EIGENVECTORS OF ORDER-PRESERVING LINEAR OPERATORS

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Abstract

Suppose that K is a closed, total cone in a real Banach space X, that $A: X \to X$ is a bounded linear operator which maps K into itself, and that A' denotes the Banach space adjoint of A. Assume that r, the spectral radius of A, is positive, and that there exist $x_0 \neq 0$ and $m \geqslant 1$ with $A^m(x_0) = r^m x_0$ (or, more generally, that there exist $x_0 \notin (-K)$ and $m \geqslant 1$ with $A^m(x_0) \geqslant r^m x_0$). If, in addition, A satisfies some hypotheses of a type used in mean ergodic theorems, it is proved that there exist $u \in K - \{0\}$ and $\theta \in K' - \{0\}$ with $A(u) = ru$, $A'(\theta) = r\theta$ and $\theta(u) > 0$. The support boundary of K is used to discuss the algebraic simplicity of the eigenvalue r. The relation of the support boundary to H. Schaefer's ideas of quasi-interior elements of K and irreducible operators A is treated, and it is noted that, if $\dim(X) > 1$, then there exists an $x \in K - \{0\}$ which is not a quasi-interior point. The motivation for the results is recent work of Toland, who considered the case in which X is a Hilbert space and A is self-adjoint; the theorems in the paper generalize several of Toland's propositions.

1. *Classical Krein–Rutman theory*

We recall some standard definitions and some classical variants of the Krein–Rutman Theorem.

By a *cone* K (with vertex at 0) in a real Banach space X we mean a convex set $K \subset X$ such that (i) $K \cap (-K) = \{0\}$, and (ii) $(\lambda K) \subset K$ for all $\lambda \geqslant 0$. We always assume that $K \neq \{0\}$. Here, $(-K) = \{-x : x \in K\}$ and $\lambda K = \{\lambda x : x \in K\}$. Usually, we assume that K is closed, and refer to a *closed cone*. A convex set $W \subset X$ is called a *wedge* if $(\lambda W) \subset W$ for all $\lambda \geqslant 0$, and a *closed wedge* if W is closed. A cone K in a Banach space X is called *total* (*in* X) if X is the norm closure of $\{x - y : x, y \in K\} := K - K$. A cone K in a Hilbert space is total if and only if $K^{\perp} = \{0\}$, where

$$K^{\perp} := \{z \mid \langle z, x \rangle = 0 \text{ for all } x \in K\}$$

and $\langle \cdot, \cdot \rangle$ denotes the inner product on H. A cone $K \subset X$ is *reproducing* (in X) if $X = \{x - y : x, y \in K\}$. A Banach space X is said to have the *bounded decomposition property* (with respect to a cone $K \subset X$) if, for every $x \in X$, there exist bounded sequences $\langle y_k : k \geqslant 1 \rangle \subset K$ and $\langle z_k : k \geqslant 1 \rangle \subset K$ such that

$$\lim_{k \to \infty} \|x - (y_k - z_k)\| = 0.$$

Clearly, if a cone $K \subset X$ is reproducing, then X has the bounded decomposition property.

A cone $K \subset X$ induces a partial ordering on X by $x \leqslant y$ if and only if $y - x \in K$. If $a, b \in X$ and $a \leqslant b$, we denote by $[a, b]$ the set $\{x \mid a \leqslant x \text{ and } x \leqslant b\}$. A cone K is called

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normal if there exists a constant M such that $\|x\| \leqslant M\|y\|$ whenever $0 \leqslant x \leqslant y$. If X is a real Banach space, we denote by X' the Banach space adjoint of X, that is, the set of continuous linear functionals $\theta: X \to \mathbb{R}$ with $\|\theta\|$ given by

$$\|\theta\| = \sup\{|\theta(x)| : \|x\| \leqslant 1\}.$$

If $K \subset X$ is a cone, we write

$$K' := \{\theta \in X' : \theta(x) \geqslant 0 \text{ for all } x \in K\}.$$

Note that K' is a closed wedge, and that K' is a cone if K is total. When $x \in X$ and $\theta \in X'$, we often write

$$\theta(x) = \langle x, \theta \rangle$$

and so $\langle \cdot, \cdot \rangle$ in this equation denotes the pairing between X and X', rather than an inner product. If X is a real Banach space and $A: X \to X$, then $A': X' \to X'$ denotes the Banach space adjoint of A, and so

$$\langle Ax, \theta \rangle = \langle x, A'\theta \rangle \quad \text{for all } x \in X, \theta \in X'. \tag{1.1}$$

We always denote by $r(A)$ the *spectral radius* of A, and so

$$r(A) = \lim_{n \to \infty} \|A^n\|^{1/n} = \inf_{n \geqslant 1} \|A^n\|^{1/n} = r(A'). \tag{1.2}$$

Recall that, if $\tilde{X} = X + \mathrm{i}X$ denotes the complexification of X, then, for $x, y \in X$, we define

$$\|x + \mathrm{i}y\| = \sup_{0 \leqslant \theta \leqslant 2\pi} \|(\cos\theta)\, x + (\sin\theta)\, y\|. \tag{1.3}$$

If X is a real Banach space, $A: X \to X$ is a bounded linear map, and $\tilde{X}$ denotes the complexification of X, then A extends to a bounded, complex linear map $\tilde{A}: \tilde{X} \to \tilde{X}$ by

$$\tilde{A}(x + \mathrm{i}y) = A(x) + \mathrm{i}A(y).$$

With respect to the norm given by (1.3), one can prove that

$$\|\tilde{A}\| = \|A\| \tag{1.4}$$

and so it follows that

$$r(\tilde{A}) = \lim_{n \to \infty} \|(\tilde{A})^n\|^{1/n} = \lim_{n \to \infty} \|A^n\|^{1/n} = r(A). \tag{1.5}$$

We write $\sigma(\tilde{A})$ for the spectrum of $\tilde{A}$, and define $\sigma(A) = \sigma(\tilde{A})$. We recall that

$$r(A) = r(\tilde{A}) = \sup\{|\lambda| : \lambda \in \sigma(\tilde{A})\}. \tag{1.6}$$

If K is a wedge in a real Banach space X and $A: X \to X$ is a bounded linear operator with $A(K) \subset K$, we write

$$\|A\|_K := \sup\{\|Ax\| : x \in K \text{ and } \|x\| \leqslant 1\}. \tag{1.7}$$

The same argument as that used to establish the usual spectral radius formula gives

$$\lim_{n \to \infty} (\|A^n\|_K)^{1/n} = \inf_{n \geqslant 1} (\|A^n\|_K^{1/n}) := r_K(A). \tag{1.8}$$

If K is a cone, we call $r_K(A)$ the *cone spectral radius* of A in K; in [**4**], it is called the *partial spectral radius*. If we define $Y = K - K$, where K is a closed cone in a Banach space X, Bonsall [**5**] has observed that Y is a Banach space in the norm defined by

$$|y| = \inf\{\|u\| + \|v\| : y = u - v \text{ and } u, v \in K\}. \tag{1.9}$$

Furthermore, one has $\| y\| = |y|$ for $y \in K$ and $\| y\| \leqslant |y|$ for all $y \in Y$. If B denotes the map A considered as a map from $(Y, |\cdot|)$ to $(Y, |\cdot|)$, then

$$\sup\{|By| : |y| \leqslant 1\} := |B| = \|A\|_K \quad \text{and} \quad r(B) = \lim_{n\to\infty} |B^n|^{1/n} = r_K(A). \tag{1.10}$$

We refer the reader to [**4**; **5**; **11**, Section 2] for further details.

