_{M_0}^{-1}(\zeta(M_0))$. Thus (4.3) holds.

We now conclude from (4.3) that whenever $M \in [\zeta(M_0), C]$ is such that $P(M) = P(M_0)$, then $\zeta(M_0) \in Z(M)$ and hence $\zeta(M) = \zeta(M_0)$. This follows directly from the relations

$$A(\zeta(M_0), M) = A(\zeta(M_0), M_0) = 0,$$

$$A(\xi, M) = A(\xi, M_0) < 0 \quad \text{for every } \xi \in (\zeta(M_0), \gamma_{M_0}^{-1}(\zeta(M_0))],$$

for such M, from the fact that $(\zeta(M_0), \gamma_M^{-1}(\zeta(M_0))] \subseteq (\zeta(M_0), \gamma_{M_0}^{-1}(\zeta(M_0))]$, and from the definition of $Z(M)$.

We thus need only prove that $P(M) = P(M_0)$ for every $M \in [\zeta(M_0), C]$. As H achieves its maximum in $[0, M_0]$ at $\zeta(M_0)$ then $P(M) = P(M_0)$ for every $M \in [\zeta(M_0), M_0]$, where we recall the definition of $P(M)$. Also, as the rightmost zero of $A(\cdot, M_0)$ in $[0, M_0]$ must belong to $Z(M_0)$ we have that $A(\xi, M_0) < 0$, equivalently $H(\xi) < P(M_0)$, throughout $(\zeta(M_0), M_0]$. It is sufficient therefore to prove that $H(\xi) < P(M_0)$ in $(M_0, C]$, and to this end we assume to the contrary that there exists $M_1 \in (M_0, C]$ such that $H(\xi) < P(M_0)$ in (M_0, M_1) but $H(M_1) = P(M_0)$. Then $P(M_1) = P(M_0)$, with the maximum of H in $[0, M_1]$ being achieved both at $\xi = \zeta(M_0)$ and at $\xi = M_1$. From the paragraph above we have that $\zeta(M_1) = \zeta(M_0)$. Also, $A(M_1, M_1) = 0$ and so $M_1 \in Z(M_1)$. But then $Z(M_1)$ contains more than one point, contradicting the fact that (H, γ) is quasimodal. This proves the proposition. □

Proof of Corollary 4.3. The first conclusion follows directly from Proposition 4.2 and the fact that γ is strictly increasing. The second conclusion follows from the first, as the contrapositive. □

We now prove the main result of this section.

Proof of Theorem 4.1. Let us establish the first line of (4.2). Several cases are considered. First, if $\zeta(C) \geqslant \gamma(C)$ then $\zeta(M) > \gamma(M)$ for every $M \in (0, C)$ by Corollary 4.3. Then for every $\xi \in [0, \gamma(M)]$ we have that $\xi \leqslant \gamma(M) < \zeta(M)$, hence $\varphi(\xi, M) = 0$, and also that $\xi \leqslant \gamma(M) < \gamma(C) \leqslant \zeta(C)$, hence $\varphi(\xi, C) = 0$. Thus $\varphi(\xi, M) = \varphi(\xi, C)$ as desired.

Next suppose that $\zeta(C) < \gamma(C)$. Then Proposition 4.2 implies that $P(M) = P(C)$ and $\zeta(M) = \zeta(C)$ for every $M \in [\zeta(C), C]$. Note in particular that $\zeta(\zeta(C)) = \zeta(C)$. Denoting $M_0 = \zeta(C)$, we have that $\gamma(M_0) \leqslant M_0 = \zeta(M_0)$ and thus $\gamma(M) < \zeta(M)$ for every $M \in (0, M_0)$ by Corollary 4.3. For this range of M we again have, for every

$\xi \in [0, \gamma(M)]$, that $\xi \leqslant \gamma(M) < \zeta(M)$ hence $\varphi(\xi, M) = 0$, and that $\xi \leqslant \gamma(M) < M < M_0 = \zeta(C)$ hence $\varphi(\xi, C) = 0$. Again $\varphi(\xi, M) = \varphi(\xi, C)$ to give the first line of (4.2).

There remains to consider the case when $\zeta(C) < \gamma(C)$ and $M \in [M_0, C]$. As noted, $P(M) = P(C)$ and so $A(s, M) = A(s, C)$, and also $\zeta(M) = \zeta(C)$. Both $\varphi(\cdot\,, M)$ and $\varphi(\cdot\,, C)$ vanish in $[0, \zeta(M)]$ and they both satisfy the same Eq. (1.8) in $[\zeta(M), \gamma(M)]$, in particular because $\gamma_M^{-1}(\xi) = \gamma_C^{-1}(\xi)$ for ξ in that interval. Thus the uniqueness result Proposition 3.2 with $[\xi_1, \xi_2] = [\zeta(M), \gamma(M)]$ implies that $\varphi(\cdot, M)$ and $\varphi(\cdot, C)$ agree in that interval. Again we have the first line of (4.2).

The equality in the second line of (4.2) is obtained from Eq. (1.6) for $\varphi(\cdot\,, M)$ upon replacing $\varphi(\gamma(s), M)$ with $\varphi(\gamma(s), C)$ in the right-hand side of that equation. This replacement is justified as these two terms agree by virtue of the first line of (4.2).

To obtain the inequality in the second line of (4.2) we consider two cases. First, if $\zeta(C) \geqslant M$ then $\varphi(\xi, C) = 0$ throughout $[0, M]$ while $\varphi(\xi, M) \leqslant 0$ there, to give the inequality. Now suppose that $\zeta(C) < M$. Then H achieves its maximum $P(C)$ in $[0, C]$ at $\zeta(C) \in [0, M)$, and so $P(C)$ is also the maximum of H in $[0, M]$, that is $P(M) = P(C)$. Thus $A(s, M) = A(s, C)$ and we have that

$$\begin{aligned}\varphi(\xi, M) &= \max_{\xi \leqslant s \leqslant M} (A(s, M) + \varphi(\gamma(s), C)) \\ &\leqslant \max_{\xi \leqslant s \leqslant C} (A(s, C) + \varphi(\gamma(s), C)) = \varphi(\xi, C)\end{aligned}$$

upon making this replacement in the equality in the second line of (4.2). This establishes (4.2), as desired. □

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