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Nonexpansive Periodic Operators in l_1 with Application to Superhigh-Frequency Oscillations in a Discontinuous Dynamical System with Time Delay

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We prove that the iterates of certain periodic nonexpansive operators in l_1 uniformly converge to zero in l_∞ norm. As a by-product we show that, for any solution $x(t)$ of the equation

$$\dot{x}(t) = -\operatorname{sign}(x(t-1)) + f(x(t)), \qquad t \geqslant 0, \quad x|_{[-1,0]} \in C[-1,0]$$

where $f: \mathbb{R} \to (-1, 1)$ is locally Lipschitz, the number of zeros of $x(t)$ on any unit interval becomes finite after a period of time, with the single exception of the case $f(0) = 0$ and $x(t) \equiv 0$.

KEY WORDS: Nonexpansive operators; differential delay equations.

1. INTRODUCTION

Recall that a map $F: D \subset X \to X$ on a Banach space $(X, \|\cdot\|)$ is called "nonexpansive" if $\|F(a) - F(b)\| \leqslant \|a - b\|$ for all $a, b \in D$. Here we shall be interested in $X = l_1(\mathbb{Z})$, the Banach space of biinfinite absolutely summable real sequences $\langle a_n \mid n \in \mathbb{Z} \rangle$, with $\|a\|_1 := \sum_{n \in \mathbb{Z}} |a_n|$. As usual, $l_1^+(\mathbb{Z}) = \{a \in l_1(\mathbb{Z}) : a_n \geqslant 0 \ \forall n \in \mathbb{Z}\}$, and we shall be interested in nonexpansive maps $F: l_1^+(\mathbb{Z}) \to l_1^+(\mathbb{Z})$. Our maps will also be "integral-preserving" (so $\|F(a)\|_1 = \|a\|_1$ for all $a \in l_1^+(\mathbb{Z})$) and "order-preserving" (defined below), and F will, in a precise sense described later, be periodic of period N, where N is a positive integer.

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If $a = \langle a_n \mid n \in \mathbb{Z} \rangle$ is a bounded, biinfinite sequence, we define $\|a\|_\infty$, as usual, by

$$\|a\|_\infty = \sup\{ |a_n| : n \in \mathbb{Z} \}$$

If F is as above, F^k will denote the kth iterate of F, the composition of F with itself k times. We shall be interested in the following questions:

Question 1. Let $D_R = \{ a \in l_1^+(\mathbb{Z}) : \|a\|_1 \leqslant R \}$ and suppose that $F: D_R \to D_R$ is as above. Is it true that $\lim_{k \to \infty} \|F^k(a)\|_\infty = 0$ for every $a \in D_R$? Is it true that $\lim_{k \to \infty} \|F^k(a)\|_\infty = 0$, uniformly with respect to $a \in D_R$?

Question 2. If the answers to Question 1 are negative, what can be said about the iterates $F^k(a)$ in general?

Our original motivation for studying such questions came from a very special case. Assume that $\sigma: [0, R] \to [0, R]$ and $\tau: [0, R] \to [0, R]$ are continuous, strictly increasing functions satisfying $\sigma(0) = \tau(0) = 0$ and

$$\lambda \leqslant \frac{\sigma(t_1) - \sigma(t_2)}{t_1 - t_2} \leqslant 1 - \lambda, \qquad \lambda \leqslant \frac{\tau(t_1) - \tau(t_2)}{t_1 - t_2} \leqslant 1 - \lambda \tag{1}$$

for some $\lambda \in (0, 1/2)$ and all $t_1 \neq t_2$ in $[0, R]$. Define $F : D_R := \{ a \in l_1^+(\mathbb{Z}) : \|a\|_1 \leqslant R \} \to D_R$ by $F(a) = b$, where

$$\begin{aligned} b_{2n} &= \sigma(a_{2n}) + \tau(a_{2n-1}), && n \in \mathbb{Z} \\ b_{2n+1} &= (\mathrm{Id} - \tau)(a_{2n+1}) + (\mathrm{Id} - \sigma)(a_{2n}), && n \in \mathbb{Z} \end{aligned} \tag{2}$$

As a very special case of our general results, we shall prove below that $\lim_{k \to \infty} \|F^k(a)\|_\infty = 0$, uniformly with respect to $a \in D_R$.

The ideas used in analyzing the above special case are sufficient to prove the absence of superhigh-frequency solutions $x(t)$ of the equation

$$\begin{aligned} \dot{x}(t) &= -\operatorname{sign}(x(t-1)) + f(x(t)), \qquad t \geqslant 0 \\ x|_{[-1, 0]} &= \varphi \in C([-1, 0]) \end{aligned} \tag{3}$$

Here $\operatorname{sign}(w) = 1$ for $w > 0$, $\operatorname{sign}(w) = 0$ for $w = 0$, $\operatorname{sign}(w) = -1$ for $w < 0$, and $f: \mathbb{R} \to (-1, 1)$ is locally Lipschitz. By the absence of superhigh-frequency solutions we mean that if either $\varphi \not\equiv 0$ or $\varphi \equiv 0$ and $f(0) \neq 0$, then there exist $T = T_\varphi \geqslant 0$ and an integer n_φ such that $x|_{(t-1, t)}$ has at most n_φ zeros for all $t \geqslant T_\varphi$. A more precise statement is given in Section 3.

In this paper we restrict ourselves for simplicity to $X = l_1(\mathbb{Z})$. In a future paper we shall consider "nonexpansive, periodic" nonlinear operators in $l_1(\mathbb{Z}^d)$ and $\mathscr{L}_1(\mathbb{R}^d)$, $d \geqslant 1$.

For the reader's convenience we mention our principal results.

- We present positive answers to Question 1 for a general, convex class of nonexpansive operators $F: l_1^+(\mathbb{Z}) \to l_1^+(\mathbb{Z})$ which is closed under composition. Theorem 7 in Section 2.4 gives sufficient conditions for pointwise convergence, $\|F^k(x)\|_\infty \to 0$ as $k \to \infty$, while Theorem 9 in Section 2.4 describes uniform convergence $\|F^k(x)\|_\infty \to 0$. Theorem 1 in Section 2.1 describes uniform convergence $\|F^k(x)\|_\infty \to 0$ for a special operator F related to discontinuous delay differential equations.
- We give new convergence results for general nonexpansive operators in $\mathbb{R}^n$ with $\|\cdot\|_1$-norm in Theorems 2, 3, 4, and 5 of Section 2.3.
- We establish the absence of infinite frequency oscillations in system (3) in Theorems 11 and 12 of Section 3.1.

The paper is organized as follows: in Section 2.1 we consider an example of a nonexpansive operator in $l_1(\mathbb{Z})$ which relates to Eq. (3) and illustrates the main ideas implying the uniform convergence $\|F^k(x)\|_\infty \to 0$. In Sections 2.2 and 2.3 we define the main classes of nonexpansive operators in $l_1(\mathbb{Z})$ which are studied later, and we prove related statements on convergence for the finite-dimensional case. In Section 2.4 we prove our main convergence results for operators in $l_1(\mathbb{Z})$. Finally, Section 3 is devoted to the study of Eq. (3).

2. NONEXPANSIVE PERIODIC OPERATORS

2.1. Nonexpansive Periodic Operators: A Basic Example

Let an operator $F: l_1^+(\mathbb{Z}) \to l_1^+(\mathbb{Z})$ be defined by (2).

For $x = \langle x_n \mid n \in \mathbb{Z} \rangle \in l_1^+(\mathbb{Z})$ denote

$$\delta(x) = \max_{n \in \mathbb{Z}} x_{2n+1}, \qquad \Delta(x) = \max_{n \in \mathbb{Z}} x_{2n}$$

Proposition 2.1. *For any $x, y \in l_1^+(\mathbb{Z})$,*

$$\|F(x)\|_1 = \|x\|_1, \qquad \|F(x) - F(y)\|_1 \leqslant \|x - y\|_1$$

$$\delta(F(x)) + \Delta(F(x)) \leqslant \delta(x) + \Delta(x)$$

Proof. The first equality is trivial. For the second relation, we denote $z = F(x) - F(y)$ and then compute

$$\begin{aligned}\|z\|_1 &= \sum_{n \in \mathbb{Z}} (|z_{2n}| + |z_{2n-1}|) \\ &\leqslant \sum_{n \in \mathbb{Z}} |\sigma(x_{2n+1}) - \sigma(y_{2n+1})| + \sum_{n \in \mathbb{Z}} |\tau(x_{2n}) - \tau(y_{2n})| \\ &\quad + \sum_{n \in \mathbb{Z}} |x_{2n+1} - \sigma(x_{2n+1}) - y_{2n+1} + \sigma(y_{2n+1})| \\ &\quad + \sum_{n \in \mathbb{Z}} |x_{2n} - \tau(x_{2n}) - y_{2n} + \tau(y_{2n})| \\ &= \sum_{n \in \mathbb{Z}} |x_{2n+1} - y_{2n+1}| + \sum_{n \in \mathbb{Z}} |x_{2n} - y_{2n}| = \|x - y\|_1\end{aligned}$$

for $y = 0$, inequalities above turn into equalities. To prove the last inequality in the assertion of Proposition 2.1, denote $z = F(x)$ and write

$$\begin{aligned}z_{2n} &= \sigma(x_{2n+1}) + \tau(x_{2n}) \leqslant \sigma(\delta(x)) + \tau(\Delta(x)) \\ z_{2n-1} &= x_{2n} - \tau(x_{2n}) + x_{2n-1} - \sigma(x_{2n-1}) \\ &\leqslant \Delta(x) - \tau(\Delta(x)) + \delta(x) - \sigma(\delta(x))\end{aligned}$$

which completes the proof. □

Theorem 1. *Under the given hypotheses, the sequence* $\langle \|F^N(x)\|_\infty \mid N \in \mathbb{N} \rangle$ *converges to zero uniformly in any ball* $\{\|x\|_1 \leqslant R\}$.

Proof. For $x = x^{(0)} \in l_1^+(\mathbb{Z})$, denote

$$\begin{aligned}x^{(N)} &= F(x^{N-1}), \qquad N \geqslant 1 \\ \delta_0 &= \delta(x), \qquad \Delta_0 = \Delta(x) \\ \Delta_N &= \sigma(\Delta_{N-1}) + \tau(\delta_{N-1}) \\ \delta_N &= (\mathrm{Id} - \sigma)(\Delta_{N-1}) + (\mathrm{Id} - \tau)(\delta_{N-1}), \qquad N \geqslant 1\end{aligned}$$

Clearly,

$$\delta_N + \Delta_N = \delta_0 + \Delta_0, \quad \delta(x^{(N)}) \leqslant \delta_N, \quad \Delta(x^{(N)}) \leqslant \Delta_N, \quad N \geqslant 1 \tag{4}$$

Let us fix $\varepsilon > 0$ and an even integer $2m \geqslant (R + \varepsilon)/\varepsilon$. We derive Theorem 1 from

Proposition 2.2. *If*

$$\delta(x)+\Delta(x)\geqslant 2\varepsilon \tag{5}$$

then

$$\delta(x^{(2m-1)})+\Delta(x^{(2m-1)})\leqslant \delta_{2m-1}+\Delta_{2m-1}-2\lambda^{2m-1}\varepsilon \tag{6}$$

where λ is defined by (1).

Proof. Relation (5) implies that $m(\delta_0+\Delta_0)\geqslant 2m\varepsilon\geqslant R+\varepsilon$. Hence, for any $n\in\mathbb{Z}$,

$$x_{2n}+x_{2n-1}+\cdots+x_{2n-2m+1}\leqslant m(\delta_0+\Delta_0)-\varepsilon \tag{7}$$

We intend to derive from (7) the inequality

$$x^{(1)}_{2n}+x^{(1)}_{2n-1}+\cdots+x^{(1)}_{2n-2m+2}\leqslant (m-1)(\delta_1+\Delta_1)+\Delta_1-\lambda\varepsilon \tag{8}$$

Denote $\sigma^*=\mathrm{Id}-\sigma$, $\tau^*=\mathrm{Id}-\tau$, and observe that the functions σ^*, τ^* satisfy inequalities (1). Then we have

$$\begin{aligned}
x^{(1)}_{2n} &= \sigma(x_{2n})+\tau(x_{2n-1})\\
&\leqslant \sigma(\Delta_0)-\lambda(\Delta_0-x_{2n})+\tau(\delta_0)-\lambda(\delta_0-x_{2n-1})\\
&= \Delta_1-\lambda(\Delta_0-x_{2n}+\delta_0-x_{2n-1})\\
&\leqslant \Delta_1-\lambda(\Delta_0-x_{2n})\\
x^{(1)}_{2n-1} &= \tau^*(x_{2n-1})+\sigma^*(x_{2n-2})\\
&\leqslant \tau^*(\delta_0)+\sigma^*(\Delta_0)-\lambda(\delta_0-x_{2n-1}+\Delta_0-x_{2n-2})\\
&\leqslant \delta_1-\lambda(\delta_0-x_{2n-1})\\
&\ \ \vdots\\
x^{(1)}_{2n-2m+3} &\leqslant \delta_1-\lambda(\delta_0-x_{2n-2m+3})\\
x^{(1)}_{2n-2m+2} &\leqslant \Delta_1-\lambda(\Delta_0+\delta_0-x_{2n-2m+2}-x_{2n-2m+1})
\end{aligned}$$

Summing up the latter inequalities, one comes to

$$\begin{aligned}
&x^{(1)}_{2n}+x^{(1)}_{2n-1}+\cdots+x^{(1)}_{2n+2m-2}\\
&\qquad\leqslant (m-1)(\Delta_1+\delta_1)+\Delta_1-\lambda(m(\Delta_0+\delta_0)-(x_{2n}+\cdots+x_{2n-2m+1}))
\end{aligned}$$

which, together with (7), implies (8).

The same sort of estimation for $x_{2n}^{(2)}, ..., x_{2n-2m+3}^{(2)}$ shows that (8) implies

$$x_{2n}^{(2)} + x_{2n-1}^{(2)} + \cdots + x_{2n-2m+3}^{(2)} \leqslant (m-1)(\delta_2 + \Delta_2) - \lambda^2 \varepsilon$$

We repeat this procedure until we come to

$$x_{2n}^{(2m-1)} \leqslant \Delta_{2m-1} - \lambda^{2m-1} \varepsilon$$

The same argument applied to the inequality

$$x_{2n-1} + x_{2n-2} + \cdots + x_{2n-2m} \leqslant m(\Delta_0 + \delta_0) - \varepsilon$$

leads to

$$x_{2n-1}^{(2m-1)} \leqslant \delta_{2m-1} - \lambda^{2m-1} \varepsilon$$

which completes the proof. □

2.2. Nonexpansive Periodic Operators in $l_1(\mathbb{Z})$

In this section we will always denote $l_1^+(\mathbb{Z})$ by K. Similarly, for n a positive integer K^n will denote the cone of the nonnegative vectors in $\mathbb{R}^n$,

$$K^n = \{x \in \mathbb{R}^n : x_i \geqslant 0,\ i = 1, ..., n\}$$

and for $x \in \mathbb{R}^n$, $\|x\|_1 := \sum_{i=1}^n |x_i|$. If $R > 0$, we shall denote by K_R and K_R^n the sets

$$K_R = \{x \in l_1(\mathbb{Z}) : \|x\|_1 \leqslant R\}, \qquad K_R = \{x \in \mathbb{R}^n : \|x\|_1 \leqslant R\}$$

The cone K (respectively, K^n) induces a partial ordering on $l_1(\mathbb{Z})$ (respectively, $\mathbb{R}^n$) by $x \leqslant y$ if and only if $y - x \in K$ (respectively, $y - x \in K^n$).

Our goal now is to prove that similar convergence theorems hold for much more general maps $F: K_R \to K_R$ than those considered in the previous sections.

