POSITIVE OPERATORS AND HAUSDORFF DIMENSION OF INVARIANT SETS

ROGER D. NUSSBAUM, AMIT PRIYADARSHI, AND SJOERD VERDUYN LUNEL

ABSTRACT. In this paper we obtain theorems which give the Hausdorff dimension of invariant sets for a family of contraction mappings which are infinitesimal similitudes on a complete, perfect metric space. We work in a setting similar to that of Mauldin and Williams [13] but the underlying space is not assumed to be finite dimensional and the maps are more general than similitudes. We use the theory of positive linear operators and generalizations of the Krein-Rutman theorem to obtain formula for the Hausdorff dimension as the nonnegative real σ for which the spectral radius of certain positive operators L_σ equals one. We also obtain results for the case of infinite iterated function systems.

1. Introduction

Given N contraction mappings θ_i, $1 \le i \le N$, on a complete metric space (X, d), there exists a unique, nonempty compact set C such that $C = \bigcup_{i=1}^{N} \theta_i(C)$. C is called an invariant set or an attractor for the family $\{\theta_i\}_{i=1}^{N}$. A general problem is to obtain theorems which allow the accurate estimation of the Hausdorff dimension of C. A well studied case is when the maps θ_i are "similitudes", i.e., when for $1 \le i \le N$, there exists r_i, $0 < r_i < 1$, with $d(\theta_i(x), \theta_i(y)) = r_i d(x, y)$ for all $x, y \in X$. If, in addition, X is a normed linear space with metric d derived from the norm on X and if the similitudes θ_i are onto maps (which is necessarily true if X is finite dimensional), then a theorem of Mazur and Ulam (see [14] or [23]) implies that each θ_i is an affine linear map. Moran [15] and Hutchinson [7] have studied the case that each θ_i, $1 \le i \le N$, is an affine linear similitude on a finite dimensional normed linear space X. Provided the "pieces" $\theta_i(C)$ do not overlap too much, they have proved that the Hausdorff dimension of C is the same as the "similarity dimension" α and is determined by the equation $\sum_{i=1}^{N} r_i^\alpha = 1$. More precisely, one needs the "open set condition", i.e., the assumption that there exists a nonempty, open set U such that the sets $\theta_i(U)$ are contained in U and are pairwise disjoint. Mauldin and Williams [13] have generalized the idea of "self-similarity" to "graph self-similarity" which allows a larger class of sets like C, but still the maps in question are affine linear similitudes. Schief [22] works in a setting similar to ours and considers similitudes on general complete metric spaces and obtains analogous

results. Interestingly, he has shown that in this generality, the open set condition is no longer sufficient and must be strengthened to the "strong open set condition".

It is of considerable interest (see [2, 3]) to allow maps θ_i which may not be affine linear. For example, in studying subsets of $\mathbb{R}$ defined by properties of their continued fraction expansions, one is led to maps $\theta_i : [0,1] \to [0,1]$ defined by $\theta_i(x) = (x + m_i)^{-1}$, m_i a positive integer.

In this paper we shall consider a complete, perfect metric space (X, d) and maps $\theta_i : X \to X$, $1 \le i \le N$, where θ_i is a contraction mapping and an "infinitesimal similitude" (instead of a similitude) for $1 \le i \le N$. If G is a bounded open subset of $\mathbb{C}$, $\theta_j : G \to G$ is analytic and $\overline{\theta_j(G)} \subset G$ for $1 \le j \le N$ and $X = \bigcup_{j=1}^{N} \overline{\theta_j(G)}$ with appropriate metric d, we obtain an important example for which the mappings $\theta_j : X \to X$ are contractions and infinitesimal similitudes. For the general definition of infinitesimal similitudes and their properties, see the beginning of Section 3. If C denotes the nonempty, compact invariant set for $\{\theta_i\}_{i=1}^{N}$ and if $\theta_i(C)$ and $\theta_j(C)$ are disjoint for $1 \le i < j \le N$, we shall obtain below a formula for the Hausdorff dimension of C. In fact, we shall obtain such a formula in a setting similar to that of Mauldin and Williams, but using contractions and infinitesimal similitudes, rather than affine linear contractions which are similitudes.

The classical Krein-Rutman theorem (see [8]) considers a positive (in the sense of mapping a suitable cone to itself), compact, linear map $T : X \to X$ which has positive spectral radius r and asserts the existence of a positive eigenvector v with $T(v) = rv$. Generalizations, particularly allowing noncompact T, can be found in [1, 9, 18, 19, 21]. Our approach in this paper will be to use generalizations of the Krein-Rutman theorem. To each nonnegative real σ we shall associate a positive linear operator L_σ on a Banach space Y of continuous functions. We shall prove that L_σ has a positive eigenvector with eigenvalue $r(L_\sigma)$, the spectral radius of L_σ. We shall prove that σ_0, the desired Hausdorff dimension, is the unique value of $\sigma \ge 0$ for which $r(L_\sigma) = 1$. We shall not use the thermodynamic formalism. Curiously, we have found no references to the Krein-Rutman theorem in the Hausdorff dimension literature, despite its relevance. Analogues of the operator L_σ we consider are sometimes called "Perron-Frobenius operators" or "Frobenius-Ruelle operators", although the theory originally developed by Perron and Frobenius is restricted to matrices with nonnegative entries, and generalizations to infinite dimensions pose substantial difficulties.

For the convenience of the reader we shall now state our main theorem in the simpler setting of iterated function systems on a compact, perfect metric space. For the more general case see section 3.

Let (S, d) be a compact, perfect metric space. If $\theta : S \to S$, we shall say that θ is an infinitesimal similitude at $t \in S$ if for any sequences $(s_k)_k$ and $(t_k)_k$ with $s_k \ne t_k$ for $k \ge 1$ and $s_k \to t$, $t_k \to t$, the limit

$$(1.1) \qquad \lim_{k \to \infty} \frac{d(\theta(s_k), \theta(t_k))}{d(s_k, t_k)} =: (D\theta)(t)$$

exists and is independent of the particular sequences $(s_k)_k$ and $(t_k)_k$. We shall say that θ is an infinitesimal similitude on S if θ is an infinitesimal similitude at t for all $t \in S$.

Assume that for $1 \le i \le N$, $\theta_i : S \to S$ is a Lipschitz map with Lipschitz constant $c_i \le c < 1$. Then we know that there exists a unique, compact, nonempty

set $C \subset S$ with

$$C = \bigcup_{i=1}^{N} \theta_i(C).$$

Assume the map $\theta_i : S \to S$ is an infinitesimal similitude on S and the map $t \to (D\theta_i)(t)$ is a strictly positive Hölder continuous function on S for $1 \leq i \leq N$. For $\sigma \geq 0$, define $L_\sigma : C(S) \to C(S)$ by

$$(1.2) \qquad\qquad (L_\sigma f)(t) = \sum_{i=1}^{N} ((D\theta_i)(t))^\sigma f(\theta_i(t)).$$

It follows (see Theorem 5.4 in [19]) that L_σ has a strictly positive eigenvector u_σ with eigenvalue equal to the spectral radius $r(L_\sigma)$ of L_σ. We also have the following lemma.

Lemma 1.1. *The map $\sigma \mapsto r(L_\sigma)$ is continuous and strictly decreasing. Furthermore, there is a unique $\sigma_0 \geq 0$ such that $r(L_{\sigma_0}) = 1$.*

We are now ready to state the theorem about the Hausdorff dimension of the invariant set C.

Theorem 1.2. *Let $\theta_i : S \to S$ for $1 \leq i \leq N$ be infinitesimal similitudes and assume that the map $t \to (D\theta_i)(t)$ is a strictly positive Hölder continuous function on S. Assume that $\theta_i : S \to S$ is a Lipschitz map with Lipschitz constant $c_i \leq c < 1$ and let C denote the unique invariant set such that*

$$C = \bigcup_{i=1}^{N} \theta_i(C).$$

Further, assume that θ_i, $1 \leq i \leq N$, satisfy

$$\theta_i(C) \cap \theta_j(C) = \emptyset \text{ for } 1 \leq i, j \leq N, i \neq j$$

and are one-to-one on an open neighborhood of C. Then the Hausdorff dimension, $\dim(C)$ of C is given by the unique σ_0 such that $r(L_{\sigma_0}) = 1$.

To see that this is a special case of our general theory of later sections, let $V = \{1\}$, $\mathcal{E} = \{1, 2, \ldots, N\}$ and $\Gamma = V \times \mathcal{E}$ in the terminology below.

We should remark that our proofs require that the pieces $\theta_i(C)$, $1 \leq i \leq N$, be pairwise disjoint. It would be very interesting to find variant arguments which allowed some overlap. For instance, we do not know if the strong open set condition is sufficient to get the results in this generality.

The paper is organised as follows. In §2 we introduce the basic set up and prove the existence of the invariant set. In §3 we discuss the Perron-Frobenius operators. In §4 we give the main theorems about the Hausdorff dimension of invariant set. In §5 the results are extended to infinite iterated function systems. In §6 we show that choosing the appropriate metric give a large class of examples of iterated function systems to which our theory can be applied. For instance, we discuss the Carathéodory-Reiffen-Finsler (CRF) metric on bounded open subsets of $\mathbb{C}$.

2. Invariant sets

Let V and $\mathcal{E}$ be finite sets and for each $v \in V$ let (S_v, d_v) be a complete metric space. Let Γ be a subset of $V \times \mathcal{E}$, and $\alpha : \Gamma \to V$. For each $(v, e) \in \Gamma$, let $\theta_{(v,e)} : S_v \to S_{\alpha(v,e)}$ be a Lipschitz map with $\mathrm{Lip}(\theta_{(v,e)}) \leq c < 1$. Recall that a map $\psi : (S_1, d_1) \to (S_2, d_2)$ is said to be Lipschitz if there is a constant c such that $d_2(\psi(s), \psi(t)) \leq c\, d_1(s, t)\ \forall s, t \in S_1$, and

$$\mathrm{Lip}(\psi) := \sup \left\{ \frac{d_2(\psi(s), \psi(t))}{d_1(s, t)} : s, t \in S_1, s \neq t \right\}.$$

Notation and assumptions will be as in the preceding paragraph for the remainder of the paper. We shall keep in mind two important particular cases.

Example 2.1. (*Mauldin-Williams graph*) Let V be the set of vertices and $\mathcal{E}$ be the set of edges of a directed multigraph. Let $i(e)$ and $t(e)$ denote the initial and terminal vertices of edge $e \in \mathcal{E}$. The set Γ is defined by $(v, e) \in \Gamma$ if and only if $v = t(e)$. The map α in this case is $\alpha(v, e) = i(e)$. See chapter 4.3 in [4] for a discussion of the Mauldin-Williams graph.

Example 2.2. Let (T, d) be a bounded complete metric space. Assume $T = \bigcup_{k=1}^{p} T_k$, where each T_k is closed subset of T and $T_k \cap T_l = \emptyset$ for $k \neq l$. For $1 \leq i \leq m$, let $\theta_i : T \to T$ be a continuous map such that $\theta_i(T_k) \subset T_{\nu(i,k)}$, $1 \leq k \leq p$, where $\nu(i, k) \in \{1, 2, \ldots, p\}$, and $\mathrm{Lip}(\theta_i | T_k) \leq c < 1$. In this case, we take $V = \{k : 1 \leq k \leq p\}$, $S_k = T_k$ for $1 \leq k \leq p$, $\mathcal{E} = \{i : 1 \leq i \leq m\}$, $\Gamma = V \times \mathcal{E}$. The map $\alpha(k, i) = \nu(i, k), 1 \leq k \leq p, 1 \leq i \leq m$, and $\theta_{(k,i)} = \theta_i | T_k$.

For $u \in V$, define

$$\Gamma_u = \{(v, e) \in \Gamma : \alpha(v, e) = u\}$$

and

$$E_u = \{e \in \mathcal{E} : (u, e) \in \Gamma\}.$$

For $n \geq 1$, define

$$\Gamma^{(n)} = \{[(v_1, e_1), \ldots, (v_n, e_n)] : (v_i, e_i) \in \Gamma, \alpha(v_{i+1}, e_{i+1}) = v_i, 1 \leq i \leq n - 1\}.$$

For $u \in V, n \geq 1$, define

$$\Gamma_u^{(n)} = \{[(v_1, e_1), \ldots, (v_n, e_n)] \in \Gamma^{(n)} : \alpha(v_1, e_1) = u\}.$$

Define $V_\infty = \{u \in V : \Gamma_u^{(n)} \neq \emptyset \quad \forall n \geq 1\}$.

Before we prove the next theorem we need to recall the definition of the *Hausdorff metric*. Let (S, d) be a complete metric space. If $A \subset S$, we define the diameter of A by

$$\mathrm{diam}(A) = \sup\{d(s, t) : s, t \in A\}.$$

We shall say that A is *bounded* if $\mathrm{diam}(A) < \infty$. For $A \subset S$ and $s \in S$ we define

$$d(s, A) = \inf\{d(s, a) : a \in A\}.$$

If $A \subset S$ and $\delta > 0$, we define $N_\delta(A)$ by

$$N_\delta(A) = \{s \in S : d(s, A) < \delta\}.$$

If A and B are nonempty, closed, bounded subsets of S, we define

$$D(A, B) = \inf\{\delta > 0 : A \subset N_\delta(B) \text{ and } B \subset N_\delta(A)\}.$$

If $\mathcal{B}(S)$ denotes the collection of nonempty, closed, bounded subsets of S, then it follows that $(\mathcal{B}(S), D)$ is a metric space. The metric D is called the Hausdorff

metric. It is known (see [16], Exercise 7, pages 280-281) that if (S, d) is a complete metric space, then $(\mathcal{B}(S), D)$ is also a complete metric space.

Theorem 2.3. *Let* $V, \mathcal{E}, \Gamma, \alpha$ *be given as before, and* $\theta_{(v,e)} : S_v \rightarrow S_{\alpha(v,e)}$ *be a Lipschitz map with* $Lip(\theta_{(v,e)}) \leq c < 1$ *for all* $(v, e) \in \Gamma$. *Assume* $\Gamma_u \neq \emptyset$ *for all* $u \in V$. *Then there exists a unique list* $(C_v)_{v \in V}$ *of nonempty closed bounded sets* $C_v \subset S_v$ *such that*

$$(2.1) \qquad C_u = \overline{\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(C_v)}$$

for all $u \in V$. *Furthermore,* C_v *is indeed compact for all* $v \in V$, *so we may remove the closure in the above equation.*

Proof. Let $\mathcal{B}(S_v)$ denote the collection of closed, bounded nonempty subsets of S_v with the Hausdorff metric D_v. Then since S_v is a complete metric space, we know that $\mathcal{B}(S_v)$ is a complete metric space. So the finite cartesian product $\prod_{v \in V} \mathcal{B}(S_v)$ with the sup metric is also a complete metric space. Define the map $\Theta : \prod_{v \in V} \mathcal{B}(S_v) \rightarrow \prod_{v \in V} \mathcal{B}(S_v)$ by

$$\Theta((A_v)_{v \in V}) = \left(\overline{\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(A_v)} \right)_{u \in V}$$

Note that for any $u \in V$, $\overline{\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(A_v)}$ is nonempty because Γ_u is nonempty by assumption, and A_v is nonempty for each $v \in V$. Also it is bounded because $\theta_{(v,e)}(A_v)$, being the image of bounded set A_v under a Lipschitz map $\theta_{(v,e)}$, is bounded, and a finite union of bounded sets is bounded. Thus the map Θ is well defined.

We claim that Θ is a contraction map. Let $A = (A_v)_{v \in V}$ and $B = (B_v)_{v \in V}$ be in $\prod_{v \in V} \mathcal{B}(S_v)$. Then $D(A, B) = \max_{v \in V} D_v(A_v, B_v)$ and

$$D(\Theta(A), \Theta(B)) = \max_{u \in V} D_u \left(\overline{\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(A_v)}, \overline{\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(B_v)} \right).$$

Let $\delta > D(A, B)$, and take any $(v, e) \in \Gamma_u$, $a_v \in A_v$. Since $D_v(A_v, B_v) \leq D(A, B) < \delta$, there exists $b_v \in B_v$ such that $d_v(a_v, b_v) < \delta$. Then

$$d_u(\theta_{(v,e)}(a_v), \theta_{(v,e)}(b_v)) \leq c d_v(a_v, b_v) < c\delta.$$

This shows that $\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(A_v)$ is contained in a $c\delta$-neighborhood of

$$\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(B_v).$$

Similarly we can prove the other way. So,

$$D_u \left(\overline{\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(A_v)}, \overline{\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(B_v)} \right) \leq c\delta$$

for all $u \in V$. Hence, $D(\Theta(A), \Theta(B)) \leq c\delta$. Since $\delta > D(A, B)$ was arbitrary, we have proved that $D(\Theta(A), \Theta(B)) \leq cD(A, B)$.

Therefore we have a contraction map Θ on a complete metric space. By the contraction mapping theorem, Θ has a unique fixed point, say $(C_v)_{v \in V}$. Thus we have $C_u = \overline{\bigcup_{(v,e) \in \Gamma_u} \theta_{(v,e)}(C_v)}$.

To see that C_v is compact for all $v \in V$ let us restrict the map Θ to $\prod_{v \in V} \mathcal{K}(S_v)$, where $\mathcal{K}(S_v)$ denotes the collection of nonempty compact subsets of S_v with the Hausdorff metric D_v. It is a straightforward exercise to prove that $\prod_{v \in V} \mathcal{K}(S_v)$ is a closed subset of the complete metric space $\prod_{v \in V} \mathcal{B}(S_v)$, so $\prod_{v \in V} \mathcal{K}(S_v)$ is itself a complete metric space. Then Θ maps the complete metric space $\prod_{v \in V} \mathcal{K}(S_v)$ into itself, and is a contraction map as seen above. Thus Θ has a unique fixed point in $\prod_{v \in V} \mathcal{K}(S_v)$. The fixed point must be the same as $(C_v)_{v \in V}$, since otherwise the original map Θ would have two fixed points. Therefore C_v is compact for all $v \in V$, and the theorem is proved. $\qquad\square$

Remark 2.4. The assumption $\Gamma_u \neq \emptyset$ for all $u \in V$ in the above theorem may be too strong for some examples. A weaker assumption under which we can prove the existence of an invariant list is $V_\infty \neq \emptyset$. Note that $\Gamma_u \neq \emptyset$ for all $u \in V$ implies $V_\infty = V$, so $V_\infty \neq \emptyset$.

