

PERIODIC POINTS OF POSITIVE LINEAR OPERATORS AND PERRON-FROBENIUS OPERATORS

ROGER NUSSBAUM*

Let $C(S)$ denote the Banach space of continuous, real-valued maps $f : S \rightarrow \mathbb{R}$ and let A denote a positive linear map of $C(S)$ into itself. We give necessary conditions that the operator A have a strictly positive periodic point of minimal period m . Under mild compactness conditions on the operator A , we prove that these necessary conditions are also sufficient to guarantee existence of a strictly positive periodic point of minimal period m . We study a class of Perron-Frobenius operators defined by

$$(Ax)(t) = \sum_{i=1}^{\infty} b_i(t)x(w_i(t)),$$

and we show how to verify the necessary compactness conditions to apply our theorems concerning existence of positive periodic points.

1 Introduction

Recently, in [10], the following question was raised: Do there exist a nonnegative, continuous function $k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ and strictly positive, continuous functions $f_0 : [0, 1] \rightarrow \mathbb{R}$ and $f_1 : [0, 1] \rightarrow \mathbb{R}$ with $f_0 \neq f_1$ such that for all $s \in [0, 1]$

$$\int_0^1 k(s, t)f_0(t)dt = f_1(s) \text{ and } \int_0^1 k(s, t)f_1(t)dt = f_0(s)?$$

A simple classical argument shows that this question does not have a positive answer if $k(s, t) > 0$ for all s and t .

In this paper we shall be interested in generalizations of the above question. Let S denote a compact, Hausdorff space, $C(S)$ the Banach space of continuous, real-valued functions $x : S \rightarrow \mathbb{R}$ with $\|x\| = \sup\{|x(s)| : s \in S\}$ and $P(S) := P$, the set of nonnegative functions in $C(S)$. In this notation, $P_o(S)$, the interior of $P(S)$, is the set of functions x such that $x(s) > 0$ for all $s \in S$. The cone P induces a partial ordering by $x \leq y$ if and only if $y - x \in P$. If $A : C(S) \rightarrow C(S)$ is a linear operator, we shall say that A is "positive" if

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$A(P(S)) \subset P(S)$. It is well known and easily proved that if $A : C(S) \rightarrow C(S)$ is a positive linear operator, then necessarily A is a bounded linear operator. As usual, if $f \in C(S)$, m is a positive integer, and $A : C(S) \rightarrow C(S)$ is a bounded, linear operator, we shall say that “ f is a periodic point of A of minimal period m ” if $A^m(f) = f$ and $A^j(f) \neq f$ for $0 < j < m$. We shall consistently use the above notation.

We can now state the main questions of interest here.

Question 1. Assume that m is a given positive integer, S is a compact, Hausdorff space and $A : C(S) \rightarrow C(S)$ is a positive, linear operator. Does there exist $f_0 \in P_o(S)$ such that f_0 is a periodic point of A of minimal period m ?

Question 2. Let m , S and A be as in Question 1. Does there exist $f_0 \in P(S)$ such that f_0 is a periodic point of A of minimal period m ?

Despite their similarity, there are important differences between these two questions. We shall find necessary conditions that Question 1 have a positive answer; and we shall prove that, under mild compactness conditions on A , these necessary conditions are also sufficient. We shall prove that a slight variant of the necessary conditions for Question 1 also leads, under mild compactness conditions on A , to sufficient conditions that Question 2 have a positive answer.

If $b_i : S \rightarrow \mathbb{R}$, $i \geq 1$, are given nonnegative, continuous functions such that $\sum_{i=1}^{\infty} b_i(s) := b(s)$ is a finite, continuous function, and if $w_i : S \rightarrow S$, $i \geq 1$, are given continuous functions, then one can define a positive linear operator $A : C(S) \rightarrow C(S)$ by

$$(Ax)(s) = \sum_{i=1}^{\infty} b_i(s)x(w_i(s)).$$

Such operators are sometimes called “Perron-Frobenius operators” and represent a major interest of this paper. Establishing appropriate compactness conditions on A so that we can apply our theorems turns out to be a delicate problem, and existing results in the literature seem inadequate for our purposes. These difficulties are already apparent in a simple example studied in Corollary 6.2 below, where $S = [0, 1]$, $k \geq 1$ is a real number and $A : C(S) \rightarrow C(S)$ is given by

$$(Ax)(t) = tx((1-t)^k) + (1-t)x(1-t^k).$$

This paper is rather long, so a guide to the principal results may be in order. In Section 2 we present the following theorem, which gives necessary conditions for Question 1 to have a positive solution.

Theorem 1.1 *Let S be a compact, Hausdorff space, $A : C(S) \rightarrow C(S)$ a positive, linear operator and m a positive integer. Assume that there exists $f_0 \in P_o(S)$ such that f_0 is a periodic point of A of minimal period m . Then the following conditions hold:*

- (1) *There exists $\theta \in P_o(S)$ with $A(\theta) = \theta$.*
- (2) *There exist closed, nonempty proper subsets $E_j \subset S$, $0 \leq j \leq m$, with $E_m = E_0$, such that (a) $\bigcap_{j=0}^{m-1} E_j = \emptyset$ and (b) whenever $f \in C(S)$ and $f|_{E_j} = 0$ for some $0 \leq j \leq m-1$, it follows that $Af|_{E_{j+1}} = 0$.*

In Corollary 2.1 below we prove that, if m is a prime, the sets E_i in Theorem 1.1 can be chosen pairwise disjoint for $0 \leq i < m$. For general m we can always arrange that, for $0 \leq i < k < m$, either $E_i = E_k$ or $E_i \cap E_k = \emptyset$.

In Section 3 we start from the necessary condition (2) in Theorem 1.1 and ask what further conditions are necessary to insure a positive answer to Question 1 or Question 2. We observe (see Theorem 3.5 below) that Questions 1 and 2 for general m can be reduced to the case that m is a prime, so we restrict attention to the case that m is a prime. The following theorem is a special case of Theorem 3.1 in Section 3.

Theorem 1.2 *Let S be a compact, Hausdorff space, $A : C(S) \rightarrow C(S)$ a positive linear operator and m a prime number. Assume that there exist closed, nonempty sets E_j , $0 \leq j \leq m$, as in condition (2) of Theorem 1.1. Suppose, also, that there exists $\theta \in P(S)$ such that $A(\theta) = \theta$ and $\theta(s_0) > 0$ for some $s_0 \in E := \cup_{j=0}^{m-1} E_j$. Finally, assume that at least one of the following compactness conditions on A is satisfied:*

- (a) *For every $M > 0$ and for every $f \in C(S)$ such that $-M\theta \leq f \leq M\theta$, the norm closure of $\{(A^m)^j(f) \mid j \geq 1\}$ is compact in the norm topology.*
- (b) *$\rho(A) < 1$, where $\rho(A)$ denotes the essential spectral radius of A .*

Then it follows that there exist positive reals a and b and $f_0 \in P(S)$ such that $a\theta \leq f_0 \leq b\theta$ and f_0 is a periodic point of A of minimal period m .

In Theorem 3.1A below we also show that a version of Theorem 1.2 holds for a class of nonlinear maps $A : P(S) \rightarrow P(S)$, although we do not pursue this point in this paper.

As a very special case of Theorems 1.1 and 1.2 (see Remark 3.2), we obtain a complete answer to our original question about integral operators.

Corollary 1.1 *Let m be a prime number, S a compact, Hausdorff space and μ a regular, Borel measure on S such that $\mu(G) > 0$ for every nonempty, open subset G of S . Let $k : S \times S \rightarrow \mathbb{R}$ be a continuous, nonnegative function and define $A : C(S) \rightarrow C(S)$ by*

$$(Ax)(s) = \int k(s, t)x(t)\mu(dt).$$

Then A has a periodic point $f_0 \in P_o(S)$ of minimal period m if and only if the following conditions are satisfied:

- (1) *There exists $\theta \in P_o(S)$ with $A(\theta) = \theta$.*
- (2) *There exist compact, nonempty, proper subsets E_i of S , $0 \leq i \leq m$, with $E_m = E_0$, such that (a) $\cap_{i=0}^{m-1} E_i = \emptyset$ and (b) $k(s, t) = 0$ for $(s, t) \in \cup_{i=1}^m E_i \times E'_{i-1}$, where E'_j denotes the complement of E_j .*

Much of the rest of this paper is devoted to finding useful hypotheses which imply the compactness condition (a) of Theorem 1.2. It is straightforward to prove that condition (b) of Theorem 1.2 implies condition (a) of Theorem 1.2. However, it will generally be the

case that for Perron-Frobenius operators A of interest, $\rho(A)$, the essential spectral radius of A , equals $r(A)$, the spectral radius of A ; and condition (b) of Theorem 1.2 cannot be satisfied. The reader's attention is particularly directed to Theorem 3.3, which gives a framework applicable to Perron-Frobenius operators and gives conditions under which condition (a) of Theorem 1.2 is satisfied.

In general, under the conditions we consider, the Krein-Rutman theorem and its generalizations only insure the existence of an eigenvector $\theta \in P(S)$ of A with eigenvalue $r(A)$, although it may well happen that $\theta \in P_\circ(S)$. Theorem 3.4 of Section 3 gives hypotheses which insure that θ satisfies the hypotheses of Theorem 1.2.

One can also ask whether A^* , the Banach space adjoint of $A : C(S) \rightarrow C(S)$, has periodic point $\mu^* \in P(S)^*$ of minimal period m . Theorem 4.1 of Section 4 provides an answer to this question. One can also argue more directly from the results of Section 3, but at the cost of unnecessary hypotheses.

Sections 5 and 6 of this paper are devoted to the theory of Perron-Frobenius operators. Section 5 treats the existence and uniqueness of positive eigenvectors of Perron-Frobenius operators and represents a slight generalization of unpublished 1989 notes which the author wrote in response to questions from Professor Jeff Geronimo. Also note that the important quantity $\rho(\lambda)$ (see equations (19)-(22) in Section 5) will in general be strictly less than $s(\lambda)$ (see eqn. (34)), which has been studied in the literature [1, 2, 6, 9, 11]. Indeed, we have $\rho(\lambda) < s(\lambda) = 1$, even for simple examples like those in Corollaries 5.2 and 6.2. Theorem 5.1 gives hypotheses under which Perron-Frobenius operators have an eigenvector in $P(S)$ with eigenvalue $r(A)$. The same theorem also gives hypotheses under which the compactness condition (a) of Theorem 1.1 is satisfied and for which every eigenvector u of \mathcal{A} , the complexification of A , with eigenvalue ζ , $|\zeta| = r(A)$, is Hölder continuous. Corollaries 5.1 and 5.2 are special cases of Theorem 5.1. We especially draw the reader's attention to Corollary 5.2, which illustrates the delicate calculus questions which can arise in verifying the hypotheses of Theorem 5.1. Theorems 5.2 and 5.3 in Section 5 deal with the question of whether $r(A)$ is the only eigenvalue α of A with $|\alpha| = r(A)$ and with the question of the algebraic multiplicity of $r(A)$.

In Theorem 6.1 of Section 6 we present a version of our fundamental Theorem 3.1 which is directly applicable to Perron-Frobenius operators. A simple, but illuminating example is given in Corollary 6.2.

2 Periodic Points of A : Necessary Conditions

Theorem 2.1 *Let S be a compact, Hausdorff space and $A : C(S) \rightarrow C(S)$ a positive, linear operator. For a given integer m , assume that there exists a strictly positive function $f_0 \in P_\circ(S)$ which is a periodic point of A of minimal period m . Then the following conditions hold:*

- (1) *There exists $\theta \in P_\circ(S)$ with $A(\theta) = \theta$.*
- (2) *There exist closed, nonempty, proper subsets $E_j \subset S$, $0 \leq j \leq m$, with $E_m = E_0$, such that (a) $\bigcap_{j=0}^{m-1} E_j = \emptyset$ and (b) whenever $f \in C(S)$ and $f|_{E_j} = 0$ for some j with*

$0 \leq j \leq m - 1$, it follows that $Af|E_{j+1} = 0$.

Proof. Define $f_j = A^j(f_0)$ for $0 \leq j \leq m$. Because $A(f_j) = f_{j+1}$ for $0 \leq j \leq m - 1$, we see that $A(\theta) = \theta$ for $\theta := \sum_{j=0}^{m-1} f_j$. Also, because $f_j \in P(S)$ for $1 \leq j \leq m - 1$ and $f_0 \in P_\circ(S)$, we conclude that $\theta \in P_\circ(S)$. If $\psi \in P_\circ(S)$, there exists $\alpha > 0$ such that $\psi - \alpha\theta \in P(S)$, so by applying A we see that $A(\psi) - \alpha\theta \in P(S)$ and $A(\psi) \in P_\circ(S)$. It follows that A maps the interior of $P(S)$ into itself, so by applying A repeatedly we see that $f_j = A^j(f_0) \in P_\circ(S)$ for $1 \leq j \leq m - 1$.

For $j \in \mathbb{Z}$, define $f_j = f_i$, where $0 \leq i \leq m - 1$ and $j \equiv i \pmod{m}$. For $i \in \mathbb{Z}$, define $r_i := \max\{r | f_i \geq r f_{i-1}\}$ and $R_i := \min\{R | f_i \leq R f_{i-1}\}$. If $r_i = R_i$ for some i , then $f_{i-1} = \lambda f_i$ for some $\lambda > 0$. By applying A repeatedly, one derives that $\lambda f_j = f_{j-1}$ for all j and that $f_0 = \lambda^m f_0$. The latter equation implies that $\lambda = 1$ and $f_1 = f_0$, a contradiction. Thus we know that $r_i \neq R_i$ for all i .

We claim that $r_i = r_{i+1} =: r$ and $R_i = R_{i+1} =: R$ for all i and that $r < 1 < R$. Recalling that $f_i \geq r_i f_{i-1}$ and applying A we see that $f_{i+1} \geq r_i f_i$, and the latter inequality implies that $r_{i+1} \geq r_i$. It follows that $r_0 \leq r_1 \leq \dots \leq r_{m-1} \leq r_m = r_0$, which implies that

$$r_{i+1} = r_i =: r \quad \text{for all } i \in \mathbb{Z}.$$

A similar argument shows that $R_i = R_{i+1} =: R$ for all i .

To see that $r < 1$, we assume not and argue by contradiction. If $r \geq 1$, then $f_{i+1} \geq f_i$ for each i ; and because $f_{i+1} \neq f_i$, there exists $s_i \in S$ with $f_{i+1}(s_i) > f_i(s_i)$. Adding these inequalities and taking $s = s_j$ for some fixed j , we obtain

$$\sum_{i=0}^{m-1} f_{i+1}(s_j) > \sum_{i=0}^{m-1} f_i(s_j),$$

which is a contradiction. The proof that $R > 1$ is similar and is left to the reader.

Let $C_i = \{s \in S | f_i(s) = r f_{i-1}(s)\}$ and $D_i = \{s \in S | f_i(s) = R f_{i-1}(s)\}$. We claim that if $f \in C(S)$ and $f|C_i = 0$ for some i , then $Af|C_{i+1} = 0$. Thus assume that $f|C_i = 0$ and $f \neq 0$ and, for $\epsilon > 0$, let $G_\epsilon = \{s \in S | |f(s)| \geq \epsilon\}$. For ϵ sufficiently small, G_ϵ is a closed, nonempty set disjoint from C_i . There is an open neighborhood H_ϵ of C_i such that the closure of H_ϵ is disjoint from G_ϵ . Urysohn's theorem implies that there is a continuous map $\psi_\epsilon : S \rightarrow \mathbb{R}$ such that $\psi_\epsilon(s) = 1$ for all $s \in G_\epsilon$, $\psi_\epsilon(s) = 0$ for all $s \in H_\epsilon$, and $0 \leq \psi_\epsilon(s) \leq 1$ for all $s \in S$. We have that $\|\psi_\epsilon f - f\| < \epsilon$, so $\|A(\psi_\epsilon f) - A(f)\| < \|A\|\epsilon$. Since $\psi_\epsilon f = 0$ on H_ϵ , $f_i - r f_{i-1} \geq 0$ and $(f_i - r f_{i-1})(s) > 0$ for all $s \notin H_\epsilon$, there exists $\lambda_\epsilon > 0$ such that

$$-\lambda_\epsilon(f_i - r f_{i-1}) \leq \psi_\epsilon f \leq \lambda_\epsilon(f_i - r f_{i-1}).$$

It follows that

$$-\lambda_\epsilon A(f_i - r f_{i-1}) \leq A(\psi_\epsilon f) \leq \lambda_\epsilon A(f_i - r f_{i-1}).$$

Because $A(f_i - r f_{i-1}) = f_{i+1} - r f_i$ and $f_{i+1} - r f_i$ vanishes on C_{i+1} , we see that $A(\psi_\epsilon f)$ vanishes on C_{i+1} . Taking the limit as $\epsilon \rightarrow 0^+$, we conclude that $A(f)$ vanishes on C_{i+1} . A similar argument shows that if f vanishes on D_i , then Af vanishes on D_{i+1} .

Notice that if $s \in \cap_{j=0}^{m-1} C_j$, then $\prod_{j=0}^{m-1} (\frac{f_j(s)}{f_{j-1}(s)}) = r^m = 1$, which contradicts that fact that $r < 1$. It follows that $\cap_{j=0}^{m-1} C_j$ is empty. A similar argument shows that $\cap_{j=0}^{m-1} D_j$ is empty. We complete the proof by either defining $E_j = C_j$ for $0 \leq j \leq m$ or by defining $E_j = D_j$ for $0 \leq j \leq m$. \square

Remark 2.1. Let assumptions and notation be as in Theorem 2.1 and define $f_{j+m} = f_j$ for $j \in \mathbb{Z}$. The sort of reasoning used in Theorem 2.1 actually proves much more. Suppose that $a_j, 0 \leq j < m$, are real numbers. If $\sum_{j=0}^{m-1} a_j f_j = 0$, then $\sum_{j=0}^{m-1} a_j f_{j+k} = 0$ for all $k \in \mathbb{Z}$. If $P = P(S)$ and $\sum_{j=0}^{m-1} a_j f_j \in \partial P$ (respectively, P_o) then $\sum_{j=0}^{m-1} a_j f_{j+k} \in \partial P$ (respectively, P_o) for all $k \in \mathbb{Z}$. If, for fixed real numbers $a_j, 0 \leq j < m$, we have that $\sum_{j=0}^{m-1} a_j f_j := g \in \partial P$, define for $k \geq 0$ $E_k = \{s \in S \mid (A^k g)(s) = 0\}$. If $f \in C(S)$ and $f|E_k = 0$, then the same reasoning used in Theorem 2.1 shows that $Af|E_{k+1} = 0$.

In fact we can provide more information about the sets E_i than is given in Theorem 2.1. In particular, as we shall now show, if m is a prime, the sets E_i can be taken to be pairwise disjoint.

Corollary 2.1 *Let hypotheses be as in Theorem 2.1. In addition to the properties listed in Theorem 2.1, the sets E_i can be selected so that for $0 \leq i < k < m$ either $E_i = E_k$ or $E_i \cap E_k = \emptyset$. If m is a prime, we can always arrange that $E_i \cap E_k = \emptyset$ for $0 \leq i < k < m$.*

Proof. Suppose that E is a closed nonempty subset of S and that there exists $f \in P(S)$ such that $E = \{s \in S \mid f(s) = 0\}$. Let $F = \{s \in S \mid (Af)(s) = 0\}$. Then the proof of Theorem 2.1 shows that for every $h \in C(S)$ with $h|E = 0$ we have $Ah|F = 0$. We shall use this observation.

Let $f_j, j \in \mathbb{Z}$, and θ be as defined in the proof of Theorem 2.1 and let

$$r_j = \sup\{r \geq 0 \mid f_j \geq r f_{j-1}\}.$$

We have seen in the proof of Theorem 2.1 that $r_j = r < 1$ is independent of j . As in the proof of Theorem 2.1, we define $E_j = \{s \in S \mid f_j(s) = r f_{j-1}(s)\}$, and we recall that E_j is a proper, closed, nonempty subset of S with $\cap_{j=0}^{m-1} E_j = \emptyset$. If J is any finite collection of integers, we define E_J by $E_J = \cap_{j \in J} E_j$. If ν is an integer, we define $J + \nu = \{j + \nu \mid j \in J\}$, so $E_{J+\nu} = \cap_{j \in J} E_{j+\nu}$. We now select $J \subset \{0, 1, \dots, m-1\}$ such that $E_J \neq \emptyset$ and such that $E_L = \emptyset$ for every subset $L \subset \{0, 1, \dots, m-1\}$ such that $|J| < |L|$. Our construction insures that $|J| := k < m$. For this choice of J , we define $\hat{E}_\nu := E_{J+\nu}$ for $\nu \in \mathbb{Z}$. For each $\nu \in \mathbb{Z}$, we define

$$J_\nu = \{i \in \{0, 1, \dots, m-1\} \mid i \equiv j + \nu \pmod{m} \text{ for some } j \in J\}.$$

We note that $|J_\nu| = k$ and $\hat{E}_\nu = E_{J_\nu}$. We shall write $J + \nu_1 \equiv J + \nu_2 \pmod{m}$ if $J_{\nu_1} = J_{\nu_2}$.

We claim that if $E_\nu, \nu \in \mathbb{Z}$, in Theorem 2.1 are replaced by \hat{E}_ν , then the properties mentioned in Theorem 2.1 and in the statement of this corollary are satisfied. To see this, first define a function g_ν by

$$g_\nu := \sum_{j \in J} (f_{j+\nu} - r f_{j+\nu-1})$$

and note that $E_{J+\nu} := \hat{E}_\nu = \{s \in S \mid g_\nu(s) = 0\}$; note also that our construction gives $E_J \neq \emptyset$. If $E_{J+\nu} = \emptyset$ for some $\nu \in \mathbb{Z}$, then $g_\nu \in P_\circ(S)$ and there exists $\delta > 0$ with $g_\nu \geq \delta\theta$. Applying A repeatedly, we see that $g_\mu \geq \delta\theta$ for all $\mu \geq \nu$. If μ is an integral multiple of m , we deduce that $g_0 \geq \delta\theta$; and the latter inequality contradicts the assumption that $E_J \neq \emptyset$. Thus we conclude that $\hat{E}_\nu \neq \emptyset$ for all $\nu \in \mathbb{Z}$. Since E_j is a closed, proper subset of S for all $j \in \mathbb{Z}$, we easily conclude that \hat{E}_ν is a closed proper subset of S for all ν and that

$$\bigcap_{\nu=0}^{m-1} \hat{E}_\nu = \bigcap_{j=0}^{m-1} E_j = \emptyset.$$

As observed above, $g_\nu \in P(S)$ and $\hat{E}_\nu = \{s \mid g_\nu(s) = 0\}$. Since $A(g_\nu) = g_{\nu+1}$ and $\hat{E}_{\nu+1} = \{s \mid g_{\nu+1}(s) = 0\}$, our original remarks imply that if $h \in C(S)$ and $h|_{\hat{E}_\nu} = 0$, then $A(h)|_{\hat{E}_{\nu+1}} = 0$.

If $0 \leq \mu < \nu < m$ and $\hat{E}_\mu \cap \hat{E}_\nu \neq \emptyset$, we see that $\hat{E}_{J_\mu} \cap \hat{E}_{J_\nu} := E_L \neq \emptyset$, where $L := J_\mu \cup J_\nu$. By our selection of J , we must have that $|L| = k$, so $J_\mu = J_\nu$ and $\hat{E}_\mu = \hat{E}_\nu$.