To motivate our class of maps, we imagine an infinite sequence of containers of sand at the seashore and we consider a procedure for shifting sand among containers. for each integer i, we suppose that there are two containers, C_i and D_i. Initially, C_i contains a volume $x_i \geqslant 0$ of sand and D_i contains no sand. We suppose that $\sum_i x_i \leqslant R < \infty$ and that the volume of each of the containers C_j and D_j, $j \in \mathbb{Z}$, is greater than R. We fix an integer $N > 1$. For each integer i, we wish to apportion all of the sand in C_i among the various containers D_j. If $i \equiv i' \bmod N$, we want the rules for shifting sand from container C_i to be the same, after shifting indices by $i'-i$, as the rules for shifting sand from $C_{i'}$. In this sense our procedure is to be periodic of period N. To be more precise, for $i, k \in \mathbb{Z}$, let $\sigma_{i,k}: [0, R] \to [0, R]$ be a continuous, increasing function with $\sigma_{i,k}(0) = 0$; we allow the possibility that $\sigma_{i,k} \equiv 0$. We assume that if container C_i contains a volume x_i of sand, then we shift a volume of sand equal to $\sigma_{i,k}(x_i)$ to container D_k for $k \in \mathbb{Z}$. Since we assume that all of the sand in C_i is shifted to some D_k, $k \in \mathbb{Z}$, we must assume that

$$\sum_{k \in \mathbb{Z}} \sigma_{i,k}(t) = t \qquad \text{for all} \quad i \in \mathbb{Z} \quad \text{and for all} \quad t \in [0, R] \tag{9}$$

We interpret our periodicity condition as meaning that

$$\sigma_{i,k}(t) = \sigma_{i+sN,\, k+sN}(t) \qquad \text{for all} \quad i, k, s \in \mathbb{Z} \quad \text{and all} \quad t \in [0, R] \tag{10}$$

We carry out the above sand shifting procedure for each container C_i, so eventually all the containers C_i, $i \in \mathbb{Z}$, are empty, and the sand has been shifted to the containers D_k. Note that D_k contains a volume y_k, where

$$y_k = \sum_{i \in \mathbb{Z}} \sigma_{i,k}(x_i) \tag{11}$$

Now we pour all the sand from D_k into C_k for $k \in \mathbb{Z}$, so all the containers D_k are empty and the container C_k contains a volume y_k of sand. We can, at this point, iterate our procedure repeatedly. Our basic question is this: What can be said about the distribution of the sand as the number of iterations approaches infinity?

We define a map $F: K_R \to K_R$ by $F(x) = y$, where

$$y_k := F_k(x) := \sum_{i \in \mathbb{Z}} \sigma_{i,k}(x_i) \tag{12}$$

If F^m denotes the mth iterate of F with itself and $x \in K_R$, we wish to investigate the behavior of $F^m(x)$ as $m \to \infty$.

For definiteness, we collect our assumptions. In the following recall that a map $\theta: [a, b] \to \mathbb{R}$ is "increasing on $[a, b]$" if $\theta(u) \leqslant \theta(v)$ whenever $a \leqslant u < v \leqslant b$, and θ is "strictly increasing on $[a, b]$" if $\theta(u) < \theta(v)$ whenever $a \leqslant u < v \leqslant b$.

Hypothesis H1.1. Let N be a given positive integer and suppose that $0 < R \leqslant +\infty$. For each pair of integers i, k, assume that $\sigma_{i,k}: [0, R] \to [0, R]$ is a continuous map which is increasing on $[0, R]$ and satisfies $\sigma_{i,k}(0) = 0$. Assume that Eqs. (9) and (10) are satisfied.

We shall always assume at least H1.1. Under further assumptions on F we shall prove that for $x \in K_R$

$$\lim_{m \to \infty} \|F^m(x)\|_\infty = 0 \tag{13}$$

Indeed, we shall prove that, under further restrictions on F, the limit in (13) is uniform, provided $R < \infty$. However, if we assume only H1.1, Eq. (13) need not be true and F may, for example, have many periodic points. Thus, if we define $F: K \to K$ by $F(x) = y$, where $y_i = x_{i-1}$ for $i \equiv j$ mod N and $1 \leqslant j \leqslant N-1$ and $y_i = x_{i+N-1}$ for $i \equiv 0$ mod N, then F is of the form given by (12), H1.1 is satisfied, and F has periodic points of period N.

We begin by establishing some general facts about F. If D is a subset of X, where $X = l_1(\mathbb{Z})$ or $\mathbb{R}^n$, a map $h: D \to X$ is called "order-preserving" if $h(x) \leqslant h(y)$ whenever $x, y \in D$ and $x \leqslant y$. The map h is called "integral-preserving" if

$$\sum_i h_i(x) = \sum_i x_i \qquad \text{for all} \quad x \in D$$

Here $h_i(x)$ denotes the ith coordinate of $h(x)$. The map h is called "nonexpansive in the l_1-norm" or simply "nonexpansive" if

$$\|h(x) - h(y)\|_1 \leqslant \|x - y\|_1 \qquad \text{for all} \quad x, y \in D$$

If $x, y \in X$, define $x \wedge y \in X$, the lattice meet of x and y, by

$$(x \wedge y)_i = \min\{x_i, y_i\}$$

If $h: D \to X$ is integral-preserving and order-preserving and if $x \wedge y \in D$ whenever $x, y \in D$, then arguments of Crandall and Tartar (1980) prove that h is l_1-norm nonexpansive.

Lemma 2.3. *Assume H1.1 and suppose that $F: K_R \to K_R$ is defined by (12). Then F is integral-preserving, order-preserving, and nonexpansive.*

Proof. By the Crandall and Tartar (1980) result, it suffices to prove that F is integral-preserving and order-preserving. To prove integral-preserving, take $x \in K_R$ and note that

$$\sum_k F_k(x) = \sum_k \sum_i \sigma_{i,k}(x_i) = \sum_i \sum_k \sigma_{i,k}(x_i) = \sum_i x_i$$

Here we have used Eq. (9). Change of the order of summation is justified, because all summands are nonnegative. We assume in H1.1 that all of the maps $\sigma_{i,k}$ are increasing on $[0, R]$, so if $u, x \in K_R$ and $u \leqslant x$, we obtain

$$F_k(u) = \sum_i \sigma_{i,k}(u_i) \leqslant \sum_i \sigma_{i,k}(x_i) = F_k(x)$$

i.e., F is order-preserving. □

Let N be as in H1.1. For $x \in l_1(\mathbb{Z})$ and an integer i, we define $\delta_i(x)$ by

$$\delta_i(x) = \sup\{|x_j|: j \in \mathbb{Z} \text{ and } j \equiv i \bmod N\} \tag{14}$$

We define $\delta(x) \in K^N$ by

$$\delta(x) = (\delta_0(x), \ldots, \delta_{N-1}(x)) \tag{15}$$

Obviously we have $\|\delta(x)\|_\infty = \|x\|_\infty$.

Assuming H1.1, we define for each pair of integers (i, j) a map $M_{i,j}: [0, R] \to [0, R]$ by the formula

$$M_{i,j}(p) = \sum_{s \in \mathbb{Z}} \sigma_{i+sN,\, j}(p) \tag{16}$$

Note that our periodicity condition (10) gives $\sigma_{i+sN,\, j}(p) = \sigma_{i,\, j-sN}(p)$, so

$$M_{i,j}(p) = \sum_s \sigma_{i,\, j-sN}(p) \leqslant \sum_k \sigma_{i,k}(p) = p$$

In fact, we have for $0 \leqslant p \leqslant R$ and $l \in \mathbb{Z}$

$$\sum_{j=0}^{N-1} M_{i,\, l+j}(p) = \sum_{j=0}^{N-1} \sum_s \sigma_{i,\, l-sN+j}(p) = \sum_{k \in \mathbb{Z}} \sigma_{i,k}(p) = p \tag{17}$$

Note also that if $i \equiv i' \bmod N$ and $l \equiv l' \bmod N$, then $M_{i,l} = M_{i',l'}$. For example, if $l' = l + tN$ for some integer t,

$$M_{i,l'}(p) = \sum_{s \in \mathbb{Z}} \sigma_{i,\, l+tN-sN}(p) = \sum_{m \in \mathbb{Z}} \sigma_{i,\, l-mN}(p) = M_{i,l}(p)$$

If $M_{i,l}$ is defined as above and H1.1 holds, we define, for $\xi = (\xi_0, \xi_1, \ldots, \xi_{N-1}) \in K_R^N$, a map $g: K_R^N \to K_R^N$ by

$$g(\xi) = (g_0(\xi), g_1(\xi), \ldots, g_{N-1}(\xi)), \qquad g_l(\xi) := \sum_{i=0}^{N-1} M_{i,l}(\xi_i), \qquad 0 \leqslant l < N \tag{18}$$

Lemma 2.4. *Assume that H1.1 is satisfied and that* $F\colon K_R \to K_R$, $g\colon K_R^N \to K_R^N$ *are defined by Eqs. (12) and (18), respectively. Then the map* g *is integral-preserving and order-preserving. If* $\delta(x)$ *is defined by (15), we have*

$$\delta(F(x)) \leqslant g(\delta(x)) \qquad \text{for all} \quad x \in K_R \tag{19}$$

Proof. Using Eqs. (18) and (17), we see that for $\xi \in K_R^N$ we have

$$\sum_{l=0}^{N-1} g_l(\xi) = \sum_{i=0}^{N-1} \sum_{l=0}^{N-1} M_{i,l}(\xi_i) = \sum_{i=0}^{N-1} \xi_i$$

so g is integral-preserving. The map $t \mapsto M_{i,l}(t)$ is increasing on $[0, R]$, because the maps $t \mapsto \sigma_{i,k}(t)$ are increasing on $[0, R]$, so it follows immediately that g is order-preserving.

If $0 \leqslant l \leqslant N-1$ is an integer, select $k \in \mathbb{Z}$ such that $k \equiv l \bmod N$ and $\delta_l(F(x)) = F_k(x)$. We have

$$F_k(x) = \sum_{j=0}^{N-1} \sum_{s \in \mathbb{Z}} \sigma_{j+sN,k}(x_{j+sN})$$

and because $x_{j+sN} \leqslant \delta_j(x)$ for all s, we find that

$$\delta_l(F(x)) = F_k(x) \leqslant \sum_{j=0}^{N-1} \sum_{s} \sigma_{j+sN,k}(\delta_j(x)) = \sum_{j=0}^{N-1} M_{j,k}(\delta_j(x))$$

By our previous remarks, $M_{j,k}(t) = M_{j,l}(t)$ for $0 \leqslant t \leqslant R$, so

$$\delta_l(F(x)) \leqslant \sum_{j=0}^{N-1} M_{j,l}(\delta_j(x)) := g_l(\delta(x))$$

which proves (19). □

At this point it is convenient to introduce some further notation. As usual, $l_\infty(\mathbb{Z})$ denotes the Banach space of bounded sequences $y = \langle y_k \mid k \in \mathbb{Z} \rangle$ with

$$\|y\|_\infty = \sup\{|y_k| : k \in \mathbb{Z}\}$$

We shall always denote by C the cone of nonnegative vectors in $l_\infty(\mathbb{Z})$, so

$$C = \{ y \in l_\infty(\mathbb{Z}) : y_i \geqslant 0 \text{ for all } i \in \mathbb{Z}\} \tag{20}$$

If n is a positive integer, we define a map $\theta_N\colon K_R^N \to C$ by

$$\theta_N(\xi) = z, \qquad \text{where} \quad z_j = \xi_i \quad \text{if } j \equiv i \bmod N \tag{21}$$

We define $S\colon l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ to be the left-hand shift, so

$$S(x) = y, \qquad \text{where} \quad y_i = x_{i+1} \quad \text{for } i \in \mathbb{Z} \tag{22}$$

The cone C induces a partial ordering on $l_\infty(\mathbb{Z})$ by $x \leqslant y$ if and only if $y - x \in C$. No confusion with the earlier partial ordering induced on $l_1(\mathbb{Z})$ by K should arise, so we shall not distinguish the partial orderings notationally. A map $F\colon D \subset l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ is called order-preserving if $F(x) \leqslant F(y)$ whenever $x, y \in D$ and $x \leqslant y$.

For a given $R > 0$ and positive integer N, the class of mappings $F\colon K_R \to K_R$ defined by (12) and H1.1 is insufficiently general in one important sense, namely, it is not closed under composition of operators. We now introduce a larger class of operators which remedies this difficulty.

Definition 2.5. Suppose that $R > 0$ and that N is an integer. Define $D = D(R, N) \subset C \subset l_\infty(\mathbb{Z})$ by

$$D := K_R \cup \theta_N(K_R^N) := K_R \cup \{\theta_N(\xi)\colon \xi \in K_R^N\}$$

where C is as in (20) and θ_N as in (21). If $F\colon D \to C$ is a map, we shall say that $F \in \mathscr{F}(R, N)$ if F satisfies the following conditions:

(a) $F\colon D \to C$ is order-preserving.

(b) The restriction of F to K_R, $F|_{K_R}$, is an integral-preserving map into K_R.

(c) For all $x \in D$, $S^{-N}FS^N(x) = F(x)$, where S is the left-shift operator given by (22).

(d) *If* $g\colon K_R^N \to \mathbb{R}^N$ is defined by

$$g(\xi) = (F_0(z), F_1(z), ..., F_{N-1}(z))$$

where $z = \theta_N(\xi)$ and F_i denote coordinate i of $F(z)$, then g is an integral-preserving map.

One can easily see that $\mathscr{F}(R, N)$ is a convex set of operators. The reader can also verify that S^N corresponds to a left-hand shift by N coordinates in $l_\infty(\mathbb{Z})$ and that condition (c) above is equivalent to

$$F_i(x) = F_{i-N}(S^N(x)) \qquad \text{for all} \quad x \in D \quad \text{and all} \quad i \in \mathbb{Z} \tag{23}$$

Also, one easily derives from condition (c) by induction on l that

$$S^{-lN}FS^{lN}(x) = F(x) \qquad \text{for all} \quad x \in D \quad \text{and all} \quad l \in \mathbb{Z} \tag{24}$$

A word about condition (d) is in order. If F satisfies conditions (a) and (c) and $\xi \in K_R^N \backslash \{0\}$, $z = \theta_N(\xi)$, one expects in general that

$$\sum_{i \in \mathbb{Z}} F_i(z) = \infty$$

However, one might hope that

$$\lim_{k \to \infty} \frac{1}{2k-1} \sum_{|j| < k} F_j(z) = \lim_{k \to \infty} \frac{1}{2k-1} \sum_{|j| < k} z_j \tag{25}$$

Because $S^{lN}(z) = z$ for all $l \in \mathbb{Z}$, Eq. (23) implies that $F_{j-lN}(z) = F_j(z)$ for all l. Thus, the right-hand side of Eq. (25) has limit equal to $(1/N) \sum_{j=0}^{N-1} z_j$, the left-hand side equals $(1/N) \sum_{j-0}^{N-1} F_j(z)$, and [assuming conditions (a) and (c)] condition (d) is equivalent to the validity of equality (25) for all $z = \theta_N(\xi)$, $\xi \in K_R^N$.

Our next lemma has essentially been proved before, but we state it for completeness.

Lemma 2.6. *Assume H1.1 and let F be defined by (12). Then (see Definition 2.5) we have $F \in \mathscr{F}(R, N)$.*

Proof. We already know (Lemma 2.3) that $F(K_R) \subset K_R$ and that $F|_{K_R}$ is integral-preserving, so condition (b) of Definition 2.5 is satisfied. If $x \in D = D(R, N)$, we must prove that $F(x) \in C$. Thus, if $z = \theta_N(\xi)$ and $\xi \in K_R^n$, we must prove that $F(z) \in C$. However, we have

$$\begin{aligned} F_i(z) &= \sum_{j \in \mathbb{Z}} \sigma_{j,i}(z_j) = \sum_{j=0}^{N-1} \sum_{s \in \mathbb{Z}} \sigma_{j-sN,i}(z_{j-sN}) \\ &= \sum_{j=0}^{N-1} \sum_{s \in \mathbb{Z}} \sigma_{j-sN,i}(z_j) = \sum_{j=0}^{N-1} \sum_{s \in \mathbb{Z}} \sigma_{j,i+sN}(z_j) \end{aligned} \tag{26}$$

Here we have used Eq. (10) and $z_{j-sN} = z_j$ for all $s \in \mathbb{Z}$; change of summation order was justified because $\sigma_{j,i}(z_j) \geqslant 0$ for all i, j. Because we have

$$\sum_{s \in \mathbb{Z}} \sigma_{j,i+sN}(z_j) \leqslant \sum_{k \in \mathbb{Z}} \sigma_{j,k}(z_j) = z_j$$

we derive from the above equations that

$$0 \leqslant F_i(z) \leqslant \sum_{j=0}^{N-1} z_j \leqslant R \qquad \text{for all} \quad i \in \mathbb{Z}$$

and $F(z) \in C$. The fact that F is order-preserving [condition (a)] now follows directly from the assumption that $\sigma_{i,j}$ is increasing for all i, j.