We also need to recall the notions of *essential spectral radius* and *cone essential spectral radius*. If S is a bounded subset of a complete metric space (X, d), Kuratowski [**9**] has defined $\alpha(S)$, the measure of noncompactness of S, as

$$\alpha(S) = \inf\{\rho > 0 \mid S = \bigcup_{i=1}^{n} S_i, n < \infty, \text{ and diameter } (S_i) \leqslant \rho \text{ for } 1 \leqslant i \leqslant n\}. \tag{1.11}$$

Kuratowski observed that $\alpha(\bar{S}) = \alpha(S)$ for all $S, \alpha(S \cup T) = \max(\alpha(S), \alpha(T))$ and $\alpha(S) = 0$ if and only if $\bar{S}$ is compact. If S_n is a decreasing sequence of closed, bounded nonempty sets and $\lim_{n\to\infty} \alpha(S_n) = 0$, then Kuratowski proved that $\bigcap_{n\geqslant 1} S_n$ is compact and nonempty. If X is a Banach space, Darbo [**7**] observed that, for all bounded sets S and T in X and all real numbers λ,

$$\alpha(\overline{co}(S)) = \alpha(S), \alpha(S+T) \leqslant \alpha(S) + \alpha(T) \text{ and } \alpha(\lambda S) = |\lambda|\, \alpha(S). \tag{1.12}$$

Here $\overline{co}(S)$ denotes the smallest closed convex set containing S, and

$$S + T = \{s + t \mid s \in S, t \in T\}.$$

If X is a real Banach space and $A : X \to X$ is a bounded linear operator, then we define $\alpha(A)$ by

$$\alpha(A) = \inf\{\lambda \geqslant 0 \mid \alpha(A(S)) \leqslant \lambda\alpha(S) \text{ for all bounded } S \subset X\}. \tag{1.13}$$

One can easily show that $\alpha(A) \leqslant \|A\|$ and $\alpha(A + C) = \alpha(A)$ for any compact linear map C. As in [**10**], we define the *essential spectral radius* of A as

$$\rho(A) = \lim_{n\to\infty} (\alpha(A^n))^{1/n} = \inf_{n\geqslant 1} (\alpha(A^n))^{1/n}. \tag{1.14}$$

We refer the reader to [**10**] for further details. If $K \subset X$ is a cone (wedge) and $A(K) \subset K$, we define $\alpha_K(A)$ as

$$\alpha_K(A) = \inf\{\lambda \geqslant 0 \mid \alpha(A(S)) \leqslant \lambda\alpha(S) \text{ for all bounded } S \subset K\}. \tag{1.15}$$

Note that $\alpha(A) = 0$ if and only if A is compact, and $\alpha_K(A) = 0$ if and only if $A \mid K$ is compact. We define the *cone* (*wedge*) *essential spectral radius* as

$$\rho_K(A) = \lim_{n\to\infty} (\alpha_K(A^n))^{1/n} = \inf_{n\geqslant 1} (\alpha_K(A^n))^{1/n} \tag{1.16}$$

and refer the reader to [**11**, Section 2] for further details.

With the preliminaries, we can recall several variants of the Krein–Rutman Theorem.

THEOREM 1.1. ([**11**]). *Let K be a closed, total cone in a real Banach space X, and let $A : X \to X$ be a bounded linear operator with $A(K) \subset K$. Assume that $\rho(A) = \rho < r = r(A)$, where $\rho(A)$ denotes the essential spectral radius of A and $r(A)$ denotes the spectral radius of A. Then there exist nonzero elements $x \in K$ and $\theta \in K'$ such that $A(x) = rx$ and $A'(\theta) = r\theta$.*

Theorem 1.1 is a generalization of the original Krein–Rutman Theorem [**8**], in which it was assumed that A is compact (and so $\rho(A) = 0$).

THEOREM 1.2. ([**11**]). *Let K be a closed cone in a real Banach space X, and let $A: X \to X$ be a bounded linear operator with $A(K) \subset K$. Assume that $\rho_K(A) < r_K(A)$, where $\rho_K(A)$ denotes the cone essential spectral radius of A and $r_K(A)$ denotes the cone spectral radius of A. Then there exists a nonzero element $x \in K$ with $A(x) = r_K(A)\,x$.*

Theorem 1.2 generalizes Bonsall's theorem [**4**, Theorem 1.1], in which it is assumed that $A \mid K$ is compact, and so $\rho_K(A) = 0$. In [**4**, Section 8], Bonsall gives interesting examples which show that, even if K is a closed, total cone and $A \mid K$ is compact, A may fail to be compact on X and $r_K(A)$ may have different values for different closed total cones. However, Theorem 1.1 shows that this can only occur if $\rho(A) = r(A)$.

THEOREM 1.3. (*Bonsall* [**4**]). *Let K be a closed, normal cone in a real Banach space X, and assume that X has the bounded decomposition property. If $A: X \to X$ is a bounded linear operator such that $A(K) \subset K$, then $r(A)$, the spectral radius of A, is in $\sigma(A)$, the spectrum of A.*

In [**4**, Section 2], Bonsall has given an elegant example which shows that Theorem 1.3 may fail for a non-normal cone with a nonempty interior. By slightly modifying Bonsall's example and working in the Hilbert space of square integrable analytic functions on the open unit disc in $\mathbb{C}$, one obtains a closed reproducing cone K in a Hilbert space H and a bounded linear operator $A: H \to H$ with $A(K) \subset K$ and $r(A) \notin \sigma(A)$. These negative examples increase the interest of a recent result by Toland [**15**]. Toland proves that, if X is a Hilbert space, K is a closed, total cone in X and $A: X \to X$ is a bounded, self-adjoint linear map with $A(K) \subset K$, then $r(A) \in \sigma(A)$. See [**17**] for further results in this direction.

Notice that Theorem 1.3 is different in character from Theorem 1.1 and Theorem 1.2. Under the hypotheses of Theorem 1.3, $r(A)$ may fail to be an eigenvalue of A.

2. *Existence and uniqueness of positive eigenvectors*

If K is a total cone in a real Banach space X, and $A: X \to X$ is a bounded linear operator such that $A(K) \subset K$, $\rho(A) = r(A)$ and $\rho_K(A) = r_K(A)$, the previously cited theorems give no information about the existence of an eigenvector $u \in K$ with eigenvalue $r(A)$ or $r_K(A)$. Recently, Toland [**15**] has given a self-contained treatment of Krein–Rutman theory for self-adjoint operators in Hilbert space, and has incidentally shed some light on the existence of positive eigenvectors when $\rho(A) = r(A)$ or $\rho_K(A) = r_K(A)$. We show here that self-adjointness and Hilbert space structure are actually irrelevant for much of the work in [**15**], and that they can be replaced by hypotheses familiar from the study of mean ergodic theorems [**16**, pp. 213–215]. However, Toland's previously cited refinement of Theorem 1.3 involves subtler questions, and falls outside the purview of our remarks.

If $A: X \to X$ is a bounded linear operator on a Banach space X, I denotes the identity operator and $\lambda \notin \sigma(A)$, we write

$$(\lambda I - A)^{-1} = R(\lambda, A). \tag{2.1}$$

We recall a definition used by H. Schaefer (see [**14**, Definition 4.7, p. 326]).