First we claim that $u \in V_\infty$ implies that there exists $v \in V_\infty$ with $(v,e) \in \Gamma_u$ for some $e \in \mathcal{E}$. Suppose not. Then for all $(v,e) \in \Gamma_u$ $v \notin V_\infty$. This implies, since V is a finite set, there exists $n \geq 1$ such that $\Gamma_v^{(n)} = \emptyset$ for all $(v,e) \in \Gamma_u$. But since $u \in V_\infty$, there exists $[(v_1,e_1),(v_2,e_2),\ldots,(v_{n+1},e_{n+1})] \in \Gamma_u^{(n+1)}$, which implies $[(v_2,e_2),\ldots,(v_{n+1},e_{n+1})] \in \Gamma_{v_1}^{(n)}$ and since $\alpha(v_1,e_1) = u, (v_1,e_1) \in \Gamma_u$. This contradicts $\Gamma_v^{(n)} = \emptyset$ for all $(v,e) \in \Gamma_u$. Hence the claim.

Now consider the map $\Theta : \prod_{v \in V_\infty} \mathcal{B}(S_v) \to \prod_{v \in V_\infty} \mathcal{B}(S_v)$ by

$$\Theta((A_v)_{v \in V_\infty}) = \left(\overline{\bigcup_{\substack{(v,e) \in \Gamma_u \\ v \in V_\infty}} \theta_{(v,e)}(A_v)} \right)_{u \in V_\infty}$$

Note that Θ is well defined because of the above claim. Again by the contraction mapping theorem, we have $(C_v)_{v \in V_\infty}, C_v \subset S_v$ is compact such that

$$C_u = \bigcup_{\substack{(v,e) \in \Gamma_u \\ v \in V_\infty}} \theta_{(v,e)}(C_v)$$

Equivalently, under the assumption that $V_\infty \neq \emptyset$, all we have done is replace V by $\hat{V} := V_\infty$, Γ by $\hat{\Gamma} := \{(v,e) \in \Gamma | v \in \hat{V}\}$ and α by $\hat{\alpha} := \alpha|\hat{\Gamma}$, and then apply Theorem 2.3.

Remark 2.5. Applying the previous result to Example 2.2, we get nonempty compact sets $C_k \subset T_k$ for $k \in V_\infty$ such that

$$C_k = \bigcup_{\substack{\nu(i,l)=k \\ l \in V_\infty}} \theta_i(C_l) \text{ for } k \in V_\infty.$$

If we let $C = \bigcup_{l \in V_\infty} C_l$, then C is a nonempty compact set and it satisfies

$$C = \bigcup_{i=1}^{m} \theta_i(C).$$

Thus we have a compact invariant set for the family of maps θ_i, $1 \leq i \leq m$.

Remark 2.6. We can relax the condition $Lip(\theta_{(v,e)}) \leq c < 1$ for all $(v,e) \in \Gamma$ in the previous theorem to the following weaker condition. Suppose for some fixed $n \geq 1$ the composition of any n of the maps $\theta_{(v,e)}$, whenever the composition is defined, is Lipschitz with Lipschitz constant $\leq c < 1$. Then it is easy to see that Θ^n is a contraction map on a complete metric space, where Θ is the map defined in the proof of the theorem. It is then well known that the map Θ has a unique fixed point. Thus the conclusion of the previous theorem holds under this weaker assumption.

3. Perron-Frobenius operators

From now on, let $V = \{1, 2, \ldots, p\}$ with $S_1, S_2, \ldots, S_p$ the corresponding complete metric spaces. We do not necessarily assume that S_j, $1 \leq j \leq p$, is compact. Let

$$X_i = C_b(S_i) = \{f : S_i \to \mathbb{R} : f \text{ is continuous and bounded}\}$$

for $1 \leq i \leq p$ with $\|f\| = \sup_{s \in S_i} |f(s)|$.

Define a linear map $A : X_1 \times X_2 \times \cdots \times X_p \to X_1 \times X_2 \times \cdots \times X_p$ by

$$(3.1) \qquad (Af)_j(s) = \sum_{e \in E_j} b_{(j,e)}(s) f_{\alpha(j,e)}(\theta_{(j,e)}(s)) \quad \text{for } s \in S_j$$

where $f = (f_1, f_2, \ldots, f_p)$ and the functions $b_{(j,e)} \in X_j$ are given. We assume throughout this section that $E_j = \{e \in \mathcal{E} : (j, e) \in \Gamma\}$ is nonempty for all $j \in V$.

Define for $M > 0, \lambda \geq 0, 1 \leq j \leq p$

$$(3.2) \quad K_j(M, \lambda) = \{f \in X_j : 0 \leq f(s) \leq f(t) \exp(M(d_j(s,t))^\lambda) \text{ for all } s, t \in S_j\}.$$

Remark 3.1. From the definition, it follows that if $f \in K_j(M, \lambda)$ and $f(t) = 0$ for some $t \in S_j$, then $f(s) = 0$ for all $s \in S_j$. Thus $f \in K_j(M, \lambda)$ implies that either f is identically zero on S_j or f is strictly positive on S_j.

If Y is a real Banach space, a closed set $K \subset Y$ is called a closed *cone* if $\lambda K + \mu K \subset K$ for all $\lambda \geq 0, \mu \geq 0$ and $K \cap (-K) = \{0\}$.

The following lemma follows by the same argument used in Lemma 5.4, p.89, in [19]. We give the proof for the reader's convenience.

Lemma 3.2. *Let $K_j := K_j(M, \lambda)$ be as defined in (3.2). Then K_j is a closed cone in $(X_j, \|.\|)$, and $\{f \in K_j : \|f\| \leq 1\}$ is equicontinuous.*

Proof. It is easy to verify that K_j is a closed cone, and the proof is left to the reader. To prove the equicontinuity of $\{f \in K_j : \|f\| \leq 1\}$ let $f \in K_j$ with $\|f\| \leq 1$. We claim that for any $s, t \in S_j$ we have

$$|f(s) - f(t)| \leq M(d_j(s,t))^\lambda.$$

According to the previous remark, either f is identically zero on S_j or f is strictly positive on S_j. The inequality is obvious in the first case. In the later case, we may assume that $0 < f(s) \leq f(t) \leq 1$. The definition of K_j implies that

$$|\ln(f(s)) - \ln(f(t))| \leq M(d_j(s,t))^\lambda.$$

The mean value theorem implies that for some ξ with $\ln(f(s)) \leq \xi \leq \ln(f(t)) \leq 0$ we have

$$|f(s) - f(t)| = \exp(\ln(f(t))) - \exp(\ln(f(s)))$$
$$= \exp(\xi)|\ln(f(t)) - \ln(f(s))| \leq M(d_j(s,t))^\lambda.$$

Since this is true for any $f \in K_j$ with $\|f\| \leq 1$, equicontinuity follows. $\qquad\square$

Lemma 3.3. *Assume for some $M_0 > 0$, $b_{(j,e)} \in K_j(M_0, \lambda)$ for all $(j,e) \in \Gamma$. Then there exists $M > 0$ so that the map A defined above maps $\prod_{i=1}^p K_i(M, \lambda)$ into itsef.*

Proof. Let $f_i \in K_i(M, \lambda)$ for $1 \leq i \leq p$ and $s, t \in S_j$. Then

$$(Af)_j(s) = \sum_{e \in E_j} b_{(j,e)}(s) f_{\alpha(j,e)}(\theta_{(j,e)}(s))$$

Since $b_{(j,e)} \in K_j(M_0, \lambda)$, $b_{(j,e)}(s) \leq b_{(j,e)}(t) \exp((M_0(d_j(s,t))^\lambda)$. Also

$$f_{\alpha(j,e)}(\theta_{(j,e)}(s)) \leq f_{\alpha(j,e)}(\theta_{(j,e)}(t)) \exp(M(d_{\alpha(j,e)}(\theta_{(j,e)}(s), \theta_{(j,e)}(t)))^\lambda)$$
$$\leq f_{\alpha(j,e)}(\theta_{(j,e)}(t)) \exp(Mc^\lambda (d_j(s,t))^\lambda).$$

Thus

$$(Af)_j(s) = \sum_{e \in E_j} b_{(j,e)}(s) f_{\alpha(j,e)}(\theta_{(j,e)}(s))$$
$$\leq \sum_{e \in E_j} b_{(j,e)}(t) f_{\alpha(j,e)}(\theta_{(j,e)}(t)) \exp((M_0 + Mc^\lambda)(d_j(s,t))^\lambda)$$
$$= (Af)_j(t) \exp((M_0 + Mc^\lambda)(d_j(s,t))^\lambda).$$

So, if we choose M such that $M_0 + Mc^\lambda \leq M$, which can be done because $c < 1$, then

$$(Af)_j(s) \leq (Af)_j(t) \exp(M(d_j(s,t))^\lambda),$$

so $(Af)_j \in K_j(M, \lambda)$ for $1 \leq j \leq p$. $\qquad\square$

We should note that observations similar to Lemma 3.3 have been made earlier by other authors. See the proof of Theorem 5.4 in [19] and [2], for example.

We shall use the following notations.

$$\bar{\Gamma}^{(n)} := \{[(j_1, e_1), \ldots, (j_n, e_n)] : (j_i, e_i) \in \Gamma, 1 \leq i \leq n, \alpha(j_i, e_i) = j_{i+1}, 1 \leq i < n\}.$$

$$\bar{\Gamma}_j^{(n)} := \{[(j_1, e_1), \ldots, (j_n, e_n)] \in \bar{\Gamma}^{(n)} : j_1 = j\}.$$

We shall also use (J, E) where $J = (j_1, \ldots, j_n)$, $E = (e_1, \ldots, e_n)$ as a shorthand notation for $[(j_1, e_1), \ldots, (j_n, e_n)] \in \bar{\Gamma}^{(n)}$.

For $(J, E) = [(j_1, e_1), \ldots, (j_n, e_n)] \in \bar{\Gamma}^{(n)}$, define for $s \in S_{j_1}$

$$b_{(J,E)}(s) := b_{(j_1, e_1)}(s) b_{(j_2, e_2)}(\theta_{(j_1, e_1)}(s)) \cdots b_{(j_n, e_n)}(\theta_{(j_{n-1}, e_{n-1})} \circ \cdots \circ \theta_{(j_1, e_1)}(s))$$

and

$$\theta_{(J,E)}(s) := \theta_{(j_n, e_n)} \circ \cdots \circ \theta_{(j_1, e_1)}(s).$$

Let us compute A^2.

$$(A^2 f)_{j_1}(s) = (A(Af))_{j_1}(s)$$
$$= \sum_{e_1 : (j_1, e_1) \in \Gamma} b_{(j_1, e_1)}(s)(Af)_{\alpha(j_1, e_1)}(\theta_{(j_1, e_1)}(s))$$

Using

$$(Af)_{\alpha(j_1, e_1)}(\theta_{(j_1, e_1)}(s)) = \sum_{\substack{e_2 : (j_2, e_2) \in \Gamma \\ j_2 = \alpha(j_1, e_1)}} b_{(j_2, e_2)}(\theta_{(j_1, e_1)}(s)) f_{\alpha(j_2, e_2)}(\theta_{(j_2, e_2)}(\theta_{(j_1, e_1)}(s))),$$

we get

$$(A^2 f)_{j_1}(s) = \sum_{(J,E) \in \bar{\Gamma}_{j_1}^{(2)}} b_{(J,E)}(s) f_{\alpha(j_2,e_2)}(\theta_{(J,E)}(s)).$$

This computation suggests the formula for A^n given in following lemma.

Lemma 3.4. *Let $n \geq 1$. Then for $f = (f_1, f_2, \ldots, f_p) \in \prod_{i=1}^p X_i$ and $1 \leq j_1 \leq p$*

$$(3.3) \qquad (A^n f)_{j_1}(s) = \sum_{(J,E) \in \bar{\Gamma}_{j_1}^{(n)}} b_{(J,E)}(s) f_{\alpha(j_n,e_n)}(\theta_{(J,E)}(s)), \quad s \in S_{j_1}$$

Also the operator norm of A^n is given by

$$(3.4) \qquad \qquad \|A^n\| = \max_{1 \leq j \leq p} \sup_{s \in S_j} \sum_{(J,E) \in \bar{\Gamma}_j^{(n)}} b_{(J,E)}(s).$$

Proof. The equation for A^n follows by a simple induction on n. If $f \in \prod_{i=1}^p X_i$ with $\|f\| \leq 1$, i.e., $|f_j(s)| \leq 1 \, \forall s \in S_j, 1 \leq j \leq p$ then the equation for $(A^n f)_j(s)$ gives

$$|(A^n f)_j(s)| \leq \sum_{(J,E) \in \bar{\Gamma}_j^{(n)}} b_{(J,E)}(s)$$

Taking supremum over $s \in S_j$ and then maximum over $1 \leq j \leq p$ gives

$$\|A^n\| \leq \max_{1 \leq j \leq p} \sup_{s \in S_j} \sum_{(J,E) \in \bar{\Gamma}_j^{(n)}} b_{(J,E)}(s).$$

If we take $f = (f_1, f_2, \ldots, f_p)$ where f_j is identically equal to one on S_j then

$$(A^n f)_j(s) = \sum_{(J,E) \in \bar{\Gamma}_j^{(n)}} b_{(J,E)}(s)$$

Therefore we get the equation for $\|A^n\|$. $\qquad \square$

Lemma 3.5. *Let $(S_j)_{j=1}^p$ be bounded complete metric spaces, assume that $\Gamma_j := \{(k,e) \in \Gamma | \alpha(k,e) = j\}$ is nonempty, and let $(C_j)_{j=1}^p$ be the unique invariant list of compact sets given by Theorem 2.3. Let $(J,E) = [(j_1, e_1), \ldots, (j_n, e_n)] \in \bar{\Gamma}^{(n)}$ and $\theta_{(J,E)}(s) = \theta_{(j_n,e_n)} \circ \cdots \circ \theta_{(j_1,e_1)}(s), s \in S_{j_1}$. Then there exists $M_1 > 0$ such that for all $n \geq 1$*

$$d_{\alpha(j_n,e_n)}(\theta_{(J,E)}(s), C_{\alpha(j_n,e_n)}) \leq M_1 c^n \quad \forall s \in S_{j_1}$$

where $c < 1$ is the constant such that $Lip(\theta_{(j,e)}) \leq c$ for all $(j,e) \in \Gamma$.

Proof. Since the metric spaces (S_j), $1 \leq j \leq p$ are bounded we can find M_1 so that $d_j(s, C_j) \leq M_1 \, \forall s \in S_j, 1 \leq j \leq p$. Let $(j,e) \in \Gamma$ and $s \in S_j$. Then we can find $t \in C_j$ such that $d_j(s, t) \leq M_1$. Since $\theta_{(j,e)}(t) \in C_{\alpha(j,e)}$,

$$d_{\alpha(j,e)}(\theta_{(j,e)}(s), C_{\alpha(j,e)}) \leq d_{\alpha(j,e)}(\theta_{(j,e)}(s), \theta_{(j,e)}(t)) \leq c d_j(s, t) \leq c M_1.$$

The result now follows easily by an induction on n. $\qquad \square$

Let us recall the definition of Kuratowski's *measure of noncompactness β*. If (S, d) is a metric space and $B \subset S$ is a bounded set, then $\beta(B)$ is defined by

$$\beta(B) = \inf\{\delta > 0 : B = \cup_{j=1}^k B_j, k < \infty \text{ and } \operatorname{diam}(B_j) \leq \delta \text{ for } 1 \leq j \leq k\}.$$

Suppose that K is a closed cone in a Banach space Y and $L : Y \to Y$ is a bounded linear map with $L(K) \subset K$. Define

$$\|L\|_K = \sup\{\|L(y)\| : y \in K, \|y\| \le 1\}.$$

Define $r_K(L)$, the *cone spectral radius* of L, and $\sigma_K(L)$, by

$$r_K(L) := \lim_{n \to \infty} \|L^n\|_K^{1/n} \text{ and}$$

$$\sigma_K(L) := \limsup_{n \to \infty} (\beta(L^n(U)))^{1/n}$$

where $U = \{y \in K : \|y\| \le 1\}$.

It is a special case of results in [10] that if $\sigma_K(L) < r_K(L)$, then there exists $y \in K \setminus \{0\}$ with $L(y) = ry$, $r = r_K(L)$. (Note that the definition in [10] of $\rho_K(L)$, the cone essential spectral radius of L, satisfies $\rho_K(L) \le \sigma_K(L)$. The definition of $\rho_K(L)$ in [10] differs from that in [9] and [18]. It is shown in [10] that the earlier definition has some serious deficiencies.) We shall use this result to prove the existence of a non-zero eigenvector for the map A given by (3.1).

Theorem 3.6. *Consider the map A defined on $\prod_{j=1}^p C_b(S_j)$ by*

$$(Af)_j(s) = \sum_{e \in E_j} b_{(j,e)}(s) f_{\alpha(j,e)}(\theta_{(j,e)}(s)) \quad \text{for } s \in S_j$$

where $f = (f_1, f_2, \ldots, f_p)$. Assume that $\Gamma_j \ne \emptyset$ for $1 \le j \le p$ and that for some $M_0 > 0$, $b_{(j,e)} \in K_j(M_0, \lambda)$ for all $(j,e) \in \Gamma$. Let K be the cone $\prod_{j=1}^p K_j(M, \lambda)$, where $M_0 + c^\lambda M \le M$. Then $\|A^n\|_K = \|A^n\|$ for all $n \ge 1$, where $\|A^n\|$ is given by equation (3.4), and $r_K(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}$. If $r_K(A) > 0$, there exists $u = (u_1, u_2, \ldots, u_p) \in K \setminus \{0\}$ with $Au = ru$, where $r = r_K(A)$. If $b_{(j,e)}(s) > 0$ for all $(j,e) \in \Gamma$ and all $s \in S_j$, then $r_K(A) > 0$.