If $\xi \in K_R^N$ and $z = \theta_N(\xi)$, condition (d) is equivalent to proving that

$$\sum_{j=0}^{N-1} F_i(z) = \sum_{j=0}^{N-1} z_i$$

However, Eqs. (26) and (9) give

$$\sum_{i=0}^{N-1} F_i(z) = \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} \sum_{s \in \mathbb{Z}} \sigma_{j,\, i+sN}(z_j) = \sum_{j=0}^{N-1} \sum_{k \in \mathbb{Z}} \sigma_{j,k}(z_j) = \sum_{j=0}^{N-1} z_j$$

so condition (d) is satisfied.

To prove condition (c), we must prove that $F_k(x) = F_{k-N}(S^N(x))$ for all $x \in D$ and $k \in \mathbb{Z}$. Using (10) and (12), we obtain for $x \in D$, $k \in \mathbb{Z}$, $F_k(x) = \sum_{j \in \mathbb{Z}} \sigma_{j,k}(x_j)$ and

$$F_{k-N}(S^N(x)) = \sum_{j \in \mathbb{Z}} \sigma_{j,\, k-N}(x_{j+N}) = \sum_{j \in \mathbb{Z}} \sigma_{j+N,\, k}(x_{j+N}) = \sum_{j \in \mathbb{Z}} \sigma_{j,k}(x_j)$$

so condition (c) is satisfied. □

We now prove the crucial property of $\mathscr{F}(R, N)$, namely, that it is closed under composition.

Lemma 2.7. *If $F \in \mathscr{F}(R, N)$ and $D = D(R, N)$ is as in Definition 2.5, then $F(\theta_N(K_N^R)) \subset \theta_N(K_R^N)$ and $F(D) \subset D$. If $g\colon K_R^N \to K_R^N$ is defined by $g(\xi) = (F_0(z),\ldots, F_{N-1}(z))$, where $z = \theta_N(\xi)$, then g is integral-preserving and order-preserving. If $x \in D$ instead of K_R we can still define $\delta(x)$ by Eqs. (14) and (15), and we have*

$$\delta(F(x)) \leqslant g(\delta(x)) \qquad \textit{for all} \quad x \in D \tag{27}$$

If $\Phi \in \mathscr{F}(R, N)$ and if $\gamma\colon K_R^N \to K_R^N$ is defined by $\gamma(\xi) = (\Phi_0(z),\ldots, \Phi_{N-1}(z))$ for $z = \theta_N(\xi)$, then $\Phi \circ F \in \mathscr{F}(R, N)$, and for all $\xi \in K_R^N$ and $z = \theta_N(\xi)$ we have

$$\gamma(g(\xi)) = (\Phi_0(F(z)),\ldots, \Phi_{N-1}(F(z)))$$

Proof. We already know that $F(K_R) \subset K_R$. If $\xi \in K_R^N$ and $z = \theta_N(\xi)$, we see that $S^{lN}(z) = z$ for all $l \in \mathbb{Z}$, so Eq. (23) implies that

$$F_j(z) = F_{j-lN}(z) \qquad \text{for all} \quad l \in \mathbb{Z}$$

If we define $\tilde{\xi} = (F_0(z),..., F_{N-1}(z)) = g(\xi)$, it follows that

$$F(z) = \theta_N(\tilde{\xi})$$

Condition (d) in Definition 2.5 implies that

$$\sum_{i=0}^{N-1} \tilde{\xi}_i = \sum_{i=0}^{N-1} F_i(z) = \sum_{i=0}^{N-1} \xi_i \leqslant R$$

so $\tilde{\xi} \in K_R^N$ and $F(z) = \theta_N(\tilde{\xi}) \in \theta_N(K_R^N)$. It follows that $F(D) \subset D$.

Condition (d) implies that g is integral-preserving, and g is order-preserving, because $F: D \to C$ is assumed order-preserving.

The definition of $\delta(x)$ implies that for $x \in D$,

$$x \leqslant \theta_N(\delta(x))$$

so, because F is order-preserving, we obtain that

$$F(x) \leqslant F(\theta_N(\delta(x)))$$

By definition of $\delta_i(F(x))$ [and because $F(x) \in D$], for each i, $0 \leqslant i \leqslant N-1$, there exists $k_i \equiv i \bmod N$ with

$$\delta_i(F(x)) = F_{k_i}(x) \leqslant F_{k_i}(\theta_N(\delta(x)))$$

By our previous argument we know that $F(\theta_N(\delta(x))) \in \theta_N(K_R^N)$, so $F_i(\theta_N(\delta(x))) = F_{k_i}(\theta_N(\delta(x)))$, and we obtain that

$$\delta_i(F(x)) \leqslant F_i(\theta_N(\delta(x))) = g_i(\delta(x))$$

which implies (27).

To prove that $\Phi \circ F \in \mathscr{F}(R, N)$, we have to prove that conditions (a)–(d) of Definition 2.5 are satisfied. Condition (a) implies that F and Φ are order-preserving on D, and we have that $F(D) \subset D$. It follows that if $x, y \in D$ and $x \leqslant y$, we have $F(x), F(y) \in D$, and $F(x) \leqslant F(y)$, so

$$\Phi(F(x)) \leqslant \Phi(F(y))$$

This proves that $\Phi \circ F$ satisfies condition (a).

We know that Φ and F are integral-preserving on K_R and that $F(K_R) \subset K_R$, $\Phi(K_R) \subset K_R$. It follows that for $x \in K_R$

$$\sum_i \Phi_i(F(x)) = \sum_i F_i(x) = \sum_i x_i$$

so $\Phi \circ F$ is integral-preserving on K_R and satisfies condition (b).

Condition (c) implies that $S^{-N} F S^N = F$ and $S^{-N} \Phi S^N = \Phi$, so we find that

$$\Phi \circ F = (S^{-N} \Phi S^N) \circ (S^{-N} F S^N) = S^{-N} (\Phi \circ F)\, S^N$$

and $\Phi \circ F$ satisfies condition (c) of Definition 2.5.

In order to prove that $\Phi \circ F$ satisfies condition (d), we have to prove that if $\xi \in K_R^N$ and $z = \theta_N(\xi)$, then

$$\sum_{j=0}^{N-1} \Phi_i(F(z)) = \sum_{j=0}^{N-1} z_j$$

However, we have already proved that $F(z) \in \theta_N(K_R^N)$ and

$$F(z) = \theta_N(F_0(z),..., F_{N-1}(z)) = \theta_N(g(\xi))$$

It follows from the definition of γ that

$$\gamma(g(\xi)) = (\Phi_0(F(z)),..., \Phi_{N-1}(F(z)))$$

Condition (d) implies that γ and g are integral-preserving, so

$$\sum_{j=0}^{N-1} \Phi_j(F(z)) = \sum_{j=0}^{N-1} \gamma_j(g(\xi)) = \sum_{j=0}^{N-1} g_j(\xi) = \sum_{j=0}^{N-1} \xi_j = \sum_{j=0}^{N-1} z_j$$

and we see that $\Phi \circ F$ satisfies condition (d). This completes the proof of Lemma 2.7. □

2.3. l_1-Norm Nonexpansive Maps in $\mathbb{R}^N$ and Weak Convergence Results

In the previous section we have defined the class $\mathscr{F}(R, N)$, which is the class of nonexpansive, periodic (or NEP) maps $F: K_R \to K_R$ which will be of interest to us. Without further assumptions, we have seen that it is false that $\lim_{k \to \infty} \|F^k(x)\|_\infty = 0$ for $x \in K_R$. Nevertheless, one can make some useful observations about the iterates $F^k(x)$, and these comments form the basis for our later, stronger convergence theorems.

To begin we need to recall some general results concerning iterates of l_1-norm nonexpansive maps $h: K_R^N \to K_R^N$. The following lemma is due to Akcoglu and Krengel (1987), but with important refinements of Scheutzow (1988, 1991). We denote below the least common multiple of the integers $\{1, 2,..., N\}$ by $\text{lcm}(N)$.

Lemma 2.8 (See Akcoglu and Krengeli, 1987; Scheutzow, 1988, 1991). *If $R > 0$ and N is a positive integer, let $h: K_R^N \to K_R^N$ be an l_1-norm nonexpansive map such that $h(0) = 0$. If $x \in K_R^N$, there exists $\xi_x \in K_R^N$ and an integer $p_x = p \geqslant 1$ such that $p \mid \text{lcm}(N)$, $\lim_{k \to \infty} h^{kp}(x) = \xi_x$, $h^p(\xi_x) = \xi_x$, and $h^j(\xi_x) \neq \xi_x$ for $1 \leqslant j < p$. If h is also integral-preserving, then $\|\xi_x\|_1 = \|x\|_1$ for all $x \in K_R^N$.*

Not every divisor p of $\text{lcm}(N)$ is the minimal period of some periodic point $\xi \in K_R^N$ of some l_1-norm nonexpansive map $h: K_R^N \to K_R^N$ with $h(0) = 0$. We refer the reader to Nussbaum (1991), Nussbaum and Scheutzow (1998), Nussbaum and Verduyn Lunel (1999), and Nussbaum *et al.* (1998) for precise results in this direction and refinements of Lemma 2.8.

Our next proposition may have some independent interest, so we isolate it as a theorem.

Theorem 2. *Suppose that $R > 0$, N is a positive integer and $h: K_R^N \to K_R^N$ is order-preserving, $h(0) = 0$, and h is l_1-norm nonexpansive. Suppose that $\langle y^k \mid k \geqslant 0 \rangle$ is a sequence of points in K_R^N such that $y^{k+1} \leqslant h(y^k)$ for all $k \geqslant 0$. Then there exists an integer p which is a divisor of $\text{lcm}(N)$ and is such that $\lim_{m \to \infty} y^{mp} = \xi$, where $h^p(\xi) = \xi$ and $h^j(\xi) \neq \xi$ for $0 < j < p$. For any integer $j \geqslant 0$, $\lim_{m \to \infty} y^{j+mp} = h^j(\xi)$.*

If h has no periodic point of minimal period $p > 1$, then $\lim_{m \to \infty} y^m = \xi$, where $h(\xi) = \xi$.

Proof. Denote $q = \text{lcm}(N)$. For any $x \in K_R^N$, Lemma 2.8 implies that there exists $\xi_x \in K_R^N$ with $\lim_{k \to \infty} h^{kq}(x) = \xi_x$ and $h^q(\xi_x) = \xi_x$. It follows that for each $n \geqslant 1$, $\lim_{k \to \infty} h^{kq}(y^n) = z^n$, where $h^q(z^n) = z^n$. Note also that

$$\|y^{k+1}\|_1 \leqslant \|h(y^k)\|_1 = \|h(y^k) - h(0)\|_1 \leqslant \|y^k\|_1 \tag{28}$$

so $\alpha_k := \|y^k\|_1$ is a decreasing sequence of nonnegative reals and $\lim_{k \to \infty} \alpha_k = \alpha$ exists. If $\alpha = 0$, we are done, so we assume that $\alpha > 0$. Note that, for $n \geqslant 1$,

$$\|z^n\|_1 = \lim_{k \to \infty} \|h^{kq}(y^n) - h^{kq}(0)\|_1 \leqslant \|y^n\|_1 \tag{29}$$

Because $y^{k+1} \leqslant h(y^k)$ and h is order-preserving, we easily see that $y^{k+m} \leqslant h^m(y^k)$ for all $k \geqslant 0$, $m \geqslant 0$, and $y^{(m+1)q} \leqslant h^q(y^{mq})$. It follows that

$$\lim_{k \to \infty} h^{kq}(y^{(m+1)q}) = z^{(m+1)q} \leqslant \lim_{k \to \infty} h^{(k+1)q}(y^{mq}) = z^{mq} \tag{30}$$

It follows from (30) that there exists $\xi \in K_R^N$ with

$$\lim_{m \to \infty} z^{mq} = \xi \tag{31}$$

and $\xi \leqslant z^{mq}$ for all $m \geqslant 1$. If $u = (1, 1,..., 1)$ and $\delta > 0$, we conclude from Eq. (31) that there exists an integer m_δ such that

$$\xi \leqslant z^{mq} \leqslant \xi + \frac{\delta}{2} u \qquad \text{for} \quad m \geqslant m_\delta \tag{32}$$

Since $z^{mq} = \lim_{k \to \infty} h^{kq}(y^{mq})$ and $h^{kq}(y^{mq}) \geqslant y^{kq+mq}$, we obtain from (32) that, given $\delta > 0$, there exists an integer n_δ such that

$$y^{nq} \leqslant \xi + \delta u \qquad \text{for} \quad n \geqslant n_\delta \tag{33}$$

We claim that $\lim_{k \to \infty} y^{kq} = \xi$. If not, there exists a sequence $n_i \to \infty$ such that $y^{n_i q} \to \xi'$ and $\xi' \neq \xi$. Equation (33) implies that $\xi' \leqslant \xi$, and since $\xi' \neq \xi$, we have $\|\xi'\|_1 \leqslant \|\xi\|_1$. However, we obtain from (29) that

$$\alpha = \lim_{i \to \infty} \|y^{n_i q}\|_1 = \|\xi'\|_1 \geqslant \lim_{i \to \infty} \|z^{n_i q}\|_1 \geqslant \|\xi\|_1$$

which contradicts $\|\xi'\|_1 < \|\xi\|_1$ and proves that $\lim_{k \to \infty} y^{kq} = \xi$. Equation (31) and the fact that $h^q(z^{mq}) = z^{mq}$ imply that $h^q(\xi) = \xi$.

Because $h(0) = 0$ and h is l_1-norm nonexpansive, we see that $\|h^m(\xi)\|_1$, $m \geqslant 1$, is a decreasing sequence; and since $\|h^{kq}(\xi)\|_1 = \|\xi\|_1$ for all $k \geqslant 1$, it follows that $\|h^m(\xi)\|_1 = \|\xi\|_1$ for all $m \geqslant 1$.

We claim that for every integer $j \geqslant 1$, $\lim_{m \to \infty} y^{mq+j} = h^j(\xi)$. The proof follows easily from the case $j = 0$. If the claim is wrong, there exists a subsequence of integers $\langle m_i \mid i \geqslant 1 \rangle$ with

$$\lim_{i \to \infty} y^{m_i q + j} = \xi', \qquad \xi' \neq h^j(\xi)$$

It follows that

$$\xi' = \lim_{i \to \infty} y^{m_i q + j} \leqslant \lim_{i \to \infty} h^j(y^{m_i q}) = h^j(\xi)$$

Since $\xi' \leqslant h^j(\xi)$ and $\xi' \neq h^j(\xi)$ we must have that

$$\|\xi'\|_1 < \|h^j(\xi)\|_1 = \|\xi\|_1$$

We already know that $\langle \|y^n\|_1 \mid n \geqslant 1 \rangle$ is a decreasing sequence with limit α, so

$$\alpha = \lim_{m \to \infty} \|y^{mq}\|_1 = \|\xi\|_1 = \lim_{i \to \infty} \|y^{m_i q + j}\|_1 = \|\xi'\|_1$$

This contradiction proves the claim.