Suppose, by way of contradiction, that m is a prime and that, for μ and ν with $0 \leq \mu < \nu < m$ we have $\hat{E}_\mu \cap \hat{E}_\nu \neq \emptyset$. As observed above, we must have $J_\mu = J_\nu$. If $j_0 \in J$, there must exist $i \in J_\mu$ with $i \equiv j_0 + \mu \pmod{m}$, and it follows that there must exist $j_1 \in J$ with $i \equiv j_1 + \nu \pmod{m}$. It follows that for any $j_0 \in J$, there exists $j_1 \in J$ with $j_0 + \rho \equiv j_1 \pmod{m}$ and $\rho := \mu - \nu$. Using this observation repeatedly, we see that for $0 \leq s < m$, there exists $j_s \in J$ with $j_s \equiv j_0 + s\rho \pmod{m}$. If $j_s = j_t$ for $0 \leq s < t < m$, we find that $(t-s)\rho \equiv 0 \pmod{m}$. However, this is impossible, because $0 < t-s < m$, $0 < -\rho < m$ and m is a prime. It follows that J has at least m distinct elements, which contradicts our earlier observation that $|J| < m$. \square

Actually, the proof of Corollary 2.1 yields more than we have stated. Let notation be as in Corollary 2.1. If $0 \leq \mu \leq \nu < m$ and $\hat{E}_\mu \cap \hat{E}_\nu \neq \emptyset$, we have seen that $J_\mu = J_\nu$. The same argument as above shows that the latter equation is impossible if $\rho := \nu - \mu$ is relatively prime to m . Furthermore, the equation $J_\mu = J_\nu$ places many other constraints. For example, if $|J| = m-1$, one must have that $J_\mu \neq J_\nu$ for $0 \leq \mu \leq \nu < m$.

Theorem 2.1 gives necessary conditions for a given positive, bounded linear operator $A : C(S) \rightarrow C(S)$ to have a periodic point $f_0 \in P_\circ$ of minimal period m . If $f_j \in P_\circ$, $0 \leq j \leq m$, are specified functions with $f_m = f_0$, one can also ask whether there exists a bounded, positive linear operator A , given by a continuous integral kernel, such that $A^j(f_0) = f_j$ for $0 \leq j \leq m$. For $m = 2$ there is a simple answer to this question.

Corollary 2.2 *Let S be a compact, Hausdorff space and μ a regular Borel measure on S such that $\mu(G) > 0$ for every nonempty, open set $G \subset S$. Suppose that $f_0 \in P_\circ(S)$ and $f_1 \in P_\circ(S)$ are given functions with $f_0 \neq f_1$. There exists a continuous, nonnegative function $k : S \times S \rightarrow [0, \infty)$ such that $\int k(s, t) f_0(t) \mu(dt) = f_1(s)$ and $\int k(s, t) f_1(t) \mu(dt) = f_0(s)$ for all $s \in S$ if and only if f_0 and f_1 satisfy the following conditions:*

(1) If $g_0(s) = \frac{f_0(s)}{f_1(s)}$ and $g_1(s) = \frac{f_1(s)}{f_0(s)}$, then

$$\min\{g_0(s) : s \in S\} = \min\{g_1(s) : s \in S\} := r,$$

where $0 < r < 1$.

(2) If $E_0 = \{s \mid f_0(s) = rf_1(s)\}$ and $E_1 = \{s \mid f_1(s) = rf_0(s)\}$, then E_0 and E_1 have nonempty interiors.

Proof. Suppose first that there exists a bounded, positive linear operator $A : C(S) \rightarrow C(S)$ such that $Af_0 = f_1$ and $Af_1 = f_0$ and $(Af)(s) = \int k(s,t)f(t)\mu(dt)$ for some continuous, nonnegative function k . By applying Remark 1 to $f_0 - rf_1$ or by recalling the proof of Theorem 2.1, we see that $\min\{g_0(s) : s \in S\} := r$ and $\min\{g_1(s) : s \in S\} = r$. Since we assume that $f_0 \neq f_1$, the proof of Theorem 2.1 shows that $0 < r < 1$.

As noted in the proof of Theorem 2.1 and in Remark 2.1. if $f \in C(S)$ and $f|_{E_0} = 0$, then $Af|_{E_1} = 0$. Because we assume that $\mu(G) > 0$ for any nonempty, open subset of S , we easily deduce that $k|_{E_1 \times E_0'} = 0$, where E' denotes the complement of a set E . Similarly, we see that $k|_{E_0 \times E_1'} = 0$. Because k is continuous, it follows that $k|_{E_1 \times F_0} = 0$ and $k|_{E_0 \times F_1} = 0$, where F_i denotes the closure of E_i' . If E_0 has empty interior, it follows that $k|_{E_1 \times S} = 0$. However, this contradicts the fact that $A(\theta) = \theta$, where $\theta := f_0 + f_1 \in P_o(S)$. Similarly, we obtain a contradiction if E_1 has empty interior.

Conversely, assume that f_0 and f_1 satisfy conditions (1) and (2) of Corollary 2.1. Because E_0 and E_1 have nonempty interiors, there exist nonnegative, continuous functions $\psi_i : S \rightarrow \mathbb{R}$, $i = 0, 1$, such that $\psi_i(t) = 0$ for all $t \notin E_i$ and $\int \psi_i(t)\mu(dt) = 1$. Define $k(s, t)$ by

$$k(s, t) = \frac{(r^{-1}f_1(s) - f_0(s))}{(r^{-1} - r)}(\psi_0(t))(f_0(t))^{-1} + \frac{(f_0(s) - rf_1(s))}{(r^{-1} - r)}(\psi_1(t))(f_0(t))^{-1}.$$

We leave it to the reader to verify that

$$\int k(s, t)f_0(t)\mu(dt) = f_1(s) \quad \text{and} \quad \int k(s, t)f_1(t)\mu(dt) = f_0(s). \quad \square$$

If m in Theorem 2.1 is a prime, we shall see in Section 3 that the necessary conditions of Theorem 2.1, together with mild compactness conditions on A , provide sufficient conditions for the existence of a strictly positive periodic point of minimal period m . In general, suppose that $\prod_{i=1}^k p_i^{a_i} = m$, where p_i , $1 \leq i \leq k$, are distinct prime numbers and a_i , $1 \leq i \leq k$, are positive integers. For $1 \leq i \leq k$, define $\nu_i = \frac{m}{p_i}$ and $B_i = A^{\nu_i}$, where A is as in Theorem 2.1. By Theorem 2.1, there exists $\theta \in P_o(S)$ with $A(\theta) = \theta$ and $B_i(\theta) = \theta$ for $1 \leq i \leq k$. Since $f_0 \in P_o(S)$ is a periodic point of A of minimal period m , $f_0 \in P_o(S)$ is also a periodic point of B_i of minimal period p_i . Thus we can apply Theorem 2.1 to the positive, bounded linear operator B_i to obtain necessary conditions that B_i have a periodic point in P_o of minimal period p_i , $1 \leq i \leq k$. We shall see in Section 3 that these necessary conditions on B_i , $1 \leq i \leq k$, together with mild compactness conditions on B_i , $1 \leq i \leq k$, insure that A has a periodic point in P_o of minimal period m .

3 Periodic Points of A : Sufficient Conditions

In this section we shall prove that if A satisfies a mild compactness condition and m is a prime, then the necessary conditions of Theorem 2.1 are also sufficient to insure that A has

a periodic point $f_0 \in P_o(S)$ of minimal period m . The case that m is not a prime can be easily analyzed and reduced to the prime case: see Theorem 3.5.

Our first lemma is a kind of extension theorem for continuous functions. It will play a crucial role in our subsequent work.

Lemma 3.1 *Let m be a positive integer and suppose that E_i , $0 \leq i \leq m-1$, are closed (possibly empty) subsets of a normal, Hausdorff space T . Assume that $h_i : E_i \rightarrow \mathbb{R}$, $0 \leq i \leq m-1$, are continuous maps, that $h : T \rightarrow \mathbb{R}$ is a continuous map and that*

$$h(x) = \sum_{i=0}^{m-1} h_i(x) \text{ for all } x \in \bigcap_{i=0}^{m-1} E_i.$$

Then there exist continuous maps $\hat{h}_i : T \rightarrow \mathbb{R}$ such that

$$(a) \hat{h}_i|_{E_i} = h_i|_{E_i}$$

$$(b) \sum_{i=0}^{m-1} \hat{h}_i(x) = h(x) \text{ for all } x \in T.$$

If, in addition, there exists $B > 0$ such that $|h(x)| \leq B$ for all $x \in T$ and $|h_i(x)| \leq B$ for all $x \in E_i$, $0 \leq i \leq m-1$, then \hat{h}_i , $0 \leq i \leq m-1$, can be chosen so that $|h_i(x)| \leq ((m+1)!)B$ for all $x \in T$, $0 \leq i \leq m-1$.

Proof. We argue by induction on m . If $m = 1$, Lemma 3.1 is obvious: define $\hat{h}_0(x) = h(x)$ for all $x \in T$ and note that if B exists, then $|\hat{h}_0(x)| \leq B < ((m+1)!)B$ for all $x \in T$. Take $m > 1$ and assume we know the Lemma for smaller values of m . Define $\hat{h}_0(x) = h_0(x)$ for $x \in E_0$ and define $\hat{h}_0(x) = h(x) - \sum_{i=1}^{m-1} h_i(x)$ for $x \in \bigcap_{i=1}^{m-1} E_i$. If $x \in E_0 \cap (\bigcap_{i=1}^{m-1} E_i)$, our assumptions imply that these definitions agree, so \hat{h}_0 is a continuous, real-valued function with domain $E_0 \cup (\bigcap_{i=1}^{m-1} E_i)$. The Tietze extension theorem implies that \hat{h}_0 can be extended to a continuous map (which we also designate \hat{h}_0) $\hat{h}_0 : T \rightarrow \mathbb{R}$. Furthermore, if B exists, we have that $|\hat{h}_0(x)| \leq mB$ for all $x \in E_0 \cup \bigcap_{i=1}^{m-1} E_i$, so the extension $\hat{h}_0 : T \rightarrow \mathbb{R}$ can also be chosen to satisfy $|\hat{h}_0(x)| \leq mB$ for all $x \in T$. By our construction, we have that $h(x) - \hat{h}_0(x) = \sum_{i=1}^{m-1} h_i(x)$ for all $x \in \bigcap_{i=1}^{m-1} E_i$. Define \hat{h} by $\hat{h}(x) = h(x) - \hat{h}_0(x)$ for $x \in T$ and notice that \hat{h} is continuous and $\hat{h}(x) = \sum_{i=1}^{m-1} h_i(x)$ for all $x \in \bigcap_{i=1}^{m-1} E_i$ and (if B exists) $|\hat{h}(x)| \leq B_1 := (m+1)B$ and $|h_i(x)| \leq B < B_1$, for all $x \in E_i$, $1 \leq i \leq m-1$. By our inductive assumption, for $1 \leq i \leq m-1$, h_i has a continuous extension $\hat{h}_i : T \rightarrow \mathbb{R}$ such that $\hat{h}(x) = \sum_{i=1}^{m-1} \hat{h}_i(x)$ for all $x \in T$ and (if B exists) $|\hat{h}_i(x)| \leq (m!)B_1 = ((m+1)!)B$ for all $x \in T$, $1 \leq i \leq m-1$. This completes the proof. \square

We shall actually only use a very special case of Lemma 3.1.

Lemma 3.2 *Let m be an integer with $m \geq 2$ and suppose that E_i , $0 \leq i \leq m$, are closed (possibly empty) subsets of a normal, Hausdorff space T and that $E_m = E_0$. Assume also that $\bigcap_{i=0}^{m-1} E_i = \emptyset$. Let r be a positive real number and set $\rho = \max(r, r^{-1})$ and $N = m(m+1)!$. Then there exist positive, continuous functions $f_i : T \rightarrow (0, \infty)$ with $f_m = f_0$ and $f_i(s) = r f_{i-1}(s)$ for all $s \in E_i$, $1 \leq i \leq m$ and $\rho^{-N} \leq f_i(s) \leq \rho^N$ for all $s \in T$, $1 \leq i \leq m$.*

Proof Define $h : T \rightarrow \mathbb{R}$ by $h(t) = 0$ for all t and define $h_i : E_i \rightarrow \mathbb{R}$ by $h_i(t) = \ln(r)$ for all $t \in E_i$, $0 \leq i \leq m$. Because $\bigcap_{i=0}^{m-1} E_i = \emptyset$, the conditions of Lemma 3.1 are satisfied (with $B := |\ln(r)| = \ln(\rho)$), and there exist continuous maps $\hat{h}_i : T \rightarrow \mathbb{R}$ such that $\hat{h}_i(t) = \ln(r)$ for all $t \in E_i$ and $\sum_{i=0}^{m-1} \hat{h}_i(t) = 0$ for all $t \in T$ and $|\hat{h}_i(t)| \leq ((m+1)!) |\ln(r)| \leq ((m+1)!) \ln(\rho)$ for all $t \in T$. We define $\hat{h}_m = \hat{h}_0$, and we define $g_i : T \rightarrow \mathbb{R}$ by $g_i(t) = \exp(\hat{h}_i(t))$ for $0 \leq i \leq m$. It follows that $g_i \in P_\circ(T)$, $g_m = g_0$, $g_i|_{E_i} = r$, $\prod_{i=0}^{m-1} g_i(t) = 1$ for all $t \in T$ and, for $n := (m+1)!$, $\rho^{-n} \leq g_i(t) \leq \rho^n$ for all $t \in T$, $1 \leq i \leq m$. We define $f_0(t) = 1$ for all $t \in T$ and we define $f_k : T \rightarrow (0, \infty)$, $1 \leq k \leq m$, by

$$f_k(t) = \left(\prod_{j=1}^k g_j(t) \right).$$

The reader can verify that, with this definition of f_k for $0 \leq k \leq m$, all the stated conditions of Lemma 3.2 are satisfied. \square

At this point we need to recall some standard results. If Y is a real Banach space and $B : Y \rightarrow Y$ is a bounded linear map, \tilde{Y} will denote the complexification of Y and $\tilde{B} : \tilde{Y} \rightarrow \tilde{Y}$ will denote the complexification of B . We define a norm on the complex Banach space \tilde{Y} by

$$\|u + iv\| = \max\{\|(\cos(\theta)u + \sin(\theta)v)\| : 0 \leq \theta \leq 2\pi\},$$

where u and v are elements of Y and $i = \sqrt{-1}$. We shall denote by $\sigma(\tilde{B})$ the spectrum of \tilde{B} , and we shall define $\sigma(B) := \sigma(\tilde{B})$. We shall always denote by $r(B)$ the spectral radius of B , and we recall that $\|B^k\| = \|B^k\|$ and

$$\lim_{n \rightarrow \infty} \|B^n\|^{\frac{1}{n}} = \inf_{n \geq 1} \|B^n\|^{\frac{1}{n}} = \sup\{|\lambda| : \lambda \in \sigma(B)\}. \quad (1)$$

If T is a bounded subset of Y , recall that $\alpha(T)$, the measure of noncompactness of T , is defined by

$$\alpha(T) = \inf\{d > 0 \mid T = \cup_{j=1}^k T_j, \text{ where } k < \infty \text{ and } \text{diameter}(T_j) \leq d \text{ for } 1 \leq j \leq k\}. \quad (2)$$

Properties of the measure of noncompactness can be found in Section 1 of [14]. If Y and Z are Banach spaces and $B : Y \rightarrow Z$ is a bounded linear map, we define $\alpha(B)$ by

$$\alpha(B) = \inf\{c > 0 \mid \alpha(B(T)) \leq c\alpha(T) \text{ for all bounded sets } T \subset Y\}. \quad (3)$$

The map $B \rightarrow \alpha(B)$ is a seminorm; $\alpha(B) \leq \|B\|$ and $\alpha(B) = 0$ if B is compact. If $B : Y \rightarrow Y$ is a bounded linear map and if there exists an integer N such that $B^N = U + C$, where $\|U\| < 1$ and C is compact, then $\alpha(B^N) = \alpha(U) < 1$. Further results about the map $B \rightarrow \alpha(B)$ are given in [12, 14].

If $B : Y \rightarrow Y$ is a bounded linear map, we shall always denote by $\rho(B)$ the radius of the essential spectrum of B ; and we recall that

$$\rho(B) = \lim_{n \rightarrow \infty} (\alpha(B^n))^{\frac{1}{n}} = \inf_{n \geq 1} (\alpha(B^n))^{\frac{1}{n}}. \quad (4)$$

Our previous remarks imply that $\rho(B) \leq r(B)$. Further results concerning the essential spectrum and the radius of the essential spectrum can be found in [12, 13, 15].

If C is a closed, bounded convex subset of a Banach space X and $L : X \rightarrow X$ is a bounded linear map such that $L(C) \subset C$, our next lemma recalls conditions which imply that L has a fixed point in C .

Lemma 3.3 *Let Y be a Banach space and $L : Y \rightarrow Y$ a bounded linear map. Assume that there exists a closed, bounded convex set $C \subset Y$ such that $L(C) \subset C$. Suppose that any one of the following conditions holds:*

- (a) *There exists $x_0 \in C$ such that the sequence $y_k = \frac{1}{k} \sum_{j=0}^{k-1} L^j(x_0)$ has a subsequence which is convergent in the weak topology on Y .*
- (b) *There exists $x_0 \in C$ such that the closure of $\{L^k(x_0) : k \geq 0\}$ in the norm topology is compact in the norm topology.*
- (c) $\rho(L) < 1$, where $\rho(L)$ denotes the essential spectral radius of L .

Then L has a fixed point in C .

Proof. Suppose that condition (a) holds and that a subsequence y_{k_i} converges weakly to z_0 . Because C is closed and convex in the norm topology, it is also closed in the weak topology, and $z_0 \in C$. The proof of the mean ergodic theorem (see [21] , pages 213-214) shows that $L(z_0) = z_0$. (We use here the fact that C is bounded, so $\{\|L^n(x)\| : n \geq 0\}$ is bounded for every $x \in C$.)

To complete the proof, it suffices to show that (c) \Rightarrow (b) \Rightarrow (a). If

$$M := \{L^k(x_0) : k \geq 0\}$$

has compact closure in the norm topology, then Mazur's theorem (see [5] , p.416) implies that $\overline{\text{co}}(M)$, the closed, convex hull of M , is compact in the norm topology. Since $y_k \in \overline{\text{co}}(M)$ for all $k \geq 1$, the sequence y_k has a subsequence which is convergent in the norm topology, and hence in the weak topology.

If $\rho(L) < 1$, there exists an integer n and a constant $c < 1$ such that $\alpha(L^n) = c < 1$. If $x \in C$ and $M := \{L^k(x) : k \geq 0\}$, then $M \subset C$ is a bounded set and $\alpha(M) < \infty$. We have that

$$L^n(M) \cup (\cup_{j=0}^{n-1} L^j(x)) = M,$$

and since $\alpha(\cup_{j=0}^{n-1} L^j(x)) = 0$, $\alpha(M) = \alpha(L^n(M)) \leq c\alpha(M)$. The latter inequality implies that $\alpha(M) = 0$, so M has compact closure in the norm topology. \square

If X_i , $0 \leq i \leq m-1$, are Banach spaces, we can form a Banach space $Y = \prod_{i=0}^{m-1} X_i$. If $y = (x_0, x_1, \dots, x_{m-1}) \in Y$, we define $\|y\| = \max\{\|x_i\|_i : 0 \leq i \leq m-1\}$. If $X_i = X_j = X$ for $1 \leq i < j \leq m-1$ and $A : X \rightarrow X$ is a bounded linear operator, we can define a bounded linear operator $\Phi : Y \rightarrow Y$ by

$$\Phi((x_0, x_1, \dots, x_{m-1})) = (Ax_{m-1}, Ax_0, Ax_1, \dots, Ax_{m-2}), \tag{5}$$

and one easily sees that $\|\Phi^k\| = \|A^k\|$ for $k \geq 0$. If C is a closed, bounded convex subset of Y and $\Phi(C) \subset C$, we need a condition on A which insures that Φ has a fixed point in C .

Lemma 3.4 *Let X be a Banach space, $m \geq 2$ a positive integer and $A : X \rightarrow X$ a bounded linear map. Let $Y = \prod_{i=0}^{m-1} X$ and let $\Phi : Y \rightarrow Y$ be defined by equation (3.5). Assume that there exists a closed, bounded, convex set $C \subset Y$ such that $\Phi(C) \subset C$ and suppose that any one of the following conditions holds:*

- (a) *For any $g = (g_0, g_1, \dots, g_{m-1}) \in C$, there exists an increasing sequence of integers $k_i \rightarrow \infty$ such that $k_i^{-1} \sum_{j=0}^{k_i-1} A^{mj}(g_0)$ is convergent in the weak topology on X as $i \rightarrow \infty$.*
- (b) *For any $g = (g_0, g_1, \dots, g_{m-1}) \in C$, the norm closure of $\{A^{mj}(g_0) : j \geq 0\}$ is compact in the norm topology.*
- (c) *$\rho(A) < 1$, where $\rho(A)$ denotes the essential spectral radius of A .*

Then there exists $h \in C$ with $\Phi(h) = h$.

Proof. Take $f = (f_0, f_1, \dots, f_{m-1}) \in C$. We shall refer to f_t , $0 \leq t \leq m-1$, as the t -coordinate of f . Define $f_j = f_t$ if $j \in \mathbb{Z}$ and $j \equiv t \pmod{m}$, $0 \leq t \leq m-1$. For a positive integer n , $f \in Y$ and $x \in X$, we define $\Phi_n : Y \rightarrow Y$ and $B_n : X \rightarrow X$ by

$$\Phi_n(f) = \frac{1}{n} \sum_{j=0}^{n-1} \Phi^j(f) \text{ and } B_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} A^{mj}(x). \quad (6)$$

A calculation shows that for $k \geq 1$, the t -coordinate of $\Phi_{mk}(f)$ is given by

$$\frac{1}{m} \sum_{s=0}^{m-1} A^s(B_k(f_{m+t-s})) = B_k\left(\frac{1}{m} \sum_{s=0}^{m-1} A^s(f_{m+t-s})\right) := B_k(g_t). \quad (7)$$

One can see that g_t in the preceding equation is the t -coordinate of $\frac{1}{m} \sum_{s=0}^{m-1} \Phi^s(f)$, so the convexity of C implies that $(g_0, g_1, \dots, g_{m-1}) \in C$. Because $\Phi^k(C) \subset C$ and $A^k(f_j)$ is one of the coordinates of $\Phi^k(f)$ and C is bounded, $\{A^k(f_j) : k \geq 0\}$ is bounded for each j . A calculation shows that for $0 \leq t \leq m-1$,

$$g_{t+1} = A(g_t) + \frac{1}{m}(f_{t+1} - A^m(f_{t+1})).$$

Applying B_k to the above equation, we obtain that for $0 \leq t \leq m-1$,

$$B_k(g_{t+1}) = A(B_k(g_t)) + \frac{1}{mk}(f_{t+1} - A^{mk}(f_{t+1})). \quad (8)$$

Assume now that condition (a) of Lemma 3.4 holds. If g_t is as above, it follows that there exists a sequence $k_i \rightarrow \infty$ such that $B_{k_i}(g_0)$ converges weakly to h_0 as $i \rightarrow \infty$. It follows from eq. (8) and the fact that $A^{mk}(f_{t+1})$ is bounded for $0 \leq t \leq m-1$ that $B_{k_i}(g_1)$ converges weakly to $A(h_0)$. By using eq. (8) and repeating the argument one shows that $B_{k_i}(g_t)$ converges weakly to $A^t(h_0)$ as $i \rightarrow \infty$ for $0 \leq t \leq m-1$. It now follows from eq. (7) that $\Phi_{mk_i}(f)$ converges weakly to $(h_0, Ah_0, \dots, A^{m-1}h_0)$ as $i \rightarrow \infty$. By using condition (a) of Lemma 3.3, we conclude that Φ has a fixed point in C .

To complete the proof, it suffices to prove that condition (c) implies condition (b) and condition (b) implies condition (a). If $f = (f_0, f_1, \dots, f_{m-1}) \in C$ and condition (b) holds,

Mazur's theorem implies that $\overline{\text{co}}(\{A^{mj}(f_0) : j \geq 0\}) := D$ is compact in the norm topology. Since $B_k(f_0) \in D$ for $k \geq 1$, the compactness of D implies that there exists a subsequence $k_i \rightarrow \infty$ such that $B_{k_i}(f_0)$ converges in the norm topology (and hence in the weak topology) as $i \rightarrow \infty$. Thus condition (a) is satisfied.

If $\rho(A) < 1$, eq. (4) implies that $\rho(A^m) < 1$, so there exists $k \geq 1$ with $\alpha(A^{mk}) = c$, $c < 1$. If $f = (f_0, f_1, \dots, f_{m-1})$, define $M = \{A^{mj}(f_0) : j \geq 0\}$ and note that our previous remarks show that M is bounded. Arguing as in Lemma 3.3, we see that

$$\alpha(A^{mk}(M)) = \alpha(M) \leq c\alpha(M),$$

so $\alpha(M) = 0$, M has compact closure, and condition (b) is satisfied. \square

We can now prove our basic theorem for the existence of periodic points of A .