Proof. It is enough to show that $\rho_K(A) < r_K(A)$. Let $(C_j)_{j=1}^p$ be the unique invariant list of nonempty compact sets given by Theorem 2.3 and let $C = \prod_{j=1}^p C_j$. Let $\mathcal{U} = \{f \in K | \|f\| \le 1\}$. From Lemma 3.2, we know that $\mathcal{U}$ is equicontinuous. Let us write $\mathcal{U}|_C = \{f|_C | f \in \mathcal{U}\}$. Then $\mathcal{U}|_C$ is a bounded equicontinuous family of functions from the compact set C into $\mathbb{R}^p$. So, by Ascoli's theorem, it is totally bounded. Therefore, given $\epsilon > 0$, we can write $\mathcal{U} = \bigcup_{l=1}^q \mathcal{U}_l$, $q < \infty$, such that $\|f|_C - g|_C\| \le \epsilon$ provided f and g are in the same $\mathcal{U}_l$. Let $f, g \in U$ and $1 \le j \le p$. Then using Lemma 3.4, we have for $s \in S_j$

$$|(A^n f)_j(s) - (A^n g)_j(s)| \le \sum_{(J,E) \in \bar{\Gamma}_j^{(n)}} b_{(J,E)}(s) |f_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s)) - g_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s))|.$$

Using Lemma 3.5, there exists $\tau \in C_{\alpha(j_n, e_n)}$ with $d(\theta_{(J,E)}(s), \tau) \le M_1 c^n$. Since $f_{\alpha(j_n, e_n)} \in K_{\alpha(j_n, e_n)}(M, \lambda)$ and $\|f\| \le 1$, we have as in the proof of Lemma 3.2

$$|f_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s)) - f_{\alpha(j_n, e_n)}(\tau)| \le M d(\theta_{(J,E)}(s), \tau)^\lambda \le M(M_1 c^n)^\lambda.$$

The same is true for g. Also, if we assume that $f, g \in \mathcal{U}_l$ for some l, $1 \le l \le q$ then $|f_{\alpha(j_n, e_n)}(\tau) - g_{\alpha(j_n, e_n)}(\tau)| \le \epsilon$. Therefore, by triangle inequality

$$|f_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s)) - g_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s))| \le \epsilon + 2M(M_1 c^n)^\lambda \text{ if } f, g \in \mathcal{U}_l.$$

So, if $f, g \in \mathcal{U}_l$ and $1 \le j \le p$,

$$|(A^n f)_j(s) - (A^n g)_j(s)| \le (\epsilon + 2M(M_1 c^n)^\lambda) \sum_{(J,E) \in \bar{\Gamma}_j^{(n)}} b_{(J,E)}(s).$$

Taking supremum over $s \in S_j$ and max over $1 \le j \le p$ and using equation (3.4), we get

$$\|A^n f - A^n g\| \le (\epsilon + 2M(M_1 c^n)^\lambda)\|A^n\|$$

for $f, g \in \mathcal{U}_l$, $1 \le l \le q$.
Thus $A^n(\mathcal{U}) = \bigcup_{l=1}^q A^n(\mathcal{U}_l)$ with

$$\text{diam}(A^n(\mathcal{U}_l)) \le (\epsilon + 2M(M_1 c^n)^\lambda)\|A^n\|$$

So, $\beta(A^n(\mathcal{U})) \le (\epsilon + 2M(M_1 c^n)^\lambda)\|A^n\|$.
Since $\epsilon > 0$ was arbitrary, $\beta(A^n(\mathcal{U})) \le 2M(M_1 c^n)^\lambda \|A^n\|$ which implies

$$(\beta(A^n(\mathcal{U})))^{\frac{1}{n}} \le (2MM_1^\lambda)^{\frac{1}{n}} c^\lambda \|A^n\|^{\frac{1}{n}}.$$

In general, it is obviously true that $\|A^n\|_K \le \|A^n\|$. On the other hand, if $f = (f_1, f_2, \ldots, f_p)$ and $f_j(s) = 1$ for all $s \in S_j$, then $f \in K$; and we have seen in the proof of Lemma 3.4 that $\|A^n(f)\| = \|A^n\|$. It follows that $\|A^n\|_K = \|A^n\|$ for all $n \ge 1$ and that $r_K(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}} = r(A)$, where $\|A^n\|$ is given by equation (3.4) and $r(A)$ denotes the spectral radius of A. Taking the limit in our estimate for $(\beta(A^n(\mathcal{U})))^{\frac{1}{n}}$ gives

$$\sigma_K(A) = \lim_{n \to \infty} (\beta(A^n(\mathcal{U})))^{\frac{1}{n}} \le (c^\lambda) r_K(A).$$

If $r_K(A) > 0$, it follows (because $0 < c < 1$) that $\sigma_K(A) < r_K(A)$, and we are done. If we assume that $b_{(j,e)}(s) > 0$ for all $(j, e) \in \Gamma$ and $s \in s_j$, then because $b_{(j,e)} \in K_j(M_0, \lambda)$, there exists $\delta > 0$ such that $b_{(j,e)}(s) \ge \delta$ for all $(j, e) \in \Gamma$ and $s \in S_j$, and it follows easily that $r_K(A) \ge \delta > 0$. $\qquad\square$

Remark 3.7. Suppose $u = (u_1, u_2, \ldots, u_p)$ is a nonzero eigenvector of the linear map A with eigenvalue $r(A)$ given by Theorem 3.6. Then for any $1 \le j \le p$, either u_j is identically zero or u_j is strictly positive on S_j. To see this note that $u_j \in K_j(M, \lambda)$, so

$$0 \le u_j(s) \le u_j(t) \exp(M(d_j(s,t))^\lambda) \text{ for all } s, t \in S_j.$$

Thus $u_j(t) = 0$ for some $t \in S_j$ will imply that $u_j(s) = 0$ for all $s \in S_j$. Also, since u is nonzero, at least one of the coordinate function u_j is strictly positive.

Remark 3.8. In general (in the context of the Krein-Rutman Theorem), if $r_K(L) = 0$, it need not be true that there exist $v \in K \setminus \{0\}$ with $L(v) = 0$. Suppose, however, that K and A are as in Theorem 3.6, that $b_{(j,e)} \in K_j(M_0, \lambda)$ for all $(j, e) \in \Gamma$ and that $r_K(A) = 0$. We claim that there exists an integer N such that $A^N = 0$ and that there exists $v \in K \setminus \{0\}$ with $A(v) = 0$. Because S_j is bounded and $b_{(j,e)} \in K_j(M_0, \lambda)$, we have already seen that either $b_{(j,e)}(s) = 0$ for all $s \in S_j$ or there exists $\delta_{(j,e)} > 0$ with $b_{(j,e)}(s) \ge \delta_{(j,e)}$ for all $s \in S_j$. Let $\mathcal{P} = \{(j, e) \in \Gamma | b_{(j,e)}(s) > 0 \text{ for all } s \in S_j\}$. Because $\mathcal{P}$ is a finite set (since Γ is finite), there exists $\delta > 0$ with $b_{(j,e)}(s) \ge \delta$ for all $s \in S_j$ and for all $(j, e) \in \mathcal{P}$. For $n \ge 1$, define $\bar{\mathcal{P}}^{(n)} \subset \bar{\Gamma}^{(n)}$ by

$$\bar{\mathcal{P}}^{(n)} = \{(J, E) \in \bar{\Gamma}^{(n)} | (j_k, e_k) \in \mathcal{P} \text{ for } 1 \le k \le n\}.$$

If $\bar{\mathcal{P}}^{(n)}$ is nonempty, it easily follows from equation (3.4) that $\|A^n\| \geq \delta^n$, so if $\bar{\mathcal{P}}^{(n)}$ is nonempty for all $n \geq 1$, $r_K(A) \geq \delta$, contrary to our assumption. Thus there must exist an integer N such that $\bar{\mathcal{P}}^{(n)}$ is empty for all $n \geq N$. However, if $(J, E) \in \bar{\Gamma}^{(n)}$ and $(J, E) \notin \bar{\mathcal{P}}^{(n)}$, $b_{(J,E)}(s) = 0$ for all $s \in S_j$, so we find that $A^n = 0$ for all $n \geq N$. If $w \in K \setminus \{0\}$, let $p \leq N$ be the least positive integer such that $A^p(w) = 0$. If we define $v = A^{p-1}(w) \in K \setminus \{0\}$, $A(v) = 0$.

4. Hausdorff dimension

Recall that a metric space (S, d) is called *perfect* if every point of S is a limit point of S, i.e., for each $s \in S$, there exists a sequence $(s_k)_k$ in S such that $s_k \neq s$ for all k and $s_k \to s$ as $k \to \infty$.

Let (S_1, d_1) and (S_2, d_2) be perfect metric spaces. A map $\theta : S_1 \to S_2$ is said to be an *infinitesimal similitude* at $s \in S_1$ if for any sequences $(s_k)_k$ and $(t_k)_k$ in S_1 with $s_k \neq t_k$ for $k \geq 1$ and $s_k \to s$, $t_k \to s$, the limit

$$(4.1) \qquad \lim_{k \to \infty} \frac{d_2(\theta(s_k), \theta(t_k))}{d_1(s_k, t_k)} =: (D\theta)(s)$$

exists and is independent of the particular sequences $(s_k)_k$ and $(t_k)_k$. We shall say that θ is an infinitesimal similitude on S_1 if θ is an infinitesimal similitude at s for all $s \in S_1$. Notice that the assumption that S_1 is perfect implies that for every $s \in S_1$, there exist sequences $(s_k)_k$ and $(t_k)_k$ as above.

We list some basic properties of infinitesimal similitudes that we shall need.

Lemma 4.1. *If $\theta : S_1 \to S_2$ is an infinitesimal similitude, then $s \mapsto (D\theta)(s)$ is continuous.*

Proof. We argue by contradiction and assume that $s \mapsto (D\theta)(s)$ is not continuous. Then there exist $\epsilon > 0$ and $s \in S_1$ and a sequence $(s_k)_k$ in S_1 with $d_1(s_k, s) \to 0$ as $k \to \infty$ such that

$$|(D\theta)(s_k) - (D\theta)(s)| > \epsilon > 0.$$

Since S_1 is perfect and θ is an infinitesimal similitude, for each $k \geq 1$, there exist t_k and w_k in S_1 with $t_k \neq w_k$, $0 < d_1(s_k, t_k) < \frac{1}{k}$, $0 < d_1(s_k, w_k) < \frac{1}{k}$ and

$$\left| \frac{d_2(\theta(t_k), \theta(w_k))}{d_1(t_k, w_k)} - (D\theta)(s_k) \right| < \frac{\epsilon}{4}.$$

Since $s_k \to s$ as $k \to \infty$, it follows that $t_k \to s$ and $w_k \to s$ as $k \to \infty$ as well. So, by definition,

$$\frac{d_2(\theta(t_k), \theta(w_k))}{d_1(t_k, w_k)} \to (D\theta)(s) \text{ as } k \to \infty.$$

So, for k large enough, $|(D\theta)(s_k) - (D\theta)(s)| < \frac{\epsilon}{2}$, a contradiction. Hence, $s \mapsto (D\theta)(s)$ is continuous. $\square$

The following lemma states an analogue of the chain rule for infinitesimal similitudes.

Lemma 4.2. *Let $\theta : S_1 \to S_2$ and $\psi : S_2 \to S_3$ be given. If θ is an infinitesimal similitude at $s \in S_1$ and ψ is an infinitesimal similitude at $\theta(s) \in S_2$, then $\psi \circ \theta$ is an infinitesimal similitude at s and*

$$(4.2) \qquad (D(\psi \circ \theta)(s) = (D\psi)(\theta(s))(D\theta)(s).$$

Proof. Let $(s_k)_k$ and $(t_k)_k$ be sequences in S_1 with $s_k \neq t_k$, $k \geq 1$, $s_k \to s$, $t_k \to s$ as $k \to \infty$. Then $\theta(s_k) \to \theta(s)$, $\theta(t_k) \to \theta(s)$ as $k \to \infty$. We consider two cases.

Case I. Assume that $(D\theta)(s) \neq 0$. We claim that there exists a positive integer k_0 with $\theta(s_k) \neq \theta(t_k)$ for all $k \geq k_0$. If not, there exists a subsequence $k_i \to \infty$ such that $\theta(s_{k_i}) = \theta(t_{k_i})$ for $i \geq 1$. Writing $\sigma_i = s_{k_i}$ and $\tau_i = t_{k_i}$, we have that $\sigma_i \to s$, $\tau_i \to s$, $\sigma_i \neq \tau_i$ and $(D\theta)(s) = \lim_{i \to \infty} \frac{d_2(\theta(\sigma_i), \theta(\tau_i))}{d_1(\sigma_i, \tau_i)} = 0$, which contradicts our assumption. It follows that, for $k \geq k_0$, we can write

$$\frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_2(\theta(s_k), \theta(t_k))} \frac{d_2(\theta(s_k), \theta(t_k))}{d_1(s_k, t_k)}.$$

As $k \to \infty$, the limit of the right hand side exists and equals $(D\psi)(\theta(s))(D\theta)(s)$, so the limit of the left hand side exists and equation (4.2) is satisfied.

Notice that if s_k and t_k are sequences with $s_k \to s$, $t_k \to s$, $s_k \neq t_k$ for all k and $\theta(s_k) \neq \theta(t_k)$ for all $k \geq k_0$, then the argument above proves that

$$\lim_{k \to \infty} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = (D\psi)(\theta(s))(D\theta)(s),$$

even if $(D\theta)(s) = 0$.

Case II. Assume that $(D\theta)(s) = 0$. Let s_k and t_k are sequences in S_1 with $s_k \to s$, $t_k \to s$ and $s_k \neq t_k$ for all $k \geq 1$. If there exists $k_0 \geq 1$ such that $\theta(s_k) \neq \theta(t_k)$ for all $k \geq k_0$, the argument above shows that

$$\lim_{k \to \infty} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = (D\psi)(\theta(s))(D\theta)(s) = 0.$$

If there exists k_1 such that $\theta(s_k) = \theta(t_k)$ for all $k \geq k_1$, we certainly that

$$\lim_{k \to \infty} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = 0.$$

Thus we can assume that $K_1 := \{k \geq 1 : \theta(s_k) \neq \theta(t_k)\}$ and $K_2 := \{k \geq 1 : \theta(s_k) = \theta(t_k)\}$ are infinite sets. However, our previous argument (Case I) shows that

$$\lim_{k \to \infty, k \in K_1} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = (D\psi)(\theta(s))(D\theta)(s) = 0,$$

and it is clear that

$$\lim_{k \to \infty, k \in K_2} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = 0,$$

so we conclude that

$$\lim_{k \to \infty} \frac{d_3(\psi(\theta(s_k)), \psi(\theta(t_k)))}{d_1(s_k, t_k)} = 0.$$

$\square$

The following lemma gives a "mean value theorem" and will be crucial in the proof of the main theorem.

Lemma 4.3. *Suppose that (S_1, d_1) and (S_2, d_2) are bounded, complete metric spaces and that $\theta : S_1 \to S_2$ is an infinitesimal similitude. Assume, also, that θ is one-to-one and Lipschitz and that $(D\theta)(s) \geq m > 0$ for all $s \in S_1$. Let $K \subset S_1$ be compact,*

nonempty set. For each $\mu > 1$, there exists an open neighborhood U_μ of K and a positive number $\epsilon = \epsilon(\mu)$ such that for every $s, t \in U_\mu$ with $0 < d_1(s,t) < \epsilon(\mu)$,

$$\mu^{-1}(D\theta)(s) \leq \frac{d_2(\theta(s), \theta(t))}{d_1(s,t)} \leq \mu(D\theta)(s)$$

Proof. For $(s,t) \in S_1 \times S_1$ with $s \neq t$, define $F(s,t) = \frac{d_2(\theta(s), \theta(t))}{d_1(s,t)}$. If $(s,s) \in S_1 \times S_1$, define $F(s,s) = (D\theta)(s)$. Because we assume that θ is Lipschitz on S_1, there is a constant M_1 with $F(s,t) \leq M_1$ for all $(s,t) \in S_1 \times S_1$. Because we assume that $(D\theta)(s) \geq m > 0$ for all $s \in S_1$ and θ is one-to-one, $F(s,t) > 0$ for all $(s,t) \in S_1 \times S_1$.

We claim that F is continuous on $S_1 \times S_1$. It suffices to prove that if $(s_k, t_k) \to (s,s)$, then $F(s_k, t_k) \to F(s,s) = (D\theta)(s)$. If $s_k \neq t_k$ for all $k \geq k_0$, we know that $F(s_k, t_k) \to (D\theta)(s)$ by the definition of $(D\theta)(s)$. If $s_k = t_k$ for all $k \geq k_1$, $F(s_k, t_k) = (D\theta)(s_k)$ for $k \geq k_1$, and Lemma 4.1 implies that $(D\theta)(s_k) \to (D\theta)(s)$. Thus we can assume that $J_1 := \{k | s_k \neq t_k\}$ and $J_2 := \{k | s_k = t_k\}$ are infinite sets. But in this case, the same reasoning implies that $\lim_{k \to \infty, k \in J_1} F(s_k, t_k) = (D\theta)(s)$ and $\lim_{k \to \infty, k \in J_2} F(s_k, t_k) = (D\theta)(s)$, so $\lim_{k \to \infty} F(s_k, t_k) = (D\theta)(s)$.

