

ESTIMATES OF THE PERIODS OF PERIODIC POINTS FOR NONEXPANSIVE OPERATORS

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ABSTRACT

Suppose that E is a finite-dimensional Banach space with a polyhedral norm $\|\cdot\|$, i.e., a norm such that the unit ball in E is a polyhedron. \mathbb{R}^n with the sup norm or \mathbb{R}^n with the l_1 -norm are important examples. If D is a bounded set in E and $T: D \rightarrow D$ is a map such that $\|T(y) - T(z)\| \leq \|y - z\|$ for all y and z in E , then T is called nonexpansive with respect to $\|\cdot\|$, and it is known that for each $x \in D$ there is an integer $p = p(x)$ such that $\lim_{j \rightarrow \infty} T^{jp}(x)$ exists. Furthermore, there exists an integer N , depending only on the dimension of E and the polyhedral norm on E , such that $p(x) \leq N$: see [1,12,18,19] and the references to the literature there. In [15], Scheutzwow has raised a question about the optimal choice of N when $E = \mathbb{R}^n$, $D = K^n$, the set of nonnegative vectors in \mathbb{R}^n , and the norm is the l_1 -norm. We provide here a reasonably sharp answer to Scheutzwow's question, and in fact we provide a systematic way to generate examples and use this approach to prove that our estimates are optimal for $n \leq 24$. See Theorem 2.1, Table 2.1 and the examples in Section 3. As we show in Corollary 2.3, these results also provide information about the case $D = \mathbb{R}^n$, i.e., $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is l_1 -nonexpansive. In addition, it is conjectured in [12] that $N = 2^n$ when $E = \mathbb{R}^n$ and the norm is the sup norm, and such a result is optimal, if true. Our theorems here show that a sharper result is true for an important subclass of nonexpansive maps $T: (\mathbb{R}^n, \|\cdot\|_\infty) \rightarrow (\mathbb{R}^n, \|\cdot\|_\infty)$.

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1. Introduction

The structure of the set of periodic points of stochastic matrices and the convergence properties of iterates of stochastic matrices are well understood. Here we shall try to obtain (among other results) sharp analogues for ℓ_1 -nonexpansive mappings of the known stochastic matrix theory. One motivation for such a study is to understand nonlinear analogues of diffusion on finite state spaces, and this is the point of view taken in [1] and [15]. However, we are also motivated by a variety of other applications described in [10, 11], where cone mappings g which are nonexpansive with respect to Hilbert's projective metric are described. After an appropriate change of coordinate (see [13, pp. 529–530]), such maps are often nonexpansive with respect to the sup norm on \mathbb{R}^n .

We begin by recalling some standard notation and basic theorems. Fix a positive integer n and define $K^n = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n\}$, the nonnegative vectors in \mathbb{R}^n . For $x, y \in \mathbb{R}^n$ we define a partial ordering by $x \leq y$ if and only if $y - x \in K^n$. If $D \subset \mathbb{R}^n$, a map $T : D \rightarrow \mathbb{R}^n$ is called "order-preserving on D " if, for all $x, y \in D$ such that $x \leq y$, one has $Tx \leq Ty$. If $\|\cdot\|$ denotes some norm on D , a map $T : D \rightarrow \mathbb{R}^n$ is called "nonexpansive" (with respect to $\|\cdot\|$) if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in D$. As usual, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the ℓ_1 -norm and the sup norm respectively, so

$$\|x\|_1 = \sum_{i=1}^n |x_i| \quad \text{and} \quad \|x\|_\infty = \max\{|x_i| : 1 \leq i \leq n\}.$$

If E is a finite-dimensional vector space, a norm $\|\cdot\|$ on E is called "polyhedral" if $B = \{x \mid \|x\| \leq 1\}$ is a polyhedron. Equivalently, a norm $\|\cdot\|$ on E is polyhedral if and only if there exist continuous linear functionals $\varphi_1, \varphi_2, \dots, \varphi_m$ on E such that for all $x \in E$

$$\|x\| = \sup\{|\varphi_i(x)| : 1 \leq i \leq m\}.$$

Obviously, the ℓ_1 -norm and the sup norm on \mathbb{R}^n are polyhedral. As already noted, if E is a finite-dimensional vector space with a polyhedral norm $\|\cdot\|$, D is a bounded subset of E and $T : D \rightarrow D$ is a nonexpansive map, then for every $x \in D$ there exists an integer $p = p(x)$ such that $\lim_{j \rightarrow \infty} T^{jp}(x)$ exists. Furthermore, there exists an integer N , dependent only on the dimension of E and the polyhedral norm, such that $p(x) \leq N$ for all $x \in D$. See [1, 12, 18] and the references cited there.

The case that $E = \mathbb{R}^n$ and $\|\cdot\| = \|\cdot\|_\infty$ plays a crucial role in [12, 18]. This is because any finite-dimensional vector space E with a polyhedral norm can be imbedded by a linear isometry into $(\mathbb{R}^m, \|\cdot\|_\infty)$ for some m . It is proved in [12] that for $(\mathbb{R}^n, \|\cdot\|_\infty)$ one has

$$(1.1) \quad N \leq 2^n \gamma(n),$$

where

$$(1.2) \quad \gamma(n) \leq n! \left(\frac{1}{\ell n(2)} \right)^n.$$

However, it is conjectured in \mathbb{R}^n that $N = 2^n$, and it is not hard to show that this estimate is, if true, best possible (see [12]).

In work in progress, R. Lyons and the author have sharpened the estimates (1.1) and (1.2) for general n and also established some weak evidence that the conjecture may be true. For example, we have proved the conjecture for $n = 3$ ($n = 1$ and $n = 2$ are fairly easy cases).

If \mathcal{F} is some subclass of the set of nonexpansive mappings $T : D \rightarrow D$, where D is a bounded set in a finite-dimensional Banach space E with a polyhedral norm, it may be possible to refine the number N . Thus it may be possible to find a number $N_* \leq N$ such that if $T \in \mathcal{F}$ and $x \in D$, there exists $p = p(x) \leq N_*$ such that $\lim_{j \rightarrow \infty} T^{jp}(x)$ exists. In our work below we shall indicate an important class \mathcal{F} of such (nonlinear) maps in $(\mathbb{R}^n, \|\cdot\|_\infty)$ for which $N_* < 2^n$ (in fact, we shall give a much sharper estimate for N_*). Specifically, we shall consider maps T which, for all vectors x and y in some subset D_1 of D , satisfy

$$(1.3) \quad T(x \wedge y) = T(x) \wedge T(y).$$

Here $x \wedge y$ denotes the minimum of two vectors x and y :

$$(1.4) \quad x \wedge y = z, \quad \text{where } z_i = x_i \wedge y_i = \min(x_i, y_i).$$

It turns out that our theorems also apply to maps $T : K^n \rightarrow K^n$ such that $T(0) = 0$ and T is nonexpansive with respect to the ℓ_1 -norm. For this class of maps our estimates for $\sup\{p_x : x \in K^n\}$ provide a reasonably sharp answer to a question raised by Scheutzow [15]. In fact we shall show with simple examples that our estimates are optimal for $n \leq 24$. A trick used by Scheutzow [16] also provides estimates on $\sup\{p_x : x \in \mathbb{R}^n\}$ when $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is ℓ_1 -nonexpansive and has a fixed point: see Corollary 2.3. However, we do not investigate the optimality of our estimates in this case.

2. Basic Estimates for the Periods of Periodic Elements

We begin by recalling some further definitions, notation and results. A map $T : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called "integral preserving (on D)" if for all $x \in D$,

$$(2.1) \quad \sum_{i=1}^n (Tx)_i = \sum_{i=1}^n x_i,$$

where z_i denotes the i th component of $z \in \mathbb{R}^n$. The map T is called "sup-decreasing (on D)" if

$$(2.2) \quad \|Tx\|_\infty \leq \|x\|_\infty \quad \text{for all } x \in D.$$

If $T : K^n \rightarrow K^n$ or $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and T is integral preserving, then it is a result of Crandall and Tartar [5] (see, also, Ackoglu and Krengel [1] and Scheutzow [15]) that T is nonexpansive with respect to the ℓ_1 -norm if and only if T is order-preserving. If $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $u = (1, 1, \dots, 1)$ and $T(x + cu) = T(x) + cu$ for all $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$, it is also proved in [5] that T is order-preserving if and only if T is nonexpansive with respect to the sup norm.

If $T : D \rightarrow \mathbb{R}^n$ and there exists $x \in D$ such that $T^p(x)$ is defined (where T^p denotes the p th iterate of T) and $T^p(x) = x$, then x is called "a periodic point (with respect to T)." The minimal such integer p is called the "period of x ". If $T : D \rightarrow D$ we shall say that " T has period p " if p is the minimal integer such that $\lim_{k \rightarrow \infty} T^{kp}(x)$ exists for all $x \in D$. As noted in [15] and as we shall see in detail later, if T has period p , it need not be true that there exists $x \in D$ such that x is periodic with respect to T and x has period p .

If $S \subset \mathbb{R}^n$ is a finite set, we shall write

$$z = \min\{x \mid x \in S\}$$

to denote the vector whose i th component z_i satisfies

$$z_i = \min\{x_i \mid x \in S\}.$$

If $S = \{y^j \in \mathbb{R}^n \mid 1 \leq j \leq p\}$ we shall also write

$$\min\{y \mid y \in S\} = \bigwedge_{j=1}^p y^j.$$

If $A \subset \mathbb{R}^n$ is a finite set, the lower semilattice generated by A is defined to be the smallest set V such that $A \subset V$ and such that for all vectors $y \in V$ and $z \in V$ one has $y \wedge z \in V$. One can easily verify that

$$V = \{z \mid z = \min\{y \mid y \in S\}, S \subset A\}.$$

Equivalently, if $A = \{y^j \in \mathbb{R}^n \mid 1 \leq j \leq p\}$, then V comprises all vectors z which can be expressed in the form

$$z = \bigwedge_{k=1}^p y^{i_k}, \quad \text{where } 1 \leq i_k \leq p \quad \text{for } 1 \leq k \leq p.$$

One can see that V has a minimal z_0 such that $z_0 \leq z$ for all $z \in V$, namely

$$z_0 = \min\{z \mid z \in A\}.$$

If y and z are vectors in \mathbb{R}^n , we write $y < z$ if $y \leq z$ and $y \neq z$.

If V is as above and $\Gamma \subset V$ and there exists $\zeta \in V$ such that $z \leq \zeta$ for all $z \in \Gamma$, then we define $\max_V(\Gamma)$, the maximum of Γ in V , by

$$\max_V(\Gamma) = \min\{\zeta \in V \mid \zeta \geq z \text{ for all } z \in \Gamma\}.$$

Notice that $\max_V(\Gamma)$ may differ from the maximum of Γ in \mathbb{R}^n . If $y \in V$, y is called irreducible if

$$(2.1) \quad y > \max_V\{\zeta \in V \mid \zeta < y\} \equiv z,$$

and, following [15], we define

$$(2.2) \quad I(y) = \{i \mid y_i > z_i\}.$$

The minimal element $z_0 \in V$ is defined to be irreducible. The height of a vector $z \in V$, $h(z)$, is defined by

$$(2.3) \quad h(z) = \sup\{k \mid \exists e_j \in V, 0 \leq j \leq k-1, \text{ such that } e_0 < e_1 < e_2 < \dots < e_{k-1} < z\}.$$

We define $h(z_0) = 0$ for the minimal element. As noted in [15], a simple induction on $h(x)$ shows that for every $x \in V$,

$$(2.4) \quad x = \max_V\{z \in V \mid z \leq x \text{ and } z \text{ irreducible}\}.$$

One reason for studying the lower semilattice V is provided by the following result of Scheutzwow [15].