Theorem 3.1 *Let S be a compact, Hausdorff space and $A : C(S) \rightarrow C(S)$ a linear operator such that $A(P) \subset P$, where $P := P(S)$ is the set of nonnegative functions in $C(S)$. Let m be a prime number. Assume that the following conditions are satisfied:*

(1) *There exist closed, nonempty subsets $E_j \subset S$, $0 \leq j \leq m$, with $E_m = E_0$, such that $(\alpha) \bigcap_{j=0}^{m-1} E_j = \emptyset$ and (β) whenever $f \in C(S)$ and $f|_{E_j} = 0$ for some j with $0 \leq j \leq m-1$, it follows that $Af|_{E_{j+1}} = 0$.*

(2) *There exists $\theta \in P$ such that $A(\theta) = \theta$ and $\theta(s_0) > 0$ for some $s_0 \in \bigcup_{j=0}^{m-1} E_j := E$.*

In addition assume that at least one of the following compactness conditions on A is satisfied:

(a) *For every $M > 0$ and every $f_0 \in C(S)$ with $-M\theta \leq f_0 \leq M\theta$, there exists a sequence $k_i \rightarrow \infty$ such that the sequence $\frac{1}{k_i} \sum_{s=0}^{k_i-1} A^{ms}(f_0)$ converges in the weak topology on $C(S)$ as $i \rightarrow \infty$.*

(b) *For every $M > 0$ and for every $f_0 \in C(S)$ such that $-M\theta \leq f_0 \leq M\theta$, the norm closure of $\{A^{mj}(f_0) | j \geq 1\}$ is compact in the norm topology.*

(c) $\rho(A) < 1$, where $\rho(A)$ denotes the essential spectral radius of A .

Then it follows that there exist positive reals a and b and $\hat{f}_0 \in P$ such that $a\theta \leq \hat{f}_0 \leq b\theta$, $A^m(\hat{f}_0) = \hat{f}_0$, and $A^j(\hat{f}_0) \neq \hat{f}_0$ for $0 < j < m$.

Proof. Let Y and Φ be as defined in Lemma 3.4 and let $r \neq 1$ be a positive real number. By Lemma 3.2, there exist functions $g_j \in P_0$, $0 \leq j \leq m$, with $g_m = g_0$, such that $g_j|_{E_j} = r g_{j-1}|_{E_j}$ for $1 \leq j \leq m$ and $g_0(s) = 1$ for all $s \in S$. We define $\hat{f}_j = \theta g_j$, so it remains true that $\hat{f}_j|_{E_j} = r \hat{f}_{j-1}|_{E_j}$ for $1 \leq j \leq m$. By the continuity and positivity of the functions g_j , there are numbers $a > 0$ and $b > 0$ such that $a \leq g_j(s) \leq b$ for all $s \in S$ and for $0 \leq j \leq m$, and this implies that $a\theta \leq \hat{f}_j \leq b\theta$ for $0 \leq j \leq m$.

Now define a set $C \subset Y$ by

$$C := \{(f_0, f_1, \dots, f_{m-1}) \in Y | a\theta \leq f_j \leq b\theta \text{ and } f_j|_{E_j} = r f_{j-1}|_{E_j} \text{ for } 1 \leq j \leq m\} \quad (9)$$

Our convention is that subscripts are taken modulo m , so $f_m = f_0$ and $f_{m-1} = f_{-1}$. Our previous remarks show that $(\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{m-1}) \in C$, so C is nonempty. The reader can verify that C is closed, bounded and convex.

We next claim that $\Phi(C) \subset C$. To see this, suppose that $f = (f_0, f_1, \dots, f_{m-1}) \in C$ and write $\Phi(f) = (h_0, h_1, \dots, h_{m-1})$, so $h_i = A(f_{i-1})$ for $0 \leq i \leq m-1$. We know that $a\theta \leq f_{i-1} \leq b\theta$ for all i , so we have

$$aA(\theta) = a\theta \leq A(f_{i-1}) = h_i \leq bA(\theta) = b\theta$$

for $0 \leq i \leq m-1$. By assumption, $(f_{i-1} - rf_{i-2})|E_{i-1} = 0$ for all i , so the defining property of the sets E_i implies that

$$A(f_{i-1} - rf_{i-2})|E_i = (h_i - rh_{i-1})|E_i = 0$$

for all i . This proves that $\Phi(f) \in C$, so $\Phi(C) \subset C$.

If $f = (f_0, f_1, \dots, f_{m-1}) \in C$, we know that $a\theta \leq f_0 \leq b\theta$; so if condition (a) of Theorem 3.1 is satisfied, there exists a sequence $k_i \rightarrow \infty$ such that $B_{k_i}(f_0)$, where B_k is given by eq. (6). A similar argument shows that if condition (b) of Theorem 3.1 is satisfied, then condition (b) of Lemma 3.4 is satisfied. It follows that if condition (a), (b) or (c) of theorem 3.1 is satisfied, then the corresponding condition of Lemma 3.4 is satisfied, and thus Lemma 3.4 implies that Φ has a fixed point in C .

Let $(\hat{f}_0, \hat{f}_1, \dots, \hat{f}_{m-1}) \in C$ denote a fixed point of Φ . The definition of Φ implies that $A\hat{f}_{i-1} = \hat{f}_i$ for $1 \leq i \leq m$ (where $\hat{f}_m := \hat{f}_0$), so we see that $A^m(\hat{f}_0) = \hat{f}_0$. We claim that $A\hat{f}_0 \neq \hat{f}_0$. Suppose, by way of contradiction, that $A\hat{f}_0 = \hat{f}_0$. Then we obtain that $\hat{f}_i = A^i(\hat{f}_0) = \hat{f}_0$ for all i . This implies that $(\hat{f}_i - r\hat{f}_{i-1})|E_i = (1-r)\hat{f}_0|E_i = 0$ for all i ; and since $r \neq 1$, we conclude that $\hat{f}_0|E_i = 0$ for all i and $\hat{f}_0|E = 0$. However, $0 \leq a\theta(s) \leq \hat{f}_0(s)$ for all s , so $\theta|E = 0$; and this contradicts the assumption that $\theta(s_0) > 0$ for some $s_0 \in E$. It follows that $A(\hat{f}_0) \neq \hat{f}_0$; and because m is a prime, we conclude that $A^j(\hat{f}_0) \neq \hat{f}_0$ for $0 < j < m$. \square

Remark 3.1. The above argument actually proves slightly more than is stated. Let hypotheses and notation be as in Theorem 3.1. Let r be any positive real, $r \neq 1$, and k an integer with $0 \leq k \leq m-1$. Then there exist positive reals a and b and $g_0 \in C(S)$ such that $a\theta \leq g_0 \leq b\theta$, g_0 is a periodic point of A of minimal period m , and $Ag_0|E_k = rg_0|E_k$.

Remark 3.2. Theorems 2.1 and 3.1 imply, as a very special case, Corollary 1.1 of the Introduction. Observe that Theorem 2.1 implies that if A in Corollary 1.1 has a periodic point $f_0 \in P_o(S)$ of minimal period m (m a prime), then $A(\theta) = \theta$ for some $\theta \in P_o(S)$ and there exist sets E_j as in condition (2) of Theorem 2.1. If $f \in C(S)$ and $f|E_j = 0$, then, for all $s \in E_j$ we have

$$(Af)(s) = \int_{S-E_j} f(t)k(s,t)\mu(dt) = 0.$$

If $k(s_0, t_0) > 0$ for some $(s_0, t_0) \in E_{j+1} \times E'_j$, we select an open neighborhood G of t_0 with $\bar{G} \subset E'_j$ such that $k(s_0, t) > 0$ for all $t \in G$. There exists a nonnegative, continuous function f which is positive on G and equal to zero on E'_j . Because we assume that $\mu(G) > 0$, this implies that $(Af)(s_0) > 0$, which is a contradiction. Thus we must have that $k(s, t) = 0$ for all $(s, t) \in \cup_{j=0}^{m-1} (E_{j+1} \times E'_j)$.

Conversely, suppose that conditions (1) and (2) of Corollary 1.1 are satisfied. Then it is clear that conditions (1) and (2) of Theorem 2.1 are satisfied. Also, A is compact, so $\rho(A) = 0 < 1$, and Theorem 3.1 implies that A has a periodic point $f_0 \in P_o(S)$ of minimal period m .

The argument in theorem 3.1 has little to do with the linearity of A , and one can give a version of Theorem 3.1 for nonlinear operators. Recall that a map $A : P(S) \rightarrow P(S)$ is called “order-preserving” if $A(x) \leq A(y)$ whenever $0 \leq x \leq y$. The map A is called “homogeneous of degree one” if $A(\lambda x) = \lambda A(x)$ for all nonnegative reals λ and all $x \in P(S)$. If $c : S \times S \rightarrow \mathbb{R}$ is a nonnegative, continuous map and $A : P(S) \rightarrow P(S)$ is defined by $(Ax)(s) = \max\{c(s, t)x(t) \mid t \in S\}$, then A provides an example of a continuous, order-preserving map which is homogeneous of degree one and takes bounded sets to sets with compact closure. Such maps arise in many applications.

If $\theta \in P(S)$ and a and b are positive reals, we shall write

$$[a\theta, b\theta] := \{f \in C(S) \mid a\theta \leq f \leq b\theta\}.$$

Theorem 3.1A *Let m be a prime, S a compact Hausdorff space, and $A : P(S) \rightarrow P(S)$ an order-preserving map which is homogeneous of degree one. Define $Q = \prod_{i=0}^{m-1} P(S)$ and define $\Phi : Q \rightarrow Q$ by $\Phi(g_0, g_1, \dots, g_{m-1}) = (h_0, h_1, \dots, h_{m-1})$, where $h_i = A(g_{i-1})$ and $g_{-1} := g_{m-1}$. Assume that there exist real numbers $a < 1 < b$, $\theta \in P(S)$ and compact, nonempty sets $E_i \subset S$, $0 \leq i \leq m$, with $E_m = E_0$, which have the following properties:*

- (a) $A(\theta) = \theta$ and $\theta(s_0) > 0$ for some $s_0 \in E := \cup_{i=0}^{m-1} E_i$.
- (b) If f and g are any two functions in $C(S)$ such that $f \in [a\theta, b\theta]$, $g \in [a\theta, b\theta]$ and $f|E_i = g|E_i$ for some i , $0 \leq i < m$, then $Af|E_{i+1} = Ag|E_{i+1}$.
- (c) $\cap_{i=0}^{m-1} E_i = \emptyset$.
- (d) For any closed, nonempty, convex set $G \subset \prod_{i=0}^{m-1} [a\theta, b\theta]$ such that $\Phi(G) \subset G$, Φ has a fixed point in G .

Then A has a periodic point $g_0 \in [a\theta, b\theta]$ of minimal period m .

Proof. Define $N = m(m + 1)!$ and select $r > 1$ such that $a < r^{-N} < r^N < b$. Let $G \subset Q$ be defined by

$$G = \{(f_0, f_1, \dots, f_{m-1}) \in Q \mid a\theta \leq f_j \leq b\theta \text{ and } f_j|E_j = rf_{j-1}|E_j \text{ for } 1 \leq j \leq m\}.$$

We know by Lemma 3.2 that there exist functions $g_j \in P_o(S)$, $0 \leq j \leq m$, $g_m = g_0$, with (1) $r^{-N} \leq g_j(s) \leq r^N$ for all $s \in S$ and for $0 \leq j \leq m$ and (2) $g_j|E_j = rg_{j-1}|E_j$ for $1 \leq j \leq m$. If we define $f_j = \theta g_j$, then $(f_0, f_1, \dots, f_{m-1}) \in G$, so $G \neq \emptyset$. It is easy to see that G is closed and convex. Essentially the same argument as in Theorem 3.1 shows that $\Phi(G) \subset G$: the fact that A is order-preserving and homogeneous of degree one and that assumption (2) in Theorem 3.1A is satisfied suffice to replace positivity and linearity of A in Theorem 3.1. By assumption (4) in the Theorem, Φ has a fixed point $f = (f_0, f_1, \dots, f_{m-1}) \in G$. By the definition of Φ we see that $A(f_i) = f_{i+1}$ for $0 \leq i \leq m - 1$ and that $A^m(f_0) = f_0$. Since m

is a prime, it follows that either $A(f_0) = f_0$ or $A^m(f_0) = f_0$ but $A^j(f_0) \neq f_0$ for $0 < j < m$. If $A(f_0) = f_0$, then $f_j = f_0$ for all j ; and since $f_j(s) - rf_{j-1}(s) = (1 - r)f_0(s) = 0$ for all $s \in E_j$, $f_0(s) = 0$ for all $s \in E_j$ and for all j . It follows that $f_0(s) = 0$ for all $s \in E$. Since $a\theta \leq f_0$, we conclude that $\theta(s) = 0$ for all $s \in E$, which contradicts assumption (1) of the theorem. \square

Theorem 3.1A can be applied to maps like $Ax(s) = \max\{c(s, t)x(t) \mid t \in S\}$. We hope to pursue these and related nonlinear questions in a future paper.

Theorem 3.1 provides sufficient conditions for the existence of a nonnegative periodic point of A of minimal period m . If one is only interested in the existence of a strictly positive periodic point of A of minimal period m , Theorems 2.1 and 3.1 can be combined to yield the following cleaner result.

Theorem 3.2 *Let S be a compact, Hausdorff space, $A : C(S) \rightarrow C(S)$ a positive linear operator and m a prime number. Assume that A satisfies at least one of the following compactness conditions:*

- (a) *For every $g \in C(S)$ there exists a sequence $k_i \rightarrow \infty$ such that $\frac{1}{k_i} \sum_{s=0}^{k_i-1} (A^{ms})(g)$ converges in the weak topology on $C(S)$ as $i \rightarrow \infty$.*
- (b) *For every $g \in C(S)$, the norm closure of $\{A^{mj}(g) : j \geq 1\}$ is compact in the norm topology.*
- (c) *$\rho(A) < 1$, where $\rho(A)$ denotes the essential spectral radius of A .*

Then A has a periodic point $f_0 \in P_o(S)$ of minimal period m if and only if the following two conditions are satisfied:

- (1) *There exists $\theta \in P_o(S)$ with $A(\theta) = \theta$.*
- (2) *There exist closed, proper, nonempty subsets $E_j \subset S$, $0 \leq j \leq m$, with $E_m = E_0$, such that $(\alpha) \bigcap_{j=0}^{m-1} E_j = \emptyset$ and (β) whenever $f \in C(S)$ and $f|_{E_j} = 0$ for some j with $0 \leq j \leq m - 1$, it follows that $Af|_{E_{j+1}} = 0$.*

Proof. The necessity of these conditions follows from Theorem 2.1 and the sufficiency from Theorem 3.1. \square

Remark 3.3. Our motivation for introducing conditions (a) and (b) in Theorems 3.1 and 3.2 instead of restricting attention to the much simpler assumption (condition (c)) that $\rho(A) < 1 = r(A)$ comes from “Perron-Frobenius operators”, which will be treated in Sections 5 and 6. For Perron-Frobenius operators it is often the case that $\rho(A) = r(A)$. In the following work we shall prove some theorems which are applicable to Perron-Frobenius operators and which allow the verification of Condition (2) of Theorem 3.1 even when $\rho(A) = r(A)$.

The existence of θ as in Theorem 3.1 or 3.2 is closely related to generalizations of the Krein-Rutman theorem and to the concept of irreducibility. If Y is a real Banach space, a closed, convex set $K \subset Y$ is called a closed cone (with vertex at 0) if $K \cap (-K) = \{0\}$ and $\lambda K \subset K$ for all $\lambda \geq 0$. The cone is called “total” if the closed linear span of K equals Y . If $L : Y \rightarrow Y$ is a bounded linear operator, K is a total cone, $L(K) \subset K$

and $\rho(L) < r(L) := r$, then it is proved in [15] that there exist $y \in K, y \neq 0$ and $y^* \in K^* := \{f \in Y^* \mid f(z) \geq 0 \text{ for all } z \in K\}, y^* \neq 0$, with $L(y) = ry$ and $L^*(y^*) = y^*$. The classical Krein-Rutman theorem treats the case that L is compact (so $\rho(L) = 0$) and $r(L) > 0$. If $K_o \neq \emptyset$, the operator L is called “irreducible” if for every $\lambda > r$ and $x \in K - \{0\}$, $(\lambda - L)^{-1}(x) = \lambda^{-1} \sum_{j=0}^{\infty} (\lambda^{-1} L)^j(x) \in K_o$. For cones with empty interior, a more general definition of irreducibility is given in the appendix of [18]. It is easy to show that if L is irreducible, $r = r(L)$, and $L(y) = ry$ for some $y \in K - \{0\}$, then $y \in K_o$.

In the context of Theorem 3.1, if $r(A) = 1$ and A is irreducible, we deduce that $\theta \in K_o$. However, the assumption that A is irreducible is frequently too restrictive. To see this, suppose that S is a compact, Hausdorff space and that $A : C(S) \rightarrow C(S)$ is a positive, bounded linear operator. Suppose (compare condition (1) of Theorem 3.1) that there exist closed, nonempty sets $E_j \subset S, 0 \leq j \leq m$, with $E_m = E_0$, such that if $f \in C(S)$ and $f|_{E_j} = 0$ then $Af|_{E_{j+1}} = 0$. Define $E = \cup_{j=0}^{m-1} E_j$ and note that if $f \in C(S)$ and $f|_E = 0$, then $Af|_E = 0$. It follows that if $f|_E = 0$, then $A^k(f)|_E = 0$ for $k \geq 0$, and so, for $\lambda > r(A)$, we must have that $(\lambda I - A)^{-1}(f)|_E = 0$. If $E \neq S$, there exists $f \in P(S) - \{0\}$ such that $f|_E = 0$. For this f we have that $(\lambda I - A)^{-1}(f)|_E = 0$ for $\lambda > r(A)$, so A is not irreducible. In particular, if condition (1) of Theorem 3.1 holds and $\cup_{j=0}^{m-1} E_j \neq S$, then A is not irreducible. Furthermore, if m is a prime and the hypotheses of Theorem 2.1 are satisfied, then we have seen in Corollary 2.1 that the sets E_i can be chosen pairwise disjoint. It follows that if A is irreducible and the hypotheses of Theorem 2.1 are satisfied with m a prime, then $S = E$ is the union of m pairwise disjoint, closed nonempty sets, so S must have at least m connected components.

In view of the difficulties described above, it seems useful to formalize the the role played by the set $E := \cup_{j=0}^{m-1} E_j$ in Theorem 3.1. First, however, it is convenient to prove the following lemma.

Lemma 3.5 *Let Y and Z be Banach spaces and assume that $\pi : Y \rightarrow Z$ is a bounded linear map of Y onto Z . Suppose that $\hat{A} : Y \rightarrow Y$ and $\hat{B} : Z \rightarrow Z$ are bounded linear maps such that $\hat{B}\pi = \pi\hat{A}$. Then it follows that $r(\hat{B}) \leq r(\hat{A})$ and $\rho(\hat{B}) \leq \rho(\hat{A})$.*

Proof. Let $\|y\|_1$ denote the norm on Y and $\|z\|_2$ denote the norm on Z . Similarly, if $S \subset Y$ (respectively, $S \subset Z$) let $diam_1(S)$ (respectively, $diam_2(S)$) denote the diameter of S with respect to the norm on Y (respectively, with respect to the norm on Z). We denote by α_1 (respectively, α_2) the measure of noncompactness on Y (respectively, on Z). Finally, for $r > 0$, we define $B_r(0) = \{y \in Y : \|y\|_1 < r\}$ and $V_r(0) = \{z \in Z : \|z\|_2 < r\}$.

We claim that $\pi\hat{A}^j = \hat{B}^j\pi$ for all $j \geq 1$. We know this is true for $j = 1$. If the equation holds for some particular $j \geq 1$, we obtain

$$(\hat{B}^j\pi)\hat{A} = (\hat{B}^j)(\pi\hat{A}) = (\hat{B}^j)(\hat{B}\pi) = \hat{B}^{j+1}\pi = (\pi\hat{A}^j)(\hat{A}) = \pi\hat{A}^{j+1},$$

and we conclude by induction that $\pi\hat{A}^k = \hat{B}^k\pi$ for all $k \geq 1$.

Because π is onto, the open mapping theorem implies that there exists $\delta > 0$ such that $\pi(B_1(0)) \supset V_\delta(0)$, so $\pi(B_{\delta^{-1}}(0)) \supset V_1(0)$. It follows that

$$\hat{B}^j(V_1(0)) \subset \hat{B}^j\pi(B_{\delta^{-1}}(0)) = \pi\hat{A}^j(B_{\delta^{-1}}(0)).$$

We deduce from these inclusions that

$$\|\hat{B}^j\|_2 = \sup\{\|\hat{B}^j(z)\|_2 : z \in V_1(0)\} \leq \sup\{\|(\pi\hat{A}^j)(y)\|_2 : y \in B_{\delta^{-1}}(0)\} \leq \|\pi\| \|\hat{A}^j\|_1 \delta^{-1}.$$

It follows that

$$r(\hat{B}) = \lim_{j \rightarrow \infty} \|\hat{B}^j\|_2^{\frac{1}{j}} \leq \lim_{j \rightarrow \infty} (\delta^{-1} \|\pi\| \|\hat{A}^j\|_1)^{\frac{1}{j}} = r(\hat{A}).$$

It remains to prove that $\rho(\hat{B}) \leq \rho(\hat{A})$. If we can prove that there exists a constant C such that $\alpha_2(\hat{B}^j) \leq C\alpha_1(\hat{A}^j)$ for all $j \geq 1$, then the desired result follows by taking j^{th} roots and taking limits as $j \rightarrow \infty$. It remains to prove the existence of C . Suppose $T \subset Z$ and $\alpha_2(T) = d$. Given $\epsilon > 0$, there exist sets T_1, T_2, \dots, T_n such that $T = \cup_{j=1}^n T_j$ and $\text{diam}_2(T_j) < d + \epsilon$ for $1 \leq j \leq n$. For each j , select $z_j \in T_j$ and $y_j \in Y$ with $\pi(y_j) = z_j$. For each $z \in T_j$ we have $\|z - z_j\|_2 < d + \epsilon$, so there exists $y = y_{z,j} \in Y$ with $\|y - y_j\|_1 < \delta^{-1}(d + \epsilon)$ and $\pi(y) = z$. Define $S_j := \{y_{z,j} : z \in T_j\}$. By our construction we have that $\pi(S_j) = T_j$ and $\text{diam}_1(S_j) \leq 2\delta^{-1}(d + \epsilon)$. It follows that

$$\alpha_2(\hat{B}^k(T_j)) = \alpha_2(\hat{B}^k\pi(S_j)) = \alpha_2(\pi\hat{A}^k(S_j)) \leq \|\pi\| \alpha_1(\hat{A}^k)\alpha_1(S_j) \leq \|\pi\| \alpha_1(\hat{A}^k)2\delta^{-1}(d + \epsilon).$$

We conclude that

$$\alpha_2(\hat{B}^k(T)) = \max\{\alpha_2(\hat{B}^k(T_j)) : 1 \leq j \leq n\} \leq \|\pi\| \alpha_1(\hat{A}^k)(2\delta^{-1})(d + \epsilon).$$

Since $\epsilon > 0$ was arbitrary, we have, taking $C := (2\delta^{-1})\|\pi\|$,

$$\alpha_2(\hat{B}^k(T)) \leq C\alpha_1(\hat{A}^k)\alpha_2(T),$$

which implies that $\alpha_2(\hat{B}^k) \leq C\alpha_1(\hat{A}^k)$ for $k \geq 1$. \square

As an immediate consequence of Lemma 3.5, we obtain

Lemma 3.6 *Let S be a compact, Hausdorff space and $A : C(S) \rightarrow C(S)$ a positive linear operator. Let E be a closed, nonempty subset of S such that whenever $g \in C(S)$ and $g|_E = 0$ it follows that $Ag|_E = 0$. If $f \in C(E)$, select $g \in C(S)$ such that $g|_E = f$ and define $B : C(E) \rightarrow C(E)$ by*

$$B(f) := A(g)|_E. \tag{10}$$

Then B is well-defined and B is a bounded linear operator such that $\|B^j\| \leq \|A^j\|$ for all $j \geq 1$, $r(B) \leq r(A)$ and $\rho(B) \leq \rho(A)$.