DEFINITION 2.1. If X is a Banach space and $A: X \to X$ is a bounded linear operator with spectral radius $r(A)$, A is said to satisfy the *growth condition* (G) if the family of operators $(\lambda - r(A))\, R(\lambda, A)$ is uniformly bounded in operator norm for $\lambda > r(A)$.

Sometimes it will be convenient to use a strengthening of growth condition (G).

DEFINITION 2.2. If X is a Banach space and $A: X \to X$ is a bounded linear operator with spectral radius $r(A) = r$, A is said to satisfy *growth condition* $(G1)$ if there exists a constant M such that

$$\|A^n\| \leqslant Mr^n \quad \text{for all } n \geqslant 1. \tag{2.2}$$

If A satisfies growth condition (G1), it clearly satisfies growth condition (G). In fact, for $\lambda > r = r(A)$, we have

$$(\lambda - r)\, R(\lambda, A) = (\lambda - r)\, \lambda^{-1} \sum_{k=0}^{\infty} \lambda^{-k} A^k$$

and so

$$\|(\lambda - r)\, R(\lambda, A)\| \leqslant (\lambda - r)\, \lambda^{-1} \sum_{k=0}^{\infty} \lambda^{-k} M r^k = M.$$

If X happens to be a Hilbert space and A is a normal operator, it is well known that $\|A\| = r(A) = r$ and $\|A^n\| = r^n$ for all $n \geqslant 1$, and so growth condition (G1) is satisfied with $M = 1$.

We also need to recall the definition of a support point and a support boundary of a closed, convex set.

DEFINITION 2.3. Let C be a closed, convex set in a real Banach space X. A point $x \in C$ is called a *support point* of C if there exist a nonzero continuous linear functional $h \in X'$ and a real number α such that $h(x) = \alpha$ and $h(y) \geqslant \alpha$ for all $y \in C$. The support boundary of C is the union of all support points of C.

It is clear that the support boundary of C is contained in the boundary of C, but in general the sets are unequal (consider the cone C of nonnegative functions in $L^p[0, 1], 1 \leqslant p < \infty$). Bishop and Phelps **[1]** have proved that the support boundary of C is always dense in the boundary of C.

With these preliminaries, we can prove our first theorem. In the following, recall that a subadditive functional on a real vector space X is a map $q: X \to \mathbb{R}$ such that, for all $x, y \in X$ and all $\lambda \geqslant 0$, one has

$$q(x+y) \leqslant q(x) + q(y) \quad \text{and} \quad q(\lambda x) = \lambda q(x). \tag{2.3}$$

THEOREM 2.1. *Let K be a closed cone in a real Banach space X, and let $A: X \to X$ be a bounded linear operator such that $A(K) \subset K$. Assume the following:*

(1) $r = r(A) > 0$.
(2) *There exists an $x_0 \in X$ with $-x_0 \notin K$ and $Ax_0 \geqslant rx_0$.*
(3) *A satisfies growth condition* (G) (*see Definition* 2.1).

Then there exists a $\theta \in K' - \{0\}$ such that $\langle x_0, \theta \rangle > 0$ and $A'(\theta) = r\theta$.

Proof. Define $q_1: X \to \mathbb{R}$ by

$$q_1(x) = \inf\{\|x+y\| : y \in K\}. \tag{2.4}$$

We leave it to the reader to verify that q_1 is a subadditive functional, that $q_1(x) = 0$ for all $x \in (-K)$, that $q_1(u) \leqslant q_1(v)$ for all $u, v \in X$ with $u \leqslant v$, and that $0 \leqslant q_1(x) \leqslant \|x\|$ for all x. Because K is closed and $x_0 \notin (-K)$, we see that $q_1(x_0) > 0$.

We next define a map $q : X \to \mathbb{R}$ as

$$q(x) = \limsup_{\lambda \to r^+} q_1((\lambda - r)\, R(\lambda, A)\, x). \tag{2.5}$$

Because A satisfies growth condition (G), there exists a constant M such that, for all $\lambda > r$,

$$(\lambda - r)\, \|R(\lambda, A)\| \leqslant M \tag{2.6}$$

and this implies that, for all x,

$$0 \leqslant q(x) \leqslant M\|x\|. \tag{2.7}$$

Because $(\lambda - r)\, R(\lambda, A)\,(K) \subset K$ for all $\lambda > r$, we also see that

$$q(x) = 0 \quad \text{for all } x \in (-K) \tag{2.8}$$

and, for all $u, v \in X$ with $u \leqslant v$,

$$q(u) \leqslant q(v). \tag{2.9}$$

Our assumptions about A imply that, for all $k > 0$,

$$A^k x_0 \geqslant r^k x_0. \tag{2.10}$$

If $\lambda > r$, it follows from (2.10) that

$$(\lambda - r)\, R(\lambda, A)\, x_0 \geqslant (\lambda - r)\, \lambda^{-1} \sum_{k=1}^{\infty} \lambda^{-k} r^k x_0 = x_0. \tag{2.11}$$

It follows from (2.11) that, for $\lambda > r$,

$$\begin{aligned} q_1((\lambda - r)\, R(\lambda, A)\, x_0) &\geqslant q_1(x_0) > 0 \\ q(x_0) &= \limsup_{\lambda \to r} q_1(\lambda - r)\, R(\lambda, A)\, x_0) \geqslant q_1(x_0). \end{aligned} \tag{2.12}$$

Finally, we leave it to the reader to deduce from (2.5) and the fact that q_1 is a subadditive functional that q is a subadditive functional.

We claim that, if $y = rx - Ax$ for some $x \in X$, then $q(y) = 0$. To see this, note that, for $\lambda > r$,

$$(\lambda - r)\, R(\lambda, A)\, y = (\lambda - r)\, x + (r - \lambda)\,(\lambda - r)\, R(\lambda, A)\, x. \tag{2.13}$$

It follows from (2.6) and (2.13) that, for $\lambda > r$,

$$\begin{aligned} q_1((\lambda - r)\, R(\lambda, A)\, y) &\leqslant \|(\lambda - r)\, x + (r - \lambda)\,(\lambda - r)\, R(\lambda, A)\, x\| \\ &\leqslant (\lambda - r)\, \|x\| + (\lambda - r)\, M\|x\| \end{aligned} \tag{2.14}$$

which implies that $q(y) = 0$. Note that this argument also shows that $q(-y) = 0$.

We now apply the Hahn–Banach Theorem (see [**6**, p. 78]) with the subadditive functional q. On the linear span of x_0, define a linear functional ϕ by

$$\phi(\beta x_0) = \beta q(x_0).$$

If $\beta \geqslant 0$, it is immediate that $\phi(\beta x_0) \leqslant q(\beta x_0)$, while we have $\phi(\beta x_0) < 0 \leqslant q(\beta x_0)$ if $\beta < 0$. It follows from the Hahn–Banach Theorem that ϕ can be extended to a linear map $\theta : X \to \mathbb{R}$ such that

$$\theta(x) \leqslant q(x) \quad \text{for all } x.$$

By using (2.7), we see that

$$\theta(x) \leqslant q(x) \leqslant M\|x\| \quad \text{and} \quad \theta(-x) = -\theta(x) \leqslant q(-x) \leqslant M\|-x\|$$

and so $|\theta(x)\| \leqslant M\|x\|$ for all x, and θ is continuous. If $x \leqslant 0$, we have

$$\theta(x) \leqslant q(x) \leqslant 0$$

and so $\theta \in K'$. Also, we see that $\theta(x_0) = q(x_0) > 0$, and so $\theta \neq 0$. If $y = rx - Ax$, then

$$\langle y, \theta \rangle \leqslant q(y) = 0.$$

As already observed, $q(-y) = 0$ also, and so

$$\langle -y, \theta \rangle \leqslant q(-y) = 0.$$

We conclude that, for all $x \in X$,

$$\langle rx - Ax, \theta \rangle = 0$$

and so $A'(\theta) = r\theta$.