LEMMA 2.1 (Scheutzow [15]): Let $T : K^n = \{x \in \mathbb{R}^n : x \geq 0\} \rightarrow K^n$ be a nonexpansive map with respect to the ℓ_1 -norm and suppose $T(0) = 0$. Let $x \in K^n$ be periodic with period p and let V be the lower semilattice generated by $A = \{T^j(x) | j \geq 0\}$. Then $T(V) \subset V$ and for all $y, z \in V$ one has

$$(2.5) \quad T(y \wedge z) = T(y) \wedge T(z).$$

Mappings which satisfy (2.5) on V need not be ℓ_1 -nonexpansive. For example, if A is a given $n \times n$ matrix, define a map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$(2.6) \quad (T(x))_i = \min_j (a_{ij} + x_j).$$

The map in (2.6) arises in many applications, e.g., statistical mechanics [3, 4, 7] and operations research [6, 9]. One can easily check that

$$T(y \wedge z) = T(y) \wedge T(z) \quad \text{for all } y, z \in \mathbb{R}^n$$

and

$$T(y + cu) = T(y) + cu \quad \text{for all } y \in \mathbb{R}^n, \quad c \in \mathbb{R},$$

where $u = (1, 1, \dots, 1)$. It follows from our previous remarks that T is nonexpansive with respect to the sup norm on \mathbb{R}^n .

Motivated by the above remarks we shall consider the following situation: $T : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a map and $\xi \in D$ is a periodic point of T of period p . Let $A = \{T^j(\xi) : 0 \leq j < p\}$ and let V be the lower semilattice generated by A and assume that T can be extended to V in such a way that (2.5) is satisfied for all $y, z \in V$. In this framework Scheutzow [15, Theorem 3.1] has proved that p divides $\text{lcm}(1, 2, \dots, n)$. (If S is a set of positive integers, $\text{lcm}(S)$ will denote the least common multiple of the integers in S and $\text{gcd}(S)$ will denote the greatest common divisor of the integers in S .) If $y \in V$ is irreducible (as an element of V), and the period of y is q , it is also proved in [15] that $q \leq n$. If $y \in V$ is irreducible and the period of y is q , then $T^j(y)$ is irreducible for all $j \geq 0$ and $I(T^j y) \cap I(T^k y)$ is empty for $0 \leq j < k < q$, where $I(\eta)$ is defined by eq. (2.2).

If, in the framework above, S denotes the set of periods p_y of irreducible elements $y \in V$, we shall now find a variety of restrictive conditions on S . We begin with a simple lemma.

LEMMA 2.2: Let X be a set with a partial ordering \leq and $T : D \subset X \rightarrow X$ a map which preserves the partial ordering: $x \leq y$ implies $Tx \leq Ty$. Assume that $\xi \in D$ is a periodic point of T with period p . Then no two distinct elements of $A = \{T^j(\xi) | 0 \leq j < p\}$ are comparable, i.e., if $x, y \in A$ and $x \neq y$, it is not true that $x \leq y$.

Proof: The partial ordering is assumed such that $a \leq b$ and $b \leq a$ imply $a = b$ and $a \leq b$ and $b \leq c$ imply $a \leq c$. If there exist $x, y \in A$, $x \neq y$, such that $x \leq y$, then there exists r , $0 < r < p$ such that $y = T^r(x) \geq x$. Using the order-preserving property of T , we find

$$(2.7) \quad T^{jr}(x) \leq T^{(j+1)r}(x) \quad \text{for all } j \geq 0.$$

On the other hand, $T|A$ is one-one, so using that $y \neq x$ we deduce that equality does not hold in (2.7) for any $j \geq 0$. It follows from this fact, from (2.7) and from properties of the partial ordering that

$$(2.8) \quad T^{jr}(x) \neq T^{kr}(x) \quad \text{for } 0 \leq j < k.$$

Equation (2.8) implies that the cardinality of A is not finite, a contradiction.

■

PROPOSITION 2.1: Let $T : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a map and suppose that $\xi \in D$ is a periodic point of T of period p . If $A = \{T^j(\xi) | j \geq 0\}$ and V is the lower semilattice generated by A , assume that T has an extension to V such that

$$T(x \wedge y) = T(x) \wedge T(y) \quad \text{for all } x, y \in V.$$

Then there cannot exist irreducible elements y^1, y^2, \dots, y^r of V , with periods p_1, p_2, \dots, p_r respectively, such that $p_i > 1$ for $1 \leq i \leq r$, $\gcd(p_i, p_j) = 1$ for $1 \leq i < j \leq r$ and $\sum_{j=1}^r p_j > n$.

Proof: Suppose, to the contrary, that there exist irreducible elements like y^1, y^2, \dots, y^r . We already know that $I(T^i y^k) \cap I(T^j y^k)$ is empty for $0 \leq i < j < p_k$. Suppose we can prove that $I(T^i y^k) \cap I(T^j y^s)$ is empty whenever $0 \leq i < p_k$, $0 \leq j < p_s$ and $0 \leq k < s \leq r$. Then the sets $I(T^i y^k)$, $0 \leq i < p_k$, $1 \leq k \leq r$, are pairwise disjoint, nonempty subsets of $\{1, 2, \dots, n\}$, and there are $\sum_{j=1}^r p_j > n$ such sets. This is impossible.

Thus fix k and s with $1 \leq k < s \leq r$ and i and j with $0 \leq i < p_k$ and $0 \leq j < p_s$. If we write $\eta = T^i(y^k), \zeta = T^j(y^s), \lambda = p_k$ and $\mu = p_s, \eta$ and ζ are irreducible elements of V of periods $\lambda > 1$ and $\mu > 1$ respectively and $\gcd(\lambda, \mu) = 1$. It remains to prove that $I(\eta) \cap I(\zeta)$ is empty. Suppose not and fix $m \in I(\eta) \cap I(\zeta)$. It must then be true that $\eta < \zeta$ or $\zeta < \eta$ (we know that $\zeta \neq \eta$ because $\lambda \neq \mu$). If not, we have $(\zeta \wedge \eta) < \zeta$ and $(\zeta \wedge \eta) < \eta$ and this implies that

$$(\zeta \wedge \eta)_m < \zeta_m \quad \text{and} \quad (\zeta \wedge \eta)_m < \eta_m,$$

which is impossible. For definiteness assume that $\eta < \zeta$. We know that T^j is order-preserving on V ; and because $T^p|_V$ is the identity, $T^{-1} = T^{p-1}$ is also defined and order-preserving on V . For any integers α and β it follows that

$$\eta = T^{\alpha\lambda}(\eta) < T^{\alpha\lambda}(\zeta) = T^{\alpha\lambda+\beta\mu}(\zeta).$$

Because λ and μ are relatively prime, for any integer ν with $0 \leq \nu < p$, we can find integers α and β so

$$\nu = \alpha\lambda + \beta\mu.$$

By replacing ξ by some power of T applied to ξ , we can assume also that

$$\zeta \leq \xi.$$

It follows that

$$(2.9) \quad \begin{aligned} \eta < T^\nu(\zeta) \leq T^\nu(\xi) \quad \text{for } 0 \leq \nu < p \quad \text{and} \\ \eta \leq \min\{T^\nu(\xi) : 0 \leq \nu < p\}. \end{aligned}$$

However, the right-hand side of (2.9) is the minimal element and a fixed point of T . It follows that η is the minimal element of V and a fixed point of T , and this contradicts the assumption that the period of η is greater than one, and we conclude that $I(\eta) \cap I(\zeta)$ is empty. ■

Remark 2.1: In the notation of Proposition 2.1, let $S = \{m \mid m \text{ is the period of an irreducible } y \in V\}$, so we know $S \subset \{1, 2, \dots, n\}$. If $q = \text{lcm}(S)$ and $y \in V$ is irreducible, it follows that

$$(2.10) \quad T^q(y) = y.$$

Since we know that

$$(2.11) \quad \xi = \max_V \{y \in V \mid y \leq \xi \text{ and } y \text{ irreducible}\},$$

it follows easily from (2.10) that

$$T^q(\xi) \geq \xi.$$

Lemma 2.2 now implies that $T^q(\xi) = \xi$, so p divides q . Since Proposition 2.1 imposes restrictions on the possible sets S , we already know more than just p divides $\text{lcm}(1, 2, \dots, n)$. ■

PROPOSITION 2.2: *Let notation and assumptions be as in Proposition 2.1. There do not exist irreducible elements w^1, w^2, \dots, w^{r+1} of V of periods p_1, p_2, \dots, p_{r+1} respectively such that $w^j < w^{j+1}$ for $1 \leq j \leq r$ and $p_j > r$ for $1 \leq j \leq r+1$ and $\text{gcd}(p_i, p_j)$ divides r for $1 \leq i < j \leq r+1$.*

Proof: Suppose, to the contrary, that irreducible elements w^j as above exist. We know that $w^{r+1} \leq T^k \xi$ for some k , and by replacing ξ by $T^k \xi$ we can assume that

$$(2.12) \quad w^{r+1} \leq \xi.$$

If $1 \leq j \leq r$ and α and β are any integers we have

$$w^j = T^{\alpha p_j}(w^j) < T^{\alpha p_j}(w^{j+1}) = T^{\alpha p_j + \beta p_{j+1}}(w^{j+1});$$

and since $\text{gcd}(p_j, p_{j+1})$ divides r , we conclude that for any integer m we have

$$(2.13) \quad w^j < T^{mr}(w^{j+1}).$$

Taking $j = r$, we conclude from (2.12) and (2.13) that

$$w^r < T^{mr}(\xi) \quad \text{for all } m \in \mathbb{Z}.$$

Now choose k , $1 \leq k < r$, and assume, by way of induction, that there exist $m_i, 1 \leq i \leq k$, with $m_1 = 0$, such that m_i is not congruent to $m_j \pmod r$ for $1 \leq i < j \leq k$ and

$$(2.14) \quad w^{r+1-k} \leq T^{m_r+m_j}(\xi) \quad \text{for all } m \in \mathbb{Z} \quad \text{and for } 1 \leq j \leq k.$$

Because $w^{r+1-k} \in V$, we know that there exist integers $i_j, 1 \leq j \leq p$, with $0 \leq i_j < p$, such that

$$(2.15) \quad w^{r+1-k} = \min\{T^{i_j}(\xi) : 1 \leq j \leq p\}.$$

If every integer $i_s, 1 \leq s \leq p$, is congruent to some $m_j, \text{ mod } r$, then (2.14) and (2.15) imply that

$$(2.16) \quad w^{r+1-k} = \min\{T^{mr+m_j}(\xi) : m \in \mathbf{Z} \text{ and } 1 \leq j \leq k\},$$

and using (2.16) and (2.5) we see that

$$(2.17) \quad T^r(w^{r+1-k}) = w^{r+1-k}.$$

Eq. (2.17) contradicts the assumption that $p_j > r$ for $1 \leq j \leq r + 1$. It follows that there exists an integer i_s , which we relabel m_{k+1} , such that

$$(2.18) \quad w^{r+1-k} \leq T^{m_{k+1}}(\xi)$$

and such that m_{k+1} is not congruent to m_j for $1 \leq j \leq k$.

Using (2.13) and (2.18) we conclude that

$$w^{r-k} \leq T^{mr+m_{k+1}}(\xi) \text{ for } m \in \mathbf{Z},$$

and since we already know that

$$w^{r-k} < w^{r+1-k} \leq T^{mr+m_j}(\xi) \text{ for } m \in \mathbf{Z}, \quad 1 \leq j \leq k,$$

the inductive step is complete. It follows that (2.14) is valid for $k = r$ which implies that

$$(2.19) \quad w^1 \leq \min\{T^j(\xi) : j \in \mathbf{Z}\}.$$

Because the right-hand side of (2.19) is the minimal element of V and because $w^1 \in V$, we have equality in (2.19) and w^1 is a fixed point of T , which contradicts the assumption that $p_1 > r$. ■

If one knows more information about the numbers $\text{gcd}(p_i, p_j)$ in Proposition 2.2, then the conclusion of Proposition 2.2 can be refined. There are many examples of such results. Our next lemma presents only one such sharpening of Proposition 2.2, but it is a refinement which we shall need later. The reader may want to concentrate on the case $\rho = 2$ in the following proposition. The proof becomes transparent then and various technicalities vanish.