Proof. Define $\pi : C(S) \rightarrow C(E)$ by $\pi(g) = g|_E$, so π is a bounded linear operator of norm one. The Tietze extension theorem implies that π is onto; and in fact, given $f \in C(E)$, there exists $g \in C(S)$ with $\pi(g) = f$ and such that for all $s \in S$

$$\inf_E f \leq g(s) \leq \sup_E f. \tag{11}$$

To show that B is well-defined, note that if $\pi(g_1) = \pi(g_2)$, then $(g_2 - g_1)|_E = 0$, so $A(g_2 - g_1)|_E = 0$, and $Ag_1|_E = Ag_2|_E$. The definition of B also shows that $B\pi = \pi A$, so Lemma 3.5 implies that $r(A) \geq r(B)$ and $\rho(A) \geq \rho(B)$. The proof of Lemma 3.5 also shows that

$B^j \pi = \pi A^j$ for all $j \geq 1$. Given $f \in C(E)$, choose $g \in C(S)$ with $\pi(g) = f$ and $\|f\| = \|g\|$. It follows that

$$\|B^j(f)\| = \|B^j(\pi g)\| = \|\pi A^j(g)\| \leq \|\pi\| \|A^j\| \|g\| \leq \|A^j\| \|f\|,$$

which implies that $\|B^j\| \leq \|A^j\|$. \square

Our next theorem is motivated by applications to Perron-Frobenius operators, for which the framework described below will be satisfied by taking Y to be a Banach space of Hölder continuous functions on a compact metric space (S, d) or a Banach space of analytic functions.

Theorem 3.3 *Let S be a compact, Hausdorff space and let X denote the real Banach space $C(S)$. Assume that $(Y, \|\cdot\|_Y)$ is a real Banach space and that there exists a continuous, one-one linear map $i : Y \rightarrow X$ such that $i(Y)$ is a dense linear subspace of X . Let $L : X \rightarrow X$ be a positive linear map and suppose that $\Lambda : Y \rightarrow Y$ is a bounded linear map such that $i\Lambda = Li$. Then we have $r(\Lambda) \geq r(L)$, and if $\rho(\Lambda) < r(L)$, $r(L) = r(\Lambda)$. If P denotes $P(S)$ and $\rho(\Lambda) < r(L) := r$, then there exists $y \in i^{-1}(P)$, $y \neq 0$, with $\Lambda(y) = ry$. If $\rho(\Lambda) < r(L) = 1$ and if $y \in Y$ is such that $\{\|i\Lambda^k(y)\| : k \geq 0\}$ is bounded, then the set $\{\|\Lambda^k(y)\|_Y : k \geq 0\}$ is bounded. If $\rho(\Lambda) < r(L) = 1$ and if there exists a constant C with $\|L^k\| \leq C$ for all $k \geq 1$, then the following results hold:*

- (a) *For every $y \in \tilde{Y}$ and every $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $\{\zeta^{-k} \tilde{\Lambda}^k(y) : k \geq 0\}$ has compact closure in the norm topology on $(\tilde{Y}, \|\cdot\|_Y)$. Here \tilde{Y} denotes the complexification of Y and $\tilde{\Lambda}$ the complexification of Λ .*
- (b) *For every $x \in \tilde{X}$ and every $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $\{\zeta^{-k} \tilde{L}^k(x) : k \geq 0\}$ has compact closure in the norm topology on \tilde{X} . Here \tilde{X} denotes the complexification of X and \tilde{L} the complexification of L .*
- (c) *If $\zeta \in \mathbb{C}$, $|\zeta| = 1$, and $M = \{y \in \tilde{Y} \mid \tilde{\Lambda}(y) = \zeta y\}$ and $N = \{x \in \tilde{X} \mid \tilde{L}(x) = \zeta x\}$, then M is a finite dimensional vector space and $i(M) = N$. Furthermore, $N = \{x \in \tilde{X} \mid (\zeta I - \tilde{L})^k(x) = 0 \text{ for some } k \geq 1\}$.*
- (d) *For every $y \in \tilde{Y}$ and every $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $y_n := n^{-1} \sum_{j=0}^{n-1} \zeta^{-j} \tilde{\Lambda}^j(y)$ converges in the $\|\cdot\|_Y$ topology to an element $Q_Y(y) \in \tilde{Y}$ with $\tilde{\Lambda}(Q_Y(y)) = Q_Y(y)$. For every $x \in \tilde{X}$ and every $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $x_n := n^{-1} \sum_{j=0}^{n-1} \zeta^{-j} \tilde{L}^j(x)$ converges in the norm topology on \tilde{X} to an element $Q(x)$ with $\tilde{L}(Q(x)) = Q(x)$. The maps $Q : \tilde{X} \rightarrow \tilde{X}$ and $Q_Y : \tilde{Y} \rightarrow \tilde{Y}$ are bounded linear projections.*

Proof. The assumption that $i : Y \rightarrow X$ is continuous implies that there is a constant C_2 with $\|iy\| \leq C_2\|y\|_Y$ for all $y \in Y$. The assumption that i is one-one and $iL = \Lambda i$ implies that if $z \in \mathbb{C}$ is an eigenvalue of $\tilde{\Lambda}$ with eigenvector $w \in \tilde{Y}$, then z is also an eigenvalue of \tilde{L} with eigenvector $i(w) \in \tilde{X}$. In particular, $\sigma_P(\tilde{\Lambda})$, the point spectrum of $\tilde{\Lambda}$, is contained in $\sigma_P(\tilde{L})$. If $B : Y \rightarrow Y$ is a bounded linear operator, we shall write $\|B\|_Y$ for its operator norm.

Let $e \in X$ denote the function identically equal to 1 and recall that $\|L^n\| = \|L^n(e)\|$ for all $n \geq 0$. Because $\{x \mid x \geq e\}$ contains an open set in X and $i(Y)$ is dense in X , there exists $y \in Y$ with $e \leq i(y)$. Because $L^n(P) \subset P$ for $n \geq 0$, we see that

$$0 \leq L^n(e) \leq L^n(i(y)) = i(\Lambda^n(y)).$$

Taking norms and recalling that i is continuous gives

$$\|L^n\| = \|L^n(e)\| \leq \|L^n(i(y))\| = \|i\Lambda^n(y)\| \leq \|i\| \|\Lambda^n\|_Y \|y\|_Y.$$

Taking n^{th} roots and letting n approach infinity we obtain

$$r(L) = \lim_{n \rightarrow \infty} \|L^n\|^{\frac{1}{n}} \leq \lim_{n \rightarrow \infty} \|\Lambda^n\|_Y^{\frac{1}{n}} = r(\Lambda).$$

Let $\sigma(\tilde{\Lambda})$ and $\sigma(\tilde{L})$ denote the spectrum of $\tilde{\Lambda}$ and \tilde{L} respectively. It is known that $\rho(\tilde{\Lambda}) = \rho(\Lambda)$ and $r(\tilde{L}) = r(L)$, so if $\rho(\Lambda) < r(L)$, it follows from results in [12, 13] that for every $z \in \sigma(\tilde{\Lambda})$ with $|z| > \rho(\Lambda)$, z is an eigenvalue of $\tilde{\Lambda}$ with finite algebraic multiplicity and with eigenvector $w_z \in \tilde{Y}$. It follows that if $z \in \sigma(\tilde{\Lambda})$ and $|z| > \rho(\Lambda)$, then $z \in \sigma(\tilde{L})$. We know that $\rho(\Lambda) < r(L) \leq r(\Lambda)$, so the above remarks imply that

$$r(\Lambda) = \sup\{|z| : z \in \sigma(\tilde{\Lambda}) \text{ and } |z| > \rho(\Lambda)\} \leq r(L).$$

Thus, if $\rho(\Lambda) < r(L)$, we have proved that $r(\Lambda) = r(L)$. Note that $i^{-1}(P_o) \subset i^{-1}(P)$, so the interior of $i^{-1}(P)$ in Y is nonempty, and one easily derives that $Y = (i^{-1}(P)) - (i^{-1}(P))$. Because i is one-one, the reader can verify that $i^{-1}(P)$ is a closed cone in Y . If one now applies the generalization of the Krein-Rutman theorem in Remark 3.3 (see [15]), one finds that there exists $y \in i^{-1}(P)$, $y \neq 0$, with $\Lambda(y) = ry$ and $r = r(L)$.

Now assume that $\rho(\Lambda) < r(L) = 1$, so $r(\Lambda) = 1$. For any $\epsilon > 0$, the set

$$\{z \in \sigma(\tilde{\Lambda}) : |z| \geq \rho(\Lambda) + \epsilon\}$$

is finite, so there exist reals β and γ with $\rho(\Lambda) < \beta < 1 < \gamma$ such that $\{z \in \mathbb{C} \mid |z| = \beta\}$ contains no element of $\sigma(\tilde{\Lambda})$. Let Γ_β and Γ_γ denote circles centered at the origin, oriented counterclockwise, and with radii β and γ respectively. Define

$$P = \frac{1}{2\pi i} \int_{\Gamma_\gamma} (z - \tilde{\Lambda})^{-1} dz - \frac{1}{2\pi i} \int_{\Gamma_\beta} (z - \tilde{\Lambda})^{-1} dz.$$

(In the previous equation i denotes $\sqrt{-1}$, of course.) Because $\rho(\Lambda) < r(\Lambda)$, it follows from standard facts about the functional calculus for bounded linear operators that $P : \tilde{Y} \rightarrow \tilde{Y}$ is a bounded linear operator with $P(Y) \subset Y$, $P^2 = P$, P is a projection with finite dimensional range, $\tilde{\Lambda}P = P\tilde{\Lambda}$ and

$$\sigma(\tilde{\Lambda}(I - P)) = \{z \in \sigma(\tilde{\Lambda}) : |z| \leq \beta\}.$$

Using this equation we see that $r(\tilde{\Lambda}(I - P)) < \beta$, so there is a constant C_2 such that for all positive integers k we have

$$\|(\tilde{\Lambda}(I - P))^k\|_{\tilde{Y}} = \|\tilde{\Lambda}^k(I - P)\|_{\tilde{Y}} \leq C_2 \beta^k.$$

Any one-one linear map from a finite dimensional, Hausdorff topological space onto another finite dimensional, Hausdorff topological space is known to be a homeomorphism. Applying this theorem to the one-one linear map $i|P(Y) : (P(Y), \|\cdot\|_Y) \rightarrow (iP(Y), \|\cdot\|)$, we see that there exist positive constants C_3 and C_4 such that

$$C_3\|iPy\| \leq \|Py\|_Y \leq C_4\|iPy\|$$

for all $y \in Y$. Since $\tilde{\Lambda}^k P = \tilde{\Lambda}^k P^2 = P\tilde{\Lambda}^k P$, $\tilde{\Lambda}^k$ maps $P(Y)$ into itself, and we obtain

$$\|\tilde{\Lambda}^k(Py)\|_Y \leq C_4\|\tilde{\Lambda}^k(Py)\|.$$

Now suppose that $y \in \tilde{Y}$ is such that $\{\|\tilde{\Lambda}^k(y)\| : k \geq 0\}$ is bounded. Because we have

$$i\tilde{\Lambda}^k(y) - i\tilde{\Lambda}^k(I - P)(y) = i\tilde{\Lambda}^k P(y),$$

and because we have proved that $\{\|\tilde{\Lambda}^k(I - P)(y)\|_Y : k \geq 1\}$ is bounded, we conclude that $\{\|i\tilde{\Lambda}^k(Py)\| : k \geq 0\}$ is bounded. It follows from our previous remarks that the set $\{\|\tilde{\Lambda}^k(Py)\|_Y : k \geq 0\}$ is bounded, so we conclude that $\{\|\tilde{\Lambda}^k(y)\|_Y : k \geq 0\}$ is bounded.

For the remainder of the proof we assume that $\rho(\Lambda) < r(L) = 1$ and that there exists a constant C with $\|L^k\| \leq C$ for all $k \geq 0$. The assumption that $\|L^k\| \leq C$ implies that $\{\|\tilde{L}^k(iy)\| : k \geq 0\}$ is bounded for every $y \in \tilde{Y}$, so our previous results imply that $\{\|\tilde{\Lambda}^k(y)\|_{\tilde{Y}} : k \geq 0\}$ is bounded for every $y \in \tilde{Y}$. We now claim that for every $y \in \tilde{Y}$ and every $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ the set $\{\zeta^{-k}\tilde{\Lambda}^k(y) : k \geq 0\}$ has compact closure in the norm topology on $(\tilde{Y}, \|\cdot\|_Y)$. If S_1 and S_2 are sets in X , recall that $S_1 + S_2 := \{x_1 + x_2 : x_i \in S_i\}$ and note that

$$\{\zeta^{-k}\tilde{\Lambda}^k(y) : k \geq 0\} \subset S_1 + S_2, \quad (12)$$

where we define $S_1 := \{\zeta^{-k}\tilde{\Lambda}^k(I - P)y : k \geq 0\}$ and $S_2 := \{\zeta^{-k}\tilde{\Lambda}^k(Py) : k \geq 0\}$. Our previous results imply that $\|\zeta^{-k}\tilde{\Lambda}^k(I - P)y\|_Y \rightarrow 0$, so, denoting the measure of noncompactness in $(\tilde{Y}, \|\cdot\|_Y)$ by $\tilde{\alpha}_Y$, we have that $\tilde{\alpha}_Y(S_1) = 0$. We also know that S_2 is bounded in $(\tilde{Y}, \|\cdot\|_Y)$ and that S_2 is a subset of $P(\tilde{Y})$, which is a finite dimensional vector subspace of \tilde{Y} , so we conclude that $\tilde{\alpha}_Y(S_2) = 0$. Equation (12) now implies that

$$\tilde{\alpha}_Y(\{\zeta^{-k}\tilde{\Lambda}^k(y) : k \geq 0\}) \leq \tilde{\alpha}_Y(S_1) + \tilde{\alpha}_Y(S_2) = 0,$$

so $\{\zeta^{-k}\tilde{\Lambda}^k(y) : k \geq 0\}$ has compact closure in $(\tilde{Y}, \|\cdot\|_Y)$.

We next select $x \in \tilde{X}$ and $\zeta \in \mathbb{C}$, $|\zeta| = 1$. We must prove that $\{\zeta^{-k}\tilde{L}^k(x) : k \geq 0\}$ has compact closure in \tilde{X} for every $x \in \tilde{X}$. Given $\delta > 0$ and $T \subset \tilde{X}$, let

$$N_\delta(T) = \{z \in \tilde{X} : \inf_{w \in T} \|z - w\| < \delta\}.$$

Select $\epsilon > 0$. Since Y is dense in X , there exists $y \in \tilde{Y}$ with $\|i(y) - x\| < \epsilon$. We have proved that T_1 , the norm closure of $\{\zeta^{-k}\tilde{\Lambda}^k(y) : k \geq 0\}$ in $(\tilde{Y}, \|\cdot\|_Y)$ is compact, and since $i : Y \rightarrow X$ is continuous, $T := i(T_1)$ is compact in X . If $\tilde{\alpha}$ denotes the measure of noncompactness on \tilde{X} , it follows that

$$0 = \tilde{\alpha}(T) \geq \tilde{\alpha}(\{\zeta^{-k}\tilde{L}^k(iy) : k \geq 0\}) \geq 0.$$

Because $\|\zeta^{-k}\tilde{L}^k(x) - \zeta^{-k}\tilde{L}^k(iy)\| \leq C\|y - x\| \leq C\epsilon$ for all $k \geq 0$, we have

$$\{\zeta^{-k}\tilde{L}^k(x) : k \geq 0\} \subset N_{C\epsilon}(T),$$

so properties of the measure of noncompactness imply that

$$\tilde{\alpha}(\{\zeta^{-k}\tilde{L}^k(x) : k \geq 0\}) \leq 2C\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\{\zeta^{-k}\tilde{L}^k(x) : k \geq 0\}$ has compact closure in \tilde{X} .

It remains to prove statements (c) and (d) of the theorem. We know (for $|\zeta| = 1$) that $\rho(\Lambda) = \rho(\tilde{\Lambda}) = \rho(\zeta^{-1}\tilde{\Lambda})$ and $r(L) = r(\tilde{L}) = r(\zeta^{-1}\tilde{L})$, so our assumptions and previous results imply that $\rho(\zeta^{-1}\tilde{\Lambda}) < r(\zeta^{-1}\tilde{\Lambda})$. The fact that M is finite dimensional now follows immediately from properties of the essential spectrum: see [12, 13]. We have proved that there exists a constant C_5 with $\|\Lambda^k\|_Y \leq C_5$ for all $k \geq 0$. We have also proved that for every $y \in \tilde{Y}$ and every $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $\{\zeta^{-k}\tilde{\Lambda}^k(y) : k \geq 0\}$ has compact closure in \tilde{Y} , so Mazur's theorem implies that the closure of the convex hull of $\{\zeta^{-k}\tilde{\Lambda}^k(y) : k \geq 0\}$ is compact in \tilde{Y} . It follows that for every $y \in \tilde{Y}$, the sequence $y_k := k^{-1} \sum_{j=0}^{k-1} \zeta^{-k}\tilde{\Lambda}^k(y)$ has a subsequence which converges in the norm topology on \tilde{Y} . We have thus proved that the hypotheses of the mean ergodic theorem are satisfied, so for every $y \in \tilde{Y}$, (y_k) converges in the norm topology on \tilde{Y} to a fixed point $u := Q_Y(y)$ of $\tilde{\Lambda}$, and Q_Y satisfies the properties in statement (d). For every $x \in \tilde{X}$ and ζ as above, $\{\zeta^{-k}\tilde{L}^k(x) : k \geq 0\}$ has compact closure in \tilde{X} and $\|L^k\| \leq C$ for all $k \geq 0$, so the same argument shows that for every $x \in \tilde{X}$, the sequence $x_k := k^{-1} \sum_{j=0}^{k-1} \zeta^{-j}\tilde{L}^j(x)$ converges in the norm topology on \tilde{X} to a fixed point $v = Q(x)$ of \tilde{L} , and Q is a bounded linear projection of \tilde{X} into itself.

Let $m = \dim(M)$. As we have already noted, $i(M) \subset N$, and since i is one-one, this implies that $m \leq \dim(N)$. If $i(M) \neq N$, there must exist a set of $m+1$ linearly independent vectors $x_j \in N$, $1 \leq j \leq m+1$. A simple argument, which we leave to the reader, shows that there exists $\delta > 0$ such that if $\{\xi_j : 1 \leq j \leq m+1\}$ is any set of $m+1$ vectors in \tilde{X} with $\|x_j - \xi_j\| < \delta$ for $1 \leq j \leq m+1$, then $\{\xi_j : 1 \leq j \leq m+1\}$ is a linearly independent set of vectors. Take $\epsilon > 0$ with $C\epsilon < \delta$ and, using the density of $i(Y)$, select $y_j \in \tilde{Y}$, $1 \leq j \leq m+1$, with $\|i(y_j) - x_j\| < \epsilon$ for $1 \leq j \leq m+1$. By our previous remarks, there exist $u_j \in M$, $1 \leq j \leq m+1$, such that

$$\lim_{k \rightarrow \infty} \|k^{-1} \sum_{s=0}^{k-1} \zeta^{-s}\tilde{\Lambda}^s(y_j) - u_j\|_Y = 0.$$

However, if we define $S_k : \tilde{Y} \rightarrow \tilde{Y}$ by $S_k(y) = k^{-1} \sum_{s=0}^{k-1} \zeta^{-s}\tilde{\Lambda}^s(y)$ and $T_k : \tilde{X} \rightarrow \tilde{X}$ by $T_k(x) = k^{-1} \sum_{s=0}^{k-1} \zeta^{-s}\tilde{L}^s(x)$ and if we recall $\zeta^{-1}\tilde{L}(x_j) = x_j$, we see that

$$\|i(S_k(y_j)) - x_j\| = \|T_k(i(y_j) - x_j)\| \leq k^{-1} \sum_{s=0}^{k-1} \|L^s\| \|i(y_j) - x_j\| < C\epsilon.$$

Taking limits as $k \rightarrow \infty$, we find that $\|i(u_j) - x_j\| \leq C\epsilon < \delta$ for $1 \leq j \leq m+1$, so our selection of δ implies that $\{i(u_j) : 1 \leq j \leq m+1\}$ is a linearly independent set of $m+1$ vectors in N , which (recalling that i is one-one) contradicts $\dim(M) = m$. Thus we must have $\dim(M) = \dim(N)$ and $i(M) = N$.

We still must prove that $N = \{x \in \tilde{X} : \text{there exists } k \geq 1 \text{ with } (\zeta I - \tilde{L})^k(x) = 0\}$. Suppose not, so $(I - \zeta^{-1}\tilde{L})^k(x) = 0$ for some $k \geq 2$ and $(I - \zeta^{-1}\tilde{L})^{k-1}(x) \neq 0$. By replacing x by $y := (I - \zeta^{-1}\tilde{L})^{k-2}(x)$, we can assume that $k = 2$. We write $y_1 := (I - \zeta^{-1}\tilde{L})(y) \neq 0$ and $B = \zeta^{-1}\tilde{L}$. Assume, by way of induction, that $B^j(y) = y - jy_1$, which is true for $j = 1$. Since $B(y) = y - y_1$ and $B y_1 = y_1$, we derive from $B^j(y) = y - jy_1$ that

$$B^{j+1}(y) = B(y) - jy_1 = y - (j + 1)y_1$$

and we conclude by induction that $B^k(y) = y - ky_1$ for all $k \geq 1$. However, this contradicts the assumption that $\|B^j\| \leq C$ for all $j \geq 1$. \square

Some general comments about Theorem 3.3 may be in order. Recall that if Z is a complex Banach space and $B : Z \rightarrow Z$ is a bounded linear map with eigenvalue α , then the geometric multiplicity of α is the dimension of $\{z \in Z : (\alpha I - B)(z) = 0\}$ and the algebraic multiplicity of α is the dimension of $\{z \in Z : (\alpha I - B)^k(z) = 0 \text{ for some } k \geq 1\}$. Part (c) of Theorem 3.3 asserts that L has finitely many eigenvalues α with $|\alpha| = 1$ and that each such eigenvalue has finite algebraic multiplicity equal to its geometric multiplicity. Nevertheless, it may happen that $\sigma(\tilde{L}) = \{z \in \mathbb{C} : |z| \leq 1\}$: see the example at the beginning of Section 5.

The assumption that L is positive in Theorem 3.3 plays a limited role. Assume that X, Y , and i are as in Theorem 3.3. Assume that $\Lambda : Y \rightarrow Y$ and $L : X \rightarrow X$ are bounded linear operators with $i\Lambda = Li$. Suppose that $\rho(\Lambda) < r(L)$ and that the set $\{\|L^k\| : k \geq 0\}$ is bounded. Then the same argument given in the proof of Theorem 3.3 shows that $r(L) \geq r(\Lambda)$. If $r(L) = 1 = r(\Lambda)$, the same argument as in Theorem 3.3 shows that statements (a)-(d) of Theorem 3.3 are satisfied.

Remark 3.4. The assumption that $\{\|L^k\| : k \geq 0\}$ is bounded plays an important role in Theorem 3.3. If $L : C(S) \rightarrow C(S)$ is a positive, bounded linear operator, and if $L(\theta) = \theta$ for some $\theta \in P_0$, then it is known that $r(L) = 1$ and that $\{\|L^k\| : k \geq 0\}$ is bounded. To see this, let $\mu = \min\{\theta(s) : s \in S\} > 0$ and note that for $f \in X := C(S)$ with $\|f\| \leq 1$ we have $-\mu^{-1}\theta \leq f \leq \mu^{-1}\theta$. It follows that

$$-\mu^{-1}A^k(\theta) = -\mu^{-1}\theta \leq A^k(f) \leq \mu^{-1}\theta = \mu^{-1}A^k(\theta)$$

for all $k \geq 0$, which implies that $\|A^k\| \leq \mu^{-1}\|\theta\|$ for all $k \geq 0$. However, one can easily see, even for 2×2 matrices, that the conditions $r(A) = 1$ and $\|A^k\| \leq C < \infty$ for all $k \geq 0$ may be satisfied even though A has no fixed point in P_0 .