Lemma 4.1 implies that $s \mapsto (D\theta)(s)$ is continuous on S_1, and $(s,t) \mapsto F(s,t)$ is continuous on $S_1 \times S_1$. Thus if we define $G(s,t)$ by $G(s,t) = \frac{F(s,t)}{(D\theta(s))}$, $(s,t) \mapsto G(s,t)$ is continuous on $S_1 \times S_1$ and $G(s,s) = 1$. Since $K \times K$ is compact, $G | K \times K$ is uniformly continuous, so given $\mu > 1$, there exists $\epsilon(\mu) > 0$ with

$$\mu^{-1} < G(s,t) < \mu$$

for all $(s,t) \in K \times K$ with $d_1(s,t) \leq \epsilon(\mu)$. We claim that there exists an open neighborhood U_μ of K such that for all $s, t \in U_\mu$ with $d_1(s,t) \leq \epsilon(\mu)$, $\mu^{-1} < G(s,t) < \mu$. We argue by contradiction and suppose not. For m a positive integer, let $V_m = \{s \in S_1 | d_1(s,K) < \frac{1}{m}\}$. By assumption, there exist $s_m, t_m \in V_m$ with $d_1(s_m, t_m) \leq \epsilon(\mu)$ and $G(s_m, t_m) < \mu^{-1}$ or $G(s_m, t_m) > \mu$. Because $d_1(s_m, K) \to 0$ and $d_1(t_m, K) \to 0$, we can, by taking a subsequence, assume that $s_m \to s \in K$ and $t_m \to t \in K$ and $d_1(s,t) \leq \epsilon(\mu)$. By continuity of G, we either have $G(s,t) \leq \mu^{-1}$ or $G(s,t) \geq \mu$. However, because $s, t \in K$ and $d_1(s,t) \leq \epsilon(\mu)$, $\mu^{-1} < G(s,t) < \mu$, a contradiction. Thus an open set U_μ exists and, in fact, we can take $U_\mu = V_m$ for some $m \geq 1$. $\square$

Remark 4.4. For each $\epsilon > 0$, define $\mu(\epsilon) \geq 1$ to be the infimum of numbers $\mu > 1$ such that $\mu^{-1}(D\theta)(s) \leq \frac{d_2(\theta(s), \theta(t))}{d_1(s,t)} \leq \mu(D\theta)(s)$ for $s, t \in K$ with $0 < d_1(s,t) \leq \epsilon$. Lemma 4.3 implies that $\lim_{\epsilon \to 0+} \mu(\epsilon) = 1$, and clearly $\mu(\epsilon)$ is an increasing function of ϵ for $\epsilon > 0$.

Throughout this section we shall make the following assumption.

H4.1 Let $V = \{1, 2, \ldots, p\}$ and $S_1, S_2, \ldots, S_p$ be bounded, complete, perfect metric spaces. Let $\mathcal{E}$ be a finite set, $\Gamma \subset V \times \mathcal{E}$ and $\alpha : \Gamma \to V$. For each $(j,e) \in \Gamma$, $\theta_{(j,e)} : S_j \to S_{\alpha(j,e)}$ is a Lipschitz map with $Lip(\theta_{(j,e)}) \leq c < 1$. Also, $\Gamma_i = \{(j,e) \in \Gamma : \alpha(j,e) = i\} \neq \emptyset$ for $1 \leq i \leq p$ and $E_j = \{e \in E : (j,e) \in \Gamma\} \neq \emptyset$ for $1 \leq j \leq p$.

If H4.1 is satisfied then Theorem 2.3 implies that there exists a unique list of nonempty compact sets $C_j \subset S_j$, $1 \leq j \leq p$ with

$$(4.3) \qquad C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \quad \text{for } 1 \leq i \leq p.$$

We shall further assume the following.

H4.2 For each $(j, e) \in \Gamma$, the map $\theta_{(j,e)} : S_j \to S_{\alpha(j,e)}$, given in H3.1, is an infinitesimal similitude and $(D\theta_{(j,e)})(s) > m > 0$ for all $s \in S_j$.

Notice that since $\theta_{(j,e)}$ is Lipschitz with $Lip(\theta_{(j,e)}) \leq c$, if $\theta_{(j,e)}$ is an infinitesimal similitude, we have $(D\theta_{(j,e)})(s) \leq c$.

Assume that H4.1 and H4.2 are satisfied. For $\sigma \geq 0$, define

$$L_\sigma : \prod_{j=1}^{p} C_b(S_j) \to \prod_{j=1}^{p} C_b(S_j)$$

by

(4.4) $(L_\sigma f)_j(s) = \sum_{e \in E_j} ((D\theta_{(j,e)})(s))^\sigma f_{\alpha(j,e)}(\theta_{(j,e)}(s))$ for $s \in S_j, 1 \leq j \leq p$.

Recall that a map $f : (S, d) \to \mathbb{R}$ is said to be *Hölder continuous with Hölder exponent* $\lambda > 0$ if there exists a constant $C \in (0, \infty)$ such that

$$|f(s) - f(t)| \leq C(d(s, t))^\lambda \text{ for all } s, t \in S.$$

Let us assume the following.

H4.3 For each $(j, e) \in \Gamma$, the map $s \mapsto (D\theta_{(j,e)})(s)$ is Hölder continuous with Hölder exponent $\lambda > 0$.

Lemma 4.5. *If H4.1, H4.2 and H4.3 hold, and $\sigma \geq 0$, then the map*

$$s \mapsto ((D\theta_{(j,e)})(s))^\sigma$$

is in the cone $K_j(M_0, \lambda)$ defined in (3.2) for some M_0 (depending on σ).

Proof. Fix $(j, e) \in \Gamma$ and $\sigma \geq 0$. Let $f(s) = (D\theta_{(j,e)})(s)$. The hypotheses H4.1 and H4.2 implies that $0 < m < f(s) \leq c$. By H4.3

$$|f(s) - f(t)| \leq C(d_j(s, t))^\lambda \text{ for all } s, t \in S_j.$$

Let $s, t \in S_j$. By the mean value theorem, there exists ξ between $f(s)$ and $f(t)$ such that

$$|\ln(f(s)) - \ln(f(t))| = \frac{1}{\xi}|f(s) - f(t)| \leq \frac{1}{\xi}C(d_j(s, t))^\lambda \leq \frac{1}{m}C(d_j(s, t))^\lambda$$

So, $f(s) \leq f(t) \exp(\frac{1}{m}C(d_j(s, t))^\lambda)$ which implies

$$f(s)^\sigma \leq f(t)^\sigma \exp(M_0(d_j(s, t))^\lambda) \text{ where } M_0 = \frac{\sigma C}{m}.$$

This completes the proof. $\square$

Now applying Theorem 3.6 to the linear map L_σ defined in (4.4), we get an eigenvector $u_\sigma \in K \setminus \{0\}$ with $L_\sigma u_\sigma = r(L_\sigma)u_\sigma$ and $r(L_\sigma) > 0$.

Lemma 4.6. *The map $\sigma \mapsto r(L_\sigma)$ is continuous and strictly decreasing. Furthermore, there is a unique $\sigma_0 \geq 0$ such that $r(L_{\sigma_0}) = 1$.*

Proof. Let u_σ be the positive eigenvector of L_σ with eigenvalue $r(L_\sigma)$. Let us write $b_{(j,e)}(t) = (D\theta_{(j,e)}(t))$ for $(j, e) \in \Gamma$. We know that $0 < m \leq b_{(j,e)}(t) \leq c < 1$ for all $t \in S_j$. Let $0 \leq \sigma < \sigma'$. Then

$$(b_{(j,e)}(t))^{\sigma'} = (b_{(j,e)}(t))^{\sigma'-\sigma}(b_{(j,e)}(t))^\sigma \leq c^{\sigma'-\sigma}(b_{(j,e)}(t))^\sigma$$

Therefore, $(b_{(j,e)}(t))^\sigma \geq \mu(b_{(j,e)}(t))^{\sigma'}$, where $\mu = (\frac{1}{c})^{\sigma'-\sigma} > 1$.

$$(L_\sigma u_{\sigma'})_j(t) = \sum_{e \in E_j} (b_{(j,e)}(t))^\sigma (u_{\sigma'})_{\alpha(j,e)}(\theta_{(j,e)}(t))$$

$$\geq \mu \sum_{e \in E_j} (b_{(j,e)}(t))^{\sigma'} (u_{\sigma'})_{\alpha(j,e)}(\theta_{(j,e)}(t))$$

$$= \mu(L'_\sigma u_{\sigma'})_j(t) = \mu r(L_{\sigma'})(u_{\sigma'})_j(t).$$

So, $L_\sigma u_{\sigma'} \geq \mu\, r(L_{\sigma'})u_{\sigma'}$. Iterating this inequality k times, we obtain

$$L_\sigma^k u_{\sigma'} \geq (\mu r(L_{\sigma'}))^k u_{\sigma'}$$

If e denotes the function identically equal to one in each component, we have $u_{\sigma'} \leq \|u_{\sigma'}\|e$. Thus

$$L_\sigma^k u_{\sigma'} \leq L_\sigma^k(\|u_{\sigma'}\|e) \leq \|u_{\sigma'}\|L_\sigma^k(e).$$

Taking norms, we get

$$\|u_{\sigma'}\|\|L_\sigma^k(e)\| \geq \|L_\sigma^k u_{\sigma'}\| \geq (\mu r(L_{\sigma'}))^k \|u_{\sigma'}\|.$$

So, $\|L_\sigma^k\| \geq \|L_\sigma^k(e)\| \geq (\mu r(L_{\sigma'}))^k$ from which it follows that

$$r(L_\sigma) = \lim_{k \to \infty} \|L_\sigma^k\|^{1/k} \geq \mu r(L_{\sigma'}).$$

Since $\mu > 1$, we have proved that $r(L_\sigma) > r(L_{\sigma'})$.

Next we prove the continuity of $\sigma \mapsto r(L_\sigma)$. Let $\sigma > 0$ be fixed. Given $\nu < 1$, select $\delta > 0$ such that

$$\nu(b_{(j,e)}(t))^\sigma \leq (b_{(j,e)}(t))^{\sigma'} \leq \nu^{-1}(b_{(j,e)}(t))^\sigma \text{ for } t \in S_j, |\sigma - \sigma'| \leq \delta.$$

Then, using the argument as above, we have $\nu r(L_\sigma) \leq r(L_{\sigma'} \leq \nu^{-1} r(L_\sigma)$ whenever $|\sigma - \sigma'| \leq \delta$. Since $\nu < 1$ was arbitrary, this proves that $\sigma \mapsto r(L_\sigma)$ is continuous.

Since $\|L_0^k\| \geq \|L_0^k e\| \geq 1$, we see that $r(L_0) \geq 1$. Also if $|\mathcal{E}|$ denote the cardinality of $\mathcal{E}$, then from the definition of L_σ and using the fact $D_{(j,e)}(t) \leq c$ for all $t \in S_j$ and $(j,e) \in \Gamma$, it is clear that $\|L_\sigma\| \leq |\mathcal{E}|c^\sigma \to 0$ as $\sigma \to \infty$. So $r(L_\sigma) \leq \|L_\sigma\| \to 0$ as $\sigma \to \infty$. It follows by the continuity and strict monotonicity of $\sigma \to r(L_\sigma)$ that there exists a unique $\sigma_0 \geq 0$ such that $r(L_{\sigma_0}) = 1$. $\qquad\square$

Definition 4.7. We define *strong connectedness* to be the property that for each pair j and k in V there exists for some $n \geq 1$, $(J,E) = [(j_1,e_1), \ldots, (j_n,e_n)]$ such that $(j_i,e_i) \in \Gamma$, for $1 \leq i \leq n$, $j_1 = j$, $\alpha(j_i,e_i) = j_{i+1}$, $1 \leq i < n$ and $\alpha(j_n,e_n) = k$. Note that in this case we have a map $\theta_{(J,E)} = \theta_{(j_n,e_n)} \circ \cdots \circ \theta_{(j_1,e_1)}$ which maps S_j into S_k. Note also (compare H4.1) that strong connectedness implies that $\Gamma_i \neq \emptyset$ for $1 \leq i \leq p$ and $E_j \neq \emptyset$ for $1 \leq j \leq p$.

From now on we shall always assume strong connectedness.

H4.4 The property of strong connectedness is satisfied.

Lemma 4.8. *Assume that the hypotheses H4.1, H4.2, H4.3 and H4.4 are satisfied and let $u_\sigma \in K \setminus \{0\}$ be a nonzero eigenvector of L_σ with eigenvalue $r(L_\sigma)$. Then each component $(u_\sigma)_j$ is a strictly positive function on S_j for $1 \leq j \leq p$. Furthermore, there are constants l and L with $0 < l \leq L < \infty$ such that for every j, $1 \leq j \leq p$*

$$(4.5) \qquad\qquad l \leq (u_\sigma)_j(t) \leq L \quad \text{for all } t \in S_j$$

Proof. Suppose for some j, $1 \leq j \leq p$, $(u_\sigma)_j$ equals zero at some point in S_j. Then, since $(u_\sigma)_j \in K_j(M, \lambda)$, it follows, as shown in Remark 2.2, that $(u_\sigma)_j$ is identically equal to zero on S_j. Fix a k, $1 \leq k \leq p$. By strong connectedness, there exist $n \geq 1$ and $(J', E') = [(j_1, e_1), \ldots, (j_n, e_n)] \in \bar{\Gamma}_j^{(n)}$ with $j_1 = j$, and $\alpha(j_n, e_n) = k$. Since $L_\sigma u_\sigma = r(L_\sigma) u_\sigma$, it follows that $L_\sigma^n u_\sigma = (r(L_\sigma))^n u_\sigma$. So, using the formula for L_σ^n given by lemma 3.4 with A replaced by L_σ, we get

$$(r(L_\sigma))^n (u_\sigma)_j(s) = \sum_{(J,E) \in \bar{\Gamma}_j^{(n)}} b_{(J,E)}(s)(u_\sigma)_{\alpha(j_n, e_n)}(\theta_{(J,E)}(s)),$$

where $b_{(j,e)}(s) = ((D\theta_{(j,e)})(s))^\sigma$. The left hand side in the above equation is zero because $(u_\sigma)_j(s) = 0$. Thus, since each term in the sum in the right hand is nonnegative, it follows that each term equals zero. In particular, $(u_\sigma)_k(\theta_{(J,E)}(s)) = 0$ since $b_{(J,E)}(s)$ is strictly positive by H4.2. This implies $(u_\sigma)_k$ is identically equal to zero on S_k. Since this is true for any k, $1 \leq k \leq p$, we arrive at a contradiction that u_σ is identically zero. Thus $(u_\sigma)_j$ is a strictly positive function on S_j for $1 \leq j \leq p$.

Since each S_j is bounded, there is a $D < \infty$ such that $\mathrm{diam}(S_j) \leq D$ for $1 \leq j \leq p$. Then, since $(u_\sigma)_j \in K_j(M, \lambda)$, it follows that

$$0 < (u_\sigma)_j(s) \leq (u_\sigma)_j(t) \exp(MD^\lambda) \quad \text{for all } s, t \in S_j.$$

From this it is easy to see that there are constants $0 < l \leq L < \infty$ such that

$$l \leq (u_\sigma)_j(t) \leq L \quad \text{for all } t \in S_j, 1 \leq j \leq p.$$

$\square$

Let $C_j \subset S_j$, $1 \leq j \leq p$ be the invariant list of nonempty compact sets such that

$$C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \text{ for } 1 \leq i \leq p.$$

Our goal is to determine the Hausdorff dimension of sets C_i.

Let us recall the definition of Hausdorff measure and Hausdorff dimension. Suppose (X, d) is a metric space and $A \subset X$. We define, for $\epsilon > 0$ and $\sigma \geq 0$,

$$\mathcal{H}_\epsilon^\sigma(A) = \inf \left\{ \sum_{k=1}^\infty (\mathrm{diam}(A_k))^\sigma : A \subset \bigcup_{k=1}^\infty A_k, \mathrm{diam}(A_k) < \epsilon \right\}.$$

It follows that $\mathcal{H}_\epsilon^\sigma$ is an outer measure. For a given $\sigma \geq 0$, the function $\epsilon \mapsto \mathcal{H}_\epsilon^\sigma$ is decreasing and we define

$$\mathcal{H}^\sigma(A) = \lim_{\epsilon \to 0+} \mathcal{H}_\epsilon^\sigma(A) = \sup_{\epsilon > 0} \mathcal{H}_\epsilon^\sigma(A).$$

It follows that $\mathcal{H}^\sigma$ is a Borel measure and is called *Hausdorff σ-dimensional measure*. It is not hard to prove that there is a unique number $\sigma_0 \geq 0$ such that $\mathcal{H}^\sigma(A) = \infty$ for $0 \leq \sigma < \sigma_0$ and $\mathcal{H}^\sigma(A) = 0$ for $\sigma > \sigma_0$. The number σ_0 is called the *Hausdorff dimension* of A. We refer the reader to [5] and [11] for the basic properties of Hausdorff measure.

First we shall prove that Hausdorff dimension of C_i is independent of i, $1 \leq i \leq p$, under the assumption of strong connectedness.

Lemma 4.9. *Assume that the hypotheses H4.1, H4.2, H4.3 and H4.4 are satisfied and let $C_j \subset S_j$, $1 \leq j \leq p$ be the unique invariant list of compact, nonempty sets such that*

$$C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \text{ for } 1 \leq i \leq p.$$

Also assume that $\theta_{(j,e)}$ is one-to-one on C_j for all $(j,e) \in \Gamma$. Then $dim(C_j)$, the Hausdorff dimension of C_j, is independent of j for $1 \leq j \leq p$.

Proof. First we claim that $\dim(\theta_{(j,e)}(C_j)) = \dim(C_j)$ for any $(j,e) \in \Gamma$. Since $\theta_{(j,e)}$ is a Lipschitz map with Lipschitz constant c, $\mathcal{H}^\sigma(\theta_{(j,e)}(C_j)) \leq c^\sigma \mathcal{H}^\sigma(C_j)$ for any $\sigma \geq 0$. This implies $\dim(\theta_{(j,e)}(C_j)) \leq \dim(C_j)$. To prove the other inequality, it is enough to show that there exists $m_0 > 0$ such that $d(\theta_{(j,e)}(s), \theta_{(j,e)}(t) \geq m_0 d(s,t)$ for all $s, t \in C_j$. We abuse notation here by letting d denote d_j and $d_{\alpha(j,e)}$. Suppose. Then for each $k \geq 1$, there exists $s_k \neq t_k \in C_j$ such that $d(\theta_{(j,e)}(s_k), \theta_{(j,e)}(t_k) < k^{-1} d(s_k, t_k)$. Since C_j is compact we may assume, by taking subsequences, that there exist $s, t \in C_j$ such that $s_k \to s$ and $t_k \to t$ as $k \to \infty$. Letting $k \to \infty$ the previous inequality implies $d(\theta_{(j,e)}(s), \theta_{(j,e)}(t)) = 0$. Since $\theta_{(j,e)}$ is one-to-one on C_j, we must have $s = t$. But then $0 = \lim_{k \to \infty} \frac{d(\theta_{(j,e)}(s_k), \theta_{(j,e)}(t_k))}{d(s_k, t_k)} = (D\theta_{(j,e)})(t)$ which contradicts H4.2.