PROPOSITION 2.3: *Let notation and assumptions be as in Proposition 2.1. There do not exist irreducible elements w^j in V of periods $p_j, 1 \leq j \leq m + 1$ with the following properties: (a) $w^j < w^{j+1}$ for $1 \leq j \leq m$, where $m = \rho^2 - \rho + 1$ and $\rho \geq 2$ is an integer, (b) $\gcd(p_i, p_j)$ divides $r = \rho^2$ for $1 \leq i < j \leq m + 1$, (c) there exists an integer $t, 0 \leq t \leq m$, such that $\gcd(p_{m+1-t}, p_j)$ divides ρ for $1 \leq j \leq m + 1$ and $j \neq m + 1 - t$ and (d) $p_j > \rho^2 = r$ for all $j \neq m + 1 - t$ and $p_{m+1-t} > \rho$.*

Proof: We can assume $w^{m+1} \leq \xi$. We shall assume that the Proposition is false and obtain a contradiction. First assume $t = 0$. For notational convenience write $\lambda = p_m$ and $\mu = p_{m+1}$. For any integers α and β we have

$$T^{\alpha\lambda}(w^m) = w^m < T^{\alpha\lambda}(w^{m+1}) = T^{\alpha\lambda+\beta\mu}(w^{m+1}).$$

Because $\gcd(p_{m+1-t}, p_j)$ divides ρ for $j \neq m + 1 - t$, we derive that for any integer ν we have

$$w^m < T^{\nu\rho}(w^{m+1}) \leq T^{\nu\rho}(\xi).$$

Writing $r = \rho^2$, it follows that there exist integers $m_j = (j - 1)\rho$ for $1 \leq j \leq \rho$ such that m_i and m_j are not congruent mod r for $1 \leq i < j \leq \rho$ and

$$w^m \leq T^{\nu r + m_j}(\xi) \quad \text{for } \nu \in \mathbb{Z} \quad \text{and} \quad 1 \leq j \leq \rho.$$

Furthermore, because $p_m > r$, the same argument used in Proposition 2.2 shows that there exists an integer $m_{\rho+1}$ such that $m_{\rho+1}$ is not congruent mod r to m_j for $1 \leq j \leq \rho$ and

$$w^m \leq T^{m_{\rho+1}}(\xi).$$

Now, arguing just as in Proposition 2.2, we see that for $0 \leq k \leq m - 1$ we have

$$w^{m-k} \leq T^{\nu r + m_j}(\xi) \quad \text{for } \nu \in \mathbb{Z}, \quad 1 \leq j \leq \rho + k,$$

where m_i and m_j are not congruent mod r for $1 \leq i < j \leq \rho + k$. Taking $k = m - 1$, so $\rho + k = r$, the above inequality implies that

$$w^1 \leq T^j(\xi) \quad \text{for all } j \in \mathbb{Z}.$$

As in Proposition 2.2, this implies that w^1 is a fixed point of T , which contradicts $p_1 > r$.

Next consider the case $t = m$. If we apply the argument of Proposition 2.2 to the points $w^j, 2 \leq j \leq m + 1$, and note that $2 = m + 1 - \rho(\rho - 1)$, we see that there exist integers $\mu_j, 1 \leq j \leq m = \rho(\rho - 1) + 1$, such that μ_i is not congruent to $\mu_j \pmod r$ for $1 \leq i < j \leq m$ and

$$(2.20) \quad w^2 \leq T^{\mu_j}(\xi) \quad \text{for } 1 \leq j \leq m.$$

Since $\gcd(p_1, p_2)$ divides ρ , the argument used in Proposition 2.2 shows that

$$(2.21) \quad w^1 \leq T^{\nu\rho + \mu_j}(\xi) \quad \text{for } 1 \leq j \leq m \quad \text{and all } \nu \in \mathbf{Z}.$$

It follows that, by replacing μ_j by $\mu_j + n_j\rho$ for an appropriate integer n_j , we can assume $0 \leq \mu_j < \rho$, and eq. (2.21) will still hold and it will be true that $\mu_j \neq \mu_k$ for $1 \leq j < k \leq m$. If we divide $S = \{\mu_j | 1 \leq j \leq m\}$ into equivalence classes by saying that μ_j is equivalent to μ_k if $\mu_j \equiv \mu_k \pmod \rho$, then each equivalence class can contain at most ρ elements because $\mu_j \leq \rho^2$ for all j . Since there are $\rho(\rho - 1) + 1$ elements in S , we conclude that for each integer k with $0 \leq k < \rho$, there exists j such that $\mu_j \equiv k \pmod \rho$. Using this fact and (2.21) we conclude that $w^1 \leq T^j(\xi)$ for all j . This implies that w^1 is the minimal element of V and that $p_1 = 1$, a contradiction.

It remains to consider the case $0 < t < m$. Arguing as in Proposition 2.2, we see that

$$w^{m+2-t} \leq T^{\nu r + m_j}(\xi) \quad \text{for all } \nu \in \mathbf{Z} \quad \text{and } 1 \leq j \leq t - 1,$$

where m_i and m_j are not congruent mod r for $1 \leq i < j \leq t - 1$. Furthermore, there exists m_t , not congruent to $m_i \pmod r$ for $1 \leq i \leq t - 1$, such that

$$w^{m+2-t} \leq T^{m_t}(\xi).$$

Let k be an integer such that $k\rho \leq t < (k + 1)\rho$. First consider the case $t = k\rho$. The same argument used in the case $t = m$ shows that there are integers $n_j, 1 \leq j \leq k$, such that n_i is not congruent to $n_j \pmod \rho$ for $1 \leq i < j \leq k$ and

$$w^{m+2-t} \leq T^{n_j}(\xi) \quad \text{for } 1 \leq j \leq k.$$

(Each n_j equals an appropriately chosen m_i .) Writing $\lambda = p_{m+1-t}$ and $\mu = p_{m+2-t}$ and recalling that $\gcd(\lambda, \mu)$ divides ρ , we find that for any integers α and β we have

$$w^{m+1-t} = T^{\alpha\lambda}(w^{m+1-t}) < T^{\alpha\lambda + \beta\mu}(w^{m+2-t})$$

and

$$w^{m+1-t} < T^{\nu\rho}(w^{m+2-t}) \leq T^{\nu\rho+n_j}(\xi),$$

for all $\nu \in \mathbb{Z}$ and $1 \leq j \leq k$. Because $\lambda > \rho$, the usual argument implies that there exists an integer n_{k+1} such that n_{k+1} is not congruent to $n_j \pmod{\rho}$ for $1 \leq j \leq k$ and

$$w^{m+1-t} \leq T^{n_{k+1}}(\xi).$$

Because $\gcd(p_{m+1-t}, p_{m-t})$ divides ρ , the same argument used above shows that

$$(2.22) \quad w^{m-t} \leq T^{\nu\rho+n_j}(\xi) \quad \text{for } \nu \in \mathbb{Z}, \quad 1 \leq j \leq k+1.$$

We clearly have $k \leq \rho - 1$, and we can assume $0 \leq n_j < \rho$ for $1 \leq j \leq k+1$. If $k = \rho - 1$, (2.22) implies that w^{m-t} is the minimal element of V , contradicting the assumption that $p_{m-t} > \rho^2$. (This part of the argument is also valid if $k\rho < t < (k+1)\rho$.) If $k < \rho - 1$, consider all integers of the form $n_j + i\rho$ for $1 \leq j \leq k+1$ and for $0 \leq i < \rho$. There are $\rho(k+1)$ such integers and if we label them ν_i for $1 \leq i \leq (k+1)\rho$, ν_i is not congruent to $\nu_j \pmod{r}$ for $1 \leq i < j \leq (k+1)\rho$. Furthermore, if we write $\tau = k\rho + 1$, we have

$$(2.23) \quad w^{m+1-\tau} < T^{\nu r + \nu_i}(\xi) \quad \text{for } 1 \leq i \leq \tau + (\rho - 1), \quad \nu \in \mathbb{Z}.$$

Now the same argument used in Proposition 2.2 shows that, for $k\rho + 1 \leq \tau \leq m$, there are integers $\nu_i, 1 \leq i \leq \tau + (\rho - 1)$, such that ν_i is not congruent to $\nu_j \pmod{r}$ for $1 \leq i < j \leq \tau + (\rho - 1)$. Taking $\tau = m$ in eq. (2.23), so $\tau + (\rho - 1) = r^2$, we obtain a contradiction as in the proof of Proposition 2.2.

It remains to consider the case $k\rho < t < (k+1)\rho, k < \rho - 1$. In this case we see by the argument used in the case $t = m$ that there are $k+1$ integers $n_j, 1 \leq j \leq k+1$, such that n_i is not congruent to $n_j \pmod{\rho}$ for $1 \leq i < j \leq k+1$ and

$$w^{m+2-t} \leq T^{n_j}(\xi) \quad \text{for } 1 \leq j \leq k.$$

The same argument as in the case $t = k\rho$ shows that

$$w^{m+1-t} \leq T^{\nu\rho+n_j}(\xi) \quad \text{for all } \nu \in \mathbb{Z}, \quad 1 \leq j \leq k+1.$$

Furthermore, because $\lambda > \rho$, there exists an integer n_{k+2} such that n_{k+2} is not congruent to $n_j \pmod{\rho}$ for $1 \leq j \leq k+1$ and

$$w^{m+1-t} \leq T^{n_{k+2}}(\xi).$$

Just as in the case $t = k\rho$ we find that

$$(2.24) \quad w^{m-t} \leq T^{\nu\rho+n_j}(\xi) \quad \text{for } \nu \in \mathbf{Z}, \quad 1 \leq j \leq k+2.$$

If $k+2 \geq \rho$, (2.24) implies that $w^{m-t} \leq T^j(\xi)$ for all j and w^{m-t} is the minimal element of V , a contradiction. Thus we can assume $k+2 < \rho$. We can also assume that $0 \leq n_j < \rho$ for $1 \leq j \leq k+2$. As before consider the $(k+2)\rho$ integers $n_j + i\rho, 1 \leq j \leq k+2, 0 \leq i < \rho$, relabel them $\nu_i, 1 \leq i \leq (k+2)\rho$, and note that ν_i is not congruent to $\nu_j \pmod r$ for $1 \leq i < j \leq (k+2)\rho$. Also, setting $\tau = (k+1)\rho$, we see that

$$(2.25) \quad w^{m+1-\tau} \leq w^{m-t} \leq T^{\nu_r+\nu_j}(\xi) \quad \text{for } \nu \in \mathbf{Z}, \quad 1 \leq j \leq \tau + \rho.$$

As in the case $t = k\rho$, eq. (2.25) is valid for $(k+1)\rho \leq \tau \leq m$ with integers $\nu_j, 1 \leq j \leq \tau + \rho$, which are pairwise incongruent mod r . Taking $\tau = m - 1$, we find that $\tau + \rho = r$ and that w^2 is the minimal element of V . Because $p_2 > 1, w^2$ is not the minimal element of V , and this contradiction completes the proof.