Remark 3.5. Let assumptions and notation be as in Theorem 3.3. (In particular we are assuming that $\rho(\Lambda) < r(L) = 1$ and that $\{\|L^k\| : k \geq 0\}$ is bounded.) Assume also that 1 is the only eigenvalue of Λ of absolute value one. We claim that, for every $x \in X$, $L^k(x)$ converges to a fixed point of L as $k \rightarrow \infty$.

In order to prove this assertion, first note that the above hypotheses imply that there exist $\beta < 1$ and $\gamma > 1$ such that 1 is the only element z of $\sigma(\Lambda)$ with $\beta \leq |z| \leq \gamma$. If P is defined as in the proof of Theorem 3.3, it follows from properties of the functional calculus for linear operators and from statement (c) of Theorem 3.3 that P is a projection with finite dimensional range $M := \{y \in Y \mid \Lambda(y) = y\}$. If $y \in Y$, the proof of Theorem 3.3

shows that $\|\Lambda^k(I - P)y\|_Y \rightarrow 0$, so

$$\lim_{k \rightarrow \infty} \|\Lambda^k(y) - P(y)\|_Y = \lim_{k \rightarrow \infty} \|\Lambda^k(Py) - Py\|_Y = \|Py - Py\|_Y = 0.$$

If $x \in X$, we claim that $(L^k x \mid k \geq 1)$ converges in the norm topology to a fixed point $Q(x)$ of L , where Q is the projection in statement (d) of Theorem 3.3. By using statements (c) and (d) of Theorem 3.3, it suffices to prove that $(L^k(x) \mid k \geq 1)$ is a Cauchy sequence in X . For $x \in X$ and $\epsilon > 0$, there exists $y \in Y$ with $\|x - i(y)\| < \epsilon$. It follows that

$$\|L^k(x) - L^k(i(y))\| = \|L^k(x) - i(\Lambda^k(y))\| \leq C\epsilon.$$

Also, we know that

$$\|i(\Lambda^k(y)) - iP(y)\| \leq \|i\| \|\Lambda^k(y) - P(y)\|_Y \rightarrow 0.$$

Using these inequalities, we conclude that for all k sufficiently large, $\|L^k(x) - iP(y)\| < (C + 1)\epsilon$, which proves that $(L^k(x) : k \geq 1)$ is a Cauchy sequence and completes the proof.

If, in the notation of Theorem 3.3, $\rho(\Lambda) < r(L) = 1$, Theorem 3.3 asserts that there exists $\theta \in P \cap Y$, $\theta \neq 0$, with $L(\theta) = \theta$. However, if $A = L$ and if E_j , $0 \leq j \leq m$, are as in Theorem 3.1, we do not have a completely satisfactory answer to the question of whether $\theta(s_0) > 0$ for some $s_0 \in E := \cup_{j=0}^{m-1} E_j$. Our next theorem will address this question, but first we need to recall some general results from the theory of positive linear operators.

If K is a closed cone in a real Banach space Y , K is called “normal” if there exists a constant M such that whenever $x \in K$, $y \in K$, and $y - x \in K$ it follows that $\|x\| \leq M\|y\|$. The cone K is “generating” or “reproducing” if $Y = \{u - v : u, v \in K\}$, and K is “total” if Y is the closed linear span of K . If K is normal and generating in Y and $L : Y \rightarrow Y$ is a bounded linear map with $L(K) \subset K$, then $r(L) \in \sigma(L)$. See [3, 17] or the appendix of [19] for a proof. Surprisingly (see [20] and [16]), this theorem is false in general if one only assumes that K is total or generating, even if Y is a Hilbert space and L is a normal operator.

Theorem 3.4 *Let S be a compact, Hausdorff space and let $A : X := C(S) \rightarrow X$ be a positive linear operator. Assume (*): For every $f \in X$, $\{A^n(f) : n \geq 0\}$ has compact closure in the norm topology on X . (Note that Theorem 3.3 gives conditions which insure that (*) holds.) Assume that $E \subset S$ is a compact, nonempty set such that whenever $f|_E = 0$ it follows that $Af|_E = 0$. Let $B : C(E) \rightarrow C(E)$ be defined as in Lemma 3.6 and assume that $r(B) = 1$. Then there exists $\theta \in P(S)$ such that $A(\theta) = \theta$ and $\theta(s_0) > 0$ for some $s_0 \in E$.*

Proof. Define $\pi : C(S) \rightarrow C(E)$ by $\pi(h) = h|_E$. Because $C_f := \overline{\{A^n(f) : n \geq 0\}}$ is compact for each $f \in X$, C_f is bounded, and the uniform boundedness principle implies that there exists M such that $\|A^n\| \leq M$ for all $n \geq 1$. Lemma 3.6 implies that $\|B^n\| \leq \|A^n\| \leq M$ for all $n \geq 1$, so we have $r(B) \leq r(A) \leq 1$. On the other hand, we assume that $r(B) = 1$, so $r(A) = r(B) = 1$. Let $e \in X$ denote the function identically equal to one and let $e_1 = \pi(e)$. Notice that $P(E)$ is a normal generating cone in $C(E)$ and $B(P(E)) \subset P(E)$, so our previous remarks imply that $r(B) = 1 \in \sigma(B)$. If we define $B_n = \frac{1}{n} \sum_{j=0}^{n-1} B^j$, the spectral mapping

theorem implies that $1 \in \sigma(B_n)$. Since $B_n(P(E)) \subset P(E)$ and $\|L\| = \|L(e_1)\|$ for any bounded, positive linear operator $L : C(E) \rightarrow C(E)$, we conclude that for all $n \geq 0$

$$1 \leq r(B_n) \leq \|B_n\| \leq \|B_n(e_1)\|.$$

We now claim that B satisfies the hypotheses of the mean ergodic theorem. If $g \in C(E)$ and $\pi(f) = g$, we know from Lemma 3.6 that $\{B^n(g) : n \geq 0\} \subset \pi(C_f)$. Since C_f is compact and π is continuous, $\{B^n(g) : n \geq 0\}$ has compact closure in the norm topology; and Mazur's theorem implies $\overline{\text{co}}\{B^n g : n \geq 0\}$ is compact. It follows that for any $g \in C(E)$, the sequence $(B_n(g) : n \geq 0)$ has a convergent subsequence. We also know that $\{\|B^n\| : n \geq 0\}$ is bounded, so the mean ergodic theorem implies that there exists $v \in C(E)$ such that $\|B_n(e_1) - v\| \rightarrow 0$ as $n \rightarrow \infty$ and $B(v) = v$. Because $\|B_n(e_1)\| \geq 1$ for all n and $B_n(P(E)) \subset P(E)$, we see that $\|v\| \geq 1$ and $v \in P(E)$.

Select $u \in P(S)$ with $\pi(u) = v$ (where $B(v) = v$) and define $A_n = \frac{1}{n} \sum_{j=0}^{n-1} A^j$. The same argument given above shows that the hypotheses of the mean ergodic theorem are also satisfied by A , so there exists $w \in P(S)$ such that $\|A_n(u) - w\| \rightarrow 0$ as $n \rightarrow \infty$ and $A(w) = w$. Since $\pi(A_n(u)) = B_n(v) = v$, $\pi(w) = v$ and $w := \theta$ satisfies the claims of the theorem. \square

Remark 3.6. Under the hypotheses of Theorem 3.4, one can see that there is fixed point $w \in P(S)$ of A with $\lim_{n \rightarrow \infty} \|A_n(e) - w\| = 0$. It follows that if $A(\psi) = \psi$, $\psi \in P(S)$ and $\|\psi\| = 1$, then $\psi \leq w$. For we have $\psi = A_n(\psi) \leq A_n(e)$, and $A_n(e)$ converges to w . Thus w is a "dominant" fixed point of A in $P(S)$.

Remark 3.7. Assume all the hypotheses of Theorem 3.4 except condition (*). It can then easily happen that B has a nonzero fixed point in $P(E)$, but A does not have a fixed point θ with $\theta(s_0) > 0$ for some $s_0 \in E$. To see this, let $S = \{1, 2\}$ with the topology inherited from \mathbb{R} . Identify $C(S)$ with \mathbb{R}^2 in the obvious way and define $A : C(S) \rightarrow C(S)$ by $A((x_1, x_2)) = (x_1, x_1 + x_2)$. If $E = \{1\}$ and $f|E = 0$ (so $f = (0, x_2)$), then $Af|E = 0$. The map B has nonzero fixed points in $P(E)$, but if $x_1 \neq 0$, (x_1, x_2) cannot be a fixed point of A .

A more interesting example is provided by a special case of a "Perron-Frobenius" type of operator. Let $S = [0, \frac{1}{2}]$, $E = \{0\}$ and define $A : C(S) \rightarrow C(S)$ by

$$(Au)(t) = b(t)u(ct),$$

where $0 < c < 1$, $b(t) := 1 - (\ln(t))^{-1}$ for $0 < t \leq \frac{1}{2}$ and $b(0) = 1$. One can prove that A is a positive linear operator. If $f|E = 0$, then we have $Af|E = 0$; and if B is defined as in Lemma 3.6 and $\theta_1(0) = 1$, then $B(\theta_1) = \theta_1$. One can prove that $\lim_{k \rightarrow \infty} \|A^k\| = \infty$ and that $r(A) = 1$, so $r(A) = r(B) = 1$. However, one can also prove that if $u \in C(S)$ and $A(u) = u$, then $u(0) = 0$. If $\psi : [\frac{c}{2}, \frac{1}{2}] \rightarrow \mathbb{R}$ is any continuous function such that $b(\frac{1}{2})\psi(\frac{c}{2}) = \psi(\frac{1}{2})$, one can prove that ψ has a unique extension to a function $u \in C(S)$ with $Au = u$. This implies (contrast Theorem 3.3) that $\{u \in C(S) : u = Au\}$ is infinite dimensional.

Until now we have restricted ourselves to the case of periodic points of prime period. However, the general case can be reduced to the case of prime periods.

Lemma 3.7 *Let Y be a Banach space and $B : Y \rightarrow Y$ a bounded linear map. Assume that u is a periodic point of B of minimal period m_1 and v is a periodic point of B of minimal*

period m_2 and let $m = \text{lcm}(m_1, m_2)$ denote the least common multiple of m_1 and m_2 . There exist real numbers $\alpha \geq 0$ and $\beta \geq 0$ such that $w := \alpha u + \beta v$ is a periodic point of B of minimal period m .

Proof. If m_1 is a divisor of m_2 , we take $w = v$; and if m_2 is a divisor of m_1 , we take $w = u$. Thus we can assume that $m > m_1$ and $m > m_2$. Obviously, $A^m(w) = w$ for any $\alpha > 0$ and $\beta > 0$. For definiteness, we take $\beta = 1$, so $w = \alpha u + v$. For $1 \leq j \leq m$ we know that $A^j(u) = u$ if and only if $j \equiv 0 \pmod{m_1}$. Thus there exists $\delta > 0$ such that $\|A^j(u) - u\| \geq \delta$ for all j such that $1 \leq j \leq m$ and j is not divisible by m_1 . Select $\alpha > 0$ so that

$$\alpha\delta > \sup\{\|A^j(v) - v\| : 1 \leq j \leq m, j \text{ is not divisible by } m_1\}.$$

For this choice of α it follows that $\alpha A^j(u) + A^j(v) \neq \alpha u + v$ for all j such that $1 \leq j \leq m$ and j is not divisible by m_1 . If $\alpha A^j(u) + A^j(v) = \alpha u + v$ and $1 \leq j \leq m$, it follows that $j \equiv m \pmod{m_1}$, and we have $A^j(v) = v$. However, the latter equation implies that j is divisible by m_2 . Since j is divisible by m_1 and m_2 , we must have $j \geq m$. It follows that for our choice of α , w is a periodic point of A of minimal period m . \square

Our next lemma casts Lemma 3.7 in a form which will be more directly useful.

Lemma 3.8 *Let Y be a Banach space and $B : Y \rightarrow Y$ a bounded linear map. Suppose that $m = \prod_{j=1}^k p_j^{\alpha_j}$, where p_j , $1 \leq j \leq k$, are distinct prime numbers and α_j , $1 \leq j \leq k$, are positive integers. Assume that for each j , $1 \leq j \leq k$, there exists a periodic point u_j of B of minimal period ν_j , where $p_j^{\alpha_j}$ is a divisor of ν_j and ν_j is a divisor of m . Then there exist nonnegative reals c_j , $1 \leq j \leq k$, such that $w = \sum_{j=1}^k c_j u_j$ is a periodic point of B of minimal period m .*

Proof. We know that $u_1 := w_1$ is a periodic point of B of minimal period ν_1 . For $1 \leq s < k$, assume that we have found nonnegative constants $\gamma_{j,s}$, $1 \leq j \leq s$, such that $w_s := \sum_{j=1}^s \gamma_{j,s} u_j$ is a periodic point of B of minimal period $\text{lcm}(\{\nu_j : 1 \leq j \leq s\})$. By using Lemma 3.7 we see that there are nonnegative reals α and β such that $w_{s+1} := \alpha w_s + \beta u_{s+1}$ is a periodic point of B of minimal period $\text{lcm}(\{\nu_j : 1 \leq j \leq k+1\})$. Continuing in this way we eventually obtain $w_k := \sum_{j=1}^k c_j u_j$, where $c_j \geq 0$, for $1 \leq j \leq k$, and w has minimal period $\text{lcm}(\{\nu_j : 1 \leq j \leq k\}) := m_1$. Since $p_j^{\alpha_j}$ is a divisor of ν_j and ν_j is a divisor of m , $m = m_1$. \square

We can now reduce the question of existence of a periodic point of period m to the question of existence of various periodic points of prime periods.

Theorem 3.5 *Let Y be a real Banach space, C a convex subset of Y and $A : Y \rightarrow Y$ a bounded linear map with $A(C) \subset C$. Let $m = \prod_{j=1}^k p_j^{\alpha_j}$, where p_j , $1 \leq j \leq k$, are distinct primes and α_j , $1 \leq j \leq k$, are positive integers, and define $\mu_j = mp_j^{-1}$ and $B_j = A^{\mu_j}$. Then A has a periodic point $f_0 \in C$ of minimal period m if and only if B_j has a periodic point $v_j \in C$ of minimal period p_j for $1 \leq j \leq k$.*

Proof. Suppose first that there exists a periodic point $f_0 \in C$ of A of minimal period m . It follows that $B_j(f_0) = A^{\mu_j}(f_0) \neq f_0$ and that $B_j^{p_j}(f_0) = A^m(f_0) = f_0$. Since p_j is a prime, this shows that $f_0 \in C$ is a periodic point of B_j of minimal period p_j for $1 \leq j \leq k$.

Conversely, suppose that for each j , $1 \leq j \leq k$, there exists a periodic point $v_j \in C$ of B_j of minimal period p_j . It follows that $A^m(v_j) = B^{p_j}(v_j) = v_j$. If ν_j denotes the minimal period of v_j as a periodic point of A , it follows that ν_j is a divisor of m ; but because $B_j(v_j) = A^{\mu_j}(v_j) \neq v_j$, ν_j is not a divisor of μ_j . These two facts imply that $p_j^{\alpha_j}$ is a divisor of ν_j and ν_j is a divisor of m . Lemma 3.8 now implies that there are nonnegative reals c_j , $1 \leq j \leq k$, such that $w = \sum_{j=1}^k c_j v_j$ is a periodic point of A of minimal period m . Because we must have $c_j > 0$ for some j , we see that $(\sum_{j=1}^k c_j)^{-1} w := \tilde{w} \in C$ and \tilde{w} is a periodic point of A of minimal period m . \square

If $A : C(S) := Y \rightarrow Y$ is a positive linear operator and the maps B_j are as in Theorem 3.5, then Theorem 3.5 reduces the question of whether A has a periodic point in $P_o(S)$ or $P(S)$ of minimal period m to the corresponding question of whether, for $1 \leq j \leq k$, the positive linear operator B_j has a periodic point in $P_o(S)$ or $P(S)$ of minimal period p_j . Since p_j is a prime, the latter question can be addressed with the aid of Theorems 3.1-3.4.

If Y is a real Banach space, recall that \tilde{Y} denotes the complexification of Y and that for a bounded linear operator $B : Y \rightarrow Y$, \tilde{B} denotes the complexification of B . We write $\sigma(\tilde{B}) = \sigma(B)$ and we call any eigenvalue of \tilde{B} an eigenvalue of B .

There is clearly a close connection between existence of certain periodic points of B and existence of certain eigenvalues. For reasons of length, we omit the proof of the following simple theorem.

Theorem 3.6 *Let Y be a real Banach space and $B : Y \rightarrow Y$ a bounded linear map. Assume that $g_0 \in Y$ is a periodic point of B of minimal period $m = \prod_{j=1}^k p_j^{\alpha_j}$, where p_j , $1 \leq j \leq k$, are distinct prime numbers and α_j , $1 \leq j \leq k$, are positive integers. Define $\nu_j = \frac{m}{p_j}$. Then, for $1 \leq j \leq k$, B has an eigenvalue $\lambda_j \in \mathbb{C}$ such that $\lambda_j^m = 1$ and $\lambda_j^{\nu_j} \neq 1$. Conversely, let m and ν_j be as above and let $B : Y \rightarrow Y$ be a bounded linear map. Suppose that, for $1 \leq j \leq k$, B has an eigenvalue $\lambda_j \in \mathbb{C}$ such that $\lambda_j^m = 1$ and $\lambda_j^{\nu_j} \neq 1$. Then B has a periodic point $h_0 \in Y$ of minimal period m .*

4 Periodic Points of the Adjoint Operator

Given a real Banach space Y , Y^* will denote the dual space of continuous, real-linear functions, $\mu : Y \rightarrow \mathbb{R}$; and if $B : Y \rightarrow Y$ is a bounded linear map, B^* will denote the Banach space adjoint of B . If S is a compact, Hausdorff space and $X = C(S)$, recall that X^* is linearly isometric to the Banach space of signed, regular Borel measures on S . If $A : X \rightarrow X$ is a bounded, positive linear operator, it is natural, in view of applications to invariant measures, to ask whether A^* has a positive periodic point μ of minimal period m . Here μ is called "positive" if

$$\mu \in P^* := \{\nu \in X^* : \nu(f) \geq 0 \forall f \in P(S)\}.$$

Alternately, $P^*(S)$ is the set of nonnegative, regular Borel measures on S . With the aid of Theorems 3.1-3.5, one can give conditions which insure that A^* has a positive periodic point, but these conditions involve unnecessary hypotheses. Thus we choose to argue more directly.

Theorem 4.1 *Let S be a compact, Hausdorff space, $A : X := C(S) \rightarrow X$ be a positive, bounded, linear operator and m be a prime number. Assume that E_j , $0 \leq j \leq m$, are compact, nonempty subsets of S with $E_m = E_0$ such that (a) $\cap_{j=0}^{m-1} E_j = \emptyset$ and (b) for $0 \leq j \leq m-1$, if $f \in X$ and $f|E_j = 0$, then $Af|E_{j+1} = 0$. Let $\cup_{j=0}^{m-1} E_j = E$ and define $B : C(E) \rightarrow C(E)$ as in Lemma 3.6, so $B(f) = A(\tilde{f})|E$, where $\tilde{f} \in X$ and $\tilde{f}|E = f$. Assume that there exists $M > 0$ such that $\|B^n\| \leq M$ for all $n \geq 0$ and that there exists $\theta \in P(E) - \{0\}$ with $B(\theta) = \theta$. Then there exists $\nu_0 \in P^*(S)$, $\nu_0 \neq 0$, such that ν_0 is a periodic point of A^* of minimal period m .*

Proof. Suppose that we can prove that there exists $\nu \in P^*(E)$ with $B^*(\nu) \neq \nu$ and $(B^*)^m(\nu) = \nu$. Because m is a prime, this will imply that ν is a periodic point of B^* of minimal period m . We know that ν is a regular Borel measure on E , and we can associate a regular Borel measure $\tilde{\nu}$ on S by defining $\tilde{\nu}(\Gamma) := \nu(\Gamma \cap E)$ for every Borel subset Γ of S . We claim that $(A^*)^m(\tilde{\nu}) = \tilde{\nu}$ and $A^*(\tilde{\nu}) \neq \tilde{\nu}$, so $\tilde{\nu}$ is a periodic point of A^* of minimal period m . For $\mu \in X^*$ and $f \in X$, we shall often write (μ, f) instead of $\mu(f)$. We have, for $\tilde{f} \in X$ and $f = \tilde{f}|E$, that

$$((A^*)^j(\tilde{\nu}), \tilde{f}) = \int_S A^j(\tilde{f}) d\tilde{\nu} = \int_E A^j(\tilde{f}) d\nu = \int_E B^j(f) d\nu = ((B^*)^j(\nu), f).$$

It follows that $((A^*)^m(\tilde{\nu}), \tilde{f}) = (\tilde{\nu}, \tilde{f})$ for all $\tilde{f} \in X$, so $(A^*)^m(\tilde{\nu}) = \tilde{\nu}$. Also, because $B^*(\nu) \neq \nu$, there exists $f \in C(E)$ with $(B^*(\nu), f) \neq (\nu, f)$. If we select $\tilde{f} \in X$ with $\tilde{f}|E = f$, it follows that $(A^*(\tilde{\nu}), \tilde{f}) \neq (\tilde{\nu}, \tilde{f})$ and $A^*(\tilde{\nu}) \neq \tilde{\nu}$.

The above remarks show that we may as well work with B from the beginning and assume that $E = S$.

We shall now use the idea of Theorem 3.1. We shall construct a closed, bounded convex set $D_{a,b}$ and a bounded linear map which takes the set $D_{a,b}$ into itself and whose fixed points in $D_{a,b}$ give periodic points of B^* of minimal period m . For $\Gamma \subset E$, let $\Gamma' = E - \Gamma$, the complement of Γ in E . Fix a number r , $0 < r < 1$, and let a and b be positive numbers which will be selected more precisely later. Let $Y = \prod_{i=0}^{m-1} C(E)$, the Banach space of ordered m -tuples $f = (f_0, f_1, \dots, f_{m-1})$ of elements of $C(E)$. Let $Z = \prod_{i=0}^{m-1} C(E)^*$ denote the Banach space of ordered m -tuples $\mu = (\mu_0, \mu_1, \dots, \mu_{m-1})$ of elements of $C(E)^*$. Recall that Z is naturally identified with Y^* by allowing μ to act on f by

$$(\mu, f) := \sum_{i=0}^{m-1} (\mu_i, f_i).$$

We define a closed, bounded, convex set $D_{a,b} \subset Y^*$ by

$$D_{a,b} = \{(\mu_0, \dots, \mu_{m-1}) \mid \forall j \forall \text{ Borel sets } \Gamma \subset E_j^c, a \leq (\mu_j, \theta) \leq b \text{ and } \mu_j(\Gamma) = r\mu_{j-1}(\Gamma)\}.$$

As usual, indices are written modulo m , so $\mu_m = \mu_0$. The condition that $\mu_j(\Gamma) = r\mu_{j-1}(\Gamma)$ for all Borel sets $\Gamma \subset E_j^c$ is equivalent to assuming that $(\mu_j - r\mu_{j-1}, f_j) = 0$ for all $f_j \in C(E)$ such that $f_j|E_j = 0$.

Define $\mathcal{B} : Y \rightarrow Y$ by $\mathcal{B}((f_0, f_1, \dots, f_{m-1})) = (g_0, g_1, \dots, g_{m-1})$, where $g_i = B(f_{i-1})$ for $1 \leq i \leq m$. Similarly, define $\mathcal{B}^* : Y^* \rightarrow Y^*$ by $\mathcal{B}^*((\mu_0, \mu_1, \dots, \mu_{m-1})) = (\nu_0, \nu_1, \dots, \nu_{m-1})$,

where $\nu_i = B^*(\mu_{i+1})$ for $0 \leq i \leq m-1$. Under the previously mentioned identification of Y^* with Z , the reader can verify that \mathcal{B}^* is the Banach space adjoint of \mathcal{B} .