Now, since $\theta_{(j,e)}(C_j) \subset C_{\alpha(j,e)}$, $\dim(C_{\alpha(j,e)}) \geq \dim(\theta_{(j,e)}(C_j)) = \dim(C_j)$ for all $(j,e) \in \Gamma$. Let $1 \leq j \leq p$ and $1 \leq k \leq p$. By strong connectedness, there exists $[(j_1, e_1), \ldots, (j_n, e_n)]$ such that $j_1 = j$, $\alpha(j_i, e_i) = j_{i+1}, 1 \leq i < n$ and $\alpha(j_n, e_n) = k$. So,

$$\dim(C_k) = \dim(C_{\alpha(j_n, e_n)}) \geq \dim(C_{j_n}) \geq \dim(C_{j_{n-1}}) \geq \cdots \geq \dim(C_{j_1}) = \dim(C_j).$$

Since j and k were arbitrary, it follows that $\dim(C_j) = \dim(C_k)$ for all $1 \leq j, k \leq p$. $\qquad \square$

We introduce a 'weighted' Hausdorff measure using the strictly positive eigenvector u_σ of L_σ with eigenvalue $r(L_\sigma)$. Let $1 \leq j \leq p$. Define for $A_j \subset S_j$ and $\epsilon > 0$,

(4.6)
$$\tilde{\mathcal{H}}^\sigma_\epsilon(A_j) = \inf \left\{ \sum_{k=1}^\infty (u_\sigma)_j(\xi_{jk})(\mathrm{diam}(A_{jk}))^\sigma : A_j \subset \bigcup_{k=1}^\infty A_{jk}, \xi_{jk} \in A_{jk}, \mathrm{diam}(A_{jk}) < \epsilon \right\}.$$

From Lemma 4.8, we know that there exist constants $0 < l \leq L < \infty$ such that for $1 \leq j \leq p$, $l \leq (u_\sigma)_j(t) \leq L$ for all $t \in S_j$. This implies that, for $A_j \subset S_j$, $\tilde{\mathcal{H}}^\sigma_\epsilon(A_j)$ and $\mathcal{H}^\sigma_\epsilon(A_j)$ are equivalent

(4.7)
$$l\mathcal{H}^\sigma_\epsilon(A_j) \leq \tilde{\mathcal{H}}^\sigma_\epsilon(A_j) \leq L\mathcal{H}^\sigma_\epsilon(A_j)$$

Theorem 4.10. *Assume that the hypotheses H4.1, H4.2, H4.3 and H4.4 are satisfied and let $C_j \subset S_j$, $1 \leq j \leq p$ be the unique invariant list of compact, nonempty sets such that*

$$C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \text{ for } 1 \leq i \leq p.$$

Also assume that the map $\theta_{(j,e)}$ is one-to-one on an open neighborhood of C_j for $(j,e) \in \Gamma_i$, $1 \leq i \leq p$. If $dim(C_i)$ denotes the Hausdorff dimension of (C_i) and σ_0 denotes the unique nonnegative real number such that $r(L_{\sigma_0}) = 1$, then $dim(C_i) \leq \sigma_0$ for $1 \leq i \leq p$.

Proof. Fix $\epsilon > 0$. Let $\delta > 0$. We can choose a covering $\{A_{jk}\}_{k=1}^{\infty}$ of C_j and points $\xi_{jk} \in A_{jk}$ such that $\operatorname{diam}(A_{jk}) < \epsilon$ and

$$(4.8) \qquad \sum_{k=1}^{\infty} (u_\sigma)_j(\xi_{jk})(\operatorname{diam}(A_{jk}))^\sigma \leq \tilde{\mathcal{H}}_\epsilon^\sigma(C_j) + \delta.$$

Since $C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j)$, we have that $\{\theta_{(j,e)}(A_{jk}) : 1 \leq k < \infty, (j,e) \in \Gamma_i\}$ is a covering of C_i with

$$\operatorname{diam}(\theta_{(j,e)}(A_{jk})) \leq c \operatorname{diam}(A_{jk}) < c\epsilon.$$

Furthermore, using Lemma 4.3 and Remark 4.4, there exists $\mu_{(j,e)}(\epsilon)$ with $\mu_{(j,e)}(\epsilon) \to 1+$ as $\epsilon \to 0+$ such that for any $\xi_{jk} \in A_{jk}$,

$$(4.9) \qquad \operatorname{diam}(\theta_{(j,e)}(A_{jk})) \leq \mu_{(j,e)}(\epsilon)(D\theta_{(j,e)})(\xi_{jk})\operatorname{diam}(A_{jk}).$$

Let $\mu(\epsilon) = \max_{(j,e) \in \Gamma} \mu_{(j,e)}(\epsilon)$.

$$\tilde{\mathcal{H}}_{c\epsilon}^\sigma(C_i) \leq \sum_{k=1}^{\infty} \sum_{(j,e) \in \Gamma_i} (u_\sigma)_i(\theta_{(j,e)}(\xi_{jk}))(\operatorname{diam}(A_{jk}))^\sigma$$

$$\leq (\mu(\epsilon))^\sigma \sum_{k=1}^{\infty} \sum_{(j,e) \in \Gamma_i} (u_\sigma)_i(\theta_{(j,e)}(\xi_{jk}))((D\theta_{(j,e)})(\xi_{jk}))^\sigma (\operatorname{diam}(A_{jk}))^\sigma.$$

Summing over i, $1 \leq i \leq p$, we have

$$\sum_{i=1}^{p} \tilde{\mathcal{H}}_{c\epsilon}^\sigma(C_i) \leq (\mu(\epsilon))^\sigma \sum_{i=1}^{p} \sum_{k=1}^{\infty} \sum_{(j,e) \in \Gamma_i} (u_\sigma)_i(\theta_{(j,e)}(\xi_{jk}))((D\theta_{(j,e)})(\xi_{jk}))^\sigma (\operatorname{diam}(A_{jk}))^\sigma.$$

Rearranging the sum, we get

$$\sum_{i=1}^{p} \tilde{\mathcal{H}}_{c\epsilon}^\sigma(C_i) \leq (\mu(\epsilon))^\sigma \sum_{j=1}^{p} \sum_{k=1}^{\infty} (\operatorname{diam}(A_{jk}))^\sigma \sum_{i:(j,e) \in \Gamma_i} (u_\sigma)_i(\theta_{(j,e)}(\xi_{jk}))((D\theta_{(j,e)})(\xi_{jk}))^\sigma$$

$$= (\mu(\epsilon))^\sigma \sum_{j=1}^{p} \sum_{k=1}^{\infty} (L_\sigma u_\sigma)_j(\xi_{jk})(\operatorname{diam}(A_{jk}))^\sigma$$

$$= (\mu(\epsilon))^\sigma r(L_\sigma) \sum_{j=1}^{p} \sum_{k=1}^{\infty} (u_\sigma)_j(\xi_{jk})(\operatorname{diam}(A_{jk}))^\sigma.$$

Thus, using (4.8), we get

$$\sum_{i=1}^{p} \tilde{\mathcal{H}}_{c\epsilon}^\sigma(C_i) \leq (\mu(\epsilon))^\sigma r(L_\sigma) \sum_{j=1}^{p} \left(\tilde{\mathcal{H}}_\epsilon^\sigma(C_j) + \delta \right).$$

Since $c < 1$, $\tilde{\mathcal{H}}_\epsilon^\sigma(C_j) \leq \tilde{\mathcal{H}}_{c\epsilon}^\sigma(C_i)$. Also $\delta > 0$ was arbitrary. Therefore,

$$(4.10) \qquad \sum_{i=1}^{p} \tilde{\mathcal{H}}_\epsilon^\sigma(C_i) \leq (\mu(\epsilon))^\sigma r(L_\sigma) \sum_{i=1}^{p} \tilde{\mathcal{H}}_\epsilon^\sigma(C_i).$$

Using Lemma 4.6, $r(L_\sigma) < 1$ for all $\sigma > \sigma_0$. Since $\mu(\epsilon) \to 1$ as $\epsilon \to 0$, given $\sigma > \sigma_0$, we can choose $\epsilon > 0$ small so that $(\mu(\epsilon))^\sigma r(L_\sigma) < 1$. By the definition, $\tilde{\mathcal{H}}_\epsilon^\sigma(C_i) < \infty$

because we can take a finite ϵ-cover of the compact set C_i. Thus, if $\sigma > \sigma_0$, (4.10) can hold only if

$$\lim_{\epsilon \to 0+} \sum_{i=1}^{p} \tilde{\mathcal{H}}_{\epsilon}^{\sigma}(C_i) = 0.$$

This implies for each i, $1 \leq i \leq p$ and $\sigma > \sigma_0$, $\lim_{\epsilon \to 0+} \tilde{\mathcal{H}}_{\epsilon}^{\sigma}(C_i) = 0$, and hence using (4.7), $\lim_{\epsilon \to 0+} \mathcal{H}_{\epsilon}^{\sigma}(C_i) = 0$, i.e., $\mathcal{H}^{\sigma}(C_i) = 0$ for all $\sigma > \sigma_0$. Thus, by the definition of Hausdorff dimension, $\dim(C_i) \leq \sigma_0$. $\qquad\square$

We define for $0 < \eta < \epsilon$, and $A_j \subset S_j$
(4.11)
$$\tilde{\mathcal{H}}_{\epsilon,\eta}^{\sigma}(A_j) = \inf\left\{ \sum_{k=1}^{\infty} (u_\sigma)_j(\xi_{jk})(\mathrm{diam}(A_{jk}))^{\sigma} : A_j \subset \bigcup_{k=1}^{\infty} A_{jk}, \xi_{jk} \in A_{jk}, \eta < \mathrm{diam}(A_{jk}) < \epsilon \right\}.$$

Lemma 4.11. *Let $1 \leq j \leq p$ and A_j be a compact subset of S_j. If $\sigma \geq 0$ and $\epsilon > 0$, then*

$$\lim_{\eta \to 0+} \tilde{\mathcal{H}}_{\epsilon,\eta}^{\sigma}(A_j) = \tilde{\mathcal{H}}_{\epsilon}^{\sigma}(A_j).$$

Proof. For $0 < \eta < \epsilon$, we have $\tilde{\mathcal{H}}_{\epsilon,\eta}^{\sigma}(A_j) \geq \tilde{\mathcal{H}}_{\epsilon}^{\sigma}(A_j)$ because the infimum is taken over a smaller set. So,

$$\lim_{\eta \to 0+} \tilde{\mathcal{H}}_{\epsilon,\eta}^{\sigma}(A_j) \geq \tilde{\mathcal{H}}_{\epsilon}^{\sigma}(A_j).$$

To prove the reverse inequality, take $\delta > 0$ and choose a covering $\{A_{jk} : 1 \leq k < \infty\}$ of A_j by sets A_{jk} with $\mathrm{diam}(A_{jk}) < \epsilon$, $1 \leq k < \infty$ such that

$$\inf\left\{ \sum_{k=1}^{\infty} (u_\sigma)_j(\xi_{jk})(\mathrm{diam}(A_{jk}))^{\sigma} : \xi_{jk} \in A_{jk} \right\} \leq \tilde{\mathcal{H}}_{\epsilon}^{\sigma}(A_j) + \delta.$$

Without loss of generality, we can assume that the sets $A_{jk}, k \geq 1$ are open. Since A_j is compact, there exists a finite open subcover of A_j, so there exists an integer $l < \infty$ such that

$$A_j \subset \bigcup_{k=1}^{l} A_{jk}.$$

Let $0 < \eta_0 < \epsilon$ be such that $\eta_0 < \min_{1 \leq k \leq l} \mathrm{diam}(A_{jk})$. Then, for $0 < \eta < \eta_0$, we have

$$\tilde{\mathcal{H}}_{\epsilon,\eta}^{\sigma}(A_j) \leq \inf\left\{ \sum_{k=1}^{l} (u_\sigma)_j(\xi_{jk})(\mathrm{diam}(A_{jk}))^{\sigma} : \xi_{jk} \in A_{jk} \right\}.$$

So, for every $\delta > 0$, there exists η_0, $0 < \eta_0 < \epsilon$, such that

$$\tilde{\mathcal{H}}_{\epsilon,\eta}^{\sigma}(A_j) \leq \tilde{\mathcal{H}}_{\epsilon}^{\sigma}(A_j) + \delta \quad \text{for } 0 < \eta < \eta_0.$$

This shows

$$\lim_{\eta \to 0+} \tilde{\mathcal{H}}_{\epsilon,\eta}^{\sigma}(A_j) \leq \tilde{\mathcal{H}}_{\epsilon}^{\sigma}(A_j)$$

and completes the proof of the lemma. $\qquad\square$

Lemma 4.12. *Let $1 \leq j \leq p$ and A_j be a compact subset of S_j. Let $\sigma > 0$ be such that $\mathcal{H}^{\sigma}(A_j) = 0$. Then for every ϵ_1 and ϵ_2 with $0 < \epsilon_1 < \epsilon_2$, there exists an $\eta_0 > 0$ such that for any $B_j \subset A_j$*

$$\tilde{\mathcal{H}}_{\epsilon_1,\eta}^{\sigma}(B_j) = \tilde{\mathcal{H}}_{\epsilon_2,\eta}^{\sigma}(B_j) \quad \text{for } 0 < \eta < \eta_0.$$

Proof. Since $\mathcal{H}^\sigma(A_j) = 0$, it follows that $\mathcal{H}^\sigma_\epsilon(A_j) = 0$ for every $\epsilon > 0$. By using (4.7), it also follows that $\tilde{\mathcal{H}}^\sigma_\epsilon(A_j) = 0$ for every $\epsilon > 0$. So, by lemma 4.11,

$$\lim_{\eta \to 0+} \tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(A_j) = 0.$$

This implies that there exists $\eta_0 > 0$ such that for $0 < \eta < \eta_0$

$$\tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(A_j) < l\epsilon_1{}^\sigma,$$

where, as before, $l > 0$ is such that $(u_\sigma)_j(t) \geq l$ for all $t \in S_j$.

If $B_j \subset A_j$, then $\tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(B_j) \leq \tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(A_j) < l\epsilon_1{}^\sigma$. Therefore, given $\delta > 0$, there exists a covering $\{B_{jk} : k \geq 1\}$ of B_j such that $\eta < \mathrm{diam}(B_{jk}) < \epsilon_2$ for $k \geq 1$ and

$$\inf\left\{\sum_{k=1}^\infty (u_\sigma)_j(\xi_{jk})(\mathrm{diam}(B_{jk}))^\sigma : \xi_{jk} \in B_{jk}\right\} \leq \tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(B_j) + \delta < l\epsilon_1{}^\sigma.$$

Next we claim that actually $\mathrm{diam}(B_{jk}) < \epsilon_1$ for all $k \geq 1$. Suppose not, then there exists an index k_1 such that $\mathrm{diam}(B_{jk_1}) \geq \epsilon_1$. By considering the term corresponding to index k_1 in the sum and using $(u_\sigma)_j(\xi_{jk}) \geq l$, we get

$$\inf\left\{\sum_{k=1}^\infty (u_\sigma)_j(\xi_{jk})(\mathrm{diam}(B_{jk}))^\sigma : \xi_{jk} \in B_{jk}\right\} \geq l\epsilon_1{}^\sigma$$

which gives a contradiction. Thus $\mathrm{diam}(B_{jk}) < \epsilon_1$ for all $k \geq 1$ and we conclude that

$$\tilde{\mathcal{H}}^\sigma_{\epsilon_1,\eta}(B_j) \leq \tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(B_j) + \delta \quad \text{for } 0 < \eta < \eta_0.$$

Since $\delta > 0$ was arbitrary, we conclude

$$\tilde{\mathcal{H}}^\sigma_{\epsilon_1,\eta}(B_j) \leq \tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(B_j) \quad \text{for } 0 < \eta < \eta_0.$$

Since $\tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(B_j)$ is a decreasing function of ϵ, the reverse inequality is obvious. Thus, we obtain

$$\tilde{\mathcal{H}}^\sigma_{\epsilon_1,\eta}(B_j) = \tilde{\mathcal{H}}^\sigma_{\epsilon_2,\eta}(B_j) \quad \text{for } 0 < \eta < \eta_0.$$

$\square$

Lemma 4.13. *Assume that the hypotheses H3.1, H3.2, H3.3 and H3.4 are satisfied and let $C_j \subset S_j$, $1 \leq j \leq p$ be the unique invariant list such that*

$$C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \text{ for } 1 \leq i \leq p.$$

Also assume that the map $\theta_{(j,e)}$ is one-to-one on an open neighborhood of C_j for $(j,e) \in \Gamma_i$, $1 \leq i \leq p$. Suppose for $\nu > 0$, $\{\Lambda^i_\alpha : \alpha \in \mathcal{A}_i(\nu)\}$ is a partition of C_i consisting of compact subsets such that $\mathrm{diam}(\Lambda^i_\alpha) < \nu$. Then there exists an $\epsilon_0 > 0$ such that for $0 < \eta < \epsilon < \epsilon_0$ and $1 \leq i \leq p$

$$(4.12) \qquad \mu(\nu) r(L_\sigma) \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(\Lambda^i_\alpha) \leq \sum_{e \in E_i} \tilde{\mathcal{H}}^\sigma_{m\epsilon,\eta}(\theta_{(i,e)}(\Lambda^i_\alpha))$$

where $\mu(\nu) \to 1$ as $\nu \to 0$ and $(D\theta_{(i,e)})(t) > m$ for all $t \in S_i$.