To complete the proof, let p be the minimal positive integer such that $h^p(\xi) = \xi$. Since $h^q(\xi) = \xi$, we know that $p \mid q$. It suffices to prove that if $0 \leqslant r < p$, then

$$\lim_{m \to \infty} y^{mp + r} = h^r(\xi)$$

Our previous results show that if $0 \leqslant j_1 < q$, $0 \leqslant j_2 < q$, and $j_1 \equiv j_2 \bmod p$, then

$$\lim_{k \to \infty} y^{kq + j_1} = h^{j_1}(\xi) = h^{j_2}(\xi) = \lim_{k \to \infty} y^{kq + j_2}$$

It follows that, given $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$\|y^{kq + j} - h^j(\xi)\|_1 < \varepsilon$$

whenever $k \geqslant k_\varepsilon$ is an integer and $0 \leqslant j < q$. Select a positive integer s such that $q = sp$ and let m be any integer with $m \geqslant sk_\varepsilon$. We can write

$$m = ls + m_0, \qquad 0 \leqslant m_0 < m$$

where l, m_0 are integers. Let r be a fixed integer, $0 \leqslant r < p$; for m as above, we have

$$mp + r = (ls + m_0)\, p + r = lq + (m_0 p + r) = lq + j$$

where $0 \leqslant j < q$ and $l \geqslant k_\varepsilon$. It follows that

$$\|y^{mp + r} - h^r(\xi)\|_1 = \|y^{lq + j} - h^j(\xi)\|_1 < \varepsilon$$

which completes the proof. □

A norm $\|\cdot\|$ on $\mathbb{R}^N$ is called "strictly monotonic" if $0 \leqslant x \leqslant y$ and $x \neq y$ implies that $\|x\| < \|y\|$. Examples are provided by the l_p-norms $\|\cdot\|_p$ for $1 \leqslant p < \infty$:

$$\|x\|_p = \left(\sum_{i=1}^{N} |x_i|^p \right)^{1/p}$$

If one uses Theorem 1.1 from Nussbaum (1994) instead of Lemma 2.8, then the following generalization of Theorem 2 follows by the same reasoning used to prove Theorem 2.

Theorem 3. *Let assumptions and notation be as in Theorem 2, but suppose that $R = \infty$ and replace the assumption that h is l_1-norm nonexpansive by the hypothesis that h is nonexpansive with respect to a strictly monotonic norm $\|\cdot\|$. Then all the conclusions of Theorem 2 remain valid.*

We can now prove our first theorem concerning iterates of operators $F\colon K_R \to K_R$, $F \in \mathscr{F}(R, N)$.

Theorem 4. *Suppose that $R > 0$, N is a positive integer and $F\colon K_R \to K_R$ is an element of $\mathscr{F}(R, N)$ (see Definition 2.5). Let $g\colon K_R^N \to K_R^N$ be defined by condition (d) of Definition 2.5 and for $y \in K_R$ define $\delta(y) \in K_R^N$ by Eq. (15). For a fixed $x \in K_R$, define $\xi^k = \delta(F^k(x))$ for $k \geqslant 0$. Then $\xi^{k+1} \leqslant g(\xi^k)$ for $k \geqslant 0$, and there exists an integer $p \geqslant 1$, $p \mid \operatorname{lcm}(N)$, and a point $\xi \in K_R^N$ such that*

(*a*) $\lim_{k \to \infty} \xi^{kp+j} = g^j(\xi)$ *for all* $j \geqslant 1$, *and*

(*b*) $g^j(\xi) \neq \xi$ *for* $0 < j < p$.

If g has no periodic points in K_R^N of minimal period > 1, then $\lim_{k \to \infty} \xi^k = \xi$ *and* $g(\xi) = \xi$.

Proof. Lemma 2.7 implies that $g\colon K_R^N \to K_R^N$ is integral-preserving and order-preserving, so we know that $g(0) = 0$ and g is l_1-norm nonexpansive. Lemma 2.7 also implies that $\xi^{k+1} \leqslant g(\xi^k)$ for $k \geqslant 0$. Theorem 4 now follows immediately from Theorem 2. □

Without further assumptions on F, one can say little more than Theorem 4. Using results from Nussbaum (1991, 1994), Nussbaum and Verduyn Lunel (1999), and Nussbaum *et al.* (1998), it is easy to construct examples which illustrate aspects of Theorem 4.

For $r \geqslant 0$, let $\Sigma_r = \{x \in K^N : \|x\|_1 = r\}$, a compact, convex set. If $g: K_R^N \to K_R^N$ is integral-preserving and order-preserving, then $g(\Sigma_r) \subset \Sigma_r$ for $0 \leqslant r \leqslant R$, and the Brouwer fixed point theorem implies that g has a fixed point in Σ_r. We shall now obtain conditions on g which ensure that g has a unique fixed point in Σ_r for $0 \leqslant r \leqslant R$ and that g has no periodic points of minimal period > 1. If these conditions are satisfied by the map g in Theorem 4, one obtains a corresponding refinement of the conclusions of Theorem 4.

Definition 2.9. Suppose that $G \subset \mathbb{R}^N$ and that $\varphi: G \to \mathbb{R}$ is a map. We shall say that "φ is increasing on G and strictly increasing on G in the ith variable" if whenever $x, y \in G$ and $y_j \leqslant x_j$ for $j \neq i$ and $y_i < x_i$, it follows that $\varphi(y) < \varphi(x)$.

An $N \times N$ matrix $A = (a_{ij})$ is called "nonnegative" if $a_{ij} \geqslant 0$ for $i, j = 1,..., N$. A nonnegative matrix A is "primitive" if there exists an integer $p \geqslant 1$ such that all entries of A^p are strictly positive.

Definition 2.10. Suppose that $G \subset \mathbb{R}^N$ and $f: G \to \mathbb{R}^N$ is a map, $f(x) = (f_1(x),..., f_N(x))$ for $x \in G$. If A is a nonnegative matrix, we shall say that "A is a strict monotonicity incidence matrix for f on G" if whenever $a_{ij} > 0$, f_i is increasing on G and strictly increasing on G in the jth variable.

Remark 2.11. Suppose that H is an open, convex subset of $\mathbb{R}^N$, that $f: H \to \mathbb{R}^N$ is an order-preserving C^1 map and that $x \mapsto f'(x) = ((\partial f_i/\partial x_j)(x))$ extends as a continuous map to $\bar{H}$, also denoted $x \mapsto f'(x)$. For each $\bar{x} \in \bar{H}$ there exists $\varepsilon > 0$ such that $f'(\bar{x})$ is a strict monotonicity incidence matrix for f on $G := \{x \in \bar{H} : \|x - \bar{x}\| \leqslant \varepsilon\}$. The straightforward argument is left to the reader.

We do not assume in Definition 2.10 that f is C^1, because many of the most interesting examples of integral-preserving, order-preserving maps on K^N are not C^1 (see Nussbaum, 1991; Nussbaum and Scheutzow, 1998; Nussbaum and Verduyn Lunel, 1999; Nussbaum *et al.*, 1998).

Lemma 2.12. *Suppose that $G \subset \mathbb{R}^N$ and that $f: G \to G$ is order-preserving and $g: G \to \mathbb{R}^N$ is order-preserving. Assume that A is a strict monotonicity incidence matrix for f on G and B is a strict monotonicity matrix for g on G. Then BA is a strict monotonicity incidence matrix for $g \circ f$ on G.*

Proof. Let $C = BA$ and suppose that $c_{ij} > 0$. Then there exists k, $1 \leqslant k \leqslant N$, such that $b_{ik} > 0$ and $a_{kj} > 0$. If $x, y \in G$, $x \geqslant y$ and $x_j > y_j$, the definition of A implies that $f(x) \geqslant f(y)$ and $f_k(x) > f_k(y)$. The definition of B now implies that $g(f(x)) \geqslant g(f(x))$ and $g_i(f(x)) > g_i(f(x))$, which completes the proof. □

Theorem 5. *Suppose that* $R > 0$, N *is a positive integer,* $g\colon K_R^N \to K_R^N$ *is integral-preserving and order-preserving, and that for each* r, $0 < r \leqslant R$, $\eta^r \in \Sigma_r := \{x \in K_R^N : \|x\|_1 = r\}$ *is a fixed pont of* g. *Assume that for each* r, $0 < r \leqslant R$, *there exist* $\varepsilon_r > 0$ *and a nonnegative primitive* $N \times N$ *matrix* A_r *such that* A_r *is a strict monotonicity incidence matrix for* $g|_{G_r}$, *where* $G_r = \{x \in K_R^N : \|x - \eta^r\|_1 \leqslant \varepsilon_r\}$ *(see Definitions 2.9 and 2.10). Then for* $0 < r \leqslant R$ *and for each* $x \in \Sigma_r$, $\lim_{k\to\infty} g^k(x) = \eta^r$. *If* $\langle \xi^k \mid k \geqslant 0 \rangle$ *is a sequence of points in* K_R^N *such that* $\xi^{k+1} \leqslant g(\xi^k)$ *for each integer* $k \geqslant 0$, *then* $\langle \|\xi^k\|_1 \mid k \geqslant 0 \rangle$ *is a decreasing sequence with limit equal to* r *and* $\lim_{k\to\infty} \xi^k = \eta^r$.

Proof. By virtue of Theorem 4, it suffices to prove that for $x \in \Sigma_r$ and $0 < r \leqslant R$, $\lim_{k\to\infty} g^k(x) = \eta^r$. We know that g is nonexpansive with respect to the l_1-norm and $g(\eta^r) = \eta^r$, so $g(G_r) \subset G_r$. It follows by applying Lemma 2.12 that for each positive integer m, A_r^m is a strict monotonicity incidence matrix for $g^m|_{G_r}$. Since A_r is assumed primitive, select m such that A_r^m has all positive entries. We define $\Gamma_r = \{x \in \Sigma_r : \|x - \eta^r\|_1 \leqslant \frac{1}{2}\varepsilon_r\}$, and we claim that for all $x, y \in \Gamma_r$ with $x \neq y$ we have

$$\|g^m(x) - g^m(y)\|_1 < \|x - y\|_1$$

To see this, suppose that $x, y \in \Gamma_r$ and $x \neq y$. The reader can verify that $\|(x \wedge y) - \eta^r\|_1 \leqslant \|x - \eta^r\|_1 + \|y - \eta^r\|_1 = \varepsilon_r$, so $x \wedge y \in G_r$. Let $S_+ = \{i\colon 1 \leqslant i \leqslant N,\ x_i > y_i\}$, $S_- = \{i\colon 1 \leqslant i \leqslant N, x_i < y_i\}$ and $S_0 = \{i\colon x_i = y_i\}$. Because $x \neq y$, we know that S_+ or S_- is nonempty. However, the assumption that $\sum_i x_i = \sum_i y_i$ implies that S_+ and S_- both are nonempty. It follows that x, y, and $x \wedge y$ are elements of G_r, $x \wedge y \leqslant x$, $x \wedge y \leqslant y$, $x \wedge y \neq x$, and $x \wedge y \neq y$. Let g_i^m denote the ith coordinate function of g^m. Because A_r^m has all positive entries and A_r^m is a strict monotonicity incidence matrix for g^m on G_r, we conclude that

$$g_i^m(x) - g_i^m(x \wedge y) > 0, \quad g_i^m(y) - g_i^m(x \wedge y) > 0 \qquad \text{for} \quad 1 \leqslant i \leqslant N \tag{34}$$

We conclude from Eq. (34) that, for $1 \leqslant i \leqslant N$,

$$\begin{aligned}|g_i^m(x) - g_i^m(y)| &= |g_i^m(x) - g_i^m(x \wedge y) + g_i^m(x \wedge y) - g_i^m(y)|\\ &< g_i^m(x) - g_i^m(x \wedge y) + g_i^m(y) - g_i^m(x \wedge y)\end{aligned}$$

so, using the integral-preserving property of g^m, we deduce that

$$\begin{aligned}\|g^m(x)-g^m(y)\|_1 &= \sum_{i=1}^{N} |g_i^m(x)-g_i^m(y)|\\ &< \sum_{i=1}^{N} (g_i^m(x)-g_i^m(x\wedge y)+g_i^m(y)-g_i^m(x\wedge y))\\ &= \|x\|_1-\|x\wedge y\|_1+\|y\|_1-\|x\wedge y\|_1=\|x-y\|_1\end{aligned}$$

For $x\in\Sigma_r$ we see that

$$\|g^{k+1}(x)-\eta^r\|_1=\|g(g^k(x))-g(\eta^r)\|_1\leqslant\|g^k(x)-\eta^r\|_1$$

so $\langle \|g^k(x)-\eta^r\|_1 \mid k\geqslant 0\rangle$ is a decreasing sequence with limit $\alpha\geqslant 0$. If $\alpha=0$, we are done, so we assume, by way of contradiction, that $\alpha>0$. Select a subsequence $k_i\to\infty$ such that $g^{k_i}(x)\to\xi\in\Sigma_r$ and $\|\xi-\eta^r\|=\alpha$. If $\alpha\leqslant\frac{1}{2}\varepsilon_r$, our previous result shows that

$$\|g^m(\xi)-g^m(\eta^r)\|_1=\|g^m(\xi)-\eta^r\|_1<\alpha$$

and this implies that $\|g^{k_i+m}(x)-\eta^r\|_1<\alpha$ for all sufficiently large i, which contradicts the fact that $\|g^k(x)-\eta^r\|_1\geqslant\alpha$ for all $k\geqslant 0$. If $\alpha>\frac{1}{2}\varepsilon_r$, define $\bar{x}=(1-t)\,\xi+t\eta^r$, $0<t<1$, where t is selected so that $\|\bar{x}-\eta^r\|_1=\frac{1}{2}\varepsilon_r$ and $\|\bar{x}-\xi\|_1=\alpha-\frac{1}{2}\varepsilon_r$. As before we have $\|g^m(\bar{x})-\eta^r\|_1<\frac{1}{2}\varepsilon_r$, so we conclude that

$$\begin{aligned}\|g^m(\xi)-\eta^r\|_1 &\leqslant \|g^m(\xi)-g^m(\bar{x})\|_1+\|g^m(\bar{x})-\eta^r\|_1\\ &< \|\xi-\bar{x}\|_1+\tfrac{1}{2}\varepsilon_r=\alpha\end{aligned}$$

We now obtain a contradiction by the same argument as before. □

Our next result follows immediately from Remark 2.11 and Theorem 5.

Corollary 6. *Suppose that* $R>0$ *and* N *is a positive integer, that* $g\colon K_R^N\to K_R^N$ *is an integral-preserving, order-preserving map and that for each* r, $0<r\leqslant R$, $\eta^r\in\Sigma_r:=\{x\in K_r^N : \|x\|_1=r\}$ *is a fixed point of* g. *For each* r, $0<r\leqslant R$, *assume that there exists* ε_r *such that (writing* $H_r=\{x\in \mathring{K}_R^N : \|x-\eta^r\|_1<\varepsilon_r\}$*)* $g|_{H_r}$ *is* C^1, *the map* $x\mapsto g'(x)$ *extends continuously to a map (also denoted* $x\mapsto g'(x)$*) on* $\bar{H}_r$ *and* $g'(x)$ *is a primitive matrix. If* $\langle\xi^k \mid k\geqslant 0\rangle$ *is a sequence of points in* K_R^N *such that* $\xi^{k+1}\leqslant g(\xi^k)$ *for all* $k\geqslant 0$, *then* $\langle\|\xi^k\|_1 \mid k\geqslant 0\rangle$ *is a decreasing sequence with limit* r *and* $\lim_{k\to\infty}\xi^k=\eta^r$.

Remark 2.13. By using results from Sections 2 and 3 of Ishikawa and Nussbaum (1990), one can refine Corollary 6 and obtain information about the rate of convergence of $g^k(x)$, $\|x\|_1 = r$, to the fixed point η^r.

2.4. Strong Convergence Results for Iterates of NEP Maps on $l_1^+(\mathbb{Z})$

If $F \in \mathscr{F}(R, N)$, we now wish to give further conditions which imply that for all $x \in K_R$,

$$\lim_{k \to \infty} \|F^k(x)\|_\infty = 0$$

As usual, we denote by $\{e_j : j \in \mathbb{Z}\}$ the standard orthonormal basis of $l_2(\mathbb{Z})$, so the coordinates of e_j all equal zero except for coordinate j, which equals one.