The proof of Theorem 2.1 is in the spirit of an argument on [**4**, p. 66].

If K is total in Theorem 2.1, $\theta(x) > 0$ for some $x \in K$. Otherwise, the facts that $\theta \in K'$ and K is total would imply that $\theta = 0$, a contradiction.

As an immediate consequence of Theorem 2.1, we obtain a more general-seeming theorem. In Theorem 2.2, note that the spectral mapping theorem implies that $(r(A))^m = r(A^m)$.

THEOREM 2.2. *Let K be a closed, total cone in a real Banach space X, and let $A: X \to X$ be a bounded linear map with $A(K) \subset K$. Assume the following:*

(1) $r = r(A) > 0$.

(2) *There exist $x_0 \in X$ and an integer $m \geqslant 1$ with $-x_0 \notin K$ and $A^m x_0 \geqslant r^m x_0$.*

(3) *A^m satisfies growth condition (G) (see Definition 2.1).*

Then there exists a $\theta \in K' - \{0\}$ with $A'(\theta) = r\theta$.

Proof. If we apply Theorem 2.1 to A^m, we see that there exists a $\psi \in K' - \{0\}$ with $\langle x_0, \psi \rangle > 0$ and $(A')^m(\psi) = r^m \psi$. Furthermore, because K is total, there exists a $u \in K$ with $\langle u, \psi \rangle > 0$. Define $\theta \in X'$ by

$$\theta = \sum_{j=0}^{m-1} r^{-j} (A')^j(\psi).$$

Because $(A')(K') \subset K'$, we see that $\theta \in K'$; and

$$\langle u, \theta \rangle \geqslant \langle u, \psi \rangle > 0$$

and so $\theta \neq 0$. An easy calculation shows that $A'(\theta) = r\theta$.

We also want to prove, under hypotheses such as those of Theorem 2.2, that A has a positive eigenvector corresponding to the eigenvalue $r(A)$. First, we need some more definitions.

DEFINITION 2.4. Let C be a closed, convex subset of a Banach space X, and let $T: X \to X$ be a bounded linear operator. We shall say that T has the *weak properness* *property* on C if, whenever $\langle x_n : n \geqslant 1 \rangle \subset C$ is a bounded sequence in C with $\lim_{n\to\infty} \| T(x_n) \| = 0$, then there exists a subsequence $\langle x_{n_j} \rangle$ which converges in the weak topology on X.

LEMMA 2.1. *Let K be a closed wedge in a Banach space X, and let $A: X \to X$ be a bounded linear operator with $A(K) \subset K$. Let $r = r_K(A)$ be defined by* (1.8), *and let $\rho = \rho_K(A)$ be defined by* (1.16). *If* (i) *X is reflexive, or* (ii) *every closed, bounded convex subset of K is weakly compact, or* (iii) *$\rho_K(A) < r_K(A)$, then $rI - A$ has the weak properness property on K.*

Proof. Assertions (i) and (ii) follow immediately from the Eberlein–Smulyan Theorem [**16**, p. 141]. Assertion (iii) follows by a standard argument, but we give the proof for completeness. Assume that $\langle x_n : n \geqslant 1 \rangle$ is a bounded sequence in K, and that $\lim_{n\to\infty} \| r x_n - A x_n \| = 0$. Define $B = r^{-1} A$ and $y_n = x_n - B x_n$, so that $\lim_{n\to\infty} \| y_n \| = 0$. By the definition of $\rho_K(A)$, there exists an integer $m \geqslant 1$ and a number λ with $0 \leqslant \lambda < 1$ such that $\alpha_K(B^m) \leqslant \lambda$ (see (1.15)). An easy induction yields

$$x_k = B^m x_k + \sum_{j=0}^{m-1} B^j y_k. \tag{2.15}$$

If we define $\Sigma = \{x_k : k \geqslant 1\}$, $\Gamma = \{y_k : k \geqslant 1\}$ and $L = \sum_{j=0}^{m-1} B^j$, then Γ has compact closure, and (2.15) implies that

$$\Sigma \subset B^m(\Sigma) + L(\Gamma). \tag{2.16}$$

Using the Kuratowski measure of noncompactness α, we derive from (2.16)

$$\alpha(\Sigma) \leqslant \alpha(B^m(\Sigma)) + \alpha(L(\Gamma)) = \alpha(B^m(\Sigma)) \leqslant \lambda \alpha(\Sigma).$$

Since $\lambda < 1$, we conclude that $\alpha(\Sigma) = 0$ and Σ has compact closure. Thus there exists a subsequence n_i such that $\langle x_{n_i} \rangle$ converges in the norm topology.

DEFINITION 2.5. Let C be a closed, convex subset of a Banach space X, and let $B: X \to X$ be a bounded linear operator with $B(C) \subset C$. We shall say that B has the *fixed point property* on C if, whenever $D \subset C$ is a closed, bounded convex subset of C with $B(D) \subset D$, then B has a fixed point in D.

As is suggested by Lemma 2.2, the weak properness property and the fixed point property are closely related.

LEMMA 2.2. *Let K be a wedge in a Banach space X, and let $A: X \to X$ be a bounded linear operator with $A(K) \subset K$. Assume the following:*

(1) $r = r(A) > 0$.

(2) *A satisfies growth condition* ($G1$).

(3) *$rI - A$ satisfies the weak properness property on K.*

Then $B := r^{-1} A$ has the fixed point property on K.

Proof. Let $D \subset K$ be a closed, bounded convex set with $B(D) \subset D$. Because A satisfies the growth condition (G1), there exists a constant M such that, for any $x \in X$,

$$\sup\{\| B^n x \| : n \geqslant 0\} \leqslant M \| x \|. \tag{2.17}$$

We define a norm $|\cdot|$ on X by

$$|x| := \sup\{\|B^n x\| : n \geqslant 0\}. \tag{2.18}$$

The reader can verify that $|\cdot|$ is indeed a norm, and (2.17) implies that $|\cdot|$ and $\|\cdot\|$ are equivalent. Furthermore, it is immediate from (2.18) that, for all $x \in X$,

$$|Bx| \leqslant |x|$$

and so $|B| \leqslant 1$.

Now select $u \in D$, and, for each integer $n \geqslant 1$, define a map $B_n : D \to D$ by

$$B_n(x) = n^{-1}u + (1-n^{-1})\, B(x).$$

Because $|B| \leqslant 1$, one can see that B_n is a Lipschitz map with Lipschitz constant less than or equal to $1-n^{-1}$. The contraction mapping principle implies that B_n has a fixed point x_n, and so

$$x_n - r^{-1}A(x_n) = n^{-1}u - n^{-1}B(x_n).$$

Because $B(x_n) \in D$, and we assume that D is bounded, we conclude that

$$\lim_{n\to\infty} \|rx_n - Ax_n\| = 0.$$

The weak properness property implies that, by taking a subsequence, we can assume that $x_n \rightharpoonup x$. Because D is closed and convex, it is weakly closed, and $x \in D$. Because $rI - A$ is continuous in the weak topology on D, then $rx - Ax = 0$, and the proof is complete.