Proof. Fix $\nu > 0, 1 \leq i \leq p$, $\alpha \in \mathcal{A}_i(\nu)$ and $e \in E_i$. Let $0 < \eta < m\epsilon$. For any $\delta > 0$, there exists an open covering $\{A_j : j \geq 1\}$ of $\theta_{(i,e)}(\Lambda_\alpha^i)$ such that $\eta < \text{diam}(A_j) < m\epsilon$ and

$$(4.13) \quad \inf\left\{\sum_{j=1}^\infty (u_\sigma)_{\alpha(i,e)}(\xi_j)(\text{diam}(A_j))^\sigma : \xi_j \in A_j\right\} \leq \tilde{\mathcal{H}}_{m\epsilon,\eta}^\sigma(\theta_{(i,e)}(\Lambda_\alpha^i)) + \delta.$$

Since $\theta_{(i,e)}(\Lambda_\alpha^i)$ is compact, there exists a finite open subcover and, by relabelling, we can assume that there exists an integer $k < \infty$ such that

$$(4.14) \quad \theta_{(i,e)}(\Lambda_\alpha^i) \subset \bigcup_{j=1}^k A_j \subset N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i))$$

The last inclusion holds because $\text{diam}(A_j) < m\epsilon$ and we may assume that each A_j, $1 \leq j \leq k$, intersects $\theta_{(i,e)}(\Lambda_\alpha^i)$. Since $\text{diam}(\theta_{(i,e)}(\Lambda_\alpha^i)) \leq c\, \text{diam}(\Lambda_\alpha^i) < c\nu$,

$$(4.15) \quad \text{diam}(N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i))) \leq c\nu + 2m\epsilon < \nu$$

provided $\epsilon < \epsilon_0 = (1-c)\nu/2m$. From (4.14), we have

$$(4.16) \quad \Lambda_\alpha^i \subset \bigcup_{j=1}^k \theta_{(i,e)}^{-1}(A_j) \subset \theta_{(i,e)}^{-1}(N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i)))$$

Since $\theta_{(i,e)}$ is one-to-one on an open neighborhood of C_i, we may assume that $\theta_{(i,e)}$ is one-to-one on $\theta_{(i,e)}^{-1}(N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i)))$ by choosing ϵ small enough. So, we have a map

$$\theta_{(i,e)}^{-1} : N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i)) \to \theta_{(i,e)}^{-1}(N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i))).$$

Using Lemma 4.3 and Remark 4.4, there exists $\mu_1(\nu) \to 1+$ as $\nu \to 0+$ and

$$(4.17) \quad \mu_1(\nu)^{-1}(D\theta_{(i,e)}^{-1})(y) \leq \frac{d(\theta_{(i,e)}^{-1}(x), \theta_{(i,e)}^{-1}(y))}{d(x,y)} \leq \mu_1(\nu)(D\theta_{(i,e)}^{-1})(y)$$

for all $x, y \in N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i)), x \neq y$ because $d(x,y) < \nu$ from (4.15). In particular, for any $x, y \in A_j$, $1 \leq j \leq k$,

$$d(\theta_{(i,e)}^{-1}(x), \theta_{(i,e)}^{-1}(y)) \leq \mu_1(\nu)(D\theta_{(i,e)}^{-1})(y)d(x,y) \leq \mu_1(\nu)\frac{1}{(D\theta_{(i,e)})(\theta_{(i,e)}^{-1}y)}\text{diam}(A_j).$$

So,

$$(4.18) \quad \text{diam}(\theta_{(i,e)}^{-1}(A_j)) \leq \mu_1(\nu)\frac{1}{(D\theta_{(i,e)})(\tau_1)}\text{diam}(A_j)$$

for some $\tau_1 \in \theta_{(i,e)}^{-1}(N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i)))$.
Since $\text{diam}(A_j) < m\epsilon$, for ν small enough,

$$\text{diam}(\theta_{(i,e)}^{-1}(A_j)) \leq \mu_1(\nu)\frac{1}{m}m\epsilon < \epsilon.$$

Also, by choosing $x_0, y_0 \in N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i))$ with $d(x_0, y_0) = \text{diam}(A_j)$, we get

$$\text{diam}(\theta_{(i,e)}^{-1}(A_j)) \geq d(\theta_{(i,e)}^{-1}(x_0), \theta_{(i,e)}^{-1}(y_0)) \geq \mu_1(\nu)^{-1}\frac{1}{(D\theta_{(i,e)})(\theta_{(i,e)}^{-1}y_0)}\text{diam}(A_j)$$

Since $\operatorname{diam}(A_j) > \eta$ and $(D\theta_{(i,e)})(\theta_{(i,e)}^{-1} y_0) < c < 1$, we get $\operatorname{diam}(\theta_{(i,e)}^{-1}(A_j)) > \eta$ if ν is small. Thus, we have obtained $\eta < \operatorname{diam}(\theta_{(i,e)}^{-1}(A_j)) < \epsilon$ and $\Lambda_\alpha^i \subset \bigcup_{j=1}^k \theta_{(i,e)}^{-1}(A_j)$ from (4.16). Therefore,

$$\tilde{\mathcal{H}}_{\epsilon,\eta}^\sigma(\Lambda_\alpha^i) \leq \inf \left\{ \sum_{j=1}^k (u_\sigma)_i(\zeta_j)(\operatorname{diam}(\theta_{(i,e)}^{-1}(A_j)))^\sigma : \zeta_j \in \theta_{(i,e)}^{-1}(A_j) \right\}.$$

Using (4.18), we get

$$\tilde{\mathcal{H}}_{\epsilon,\eta}^\sigma(\Lambda_\alpha^i) \leq \frac{\mu_1(\nu)^\sigma}{((D\theta_{(i,e)})(\tau_1))^\sigma} \inf \left\{ \sum_{j=1}^k \frac{(u_\sigma)_{\alpha(i,e)}(\theta_{(i,e)}(\zeta_j))(\operatorname{diam}(A_j))^\sigma (u_\sigma)_i(\zeta_j)}{(u_\sigma)_{\alpha(i,e)}(\theta_{(i,e)}(\zeta_j))} : \zeta_j \in \theta_{(i,e)}^{-1}(A_j) \right\}.$$

Choose $\tau_2 \in \theta_{(i,e)}^{-1}(N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i)))$ such that

$$\frac{(u_\sigma)_i(\tau_2)}{(u_\sigma)_{\alpha(i,e)}(\theta_{(i,e)}(\tau_2))} \geq \frac{(u_\sigma)_i(\zeta_j)}{(u_\sigma)_{\alpha(i,e)}(\theta_{(i,e)}(\zeta_j))}$$

for all $\zeta_j \in \theta_{(i,e)}^{-1}(A_j)$, $1 \leq j \leq k$. Using this together with (4.13), we get

$$\tilde{\mathcal{H}}_{\epsilon,\eta}^\sigma(\Lambda_\alpha^i) \leq \frac{\mu_1(\nu)^\sigma}{((D\theta_{(i,e)})(\tau_1))^\sigma} \frac{(u_\sigma)_i(\tau_2)}{(u_\sigma)_{\alpha(i,e)}(\theta_{(i,e)}(\tau_2))} (\tilde{\mathcal{H}}_{m\epsilon,\eta}^\sigma(\theta_{(i,e)}(\Lambda_\alpha^i)) + \delta).$$

Since $\delta > 0$ was arbitrary,

$$(4.19) \qquad \tilde{\mathcal{H}}_{\epsilon,\eta}^\sigma(\Lambda_\alpha^i) \leq \frac{\mu_1(\nu)^\sigma}{((D\theta_{(i,e)})(\tau_1))^\sigma} \frac{(u_\sigma)_i(\tau_2)}{(u_\sigma)_{\alpha(i,e)}(\theta_{(i,e)}(\tau_2))} \tilde{\mathcal{H}}_{m\epsilon,\eta}^\sigma(\theta_{(i,e)}(\Lambda_\alpha^i)).$$

The final step consists of replacing τ_1, τ_2 by a $\xi \in \Lambda_\alpha^i$. From (4.17), it follows that

$$\operatorname{diam}(\theta_{(i,e)}^{-1}(N_{m\epsilon}(\theta_{(i,e)}(\Lambda_\alpha^i)))) \leq \mu_1(\nu)\frac{1}{m}\nu \leq \kappa\nu,$$

where κ is independent of ν. In particular, we have that $d(\tau_1, \tau_2) \leq \kappa\nu$, so by continuity, there exists a function $\mu_2(\nu)$ such that $\mu_2(\nu) \to 1$ as $\nu \to 0$ and

$$\mu_2(\nu)((D\theta_{(i,e)})(\xi))^\sigma \frac{(u_\sigma)_{\alpha(i,e)}(\theta_{(i,e)}(\xi))}{(u_\sigma)_i(\xi)} \leq ((D\theta_{(i,e)})(\tau_1))^\sigma \frac{(u_\sigma)_{\alpha(i,e)}(\theta_{(i,e)}(\tau_2))}{(u_\sigma)_i(\tau_2)},$$

$\xi \in \Lambda_\alpha^i$. Using this (4.19) implies

$$\tilde{\mathcal{H}}_{m\epsilon,\eta}^\sigma(\theta_{(i,e)}(\Lambda_\alpha^i)) \geq \mu(\nu)\frac{(u_\sigma)_{\alpha(i,e)}(\theta_{(i,e)}(\xi))}{(u_\sigma)_i(\xi)}((D\theta_{(i,e)})(\xi))^\sigma \tilde{\mathcal{H}}_{\epsilon,\eta}^\sigma(\Lambda_\alpha^i),$$

where $\mu(\nu) = \mu_2(\nu)(\mu_1(\nu))^{-\sigma}$. Now, we sum over $e \in E_i$, and use that

$$\sum_{e \in E_i} ((D\theta_{(i,e)})(\xi))^\sigma (u_\sigma)_{\alpha(i,e)}(\theta_{(i,e)}(\xi)) = (L_\sigma u_\sigma)_i(\xi) = r(L_\sigma)(u_\sigma)_i(\xi)$$

to obtain (4.12). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Now we are ready to prove the remaining inequality.

Theorem 4.14. *Assume that the hypotheses H3.1, H3.2, H3.3 and H3.4 are satisfied and let $C_j \subset S_j$, $1 \leq j \leq p$ be the unique invariant list such that*

$$C_i = \bigcup_{(j,e) \in \Gamma_i} \theta_{(j,e)}(C_j) \text{ for } 1 \leq i \leq p.$$

Also assume that the map $\theta_{(j,e)}$ is one-to-one on an open neighborhood of C_j for $(j,e) \in \Gamma_i$, $1 \le i \le p$, and $\theta_{(j,e)}(C_j) \cap \theta_{(j',e')}(C'_j) = \emptyset$ for $(j,e) \in \Gamma_i, (j',e') \in \Gamma_i, (j,e) \ne (j',e')$. Let σ_0 be the unique nonnegative real number such that $r(L_{\sigma_0}) = 1$ and β_0 denote the common Hausdorff dimension of C_i for $1 \le i \le p$. Then $\beta_0 \ge \sigma_0$.

Proof. Suppose $\beta_0 < \sigma_0$. Then there exists a $\sigma < \sigma_0$ such that $\mathcal{H}^\sigma(C_i) = 0$ for $1 \le i \le p$. This implies for every $\epsilon > 0$, $\mathcal{H}^\sigma_\epsilon(C_i) = 0$ and using (4.7), we have

$$(4.20) \qquad \tilde{\mathcal{H}}^\sigma_\epsilon(C_i) = 0 \quad \text{for } \epsilon > 0.$$

Let $(J,E) = [(j_1,e_1), \cdots, (j_n,e_n)] \in \Gamma^{(n)}$ and $\theta_{(J,E)} = \theta_{(j_1,e_1)} \circ \ldots \circ \theta_{(j_n,e_n)}$. Then $\operatorname{diam}(\theta_{(J,E)}(C_{j_n})) \le c^n \operatorname{diam}(C_{j_n})$. So, given $\nu > 0$, we can choose $n = n(\nu)$ large enough such that

$$\operatorname{diam}(\theta_{(J,E)}(C_{j_n})) < \nu \text{ for all } (J,E) \in \Gamma^{(n)}.$$

For $1 \le i \le p$, we have

$$(4.21) \qquad C_i = \bigcup_{(J,E) \in \Gamma_i^{(n)}} \theta_{(J,E)}(C_{j_n})$$

with the union being pairwise disjoint using the disjointness assumption. By the previous lemma,

$$\mu(\nu) r(L_\sigma) \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(\theta_{(J,E)}(C_{j_n})) \le \sum_{e \in E_i} \tilde{\mathcal{H}}^\sigma_{m\epsilon,\eta}(\theta_{(i,e)}(\theta_{(J,E)}(C_{j_n})))$$

where $\mu(\nu) \to 1$ as $\nu \to 0$. Since $\mathcal{H}^\sigma(C_j) = 0$ for $1 \le j \le p$, using Lemma 4.12, we get $\eta_0 > 0$ such that for $0 < \eta < \eta_0$

$$(4.22) \qquad \tilde{\mathcal{H}}^\sigma_{m\epsilon,\eta}(\theta_{(i,e)}(\theta_{(J,E)}(C_{j_n}))) = \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(\theta_{(i,e)}(\theta_{(J,E)}(C_{j_n})))$$

Therefore, the previous inequality becomes

$$(4.23) \qquad \mu(\nu) r(L_\sigma) \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(\theta_{(J,E)}(C_{j_n})) \le \sum_{e \in E_i} \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(\theta_{(i,e)}(\theta_{(J,E)}(C_{j_n})))$$

Now from (4.21), since the union is disjoint, we can choose $\epsilon > 0$ so small that

$$N_\epsilon(\theta_{(J,E)}(C_{j_n})) \cap N_\epsilon(\theta_{(J',E')}(C_{j_n})) = \emptyset$$

for all $(J,E), (J',E') \in \Gamma_i^{(n)}, (J,E) \ne (J',E')$. This implies that

$$\tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(C_i) = \sum_{(J,E) \in \Gamma_i^{(n)}} \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(\theta_{(J,E)}(C_{j_n})).$$

Therefore, we can sum (4.23) over all $(J,E) \in \Gamma_i^{(n)}$ to obtain

$$\mu(\nu) r(L_\sigma) \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(C_i) \le \sum_{(J,E) \in \Gamma_i^{(n)}} \sum_{e \in E_i} \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(\theta_{(i,e)}(\theta_{(J,E)}(C_{j_n}))).$$

Now we sum over $i = 1, 2, \cdots, p$ to get

$$\mu(\nu) r(L_\sigma) \sum_{i=1}^{p} \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(C_i) \le \sum_{i=1}^{p} \sum_{(J,E) \in \Gamma_i^{(n)}} \sum_{e \in E_i} \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(\theta_{(i,e)}(\theta_{(J,E)}(C_{j_n}))).$$

Note that $\theta_{(i,e)}(\theta_{(J,E)}(C_{j_n})) \subset C_{\alpha(i,e)}$, so collecting the terms with $\alpha(i,e) = j$, $1 \leq j \leq p$, we get

$$\mu(\nu)r(L_\sigma) \sum_{i=1}^p \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(C_i) \leq \sum_{j=1}^p \sum_{(i,e)\in\Gamma_j} \sum_{(J,E)\in\Gamma_i^{(n)}} \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(\theta_{(i,e)}(\theta_{(J,E)}(C_{j_n}))).$$

Since $C_j = \bigcup_{(i,e)\in\Gamma_j} \bigcup_{(J,E)\in\Gamma_i^{(n)}} \theta_{(i,e)}(\theta_{(J,E)}(C_{j_n}))$ with disjoint union, we get

$$(4.24) \qquad \mu(\nu)r(L_\sigma) \sum_{i=1}^p \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(C_i) \leq \sum_{j=1}^p \tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(C_j)$$

Since $\sigma < \sigma_0$, Lemma 4.6 implies $r(L_\sigma) > 1$, so we can choose $\nu > 0$ small enough so that $\mu(\nu)r(L_\sigma) > 1$. But, for $1 \leq i \leq p$, we have $\tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(C_i) > 0$ by the definition of $\tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}$. Also using Lemma 4.11 and (4.20), $\tilde{\mathcal{H}}^\sigma_{\epsilon,\eta}(C_i) < \infty$ for η small enough. So, (4.24) cannot be true. Therefore, our initial assumption must be wrong. Thus, $\beta_0 \geq \sigma_0$. $\qquad \square$

Combining Theorem 4.10 and Theorem 4.14, we have proved the following theorem.

Theorem 4.15. *Assume that the hypotheses H3.1, H3.2, H3.3 and H3.4 are satisfied and let $C_j \subset S_j$, $1 \leq j \leq p$ be the unique invariant list such that*

$$C_i = \bigcup_{(j,e)\in\Gamma_i} \theta_{(j,e)}(C_j) \text{ for } 1 \leq i \leq p.$$

Also assume that the map $\theta_{(j,e)}$ is one-to-one on an open neighborhood of C_j for $(j,e) \in \Gamma_i$, $1 \leq i \leq p$, and $\theta_{(j,e)}(C_j) \cap \theta_{(j',e')}(C'_j) = \emptyset$ for $(j,e) \in \Gamma_i, (j',e') \in \Gamma_i, (j,e) \neq (j',e')$. Let σ_0 be the unique nonnegative real number such that $r(L_{\sigma_0}) = 1$. Then the Hausdorff dimension of each C_i for $1 \leq i \leq p$ is the same, and if β_0 denote the common Hausdorff dimension of C_i, then $\beta_0 = \sigma_0$.

5. INFINITE ITERATED FUNCTION SYSTEMS

Let (S,d) be a compact metric space, and for $1 \leq i < \infty$, $\theta_i : S \to S$ be a Lipschitz map with $\mathrm{Lip}(\theta_i) \leq c < 1$. Then, following the proof of Theorem 2.3, we can prove that there exists a unique nonempty compact set K such that

$$(5.1) \qquad K = \overline{\bigcup_{i=1}^\infty \theta_i(K)}$$

We wish to find a formula for the Hausdorff dimension of K. As in the case of finite iterated function systems, we study Perron-Frobenius operator $L : C(S) \to C(S)$ of the form

$$(5.2) \qquad (Lf)(t) = \sum_{i=1}^\infty b_i(t)f(\theta_i(t))$$

We shall make the following assumptions.

H5.1 For $1 \leq i < \infty$, the function $b_i : S \to \mathbb{R}$ is nonnegative and continuous. Furthermore, for each $t \in S$, $b(t) = \sum_{i=1}^\infty b_i(t) < \infty$ and $b : S \to \mathbb{R}$ is continuous.

H5.2 For $1 \leq i < \infty$, $\theta_i : S \to S$ is a Lipschitz map with $\mathrm{Lip}(\theta_i) \leq c < 1$.

If H5.1 is satisfied and the maps θ_i are all continuous, it is easy to verify that L defines a bounded linear map on $C(S)$. We refer the reader to Section 5 of [19] for a detailed discussion of such operators.

Let $K(M, \lambda) = \{f \in C(S) : 0 \le f(s) \le f(t) \exp(Md(s,t)^\lambda) \text{ for all } s, t \in S\}$.