Definition 2.14. Suppose that $D \subset l_\infty(\mathbb{Z})$. A map $\varphi: D \to \mathbb{R}$ is "increasing on D and strictly increasing on D in variable j" if for all $\delta > 0$, there exists $\varepsilon > 0$ such that for all $x, y \in D$ with $x \leqslant y$ and $y_j - x_j \geqslant \delta$, one has

$$g(y) - g(x) \geqslant \varepsilon$$

In Definition 2.14, $x \leqslant y$ means that $x_k \leqslant y_k$ for all $k \in \mathbb{Z}$.

Definition 2.15. Suppose that $D \subset l_\infty(\mathbb{Z})$ and $F: D \to l_\infty(\mathbb{Z})$ is an order-preserving map. A bounded linear map $A: l_1(\mathbb{Z}) \to l_1(\mathbb{Z})$ will be called "a strict monotonicity incidence matrix for F on D" if

(1) for all $j \in \mathbb{Z}$, $A(e_j) = \sum_{i \in \mathbb{Z}} a_{ij} e_i$, where $a_{i,j} \geqslant 0$ for all $i, j \in \mathbb{Z}$ and $\sum_{i \in \mathbb{Z}} a_{ij} \leqslant 1$, and

(2) if $a_{ij} > 0$, then $F_i: D \to \mathbb{R}$ is increasing on D and strictly increasing on D in variable j.

If A is as in Definition 2.15, we shall refer to a_{ij} as the (i, j) entry of A.

We shall need a slightly stronger assumption than that a map $\varphi: D \to \mathbb{R}$ is increasing and strictly increasing on D in variable j.

Definition 2.16. Suppose that $D \subset l_\infty(\mathbb{Z})$ and $F: D \to l_\infty(\mathbb{Z})$ is order-preserving. Assume that A is a strict monotonicity incidence matrix for F on D. Given a constant $c > 0$, we shall say that "A is a strict monotonicity incidence matrix with constant c for F on D" if for all $x, y \in D$ with $x \leqslant y$, we have

$$F_i(y) - F_i(x) \geqslant c \sum_{j,\, a_{ij} > 0} (y_j - x_j) \tag{35}$$

Remark 2.17. Suppose that $F, H \in \mathscr{F}(R, N)$ (see Definition 2.5) and A is a strict monotonicity incidence matrix with constant c for F on D. Then for any $t \in (0, 1]$, A is a strict monotonicity incidence matrix with constant $tc > 0$ for $tF + (1-t)\, H$ on D.

Remark 2.18. Suppose that $\varphi: G \subset K_R^N \to \mathbb{R}$ is increasing on G and strictly increasing on G in variable j in the sense of Definition 2.9. If G is closed, $R < \infty$, and φ is continuous, a simple compactness argument shows that for each $\delta > 0$ there exists $\varepsilon > 0$ such that for all $x, y \in G$ with $x \leqslant y$ and $y_j - x_j \geqslant \delta$, one has $\varphi(y) - \varphi(x) \geqslant \varepsilon$. Thus, Definitions 2.9 and 2.14 are consistent.

Remark 2.19. Suppose that H1.1 is satisfied and that $F: K_R \to K_R$ is defined by Eq. (12). We have proved (Lemmas 2.6 and 2.7) that $F \in \mathscr{F}(R, N)$ and $F: D(R, N) := D \to D$, where $D(R, N)$ is as in Definition 2.5. In particular, we have proved that

$$F_i(x) := \sum_{j \in \mathbb{Z}} \sigma_{j,\, i}(x_j) < \infty$$

for all $x \in D$. For each i, let $J_i = \{ j \in \mathbb{Z} : \sigma_{j,\, i}$ is strictly increasing on $[0, R] \}$. Define $a_{ij} = 0$ of $j \notin J_i$ and select $a_{ij} > 0$ if $j \in J_i$. We can also arrange that $\sum_i a_{ij} \leqslant 1$. Define a bounded linear map $A: l_1(\mathbb{Z}) \to l_1(\mathbb{Z})$ by $A(e_j) = \sum_i a_{ij} e_i$. Then A is a strict monotonicity incidence matrix for F on $D(R, N)$. To see this, suppose $a_{ij} > 0$ and, given $\delta > 0$, suppose $x, y \in D$, $x \leqslant y$, and $y_j - x_j \geqslant \delta$. Because $\sigma_{j,\, i}$ is strictly increasing on $[0, R]$, there exists $\varepsilon > 0$ such that $\sigma_{j,\, i}(t) - \sigma_{j,\, i}(s) \geqslant \varepsilon$ for all $x, t \in [0, R]$ with $t - s \geqslant \delta$. For this $\varepsilon > 0$ we have

$$\begin{aligned} F_i(y) - F_i(x) &= \sum_{k \neq j} (\sigma_{k,\, i}(y_k) - \sigma_{k,\, i}(x_k)) + \sigma_{j,\, i}(y_j) - \sigma_{j,\, i}(x_j) \\ &\geqslant \sigma_{j,\, i}(y_j) - \sigma_{j,\, i}(x_j) \geqslant \varepsilon \end{aligned}$$

Next, suppose that $c > 0$. For each $i \in \mathbb{Z}$, let

$$J_i^c = \{ j \in \mathbb{Z} : |\sigma_{j,\, i}(t) - \sigma_{j,\, i}(s)| \geqslant c\, |t - s| \text{ for all } t, s \in [0, R] \}$$

Define $b_{ij}=0$ if $j \notin J_i^c$. Select $b_{ij}>0$ if $j \in J_i^c$. We can also arrange that $\sum_i b_{ij} \leqslant 1$ for all $j \in \mathbb{Z}$. If we define a bounded linear operator B: $l_1(\mathbb{Z}) \to l_1(\mathbb{Z})$ by $B(e_j)=\sum_i b_{ij} e_i$, then B is a strict monotonicity incidence matrix with constant c for F on $D(R, N)$. To see this, note that if $x, y \in D(R, N)$ and $x \leqslant y$, we have

$$
\begin{aligned}
F_i(y)-F_i(x) &= \sum_{k,\, b_{ik}=0} (\sigma_{k,i}(y_k)-\sigma_{k,i}(x_k)) + \sum_{k,\, b_{ik}>0} (\sigma_{k,i}(y_k)-\sigma_{k,i}(x_k)) \\
&\geqslant \sum_{k,\, b_{ik}>0} (\sigma_{k,i}(y_k)-\sigma_{k,i}(x_k)) \\
&\geqslant c \sum_{k,\, b_{ik}>0} (y_k - x_k)
\end{aligned}
$$

Remark 2.20. Suppose that $F \in \mathscr{F}(R, N)$ and that for $x, y \in D(R, N)$ with $x \leqslant y$ and $y_j - x_j \geqslant \delta$ we have $F_i(y)-F_i(x) \geqslant \varepsilon$. If m is an integer and $\tilde{y}=S^{mN}y$, $\tilde{x}=S^{mN}x$, we have that $\tilde{x} \leqslant \tilde{y}$ and $\tilde{y}_{j-mN}-\tilde{x}_{j-mN} \geqslant \delta$; and we derive from (24) that

$$
F_{i-mN}(\tilde{y})-F_{i-mN}(\tilde{x})=F_i(y)-F_i(x) \geqslant \varepsilon
$$

It follows that if A is a strict monotonicity incidence matrix for F on $D(R, N)$, we can assume that $a_{ij}=a_{i-mN,\, j-mN}$ for all integers i, j, m. The same sort of argument shows that if A is a strict monotonicity incidence matrix with constant c for F on D, we can assume that $a_{ij}=a_{i-mN,\, j-mN}$ for all integers i, j, m. Thus, we shall always assume that our incidence matrices for $F \in \mathscr{F}(R, N)$ satisfy $a_{ij}=a_{i-mN,\, j-mN}$ for all integers i, j, m. Note that if $F \in \mathscr{F}(R, N)$ and A is a strict monotonicity incidence matrix with constant $c>0$ for F on $D(R, N)=D$, then necessarily, for each i, $a_{ij}>0$ for at most finitely many j.

Lemma 2.21. *Suppose that $F, G \in \mathscr{F}(R, N)$ (see Definition 2.5). Assume that A is a strict monotonicity incidence matrix for F on $D(R, N)$ and B is a strict monotonicity incidence matrix for G on D. Then BA is a strict monotonicity incidence matrix for $G \circ F$ on D. If $\tilde{A}$ is a strict monotonicity incidence matrix with constant $c>0$ for F on D and $\tilde{B}$ is a strict monotonicity incidence matrix with constant $d>0$ for G on D, then $\tilde{B}\tilde{A}$ is a strict monotonicity incidence matrix with constant $dc>0$ for $G \circ F$ on D.*

Proof. If $C=BA$, one easily checks that $C(e_j)=\sum_j c_{ij} e_i$, where $c_{ij}=\sum_k b_{ik} a_{kj}$. One easily derives from the corresponding properties of A and B that C is a bounded linear map of $l_1(\mathbb{Z})$ into itself, that $c_{ij} \geqslant 0$ for all integers i, j, that $c_{ij} = c_{i-mN,\, j-mN}$ for all integers, i, j, m, and that $\sum_i c_{ij} \leqslant 1$. Similar remarks apply to $\widetilde{C} = \widetilde{B}\widetilde{A}$. To show that C is a strict monotonicity incidence matrix for $G \circ F$ on D, suppose that $c_{ij} = \sum_k b_{ik} a_{kj} > 0$. Then there exists $k \in \mathbb{Z}$ such that $b_{ik} > 0$ and $a_{kj} > 0$. Select $\delta > 0$ and suppose that $x, y \in D$, $x \leqslant y$, and $y_j - x_j \geqslant \delta$. By definition, there exists $\eta > 0$, dependent only on δ, k, j, such that $F(y) \geqslant F(x)$ and $F_k(y) - F_k(x) \geqslant \eta$. By definition of the strict monotonicity incidence matrix for G, there exists $\varepsilon > 0$, dependent only on η, i, k, such that $G(F(y)) \geqslant G(F(x))$ and $G_i(F(y)) - G_i(F(x)) \geqslant \varepsilon$. This proves that BA is a strict monotonicity incidence matrix for $G \circ F$ on D.

Suppose that $\tilde{a}_{kj}$ is the (k, j) entry of $\widetilde{A}$ and $\tilde{b}_{kj}$ is the (i, k) entry of $\widetilde{B}$. If $x, y \in D$ and $x \leqslant y$, we have

$$F_k(y) - F_k(x) \geqslant c \sum_{j,\, \tilde{a}_{kj} > 0} (y_j - x_j) \qquad \text{and}$$

$$G_i(F(y)) - G_i(F(x)) \geqslant d \sum_{k,\, \tilde{b}_{ik} > 0} (F_k(y) - F_k(x))$$

It follows that

$$G_i(F(y)) - G_i(F(x)) \geqslant cd \sum_{k,\, \tilde{b}_{ik} > 0} \left(\sum_{j,\, \tilde{a}_{k,j} > 0} (y_j - x_j) \right) \tag{36}$$

Define $\varepsilon_{kj} = 1$ if $\tilde{a}_{kj} > 0$ and $\varepsilon_{kj} = 0$ if $\tilde{a}_{kj} = 0$; define $\eta_{ij} = 1$ if $\tilde{b}_{ik} > 0$ and $\eta_{ik} = 0$ if $\tilde{b}_{ik} = 0$. Equation (36) gives that

$$\begin{aligned} G_i(F(y)) - G_i(F(x)) &\geqslant cd \sum_k \left(\sum_j \eta_{ik} \varepsilon_{kj} (y_j - x_j) \right) \\ &= cd \sum_j \left((y_j - x_j) \sum_k \eta_{ik} \varepsilon_{kj} \right) \end{aligned} \tag{37}$$

Changing the order of summation in (37) is justified, because all summands are nonnegative. Note that $\sum_k \eta_{ik} \varepsilon_{kj} \geqslant 1$ if $\tilde{c}_{ij} = \sum_k \tilde{b}_{ik} \tilde{a}_{kj} > 0$, so we obtain from (33) that

$$\begin{aligned} G_i(F(y)) - G_i(F(x)) &\geqslant cd \sum_j \left((y_j - x_j) \sum_k \eta_{ik} \varepsilon_{kj} \right) \\ &\geqslant cd \sum_{j,\, \tilde{c}_{ij} > 0} (y_j - x_j) \end{aligned}$$

This proves that $\widetilde{B}\widetilde{A}$ is a strict monotonicity incidence matrix with constant $cd > 0$ for $G \circ F$ on D. $\square$

Suppose now that $F \in \mathscr{F}(R, N)$ and that $g: K_R^N \to K_R^N$ is defined by $g(\xi) = F(\theta_N(\xi))$ (see Lemma 2.7). We know that $\delta(F^{k+1}(x)) \leqslant g(\delta(F^k(x)))$ and g is integral-preserving, so

$$\begin{aligned} \|\delta(F^{k+1}(x))\|_1 := \sum_{i=0}^{N-1} \delta_i(F^{k+1}(x)) &\leqslant \|g(\delta(F^k(x)))\|_1 \\ &= \sum_{i=0}^{N-1} \delta_i(F^k(x)) \end{aligned} \tag{38}$$

and $\|\delta(F^k(x))\|_1$ is a decreasing sequence.

We want to prove (under further assumptions on F):

(a) For each $x \in K_R$, $\lim_{k\to\infty} \|F^k(x)\|_\infty = 0$

We also want to prove a stronger, uniform version of (a):

(b) For each $\eta > 0$ there exists $k(\eta)$ such that $\|F^j(x)\|_\infty \leqslant \eta$ for all $x \in K_R$ and all $j \geqslant k(\eta)$.

Because $\|\delta(F^k(x))\|_1$ is decreasing, (a) is equivalent to

(a′) For each $x \in K_R$, there exists an integer $m \geqslant 1$ such that $\lim_{k\to\infty} \|\delta(F^{km}(x))\|_1 = 0$.

and (b) is equivalent to

(b′) For each $\eta > 0$ there exists $m = m(\eta)$ and $k = k(\eta)$ such that $\|\delta(F^{km}(x))\|_1 \leqslant \eta$ for all $x \in K_R$.

***Theorem* 7.** *Suppose that $F \in \mathscr{F}(R, N)$ and that A is a strict monotonicity incidence matrix for F on $D = D(R, N)$ (see Definition 2.15). Suppose that for each i, $0 \leqslant i < N$, there exist integers m_i and $n_i \neq 0$ such that A^{m_i} has positive entries in the (i, i) position and the $(i, i + n_i N)$ position. Then, for every $x \in K_R$, we have $\lim_{k\to\infty} \|F^k(x)\|_\infty = 0$.*

Proof. Let $m_* = \operatorname{lcm}(\{m_i: 0 \leqslant i \leqslant N-1\})$, so $m_* = t_i m_i$ for some positive integer t_i. If α_{kj} denotes the k, j entry of A^{m_i} and we recall that $\alpha_{kj} > 0$ implies $\alpha_{k+mN, j+mN} > 0$ for all integers m, we see that $A^{m_*} = (A^{m_i})^{t_i}$ has positive entries at the (i, j) position for $0 \leqslant i < N$ and $j = i + t n_i N$, for $0 \leqslant t \leqslant t_i$. Let $g(w) = F(\theta_N(w))$ for $w \in K_R^N$. Because $\delta(F^{k+1}(x)) \leqslant g(\delta(F^k(x)))$, Theorem 2 implies that, for $q = \operatorname{lcm}(N)$ and $x \in K_R$, $\lim_{k\to\infty} \delta(F^{kq}(x))$ exists and equals $\xi = \xi_x$, where $g^q(\xi) = \xi$. Let $p = \operatorname{lcm}(q, m_*)$ and write $p = s_i m_i$. Lemma 2.21 implies that $\tilde{A} := A^p$ is a strict monotonicity incidence matrix for $\tilde{F} := F^p$. Lemma 2.7 implies that

$\widetilde{F} \in \mathscr{F}(R, N)$ and that $\tilde{g} = \widetilde{F} \circ \theta_N = g^p$. Our selection of p ensures that $\lim_{k \to \infty} \delta(\widetilde{F}^k(x)) = \xi$, where $\tilde{g}(\xi) = \xi$. Also, we see that $\widetilde{A}$ has positive entries at the (i, j) position for $i \in \mathbb{Z}$ and $j \in \{i + sn_i N : 0 \leqslant s \leqslant s_i\}$ (where we put $n_{i_1} = n_{i_2}$ if $i_1 \equiv i_2 \bmod N$). For convenience, we write F, g, and A instead of $\widetilde{F}$, $\tilde{g}$, and $\widetilde{A}$, and we use the above properties.