THEOREM 2.3. *Let the assumptions and notation be as in Theorem* 2.2. *In addition, assume that $rI - A$ satisfies the weak properness property on K* (*see Definition* 2.4). *Then there exist $\theta \in K' - \{0\}$ with $A'(\theta) = r\theta$ and $y \in K - \{0\}$ with $A(y) = ry$ and $\langle y, \theta\rangle > 0$.*

Proof. We have already proved the existence of θ. Select $x \in K$ with $\langle x, \theta\rangle > 0$. For each integer $k \geqslant 1$, define $\lambda_k = r + k^{-1}$ and define x_k as

$$x_k = (\lambda_k - r)\, R(\lambda_k, A)\, x.$$

Because $\lambda_k > r$, $x_k \in K$, and growth condition (G) implies that $\langle x_k\rangle$ is a bounded sequence. A calculation gives

$$\langle x_k, \theta\rangle = \langle x, (\lambda_k - r)\, R(\lambda_k, A')\, \theta\rangle = \langle x, \theta\rangle.$$

Using (2.13), we see that

$$(rI - A)\, x_k = (\lambda_k - r)\, x + (r - \lambda_k)\, x_k$$

and so we conclude that

$$\lim_{k\to\infty} \|rx_k - Ax_k\| = 0.$$

It follows from the weak properness property that there exists a weakly convergent subsequence $x_{k_i} \rightharpoonup y \in K$ and that

$$\langle y, \theta\rangle = \lim_{i\to\infty} \langle x_{k_i}, \theta\rangle = \langle x, \theta\rangle > 0.$$

Finally, we see that

$$0 = \lim_{i\to\infty} rx_{k_i} - Ax_{k_i} = ry - Ay.$$

REMARK 2.1. Instead of it being assumed that a point x_0 exists as in Theorem 2.2(2), suppose that there exist $m \geqslant 1$ and $u \neq 0$ so that $A^m u = r^m u$. If $-u \notin K$, we can take $x_0 = u$, and otherwise we can take $x_0 = -u$. Thus the conclusions of Theorem 2.2 and Theorem 2.3 are satisfied if, instead of assuming the existence of x_0 as in Theorem 2.2, we assume that, for some m, A^m has an eigenvector u with eigenvalue r^m. If $\tilde{X}$ denotes the complexification of X, and $\tilde{A}$ is the usual extension of A to $\tilde{X}$, then, if there exists an eigenvalue λ of $\tilde{A}$ such that $\lambda^m = r^m$, where $r = r(A)$ and $m \geqslant 1$, then it follows that $(\tilde{A})^m(w) = r^m w$ for some $w \in \tilde{X}$, $w \neq 0$. This in turn implies that there exists a $u \in X$, $u \neq 0$, with $A^m u = r^m u$.

If X is a real Banach lattice, Remark 2.1 can be refined. As usual, $\tilde{X}$ denotes the complexification of X.

COROLLARY 2.1. *Let X be a real Banach lattice in the ordering induced by a closed cone $K \subset X$, and assume that the absolute value on X extends to $\tilde{X}$ (see* [**13**, *p.* 274]). *Let $A: X \to X$ be a bounded linear operator with $A(K) \subset K$. Assume the following:*

(1) $r(A) > 0$.
(2) *There exists an eigenvalue $\lambda \in \sigma(A) = \sigma(\tilde{A})$ with $|\lambda| = r(A)$.*
(3) *A satisfies growth condition (G).*
(4) *$rI - A$ satisfies the weak properness property on K.*

Then there exist $\theta \in K' - \{0\}$ with $A'(\theta) = r\theta$ and $y \in K - \{0\}$ with $Ay = ry$ and $\langle y, \theta \rangle > 0$.

Proof. By Theorem 2.3, it suffices to prove that there exists an $x_0 \in X$ which satisfies Theorem 2.2(2). If $\tilde{A}$ is the complexification of A, then by assumption there exist $u \in \tilde{X}$, $u \neq 0$, and $\lambda \in \mathbb{C}$, $|\lambda| = r$, with $\tilde{A}(u) = \lambda u$. It follows that

$$|\tilde{A}(u)| = |\lambda|\,|u| \leqslant \tilde{A}(|u|) = A(|u|).$$

Taking $x_0 = |u|$, we see that Theorem 2.2(2) is satisfied.

The hypotheses of Theorem 2.1, Theorem 2.2 and Theorem 2.3 are close to optimal, as is indicated by the following examples.

EXAMPLE 2.1. Let l_1 denote the real Banach space of absolutely summable sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$. If $x \in l_1$, we write

$$\|x\|_1 = \sum_{j=1}^{\infty} |x_j|.$$

Define $X = \mathbb{R} \times l_1$. X is a real Banach space, and, for $(s, x) \in X$, we define

$$\|(s, x)\| = |s| + \|x\|_1.$$

Define a set $K \subset X$ by

$$K = \{(s, x) \in X : |s| \leqslant \|x\|_1 \text{ and } x_i \geqslant 0 \text{ for } 1 \leqslant i < \infty, \text{ where } x = (x_1, x_2, \ldots)\}.$$

We leave to the reader the verification that K is a closed cone. If $(s, x) \in K$, then $(-s, x) \in K$, and so $K - K$ contains all points of the form $(2s, 0)$, $s \in \mathbb{R}$. Similarly, one can verify that $K - K$ contains all points of the form $(0, x)$ for $x \in l_1$, and one concludes that $X = K - K$. The reader can also verify that K is normal.

Define U to be the shift to the right on l_1:

$$U(x) = (0, x_1, x_2, \dots, x_k, \dots) \quad \text{where } x = (x_1, x_2, \dots, x_k, \dots).$$

Define $A: X \to X$ by

$$A(s, x) = (s, Ux).$$

We leave to the reader the verification that $A(1,0) = (1,0)$ and that $\|A^n\| = 1$ for all $n \geqslant 1$. It follows that $r(A) = 1$, A satisfies growth condition (G1), and all hypotheses of Theorem 2.2 and Theorem 2.1 are satisfied, and so there exists a $\theta \in K' - \{0\}$ with $A'(\theta) = \theta$ and $\theta((1,0)) > 0$. Recall that X' can be isometrically identified with $\mathbb{R} \times l_\infty$, where l_∞ is the real Banach space of bounded sequences in the sup norm and, for $(t, y) \in \mathbb{R} \times l_\infty$,

$$\|(t, y)\| = \max(|t|, \|y\|_{l_\infty}).$$

If $(t, y) \in \mathbb{R} \times l_\infty$, then (t, y) gives an element of X' as follows:

$$((t, y))(s, x) = st + \sum_{i=1}^{\infty} x_i y_i.$$

Define $V: l_\infty \to l_\infty$ to be the shift to the left, so that

$$V(y) = (y_2, y_3, \dots, y_k, \dots) \quad \text{where } y = (y_1, y_2, \dots, y_k, \dots).$$

With the above-mentioned identification of $\mathbb{R} \times l_\infty$ and X', we have

$$A'(t, y) = (t, V(y)).$$

If e is the element of l_∞ all of the components of which equal 1, and $\theta = (1, e)$, one can verify that $\theta((1,0)) > 0$, $\theta \in K'$ and $A'(\theta) = \theta$, as is insured by Theorem 2.1. Note, however, that $(1,0)$ is also a fixed point of A' but that $(1,0) \notin K'$.