Arguing as in Theorem 3.1, the reader can verify that $\mathcal{B}^*(D_{a,b}) \subset D_{a,b}$. The problem is to prove that $D_{a,b}$ is nonempty. Suppose we can prove that for some choice of positive reals a and b , $D_{a,b} \neq \emptyset$. Select $\mu = (\mu_0, \mu_1, \dots, \mu_{m-1}) \in D_{a,b}$ and define $\mu^k \in D_{a,b}$ by

$$\mu^k = \frac{1}{k} \sum_{j=0}^{k-1} (\mathcal{B}^*)^j(\mu).$$

We know that there exists a constant M such that $\|B^m\| \leq M$ for all $m \geq 0$; so if we define the norm on Y by

$$\|f_0, f_1, \dots, f_{m-1}\| = \max\{\|f_i\| : 1 \leq i \leq m-1\},$$

we see that $\|\mathcal{B}^m\| = \|B^m\| = \|(\mathcal{B}^*)^m\|$ and $\|(\mathcal{B}^*)^m\| \leq M$ for all $m \geq 0$. It follows that $(\mu^k : k \geq 1)$ is bounded in Y^* , and the Banach-Alaoglu theorem implies that there exists a subsequence (μ^{k_i}) , $k_i \rightarrow \infty$, such that μ^{k_i} converges in the weak star (w^*) topology to ν , $\mu^{k_i} \rightharpoonup \nu$. Notice that by our construction, $D_{a,b}$ is the intersection of w^* closed sets, so $D_{a,b}$ is w^* closed and $\nu \in D_{a,b}$. We have that

$$\|\mathcal{B}^*(\mu^{k_i}) - \mu^{k_i}\| = \frac{1}{k_i} \|(\mathcal{B}^*)^{k_i}(\mu) - \mu\| \rightarrow 0,$$

so $\mathcal{B}^*(\mu^{k_i}) \rightharpoonup \nu$. It follows that for any $f \in Y$,

$$\lim_{i \rightarrow \infty} (\mathcal{B}^*(\mu^{k_i}), f) = (\nu, f) = \lim_{i \rightarrow \infty} (\mu^{k_i}, \mathcal{B}(f)) = (\nu, \mathcal{B}f).$$

This implies that $(\mathcal{B}^*(\nu), f) = (\nu, f)$ for all $f \in Y$, so $\mathcal{B}^*(\nu) = \nu$. If $\nu = (\nu_0, \nu_1, \dots, \nu_{m-1})$ and $\mathcal{B}^*(\nu) = \nu$, we easily see that $\mathcal{B}^*(\nu_i) = \nu_{i-1}$ for $1 \leq i \leq m$, and this implies that $(\mathcal{B}^*)^m(\nu_0) = \nu_0$. If $\mathcal{B}^*(\nu_0) = \nu_0$, a calculation shows that $\nu_0 = \nu_j$ for $0 \leq j \leq m-1$. Since $(\nu_j - r\nu_{j-1})(f) = 0$ for all $f \in C(E)$ such that $f|_{E_j} = 0$, it follows that $(1-r)\nu_0(f) = 0$ for all $f \in C(E)$ such that $f|_{E_j} = 0$, i.e., $\nu_0(f) = 0$ for all $f \in C(E)$ such that $f|_{E_j} = 0$. Because E'_j is open in E and $(\cup_{j=0}^{m-1} E'_j) = (\cap_{j=0}^{m-1} E_j)' = E$, $\{E'_j : 0 \leq j \leq m-1\}$ is an open covering of E and there exists a partition of unity $\{\psi_j | 0 \leq j \leq m-1\}$ subordinate to $\{E'_j : 0 \leq j \leq m-1\}$. If $f \in C(E)$, we can write $f = \sum_{j=0}^{m-1} \psi_j f$, and because $\psi_j f|_{E_j} = 0$, we obtain

$$\nu_0(f) = \sum_{j=0}^{m-1} \nu_0(\psi_j f) = 0,$$

which contradicts $\nu_0(\theta) \geq a$ and proves that $\mathcal{B}^*(\nu_0) \neq \nu_0$.

Thus it only remains to show that $D_{a,b} \neq \emptyset$ for some positive reals a and b . Because $\theta \in P(E) - 0$, $G_\theta := \{s \in E : \theta(s) > 0\}$ is nonempty and open. It is easy to see that there exists a nonnegative, regular Borel measure μ with $\mu(G_\theta) > 0$ (for example, define μ by $\mu(f) = f(s_0)$, where $f \in C(S)$ and $s_0 \in G_\theta$).

If μ is as above, the outer regularity of μ implies that

$$\inf\{\mu(E'_j \cap U) : U \text{ open}, E_j \subset U\} = 0, \quad j = 0, \dots, m-1,$$

so that

$$\inf\{\mu(\cup_{i=0}^{m-1} E'_i \cap U_i) : U_i \text{ open, } E_i \subset U_i, i = 0, \dots, m-1\} = 0.$$

Since $\mu(G_\theta) > 0$, the previous equation implies that for appropriate open sets U_j with $E_j \subset U_j$ for $j = 0, 1, \dots, m-1$, we have

$$\mu(G_\theta \cap (\cup_{i=0}^{m-1} E'_i \cap U_i)') > 0.$$

Select such sets U_j and define $F_j := U'_j$ for $j \geq 0$. Because $F_i \subset E$ is compact and $\cap_{i=0}^{m-1} F_i = \emptyset$, we can apply Lemma 3.2; and we find that there exist $h_i \in P_\circ(E)$, $0 \leq i \leq m$, with $h_m = h_0$, such that $h_i(s) = rh_{i-1}(s)$ for all $s \in F_i$. Define $\chi(s) = 0$ if $s \in \cup_{j=0}^{m-1} (E'_j \cap U_j)$ and $\chi(s) = 1$ otherwise. Define $g_i(s) := \chi(s)h_i(s)$. Our construction insures that g_i is a bounded, nonnegative, Borel measurable function, $g_i(s) = rg_{i-1}(s)$ for all $s \in E'_i$, and g_i is strictly positive on the complement of $\cup_{j=0}^{m-1} (E'_j \cap U_j)$. Thus it makes sense to define $\mu_i \in P^*(E)$ by

$$\mu_i(f) := \mu(g_i f) = \int_E f(s)g_i(s)\mu(ds).$$

Because $\theta(s)g_i(s) > 0$ for all $s \in H := G_\theta \cap (\cup_{j=0}^{m-1} (E'_j \cap U_j))'$ and $\mu(H) > 0$, it follows that $\mu_i(\theta) = \mu(g_i\theta) > 0$. Thus there exist positive numbers a and b with $a \leq \mu_i(\theta) \leq b$ for $0 \leq i \leq (m-1)$.

We claim that $\mu = (\mu_0, \mu_1, \dots, \mu_{m-1}) \in D_{a,b}$. To prove this it suffices to show that if $f \in C(E)$ and $f|_{E_i} = 0$, then $\mu_i(f) = r\mu_{i-1}(f)$. However, we have

$$\mu_i(f) - r\mu_{i-1}(f) = \mu((g_i - rg_{i-1})f).$$

By our construction, we have $(g_i - rg_{i-1})|_{E'_i} = 0$ and $f|_{E_i} = 0$, so $(g_i - rg_{i-1})f = 0$ and $\mu_i(f) = r\mu_{i-1}(f)$. \square

We assume in Theorem 4.1 that m is a prime. The case that m is not a prime can be handled by combining Theorems 4.1 and 3.5.

5 Perron-Frobenius Operators: Existence and Uniqueness of Positive Eigenvectors

In this section (S, d) will denote a compact metric space with metric d and $A : C(S) \rightarrow C(S)$ will be a bounded, positive linear operator of the form

$$(Au)(s) = \sum_{i=1}^{\infty} b_i(s)u(w_i(s)). \quad (13)$$

Here $b_i : S \rightarrow \mathbb{R}$ and $w_i : S \rightarrow S$ are given maps, and we shall usually make at least the following assumptions:

H5.1. For $1 \leq i < \infty$, $b_i : S \rightarrow \mathbb{R}$ is a nonnegative, continuous function. For each $s \in S$, $\sum_{i=1}^{\infty} b_i(s) := b(s) < \infty$ and $b : S \rightarrow \mathbb{R}$ is continuous.

H5.2. For $1 \leq i < \infty$, the maps $w_i : S \rightarrow S$ are uniformly Lipschitz, i.e., there exists a constant C , independent of i , such that $d(w_i(t), w_i(s)) \leq Cd(t, s)$ for all $s, t \in S$ and all $i \geq 1$.

By Dini's theorem, the assumption in H5.1 that b is continuous is equivalent to assuming that $\sum_{i=1}^n b_i(s)$ converges uniformly to $b(s)$ as $n \rightarrow \infty$. If H5.1 is satisfied and the w_i are all continuous, it is easy to verify that A defines a bounded linear map from $X := C(S)$ to X . Operators of the form given by eq. (13) have been called "Perron-Frobenius operators". Such operators arise in many contexts, e.g., in the study of invariant measures and in finding the Hausdorff dimension of various sets. See [1, 2], [4], [6], [7, 8], [9, 11] for further references.

As was noted in Remark 3.3, the inequality $\rho(A) < r(A)$ frequently fails for Perron-Frobenius operators. This point was already implicitly observed by F.F. Bonsall [3], who considered the case $S = [0, 1] \subset \mathbb{R}$ and $A : C(S) \rightarrow C(S)$ defined by

$$(Ax)(s) = x\left(\frac{1}{2}s\right).$$

If e denotes the function identically equal to one, $Ae = e$ and $r(A) = 1$; but Bonsall observed that if $x_\gamma \in C(S)$ is defined by $x_\gamma(t) = t^\gamma$, where $\gamma \in \mathbb{C}$ and $Re(\gamma) > 0$, then one has

$$A(x_\gamma) = \left(\frac{1}{2}\right)^\gamma x_\gamma,$$

so the spectrum of A contains the closed unit disc in \mathbb{C} . Since $r(A) = 1$, this shows that the spectrum of A equals the closed unit disc in \mathbb{C} : see Remark 2.7 in [15]. More generally, if S is a finite union of intervals in \mathbb{R} and the functions b_i and w_i in eq. (13) are suitably differentiable, it was observed in section 2 of [15] that A can be considered as a map $A_n : C^n(S) \rightarrow C^n(S)$ and that the spectrum of A_n varies with n . See Theorem 2.3 and Remarks 2.5-2.7 in [15].

For most of the work here we shall need more than just the continuity of the maps b_i , $i \geq 1$. For simplicity, we shall usually restrict ourselves to the following Hölder continuity assumption, although our results can be extended to the case that there exists a modulus of continuity for the maps b_i , $i \geq 1$, which satisfies a Caratheodory condition as in [2].

H5.3. For $1 \leq i < \infty$, $b_i : S \rightarrow \mathbb{R}$ is a continuous function. There exist constants $M_0 > 0$ and λ_0 , $0 < \lambda_0 \leq 1$, such that

$$\sup\left\{\sum_{i=1}^{\infty} \frac{|b_i(s) - b_i(t)|}{d(s, t)^{\lambda_0}} : s, t \in S, s \neq t\right\} \leq M_0 < \infty. \tag{14}$$

Remark 5.1. For simplicity we shall restrict ourselves to operators of the form given by eq. (13), but this excludes some interesting examples. To see this, suppose that S is the unit circle in \mathbb{R}^2 , so $C(S)$ can be identified with Y , the Banach space of continuous, 2π -periodic maps $y : \mathbb{R} \rightarrow \mathbb{R}$. Let $b : Y \rightarrow Y$ be a nonnegative 4π -periodic, continuous function and define $A : Y \rightarrow Y$ by

$$(Ay)(\theta) = b(\theta)y\left(\frac{1}{2}\theta\right) + b(\theta + 2\pi)y\left(\frac{1}{2}\theta + \pi\right).$$

The methods of this section apply to A ; but considered as a map on $C(S)$, A is not of the proper form.

To remedy this deficiency, one can assume that, for a compact metric space (S, d) , S is given locally by an equation like eq. (13). More precisely, for each $s \in S$, assume that there exists an open neighborhood U of s , continuous, nonnegative functions $b_{i,U} : U \rightarrow \mathbb{R}$, and Lipschitzian functions $w_{i,U} : U \rightarrow S$, which satisfy analogues of H5.1, H5.2, and H5.3 and for which

$$(Ax)(t) = \sum_{i=1}^{\infty} b_{i,U} x(w_{i,U}(t))$$

for all $t \in U$ and all $x \in C(S)$. The results of this section extend to this more general class of operators, but we omit details.

To continue, we shall need some notation. We shall denote by \mathcal{I}_m the collection of ordered m -tuples $I = (i_1, i_2, \dots, i_m)$ of positive integers. If $I \in \mathcal{I}_m$, $I = (i_1, i_2, \dots, i_m)$, and H5.1 and H5.2 are satisfied, we define functions $b_I : S \rightarrow \mathbb{R}$ and $w_I : S \rightarrow S$ by

$$b_I(t) = b_{i_1}(t) b_{i_2}(w_{i_1}(t)) b_{i_3}(w_{i_2} w_{i_1}(t)) \cdots b_{i_m}(w_{i_{m-1}} w_{i_{m-2}} \cdots w_{i_1}(t)) \quad (15)$$

$$\text{and } w_I(t) = w_{i_m} w_{i_{m-1}} \cdots w_{i_1}(t), \quad (16)$$

where $w_i w_{j-1} \cdots w_{i_1}$ denotes the composition of functions. If $J = (i_1, i_2, \dots, i_s) \in \mathcal{I}_s$ and $K = (i_{s+1}, i_{s+2}, \dots, i_t) \in \mathcal{I}_{t-s}$, we can write $(J, K) = (i_1, i_2, \dots, i_t) \in \mathcal{I}_t$, and we have

$$b_{(J,K)}(t) = b_J(t) b_K(w_J(t)) \text{ and } b_{(J,i_{s+1})}(t) = b_J(t) b_{i_{s+1}}(w_J(t)) \text{ and } w_{(J,K)}(t) = w_K(w_J(t)).$$

If $J = \emptyset$ and $s \in S$, we define $b_J(s) = 1$ and $w_J(s) = s$; and we define \mathcal{I}_0 to be the set whose only element is \emptyset . If $I = (i_1, i_2, \dots, i_m) \in \mathcal{I}_m$, we define $J_0(I) = \emptyset$ and

$$J_r(I) = (i_1, i_2, \dots, i_r) \in \mathcal{I}_r \quad \text{for } 1 \leq r \leq m.$$

Similarly, we define $K_m(I) = \emptyset$ and $K_r(I) = (i_{r+1}, i_{r+2}, \dots, i_m) \in \mathcal{I}_{m-r}$ for $0 \leq r < m$. Finally, we define $i_\nu(I) = i_\nu$, the ν th element of I , for $1 \leq \nu \leq m$. In this notation, the reader can check that for $I \in \mathcal{I}_m$,

$$b_I(t) - b_I(s) = \sum_{\nu=1}^m b_{J_{\nu-1}(I)}(s) [b_{i_\nu(I)}(w_{J_{\nu-1}(I)}(t)) - b_{i_\nu(I)}(w_{J_{\nu-1}(I)}(s))] b_{K_\nu(I)}(w_{J_\nu(I)}(t)). \quad (17)$$

By using the previous equation and changing order of summation (assuming H5.1 and H5.2), we find that

$$\sum_{I \in \mathcal{I}_m} |b_I(t) - b_I(s)| \leq \sum_{\nu=1}^m \left(\sum_{J \in \mathcal{I}_{\nu-1}} \sum_{K \in \mathcal{I}_{m-\nu}} \sum_{i=1}^{\infty} b_J(s) |b_i(w_J(t)) - b_i(w_J(s))| b_K(w_{(J,i)}(t)) \right). \quad (18)$$

This inequality will prove useful later.

The following lemma provides the motivation for the above notation.

Lemma 5.1 *Assume H5.1 and H5.2 and let A be defined by eq. (13). Then for all $f \in C(S)$ we have that*

$$(A^m f)(s) = \sum_{I \in \mathcal{I}_m} b_I(s) f(w_I(s)) \text{ and}$$

$$\|A^m\| = \sup\left\{ \sum_{I \in \mathcal{I}_m} b_I(s) : s \in S \right\}.$$

If H5.3 is also satisfied and if $f \in C(S)$ is Hölder continuous with

$$|f(\sigma) - f(\tau)| \leq M d(\sigma, \tau)^\lambda \text{ for all } \sigma, \tau \in S,$$

(where $0 < \lambda \leq 1$), then for $m \geq 1$ and $s, t \in S$ we have

$$|(A^m f)(t) - (A^m f)(s)| \leq M \sum_{I \in \mathcal{I}_m} b_I(t) d(w_I(t), w_I(s))^\lambda$$

$$+ M_0 \|f\| \sum_{\nu=1}^m \sum_{J \in \mathcal{I}_{\nu-1}} \|A^{m-\nu}\| b_J(s) d(w_J(t), w_J(s))^{\lambda_0},$$

where M_0 and λ_0 are as in H5.3 and $\|f\| = \sup_{s \in S} |f(s)|$.

Proof. The equation for $A^m(f)$ follows by a simple induction on m , which we leave to the reader. If e is the function identically equal to one, we know that $\|A^m\| = \|A^m(e)\|$, and this directly yields the equation for $\|A^m\|$.

To obtain the final statement of the lemma, we observe that

$$|(A^m f)(t) - (A^m f)(s)| \leq \sum_{I \in \mathcal{I}_m} b_I(t) |f(w_I(t)) - f(w_I(s))| + \sum_{I \in \mathcal{I}_m} |f(w_I(s))| |b_I(t) - b_I(s)|.$$

By using the Hölder estimate for f , we see that

$$\sum_{I \in \mathcal{I}_m} b_I(t) |f(w_I(t)) - f(w_I(s))| \leq M \sum_{I \in \mathcal{I}_m} b_I(t) d(w_I(t), w_I(s))^\lambda.$$

By using eq. (18) and H5.3 we see that

$$\begin{aligned} \sum_{I \in \mathcal{I}_m} |f(w_I(s))| |b_I(t) - b_I(s)| &\leq \|f\| \sum_{I \in \mathcal{I}_m} |b_I(t) - b_I(s)| \\ &\leq \|f\| \sum_{\nu=1}^m \sum_{J \in \mathcal{I}_{\nu-1}} \sum_{i=1}^{\infty} b_J(s) |b_i(w_J(t)) - b_i(w_J(s))| \|A^{m-\nu}\| \\ &\leq M_0 \|f\| \sum_{\nu=1}^m \sum_{J \in \mathcal{I}_{\nu-1}} b_J(s) (d(w_J(t), w_J(s))^{\lambda_0} \|A^{m-\nu}\|. \end{aligned}$$

Combining these inequalities gives the estimate in the lemma. \square

We now ask whether the operator A given by eq. (13) has a nonnegative eigenvector with eigenvalue $r(A)$. Without further assumptions than H5.1, H5.2 and H5.3, one can see that A may fail to have such an eigenvector: take $S = [0, 1]$ and $(Ax)(t) = (\frac{1}{2} + \frac{1}{2}t)x(t)$ and note that $r(A) = 1$ and $Ax \neq x$ for $x \in P(S) - 0$.

To discuss a suitable framework, we shall need to consider Banach spaces of Hölder continuous functions on (S, d) . For $0 < \lambda \leq 1$ a function $f \in C(S)$ is called Hölder continuous with Hölder exponent λ if

$$\sup\left\{\frac{|f(s) - f(t)|}{d(s, t)^\lambda} : s, t \in S \text{ and } d(s, t) > 0\right\} < \infty.$$

We define X_λ to be the real vector space of Hölder continuous functions $f : S \rightarrow \mathbb{R}$ with Hölder exponent λ . We define a norm $\|\cdot\|_\lambda$ on X_λ by

$$\|f\|_\lambda = \sup\{|f(s)| : s \in S\} + \sup\left\{\frac{|f(s) - f(t)|}{d(s, t)^\lambda} : s, t \in S \text{ and } d(s, t) > 0\right\},$$

and we recall that X_λ is a Banach space in this norm.

If $S = \overline{G}$, where G is a bounded open subset of \mathbb{R}^n , we could also consider certain real Banach spaces of analytic functions on G or the Banach spaces $C^{m, \lambda}(S)$. If the functions b_i and w_i were sufficiently smooth, we could improve the results we shall give here by working in such Banach spaces.

For $\theta > 0$, it will be convenient to consider an equivalent norm $\|\cdot\|_{\lambda, \theta}$ defined by

$$\|f\|_{\lambda, \theta} = \sup\{|f(s)| : s \in S\} + \sup\left\{\frac{|f(s) - f(t)|}{d(s, t)^\lambda} : s, t \in S, 0 < d(s, t) \leq \theta\right\}.$$

Obviously, we have

$$\|f\|_{\lambda, \theta} \leq \|f\|_\lambda \leq \|f\|_{\lambda, \theta} + 2\theta^{-\lambda}\|f\|,$$

where $\|f\|$ will henceforth denote $\sup_{s \in S} |f(s)|$. The above inequality shows that $\|\cdot\|_\lambda$ and $\|\cdot\|_{\lambda, \theta}$ are equivalent norms on X_λ .

If $\{U_i | 1 \leq i \leq m\}$ is an open covering of the compact metric space (S, d) , there exists a Lipschitzian partition of unity $\{\phi_i | 1 \leq i \leq m\}$ subordinate to $\{U_i | 1 \leq i \leq m\}$. Using such partitions of unity, it is not hard to see that X_1 is dense in $X := C(S)$; and since $X_1 \subset X_\lambda$ for $0 < \lambda \leq 1$, X_λ is also dense in X . Clearly, the inclusion map $i : X_\lambda \rightarrow X$ is continuous. We shall need these facts later.

In the Banach space X_λ one can consider the measure of noncompactness α_λ derived from $\|\cdot\|_\lambda$ or the measure of noncompactness $\alpha_{\lambda, \theta}$ derived from $\|\cdot\|_{\lambda, \theta}$. The next lemma shows that these two measures of noncompactness are actually equal.

Lemma 5.2 *Let (S, d) be a compact metric space and, for $0 < \lambda \leq 1$, let X_λ be the Banach space defined above. Let α_λ and $\alpha_{\lambda, \theta}$ denote the measures of noncompactness on X_λ derived respectively from the norms $\|\cdot\|_\lambda$ and $\|\cdot\|_{\lambda, \theta}$. Then for any bounded set Γ in X_λ , we have $\alpha_\lambda(\Gamma) = \alpha_{\lambda, \theta}(\Gamma)$.*

Proof. Because $\|f\|_\lambda \geq \|f\|_{\lambda, \theta}$ for all $f \in X_\lambda$, we certainly have that $\alpha_\lambda(\Gamma) \leq \alpha_{\lambda, \theta}(\Gamma)$. Because Γ is a bounded set in X_λ , the Ascoli-Arzelà theorem implies that Γ is precompact as a subset of $X := C(S)$. Thus, given $\epsilon > 0$, there exist sets $\Gamma_j \subset \Gamma$, $1 \leq j \leq m$, such that $\bigcup_{j=1}^m \Gamma_j = \Gamma$ and $\sup_{s \in S} |u(s) - v(s)| \leq \min(\epsilon, \epsilon\theta^\lambda)$ for all $u, v \in \Gamma_j$, $1 \leq j \leq m$. If $\alpha_{\lambda, \theta}(\Gamma) = \rho$, then by definition of the measure of noncompactness, there exist sets B_i , $1 \leq i \leq n$, such

that $\Gamma = \cup_{i=1}^n B_i$ and the diameter of B_i with respect to the norm $\|\cdot\|_{\lambda, \theta}$ is less than or equal to $\rho + \epsilon$. Thus, for all $u, v \in B_i$, $1 \leq i \leq n$, we have

$$\|u - v\| + \sup\{|(u(s) - v(s)) - (u(t) - v(t))|d(s, t)^{-\lambda} : 0 < d(s, t) \leq \theta\} \leq \rho + \epsilon.$$

We write $\Gamma = \cup_{i,j}(B_i \cap \Gamma_j)$ and consider the diameter of $B_i \cap \Gamma_j$ with respect to the $\|\cdot\|_{\lambda}$ norm. If $u, v \in B_i \cap \Gamma_j$ we have

$$\sup_{s \in S} |u(s) - v(s)| \leq \epsilon.$$

If $u, v \in B_i \cap \Gamma_j$ and $d(s, t) \geq \theta$, our definition of Γ_j implies that

$$|(u(s) - v(s)) - (u(t) - v(t))|d(s, t)^{-\lambda} \leq \theta^{-\lambda} \|u(s) - v(s)\| + \|u(t) - v(t)\| \leq 2\epsilon.$$

If $u, v \in B_i \cap \Gamma_j$ and $d(s, t) \leq \theta$, we have

$$|(u(s) - v(s)) - (u(t) - v(t))|d(s, t)^{-\lambda} \leq \rho + \epsilon.$$

Combining these estimates, we find that if $u, v \in B_i \cap \Gamma_j$, we have

$$\begin{aligned} \|u - v\|_{\lambda} &= \|u - v\| + \sup\{|(u(t) - v(t)) - (u(s) - v(s))|d(s, t)^{-\lambda} : 0 < d(s, t)\} \\ &\leq \epsilon + \max(2\epsilon, \rho + \epsilon) \leq \rho + 3\epsilon. \end{aligned}$$

Thus Γ can be expressed as a finite union of sets $B_i \cap \Gamma_j$, and $\text{diameter}(B_i \cap \Gamma_j) \leq \rho + 3\epsilon$ (in the norm $\|\cdot\|_{\lambda}$). This implies that $\alpha_{\lambda}(\Gamma) \leq \rho + 3\epsilon$, and since $\epsilon > 0$ is arbitrary, we have proved that $\alpha_{\lambda}(\Gamma) \leq \alpha_{\lambda, \theta}(\Gamma) = \rho$. \square

Assume that H5.1, H5.2 and H5.3 hold and that $A : X := C(S) \rightarrow X$ is defined by eq. (13). Let λ_0 and M_0 be as in H5.3 and select λ with $0 < \lambda \leq \lambda_0$. Then by using Lemma 5.1 one can see that $A(X_{\lambda}) \subset X_{\lambda}$ and A induces a bounded linear map $A_{\lambda} : X_{\lambda} \rightarrow X_{\lambda}$ by $A_{\lambda}(f) = A(f)$ for $f \in X_{\lambda}$. Henceforth A_{λ} will denote this map.