To complete the proof, it suffices to prove that $\lim_{k \to \infty} \delta(F^k(x)) = 0$. We assume not, so $\xi \neq 0$, and we argue by contradiction. Select a fixed integer i, $0 \leqslant i < N$, such that $\xi_i = \max_k \xi_k$. Write $x^j = F^j(x)$. Suppose, for each j, that $k(j)$ is an integer such that $k(j) \equiv i \bmod N$ and $\lim_{j \to \infty} x^j_{k(j)} = \xi_i$. By our construction, such a sequence $k(j)$, $j \geqslant 1$, exists. If $F_k(x)$ denotes the kth coordinate of $F(x)$, we have

$$\begin{aligned} x^j_{k(j)} &= F_{k(j)}(x^{j-1}) = g_i(\delta(x^{j-1})) - (g_i(\delta(x^{j-1})) - F_{k(j)}(x^{j-1})) \\ &= F_{k(j)}(\theta_N(\delta(x^{j-1}))) - (F_{k(j)}(\theta_N(\delta(x^{j-1}))) - F_{k(j)}(x^{j-1})) \end{aligned} \tag{39}$$

We claim that $\lim_{j \to \infty} x^{j-1}_{k(j) + sn_i N} = \xi_i$ for $0 \leqslant s \leqslant s_i$. For, suppose not. By taking a subsequence $j_l \to \infty$ as $l \to \infty$ and an appropriate s, $0 \leqslant s \leqslant s_i$, we can assume that $\lim_{j \to \infty} x^{j-1}_{k(j) + sn_i N} = \alpha < \xi_i$. If we write $k(j) = i + \mu_j M$, where $\mu_j \in \mathbb{Z}$, Eq. (23) gives

$$\begin{aligned} &F_{k(j)}(\theta_N(\delta(x^{j-1}))) - F_{k(j)}(x^{j-1}) \\ &\qquad = F_i(\theta_N(\delta(x^{j-1}))) - F_i(S^{\mu_j N}(x^{j-1})) \end{aligned} \tag{40}$$

We know that F_i is strictly increasing on D in the $i + sn_i N$ coordinate, because we write A for $\widetilde{A}$ and the $(i, i + sn_i N)$ entry of $\widetilde{A}$ is positive. By our construction, $\theta_N(\delta(x^{j-1})) \geqslant S^{\mu_j N}(x^{j-1})$ for all j; and the limit as j approaches ∞ of the $i + sn_i N$ coordinate of $\theta_N(\delta(x^{j-1}))$ [respectively, $S^{\mu_j N}(x^{j-1})$] is ξ_i (respectively, α). It follows from the definition of a function being strictly increasing on D that there exists $\varepsilon > 0$ such that for all sufficiently large j,

$$F_i(\theta_N(\delta(x^{j-1}))) - F_i(S^{\mu_j N}(x^{j-1})) \geqslant \varepsilon \tag{41}$$

If we use Eqs. (40) and (41) in Eq. (39), we see that for all sufficiently large j,

$$x^j_{k(j)} \leqslant g_i(\delta(x^{j-1})) - \varepsilon \tag{42}$$

Because $\lim_{j \to \infty} x^j_{k(j)} = \xi_i = \lim_{j \to \infty} g_i(\delta(x^{j-1}))$, Eq. (42) gives a contradiction.

Thus, the assumption that $\lim_{j\to\infty} x^j_{k(j)} = \xi_i = \max_k \xi_k$ implies that $\lim_{j\to\infty} x^{j-1}_{k(j)+sn_iN} = \xi_i$ for $0 \leqslant s \leqslant s_i$. But now we can apply the same observation to conclude that

$$\lim_{j\to\infty} x^{j-2}_{k(j)+sn_iN} = \xi_i \qquad \text{for} \quad 0 \leqslant s \leqslant 2s_i$$

In general, for $m \geqslant 1$, we find that

$$\lim_{j\to\infty} x^{j-m}_{k(j)+sn_iN} = \xi_i \qquad \text{for} \quad 0 \leqslant s \leqslant ms_i \tag{43}$$

Because $s_i \geqslant 1$ and we assume that $n_i \neq 0$, it follows that

$$\liminf_{j\to\infty} \|x^{j-m}\|_1 = \liminf_{j\to\infty} \|x^j\|_1 \geqslant (m+1)\,\xi_i$$

However, $\|x^j\|_1 = \|x\|_1$, because F is integral-preserving, so if we select m so that $(m+1)\,\xi_i > \|x\|_1$, we obtain a contradiction. Thus, the assumption that $\xi \neq 0$ leads to a contradiction. □

As a simple example of how Theorem 7 can be used, we give the following.

Corollary 8. *Assume hypothesis H1.1 and let $F \in \mathscr{F}(R, N)$ be defined by (12). For every $i \in \mathbb{Z}$, assume that $\sigma_{i-1,i}$ and σ_{ii} are strictly increasing on $[0, R]$. Then for every $x \in K_R$, $\lim_{k\to\infty} \|F^k(x)\|_\infty = 0$.*

Proof. Our assumption imply that for every $i \in \mathbb{Z}$, F_i, the ith coordinate function of F, is increasing on $D = D(R, N)$ and strictly increasing on D in variable j for $j = i$ and $j = i-1$. If A is a strict monotonicity incidence matrix for F on D, it follows that $a_{i,i-1} > 0$ and $a_{ii} > 0$. This implies that the (i, j) entry of A^k is positive for $i-k \leqslant j \leqslant i$. In particular, the (i, j) entry of A^N is positive for $i-N \leqslant j \leqslant i$, and the hypotheses of Theorem 7 are satisfied with $m_i = N$ and $n_i = -1$ for $0 \leqslant i < N$. □

It remains to give conditions under which $\|F^k(x)\|_\infty$ converges to zero uniformly for $x \in K_R$.

Theorem 9. *Suppose that $F \in \mathscr{F}(R, N)$ and that A is a strict monotonicity incidence matrix with constant $c > 0$ for F on $D(R, N)$ [see Definition 2.16, where $\mathscr{F}(R, N)$ and $D(R, N) = D$ are as in Definition 2.5]. For each i, $0 \leqslant i < N$, assume that there exists an integer $m_i \geqslant 1$ and an integer $n_i \neq 0$ such that A^{m_i} has positive entries in the (i, i) and $(i, i+n_iN)$ positions. Then, given any $\eta > 0$, there exists an integer $k(\eta)$ such that $\|F^k(x)\|_\infty \leqslant \eta$ for all $x \in K_R$ and all $k \geqslant k(\eta)$.*

Proof. Define $m = \text{lcm}(\{m_i\colon 0 \leqslant i < N\})$ and write $m = s_i m_i$. As in the proof of Theorem 7, A^m has positive entries at the (i, j) position for $0 \leqslant i < N$ and for $j \in \{i + s n_i N : 0 \leqslant s \leqslant s_i\}$. If b_{kj} denotes the (k, j) entry of A^m, we know that $b_{k+\mu N,\, j+\mu N} = b_{kj}$ for all integers μ; so if $n_{i_1} = n_{i_2}$ for $i_1 \equiv i_2 \bmod N$, we find that A^m has a positive entry at (i, j) if $i \in \mathbb{Z}$ and $j \in \{i + s n_i N : 0 \leqslant s \leqslant s_i\}$.

Because $\|\delta(F^k(x))\|_1 = \sum_{i=0}^{N-1} \delta_i(F^k(x))$ is a decreasing function of k, it suffices to prove that, given $\varepsilon > 0$, there exists $k(\varepsilon) = k$ such that $\sum_{i=0}^{N-1} \delta_i(F^{mk}(x)) \leqslant N\varepsilon$ for all $x \in K_R$. Recall (Lemma 2.21) that A^{mk} is a strict monotonicity incidence matrix with constant c^{mk} for F^{mk} on $D(R, N)$ and that A^{mk} has positive entries at the (i, j) positions for $i \in \mathbb{Z}$ and $j \in \{i + s n_i N : 0 \leqslant s \leqslant k s_i\}$.

Define l to be the smallest positive integer such that $lN\varepsilon \geqslant R + \varepsilon$. Select an arbitrary $x \in K_R$. If $\sum_{i=0}^{N-1} \delta_i(x) \leqslant N\varepsilon$, then we know that $\sum_{i=0}^{N-1} \delta_i(F^k(x)) \leqslant N\varepsilon$ for all $k \geqslant 1$, and we are done. Thus assume that $\sum_{i=0}^{N-1} \delta_i(x) > N\varepsilon$. Define $y = \theta_N(\delta_0(x),..., \delta_{N-1}(x))$, so $y \in D(R, N)$ and $y \geqslant x$. For each i, let F_i^k be the ith coordinate function of F^k, and let $x^k = F^k(x)$ and $x_i^k = F_i^k(x)$. Choose integers μ_i, $i = 0,..., N-1$, such that

$$F^{ml}_{i+\mu_i N}(x) := x^{ml}_{i+\mu_i N} = \delta_i(F^{ml}(x))$$

Lemma 2.7 implies that $F^{ml} \in \mathscr{F}(R, N)$, $\sum_{i=0}^{N-1} F_i^{ml}(y) = \sum_{i=0}^{N-1} \delta_i(x)$, and $F^{ml}_{i+\mu_i N}(y) = F_i^{ml}(y)$. Putting this information together we find that

$$\begin{aligned} F_i^{ml}(y) - F^{ml}_{i+\mu_i N}(x) &= F^{ml}_{i+\mu_i N}(y) - F^{ml}_{i+\mu_i N}(x) \\ &\geqslant c^{ml} \left(\sum_{s=0}^{l-1} (\delta_i(x) - x^{ml}_{i+\mu_i N + s n_i N}) \right) \end{aligned} \tag{44}$$

Adding (44) for $0 \leqslant i \leqslant N-1$, we obtain

$$\begin{aligned} \sum_{i=0}^{N-1} (F_i^{ml}(y) - F^{ml}_{i+\mu_i N}(x)) &= \sum_{i=0}^{N-1} \delta_i(x) - \sum_{i=0}^{N-1} \delta_i(F^{ml}(x)) \\ &\geqslant c^{ml} \left(l \sum_{i=0}^{N-1} \delta_i(x) - \sum_{i=0}^{N-1} \sum_{s=0}^{l-1} x^{ml}_{i+\mu_i N + s n_i N} \right) \\ &\geqslant c^{ml} \left(l \sum_{i=0}^{N-1} \delta_i(x) - \sum_{j \in \mathbb{Z}} x_j^{ml} \right) \\ &= c^{ml} \left(l \sum_{i=0}^{N-1} \delta_i(x) - \sum_{j \in \mathbb{Z}} x_j \right) \end{aligned} \tag{45}$$

Since we assume that $\sum_{i=0}^{N-1} \delta_i(x) > N\varepsilon$, we have that

$$l \sum_{i=0}^{N-1} \delta_i(x) - \sum_{j \in \mathbb{Z}} x_j > lN\varepsilon - R \geqslant \varepsilon$$

and we have proved that for any $x \in K_R$ such that $\sum_{i=0}^{N-1} \delta_i(x) > N\varepsilon$, we have

$$\sum_{i=0}^{N-1} \delta_i(F^{ml}(x)) \leqslant \sum_{i=0}^{N-1} \delta_i(x) - c^{ml}\varepsilon \tag{46}$$

If $\|\delta(F^{ml}(x))\|_1 \leqslant N\varepsilon$, we know that $\|\delta(F^j(x))\|_1 \leqslant N\varepsilon$ for all $j \geqslant ml$. Otherwise, we apply the same argument to x^{ml} and conclude that

$$\begin{aligned}\sum_{i=0}^{N-1} \delta_i(F^{ml}(x^{ml})) = \sum_{i=0}^{N-1} \delta_i(F^{2ml}(x)) &\leqslant \sum_{i=0}^{N-1} \delta_i(F^{ml}(x)) - c^{ml}\varepsilon \\ &\leqslant \sum_{i=0}^{N-1} \delta_i(x) - 2c^{ml}\varepsilon\end{aligned}$$

In general, if $\sum_{i=0}^{N-1} \delta_i(F^{jml}(x)) > N\varepsilon$ for $0 \leqslant j \leqslant k-1$, we find that

$$\sum_{i=0}^{N-1} \delta_i(F^{kml}(x)) \leqslant \sum_{i=0}^{N-1} \delta_i(x) - kc^{ml}\varepsilon \leqslant R - kc^{ml}\varepsilon \tag{47}$$

If k is the first positive integer such that $R - kc^{ml}\varepsilon \leqslant N\varepsilon$, it follows from (47) that there must exist j, $0 \leqslant j \leqslant k$, such that $\sum_{i=0}^{N-1} \delta_i(F^{jml}(x)) \leqslant N\varepsilon$, and this completes the proof. □

The following is a simple application of Theorem 9 to our original class of operators. The proof follows from Remark 2.19 and Theorem 9 and is left to the reader.

Corollary 10. *Assume hypothesis H1.1 and let $F \in \mathscr{F}(R, N)$ be defined by (12). Assume that there exists $c > 0$ such that for integers i with $0 \leqslant i \leqslant N-1$ and all real numbers $s, t \in [0, R]$, we have*

$$|\sigma_{i-1,i}(t) - \sigma_{i-1,i}(s)| \geqslant c\,|t-s| \qquad \text{and} \qquad |\sigma_{i,i}(t) - \sigma_{i,i}(s)| \geqslant c\,|t-s|$$

Then for every $\eta > 0$ there exists $k(j) \geqslant 0$ such that $\|F^k(x)\|_\infty \leqslant \eta$ for all $x \in K_R$ and all $k \geqslant k(j)$.

Note that Corollary 10 contains as a very special case the example studied in Section 2.1.

3. DISCONTINUOUS DELAY EQUATIONS

3.1. Statement of the Problem

We study the equation

$$\dot{x}(t) = -\mathrm{sign}(x(t-1)) + f(x(t)), \qquad t \geqslant 0 \tag{48}$$

with a locally Lipschitz function $f: \mathbb{R} \to \mathbb{R}$ satisfying $|f(x)| \leqslant p < 1$, and $\mathrm{sign}(x) = 1$, 0, or -1 according to whether x is positive, zero, or negative. This is a model of an autonomous system with a retarded relay control element. For a general theory of delay equations we refer to Dieckmann *et al.* (1995), Hale (1971), and Hale and Verduyn Lunel (1993), and for discontinuous delay equations to Utkin (1992). Models like (48) were used in some control systems (see Andre and Seibert, 1956; Bartolini *et al.*, 1997; Choi and Hedrick, 1996; Moskwa and Hedrick, 1989).

The Cauchy problem $x(t) = \varphi(t)$, $t \in [-1, 0]$, has a unique continuous solution x_φ for any $\varphi \in C[-1, 0]$. All these solutions oscillate around the zero level, and the *frequency function*,

$$v_\varphi(t) = \#(x_\varphi^{-1}(0) \cap (t^* - 1, t^*)) \qquad t^* = \max\{\tau \leqslant t: x_\varphi(\tau) = 0\}$$

is *decreasing* (see Shustin *et al.*, 1993). Hence there always exists the *limit frequency*

$$N_\varphi = \lim_{t \to \infty} v_\varphi(t) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$$

In particular, once the frequency becomes finite, it will be finite thereafter.