■

Our interest in Propositions 2.2 and 2.3 derives from the fact that one can determine conditions which imply that some given set of irreducible elements in V is totally ordered. Our next lemma describes one such set of conditions. For notational convenience, define, for $y \in V$,

$$(2.26) \quad \Omega(y; T) = \Omega(y) = \{T^j(y) \mid j \geq 0\}.$$

PROPOSITION 2.4: *Let notation and assumptions be as in Proposition 2.1. Assume that $y^j, 1 \leq j \leq s$, is a set of irreducible elements of periods $q_j > 1, 1 \leq j \leq s$. (We allow $s = 0$, in which case $\{y^j \mid 1 \leq j \leq s\}$ is empty.) Assume that $z^j, 1 \leq j \leq r+1$, is a set of irreducible elements of periods $p_j > 1, 1 \leq j \leq r+1$. Assume that $\gcd(q_i, q_j) = 1$ for $1 \leq i < j \leq s$ and $\gcd(q_i, p_k) = 1$ for $1 \leq i \leq s$ and $1 \leq k \leq r+1$. Suppose that $\Omega(z^i) \cap \Omega(z^j)$ is empty for $1 \leq i < j \leq r+1$. (Note that the latter condition is automatically satisfied if $p_i \neq p_j$ for $1 \leq i < j \leq r+1$.) Finally, suppose that*

$$p_i + p_j > n - \sum_{k=1}^s q_k$$

for $1 \leq i < j \leq r + 1$. Then there exists a permutation σ of $\{1, 2, \dots, r + 1\}$ and integers k_j with $0 \leq k_j < p_j$ for $1 \leq j \leq r + 1$, such that if $w^i = T^{k_j}(z^j)$ for $j = \sigma(i)$, then $w^i < w^{i+1}$ for $1 \leq i \leq r$. (Note that w^i is an irreducible element of V of period $p_{\sigma(i)}$.)

Proof: The argument used in Proposition 2.1 shows that if ζ and η are irreducible elements of V of periods ρ_1 and ρ_2 respectively with $\rho_i > 1$ for $i = 1, 2$ and $\text{gcd}(\rho_1, \rho_2) = 1$, then $I(\zeta) \cap I(\eta)$ is empty. Applying this observation we see that the sets $I(T^i(y^j)), 0 \leq i < q_j, 1 \leq j \leq s$, are pairwise disjoint and nonempty, so

$$\Gamma = \bigcup_{j=1}^s \left[\bigcup_{i=0}^{q_j-1} T^i(y^j) \right]$$

contains at least $m = \sum_{k=1}^s q_k$ elements. Furthermore, the same argument shows that $I(T^k(z^j)) \cap \Gamma$ is empty for $k \in \mathbb{Z}$ and $1 \leq j \leq r + 1$. Thus $I(T^k(z^j))$ is contained in Γ' , where

$$\Gamma' = \{j \mid 1 \leq j \leq n, j \notin \Gamma\},$$

and Γ' has $n - m$ elements.

We first prove the Proposition in the case $r = 1$. We know that if $x = z^k, k = 1$ or 2 , then

$$I(T^i(x)) \cap I(T^j(x)) = \emptyset \quad \text{for } 0 \leq i < j < p_k.$$

Suppose that for all i and j with $0 \leq i < p_1$ and $0 \leq j < p_2$ it is not true that $T^i(z^1) \leq T^j(z^2)$ and it is not true that $T^j(z^2) \leq T^i(z^1)$. For convenience, fix i and j and put $\zeta = T^j(z^2)$ and $\eta = T^i(z^1)$. Since we know that $\zeta \neq \eta$, we are assuming that $\zeta \wedge \eta < \zeta$ and $\zeta \wedge \eta < \eta$ (otherwise $\zeta \leq \eta$ or $\eta \leq \zeta$). It follows that $I(\zeta) \cap I(\eta)$ is empty: if $k \in I(\zeta) \cap I(\eta)$, we would have $\zeta_k \wedge \eta_k < \zeta_k$ and $\zeta_k \wedge \eta_k < \eta_k$. Thus we are assuming that the nonempty sets $I(T^i(z^j)), 0 \leq i < p_j, j = 1$ or 2 , are pairwise disjoint. However, there are $p_1 + p_2 > n - m$ such sets and each such set is contained in Γ' , which has $n - m$ elements. This is impossible, so there exist $i, 0 \leq i < p_1$ and $j, 0 \leq j < p_2$, such that

$$(2.27) \quad T^i(z^1) < T^j(z^2) \quad \text{or} \quad T^j(z^2) < T^i(z^1).$$

Thus we have proved the lemma for $r = 2$. By applying T^{p_1-i} to both sides of (2.27), we see that we can also take $i = 0$ in (2.27). Also, the same argument

shows that for z^i and z^j with $1 \leq i < j \leq r + 1$, there exists $\nu = \nu(i, j)$ such that $0 \leq \nu < p_j$ and

$$(2.28) \quad T^\nu z^j < z^i \quad \text{or} \quad z^i < T^\nu z^j.$$

By using mathematical induction, we can now assume, by permuting $z_j, 2 \leq j \leq r + 1$, and applying powers of T , that $z^j < z^{j+1}$ for $2 \leq j \leq r$. For every $j, 2 \leq j \leq r$, there exists $k_j, 0 \leq k_j < p_1$, such that

$$T^{k_j}(z^1) < z^j \quad \text{or} \quad z^j < T^{k_j}(z^1).$$

If $T^{k_2}(z^1) < z^2$, we let $w^1 = T^{k_2}(z^1)$ and $w^j = z^j$ for $j \geq 2$ and we are done. Similarly, we are done if $T^{k_{r+1}}(z^1) > z^{r+1}$. Otherwise, let $m > 2$ be the first integer such that

$$T^{k_m}(z^1) < z^m.$$

By definition we have that

$$T^{k_j}(z^1) > z^j \quad \text{for} \quad 2 \leq j < m,$$

so

$$T^{k_m}(z^1) \equiv w^{m-1} = T^{k_m - k_{m-1}}(T^{k_{m-1}}z^1) > T^{k_m - k_{m-1}}(z^{m-1}) \equiv w^{m-2}.$$

Thus, if we define $w^j = z^j$ for $m \leq j \leq r + 1$ and

$$T^{k_m}(z^1) = w^{m-1}$$

and

$$T^{k_m - k_{m-1}}(z^{m-j}) = w^{m-j-1} \quad \text{for} \quad 1 \leq j \leq m - 2,$$

the conditions of the Proposition are satisfied. ■

The condition in Proposition 2.4 that

$$p_i + p_j > n - \sum_{k=1}^s q_k$$

for all i, j such that $1 \leq i < j \leq r + 1$ is too restrictive for some of our applications. Our next proposition gives another of the many possible variations of the theme in Proposition 2.4.

PROPOSITION 2.5: *Let notation and assumptions be as in Proposition 2.1. Assume that $m \geq 2$ and $r \geq 2$ and that, for $1 \leq j \leq m+r-1$, z^j is an irreducible element of V of period $p_j > 1$. Assume that $\Omega(z^i) \cap \Omega(z^j)$ (see eq. (2.26)) is empty for $1 \leq i < j \leq m+r-1$. Suppose that*

$$\sum_{j=1}^m p_j > n \text{ and } p_j + p_k > n \text{ for all } j \neq k$$

with $1 \leq j \leq m+r-1, m < k \leq m+r-1$.

Then there exists a one-one map σ from $\{j|1 \leq j \leq r+1\}$ to $\{j|1 \leq j \leq m+r-1\}$ and integers $k_j, 0 \leq k_j < p_j$, for $1 \leq j \leq m+r-1$, such that if $w^i = T^{k_j}(z^j)$, where $j = \sigma(i)$, then $w^i < w^{i+1}$ for $1 \leq i \leq r$.

Proof: Consider the sets $I(T^\alpha(z^i))$ for $1 \leq i \leq m$ and $0 \leq \alpha < p_i$. By assumption there are $\sum_{j=1}^m p_j > n$ such sets, and they are all nonempty. We also know that $I(T^\alpha(z^i)) \cap I(T^\beta(z^i))$ is empty for $0 \leq \alpha < \beta < p_i$. A simple counting argument implies that there are integers i and j with $1 \leq i < j \leq m$ and integers $0 \leq \alpha < p_i$ and $0 \leq \beta < p_j$ such that

$$I(T^\alpha(z^i)) \cap I(T^\beta(z^j)) \neq \emptyset.$$

As we have seen this implies that

$$T^\alpha(z^i) < T^\beta(z^j) \text{ or } T^\beta(z^j) < T^\alpha(z^i).$$

Assuming for definiteness that the first inequality holds, we define $v^1 = T^\alpha(z^i)$ and $v^2 = T^\beta(z^j)$ and write $q_1 = p_i$ and $q_2 = p_j$. Thus we have $v^1 < v^2$ and v^k is irreducible with period $q_k, k = 1, 2$.

We now work only with v^1, v^2 and the $r-1$ elements z^k for $m < k \leq m+r-1$. We know that $p_s + p_k > n$ for $m < s < k \leq m+r-1$ and

$$q_1 + p_k > n \text{ and } q_2 + p_k > n \text{ for } m < k \leq m+r-1.$$

It may happen that $q_1 + q_2 \leq n$, but this will be irrelevant because we already know that $v^1 < v^2$. If we now apply the argument of Proposition 2.4, starting with z^{m+1} , we find that after relabeling v^1, v^2 , and $z^k, m < k \leq m+r-1$ and applying powers of T we obtain the conclusion of Proposition 2.5. ■

If S is a nonempty subset of $\{j : 1 \leq j \leq n\}$, we now place several conditions on S which are motivated by Propositions 2.1-2.5.

CONDITION A: S does not contain a subset Q such that (1) $\gcd(i, j) = 1$ for all $i, j \in Q$ with $i \neq j$ and (2) $\sum_{i \in Q} i > n$.

CONDITION B: S does not contain disjoint subsets Q and R which satisfy the following properties: (1) $\gcd(i, j) = 1$ for $i, j \in Q$ with $i \neq j$. (2) $\gcd(i, k) = 1$ for all $i \in Q$ and $k \in R$. (3) R has $r + 1$ elements, $r \geq 1, i > r$ for all $i \in R$, and $\gcd(i, j)$ divides r for all $i, j \in R$ with $i \neq j$. (4) $i + j > n - (\sum_{k \in Q} k)$ for all $i, j \in R$ with $i \neq j$.

The possibility that Q is the empty set in Condition B is allowed. In that case conditions (1) and (2) in B are vacuous and $\sum_{k \in Q} k = 0$.

A little thought shows that if S satisfies condition B (with $r = 1$) then S satisfies condition A, but we prefer to state condition A separately for simplicity.

CONDITION C: S does not contain disjoint subsets Q and R with the following properties: (1) $\gcd(i, j) = 1$ for all $i, j \in Q$ such that $i \neq j$. (2) $\gcd(i, k) = 1$ for all $i \in Q$ and $k \in R$. (3) R has $m + 1$ elements where $m = \rho^2 - \rho + 1$ and $\rho \geq 2$ is an integer and $\gcd(i, j)$ divides $r = \rho^2$ for all $i, j \in R$ with $i \neq j$. (4) There exists $\gamma \in R$ such that $\gcd(\gamma, j)$ divides ρ for all $j \in R, j \neq \gamma$, and $\gamma > \rho$ and $j > \rho^2$ for $j \neq \gamma$. (5) $i + j > n - (\sum_{k \in Q} k)$ for all $i, j \in R$ with $i \neq j$.

CONDITION D: S does not contain a set R with the following properties: (1) $R = \{p_j | 1 \leq j \leq m + r - 1\}$, where $m \geq 2, r \geq 2$, and $p_i \neq p_j$ for $1 \leq i < j \leq m + r - 1$. (2) $\gcd(p_i, p_j)$ divides r for $1 \leq i < j \leq m + r - 1$ and $p_i > r$ for $1 \leq i \leq m + r - 1$. (3) $\sum_{j=1}^m p_j > n$ and $p_j + p_k > n$ for all $j \neq k$ such that $1 \leq j \leq m + r - 1$ and $m < k \leq m + r - 1$.

If $S \subset \{j | 1 \leq j \leq n\}$ we say that S is "admissible for n " if S satisfies conditions A, B, C and D. If $S \subset \{j | 1 \leq j \leq n\}$ and S is admissible for n , then one can check that S is admissible for $n + 1$; but it may happen that S is admissible for $n + 1$ and not admissible for n .