To state our next theorem, we need to define some constants. If m is a positive integer and θ and λ are positive reals, we define reals $\rho_m(\theta, \lambda)$, $\rho_m(\lambda)$ and r_m by

$$\rho_m(\theta, \lambda) = \sup\left\{\sum_{I \in \mathcal{I}_m} b_I(t) \frac{d(w_I(s), w_I(t))^{\lambda}}{d(s, t)^{\lambda}} : 0 < d(s, t) \leq \theta\right\}, \quad (19)$$

$$\rho_m(\lambda) = \lim_{\theta \rightarrow 0^+} \rho_m(\theta, \lambda) \text{ and} \quad (20)$$

$$r_m = \sup\left\{\sum_{I \in \mathcal{I}_m} b_I(t) \mid t \in S\right\}. \quad (21)$$

If $\lambda = 0$, we define $\rho_m(\theta, 0) = \rho_m(0) = r_m$. If $r(A)$ denotes the spectral radius of A , Lemma 5.1 implies that

$$r(A) = \lim_{m \rightarrow \infty} r_m^{\frac{1}{m}} = \inf_{m \geq 1} r_m^{\frac{1}{m}}.$$

We shall need a simple lemma concerning the functions $\rho_m(\theta, \lambda)$ and $\rho_m(\lambda)$.

Lemma 5.3 *Assume H5.1 and H5.2 and for $\theta > 0$ and $\lambda \geq 0$ let $\rho_m(\theta, \lambda)$, $\rho_m(\lambda)$ and r_m be defined by equations (19)-(21). Then for all integers $m, n \geq 1$, we have*

$$0 \leq \rho_{m+n}(\lambda) \leq \rho_m(\lambda)\rho_n(\lambda) \text{ and}$$

$$\bar{\rho}(\lambda) := \inf\{\rho_m(\lambda)^{\frac{1}{m}} : m \geq 1\} = \lim_{m \rightarrow \infty} \rho_m(\lambda)^{\frac{1}{m}}. \quad (22)$$

If $0 \leq \lambda_1 \leq \lambda_2$, $0 \leq \sigma \leq 1$, and $\theta > 0$, then we have

$$\rho_m(\theta, (1-\sigma)\lambda_1 + \sigma\lambda_2) \leq \rho_m(\theta, \lambda_1)^{1-\sigma} \rho_m(\theta, \lambda_2)^\sigma \text{ and} \quad (23)$$

$$\bar{\rho}((1-\sigma)\lambda_1 + \sigma\lambda_2) \leq (\bar{\rho}(\lambda_1))^{1-\sigma} (\bar{\rho}(\lambda_2))^\sigma. \quad (24)$$

Proof. If $K \in \mathcal{I}_p$ and $s, t \in S$, we define a function $g_K(s, t)$ by

$$g_K(s, t) = \frac{d(w_K(s), w_K(t))}{d(s, t)} \text{ for } s \neq t, \quad g_K(s, s) = 0.$$

In this notation, we can write

$$\begin{aligned} \rho_{m+n}(\theta, \lambda) &= \sup\left\{ \sum_{I \in \mathcal{I}_m, J \in \mathcal{I}_n} b_{(I,J)}(t) g_{(I,J)}(s, t)^\lambda : 0 < d(s, t) \leq \theta \right\} \\ &= \sup\left\{ \sum_{I \in \mathcal{I}_m, J \in \mathcal{I}_n} b_I(t) b_J(w_I(t)) g_J(w_I(t), w_I(s))^\lambda g_I(t, s)^\lambda : 0 < d(s, t) \leq \theta \right\} \\ &= \sup\left\{ \sum_{I \in \mathcal{I}_m} b_I(t) g_I(t, s)^\lambda \left(\sum_{J \in \mathcal{I}_n} b_J(w_I(t)) g_J(w_I(t), w_I(s))^\lambda \right) : 0 < d(s, t) \leq \theta \right\}. \end{aligned}$$

Because $d(w_I(t), w_I(s)) \leq C^m d(t, s) \leq C^m \theta$ for C as in H5.2, we have

$$\sum_{J \in \mathcal{I}_n} b_J(w_I(t)) g_J(w_I(t), w_I(s))^\lambda \leq \rho_N(C^m \theta, \lambda).$$

Substituting this estimate in the above equations gives

$$\rho_{m+n}(\theta, \lambda) \leq \sup\left\{ \sum_{I \in \mathcal{I}_m} b_I(t) g_I(t, s)^\lambda \rho_n(C^m \theta, \lambda) : 0 < d(s, t) \leq \theta \right\} \leq \rho_m(\theta, \lambda) \rho_n(C^m \theta, \lambda).$$

Taking the limit as $\theta \rightarrow 0$ in the above inequality gives the first inequality in Lemma 5.3. The standard proof which gives the formula for the spectral radius of a bounded linear operator shows that a sequence of nonnegative reals $(\rho_m(\lambda) : m \geq 1)$ which satisfies the first inequality in Lemma 5.3 necessarily satisfies eq. (22).

If $0 \leq \lambda_1 < \lambda_2$, $0 < \sigma < 1$ and $\theta > 0$, Hölder's inequality implies that, for $0 < d(s, t) \leq \theta$ and $\lambda = (1-\sigma)\lambda_1 + \sigma\lambda_2$,

$$\begin{aligned} \sum_{I \in \mathcal{I}_m} b_I(t) g_I(s, t)^\lambda &= \sum_{I \in \mathcal{I}_m} b_I(t)^{1-\sigma} g_I(s, t)^{\lambda_1(1-\sigma)} b_I(t)^\sigma g_I(s, t)^{\lambda_2\sigma} \\ &\leq \left(\sum_{I \in \mathcal{I}_m} b_I(t) g_I(s, t)^{\lambda_1} \right)^{1-\sigma} \left(\sum_{I \in \mathcal{I}_m} b_I(t) g_I(s, t)^{\lambda_2} \right)^\sigma. \end{aligned}$$

Taking the supremum over s, t with $d(s, t) \leq \theta$ on both sides of this inequality gives

$$\rho_m(\theta, \lambda) \leq \rho_m(\theta, \lambda_1)^{1-\sigma} \rho_m(\theta, \lambda_2)^\sigma,$$

and taking limits as $\theta \rightarrow 0^+$ yields

$$\rho_m((1-\sigma)\lambda_1 + \sigma\lambda_2) \leq \rho_m(\lambda_1)^{1-\sigma} \rho_m(\lambda_2)^\sigma.$$

If we take m^{th} roots and let m approach infinity in the above inequality and use eq. (22), we obtain eq. (24). \square

Theorem 5.1 *Assume that H5.1, H5.2 and H5.3 hold and that $A : X = C(S) \rightarrow X$ is defined by eq. (13). If λ_0 is as in H5.3 and $0 < \lambda \leq \lambda_0$, then $A(X_\lambda) \subset X_\lambda$ and A induces a bounded linear map $A_\lambda : X_\lambda \rightarrow X_\lambda$ by $A_\lambda(f) = A(f)$ for $f \in X_\lambda$. Let $\rho(A_\lambda)$ denote the essential spectral radius of A_λ , $r(A_\lambda)$ the spectral radius of A_λ and $r(A)$ the spectral radius of A . If $\rho_m(\lambda)$, r_m and $\tilde{\rho}(\lambda)$ are defined by equations (19)-(22), then we have, for $0 < \lambda \leq \lambda_0$,*

$$\rho(A_\lambda) \leq \tilde{\rho}(\lambda) = \inf_{m \geq 1} \rho_m(\lambda)^{\frac{1}{m}} = \lim_{m \rightarrow \infty} \rho_m(\lambda)^{\frac{1}{m}} \text{ and} \quad (25)$$

$$r(A_\lambda) \geq r(A) = \lim_{m \rightarrow \infty} r_m^{\frac{1}{m}} = \inf_{m \geq 1} r_m^{\frac{1}{m}}. \quad (26)$$

If we have that $\rho(A_\lambda) < r(A)$ (which will be true if $\tilde{\rho}(\lambda) < r(A) := r$), then it follows that $r(A_\lambda) = r(A) := r$, and there exists $u \in P(S) \cap X_\lambda$, $u \neq 0$, with $A_\lambda(u) = ru$. Furthermore, $E = \{y \in X_\lambda : (rI - A_\lambda)^k(y) = 0 \text{ for some } k \geq 1\}$ is finite dimensional. If $\rho(A_\lambda) < r(A)$, $i : X_\lambda \rightarrow X$ denotes the inclusion map, and $y \in X_\lambda$ is such that $\{\|r^{-k}A^k(i(y))\| : k \geq 0\}$ is bounded, then $\{\|r^{-k}A_\lambda^k(y)\|_\lambda : k \geq 0\}$ is bounded. If $\tilde{\rho}(\lambda) < r(A) := r$ and $0 < \mu \leq \lambda$, then $\tilde{\rho}(\mu) < r$ and

$$\tilde{\rho}(\mu) \leq \tilde{\rho}(\lambda)^\sigma r^{1-\sigma}, \quad \sigma := \frac{\mu}{\lambda}.$$

If $\rho(A_\lambda) < r(A)$ and if there exists $D < \infty$ such that $\|A^k\| \leq Dr^k$ for all $k \geq 1$ (which will be true if $A(x) = rx$ for some $x \in P_o(S)$), then the following results hold:

- For every $y \in \tilde{X}_\lambda$ and every $\zeta \in \mathbb{C}$ with $|\zeta| = r$, $\{\zeta^{-k}\tilde{A}_\lambda^k(y) : k \geq 0\}$ has compact closure in the norm topology on \tilde{X}_λ . (Here \tilde{X}_λ denotes the complexification of X_λ and \tilde{A}_λ denotes the complexification of A_λ .)
- For every $x \in \tilde{X}$ and every $\zeta \in \mathbb{C}$ with $|\zeta| = r$, $\{\zeta^{-k}\tilde{A}^k(x) : k \geq 0\}$ has compact closure in the norm topology on \tilde{X} . (Here \tilde{X} is the complexification of X and \tilde{A} is the complexification of A .)
- If $\zeta \in \mathbb{C}$, $|\zeta| = r$, and $N = \{x \in \tilde{X} \mid \tilde{A}(x) = \zeta x\}$ and $M = \{y \in \tilde{X}_\lambda \mid \tilde{A}_\lambda(y) = \zeta y\}$, then M is finite dimensional and $N = M$. Furthermore, $N = \{x \in \tilde{X} \mid (\zeta I - \tilde{A})^k(x) = 0 \text{ for some } k \geq 1\}$.
- For every $y \in \tilde{X}_\lambda$ and every $\zeta \in \mathbb{C}$ with $|\zeta| = r$, $y_n := n^{-1} \sum_{j=0}^{n-1} (\zeta^{-1} \tilde{A}_\lambda)^j(y)$ converges in the norm topology on \tilde{X}_λ to an element $z := Q_\lambda(y)$, and $\tilde{A}_\lambda(Q_\lambda(y)) = \zeta Q_\lambda(y)$. For every $x \in \tilde{X}$ and every $\zeta \in \mathbb{C}$ with $|\zeta| = r$, $x_n := n^{-1} \sum_{j=0}^{n-1} (\zeta^{-1} \tilde{A})^j(x)$ converges in the norm topology on \tilde{X} to an element $Q(x)$, and $\tilde{A}(Q(x)) = \zeta Q(x)$. The maps $Q_\lambda : \tilde{X}_\lambda \rightarrow \tilde{X}_\lambda$ and $Q : \tilde{X} \rightarrow \tilde{X}$ are bounded linear projections.

Proof. We shall use I to denote the identity operator on X or X_λ and also to denote $I \in \mathcal{I}_m$, but the meaning should be clear from the context. Other notation will be consistent with that in H5.1 and H5.2.

Let Γ be a bounded set in X_λ , where $0 < \lambda \leq \lambda_0$, and let $0 < \epsilon$ and $0 < \theta \leq 1$ be reals. Lemma 5.2 implies that $\gamma := \alpha_{\lambda,\theta}(\Gamma) = \alpha_\lambda(\Gamma)$ and $\alpha_{\lambda,\theta}(A_\lambda(\Gamma)) = \alpha_\lambda(A_\lambda(\Gamma))$. By the argument used in Lemma 5.2, there exist sets C_i , $1 \leq i \leq p < \infty$, such that $\Gamma = \cup_{i=1}^p C_i$ and if $u, v \in C_i$, then $\|u - v\| \leq \epsilon$ and $\|u - v\|_\lambda \leq \gamma + \epsilon$. For $u, v \in C_i$, write $f = u - v$, so $\|f\| \leq \epsilon$ and $\|f\|_\lambda \leq \gamma + \epsilon$. If M_0 and λ_0 are as in H5.3 and $m \geq 1$, we obtain from Lemma 5.1 that

$$\begin{aligned} \sup_{0 < d(s,t) \leq \theta} \left\{ \frac{|(A_\lambda^m f)(t) - (A_\lambda^m f)(s)|}{d(s,t)^\lambda} \right\} &\leq (\gamma + \epsilon)\rho_m(\theta, \lambda) + \\ &+ \sup_{0 < d(s,t) \leq \theta} M_0 \epsilon \sum_{\nu=1}^m \|A^{m-\nu}\| \sum_{J \in \mathcal{I}_{\nu-1}} b_J(s) \left(\frac{d(w_J(t), w_J(s))^{\lambda_0}}{d(s,t)^\lambda} \right). \end{aligned}$$

Because of H5.2, we have that for $d(s, t) \leq \theta$ and $J \in \mathcal{I}_{\nu-1}$,

$$\frac{d(w_J(t), w_J(s))^{\lambda_0}}{d(s,t)^\lambda} \leq C^{\nu-1} \frac{d(s,t)^{\lambda_0}}{d(s,t)^\lambda} \leq C^{\nu-1} \theta^{\lambda_0-\lambda} \leq C^{\nu-1}.$$

By using H5.1 we see that

$$\sum_{J \in \mathcal{I}_{\nu-1}} b_J(s) \leq \|b\|^{\nu-1}.$$

Using these estimates in eq. (22), we see that

$$\sup_{0 < d(s,t) \leq \theta} \left\{ \frac{|(A_\lambda^m f)(t) - (A_\lambda^m f)(s)|}{d(s,t)^\lambda} \right\} \leq (\gamma + \epsilon)\rho_m(\theta, \lambda) + M_0 \epsilon \sum_{\nu=1}^m \|A^{m-\nu}\| \|b\|^{\nu-1} C^{\nu-1}$$

If we take the supremum of the lefthand side of this equation over all $f = u - v$, $u, v \in C_i$, we find that for all $u, v \in C_i$ we have

$$\|A_\lambda^m u - A_\lambda^m v\|_{\lambda,\theta} \leq \|A^m\| \epsilon + (\gamma + \epsilon)\rho_m(\theta, \lambda) + M_0 \epsilon \sum_{\nu=1}^m \|A^{m-\nu}\| \|b\|^{\nu-1} C^{\nu-1}.$$

It follows that

$$\begin{aligned} \alpha_{\lambda,\theta}(A_\lambda^m(\Gamma)) = \alpha_\lambda(A_\lambda^m(\Gamma)) &= \sup_{1 \leq i \leq p} \alpha_{\lambda,\theta}(A_\lambda^m(C_i)) \\ &\leq \|A^m\| \epsilon + (\gamma + \epsilon)\rho_m(\theta, \lambda) + M_0 \epsilon \sum_{\nu=1}^m \|A^{m-\nu}\| \|b\|^{\nu-1} C^{\nu-1}. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$\alpha_\lambda(A_\lambda^m(\Gamma)) \leq \rho_m(\theta, \lambda) \alpha_\lambda(\Gamma).$$

By taking limits as $\theta \rightarrow 0^+$ we obtain from the previous equation that

$$\alpha_\lambda(A_\lambda^m(\Gamma)) \leq \rho_m(\lambda) \alpha_\lambda(\Gamma),$$

which implies (see eq. (3)) that

$$\alpha_\lambda(A_\lambda^m) \leq \rho_m(\lambda).$$

Since we know (see eq. (4)) that

$$\rho(A_\lambda) = \lim_{m \rightarrow \infty} (\alpha_\lambda(A_\lambda^m))^{\frac{1}{m}} = \inf_{m \geq 1} (\alpha_\lambda(A_\lambda^m))^{\frac{1}{m}},$$

we conclude, using Lemma 5.3, that

$$\rho(A_\lambda) \leq \inf_{m \geq 1} \rho_m(\lambda)^{\frac{1}{m}} = \lim_{m \rightarrow \infty} \rho_m(\lambda)^{\frac{1}{m}} := \bar{\rho}(\lambda).$$

Theorem 3.3 implies that $r(A_\lambda) \geq r(A)$, so, using Lemma 5.1, we have proved the first two displayed equations of Theorem 5.1. If $\rho(A_\lambda) < r(A) := r$, then Theorem 3.3 implies that $r(A_\lambda) = r(A)$. Also, if $\rho(A_\lambda) < r$, Theorem 3.3 implies the existence of u as in the statement of Theorem 5.1, and the fact that $\rho(A_\lambda) < r$ implies that E is finite dimensional. If $\bar{\rho}(\lambda) < r$, the final displayed equation of Theorem 5.1 follows from Lemma 5.3.

If we define $L = r^{-1}A$, the remaining assertions in Theorem 5.1 follow directly from Theorem 3.3. \square

If A is as in Theorem 5.1 and if $\rho(A_\lambda) < r(A)$ and $\|A^k\| \leq Dr^k$ for all $k \geq 1$, we have shown that $L := r^{-1}A$ satisfies all hypotheses of Theorem 3.3. Thus, by using the comments immediately preceding Remark 3.4 in Section 3, we conclude that there exist at most finitely many $z \in \mathbb{C}$ such that $|z| = r(A)$ and z is an eigenvalue of A . Furthermore, every such eigenvalue has finite algebraic multiplicity.

Verifying that $\rho(A_\lambda) < r(A)$ plays a crucial role in applying theorem 5.1. The following corollary gives a trivial case for which $\rho(A_\lambda) < r(A)$ is satisfied.

Corollary 5.1 *Assume that H5.1, H5.2 and H5.3 are satisfied. In addition assume that there exists an integer $m \geq 1$, a constant c with $0 \leq c < 1$, and a positive real θ such that for all $s, t \in S$ with $d(s, t) \leq \theta$ and all $I \in \mathcal{I}_m$,*

$$d(w_I(s), w_I(t)) \leq c^m d(s, t). \quad (27)$$

Assume also that $r(A) > 0$. Then if $0 < \lambda \leq \lambda_0$ (for λ_0 as in H5.3), it follows that $\rho(A_\lambda) < r(A) = r(A_\lambda) := r$. The condition $r(A) > 0$ will be satisfied if

$$\inf_{i \in S} b(t) := \inf_{i \in S} \left(\sum_{i=1}^{\infty} b_i(t) \right) > 0. \quad (28)$$

If $r(A) > 0$, there exists $u \in P(S) \cap X_{\lambda_0}$, $u \neq 0$, with $Au = ru$; and for $0 < \lambda \leq \lambda_0$, $E_\lambda := \{y \in X_\lambda : (rI - A_\lambda)^k y = 0 \text{ for some } k \geq 0\}$ is finite dimensional. If, in addition, there exists $D > 0$ such that $\|A^k\| \leq Dr^k$ for all $k \geq 1$, then statements (a)-(d) of Theorem 5.1 are satisfied.

Proof. Assume first that $r(A) := r > 0$. It suffices, by virtue of Theorem 5.1, to prove that $\bar{\rho}(\lambda) < r$ for $0 < \lambda \leq \lambda_0$. If m is as in eq. (27) and $J \in \mathcal{I}_{mp}$, one derives easily from eq. (27), that, for $0 < d(s, t) \leq \theta$,

$$d(w_J(s), w_J(t)) \leq c^{mp} d(s, t).$$

Using this inequality in eq. (19), we obtain

$$\rho_{mp}(\theta, \lambda) \leq c^{\lambda mp} \sup\left\{ \sum_{J \in \mathcal{I}_{mp}} b_J(t) : 0 < d(s, t) \leq \theta \right\} := c^{\lambda mp} r_{mp}.$$

Since $\rho_{mp}(\theta, \lambda) \geq \rho_{mp}(\lambda)$, we deduce that

$$\rho_{mp}(\lambda) \leq c^{\lambda mp} r_{mp}.$$

Taking n^{th} roots, $n = mp$, letting p approach infinity and using Theorem 5.1, we obtain

$$\rho(A_\lambda) \leq \bar{\rho}(\lambda) = \inf_{m \geq 1} (\rho_m(\lambda))^{\frac{1}{m}} \leq c^\lambda \inf_{m \geq 1} r_m^{\frac{1}{m}} = c^\lambda r(A).$$

Since $r(A) > 0$, this shows that $\bar{\rho}(\lambda) < r(A)$.

To complete the proof, we must also prove that $r(A) > 0$ if $\inf_{t \in S} b(t) := \delta > 0$. However, if e is the function identically equal to one, we have $A(e) \geq \delta e$, which implies that $A^m(e) \geq \delta^m e$ and that $r(A) \geq \delta$. \square

Remark 5.2. Corollary 5.1 is a crude result compared to Theorem 5.1. Suppose, for example, that S is a compact subset of \mathbb{R}^n and that the metric d on S comes from a norm $\|\cdot\|$ on \mathbb{R}^n . Assume that H5.1, H5.2, and H5.3 are satisfied and that each map w_i extends to a C^1 map \tilde{w}_i defined on some open neighborhood U_i of S . If $I = (i_1, i_2, \dots, i_m)$, w_I also has a C^1 extension \tilde{w}_I . Let $\tilde{w}'_I(t)$ denote the Fréchet derivative of \tilde{w}_I at t . If $\rho_m(\lambda)$ is defined by eq. (20), the reader can easily verify that

$$\rho_m(\lambda) \leq \sup\left\{ \sum_{I \in \mathcal{I}_m} b_I(t) \|\tilde{w}'_I(t)\|^\lambda : t \in S \right\} \quad (29)$$

The proof of Theorem 5.1 shows that $\rho(A_\lambda^m) \leq \rho_m(\lambda)$. We know that $\rho(A_\lambda^m) = (\rho(A_\lambda))^m$ and that $r(A^m) = (r(A))^m$, so to prove that $\rho(A_\lambda) < r(A)$, it suffices to prove that there exists $m \geq 1$ with

$$\sup\left\{ \sum_{I \in \mathcal{I}_m} b_I(t) \|\tilde{w}'_I(t)\|^\lambda : t \in S \right\} < r(A^m) = (r(A))^m. \quad (30)$$

It may easily happen that eq. (30) is satisfied for some $m \geq 1$, even though the maps w_i do not satisfy the hypotheses of Corollary 5.1.