Lemma 5.1. *Suppose that H5.1 and H5.2 are satisfied, and let $L : C(S) \to C(S)$ be defined by eq.(5.2). Assume that there exist $M_0 > 0$ and $\lambda > 0$ such that for each $i \ge 1$, $b_i \in K(M_0, \lambda)$. Also assume that for all $t \in S$, $\sum_{i=1}^\infty b_i(t) > 0$. Then there exists $M > 0$ such that $L(K(M, \lambda)) \subset K(M, \lambda)$ and L has a strictly positive eigenvector $u \in K(M, \lambda)$ with eigenvalue $r(L) > 0$.*

Proof. Since $c < 1$, we can choose $M > 0$ so that $M_0 + Mc^\lambda \le M$. We claim that $L(K) \subset K$, where $K := K(M, \lambda)$. Let $f \in K$, $i \ge 1$ and $s, t \in S$. Then we have

$$f(\theta_i(s)) \le f(\theta_i(t)) \exp(Md(\theta_i(s), \theta_i(t))^\lambda) \le f(\theta_i(t)) \exp(Mc^\lambda d(s,t)^\lambda) \text{ and}$$
$$b_i(s) \le b_i(t) \exp(M_0 d(s,t)^\lambda).$$

Thus

$$(Lf)(s) = \sum_{i=1}^\infty b_i(s) f(\theta_i(s))$$
$$\le \sum_{i=1}^\infty b_i(t) f(\theta_i(t)) \exp((M_0 + Mc^\lambda)d(s,t)^\lambda)$$
$$\le (Lf)(t) \exp(Md(s,t)^\lambda).$$

This proves that $L(K) \subset K$.

By Lemma 3.2, $\{f \in K : \|f\| \le 1\}$ is equicontinuous, and hence it is compact by Ascoli's theorem. It follows that $\rho_K(L) = 0$, where $\rho_K(L)$ denotes the cone essential spectral radius of L. The constant function $e = 1$ is in K, so

$$r_K(L) \ge \lim_{n \to \infty} \|L^n(e)\|^{\frac{1}{n}} = r(L)$$

The opposite inequality is obviously true, so $r_K(L) = r(L)$. By assumption $b(t) = \sum_{i=1}^\infty b_i(t) > 0$ for all $t \in S$, and $b : S \to \mathbb{R}$ is continuous by H5.1. Therefore, there exists $\delta > 0$ with $\sum_{i=1}^\infty b_i(t) \ge \delta$. So $L(e) \ge \delta e$ from which it follows that $r(L) \ge \delta > 0$. Thus $\rho_K(L) < r_K(L)$, which implies that there exists $u \in K$, $\|u\| = 1$, with $L(u) = ru$, $r = r(L)$. From the definition of $K(M, \lambda)$ it is clear that $u \in K \setminus \{0\}$ implies $u(t) > 0$ for all $t \in S$. $\qquad \square$

Throughout this section we shall make the following assumption.

H5.3 Let (S, d) be a compact metric space and assume that for $1 \le i < \infty$, $\theta_i : S \to S$ is an infinitesimal similitude on S and is a Lipschitz map with $\mathrm{Lip}(\theta_i) \le c < 1$. Assume that $D\theta_i(t) > 0$ for all $t \in S$ and that $D\theta_i \in K(M_0, \lambda)$ for some $M_0 > 0$, $\lambda > 0$.

Lemma 5.2. *If $\sum_{i=1}^\infty (D\theta_i(t_*))^s < \infty$ for some $t_* \in S$ and $s > 0$, then for any $\sigma \ge s$, $\sum_{i=1}^\infty (D\theta_i(t))^\sigma < \infty$ for all $t \in S$.*

Proof. Since $D\theta_i(t) < 1$ for all $t \in S$, $(D\theta_i(t))^\sigma \le (D\theta_i(t))^s$ for $\sigma \ge s$. So $\sum_{i=1}^\infty (D\theta_i(t_*))^s < \infty$ implies that $\sum_{i=1}^\infty (D\theta_i(t_*))^\sigma < \infty$ for $\sigma \ge s$. Thus we only have to prove that for any $t \in S$, $\sum_{i=1}^\infty (D\theta_i(t))^s < \infty$. Let $t \in S$. Since $D\theta_i \in K(M_0, \lambda)$, $D\theta_i(t) \le D\theta_i(t_*) \exp(M_0 d(t, t_*)^\lambda)$. Therefore,

$$(D\theta_i(t))^s \le (D\theta_i(t_*))^s \exp(sM_0 d(t, t_*)^\lambda)$$

from which the result follows. $\qquad\square$

Assume H5.3 is satisfied. Define for $\sigma > 0$ with $\sum_{i=1}^{\infty}(D\theta_i(t))^{\sigma} < \infty$,

$$(L_{\sigma}f)(t) = \sum_{i=1}^{\infty}(D\theta_i(t))^{\sigma}f(\theta_i(t)).$$

Let $\sigma > 0$ satisfying above be fixed. If H5.1 is satisfied with $b_i(t) := (D\theta_i(t))^{\sigma}$ for $i \geq 1$, we know that L_{σ} defines a bounded linear map on $C(S)$. By H5.3, $D\theta_i \in K(M_0, \lambda)$ for some $M_0 > 0$, $\lambda > 0$. This implies that $b_i \in K(\sigma M_0, \lambda)$ for all $i \geq 1$. Therefore, if we choose $0 < M < \infty$ such that $\sigma M_0 + M c^{\lambda} \leq M$, L_{σ} maps $K(M, \lambda)$ into itself and has an eigenvector $u_{\sigma} \in K(M, \lambda)$ with eigenvalue $r(L_{\sigma})$.

Let $\sigma_0 = \inf\{\sigma > 0 : \sum_{i=1}^{\infty}(D\theta_i(t))^{\sigma} < \infty \text{ for all } t \in S\}$. There are two possible cases.

Case I: $\sum_{i=1}^{\infty}(D\theta_i(t))^{\sigma} < \infty$ for $\sigma > \sigma_0$ but $\sum_{i=1}^{\infty}(D\theta_i(t))^{\sigma_0} = \infty$.

Case II: $\sum_{i=1}^{\infty}(D\theta_i(t))^{\sigma} < \infty$ for $\sigma \geq \sigma_0$ and $\sum_{i=1}^{\infty}(D\theta_i(t))^{\sigma} = \infty$ for $\sigma < \sigma_0$.

Note that L_{σ} is defined for $\sigma > \sigma_0$ in case I and for $\sigma \geq \sigma_0$ in case II.

Lemma 5.3. *The map $\sigma \mapsto r(L_{\sigma})$ is strictly decreasing and continuous for $\sigma > \sigma_0$ in case I and for $\sigma \geq \sigma_0$ in case II. Also, $r(L_{\sigma}) \to 0$ as $\sigma \to \infty$.*

Proof. The proof of strictly decreasing and continuity is exactly the same as the proof of Lemma 4.6 and we omit the details. To see the last part, fix $s > 0$ with $\sup_{t \in S}\sum_{i=1}^{\infty}(D\theta_i(t))^s \leq K < \infty$. Let $\sigma > s$. Then since $D\theta_i(t) \leq c$, $\sum_{i=1}^{\infty}(D\theta_i(t))^{\sigma} \leq c^{\sigma-s}\sum_{i=1}^{\infty}(D\theta_i(t))^s$. Therefore, $||L_{\sigma}|| \leq \sup_{t \in S}\sum_{i=1}^{\infty}(D\theta_i(t))^{\sigma} \leq c^{\sigma-s}K$ which implies $||L_{\sigma}|| \to 0$ as $\sigma \to \infty$ because $c < 1$. Since $r(L_{\sigma}) \leq ||L_{\sigma}||$, the result follows. $\qquad\square$

We should note that in the case of infinite iterated function systems there need not be a value of σ for which $r(L_{\sigma}) = 1$ because we cannot guarantee a σ for which $r(L_{\sigma}) \geq 1$. It is possible that $r(L_{\sigma}) < 1$ for all the values of σ for which L_{σ} is defined. Let

$$(5.3) \qquad\qquad \sigma_{\infty} = \inf\{\sigma > 0 : r(L_{\sigma}) < 1\}$$

We want to claim that, under suitable disjointness condition, the Hausdorff dimension of the invariant set K is equal to σ_{∞}.

By Lemma 4.3 and Remark 4.4 we know that given $\epsilon > 0$, there exists a $\mu_i(\epsilon)$ such that for every $t, s \in S$ with $0 < d(s, t) < \epsilon$,

$$\mu_i(\epsilon)^{-1}(D\theta_i)(t) \leq \frac{d(\theta_i(s), \theta_i(t))}{d(s, t)} \leq \mu_i(\epsilon)(D\theta_i)(t)$$

and $\lim_{\epsilon \to 0+}\mu_i(\epsilon) = 1$. In the case of finitely many θ_i's a uniform $\mu(\epsilon)$ satisfying the above property could be chosen by taking the maximum over i. But for the infinite case we cannot guarantee a uniform $\mu(\epsilon)$ which would work for each θ_i. So instead we shall make the assumption that a uniform $\mu(\epsilon)$ can be chosen. For a specific problem we would have to check that this condition is indeed satisfied. For some important examples like complex continued fractions, which has been studied by other authors (see Section 6 of [12]), this condition can be easily verified.

H5.4 Given $\epsilon > 0$, there exists a $\mu(\epsilon) \geq 1$ such that for every $t, s \in S$ with $0 < d(s, t) < \epsilon$,

$$\mu(\epsilon)^{-1}(D\theta_i)(t) \leq \frac{d(\theta_i(s), \theta_i(t))}{d(s, t)} \leq \mu(\epsilon)(D\theta_i)(t)$$

and $\lim_{\epsilon \to 0+} \mu(\epsilon) = 1$.

Now we are ready to prove the upper bound for the Hausdorff dimension of K.

Theorem 5.4. *Assume that H5.3 and H5.4 are satisfied. Also assume that H5.1 is satisfied with $b_i(t) = (D\theta_i(t))^\sigma$ for $\sigma > \sigma_0$. Let $dim(K)$ denote the Hausdorff dimension of the invariant set K and σ_∞ be as defined in eq. (5.3). Then $dim(K) \leq \sigma_\infty$.*

Proof. Once we assume the existence of a uniform $\mu(\epsilon)$ as defined above, the proof is exactly similar (in fact simpler since we are in the setting of iterated function systems) to the proof of Theorem 4.10, and is left to the reader. $\square$

To prove the other half, $dim(K) \geq \sigma_\infty$, we shall consider the infinite iterated system as the limit of finite iterated systems and use the result that we have for the finite case.

Define for $N \geq 1$ and $f \in C(S)$,

$$(L_{\sigma,N} f)(t) = \sum_{i=1}^{N} (D\theta_i(t))^\sigma f(\theta_i(t)).$$

For $N \geq 1$, let K_N be the unique nonempty compact invariant set satisfying

$$K_N = \bigcup_{i=1}^{N} \theta_i(K_N)$$

and let σ_N be the unique positive real number such $r(L_{\sigma_N,N}) = 1$. We shall assume the following hypothesis.

H5.5 For each $N \geq 1$, $\theta_i(K_N) \cap \theta_j(K_N) = \emptyset$ for $1 \leq i < j \leq N$, and θ_i is one-to-one in a neighborhood of K_N for $1 \leq i \leq N$.

By Theorem 4.15 we know that, assuming H5.5, $dim(K_N) = \sigma_N$.

Lemma 5.5. *For each $N \geq 1$, $K_N \subset K_{N+1}$ and $K_N \subset K$. Hence, $dim(K) \geq dim(K_N) = \sigma_N$.*

Proof. For any closed bounded nonempty set $A \subset S$, and $N \geq 1$, let $\Theta(A) = \overline{\bigcup_{i=1}^{\infty} \theta_i(A)}$ and $\Theta_N(A) = \overline{\bigcup_{i=1}^{N} \theta_i(A)}$. Then $\Theta(A) \supset \Theta_N(A)$ and $\Theta_{N+1}(A) \supset \Theta_N(A)$. Also $D(\Theta^k(A), K) \to 0$ as $k \to \infty$ where D denotes the Hausdorff metric. Taking $A = K_N$ we get $\Theta^k(K_N) \supset \Theta_N^k(K_N) = K_N$, and since $D(\Theta^k(K_N), K) \to 0$ as $k \to \infty$, we get that $K \supset K_N$. Similarly, $K_{N+1} \supset K_N$. $\square$

It is easy to see that $\|L_\sigma - L_{\sigma,N}\| \to 0$ as $N \to \infty$ for $\sigma > \sigma_0$ in case I and for $\sigma \geq \sigma_0$ in case II.

Remark 5.6. Let X be a real or complex Banach space and $L : X \to X$, $L_k : X \to X$, $k \geq 1$ be bounded linear operators. Assume that $\lim_{k \to \infty} \|L_k - L\| = 0$. Then we have that $\limsup_{k \to \infty} r(L_k) \leq r(L)$. But, in general, it is not true that $\lim_{k \to \infty} r(L_k) \to r(L)$. In fact, Kakutani gave an example of a sequence of bounded linear operators L_k on a Hilbert space which converges in the operator norm to an operator L satisfying $r(L_k) = 0$ for all $k \geq 1$ and $r(L) > 0$. The example can be found on pages 282-283 of [20]. If, in addition, we know that $\rho(L) < r(L)$, where $\rho(L)$ is the essential spectral radius of L, then it is true that $r(L_k) \to r(L)$. By the natural extention of L to the complexification of X, we can assume that X is a complex Banach space. Then, if $\sigma(L)$ denotes the spectrum of L, recall that

$\sigma(L) \cap \{z \in \mathbb{C} : |z| > \rho(L)\}$ consists of isolated points each of which is an eigenvalue of L of finite algebraic multiplicity. Then exactly the argument on pages 227-228 of [17] proves that $r(L_k) \to r(L)$.

Lemma 5.7. *Let (S, d) be a compact metric space and suppose that $L : X = C(S) \to X$ be a positive bounded linear map, i.e., $f(t) \geq 0$ for all $t \in S$ implies that $(Lf)(t) \geq 0$ for all $t \in S$. Let e denotes the function identically equal to 1. If $r(L)$ denotes the spectral radius of L, we have $r(L) = \lim_{n \to \infty} \|L^n e\|^{\frac{1}{n}}$. Furthermore, if $u \in X$ such that $u(t) > 0$ for all $t \in S$, then $r(L) = \lim_{n \to \infty} \|L^n u\|^{\frac{1}{n}}$. Finally, if $Lu = ru$ with $u(t) > 0$ for all $t \in S$, then $r(L) = r$.*

Proof. We shall write $f \leq g$ to mean $f(t) \leq g(t)$ for all $t \in S$. Since L is linear and maps nonnegative functions to nonnegative functions, it follows that $Lf \leq Lg$ whenever $f \leq g$. If $f \in X$ with $\|f\| \leq 1$, we have $-e \leq f \leq e$. So, $-L^n e \leq L^n f \leq L^n e$ which implies that $|(L^n f)(t)| \leq |L^n e(t)|$ for all $t \in S$. Thus $\|L^n f\| \leq \|L^n e\|$ whenever $\|f\| \leq 1$ which gives $\|L^n\| = \|L^n e\|$. Taking the nth root and taking the limit as n goes to ∞, we get $r(L) = \lim_{n \to \infty} \|L^n e\|^{\frac{1}{n}}$.

Now let $u \in X$ such that $u(t) > 0$ for all $t \in S$. Since S is compact, there exist $0 < m < M < \infty$ such that $me \leq u \leq Me$. This implies $mL^n e \leq L^n u \leq ML^n e$, so $m\|L^n e\| \leq \|L^n u\| \leq M\|L^n e\|$. Taking the nth root and taking the limit, we get $\lim_{n \to \infty} \|L^n u\|^{\frac{1}{n}} = \lim_{n \to \infty} \|L^n e\|^{\frac{1}{n}} = r(L)$. To see the last part, note that $Lu = ru$ implies $L^n u = r^n u$. So, $\|L^n u\|^{\frac{1}{n}} = r\|u\|^{\frac{1}{n}}$. Since $\|u\| > 0$, $\|u\|^{\frac{1}{n}} \to 1$. Thus we get $r(L) = r$. $\square$

Corollary 5.8. *Let (S, d) be a compact metric space and suppose that $L : X = C(S) \to X$ and $L_k : X \to X$ be positive bounded linear maps. Assume that $\|L_k - L\| \to 0$ as $k \to \infty$. Suppose $Lu = ru$ with $u(t) > 0$ for all $t \in S$. Then $r(L_k) \to r(L) = r$.*

Proof. First we know that $r = r(L)$ by the previous lemma. Now we have that $\|L_k u - Lu\| \to 0$ as $k \to \infty$. Because u is strictly positive, given $\delta > 0$, there exists k_0 such that $L_k u \geq (1 - \delta)ru$ for all $k \geq k_0$. This implies for any $n \geq 1$, $\|L_k^n u\| \geq (1 - \delta)^n r^n \|u\|$ for $k \geq k_0$. Using the previous lemma, $r(L_k) = \lim_{n \to \infty} \|L_k^n u\|^{\frac{1}{n}} \geq (1 - \delta)r$ for $k \geq k_0$. Since $\delta > 0$ was arbitrary, $\liminf_{k \to \infty} r(L_k) \geq r$. Thus we are done because we always have $\limsup_{k \to \infty} r(L_k) \leq r(L)$. $\square$

Lemma 5.9. *Assume that H5.3 and H5.4 are satisfied. Also assume that H5.1 is satisfied with $b_i(t) = (D\theta_i(t))^\sigma$ when $L\sigma$ is defined. $r(L_{\sigma,N}) \uparrow r(L_\sigma)$ as $N \to \infty$ for $\sigma > \sigma_0$ in case I and for $\sigma \geq \sigma_0$ in case II.*

Proof. Let $\sigma > \sigma_0$ if we are in case I or $\sigma \geq \sigma_0$ if we are in case II. Then L_σ defines a positive bounded linear operator on $C(S)$. Clearly $L_{\sigma,N} e \leq L_{\sigma,N+1} e$, so $\|L_{\sigma,N} e\| \leq \|L_{\sigma,N+1} e\|$. Using the previous Lemma, we get that $r(L_{\sigma,N} \leq r(L_{\sigma,N+1}$. It is easy to see that $\|L_\sigma - L_{\sigma,N}\| \to 0$ as $N \to \infty$ using H5.1. Since we know that L_σ has a strictly positive eigenvector with eigenvalue $r(L_\sigma)$, using the previous corollary, we must have $r(L_{\sigma,N}) \to r(L_\sigma)$. $\square$

Now we can prove the lower bound for the Hausdorff dimension of K.