We define an *ad hoc* function $\varphi(n)$ by

$$(2.29) \quad \varphi(n) = \sup\{\text{lcm}(S) | S \subset \{1, 2, \dots, n\} \text{ and } S \text{ satisfies conditions A, B, C and D}\}.$$

We have $\varphi(1) = 1$, and the values of $\varphi(n)$ are tabulated in Table 2.1 below for $n \leq 24$. Because the fact that S is admissible for n implies that S is admissible for $n + 1$, we have that $\varphi(n) \leq \varphi(n + 1)$. It is also clear that if $S \subset \{j : 1 \leq j \leq n\}$

and $\sum_{j \in S} j \leq n$, then S is admissible for n , and this implies that

$$(2.30) \quad \varphi(n) \geq \sup\{\text{lcm}(S) \mid S \subset \{1, 2, \dots, n\}, \sum_{j \in S} j \leq n\} = \psi(n).$$

The function $\psi(n)$ in (2.30) is the maximal order of an element of the permutation group on n letters.

THEOREM 2.1 : *Let $T : D \subset \mathbb{R}^n$ be a map and suppose that $\xi \in D$ is a periodic point of T of (minimal) period p . If V is the lower semilattice generated by*

$$A = \{T^j(\xi) \mid j \geq 0\},$$

assume that $V \subset D$ and that

$$T(x \wedge y) = T(x) \wedge T(y) \quad \text{for all } x, y \in V.$$

Then p divides $\text{lcm}(S)$, where $S \subset \{j \mid 1 \leq j \leq n\}$ is some set which satisfies conditions A, B, C and D. Furthermore, $p \leq \varphi(n)$, where $\varphi(n)$ is defined in (2.29).

Proof: For each $x \in V$, let p_x denote the period of x . Let

$$S_1 = \{p_x \mid x \in V, x \leq \xi \text{ and } x \text{ is irreducible}\}.$$

We know (see Remark 2.1) that p divides $\text{lcm}(S_1)$. If $\text{lcm}(S_1) = 1, S_1 = \{1\} = S$, and we are done. Otherwise, take $S = S_1 - \{1\}$ and note that p divides $\text{lcm}(S)$.

We claim first that S satisfies condition A. If not, there are irreducible elements y^j in $V, 1 \leq j \leq m$, of periods $p_j, 1 \leq j \leq m$, such that $\text{gcd}(p_i, p_j) = 1$ for $1 \leq i < j \leq m$ and $p_j > 1$ for $1 \leq j \leq m$ and $\sum_{j=1}^m p_j > n$. (In the notation of condition A, $Q = \{p_j : 1 \leq j \leq m\}$.) However, this contradicts Proposition 2.1.

We next claim that S satisfies condition B. If not, let

$$Q = \{q_j \mid 1 \leq j \leq s\} \quad \text{and} \quad R = \{p_i \mid 1 \leq i \leq r + 1\}$$

be sets satisfying (1), (2), (3) and (4) in condition B and let $y^j, 1 \leq j \leq s$, be irreducible elements in V of period $q_j, 1 \leq j \leq s$, and $z^i, 1 \leq i \leq r + 1$, be irreducible elements of periods $p_i, 1 \leq i \leq r + 1$. Proposition 2.4 implies that, by permuting the z^i and applying appropriate powers of T , we can assume $z^i < z^{i+1}$ for $1 \leq i \leq r$. Under these operations the corresponding periods are permuted

and it remains true that $\gcd(p_i, p_j)$ divides r for $1 \leq i \leq j \leq r+1$ and $p_i > r$ for $1 \leq i \leq r+1$. However, Proposition 2.2 asserts that this is impossible.

The proof that S must satisfy condition C follows as above if one uses Proposition 2.3 instead of Proposition 2.2. Details are left to the reader.

If S fails to satisfy condition D and R is as in condition D, then there are irreducible elements $z^j, 1 \leq j \leq m+r-1$ with periods $p_j, 1 \leq j \leq m+r-1$, such that $R = \{p_j\}$ satisfies conditions (1), (2) and (3) in D. By using Proposition 2.5, we obtain irreducible elements $w^i, 1 \leq i \leq r+1$, of periods $p_i > r$ such that $w^i < w^{i+1}$ for $1 \leq i \leq r$ and $\gcd(p_i, p_j)$ divides r for $1 \leq i < j \leq r+1$. This contradicts Proposition 2.2.

It follows that S is admissible for n and obviously $p \leq \text{lcm}(S) \leq \varphi(n)$. ■

We tabulate (Table 2.1) the values of $\varphi(n)$ for $1 \leq n \leq 24$ and indicate admissible sets S for which $\varphi(n) = \text{lcm}(S)$.

Even for relatively small values of n like 23 or 24 it is not a trivial matter to compute $\varphi(n)$ by hand, so it may be worthwhile to indicate a verification procedure for this table. One can check that $S = \{6, 14, 16, 20\}$ is admissible for $n = 20$, so $\varphi(20) \geq 1680$. Because φ is monotonic, it suffices to prove that $\varphi(23) = 1680$ in order to verify that $\varphi(j) = 1680$ for $20 \leq j \leq 23$. If T is a set which is admissible for $n = 23$ and $T \neq \{1\}$, $\text{lcm}(T)$ is a product of terms chosen from $2^\alpha, 0 \leq \alpha \leq 4, 3^\beta, 0 \leq \beta \leq 2, 5, 7, 11, 13, 17, 19$ and 23. By a laborious case-by-case analysis, one can show that if T contains an odd number $j \geq 9$, then $\text{lcm}(T) < 1680$; it is easiest successively to eliminate odd numbers in order of decreasing size. It follows that if $\text{lcm}(T) \geq 1680$, T contains only even numbers and odd numbers $j \leq 7$. If $22 \in T$ and $\text{lcm}(T) \geq 1680$, note that condition A and the fact $11 \notin T$ imply that T contains only even elements. Again, a case-by-case analysis shows that, if $22 \in T$ and T contains only even integers, then $\text{lcm}(T) < 1680$, so if $\text{lcm}(T) \geq 1680, 22 \notin T$. Note that $\text{lcm}(6, 8, 14, 22) = 1848$, and $T = \{6, 8, 14, 22\}$ satisfies conditions A, B and C, but not D, so condition D is needed. Similarly, condition C is needed to eliminate $T = \{12, 16, 20, 22\}$. Another case-by-case analysis shows that if $\text{lcm}(T) \geq 1680$, then $18 \notin T$. Thus one can show that if $\text{lcm}(T) \geq 1680$ and T is admissible for $n = 23$, then T contains no odd integer $j \geq 9$ and $18, 22 \notin T$. It follows that 9, 11, 13, 17, 19 and 23 cannot be factors of $\text{lcm}(T)$, so

$$\text{lcm}(T) \leq 16 \times 3 \times 5 \times 7 = 1680.$$

Table 2.1: Values of $\varphi(n)$ for $1 \leq n \leq 24$

n	$\varphi(n)$	n	$\varphi(n)$
1	1	13	120
2	$2 = \text{lcm}(2)$	14	$168 = \text{lcm}(8, 12, 14)$
3	$3 = \text{lcm}(3)$	15	$180 = \text{lcm}(9, 12, 15)$
4	$4 = \text{lcm}(4)$	16	$336 = \text{lcm}(12, 14, 16)$
5	$6 = \text{lcm}(2, 3)$	17	$420 = \text{lcm}(3, 4, 10, 14)$
6	$12 = \text{lcm}(4, 6)$	18	420
7	12	19	$840 = \text{lcm}(5, 8, 12, 14)$
8	$24 = \text{lcm}(6, 8)$	20	$1680 = \text{lcm}(6, 14, 16, 20)$
9	24	21	1680
10	$60 = \text{lcm}(4, 6, 10)$	22	1680
11	60	23	1680
12	$120 = \text{lcm}(8, 10, 12)$	24	$2640 = \text{lcm}(16, 20, 22, 24)$

Theorem 2.1 provides constraints on the possible period $p \leq \varphi(n)$ of a periodic point ξ of T . We illustrate one such constraint.

COROLLARY 2.1: *Let notation and assumptions be as in Theorem 2.1. If p_1 and p_2 are distinct prime numbers and α and β are positive integers such that $p_1^\alpha + p_2^\beta > n$ and $p_1 p_2^\beta > n$ and $p_1^\alpha \geq p_2^\beta$, it follows that $p \neq p_1^\alpha p_2^\beta$.*

Proof: Suppose not, so $p = p_1^\alpha p_2^\beta$. We know that p divides $\text{lcm}(S)$, where $S = \{p_x \mid x \in V, x \leq \xi, x \text{ is irreducible of period } p_x\}$. It follows that there must be irreducible elements x and y in V of periods $m_1 p_1^\alpha$ and $m_2 p_2^\beta$ respectively, where m_1 and m_2 are integers and $m_1 p_1^\alpha \leq n$ and $m_2 p_2^\beta \leq n$. We know that $2p_1^\alpha \geq p_1^\alpha + p_2^\beta > n$, so $m_1 = 1$. Condition A implies that p_1^α and $m_2 p_2^\beta$ must have a common factor. It follows that p_1 divides m_2 , which contradicts the assumption that $p_1 p_2^\beta > n$. ■

A typical application of Corollary 2.1 and Theorem 2.1 is that, for $n = 12$, the number p in Theorem 2.1 is never equal to $11j$ for $j \geq 2, j$ an integer.

In view of the difficulty of computing $\varphi(n)$ precisely, it is important to have upper and lower bounds for $\varphi(n)$. As usual, we denote by $[x]$ the greatest integer

$m \leq x$. We define $\theta(m) = \text{lcm}(\{j | 1 \leq j \leq m\})$.

PROPOSITION 2.6: If $\varphi(n)$ is defined by eq. (2.29), then

$$(2.31) \quad \varphi(n) \leq n \text{lcm}(\{j | 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\}) \equiv n\theta(\lfloor \frac{n}{2} \rfloor),$$

$$(2.32) \quad \varphi(n) \geq 2 \text{lcm}(\{j | 1 \leq j \leq \lfloor \frac{n}{4} \rfloor + 1\}) \equiv 2\theta(\lfloor \frac{n}{4} \rfloor + 1).$$

Furthermore, we have, for all n ,

$$(2.33) \quad \varphi(n) \leq n \exp(1.03883 \lfloor \frac{n}{2} \rfloor) \quad \text{and} \quad \varphi(n) < 2^n.$$

Given any $\epsilon > 0$, we have, for n sufficiently large,

$$(2.34) \quad \varphi(n) \leq n \exp((1 + \epsilon) \lfloor \frac{n}{2} \rfloor).$$

Proof: Suppose that $S \subset \{j | 1 \leq j \leq n\}$ satisfies condition A. It suffices to prove that

$$\text{lcm}(S) \leq n\theta(\lfloor \frac{n}{2} \rfloor).$$

If $j \in S$ and $j \leq \lfloor n/2 \rfloor$, then j divides $\theta(\lfloor n/2 \rfloor)$. If $\lfloor n/2 \rfloor < j \leq n$ and $j = m_1 m_2$ where $m_1 > 1, m_2 > 1$ and $\text{gcd}(m_1, m_2) = 1$, it follows that $m_i \leq \lfloor n/2 \rfloor$ and m_i divides $\theta(\lfloor n/2 \rfloor)$ for $i = 1$ or $i = 2$ and therefore $j = m_1 m_2$ divides $\theta(\lfloor n/2 \rfloor)$. If $j \in S$, the only remaining possibility is that $j = p_1^\alpha$, where p_1 is a prime and $\lfloor n/2 \rfloor < p_1^\alpha \leq n$. However, there can be at most one such element in S ; for if $k \in S$ and $k = p_2^\beta$, where p_2 is a prime and $\lfloor n/2 \rfloor < k \leq n$ and $k \neq j$, then $p_1 \neq p_2$ and $\text{gcd}(j, k) = 1$ and $j + k > n$, which contradicts condition A. It follows that either (a) $\text{lcm}(S)$ divides $\theta(\lfloor n/2 \rfloor)$ (if S does not contain an element $j = p_1^\alpha$ such that p_1 is a prime and $\lfloor n/2 \rfloor \leq j \leq n$) or (b) $\text{lcm}(S)$ divides $p_1^\alpha \theta(\lfloor n/2 \rfloor)$, where p_1 is a prime and $\lfloor n/2 \rfloor < p_1^\alpha \leq n$. In either case, (2.31) is satisfied.