However, $I - A$ does *not* have the weak properness property on K, and there does *not* exist a $u \in K - \{0\}$ with $A(u) = u$. Indeed, if $u = (s, x) \in K - \{0\}$ and $Au = u$, we find that

$$(s, x) = A((s, x)) = (s, Ux).$$

This implies that $x = Ux$, from which one sees that $x = 0$. Because $|s|$ is smaller than or equal to $\|x\|$, for $(s, x) \in K$, we conclude that $s = 0$, which contradicts $u \neq 0$. This shows that the weak properness property, or something like it, is necessary in Theorem 2.3.

If $x = (1, 0)$ in this example, one can verify that $d(x, K)$, the distance of x to K, equals $\|x\|$, and $d(x, -K) = \|x\|$. Such an example is impossible if the map $y \to \|y\|$ is Fréchet-differentiable away from 0. One can prove that, if K is a closed, total cone in a real Banach space X, $x \neq 0$, and the map $y \to \|y\|$ is Fréchet-differentiable away from 0, then either $d(x, K) < \|x\|$ or $d(x, -K) < \|x\|$.

In our next example, we show that, if all hypotheses of Theorem 2.1 except growth condition (G) are satisfied, then, in the notation of Theorem 2.1, it may not be possible to select a positive eigenvector θ of A' with $\langle x_0, \theta \rangle > 0$.

EXAMPLE 2.2. For $i \geqslant 1$, define $\varepsilon_i = i^{-1}$, and note that

$$\prod_{i=1}^{n} (1 + \varepsilon_i) = n + 1 \quad \text{and} \quad \lim_{n \to \infty} (n+1)^{1/n} = 1. \tag{2.19}$$

Define a weighted shift operator $U: l_1 \to l_1$ by

$$U(x) = (0, (1+\varepsilon_1)x_1, (1+\varepsilon_2)x_2, \ldots, (1+\varepsilon_k)x_k, \ldots).$$

Let $X = \mathbb{R} \times l_1$, let K be defined as in Example 2.1, and define $A: X \to X$ by $A(s, x) = (s, Ux)$. The reader can verify that $A(K) \subset K$. If $x \in l_1$ and $U^k(x) = y$, one can see that $y_j = 0$ for $j \leqslant k$ and

$$y_{j+k} = x_j \left(\prod_{i=j}^{j+k-1} (1+\varepsilon_i) \right) \quad j \geqslant 1. \tag{2.20}$$

It follows from (2.20) that

$$\| U^k \| = \prod_{i=1}^{k} (1+\varepsilon_i) = k+1$$

and so $r(A) = 1$, but A does not satisfy growth condition (G1). Also, using (2.20), one can see that, for any nonzero $x \in l_1$, $\lim_{k\to\infty} \| U^k x \| = \infty$, and this implies that A has no nonzero fixed point in K.

Define $V: l_\infty \to l_\infty$ by

$$V(y) = ((1+\varepsilon_1)y_2, (1+\varepsilon_2)y_3, \ldots, (1+\varepsilon_k)y_{k+1}, \ldots) \quad \text{where } y = (y_1, y_2, \ldots).$$

Identifying X' with $\mathbb{R} \times l_\infty$, one can see that

$$A'((t, y)) = (t, V(y)).$$

If $A'(t, y) = (t, y)$, the formula for V implies that

$$y_j = \left(\frac{1}{j}\right) y_1 \quad \text{for } j \geqslant 2. \tag{2.21}$$

If $t = 0$, $y_1 > 0$ and y_j is given by (2.21), then $A'(t, y) = (t, y)$ and $(t, y) \in K'$, but $((t, y))(1, 0) = 0$. If $A'(t, y) = (t, y)$ and $t > 0$, then $((t, y))(1, 0) = t > 0$ and (2.21) is satisfied. However, if we select j so large that $y_1 j^{-1} < t$ and define $x \in l_1$ by $x_i = 0$ for $i \neq j$ and $x_j = 1$, we see that $(-1, x) \in K$ and $((t, y))(-1, x) = -t + y_1 j^{-1} < 0$, and so $(t, y) \notin K'$. We have proved that there does not exist a $\theta \in K' - \{0\}$ such that $A'(\theta) = \theta$ and $\theta((1, 0)) > 0$, and, comparing Theorem 2.1, we conclude that A does not satisfy growth condition (G) (as can be seen directly).

Theorem 2.1, Theorem 2.2 and Theorem 2.3 concern the existence of positive eigenvectors. One can also ask about the uniqueness of such eigenvectors. Classically, this question has been studied via the concept of *irreducibility* (see [**13**, pp. 269, 270]). Recall that, if K is a closed cone in a Banach space X, then a point $z \in K$ is called a *quasi-interior point* of K if X is the closed linear span of $[0, z]$. The reader can verify that, if z is a quasi-interior point of K, then z is not in the support boundary of K. If $A: X \to X$ is a bounded linear operator with $A(K) \subset K$ and $r = r(A)$, then A is called irreducible if, for every $\lambda > r$ and every $x \in K - \{0\}$, $AR(\lambda, A)x$ is a quasi-interior point of K. Note that, if $x \in K$ is an eigenvector of A with eigenvalue r, $\lambda > r$, and A is irreducible, then $(\lambda - r)\lambda^{-1}[x + AR(\lambda, A)x] = x$ is a quasi-interior point, and therefore x is not in the support boundary of K.

If K is a closed, total cone in a Banach space X of dimension greater than 1, then the previously mentioned theorem of Bishop and Phelps [**1**] asserts that the support points of K are dense in the boundary of K. Since no support point can be a quasi-interior point of K, it follows that K contains nonzero points which are not quasi-interior points. Surprisingly, this simple observation seems new. As is remarked on

[**13**, p. 270], the existence of nonzero elements of K which are not quasi-interior points leads to a refinement of a uniqueness result for positive eigenvectors (see [**13**, Theorem 3.2, p. 270]). Thus the support boundary is useful even in the 'classical' approach to the uniqueness of positive eigenvectors. Here we shall generalize irreducibility by directly assuming that our positive operator A has no eigenvector which lies in the support boundary of K and has eigenvalue $r(A)$, and we shall exploit this assumption to prove the uniqueness of positive eigenvectors.

If X is a Banach space and $B: X \to X$ is a bounded linear operator, we denote the null space of B by $N(B)$:

$$N(B) = \{x \in X \mid Bx = 0\}. \tag{2.22}$$

If λ is an eigenvalue of B, recall that λ is called algebraically simple if $\bigcup_{k \geq 1} N((\lambda I - B)^k)$ is 1-dimensional.

Lemma 2.3 is a simple exercise.

LEMMA 2.3. *Let X be a Banach space, and let $A: X \to X$ be a bounded linear operator which satisfies growth condition* (*G*) (*see Definition* 2.1). *Then, if $r = r(A)$, we have*

$$\bigcup_{k=1}^{\infty} N((rI - A)^k) = N(rI - A).$$

Proof. Suppose that this is not true. Then there exist $x \neq 0$ and y with

$$ry - Ay = x \quad \text{and} \quad rx - Ax = 0.$$

By a simple induction argument, we find that

$$A^k y = r^k y - k r^{k-1} x \quad \text{for } k \geq 1. \tag{2.23}$$

If $\lambda > r$, (2.23) implies that

$$\begin{aligned}(\lambda - r) R(\lambda, A) y &= (\lambda - r) \lambda^{-1} \sum_{k=0}^{\infty} \lambda^{-k} A^k y = (\lambda - r) \lambda^{-1} \sum_{k=0}^{\infty} \lambda^{-k} [r^k y - k r^{k-1} x] \\ &= y - (\lambda - r)^{-1} x.\end{aligned}$$

It follows that

$$\limsup_{\lambda \to r^+} \|(\lambda - r) R(\lambda, A) y\| = \limsup_{\lambda \to r^+} (\lambda - r)^{-1} \|(\lambda - r) y - x\| = \infty$$

which contradicts growth condition (G).