Properties of solutions to (48) with a finite limit frequency are basically known (see Mallet-Paret and Nussbaum, 1986, 1992; Mallet-Paret, 1988; Shustin *et al.*, 1993; Walther, 1981, 1991, 1995). For the reader's convenience we present a proof of the existence of periodic solutions with arbitrary even limit frequency N_φ. Our proof is simpler than earlier methods.

Assume that $\varphi \in C[-1, 0]$ is negative everywhere but $\varphi(0) = 0$. Then, in the interval $[0, 1]$, $x_\varphi(t)$ coincides with the solution of the equation

$$\dot{x}(t) = 1 + f(x(t)), \qquad x(0) = 0$$

In the interval $[1, z+1]$, $x_\varphi(t)$ coincides with the solution of the equation

$$\dot{x}(t) = -1 + f(x(t)), \qquad x(1) = x_\varphi(1)$$

where $z > 1$ is the unique zero of the latter solution. Finally, in the interval $[z+1, w]$, $x_\varphi(t)$ coincides with the solution of the equation

$$\dot{x}(t) = 1 + f(x(t)), \qquad x(z+1) = x_\varphi(z+1)$$

where w is the unique zero of the latter solution. Since the constant sign intervals for $x_\varphi(t)$, $t \in [0, w]$, are of length > 1, $x_\varphi(t)$ extends for $t \geqslant w$ periodically with period w, and it has $N_\varphi = 0$. Note that

$$\frac{4}{1+p} \leqslant w \leqslant \frac{4}{1-p} \tag{49}$$

Given any $\lambda > 0$, one can similarly construct a solution [which we denote $X_\lambda(t)$] to the equation

$$\dot{x}(t) = -\operatorname{sign}(x(t-1)) + f(\lambda x(t)), \qquad t \geqslant 0$$

which has $N_\varphi = 0$ and a period $w(\lambda)$ satisfying (49). Now, for arbitrary positive integer n, put

$$Y_{\lambda, n}(t) = \frac{1}{nw(\lambda)+1} \cdot X_\lambda((nw(\lambda)+1)\, t), \qquad t \geqslant 0$$

Hence

$$\begin{aligned}
\dot{Y}_{\lambda, n}(t) &= \dot{X}_\lambda((nw(\lambda)+1)\, t) \\
&= -\operatorname{sign}(X_\lambda((nw(\lambda)+1)\, t - 1)) + f(\lambda X_\lambda((nw(\lambda)+1)\, t)) \\
&= -\operatorname{sign}(X_\lambda((nw(\lambda)+1)\, t - nw(\lambda) - 1)) + f(\lambda(nw(\lambda)+1) \cdot Y_{\lambda, n}(t)) \\
&= -\operatorname{sign}(Y_{\lambda, n}(t-1)) + f(\lambda(nw(\lambda)+1) \cdot Y_{\lambda, n}(t))
\end{aligned}$$

Due to (49) there exists a positive λ_n satisfying $\lambda_n(nw(\lambda_n)+1) = 1$, which thereby defines a solution Y_n to (48) with a period

$$\frac{w(\lambda_n)}{nw(\lambda_n)+1} \in \left(\frac{1}{n}, \frac{1}{n+1}\right)$$

Its limit frequency is $2n$.

We state the question: *Do there exist solutions to (48) with the infinite limit frequency* $N_\varphi = \infty$? We prove here the following result by means of the above theory of nonexpansive operators.

Theorem 11. *There are no solutions x_φ to (48) with $N_\varphi = \infty$ with the single exception of the case $f(0) = 0$, $x_\varphi(t) \equiv 0$.*

Moreover, we show that if $\varphi \in C[-1, 0]$ has infinitely many zeros (but is not identically zero), then the length of the interval with infinite frequency of oscillations can be uniformly estimated.

Theorem 12. *For any $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that*

$$v_\varphi(t) < \infty \qquad \textit{as} \quad t \geqslant T_\varepsilon$$

provided

$$\max\{\text{length}(I)\text{: } I \textit{ is a connected component of } [-1, 1]\backslash\varphi^{-1}(0)\} \geqslant \varepsilon$$

Independently Theorems 11 and 12 were proved by Akian and Bliman (2000) in a way similar to that of Shustin (1995).

Some results in this direction were obtained before.

- Shustin (1995) considered the equation

  $$\dot{x}(t) = -\text{sign}(x(t-1)) + f(x(t), t), \qquad t \geqslant 0$$

  with a twice differentiable function $f(x, t)$, satisfying $|f(x, t)| \leqslant p < 1$. It was shown that for any $p_0 \in (-1, 1)$ there exists $\varepsilon(p_0) > 0$ such that the statement of Theorem 12 holds under the condition

  $$f(x, t) = p_0 + xg(x, t), \qquad \max\{g(x, t), g_x(x, t), g_t(x, t)\} \leqslant \varepsilon(p_0)$$

- Dix (1998) considered the equation

  $$\dot{x}(t) = s(x(t - \tau(t))) \tag{50}$$

  where $s(0) = 0$, $s(x) = -p_0 < 0$ when $x > 0$, $s(x) = p_1 > 0$ when $x < 0$, and $\tau(t)$ is a nonincreasing bounded positive function with $\tau'(t) < 1$. He showed that the only infinite frequency solution of this equation is identically zero.

3.2. NEP Operators Associated with Delay Equations

The proof of Theorems 11 and 12 is strongly based on properties of the operators in $l_1^+(\mathbb{Z})$ studied in Section 2.1. In the sequel it is more convenient to define these operators not in $l_1^+(\mathbb{Z})$ but on the set of increasing bounded biinfinite sequences in $\mathbb{R}$.

Let $t_n \in \mathbb{R}$, $n \in \mathbb{Z}$, be such that

$$t_n \leqslant t_{n+1}, \quad n \in \mathbb{Z}, \qquad \lim_{n \to -\infty} t_n = \alpha \in \mathbb{R}, \qquad \lim_{n \to \infty} t_n = \beta \in \mathbb{R}$$

These data define, in particular, an element $a = \langle a_n : n \in \mathbb{Z} \rangle \in l_1^+(\mathbb{Z})$:

$$a_n = t_{n+1} - t_n, \quad n \in \mathbb{Z}, \qquad \|a\|_1 = \beta - \alpha$$

The image $\Psi(\{t_n\}) = \{t'_n\} \subset [\alpha, \beta]$ is constructed as follows. For any $n \in \mathbb{Z}$ we take the solutions $x = \xi_n(t)$, $t \in \mathbb{R}$, of the Cauchy problem

$$\dot{x} = 1 + f(x), \qquad x(t_{2n}) = 0$$

and the solution $x = \eta_n(t)$, $t \in \mathbb{R}$, of the Cauchy problem

$$\dot{x} = -1 + f(x), \qquad x(t_{2n-1}) = 0$$

All functions $\xi_n(t)$ are strongly increasing, and all functions $\eta_n(t)$ are strongly decreasing, hence there is a unique point $t'_{2n} \in [t_{2n}, t_{2n+1}]$ such that $\xi_n(t'_{2n}) = \eta_{n+1}(t'_{2n})$, $n \in \mathbb{Z}$, and there is a unique point $t'_{2n-1} \in [t_{2n-1}, t_{2n}]$ such that $\eta_n(t'_{2n-1}) = \xi_n(t'_{2n-1})$, $n \in \mathbb{Z}$. One can easily see that

$$t'_{2n} - t_{2n} = \sigma(a_{2n+1}), \qquad t_{2n} - t'_{2n-1} = \tau(a_{2n})$$

where σ, τ are nonnegative functions which can be found from the equations

$$\int_0^s \frac{dx}{1 + f(x)} = \sigma(t), \qquad \int_0^s \frac{dx}{1 - f(x)} = t - \sigma(t) \tag{51}$$

where $0 \leqslant s \leqslant t$,

$$\int_0^s \frac{dx}{1 + f(x)} = -\tau(-t), \qquad \int_0^s \frac{dx}{1 - f(x)} = t + \tau(-t) \tag{52}$$

where $t \leqslant s \leqslant 0$. One can easily see that $b = \langle b_n \mid n \in \mathbb{Z} \rangle \in l_1^+(\mathbb{Z})$, $b_n = t'_{n+1} - t'_n$, $n \in \mathbb{Z}$, satisfies

$$b_{2n} = \sigma(a_{2n}) + \tau(a_{2n-1}), \qquad b_{2n+1} = a_{2n+1} - \tau(a_{2n+1}) + a_{2n} - \sigma(a_{2n})$$

Due to (51), (52), the functions σ, τ are differentiable and satisfy

$$\sigma'(a) = \frac{1 - f(\xi)}{2}, \qquad \tau'(a) = \frac{1 - f(\eta)}{2}$$

with some $-a \leqslant \eta \leqslant 0 \leqslant \xi \leqslant a$. Hence σ, τ satisfy (1) with $\lambda = (1-p)/2$, $p = \sup |f(x)|$. Then the operator $b = F(a)$ satisfies the hypotheses of Theorem 1.

3.3. Uniform Convergence to Zero of Certain Sequences

Here we pass from biinfinite bounded sequences to bounded closed subsets which later will be interpreted as zero sets of solutions to (48) on unit segments.

For a closed set $G \subset \mathbb{R}$ we denote by $G_a \subset G$ the set of points in G which are not isolated.

Definition 3.1. (1) Denote by $\mathscr{G}$ the set of pairs (G, E), where $G \subsetneq [0, 1]$ is closed, and $E\colon [0, 1] \to \{-1, 0, 1\}$ is such that $E|_G = 0$, and $E\colon [0, 1] \backslash G \to \{\pm 1\}$ is locally constant. Denote by $\mathscr{G}_0$ the set of $(G, E) \in \mathscr{G}$ such that $0, 1 \in G$, and the measure of G is zero if $f(0) \neq 0$.

(2) Define a map $\pi\colon C[0, 1] \to \mathscr{G}$ by

$$\varphi \in C[0, 1] \mapsto (G, E), \qquad G = \varphi^{-1}(0), \qquad E(t) = \operatorname{sign}(\varphi(t))$$

Denote $C_0[0, 1] = \pi^{-1}(\mathscr{G}_0)$.

(3) Define an operator $J\colon \mathscr{G}_0 \to C[0, 1]$, $J(G, E) = \psi$, as follows:

$$\psi(a) = \int_0^a (-E(t) + f(\psi(t)))\, dt$$

The operator J is well defined, since the latter equation is uniquely solvable with respect to ψ. Moreover, it is injective.

Now we shall construct an operator $\Gamma\colon \mathscr{G}_0 \to C_0[0, 1]$ such that $\pi \circ \Gamma = \mathrm{Id}$ and $\mathrm{Im}(\Gamma) \subset \mathrm{Im}(J)$. First, for $x = \Gamma(G, E)$, put $x(t) = 0$, $t \in G$. Let $I = (\alpha, \beta)$ be a component of $[0, 1] \backslash G$. Take the solution $y(t)$ of the Cauchy problem

$$\dot{x}(t) = E(t) + f(x(t)), \qquad x(\alpha) = 0, \qquad \alpha \leqslant t \leqslant \beta$$

and the solution $z(t)$ of the Cauchy problem

$$\dot{x}(t) = -E(t) + f(x(t)), \qquad x(\beta) = 0, \qquad \alpha \leqslant t \leqslant \beta$$

There exists a unique $\xi = \xi(I) \in (\alpha, \beta)$ such that $y(\xi) = z(\xi)$. Put

$$x(t) = y(t), \qquad \alpha < t \leqslant \xi, \qquad x(t) = z(t), \qquad \xi < t < \beta$$

The function x constructed above belongs to $\pi^{-1}(G, E)$, and is $J(\hat{G}, \hat{E})$ for the following $(\hat{G}, \hat{E}) \in \mathcal{G}_0$:

$$\hat{G} = G_a \cup G^{\text{even}} \cup \{\xi(I): I \text{ is a connected component of } [0, 1] \backslash G\}$$

and G^{even} is the set of isolated points $t \in G$ such that $E(t-0) = E(t+0)$. By construction, $x(t)$ is differentiable in $[0, 1] \backslash \hat{G}$. For any $t \in [0, 1] \backslash \hat{G}$ we define $\hat{E}(t) = -\text{sign}(\dot{x}(t))$.

We can partially order the set $\pi^{-1}(G, E) \cap \text{Im}(J)$: for $x_1, x_2 \in \pi^{-1}(G, E) \cap \text{Im}(J)$,

$$x_1 \succ x_2 \Leftrightarrow x_1(t)\, E(t) \geqslant x_2(t)\, E(t), \qquad t \in [0, 1] \backslash G \tag{53}$$

One can easily see that $\Gamma(G, E)$ is the unique maximal element in $\pi^{-1}(G, E) \cap \text{Im}(J)$.

The element $(\hat{G}, \hat{E})$ can be viewed as an image of the operator Ψ, described in the previous subsection. Note that $\hat{G}_a = G_a$. Let $I = (\alpha, \beta)$ be a component of $[0, 1] \backslash G_a$. The set $[\alpha, \beta] \cap G$ yields an increasing biinfinite sequence [referred to below as *a tame sequence of* (G, E)]. We shall obtain such a sequence by numbering the points of the set $[\alpha, \beta] \cap G$ in increasing order so that on any interval (t_{2n}, t_{2n+1}) we have $E = 1$, and on any interval (t_{2n-1}, t_{2n}) we have $E = -1$. Let us describe the construction of a tame sequence in detail. If $I \cap G = \varnothing$, $E(t) = 1$, $t \in I$, then put

$$t_n = \alpha, \qquad n \leqslant 0, \qquad t_n = \beta, \qquad n > 0$$

If $I \cap G = \varnothing$, $E(t) = -1$, $t \in I$, then put

$$t_n = \alpha, \qquad n \leqslant 1, \qquad t_n = \beta, \qquad n > 1$$

If $I \cap G \neq \varnothing$, we put $t_0 = t$ for some $t \in I \cap G$, and then define t_n, $n \neq 0$, in the following recursive procedure:

- if $t_n = \beta$, put $t_m = \beta$, $m > n$;
- if $t_n = \alpha$, put $t_m = \alpha$, $m < n$;
- if $t_{2n} = \xi \in I$, $E(\xi + 0) = 1$, put $t_{2n+1} = \eta = \min\{t \in [\alpha, \beta] \cap G,\ t > \xi\}$;
- if $t_{2n} = \xi \in I$, $E(\xi + 0) = -1$, put $t_{2n+1} = \xi$;
- if $t_{2n-1} = \xi \in I$, $E(\xi + 0) = 1$, put $t_{2n} = \xi$;
- if $t_{2n-1} = \xi \in I$, $E(\xi + 0) = -1$, put $t_{2n} = \eta = \min\{t \in [\alpha, \beta] \cap G,\ t > \xi\}$;
- if $t_{2n} = \xi \in I$, $E(\xi - 0) = -1$, put $t_{2n-1} = \eta = \max\{t \in [\alpha, \beta] \cap G,\ t < \xi\}$;

- if $t_{2n}=\xi\in I$, $E(\xi-0)=1$, put $t_{2n-1}=\xi$;
- if $t_{2n+1}=\xi\in I$, $E(\xi-0)=-1$, put $t_{2n}=\xi$;
- if $t_{2n+1}=\xi\in I$, $E(\xi-0)=1$, put $t_{2n}=\eta=\max\{t\in[\alpha,\beta]\cap G,\ t<\xi\}$.

Then $[\alpha,\beta]\cap\hat{G}$ is the set of points of the sequence $\{t'_n\}=\Psi(\{t_n\})$ with added (if necessary) points α, β.

Definition 3.2. Let $(G,E)\in\mathcal{G}_0$. Define $\|G,E\|_\infty$ to be the maximum length of a component I of $[0,1]\backslash G$.

Lemma 9.9. *All sequences* $\langle\|G^{(N)},E^{(N)}\|_\infty\mid N\in\mathbb{N}\rangle$ *such that*

$$(G^{(0)},E^{(0)})\in\mathcal{G}_0,\qquad (G^{(m-1)},E^{(m-1)})=\pi(J(G^{(m)},E^{(m)})),\qquad m\geqslant 1 \tag{54}$$

converge to zero, uniformly with respect to $(G^{(0)},E^{(0)})\in\mathcal{G}_0$.