Next define $T = \{2j | 1 \leq j \leq \lfloor n/4 \rfloor + 1\}$. In order to prove (2.31), it suffices to prove that T satisfies conditions A, B, C and D because then $\varphi(n) \geq \text{lcm}(T)$. Condition A is obvious, since all elements of T are divisible by 2. If condition B fails the set Q in condition B must be empty and the number r in B satisfies $r \geq 2$. But then the set R must contain at least 3 elements and at least two of these elements must be less than or equal to $2\lfloor n/4 \rfloor$. However, this contradicts the assumption that $i + j > n$ for all $i, j \in R, i \neq j$, and condition B is satisfied.

Essentially the same argument shows that condition C is satisfied; details are left to the reader. To see that condition D is satisfied, suppose not and let $p_j \in T, 1 \leq j \leq m+r-1$, be as in condition D. Because $m \geq 2$ and $r \geq 2$ it follows that $p_j \leq 2(\lfloor n/4 \rfloor - 1)$ for some $j, 1 \leq j \leq m+r-1$. If $j \leq m$, select k so that $m < k \leq m+r-1$ and observe that $p_j + p_k \leq n$. If $m < j \leq m+r-1$, select k so that $1 \leq k \leq m$ and again observe that $p_j + p_k \leq n$. Thus we obtain a contradiction in either case, and we conclude that condition D is satisfied.

It is a classical result (see Theorem 12 in [14]) that

$$\theta(m) \leq \exp(1.03883m) \quad \text{for all } m \geq 1.$$

Furthermore, given $\varepsilon > 0$, there exists an integer m_ε such that

$$\theta(m) \leq \exp((1 + \varepsilon)m) \quad \text{for all } m \geq m_\varepsilon.$$

Thus, the first part of eq. (2.33) and eq. (2.34) are immediate consequences of eq. (2.31).

Table 2.1 shows that $\varphi(n) < 2^n$ for $n \leq 24$, and it is a simple calculus exercise (left to the reader) to prove that

$$n \exp(1.03883\lfloor n/2 \rfloor) < 2^n \quad \text{for all } n \geq 16.$$

We conclude that $\varphi(n) < 2^n$ for all n . ■

As an immediate consequence of Theorem 2.1 and our earlier remarks we have

COROLLARY 2.2: *Let $K^n = \{x \in \mathbb{R}^n \mid x \geq 0\}$ and let $T : K^n \rightarrow K^n$ be a map such that $T(0) = 0$ and T is nonexpansive with respect to the ℓ_1 -norm on \mathbb{R}^n . Then, for every $x \in K^n$, there exists an integer $p = p_x$ such that $\lim_{k \rightarrow \infty} T^{kp}(x)$ exists. The integer p divides $\text{lcm}(S)$ where $S \subset \{j \mid 1 \leq j \leq n\}$ is a set which satisfies conditions A, B, C and D. It is also true that $p \leq \varphi(n)$, where $\varphi(n)$ is defined by eq. (2.28), tabulated in Table 2.1 for $n \leq 24$ and estimated in Proposition 2.6 for general n .*

Theorems 4.1 and 4.3 in [15] suggest that results like Corollary 2.2 can be used to provide information about the case of an ℓ_1 -nonexpansive map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Unfortunately, this author and Michael Scheutzwow have independently observed an error in the proof of Theorem 4.1 in [15]. However, Scheutzwow has provided an elegant correction in [16], and with the results of [16] and Corollary 2.2, we can study the case of an ℓ_1 -nonexpansive map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

COROLLARY 2.3: *Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a map which is ℓ_1 -nonexpansive and has a fixed point x_0 . Then, for every $x \in \mathbb{R}^n$, there exists a minimal integer $p = p_x$ such that $\lim_{k \rightarrow \infty} T^{kp}(x)$ exists. The integer p divides $\text{lcm}(S)$, where $S \subset \{j : 1 \leq j \leq 2n\}$ and X is admissible for $2n$. It is also true that $p \leq \varphi(2n)$, where φ is defined by eq. (2.28).*

Proof: By replacing T by $S^{-1}TS$, where $S(x+x_0)$, we can assume that $T(0) = 0$. For notational convenience we write

$$K^{2n} = \{(y, z) : y, z \in K^n\}.$$

Scheutzwow defines an isometry γ of \mathbb{R}^n into K^{2n} by

$$\gamma(x) = (x \vee 0, -(x \wedge 0)).$$

If $\Gamma \subset K^{2n}$ denotes the range of γ , it is observed in [16] that there is an ℓ_1 -nonexpansive retraction ρ of K^{2n} onto Γ given by

$$\rho(y, z) = ((y \vee z) - z, (y \vee z) - y).$$

Thus, if we define $T_1 : K^{2n} \rightarrow K^{2n}$ by

$$T_1(\xi) = (\gamma T \gamma^{-1} \rho)(\xi),$$

$T_1(0) = 0, T_1$ is ℓ_1 -nonexpansive and

$$T_1^j = \gamma T^j \gamma^{-1} \rho.$$

Using these observations and Corollary 2.2, we obtain the conclusions of Corollary 2.3 from the known results for T_1 . ■

The next corollary also follows immediately from our previous results.

COROLLARY 2.4: *Let $E = (\mathbb{R}^n, \|\cdot\|)$ be a finite-dimensional Banach space of dimension n with a polyhedral norm $\|\cdot\|$. Assume that $D \subset E$ is a bounded set and $T : D \rightarrow D$ is a map which is nonexpansive with respect to $\|\cdot\|$. Let $\hat{D} \supset D$ be a set such that for all $x, y \in \hat{D}, x \wedge y \in \hat{D}$. Assume that T has an extension $\hat{T} : \hat{D} \rightarrow \hat{D}$ such that $\hat{T}(x \wedge y) = \hat{T}(x) \wedge \hat{T}(y)$ for all $x, y \in \hat{D}$. Then for every $x \in D$, there exists an integer $p = p_x$ such that $\lim_{k \rightarrow \infty} T^{kp}(x)$ exists. The integer p_x divides $\text{lcm}(S)$, where S is some set which is admissible for n . Furthermore, we have $p_x \leq \varphi(n) < 2^n$ for all n where $\varphi(n)$ is defined by eq. (2.28). (See also Table 2.1 and Proposition 2.6.)*

Remark 2.2: If $T : E \rightarrow E$ is nonexpansive with respect to $\|\cdot\|$ and T has a fixed point w , then the bounded set D in Corollary 2.3 can be taken to be $D = B_R(w)$, the ball of radius R and center w .

As a very special case of Corollary 2.4, we mention a class of examples which arises in many applications [3, 4, 6, 7]. ■

COROLLARY 2.5: Let A be an $n \times n$ matrix, define λ by

$$(2.35) \quad \lambda = \inf \left\{ \frac{1}{k} \sum_{i=1}^k a_{j_i j_{i+1}} \mid 1 \leq j_i \leq n \text{ for } 1 \leq i \leq k+1 \text{ and } j_{k+1} = j_1 \right\}$$

and assume $\lambda = 0$. (The inf in eq. (2.35) is taken over all $k \geq 1$ and all sequences $\{j_1, j_2, \dots, j_{k+1}\}$ satisfying the given conditions.) Define $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$(2.36) \quad (T(x))_i = \min_j (a_{ij} + x_j).$$

Then for every $x \in \mathbb{R}^n$ there exists an integer $p = p_x$ such that $\lim_{k \rightarrow \infty} T^{kp}(x)$ exists. There exists a set S , admissible for n , such that p divides $\text{lcm}(S)$, and $p_x \leq \varphi(n)$, where $\varphi(n)$ is defined in eq. (2.29) and $\varphi(n) < 2^n$.

Proof: We have already noted that T is nonexpansive with respect to $\|\cdot\|_\infty$, that $T(x + cu) = T(x) + cu$ for all $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$ ($u = (1, 1, \dots, 1)$), and $T(x \wedge y) = T(x) \wedge T(y)$ for all x and y in \mathbb{R}^n . If λ is defined by (2.35), it is known [3] that there exists $w \in \mathbb{R}^n$ such that

$$T(w) = \lambda u + w, \quad u = (1, 1, \dots, 1),$$

so if $\lambda = 0$, T has a fixed point in \mathbb{R}^n . Corollary 2.5 now follows immediately from Corollary 2.3 and Remark 2.2. ■

Remark 2.3: If $\min_j (a_{ij}) = 0$ for $1 \leq i \leq n$, one can easily check that $T(0) = 0$ and $\lambda = 0$, and the general situation (for $\lambda = 0$) reduces to this case by a change of coordinates. ■

Remark 2.4: The estimate $p_x \leq \varphi(n)$ is *not* best possible for the class of maps in Corollary 2.4. It is proved in [13] that if T is given by eq. (2.35) and $\lambda = 0$, so T has a fixed point, then there is a set $S \subset \{k \mid 1 \leq k \leq n\}$ such that $\sum_{j \in S} j \leq n$ and such that if $p = \text{lcm}(S)$, then $\lim_{k \rightarrow \infty} T^{kp}(x)$ exists for all $x \in \mathbb{R}^n$. In [13] the set S is described in terms of the matrix $A = (a_{ij})$. In particular this proves that $p \leq \psi(n)$ for $\psi(n)$ as in eq. (2.30). This also proves a conjecture which was made by R. B. Griffiths [8] in response to a question from this author. ■

3. Examples Concerning the Optimality of Estimates for Periods

In this section we wish to give some systematic ways of constructing integral-preserving, order-preserving maps $T : K^n \rightarrow K^n$. We shall use these examples to prove that $\varphi(n)$ is a best possible upper bound (in the context of Theorem 2.1) for $1 \leq n \leq 24$.

In order to save space we collect our hypotheses. We shall always denote by u the vector in \mathbb{R}^n such that $u_i = 1$ for $1 \leq i \leq n$.

Definition 3.1: A map $T : K^n \rightarrow K^n$ satisfies H3.1 if T is integral-preserving and order-preserving and $T(cu) = cu$ for all $c \geq 0$. ■

All of our subsequent examples will satisfy H3.1. Recall that if T satisfies H3.1, T is nonexpansive with respect to the ℓ_1 -norm and T is sup decreasing.

LEMMA 3.1: Suppose that $n = mr$, where m and r are positive integers, and consider K^n as the r -fold Cartesian product of K^m . If $f : K^m \rightarrow K^m$ satisfies H3.1 and $T : K^n \rightarrow K^n$ is defined by

$$(3.1) \quad T(y^1, y^2, \dots, y^r) = (f(y^r), y^1, y^2, \dots, y^{r-1}),$$

where $y^j \in \mathbb{R}^m$ for $1 \leq j \leq r$, then T satisfies H3.1. If $z^j \in K^m$ is a periodic point of f of period p_j for $1 \leq j \leq r$ and if $\Omega(z^j) \cap \Omega(z^k)$ is empty for $1 \leq j < k \leq r$ (see eq. (2.26)), then $z = (z^1, z^2, \dots, z^r)$ is a periodic point of T of period $\text{rlcm}(\{p_j | 1 \leq j \leq r\})$.

Proof: We leave to the reader the easy verification that T satisfies H3.1. Because $\Omega(z^j) \cap \Omega(z^k)$ is empty for $j \neq k$, one can see that $T^j(z)$ cannot equal z unless j is a multiple of r . Furthermore, we have that

$$T^{sr}(z) = (f^s(z^1), f^s(z^2), \dots, f^s(z^r)).$$

Thus the smallest value of s such that $T^{sr}(z) = z$ must be divisible by p_j for $1 \leq j \leq r$, so $s = \text{lcm}(\{p_j | 1 \leq j \leq r\})$. ■

Note that if $p_j \neq p_k$ for $j \neq k$, then $\Omega(z^j) \cap \Omega(z^k)$ is empty for $j \neq k$.

The proof of the next lemma is trivial and is left to the reader.