We shall also need a simple geometric lemma.

LEMMA 2.4. *Let K be a closed cone in a Banach space X. Suppose that x_0 and x_1 are distinct vectors with $x_1 \in K$. If we define $x_t = (1-t)x_0 + t x_1$, then either* (i) *there exists a $\tau < \infty$ such that $x_\tau \in K$ and $x_t \notin K$ for all $t > \tau$, or* (ii) *there exists a $\sigma > -\infty$ such that $x_\sigma \in K$ and $x_t \notin K$ for all $t < \sigma$.*

Proof. Define $\tau = \sup\{t : x_t \in K\}$ and $\sigma = \inf\{t : x_t \in K\}$. Because $x_1 \in K$, then τ and σ are well defined. We must prove that $\tau < \infty$ or $\sigma > -\infty$. If $\tau = +\infty$, there exists a sequence $t_k \to +\infty$ with $x_{t_k} \in K$. It follows that $\lim_{k \to \infty} t_k^{-1} x_{t_k} = (x_1 - x_0) \in K$. If

$\sigma = -\infty$, a similar argument shows that $x_0 - x_1 \in K$. If $u = (x_1 - x_0)$, then we have assumed that $u \neq 0$, and we have shown that $\pm u \in K$ if $\sigma = -\infty$ and $\tau = +\infty$. This contradicts the fact that K is a cone, and so $\sigma > -\infty$ or $\tau < \infty$.

THEOREM 2.4. *Let K be a closed, total cone in a real Banach space X, and let $A: X \to X$ be a bounded linear operator with $A(K) \subset K$. Assume the following:*

(1) $r(A) = r > 0$.

(2) *There exist $y \in X$ and $m \geqslant 1$ with $-y \notin K$ and $A^m y \geqslant r^m y$.*

(3) *A satisfies the growth condition (G1) (see Definition 2.2).*

(4) *$rI - A$ satisfies the weak properness property on K (see Definition 2.4).*

Then there exist $x_1 \in K - \{0\}$ and $\theta \in K' - \{0\}$ with $A(x_1) = rx_1$ and $A'(\theta) = r\theta$ and $\langle x_1, \theta \rangle > 0$. Furthermore, we have

$$\bigcup_{k=1}^{\infty} N((rI - A)^k) = N(rI - A) \tag{2.24}$$

and, if the dimension of $N(rI - A)$ is greater than 1, then A has an eigenvector u which lies in the support boundary of K and has eigenvalue r. If A has no eigenvector which lies in the support boundary of K and has eigenvalue r, then r is an algebraically simple eigenvalue of A.

Proof. The existence of x_1 and θ follow from Theorem 2.3, because growth condition (G1) is less general than growth condition (G). Equation (2.24) follows from Lemma 2.3. Thus it suffices to assume that $\dim(N(rI - A)) \geqslant 2$ and prove the existence of an eigenvector of A with eigenvalue r in the support boundary of K. Take $x_0 \in N((rI - A))$ such that x_0 and x_1 are linearly independent. Define $x_t = (1-t)x_0 + tx_1$. By Lemma 2.4, there exists a number γ such that $x_\gamma \in K$ and either $x_t \notin K$ for all $t > \gamma$ or $x_t \notin K$ for all $t < \gamma$. For definiteness, assume that $x_t \notin K$ for $\gamma < t$. Because x_0 and x_1 are linearly independent, we know that $x_\gamma \neq 0$. It follows that there exists a $\delta > 0$ with $x_t \notin K \cup (-K)$ for $\gamma < t < \gamma + \delta$.

Let $B = r^{-1}A$ and let $|x|$ be the norm defined by (2.18) so we know that $|B(x)| \leqslant |x|$ for all x and $|x| \leqslant M\|x\|$, where M is a constant as in Definition 2.2. Choose a fixed number t with $\gamma < t < \gamma + \delta$ such that

$$\rho(x_t) := \inf\{|x_t - y| : y \in K\} < |x_t|.$$

(Such a choice is possible because $\rho(x_\gamma) = 0 < |x_\gamma|$). If $\rho(x_t) < \alpha < |x_t|$, then define a set C_α as

$$C_\alpha = \{y \mid y \in K \text{ and } |y - x_t| \leqslant \alpha\}.$$

The reader can verify that C_α is a closed, bounded convex set; C_α is nonempty and $0 \notin C_\alpha$ because $\rho(x_t) < \alpha < |x_t|$. If $y \in C_\alpha$, then $B(y) \in K$ (because $y \in K$) and

$$|B(y) - x_t| = |B(y - x_t)| \leqslant |y - x_t| \leqslant \alpha.$$

Thus we have shown that $B(C_\alpha) \subset C_\alpha$, and Lemma 2.2 implies that B has a fixed point in C_α. Select a sequence α_k with $\alpha_k \to \rho(x_t)^+$, and, for each α_k, let u_k be a fixed point of B in C_{α_k}. By the weak properness property, we can assume by taking a subsequence that $u_k \rightharpoonup u$. Because C_α is closed in the weak topology, we have $u \in C_\alpha$ for $\rho(x_t) < \alpha < |x_t|$. In particular, we have $u \neq 0$, $u \in K$ and $|x_t - u| \leqslant \rho(x_t)$, and so the definition of $\rho(x_t)$ implies that $|x_t - u| = \rho(x_t)$. The weak continuity of B implies that $Bu = u$.

It remains to prove that u is in the support boundary of K. Let $V = \{y \mid |y - x_t| < \rho(x_t)\}$ so that V is an open convex set disjoint from the closed, convex set K. The Hahn–Banach Theorem implies that there exists a nonzero continuous linear functional h on X with $h(y) \geqslant 0$ for all $y \in K$ and $h(z) < 0$ for all $z \in V$. Because $u \in \bar{V} \cap K$, we must have $h(u) = 0$, and u is in the support boundary of K.

If X is reflexive or K is locally weakly compact or $\rho_K(A) < r_K(A)$, Lemma 2.1 implies that $rI - A$ satisfies the weak properness property on K. If X is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, the Riesz–Fischer Theorem implies that there is a conjugate linear isometry $J: X \to X'$ defined by $J(y)(x) := \langle x, y \rangle$. If $A: X \to X$ is a bounded linear operator and A^* denotes the Hilbert space adjoint of A, then $A^* = J^{-1}A'J$. Using these observations, one easily obtains a Hilbert space version of Theorem 2.4.