Theorem 5.10. *Assume that H5.3, H5.4 and H5.5 are satisfied. Also assume that H5.1 is satisfied with $b_i(t) = (D\theta_i(t))^\sigma$ for $\sigma > \sigma_0$. Let $dim(K)$ denote the Hausdorff dimension of K and σ_∞ be as defined in (5.3). Then $dim(K) \geq \sigma_\infty$.*

Proof. If $\sigma_0 < \sigma_\infty$ then for $\sigma_0 < \sigma < \sigma_\infty$, L_σ is defined and $r(L_\sigma) > 1$ by the definition of σ_∞ and the fact that $\sigma \mapsto r(L_\sigma)$ is strictly decreasing. So, using the previous lemma, there exists N_σ such that $r(L_{\sigma,N}) > 1$ for all $N \geq N_\sigma$. Since $r(L_{\sigma_N,N}) = 1$, $\sigma_N > \sigma$ for all $N \geq N_\sigma$. Therefore, $\dim(K) \geq \dim(K_{N_\sigma}) = \sigma_{N_\sigma} > \sigma$. Since this is true for any σ with $\sigma_0 < \sigma < \sigma_\infty$, it follows that $\dim(K) \geq \sigma_\infty$. If $\sigma_0 = \sigma_\infty$ then for $\sigma < \sigma_\infty$, $\sum_{i=1}^\infty (D\theta_i(t))^\sigma = \infty$, so for large N (depending on σ), $\sum_{i=1}^N (D\theta_i(t))^\sigma > 1$ for all $t \in S$. This implies that $r(L_{\sigma,N}) > 1$, so $\sigma_N > \sigma$. Thus $\dim(K) > \sigma$ for all $\sigma < \sigma_\infty$, and hence $\dim(K) \geq \sigma_\infty$. This completes the proof of the theorem. □

Combining the previous two theorems we have the following theorem.

Theorem 5.11. *Assume that H5.3, H5.4 and H5.5 are satisfied. Also assume that H5.1 is satisfied with $b_i(t) = (D\theta_i(t))^\sigma$ for $\sigma > \sigma_0$. Let $\dim(K)$ denote the Hausdorff dimension of K and σ_∞ be as defined in (5.3). Then $\dim(K) \geq \sigma_\infty$*

In the next example we discuss a special infinite iterated function system that is generated by complex continued fractions. This has been studied in section 6 of [12]. We show how our theory is applicable to this particular example.

Example 5.12. (*Complex continued fractions*) Let $I = \{m + ni : m \in \mathbb{N}, n \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of positive integers. Let $X \subset \mathbb{C}$ be the closed disc centered at the point $\frac{1}{2}$ with radius $\frac{1}{2}$. For $b \in I$ define $\theta_b : X \to X$ by

$$\theta_b(z) = \frac{1}{b+z}.$$

The collection of mappings $\{\theta_b : b \in I\}$ is not really a conformal iterated function system because θ_1 is not a strict contraction as $|\theta_1'(0)| = 1$. Therefore we consider the system $\{\theta_b \circ \theta_c : b, c \in I\}$. It is easy to verify that $\theta_b \circ \theta_c$ is a strict contraction for each $b, c \in I$ with a uniform Lipschitz constant $c < 1$. Let K be the unique compact invariant set for this system. First note that $\theta_b(z) = \theta_c(w)$ implies that $|z - w| = |b - c|$. So, if $|b - c| > 1$ then $\theta_b(X) \cap \theta_c(X) = \emptyset$. Furthermore, if $|b - c| = 1$ then $\theta_b(z) = \theta_c(w)$ implies that z and w belong to the boundary of X and $|z - w| = 1$.

Lemma 5.13. $\theta_b \circ \theta_c(X)$ *is contatined in the interior of X.*

Proof. First we claim that $\theta_b(z) \in \partial X$ implies $z = 0$. Let $b = m + ni$, $m \in \mathbb{N}$, $n \in \mathbb{Z}$ and $z = x + yi \in X$. Then $\theta_b(z) \in \partial X$ implies $|\frac{1}{b+z} - \frac{1}{2}| = \frac{1}{2}$ which implies $|2 - b - z| = |b + z|$. Therefore, $(2 - m - x)^2 + (n + y)^2 = (m + x)^2 + (n + y)^2$ which implies $m + x = 1$, i.e., $x = 1 - m$. Since $m \geq 1$ and $x \geq 0$ for $z \in X$, it follows that $m = 1$ and $x = 0$. But $x = 0$ implies that $z = 0$.

Now suppose that $\theta_b \circ \theta_c(z) \in \partial X$ for some $z \in X$. Then, by the above claim, $\theta_c(z) = 0$, which is impossible by the definition of $\theta_c(z)$. □

Let us verify H5.5. By the previous lemma we know that for any $b, c \in I$, $\theta_b \circ \theta_c(X)$ is a compact set contained in the interior of X. So if we take finitely many b_i, c_i, the union of the images would still be a compact subset of the interior of X. So, for any N, K_N is a compact subset of the interior of X which means $\text{diam}(K_N) < 1$. We claim that if $\hat{X} \subset \text{int}(X)$ with $\text{diam}(\hat{X}) < 1$ then $\theta_{b_1} \circ \theta_{c_1}(\hat{X}) \cap \theta_{b_2} \circ \theta_{c_2}(\hat{X}) = \emptyset$ for any $(b_1, c_1) \neq (b_2, c_2)$. Suppose instead that $\theta_{b_1}(\theta_{c_1}(z)) = \theta_{b_2}(\theta_{c_2}(w))$ with $z, w \in \hat{X}$. This implies that $b_1 + \theta_{c_1}(z) = b_2 + \theta_{c_2}(w)$. If $b_1 = b_2$,

this would imply that $\theta_{c_1}(z) = \theta_{c_2}(w)$, i.e., $c_1 + z = c_2 + w$, which is impossible because $|z - w| < 1$ and $|c_1 - c_2| \geq 1$. If $b_1 \neq b_2$, we must have $|\theta_{c_1}(z) - \theta_{c_2}(w)| = 1$ which is possible only if both $\theta_{c_1}(z)$ and $\theta_{c_2}(w)$ belong to the boundary of X, which is possible only if $z = w = 0$. This is a contradiction to the fact that $\hat{X}$ is in the interior of X. Thus the disjointness condition in H5.5 is satisfied. Also for any $b \in I$, the map θ_b is clearly one-to-one.

For $b \in I$, we have $D\theta_b(z) = |\theta'(z)| = \frac{1}{|z+b|^2}$. So, $D\theta_b(z) > 0$ for all $z \in X$. We claim that there exists $0 < M_0 < \infty$ such that $D\theta_b \in K(M_0, \lambda)$ with $\lambda = 1$. Let $z, w \in X$. We have

$$(D\theta_b)(z) \leq (D\theta_b)(w) \exp(M_0 |z - w|)$$

$$\Leftrightarrow \frac{1}{|z+b|^2} \leq \frac{1}{|w+b|^2} \exp(M_0 |z - w|)$$

$$\Leftrightarrow M_0 \geq 2 \frac{|\ln|\frac{w+b}{z+b}||}{|w - z|}$$

But $\ln|\frac{w+b}{z+b}| = \ln|1 + \frac{w-z}{z+b}| \leq \ln(1 + \frac{|w-z|}{|z+b|}) \leq \frac{|w-z|}{|z+b|}$. Therefore, $2\frac{|\ln|\frac{w+b}{z+b}||}{|w-z|} \leq \frac{2}{|z+b|} \leq$ 2. So, we can choose M_0 independent of $b \in I$. Thus H5.3 is satisfied.

To verify H5.1 note that

$$\sum_{b \in I} (D\theta_b)^\sigma(0) = \sum_{b \in I} \frac{1}{|b|^{2\sigma}} = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{1}{(m^2 + n^2)^{2\sigma}}$$

which converges for $\sigma > \frac{1}{2}$.

To verify H5.4 it is enough to show that given $\epsilon > 0$ there exist $\mu_1(\epsilon)$ and $\mu_2(\epsilon)$ such $\lim_{\epsilon \to 0+} \mu_1(\epsilon) = \lim_{\epsilon \to 0+} \mu_2(\epsilon) = 1$ and if $0 < |z - w| < \epsilon$ then $\mu_1(\epsilon) \leq \frac{1}{(D\theta_b)(z)} \frac{|\theta_b(z) - \theta_b(w)|}{|z-w|} \leq \mu_2(\epsilon)$ for all $b \in I$. But using $\theta_b(z) = \frac{1}{z+b}$, we get $\frac{1}{(D\theta_b)(z)} \frac{|\theta_b(z) - \theta_b(w)|}{|z-w|} = |\frac{z+b}{w+b}| = |1 + \frac{z-w}{w+b}|$, which is bounded between $1 - \frac{|z-w|}{|w+b|}$ and $1 + \frac{|z-w|}{|w+b|}$. Since for any $w \in X$ $|w + b|^{-1} = |\theta_b(w)| \leq 1$, taking $\mu_1(\epsilon) = 1 - \epsilon$ and $\mu_2(\epsilon) = 1 + \epsilon$ does the job.

Thus we see that all the hypotheses of the theorem are satisfied for this particular example and hence the Hausdorff dimension of the invariant set is given by the value of σ_∞.

In the next section we shall see that sometimes it is important to look at another metric rather than the Euclidean metric.

6. Choice of appropriate metric

We need to recall the definition of the Carathéodory-Reiffen-Finsler (CRF) metric on bounded domains in Banach spaces. Let G be a bounded domain in a Banach space X and let U denote the open unit disc in $\mathbb{C}$. Let $Hol(G, U)$ be the family of all holomorphic functions $f : G \to U$. Define $\alpha : G \times X \to \mathbb{R}$ by

$$\alpha(x, v) = \sup\{|Dg(x)v| : g \in Hol(G, U)\}$$

where $Dg(x)$ denotes the Fréchet derivative of g at x. Given any two points x and y in G, consider the family of curves $\gamma : [0, 1] \to G$ that have piecewise continuous

derivatives and $\gamma(0) = x$, $\gamma(1) = y$. Call such a curve admissible and define its length by

$$L(\gamma) = \int_0^1 \alpha(\gamma(t), \gamma'(t)) \, dt.$$

We now define the distance between x and y by

$$\rho(x, y) = \inf\{L(\gamma) : \gamma \text{ is admissible with } \gamma(0) = x \text{ and } \gamma(1) = y\}.$$

ρ is called the CRF metric on G. For a detailed discussion of the CRF metric we refer the reader to [6].

Let G be a bounded open set in $\mathbb{C}$ and let $\theta : G \to G$ be a holomorphic map such that $\overline{\theta(G)}$ is a compact subset of G. If ρ denotes the CRF metric on G then it is known (see Theorem 13.1 in [6]) that θ is a strict contraction on G with respect to the CRF metric ρ. Also, on a compact subset C of G, ρ is a complete metric and is equivalent to the standard Euclidean metric, i.e., there exist positive constants m and M such that

$$m|z - w| \leq \rho(z, w) \leq M|z - w| \text{ for all z,w in C.}$$

Let G be a bounded open set in $\mathbb{C}$ and assume for $1 \leq i \leq N$ that $\theta_i : G \to G$ is a holomorphic map such that $C = \overline{\theta_i(G)}$ is a compact subset of G and $\theta_i'(z) \neq 0$ for all $z \in G$. Then there exists a unique nonempty compact set K with $K = \bigcup_{i=1}^N \theta_i(K)$. For $k \geq 1$, define $\mathcal{I}_k = \{I = (i_1, i_2, \ldots, i_k) | 1 \leq i_j \leq N \text{ for } 1 \leq j \leq k\}$. For $I = (i_1, i_2, \ldots, i_k) \in \mathcal{I}_k$, define $\theta_I = \theta_{i_k} \circ \cdots \circ \theta_{i_2} \circ \theta_{i_1}$. It is easy to see that $K = \bigcup_{I \in \mathcal{I}_k} \theta_I(K)$. We claim that, for large k, θ_I is a strict contraction map with respect to the Euclidean metric. Suppose $z, w \in C$, $z \neq w$. Then

$$\frac{|\theta_I(z) - \theta_I(w)|}{|z - w|} \leq \frac{\frac{1}{m}\rho(\theta_I(z), \theta_I(w))}{\frac{1}{M}\rho(z, w)} \leq \frac{M}{m}c^k,$$

where $c < 1$ is the maximum of the contraction ratios of the maps θ_i, $1 \leq i \leq N$, with respect to the metric ρ. If we choose k large enough so that $\frac{M}{m}c^k < 1$, then it follows that θ_I is a contraction map for all $I \in \mathcal{I}_k$ with respect to the Euclidean metric. Thus, if $\theta_I(K) \cap \theta_J(K) = \emptyset$ for $I, J \in \mathcal{I}_k$, $I \neq J$ (which is certainly true if $\{\theta_i(K)\}_{i=1}^N$ are pairwise disjoint), the Hausdorff dimension of the invariant set K is given by Theorem 1.2 by considering the iterated function system given by the maps $\{\theta_I\}_{I \in \mathcal{I}_k}$ and the standard Euclidean metric. Note that in this case $(D\theta_I)(z)$ is nothing but $|\theta_I'(z)|$. If we write, for $\sigma \geq 0$, $(L_\sigma f)(z) = \sum_{i=1}^N |\theta_i'(z)|^\sigma f(\theta_i(z))$ and $(\tilde{L}_\sigma f)(z) = \sum_{I \in \mathcal{I}_k} |\theta_i'(z)|^\sigma f(\theta_i(z))$, where k is as chosen above, then it is easy to see that $\tilde{L}_\sigma = L_\sigma^k$. It follows from the following lemma that $r(\tilde{L}_{\sigma_0}) = 1$ if and only if $r(L_{\sigma_0}) = 1$. Thus, to find the Hausdorff dimension of the invariant set K it is enough to find σ_0 such that $r(L_{\sigma_0}) = 1$.

Lemma 6.1. *Let X be a Banach space, $L : X \to X$ be a bounded linear map and $k \geq 1$ be a positive integer. Then $r(L^k) = (r(L))^k$, where $r(L)$ denotes the spectral radius of L.*

Proof. We have that

$$
\begin{aligned}
r(L^k) &= \lim_{n\to\infty} \|(L^k)^n\|^{1/n} \\
&= \lim_{n\to\infty} (\|L^{kn}\|^{1/kn})^k \\
&= (\lim_{n\to\infty} \|L^{kn}\|^{1/kn})^k \\
&= (r(L))^k.
\end{aligned}
$$

$\square$

References

1. F. F. Bonsall, *Linear operators in complete positive cones*, Proc. London Math. Soc. **8** (1958), 53–75.
2. R. T. Bumby, *Hausdorff dimensions of Cantor sets*, J. Reine Angew. Math. **331** (1982), 192–206.
3. _____, *Hausdorff dimension of sets arising in number theory*, Number theory (New York, 1983–84), Lecture Notes in Math., vol. 1135, Springer, Berlin, 1985, pp. 1–8.
4. G. Edgar, *Measure, topology and fractal geometry*, second ed., Undergraduate Texts in Mathematics, Springer, New York, 2008.
5. K. J. Falconer, *The goemetry of fractal sets*, Cambridge Univ. Press, 1985.
6. K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 83, Marcel Dekker Inc., New York, 1984.
7. J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747.
8. M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space (in Russian)*, Uspehi Matem. Nauk **3** (1948), no. 1(23), 3–95.
9. J. Mallet-Paret and R. D. Nussbaum, *Eigenvalues for a class of homogeneous cone maps arising from max-plus operators*, Discrete Contin. Dyn. Syst. **8** (2002), no. 3, 519–562.
10. _____, *Eigenvectors for homogeneous cone mappings and the cone essential spectral radius*, in preparation, 2009.
11. P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Univ. Press, 1995.
12. R. D. Mauldin and M. Urbański, *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Soc. (3) (1996), no. 1, 105–154.
13. R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 811–829.
14. S. Mazur and S. Ulam, *Sur les transformationes isomtriques d'espaces vectoriels norms*, C. R. Acad. Sci. Paris **194** (1932), 946–948.
15. P. A. P. Moran, *Additive functions of intervals and Hausdorff measure*, Proc. Cambridge Philos. Soc. **42** (1946), 15–23.
16. J. R. Munkres, *Topology*, second ed., Prentice Hall, 2000.
17. R. D. Nussbaum, *Periodic solutions of some nonlinear integral equations*, Dynamical systems (Proc. Internat. Sympos., Univ. Florida, Gainesville, Fla., 1976), Academic Press, New York, 1977, pp. 221–249.
18. _____, *Eigenvectors of nonlinear positive operators and the linear Krein-Rutman theorem*, Fixed point theory, Lecture Notes in Math., vol. 886, Springer, Berlin, 1981, pp. 309–330.
19. _____, *Periodic points of positive linear operators and Perron-Frobenius operators*, Integral Equations Operator Theory **39** (2001), no. 1, 41–97.
20. C. E. Rickart, *General theory of banach algebras*, Robert E. Krieger Publishing Co., New York, 1974.
21. H. H. Schaefer, *Banach lattices and positive operators*, Springer Verlag, New York, 1974.
22. A. Schief, *Self-similar sets in complete metric spaces*, Proc. Amer. Math. Soc. **124** (1996), no. 2, 481–490.
23. J. Väisälä, *A proof of the Mazur-Ulam theorem*, Amer. Math. Monthly **110** (2003), no. 7, 633–635.

Department of Mathematics, Rutgers University, New Jersey 08854
E-mail address: nussbaum@math.rutgers.edu

Department of Mathematics, Rutgers University, New Jersey 08854
E-mail address: amitpriy@math.rutgers.edu

Mathematisch Instituut, Universiteit Leiden, Leiden, the Nedherlands
E-mail address: verduyn@math.leidenuniv.nl