LEMMA 3.2: Suppose that $n = m + r$, where m and r are positive integers, and consider K^n as the Cartesian product of K^m and K^r . If $f : K^m \rightarrow K^m$ and $g : K^r \rightarrow K^r$ both satisfy H3.1 and if $T : K^n \rightarrow K^n$ is defined by

$$T(y, z) = (f(y), g(z)), \quad y \in K^m, \quad z \in K^r,$$

then T satisfies H3.1. If $\bar{y} \in K^m$ is a periodic point of f of period p , and $\bar{z} \in K^r$ is a periodic point of g of period p_2 , then $\bar{x} = (\bar{y}, \bar{z})$ is a periodic point of T of period $\text{lcm}(p_1, p_2)$.

With the aid of Lemmas 3.1 and 3.2, we can already show that the estimate $p \leq \varphi(n)$ is optimal for $1 \leq n \leq 11$.

LEMMA 3.3: For $1 \leq n \leq 11$, there exists a map $T : K^n \rightarrow K^n$ such that T satisfies H3.1 (see Definition 3.1), and T has a periodic point x of period $\varphi(n)$ (see Table 2.1).

Proof: For $1 \leq n \leq 5$ it suffices to take T to be an appropriate permutation. Since $\varphi(6) = \varphi(7) = 12, \varphi(8) = \varphi(9) = 24$ and $\varphi(10) = \varphi(11) = 60$, it suffices, by using Lemma 3.2 with $r = 1$ and g the identity on K^1 , to prove Lemma 3.3 for $n = 6, 8$ and 10 .

By Lemma 3.1, it suffices, for $n = 6$, to find a map $f : K^3 \rightarrow K^3$ which satisfies H3.1 and has periodic points \bar{y} and \bar{z} in K^3 of period 3 and 2 respectively. For then the map T in eq. (3.1) will have period $2 \text{lcm}(3, 2) = 12$. If f is defined by

$$f(y_1, y_2, y_3) = ((y_3 \vee 1) + (y_2 \wedge 1) - 1, y_1, (y_3 \wedge 1) + (y_2 \vee 1) - 1),$$

f satisfies H3.1, and if $\bar{y} = (2, 1, 1)$ and $\bar{z} = (1, 0, 1)$, \bar{y} and \bar{z} have period 3 and 2 respectively. (Recall that $a \vee b = \max(a, b)$.)

Since $24 = 2 \text{lcm}(4, 3)$, Lemma 3.1 shows that to prove Lemma 3.3 for $n = 8$, it suffices to find a map $f : K^4 \rightarrow K^4$ which satisfies H3.1 and has periodic points \bar{y} and \bar{z} of periods 4 and 3 respectively. Such a map is provided by

$$f(y_1, y_2, y_3, y_4) = ((y_3 \wedge 1) + (y_4 \vee 1) - 1, y_1, y_2, (y_3 \vee 1) + (y_4 \wedge 1) - 1),$$

with $\bar{y} = (2, 1, 1, 1)$ and $\bar{z} = (1, 0, 0, 1)$.

Finally, since $60 = 2 \text{lcm}(6, 5)$, Lemma 3.1 shows that to prove Lemma 3.3 for $n = 10$, it suffices to find a map $f : K^5 \rightarrow K^5$ which satisfies H3.1 and has periodic points \bar{y} and \bar{z} of periods 6 and 5 respectively. A map of this type is given by

$$f(x_1, x_2, x_3, x_4, x_5) = ((x_3 \vee 1) + (x_5 \wedge 1) - 1, x_1, x_2, (x_3 \wedge 1) + (x_5 \vee 1) - 1, x_4)$$

with $\bar{y} = (2, 1, 1, 2, 1)$ and $\bar{z} = (1, 0, 0, 0, 0)$. ■

The example of a map T which we have constructed here for $n = 6$ can be shown to be equivalent, by a change of variables, to an example in [15].

To continue further it seems useful to have a more systematic way of generating examples of maps satisfying H3.1. In the following work we modify our notation slightly and define

$$(3.2) \quad K_+^n = \{x \in \mathbb{R}^n \mid x \geq 0\} \quad \text{and} \quad K_-^n = \{x \in \mathbb{R}^n \mid x \leq 0\}.$$

LEMMA 3.4: *If $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by*

$$(3.3) \quad S(x_1, x_2) = ((x_1 \vee 0) + (x_2 \wedge 0), (x_1 \wedge 0) + (x_2 \vee 0)),$$

then $S^2 = S \circ S$ is an integral-preserving, order-preserving, sup-decreasing retraction of \mathbb{R}^2 onto $K_+^2 \cup K_-^2 = D^2$.

Proof: If f and g are any two integral-preserving, order-preserving maps of \mathbb{R}^n into \mathbb{R}^n or K^n into K^n , it is easy to see that $g \circ f$ is integral-preserving and order-preserving. One can check that S is integral-preserving and order-preserving, so S^2 is also. A direct calculation shows that if $x \in K_+^2$ or K_-^2 , then $S^2(x) = x$. It remains to prove that $S^2(x) \in K_+^2 \cup K_-^2$ for all $x \in \mathbb{R}^2$. If $x \in \mathbb{R}^2$ and $x_1 \geq 0$ and $x_2 \leq 0$, a calculation gives

$$S^2(x) = \begin{cases} (x_1 + x_2, 0) & \text{if } x_1 + x_2 \geq 0, \\ (0, x_1 + x_2) & \text{if } x_1 + x_2 \leq 0, \end{cases}$$

so $S^2(x) \in D^2$. A similar calculation applies if $x_1 \leq 0$ and $x_2 \geq 0$, so S^2 is a retraction onto D^2 .

Because $S^2(cu) = cu$ for all $c \in \mathbb{R}, u = (1, 1)$, and S^2 is order-preserving, one obtains that $\|S^2(x)\|_\infty \leq \|x\|_\infty$ for all $x \in \mathbb{R}^2$. ■

Our real interest, of course, is in proving the analogue of Lemma 3.4 for \mathbb{R}^n .

THEOREM 3.1: *If K_+^n and K_-^n are defined as in eq. (3.2) and if $D^n = K_+^n \cup K_-^n$, there exists an integral-preserving, order-preserving, sup-decreasing retraction ρ of \mathbb{R}^n onto D^n .*

Proof: For $1 \leq j, k \leq n$ and $j \neq k$, define a map $S_{jk} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$S_{jk}(x) = y,$$

where

$$y_i = x_i \text{ for } i \neq j, k, y_j = (x_j \vee 0) + (x_k \wedge 0) \text{ and } y_k = (x_j \wedge 0) + (x_k \vee 0).$$

Because S_{jk} is integral-preserving and order-preserving, S_{jk}^2 is also, and a calculation as in Lemma 3.4 shows that $S_{jk}^2|D^n$ is the identity map. If $\xi \in \mathbb{R}^n$ and $\eta = S_{jk}^2(\xi)$, one can also verify as in Lemma 3.4 that if $\xi_r = 0$ for some r , then $\eta_r = 0$; and if $\xi_j \xi_k < 0$, either $\eta_j = 0$ or $\eta_k = 0$.

We have already proved Theorem 3.1 in the case $n = 2$ in Lemma 3.4, so assume by induction that the theorem is true for \mathbb{R}^{n-1} , $n \geq 3$, and consider \mathbb{R}^n as $\mathbb{R}^1 \times \mathbb{R}^{n-1}$. Let σ be the integral-preserving, order-preserving retraction of \mathbb{R}^{n-1} onto D^{n-1} and define $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\tau(x_1, x') = (x_1, \sigma(x')), \quad x \in \mathbb{R}, \quad x' \in \mathbb{R}^{n-1}.$$

Note that τ is integral-preserving and order-preserving and $\tau|D^n$ is the identity map. Similarly S_{1j}^2 is integral-preserving and order-preserving for $2 \leq j \leq n$ and $S_{1j}^2|D^n$ is the identity map. Define a map ρ by

$$(3.4) \quad \rho = S_{12}^2 S_{13}^2 S_{14}^2 \dots S_{1n}^2 \tau,$$

where composition of maps is indicated in (3.4). We know that ρ is integral-preserving and order-preserving and $\rho|D^n$ is the identity, so it suffices to prove $\rho(\mathbb{R}^n) \subset D^n$. For $x \in \mathbb{R}^n$, define $\tau(x) = \xi$, so we know $\xi_j \geq 0$ for all $j \geq 2$ or $\xi_j \leq 0$ for all $j \geq 2$. For definiteness assume that $\xi_j \geq 0$ for $2 \leq j \leq n$. If $\xi_1 \geq 0$, $\rho(x) \in K_+^n$ and we are done, so assume $\xi_1 < 0$. If $\xi_1 + \xi_n \geq 0$, a calculation gives

$$S_{1n}^2(\xi) = (0, \xi_2, \xi_3, \dots, \xi_{n-1}, \xi_1 + \xi_n) \in D^n,$$

so $\rho(x) \in D^n$ and we are done. Thus we assume that $\xi_1 + \xi_n < 0$, so

$$S_{1n}^2(\xi) = (\xi_1 + \xi_n, \xi_2, \dots, \xi_{n-1}, 0).$$

Applying the map S_{1n-1}^2 we see that if $\xi_1 + \xi_n + \xi_{n-1} \geq 0$, then

$$S_{1n-1}^2 S_{1n}^2(\xi) = (0, \xi_2, \xi_3, \dots, \xi_{n-2}, \xi_1 + \xi_{n-1} + \xi_n, 0) \in D^n,$$

so $\rho(x) \in D^n$ and we are done. Thus we can assume that $\xi_1 + \xi_{n-1} + \xi_n < 0$ and

$$S_{1n-1}^2 S_{1n}^2(\xi) = (\xi_1 + \xi_{n-1} + \xi_n, \xi_2, \dots, \xi_{n-2}, 0, 0).$$

Continuing in this way we see that if, for $j \geq 3$,

$$S^2_{1j} \dots S^2_{1n}(\xi) \notin D,$$

then

$$(3.5) \quad S^2_{1j} \dots S^2_{1n}(\xi) = \left(\xi_1 + \sum_{k=j}^n \xi_k, \xi_2, \xi_3, \dots, \xi_{j-1}, 0, 0, \dots, 0 \right),$$

where the final $n - j + 1$ coordinates on the right side of (3.5) equal zero and $\xi_1 + \sum_{k=j}^n \xi_k < 0$. Thus we either obtain that $S^2_{1j} \dots S^2_{1n}(\xi) \in D$ for some $j \geq 3$, so $\rho(x) \in D$, or

$$S^2_{13} \dots S^2_{1n}(\xi) \notin D \quad \text{and}$$

$$S^2_{13} \dots S^2_{1n}(\xi) = \left(\xi_1 + \sum_{k=3}^n \xi_k, \xi_2, 0, \dots, 0 \right) = (\eta_1, \eta_2, 0, \dots, 0) = \eta,$$

where $\eta_1 < 0$ and $\eta_2 \geq 0$. But now a calculation gives

$$S^2_{2n}(\eta) = \begin{cases} (0, \eta_1 + \eta_2, 0, \dots, 0) & \text{if } \eta_1 + \eta_2 \geq 0, \\ (\eta_1 + \eta_2, 0, 0, \dots, 0) & \text{if } \eta_1 + \eta_2 \leq 0. \end{cases}$$

In either case $S^2_{2n}(\eta) = \rho(x) \in D^n$, and the proof is complete. (The map ρ is sup-decreasing because ρ is order-preserving and $\rho(cu) = cu$ for all real c .) ■

Remark 3.1: An examination of the proof of Theorem 3.1 shows that we have proved that the retraction can be chosen to be a composition of some of the maps S^2_{jk} . ■

Our next theorem is an extension of Theorem 3.1.

THEOREM 3.2: *Let $\alpha_j, 0 \leq j \leq k$, be a strictly increasing sequence of real numbers with $\alpha_0 = 0$. For $0 \leq j < k$ define*

$$B_j = \{x \in K^n | \alpha_j u \leq x \leq \alpha_{j+1} u\} \quad \text{and} \quad B_k = \{x \in K^n | \alpha_k u \leq x\}$$

and

$$B = \bigcup_{j=0}^k B_j.$$

Then there exists an integral-preserving, order-preserving retraction r of K^n onto B .