THEOREM 2.5. *Let K be a closed, total cone in a real Hilbert space H, and let $A: H \to H$ be a bounded linear operator with $A(K) \subset K$. Assume the following:*

(1) $r(A) := r > 0$.
(2) *There exist $y \in X$ and $m \geqslant 1$ with $-y \notin K$ and $A^m y \geqslant r^m y$.*
(3) *A satisfies growth condition (G1) (which is true if A is normal).*

Then there exist $x_1 \in K - \{0\}$ and $\theta \in K^* - \{0\}$ with $A(x_1) = rx_1$, $A^*(\theta) = r\theta$ and $\langle x_1, \theta \rangle > 0$. (Here $K^* := \{y \mid \langle x, y \rangle \geqslant 0$ for all $x \in K\}$). Furthermore, (2.24) is satisfied, and, if the dimension of $N(rI - A)$ is greater than 1, then A has an eigenvector u which lies in the support boundary of K and has eigenvalue r.

3. *The cone spectral radius and the functional $\chi_K(A)$*

In [**15**], Toland restricts himself to self-adjoint operators in a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, and he never introduces the cone spectral radius. Instead, if $A: H \to H$ is a bounded linear operator and K is a closed wedge, Toland defines

$$\chi_K(A) = \sup\{\langle Ax, x \rangle : x \in K \text{ and } \|x\| \leqslant 1\} \tag{3.1}$$

and works with $\chi_K(A)$. We shall now relate $\chi_K(A)$ to $\|A\|_K$ and $r_K(A)$ (see equations (1.7) and (1.8)).

LEMMA 3.1. *Let K be a closed wedge in a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. Let $A: H \to H$ be a bounded linear operator with $A(K) \subset K$. Then one has $\chi_K(A) \leqslant \|A\|_K$ and*

$$\limsup_{n \to \infty} (\chi_K(A^n))^{1/n} \leqslant \limsup_{n \to \infty} (\|A^n\|_K)^{1/n} = r_K(A). \tag{3.2}$$

If K is a closed cone and $\rho_K(A) < r_K(A)$ (see (1.16)), then equality holds in (3.2). (Recall that $\rho_K(A) = 0$ if $A \mid K$ is compact). If A is self-adjoint and $\langle Ax, x \rangle \geqslant 0$, for all x in the closed linear span of K, then

$$\chi_K(A) = \|A\|_K. \tag{3.3}$$

Proof. The inequality $\chi_K(A) \leqslant \|A\|_K$ follows from the Cauchy–Schwarz inequality, and (3.2) is then an immediate consequence. If $\rho_K(A) < r_K(A) := \alpha$, Theorem 1.2 implies that there exists an $x \in K$, $\|x\| = 1$, with $A(x) = \alpha x$, and so

$$\chi_K(A^n) \geqslant \langle A^n x, x \rangle = \alpha^n$$

which implies equality in (3.2).

If A is self-adjoint and $\langle Ax, x\rangle \geqslant 0$, for x in H_0, the closed linear span of K, then define a bilinear form (u, v) as $(u, v) := \langle Au, v\rangle$. Because we assume that $(u, u) \geqslant 0$ for all $u \in H_0$, the Cauchy–Schwartz inequality applies and gives

$$(u, v)^2 \leqslant (u, u)(v, v).$$

Taking $u = x \in K$ with $\|x\| \leqslant 1$ and $v = Ax$, we obtain

$$\begin{aligned} \langle Ax, Ax\rangle^2 = \|Ax\|^4 \leqslant \langle Ax, x\rangle \langle A^2x, Ax\rangle &\leqslant \chi_K(A)\,\|A^2x\|\,\|Ax\| \\ &\leqslant \chi_K(A)\,(\|A\|_K)^3. \end{aligned} \tag{3.4}$$

If we select a sequence $x_k \in K$ with $\|x_k\| = 1$ and $\|Ax_k\| \to \|A\|_K$, we obtain from (3.4)

$$(\|A\|_K)^4 \leqslant \chi_K(A)\,(\|A\|_K)^3$$

which implies that $\|A\|_K \leqslant \chi_K(A)$. We already know the opposite inequality, and so we obtain (3.3).

LEMMA 3.2. *Suppose that K is a closed wedge in a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. Let $A: H \to H$ be a bounded, self-adjoint linear operator with $A(K) \subset K$. Then, for all $n \geqslant 1$, we have $\|A^n\|_K = (\|A\|_K)^n$, and so $r_K(A) = \|A\|_K$.*

Proof. We know that

$$r_K(A) = \inf\{(\|A^n\|_K)^{1/n} : n \geqslant 1\} \quad \text{and} \quad \|A^n\|_K \leqslant (\|A\|_K)^n$$

and so it suffices to prove that $r_K(A) = \|A\|_K$. Notice that, for any $n \geqslant 1$, A^{2n} is self-adjoint, $A^{2n}(K) \subset K$, and $\langle A^{2n}y, y\rangle \geqslant 0$ for all y in the closed linear span of K. It follows from Lemma 3.1 that

$$\begin{aligned} \|A^{2n}\|_K = \chi_K(A^{2n}) &= \sup\{\langle A^{2n}x, x\rangle : x \in K, \|x\| = 1\} \\ &= \sup\{\|A^n x\|^2 : x \in K, \|x\| = 1\} = (\|A^n\|_K)^2. \end{aligned}$$

If we apply this observation repeatedly, then we see that

$$(\|A\|_K)^m = \|A^m\|_K \quad \text{for } m = 2^j, j \geqslant 1.$$

It follows that

$$r_K(A) = \lim_{m\to\infty} (\|A^m\|_K)^{1/m} = \|A\|_K$$

and the proof is complete.

LEMMA 3.3. *Suppose that K is a closed wedge in a real Hilbert space H and that $B: H \to H$ is a bounded, normal linear operator such that $(B^*B)(K) \subset K$. Then it follows that $\|B^k\|_K = (\|B\|_K)^k$ for all $k \geqslant 1$.*

Proof. B^*B is self-adjoint, and so Lemma 3.2 implies that

$$(\|B^*B\|_K)^k = \|(B^*B)^k\|_K. \tag{3.5}$$

However, we obtain from Lemma 3.1

$$\begin{aligned} \|(B^*B)^k\|_K = \|(B^*)^k B^k\|_K &= \sup\{\langle (B^*)^k B^k x, x\rangle : x \in K, \|x\| = 1\} \\ &= \|B^k\|_K^2. \end{aligned} \tag{3.6}$$

Using (3.5) and (3.6), we see that

$$(\|B\|_K)^k = \|B^k\|_K \quad \text{for all } k \geqslant 1.$$

Lemma 3.1, Lemma 3.2 and Lemma 3.3 enable us to give a refinement of Theorem 1.2.

THEOREM 3.1. *Let K be a closed cone in a real Hilbert space H, and let $A: H \to H$ be a bounded, normal linear operator such that $A(K) \subset K$ and $A^*(K) \subset K$. If $\rho_K(A) < \|A\|_K$ (where $\rho_K(A)$ is defined by* (1.16)), *then there exists a $u \in K - \{0\}$ with $Au = (\|A\|_K)u$ and $\chi_K(A) = \|A\|_K$. In particular, these conclusions hold if $A \mid K$ is compact.*

Proof. It follows from Lemma 3.3 that $r_K(A) = \|A\|_K := \alpha$, and Lemma 3.1 implies that $\chi_K(A) \leqslant \alpha$. If $\rho_K(A) < \alpha$, then Theorem 1.2 implies that there exists a $u \in K$ with $\|u\| = 1$ and $Au = \alpha u$. If $A \mid K$ is compact and $\alpha = 0$, then $A(x) = 0$ for all $x \in K$, and such a u certainly exists. It follows that

$$\chi_K(A) \geqslant \langle Au, u \rangle = \alpha$$

and so $\chi_K(A) = \|A\|_K$.

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