Proof. We exploit the same idea as in the proof of Theorem 1.

Put $\delta(G,E)$ [resp. $\Delta(G,E)$] to be the maximum length of a component I of $[0,1]\backslash G$ with $E(t)=1$, $t\in I$ [resp. $E(t)=-1$, $t\in I$]. As has been shown in the proof of Proposition 2.1,

$$\delta(J^{-1}\Gamma(G,E))\leqslant\sigma(\delta(G,E))+\tau(\Delta(G,E)) \tag{55}$$

$$\Delta(J^{-1}\Gamma(G,E))\leqslant\delta(G,E)-\sigma(\delta(G,E))+\Delta(G,E)-\tau(\Delta(G,E)) \tag{56}$$

For an arbitrary element $(\tilde{G},\tilde{E})\in J^{-1}\pi^{-1}(G,E)$,

$$\delta(\tilde{G},\tilde{E})\leqslant\delta(J^{-1}\Gamma(G,E)),\qquad \Delta(\tilde{G},\tilde{E})\leqslant\Delta(J^{-1}\Gamma(G,E))$$

We shall prove even stronger statement. Let $(\tilde{\alpha},\tilde{\beta})$ be a component of $[0,1]\backslash\tilde{G}_a$, and the set $[\tilde{\alpha},\tilde{\beta}]\cap\tilde{G}$ yields a tame sequence $\langle\tilde{t}_n\mid n\in\mathbb{Z}\rangle$. The interval $(\tilde{\alpha},\tilde{\beta})$ is contained in a component (α,β) of $[0,1]\backslash G_a$ [see Shustin (1995) or Section 3.4 below]. Let $\langle t_n\mid n\in\mathbb{Z}\rangle$, $\langle t'_n\mid n\in\mathbb{Z}\rangle$ be tame sequences of points of the sets $[\alpha,\beta]\cap G$ and $[\alpha,\beta]\cap\hat{G}$, respectively, where $(\hat{G},\hat{E})=J^{-1}\Gamma(G,E)$.

Proposition 3.4. *In the above notation, for any* $n\in\mathbb{Z}$, $k\in\mathbb{N}$, *there exists* $m\in\mathbb{Z}$ *such that*

$$\tilde{t}_{n+k}-\tilde{t}_n\leqslant t'_{n+k+2m}-t'_{n+2m}$$

Proof. Since the function $\psi=J(\tilde{G},\tilde{E})$ vanishes at G, we have the following:

- $G^{\text{even}} \cap (\tilde{\alpha}, \tilde{\beta}) \subset \tilde{G}$, since any point $t^* \in G^{\text{even}}$ is an extremum of ψ;
- any nonempty interval (t_n, t_{n+1}) contains a point of $\tilde{G}$, since ψ must have an extremum in (t_n, t_{n+1});
- if $t_n < t'_n < t_{n+1}$, then $(t_n, t'_n] \cap \tilde{G}$ and $[t'_n, t_{n+1}) \cap \tilde{G}$ are nonempty; in this case, if $t'_n = \tilde{t}_s$, then $n \equiv s \bmod 2$: this follows from the fact (pointed out before) that the graph of ψ in $[t_n, t_{n+1}]$, glued out of trajectories of the equations $\dot{x} = \pm 1 + f(x)$ lies in the domain bounded by the graph of $\Gamma(G, E)$ and the t-axis;
- if $t'_n < t'_{n+1}$, then $[t'_n, t'_{n+1}]$ contains at least two points of $\tilde{G}$, which is a consequence of the above reasoning.

Without loss of generality we can suppose that $\tilde{t}_n < \tilde{t}_{n+k}$. We consider several cases.

Case 1. Let $k = 1$. Then the graph of the function $x(t) = J(\tilde{G}, \tilde{E})$ restricted to the segment $[\tilde{t}_n, \tilde{t}_{n+1}]$ lies in a curvilinear triangle bounded by the t-axis and the graph of the function $\Gamma(G, E)$ on $[t_s, t_{s+1}]$ for some $s \in \mathbb{Z}$ (see Fig. 1). Since all these graphs consist of integral lines of the differential equations $\dot{x} = \pm 1 + f(x)$, one can easily derive that

$$\tilde{t}_{n+1} - \tilde{t}_n \leqslant t'_{s+1} - t'_s \qquad \text{or} \qquad t'_s - t'_{s-1}$$

according to whether $n - s$ is even or odd.

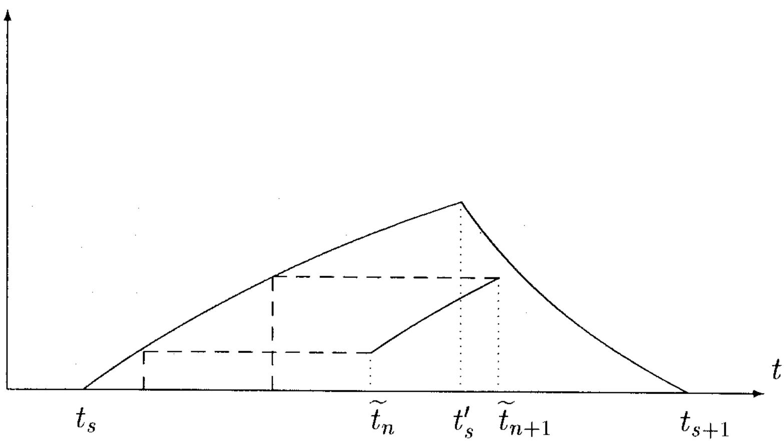

Fig. 1. Functions $J(\tilde{G}, \tilde{E})$ and $\Gamma(G, E)$.

Case 2. Assume that $k>1$ and $t'_{n+2m} \leqslant \tilde{t}_n < t'_{n+2m+1}$. If $t'_{n+2m+k} = \beta$, we are done. If $t'_{n+2m+k} < \beta$, then by construction,

$$t'_{n+2m} < t'_{n+2m+1} < \cdots < t'_{n+2m+k}$$

Since any segment $[t'_{n+2m+i}, t'_{n+2m+i+1}]$, $1 \leqslant i < k$, contains at least two points of $\tilde{G}$, one easily derives that there are at least k points of $\tilde{G}$ in $[t'_{n+2m}, t'_{n+2m+k}]$; hence $\tilde{t}_{n+k} \leqslant t'_{n+2m+k}$, and we are done.

Case 3. Assume that $k>1$ and

$$t'_{n+2m+1} \leqslant \tilde{t}_n < t'_{n+2m+2} \tag{57}$$

If $t'_{n+2m+k} = \beta$, we are done. So, assume that $t'_{n+2m+k} < \beta$; hence by construction,

$$t'_{n+2m+1} < t'_{n+2m+2} < \cdots < t'_{n+2m+k}$$

Due to (57), the derivatives of the functions $\Gamma(G, E)$ and $J(\tilde{G}, \tilde{E})$ at $\tilde{t}_n + 0$ have different signs. Therefore the segment $(\tilde{t}_n, t'_{n+2m+2}]$ contains at least two points of $\tilde{G}$. As above, since any segment $[t'_{n+2m+i}, t'_{n+2m+i+1}]$, $1 < i < k$, contains at least two points of $\tilde{G}$, one obtains at least k points of $\tilde{G}$ in $(\tilde{t}_n, t'_{n+2m+k}]$, which completes the proof. □

Proposition 3.4 together with (55) and (56) implies that the sequence $\langle \delta(G^{(N)}, E^{(N)}) + \Delta(G^{(N)}, E^{(N)}) \mid N \in \mathbb{N} \rangle$ is decreasing. Its uniform convergence to zero follows from a statement similar to Proposition 2.2.

Proposition 3.5. *Let $\varepsilon > 0$ and $2m \geqslant (1+\varepsilon)/\varepsilon$, $m \in \mathbb{N}$. If*

$$\delta(G^{(0)}, E^{(0)}) + \Delta(G^{(0)}, E^{(0)}) \geqslant 2\varepsilon \tag{58}$$

then

$$\begin{aligned}\delta(G^{(2m-1)}, E^{(2m-1)}) &+ \Delta(G^{(2m-1)}, E^{(2m-1)}) \\ &\leqslant \delta(G^{(0)}, E^{(0)}) + \Delta(G^{(0)}, E^{(0)}) - 2\lambda^{2m-1}\varepsilon\end{aligned} \tag{59}$$

where $\lambda = (1-p)/2$ and $p = \sup |f(x)|$.

Proof. Relation (58) implies that

$$m(\delta(G^{(0)}, E^{(0)}) + \Delta(G^{(0)}, E^{(0)})) \geqslant 1 + \varepsilon$$

Hence, for any component (α, β) of $[0,1] \backslash G_a^{(0)}$ and any tame sequence $\langle t_n \mid n \in \mathbb{Z} \rangle$ of $(G^{(0)}, E^{(0)})$, it holds that

$$t_{n+2m} - t_n \leqslant m(\delta(G^{(0)}, E^{(0)}) + \Delta(G^{(0)}, E^{(0)})) - \varepsilon, \qquad n \in \mathbb{Z}$$

Using this was a starting point, we derive (59) from the following implications, which are immediate consequences of (55), (56), Proposition 3.4, and the computation performed in the proof of Proposition 2.2 with $\lambda = (1-p)/2$, $p = \sup |f(x)|$.

(1) If, for some $N \geqslant 0$, $k \geqslant 1$, and any tame sequence $\langle t_n |, n \in \mathbb{Z} \rangle$ of $(G^{(N)}, E^{(N)})$ in the closure of any component of $[0, 1] \backslash G_a^{(N)}$, there holds

$$t_{n+2k} - t_n \leqslant k(\delta(G^{(N)}, E^{(N)}) + \Delta(G^{(N)}, E^{(N)})) - \xi, \qquad n \in \mathbb{Z}$$

then any tame sequence $\langle \tilde{t}_n \mid n \in \mathbb{Z} \rangle$ of $(G^{(N+1)}, E^{(N+1)})$ satisfies

$$\begin{aligned} \tilde{t}_{2r+2k-1} - \tilde{t}_{2r} &\leqslant (k-1)(\delta(G^{(N+1)}, E^{(N+1)}) + \Delta(G^{(N+1)}, E^{(N+1)})) \\ &\quad + \delta(G^{(N+1)}, E^{(N+1)}) - \xi\lambda, \qquad r \in \mathbb{Z} \\ \tilde{t}_{2r+2k} - \tilde{t}_{2r+1} &\leqslant (k-1)(\delta(G^{(N+1)}, E^{(N+1)}) + \Delta(G^{(N+1)}, E^{(N+1)})) \\ &\quad + \Delta(G^{(N+1)}, E^{(N+1)}) - \xi\lambda, \qquad r \in \mathbb{Z} \end{aligned}$$

(2) If, for some $N \geqslant 0$, $k \geqslant 1$, and any tame sequence $\langle t_n \mid n \in \mathbb{Z} \rangle$ of $(G^{(N)}, E^{(N)})$ in the closure of any component of $[0, 1] \backslash G_a^{(N)}$, there holds

$$\begin{aligned} t_{2r+2k+1} - t_{2r} &\leqslant k(\delta(G^{(N)}, E^{(N)}) + \Delta(G^{(N)}, E^{(N)})) \\ &\quad + \delta(G^{(N)}, E^{(N)}) - \xi, \qquad r \in \mathbb{Z} \\ t_{2r+2k} - t_{2r-1} &\leqslant k(\delta(G^{(N)}, E^{(N)}) + \Delta(G^{(N)}, E^{(N)})) \\ &\quad + \Delta(G^{(N)}, E^{(N)}) - \xi, \qquad r \in \mathbb{Z} \end{aligned}$$

then any tame sequence $\langle \tilde{t}_n \mid n \in \mathbb{Z} \rangle$ of $(G^{(N+1)}, E^{(N+1)})$ satisfies

$$\begin{aligned} \tilde{t}_{n+2k} - \tilde{t}_n &\leqslant k(\delta(G^{(N+1)}, E^{(N+1)}) \\ &\quad + \Delta(G^{(N+1)}, E^{(N+1)})) - \xi\lambda, \qquad n \in \mathbb{Z} \end{aligned}$$

□

3.4. Infinite Frequency Oscillations

Now we are ready to complete the proof of Theorems 11 and 12. We start with the following observation. any solution $x_\varphi(t)$ to (48) is absolutely continuous, hence is differentiable almost everywhere. In particular, this implies that if $f(0) \neq 0$, then the zero set of any solution $x_\varphi(t)$ to (48) has zero measure in $[1, \infty)$. Indeed, if $x_\varphi(t^*) = 0$, where $\dot{x}_\varphi(t^*)$ exists, then $\dot{x}_\varphi(t^*) \neq 0$, which says that t^* is an isolated zero of x_φ. Therefore, $x_\varphi^{-1}(0) \cap [1, \infty)$ is contained in the union of a zero measure set and a countable set.

We argue by contradiction. Assume that there exists a nonzero solution $x(t)$, $t \geqslant -1$, of Eq. (48) which has infinitely many zeros in any unit interval of $[0, \infty)$. To describe the structure of the set $x^{-1}(0)$, we recall here observations made by Shustin (1995). Consider the sets

$$S_n = x^1(0) \cap [n, n+1] - n \subset [0, 1], \qquad n \in \mathbb{N}$$

where $-n$ means the shift by n to the left. Note that $S_n \neq [0, 1]$ for all $n \in \mathbb{N}$: for $f(0) \neq 0$ this as explained above; for $f(0) = 0$, $S_n = [0, 1]$ would imply $x(t) \equiv 0$, contrary to the assumption on $x(t)$. Next consider the sets $S_{n,a}$ of accumulation points of S_n, $n \geqslant 1$. They are closed and nonempty, since all S_n are supposed infinite. For any two points $t' < t'' \in S_n$ the function $x(t)$ has an extremum $t^* + n \in (t' + n, t'' + n)$ which must correspond to a zero $t^* + n - 1$ of $x(t)$; hence $t^* \in S_{n-1}$. It follows that $S_{n-1,a} \supset S_{n,a}$, $n > 1$, so the set $S_{\infty,a} = \bigcap_n S_{n,a}$ is nonempty. Let $t_0 \in S_{\infty,a}$. The above argument gives, in particular, that all the points $t_0 + n$, $n \in \mathbb{N}$, $n \geqslant 1$, are zeros of $x(t)$.

Now we introduce a sequence $\langle (G^{(N)}, E^{(N)}) \mid N = 0, -1, -2,... \rangle$ of elements of $\mathcal{G}$ (see Definition 3.1):

$$G^{(N)} = x^{-1}(0) \cap [t_0 + 1 - N, t_0 + 2 - N] - (t_0 + 1 - N)$$

$$E^{(N)}(t) = \operatorname{sign}(x(t + t_0 + 1 - N)), \qquad t \in [0, 1]$$

where $-(t_0 + 1 - N)$ means the corresponding shift to the left. By construction, $(G^{(N)}, E^{(N)}) \in \mathcal{G}_0$, and any two elements $(G^{(N)}, E^{(N)})$ and $(G^{(N-1)}, E^{(N-1)})$ are related as

$$(G^{(N-1)}, E^{(N-1)}) = \pi J(G^{(N)}, E^{(N)}), \qquad N \leqslant 0$$

which coincides with (54) up to shifts of index in $\mathbb{Z}$. Fix positive $\varepsilon < \|(G^{(0)}, E^{(0)})\|_\infty$. There exists $N_\varepsilon \in \mathbb{N}$ such that for any sequence in Lemma 3.3 the $\|\cdot\|_\infty$-norm of the element with number N_ε is less than ε. This conclusion applied to the finite sequence of elements constructed above,

$$(G^{(N)}, E^{(N)}), \qquad 0 \geqslant N \geqslant -N_\varepsilon$$

says that

$$\|G^{(0)}, E^{(0)}\|_\infty < \varepsilon$$

which contradicts the assumption made and, thereby, proves Theorem 12 with $T_\varepsilon = N_\varepsilon + 2$.

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