Proof: First assume that $k = 1$, let ρ be a retraction as in Theorem 3.1 and define $r(x)$ by

$$r(x) = \rho(x - \alpha_1 u) + \alpha_1 u.$$

Because ρ is integral-preserving and order-preserving, r is also. Thus we have, for $x \in K^n$,

$$r(x) \geq r(0) = \rho(-\alpha_1 u) + \alpha_1 u = -\alpha_1 u + \alpha_1 u = 0.$$

Because $\rho(x - \alpha_1 u) \in D_+^n \cup D_-^n$, we can thus say that $r(x) \in B_0$ or $r(x) \in B_1$. Also if $x \in B_0$ or B_1 , then $x - \alpha_1 u \in D_-^n$ or D_+^n and $\rho(x - \alpha_1 u) = x - \alpha_1 u$ and $r(x) = x$. Thus r is the desired retraction.

Assume, by way of mathematical induction, that we have proved the theorem for some $k \geq 1$. Let $\alpha_j, 0 \leq j \leq k + 1$ be a strictly increasing sequence of real numbers with $\alpha_0 = 0$. Let B_j be as defined in the theorem and define B'_j by

$$B'_j = \{x \in K^n \mid \alpha_j u \leq x \leq \alpha_{j+1} u\} \quad \text{for } 0 \leq j \leq k, \quad B'_{k+1} = \{x \mid x \geq \alpha_{k+1} u\}.$$

By induction there exists an integral-preserving, order-preserving retraction

$$\sigma : K^n \rightarrow \bigcup_{j=0}^k B_j.$$

Let ρ be the retraction in Theorem 3.1 and define ρ_1 by

$$\rho_1(x) = \rho(x - \alpha_{k+1} u) + \alpha_{k+1} u.$$

Define a map r by

$$r(x) = \sigma(\rho_1(x)).$$

Because ρ_1 and σ are integral-preserving, order-preserving maps of K^n to K^n , r is also. We claim that r is a retraction of K^n onto $B' = \cup_{j=0}^{k+1} B'_j$.

If $x \in K^n$, we must prove that $r(x) \in B'$. If $\xi = \rho_1(x)$, we know that $\xi \geq \alpha_{k+1} u$ or $\xi \leq \alpha_{k+1} u$. In the first case, $\sigma(\xi) = \xi \in B'_{k+1}$, because σ leaves elements of B_k fixed. If $\xi \leq \alpha_{k+1} u$, we know that $\sigma(\xi) \in B$ and $\sigma(\xi) \leq \sigma(\alpha_{k+1} u) = \alpha_{k+1} u$, and these two facts imply that $\sigma(\xi) = r(x)$ is an element of $B \cup_{j=0}^k B'_j \subset B'$.

We leave to the reader the routine verification that $r|_{B'}$ is the identity map. By induction, the proof is complete. ■

Our immediate motivation for proving Theorems 3.1 and 3.2 is the following result.

THEOREM 3.3: *Suppose that for $1 \leq j \leq r$, $f_j : K^n \rightarrow K^n$ is a map which satisfies H3.1 (see Definition 3.1) and f_j has a periodic point x^j of period p_j . Then there exists a map $T : K^n \rightarrow K^n$ which satisfies H3.1 and has periodic points ξ^j of period p_j for $1 \leq j \leq r$. Furthermore, $\Omega(\xi^j, T) \cap \Omega(\xi^k, T)$ is empty for $1 \leq j < k \leq r$.*

Proof: Select a positive number β_j such that

$$x^j < \beta_j u \quad \text{for } 1 \leq j \leq r - 1.$$

Define $\alpha_0 = 0$ and $\alpha_k = \sum_{j=1}^k \beta_j$ for $1 \leq k \leq r - 1$. Define B_j by

$$B_j = \{x | \alpha_j u \leq x \leq \alpha_{j+1} u\} \quad \text{for } 0 \leq j < r - 1 \quad \text{and} \quad B_{r-1} = \{x | \alpha_{r-1} u \leq x\}.$$

Define $\xi^j = x^j + \alpha_{j-1} u$ for $1 \leq j \leq r$ and for $x \in B_{j-1}$ define

$$F(x) = f_j(x - \alpha_{j-1} u) + \alpha_{j-1} u, \quad 1 \leq j \leq r.$$

Because $B_j \cap B_k$ is either empty or consists of a single point of the form cu and because $f_j(cu) = cu$ for all $c \geq 0$ and $1 \leq j \leq r$, the map F is well-defined on $B = \cup_{j=0}^{r-1} B_j$. Using in addition the order-preserving property of f_j , one sees that $F(B_j) \subset B_j$ for $0 \leq j \leq r - 1$. It is also not hard to verify that F is integral-preserving and order-preserving as a map of B to B and that ξ^j is a periodic point of F of period p_j . By Theorem 3.2 there exists an integral-preserving, order-preserving retraction r of K^n onto B . If one defines $T = F \circ r$, T satisfies H3.1 and ξ^j is a periodic point of period p_j of T . ■

With the aid of Theorem 3.3 we can complete the proof that the upper bound $\varphi(n)$ is optimal for $1 \leq n \leq 24$.

THEOREM 3.4: *For each $n, 1 \leq n \leq 24$, there exists an integral-preserving, order-preserving map $T : K^n \rightarrow K^n$ such that*

$$T(cu) = cu \quad \text{for all } c \geq 0,$$

$u = (1, 1, \dots, 1)$, and T has a periodic point x of period $\varphi(n)$. (The function $\varphi(n)$ is defined in eq. (2.29) and tabulated in Table 2.1.)

Proof: We have already proved Theorem 3.4 for $1 \leq n \leq 11$ in Lemma 3.3.

For $n = 12$, note that $\varphi(12) = 120 = 2 \times (12 \times 5)$. By Lemma 3.1 and Theorem 3.3, it suffices to find a map $T : K^6 \rightarrow K^6$ which satisfies H3.1 and has periodic points y and z of period 12 and 5 respectively. We have already proved that there exists $f_1 : K^6 \rightarrow K^6$ which satisfies H3.1 and has a periodic point of period 12, and there is clearly a permutation $f_2 : K^6 \rightarrow K^6$ which has a periodic point of period 5. The existence of T now follows from Theorem 3.3.

Since $\varphi(13) = \varphi(12) = 120$, the existence of $T : K^{13} \rightarrow K^{13}$ which satisfies H3.1 and has a periodic point of period 120 follows from Lemma 3.2 and the corresponding result for $n = 12$.

Since $\varphi(14) = 168 = 2 \times (7 \times 12)$, it suffices by Lemma 3.1 and Theorem 3.3 to find maps f_1 and f_2 of K^7 into K^7 which satisfy H3.1 and possess periodic points of period 12 and 7 respectively. However, we already know there is such a map with periodic point of period 12 and there is a permutation f_2 which has a periodic point of period 7.

Since $\varphi(15) = 180 = 3 \times \text{lcm}(4, 5, 6)$, it suffices, with the aid of Lemma 3.1 and Theorem 3.3, to find maps f_1, f_2 and f_3 of K^5 into K^5 which satisfy H3.1 and have periodic points of periods 4, 5 and 6 respectively. Clearly, the maps f_1, f_2 and f_3 can be taken to be appropriate permutations.

For $n = 16$, we know that $\varphi(16) = 336 = 2 \times (7 \times 24)$, so by the same argument already used, it suffices to find maps f_1, f_2 of K^8 into K^8 which satisfy H3.1 and have periodic points of period 24 and 7 respectively. Again, we already know there is such a map f_1 , and the map f_2 can be taken to be an appropriate permutation.

Since $\varphi(17) = \varphi(18) = 420$, the usual argument using Lemma 3.2 shows that it suffices to find an example for $n = 17$. By Lemma 3.2 it suffices to find a map $f_1 : K^{10} \rightarrow K^{10}$ which satisfies H3.1 and has a periodic point of period 60 and a map $f_2 : K^7 \rightarrow K^7$ which satisfies H3.1 and has a periodic point of period 7. We already know that f_1 exists and f_2 can be taken to be an appropriate permutation.

We know that there is a map $f_1 : K^{12} \rightarrow K^{12}$ which satisfies H3.1 and has a periodic point of period 120, and there is clearly a permutation $f_2 : K^7 \rightarrow K^7$ which has a periodic point of period 7. It follows by Lemma 3.2 that there is a map $T : K^{19} \rightarrow K^{19}$ which satisfies H3.1 and has a periodic point of period $840 = \varphi(19)$.

Since $\varphi(n) = 1680$ for $20 \leq n \leq 23$, it suffices as usual to find a suitable map

$T : K^{20} \rightarrow K^{20}$. Because $1680 = 2 \times (40 \times 21)$, Lemma 3.1 and Theorem 3.3 show that it suffices to find maps f_1 and f_2 of K^{10} to K^{10} such that each map satisfies H3.1, f_1 has a periodic point of period 40 and f_2 has a periodic point of period 21. The map f_2 can be taken to be an appropriate permutation. To prove the existence of a map like f_1 , we apply Lemma 3.1 and Theorem 3.3 again. Since $40 = 2 \times (5 \times 4)$, it suffices to find maps g_1 and g_2 of K^5 into K^5 which satisfy H3.1 and possess periodic points of period 4 and 5 respectively. The maps g_1 and g_2 can be taken to be permutations of order 4 and order 5 respectively.

We know that $\varphi(24) = 2640 = 2 \times (120 \times 11)$, so by Lemma 3.1 and Theorem 3.3, it suffices to find maps f_1 and f_2 of K^{12} into itself which satisfy H3.1 and have periodic points of period 120 and 11 respectively. A map like f_1 is already known to exist and f_2 can be taken to be a permutation of order 11. ■

Remark 3.2: Lemmas 3.1 and 3.2 and Theorem 3.3 can be used in general to construct maps $T : K^n \rightarrow K^n$ which satisfy H3.1 and have periodic points of a given period $p \leq \varphi(n)$. In this way one obtains a lower bound for $\varphi(n)$. Thus, if $n = 28, 9240 = 2 \text{ lcm}(140, 33)$, and using Lemma 3.1 one finds that there exists a map $T : K^{28} \rightarrow K^{28}$ which satisfies H3.1 and has a periodic point of period 9240 if there exist maps $f_i : K^{14} \rightarrow K^{14}, i = 1, 2$ which satisfy H3.1 and have periodic points of period 140 and 33 respectively. There is a permutation map f_2 and in \mathbb{R}^{14} of order 3×11 . Also, there are permutation maps g_1 and g_2 in \mathbb{R}^7 of orders 10 and 7 respectively, so Lemma 3.1 implies that there is a map $f_1 : K^{14} \rightarrow K^{14}$ which satisfies H3.1 and has a periodic point of period $140 = 2 \text{ lcm}(10, 7)$. It follows that $\varphi(28) \geq 9240$, and with further effort one can actually prove that $\varphi(28) = 9240$. Similarly, one finds that $\varphi(48) \geq 2((2640) \times 17 \times 7) = 628320$.

■

Remark 3.3: For $n \geq 1$, let $P(n)$ denote the set of positive integers p such that there exists $T : K^n \rightarrow K^n$ which satisfies H3.1 and has a periodic point of period p . If $r \geq 2$ divides n and p_1, p_2, \dots, p_r are elements of $P(n/r)$, we know that $r \text{ lcm}(\{p_j | 1 \leq j \leq r\}) \in P(n)$. Similarly, if n_1 and n_2 are positive integers such that $n_1 + n_2 = n$ and $p_1 \in P(n_1)$ and $p_2 \in P(n_2)$, then $\text{lcm}(p_1, p_2) \in P(n)$. Finally, of course $n \in P(n)$. Is it true that every element of $P(n)$ arises in one of these ways? ■

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