To illustrate the point of Remark 5.2, we discuss an example. Select a real number $k \geq 1$, let $S = [0, 1] \subset \mathbb{R}$ and for $t \in S$ define $b_1(t) = t$, $b_2(t) = 1 - t$, $w_1(t) = (1 - t)^k$, $w_2(t) = t^k$ and $b_i(t) = w_i(t) = 0$ for $i > 2$. Then $A : X = C(S) \rightarrow X$ is given by

$$(Ax)(t) = tx((1 - t)^k) + (1 - t)x(t^k), \quad 0 \leq t \leq 1; \quad (31)$$

and one computes that

$$\begin{aligned} (A^2x)(t) &= (1 - t)(1 - t^k)x(t^{k^2}) + (1 - t)t^kx((1 - t^k)^k) + t(1 - (1 - t)^k)x((1 - t)^{k^2}) \\ &\quad + t(1 - t)^kx((1 - (1 - t)^k)^k). \end{aligned} \quad (32)$$

As usual, $A_\lambda : X_\lambda \rightarrow X_\lambda$, $0 < \lambda \leq 1$, is the map induced by A .

Corollary 5.2 *Let $A : X \rightarrow X$ be the map induced by equation (31). For all $k \geq 1$, we have $r(A) = 1$. There exists $\delta > 0$ such that if $k > 2 - \delta$ and $0 < \lambda \leq 1$, then $\rho(A_\lambda) < 1 = r(A)$, A satisfies conditions (a), (b), (c) and (d) of Theorem 5.1 and $N := \{x \in X : (I - A)^j(x) = 0 \text{ for some } j \geq 1\}$ is finite dimensional.*

Proof. We have $A(e) = e$, where e is the function identically equal to one, so $r(A) = 1$ for all $k \geq 1$ and $\|A^m\| \leq 1$ for all $m \geq 1$ and for all $k \geq 1$. Theorem 5.1 shows that in order to prove that $\rho(A_\lambda) < 1$ for $0 < \lambda \leq 1$, it suffices to prove that $\bar{\rho}(1) < 1$, where $\bar{\rho}(\lambda)$ is given by eq. (22). Thus, by eq. (20) and eq. (22), it suffices to prove that $\rho_m(1) < 1$ for some $m \geq 1$. We shall only consider $m = 1$ or $m = 2$. Using Remark 5.2 and eq. (30) for $m = 1$ or $m = 2$ and $\lambda = 1$, we define

$$k\sigma_1(t; k) := kt(1-t)^{k-1} + k(1-t)t^{k-1} \text{ and}$$

$$\begin{aligned} k^2\sigma_2(t; k) &= k^2(1-t)(1-t^k)(t^{k^2-1}) + k^2(1-t)t^k(1-t^k)^{k-1}t^{k-1} \\ &\quad + k^2t[1 - (1-t)^k](1-t)^{k^2-1} + k^2t(1-t)^k[1 - (1-t)^k]^{k-1}(1-t)^{k-1}; \end{aligned}$$

and we note that it suffices to prove that

$$\max\{k\sigma_1(t; k) : 0 \leq t \leq 1\} < 1 \text{ or } \max\{k^2\sigma_2(t; k) : 0 \leq t \leq 1\} < 1.$$

If $k = 2$, we have that $\max\{2\sigma_1(t; 2) : 0 \leq t \leq 1\} = 1$. A calculation shows that

$$4\sigma_2(t; 2) = 16v^2(1-v), \quad v := t(1-t).$$

Since $0 \leq v \leq \frac{1}{4}$ for $0 \leq t \leq 1$ and $v \rightarrow v^2(1-v)$ is increasing for $0 \leq v \leq \frac{2}{3}$, we conclude that

$$\max\{4\sigma_2(t; 2) \mid 0 \leq t \leq 1\} = 16\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) = \frac{3}{4} < 1.$$

By continuity of $k^2\sigma_2(t; k)$ in k , there exists $\delta > 0$ such that if $|k - 2| < \delta$, then

$$\max\{k^2\sigma_2(t; k) : 0 \leq t \leq 1\} < 1.$$

We now assume that $k > 2$. Because $\sigma_1(t; k) = \sigma_1(1-t; k)$, $\sigma_1(\cdot; k)$ achieves its maximum on $[0, 1]$ on $[0, \frac{1}{2}]$ and $\sigma'_1(\frac{1}{2}; k) = 0$. If $2 < k \leq 4$, we claim that $\sigma'_1(t; k) > 0$ for $0 \leq t < \frac{1}{2}$. First assume that $2 < k \leq 3$. Then we have that for $0 \leq t < \frac{1}{2}$

$$\sigma'_1(t; k) = [(1-t)^{k-1} - t^{k-1}] + (k-1)(1-t)^{k-1}\left[\left(\frac{t}{1-t}\right)^{k-2} - \left(\frac{t}{1-t}\right)\right].$$

Because $0 \leq t < \frac{1}{2}$ and $0 < k-2 \leq 1$, we see that $\frac{t}{1-t} < 1$ and $\left(\frac{t}{1-t}\right)^{k-2} \geq \frac{t}{1-t}$. It follows that both bracketed terms in the above equation are nonnegative, and the first term is strictly positive for $0 \leq t < \frac{1}{2}$, so $\sigma'_1(t; k) > 0$ for $0 \leq t < \frac{1}{2}$. If $3 < k \leq 4$, it suffices to prove (since $\sigma'_1(\frac{1}{2}; k) = 0$) that $\sigma''_1(t; k) < 0$ for $0 < t < \frac{1}{2}$. A calculation gives

$$\sigma''_1(t; k) = (k-1)t^{k-3}[-2t + (k-2)(1-t)] + (k-1)(1-t)^{k-3}[-2(1-t) + (k-2)t].$$

Because $(1-t) > t$ for $0 < t < \frac{1}{2}$ and $2 \geq k-2$, we see that $-2(1-t) + (k-2)t < 0$ for $0 < t < \frac{1}{2}$. If $t \in (0, \frac{1}{2})$ is such that $-2t + (k-2)(1-t) \leq 0$, it follows that $\sigma_1''(t; k) < 0$. However, if $t \in (0, \frac{1}{2})$ is such that $-2t + (k-2)(1-t) > 0$, we see that

$$\begin{aligned}\sigma_1''(t; k) &< (k-1)(1-t)^{k-3}[-2t + (k-2)(1-t)] \\ &\quad + (k-1)(1-t)^{k-3}[-2(1-t) + (k-2)t] \\ &= (k-1)(k-4)(1-t)^{k-3} \leq 0.\end{aligned}$$

It follows that for any t with $0 < t < \frac{1}{2}$ and for $3 < k \leq 4$, we have $\sigma_1''(t; k) < 0$. Thus we have proved that for $2 < k \leq 4$,

$$\max\{k\sigma_1(t; k) : 0 \leq t \leq 1\} = k\sigma_1(\frac{1}{2}; k) = k(\frac{1}{2})^{k-1}.$$

We leave to the reader the calculus exercise of proving that $k(\frac{1}{2})^{k-1} < 1$ for all $k > 2$.

It is easy to see that $k \rightarrow \sigma_1(t; k)$ is strictly decreasing for each fixed t with $0 < t < 1$, so we conclude that for $k > 4$ we have

$$\max\{\sigma_1(t; k) : 0 \leq t \leq 1\} < \max\{\sigma_1(t; 4) : 0 \leq t \leq 1\} = \frac{1}{8}.$$

This implies that for $4 \leq k \leq 8$ we have

$$\max\{k\sigma_1(t; k) : 0 \leq t \leq 1\} < 8(\frac{1}{8}) = 1.$$

For $k \geq 8$, it suffices to estimate crudely. We have, for $k \geq 8$,

$$\begin{aligned}\max\{k\sigma_1(t; k) : 0 \leq t \leq 1\} &= \max\{k\sigma_1(t; k) : 0 \leq t \leq \frac{1}{2}\} \\ &\leq \max\{kt(1-t)^{k-1} : 0 \leq t \leq \frac{1}{2}\} \\ &\quad + \max\{k(1-t)t^{k-1} : 0 \leq t \leq \frac{1}{2}\}.\end{aligned}$$

The reader can verify that

$$\max\{kt(1-t)^{k-1} : 0 \leq t \leq \frac{1}{2}\} = (1 - k^{-1})^{k-1}$$

and

$$\max\{k(1-t)t^{k-1} : 0 \leq t \leq \frac{1}{2}\} = k(\frac{1}{2})^k.$$

One can check that $k \rightarrow (1 + xk^{-1})^k$ is an increasing function of k for $k > |x|$, so certainly $k \rightarrow (1 - k^{-1})^k$ is an increasing function of k for $k \geq 8$. Since $\lim_{k \rightarrow \infty} (1 - k^{-1})^k = e^{-1}$, we obtain for $k \geq 8$ that

$$(1 - k^{-1})^{k-1} \leq e^{-1}(\frac{k}{k-1}) \leq (\frac{8}{7})e^{-1}.$$

Similarly, one can prove that $k \rightarrow k(\frac{1}{2})^k$ is a decreasing function of k for $k \geq 8$, so, for $k \geq 8$, we obtain

$$k(\frac{1}{2})^k \leq 8(\frac{1}{2})^8 = \frac{1}{32}.$$

It follows that for $k \geq 8$ we have

$$\max\{k\sigma_1(t; k) : 0 \leq t \leq 1\} \leq \left(\frac{1}{32}\right) + \left(\frac{8}{7}\right)e^{-1} < 1.$$

□

If $k > 1$ and A is given by eq. (31), we conjecture that there exists $\lambda > 0$ such that $\rho(A_\lambda) < 1 = r(A)$. If $k = 1$, A is a projection operator onto $Z := \{x \in X \mid x(1 - s) = x(s) \text{ for } 0 \leq s \leq 1\}$. Since the fixed point set of A is infinite dimensional for $k = 1$, it follows that $\rho(A_\lambda) \geq 1$ for all $\lambda > 0$.

As a contrast to the case that $A : X \rightarrow X$ has periodic points, we now describe conditions which insure that if λ is an eigenvalue of A and $|\lambda| = r(A)$, then $\lambda = r(A)$ and $r(A)$ has algebraic multiplicity one. Assume H5.1, H5.2 and H5.3; and for $0 < \lambda \leq \lambda_0$ and an integer $m \geq 1$ define

$$s_m(\lambda) = \sup\left\{ \sum_{I \in \mathcal{I}_m} b_I(t) \frac{d(w_I(t), w_I(s))^\lambda}{d(s, t)^\lambda} : s, t \in S, s \neq t \right\}. \tag{33}$$

For $\lambda = 1$ the numbers $s_m(1)$ were used by Hennion [6]. If $\theta \geq \text{diameter}(S)$, we have $s_m(\lambda) = \rho_m(\theta, \lambda)$. The same argument used in Lemma 5.3 proves that

$$s_{m+n}(\lambda) \leq s_m(\lambda)s_n(\lambda) \text{ and}$$

$$\tilde{s}(\lambda) := \inf_{m \geq 1} s_m(\lambda)^{\frac{1}{m}} = \lim_{m \rightarrow \infty} s_m(\lambda)^{\frac{1}{m}}. \tag{34}$$

Clearly, we always have $\rho_m(\lambda) \leq s_m(\lambda)$ and $\bar{\rho}(\lambda) \leq \tilde{s}(\lambda)$ for $\rho_m(\lambda)$ and $\bar{\rho}(\lambda)$ as in eq. (20) and eq. (22). However, it may easily happen that $\rho_m(\lambda) < s_m(\lambda)$ or $\bar{\rho}(\lambda) < \tilde{s}(\lambda)$. To see this, let A denote the operator given by eq. (31) and studied in Corollary 5.2. By taking $t = 1$ and $s = 0$ in eq. (33), one can prove that $s_m(\lambda) \geq 1$ for $0 < \lambda \leq \lambda_0$ and $m \geq 1$, so $\tilde{s}(\lambda) \geq 1$ for $0 < \lambda \leq 1 := \lambda_0$. However, if $k \geq 2$, we proved in Corollary 5.2 that either $\rho_1(1) < 1$ or $\rho_2(1) < 1$; and it follows that $\bar{\rho}(1) < 1$. Lemma 5.3 now implies that $\bar{\rho}(\lambda) < 1$ for $0 < \lambda \leq 1$, so, for $k \geq 2$ and $0 < \lambda \leq 1$ we have $\bar{\rho}(\lambda) < \tilde{s}(\lambda)$.

A more straightforward class of examples with $\bar{\rho}(\lambda) < \tilde{s}(\lambda)$ for $0 < \lambda \leq \lambda_0$ can be constructed as follows. Assume H5.1, H5.2 and H5.3 hold. Assume, moreover, that $S = \cup_{j=1}^n S_j$, where S_j is compact and nonempty for $1 \leq j \leq n$ and $S_j \cap S_k = \emptyset$ for $1 \leq j < k \leq n$. Assume that $w_i(S_j) \subset S_j$ and that $w_i|_{S_j}$ is a Lipschitz map with Lipschitz constant $c < 1$ for $1 \leq i$ and $1 \leq j \leq n$. Finally, suppose that there exists $u \in P_c(S)$ and $r = r(A) > 0$ with $Au = ru$. Then we claim that $\tilde{s}(\lambda) \geq r$ for $0 < \lambda \leq \lambda_0$ and $\bar{\rho}(\lambda) < r$ for $0 < \lambda \leq \lambda_0$. If we select $\theta > 0$ such that

$$d(S_j, S_k) := \inf\{d(\sigma, \tau) : \sigma \in S_j, \tau \in S_k\} > \theta$$

for $1 \leq j < k \leq m$, the proof of Corollary 5.1 shows that $\bar{\rho}(\lambda) \leq c^\lambda r$ for $0 < \lambda \leq \lambda_0$. To see that $\tilde{s}(\lambda) \geq r$, select $\delta > 0$ so that $\delta \leq \frac{u(s)}{u(t)} \leq \delta^{-1}$ for all $s, t \in S$. The equation

$$r^k u(t) = \sum_{I \in \mathcal{I}_k} b_I(t) u(w_I(t))$$

then easily implies that for all $t \in S$

$$\delta \sum_{I \in \mathcal{I}_k} b_I(t) \leq r^k \leq \delta^{-1} \sum_{I \in \mathcal{I}_k} b_I(t).$$

One derives from this equation that for all $t, s \in S$ and for all $k \geq 1$,

$$\sum_{I \in \mathcal{I}_k} b_I(t) \leq \delta^{-2} \sum_{I \in \mathcal{I}_k} b_I(s).$$

Using the previous equation and equation (21), we see that for any sequence $(t_m \in S \mid m \geq 1)$ we have

$$\lim_{m \rightarrow \infty} \left(\sum_{I \in \mathcal{I}_m} b_I(t_m) \right)^{\frac{1}{m}} = \lim_{m \rightarrow \infty} r^{\frac{1}{m}} = r.$$

Fix j, k with $1 \leq j < k \leq n$ and select $\sigma \in S_j$ and $\tau \in S_k$ with $d(\sigma, \tau) = d(S_j, S_k)$. For $I \in \mathcal{I}_m$ we know that $w_I(\tau) \in S_k$ and $w_I(\sigma) \in S_j$ so we obtain

$$s_m(\lambda) \geq \sum_{I \in \mathcal{I}_m} b_I(\tau) \frac{d(w_I(\tau), w_I(\sigma))^\lambda}{d(\tau, \sigma)^\lambda} \geq \sum_{I \in \mathcal{I}_m} b_I(\tau).$$

It now follows from our previous remarks that

$$\lim_{m \rightarrow \infty} s_m(\lambda)^{\frac{1}{m}} := \tilde{s}(\lambda) \geq r.$$

Theorem 5.2 *Assume that H5.1, H5.2 and H5.3 hold and let $A : X := C(S) \rightarrow X$ be given by eq. (13). Let $\tilde{s}(\lambda)$ be defined by eq. (34), suppose that there exists $\lambda_1 > 0$, $\lambda_1 \leq \lambda_0$, with $\tilde{s}(\lambda_1) < r(A) := r$ and assume that there exists $u \in P_o(S)$ with $Au = ru$. For each $i \geq 1$ assume that $b_i = 0$ or $b_i \in P_o(S)$. Then if z is an eigenvalue of A and $|z| = r(A)$, it follows that $z = r(A)$. Furthermore, $r(A)$ has algebraic multiplicity one and $u \in X_{\lambda_1}$.*

Proof. By possibly replacing the infinite sum in the definition of $A(x)$ by a finite sum, we can assume that $b_i \in P_o(S)$ for all relevant i and write, for some N with $1 \leq N \leq \infty$

$$(Ax)(t) = \sum_{i=1}^N b_i(t)x(w_i(t)).$$

By replacing A by $r^{-1}A$, we can assume that $r = 1$. We shall assume that $N = \infty$; the proof is the same for $N < \infty$.

Because $\tilde{\rho}(\lambda_1) \leq \tilde{s}(\lambda_1) < 1 = r(A)$ and $u \in P_o(S)$ satisfies $Au = u$, Theorem 5.1 implies that $u \in X_{\lambda_1}$. Define a map $M = M_u : X \rightarrow X$ by $(M_u(x))(t) = u(t)x(t)$, so $(M_u^{-1}(x))(t) = u(t)^{-1}x(t)$. One can easily check that M_u defines a bounded, linear, one-one map of X onto X and of X_{λ_1} onto X_{λ_1} . Furthermore, z is an eigenvalue of A of algebraic multiplicity ν if and only if z is an eigenvalue of $M^{-1}AM$ of algebraic multiplicity ν . A calculation shows that

$$(\bar{A}x)(t) := ((M^{-1}AM)(x))(t) = \sum_{i=1}^N u(t)^{-1}b_i(t)u(w_i(t))x(w_i(t)) := \sum_{i=1}^N \bar{b}_i(t)x(w_i(t)).$$

The reader can verify that \bar{b}_i and w_i , $1 \leq i < \infty$, satisfy H5.1, H5.2, and H5.3, with λ_0 in H5.3 replaced by λ_1 and M_0 by some constant M_1 .

We assume that $\bar{s}(\lambda_1) < 1$, so there exists $n \geq 1$ such that $s_n(\lambda_1) = \gamma^n < 1$. It follows that $s_{np}(\lambda_1) \leq \gamma^{np}$ for $p \geq 1$. Because $\bar{A}^k = M^{-1}A^kM$, we have that

$$(\bar{A}^k x)(t) := \sum_{I \in \mathcal{I}_k} \bar{b}_I(t) x(w_I(t)) = \sum_{I \in \mathcal{I}_k} u(t)^{-1} b_I(t) u(w_I(t)) x(w_I(t)),$$

and $\bar{b}_I(t) = u(t)^{-1} b_I(t) u(w_I(t))$. If $C = \sup\{u(t)^{-1} u(s) : s, t \in S\}$, we conclude that for $t, s \in S$ with $t \neq s$ we have

$$\sum_{I \in \mathcal{I}_{np}} \bar{b}_I(t) \frac{d(w_I(t), w_I(s))^{\lambda_1}}{d(s, t)^{\lambda_1}} \leq C \sum_{I \in \mathcal{I}_{np}} b_I(t) \frac{d(w_I(t), w_I(s))^{\lambda_1}}{d(s, t)^{\lambda_1}} \leq C \gamma^{np}.$$

If $\bar{s}_m(\lambda)$ is obtained by substituting $\bar{b}_I(t)$ for $b_I(t)$ in eq. (33), it follows that $\bar{s}_{np}(\lambda_1) \leq C \gamma^{np}$ and

$$\bar{s}(\lambda_1) := \inf_{m \geq 1} \bar{s}_m(\lambda_1)^{\frac{1}{m}} = \lim_{m \rightarrow \infty} \bar{s}_m(\lambda_1)^{\frac{1}{m}} \leq \gamma < 1.$$

It follows from the above calculations that \bar{A} satisfies the same hypotheses as A but that, in addition, $\bar{A}(e) = e$ and $\sum_{i=1}^{\infty} \bar{b}_i(t) = 1$ for all $t \in S$. Now suppose that $\bar{A}(y) = \alpha y$, where $|\alpha| = 1$, $\|y\| = 1$ and y is complex-valued. Define $S_0 := \{t \in S \mid |y(t)| = 1\}$. If $t \in S_0$, then by using the facts that $|y(w_i(t))| \leq 1$ for all i , $\bar{b}_i(t) > 0$ for all i , and $\sum_{i=1}^{\infty} \bar{b}_i(t) = 1$, we see that $y(w_i(t)) = \alpha y(t)$ for $1 \leq i < \infty$. It follows that $w_i(S_0) \subset S_0$ for $1 \leq i < \infty$ and that if $\alpha \neq 1$, y is not constant on S_0 . If $\alpha = 1$, we assume, by way of contradiction, that y is not constant on S_0 . By using Remark 3.4 and Theorem 3.3 we see that $y \in \tilde{X}_{\lambda_1}$.

Because $y \in \tilde{X}_{\lambda_1}$, we can define $\delta > 0$ by

$$\delta := \sup\left\{ \frac{|y(s) - y(t)|}{d(s, t)^\lambda} : s, t \in S_0, s \neq t, \lambda = \lambda_1 \right\} < \infty.$$

If $s, t \in S_0$, we have $y(w_I(s)) = \alpha^m y(s)$ and $y(w_I(t)) = \alpha^m y(t)$ for all $I \in \mathcal{I}_m$, so

$$|y(s) - y(t)| = \sum_{I \in \mathcal{I}_m} \bar{b}_I(s) |y(w_I(s)) - y(w_I(t))|.$$

Writing $\lambda := \lambda_1$ we know that for all $s, t \in S_0$ with $s \neq t$

$$\frac{|y(w_I(t)) - y(w_I(s))|}{d(s, t)^\lambda} \leq \delta \frac{d(w_I(t), w_I(s))^\lambda}{d(s, t)^\lambda},$$

so we conclude that for $s, t \in S_0$ with $s \neq t$

$$\frac{|y(t) - y(s)|}{d(s, t)^\lambda} \leq \delta \sum_{I \in \mathcal{I}_m} \bar{b}_I(s) \frac{d(w_I(t), w_I(s))^\lambda}{d(s, t)^\lambda} \leq \bar{s}_m(\lambda) \delta.$$

Taking the supremum over $s, t \in S_0$ gives $\delta \leq \bar{s}_m(\lambda) \delta$. Since we know that $\delta > 0$ and we assume that $\bar{s}_m(\lambda) < 1$ for m large, we have obtained a contradiction. It follows that $\alpha = 1$ and that $y|_{S_0}$ is constant. If y is not a scalar multiple of e , then by choosing appropriate scalars

a and b , we can arrange that $y_1 := a(y - be)$ satisfies $\max_{s \in S} y_1(s) = -\min_{s \in S} y_1(s) = 1$. We see then that y_1 is an eigenvector of A with eigenvalue 1 and that y_1 is not constant on $S_1 := \{t \in S : |y_1(t)| = 1\}$, a contradiction. Thus we conclude that $\{v \in X : Av = v\}$ is one dimensional. Since we assume that there exists $u \in P_o(S)$ with $Au = u$, our previous results imply that

$$\{v \in X : Av = v\} = \{v \in X : \exists k \geq 1 \text{ with } (I - A)^k(v) = 0\}.$$

□

It seems difficult to give hypotheses which yield the conclusions of Theorem 5.2 and cover all interesting examples. The following theorem complements Theorem 5.2. As usual, \tilde{X} denotes the complexification of X .

Theorem 5.3 *Assume H5.1, let $w_i : S \rightarrow S$ be continuous maps for $1 \leq i < \infty$ and let $A : X := C(S) \rightarrow X$ be given by eq. (13). Assume the following:*

- (a) *There exists $u \in P_o(S)$ with $Au = ru$, $r = r(A)$.*
- (b) *There exists $m \geq 1$ and $I_* \in \mathcal{I}_m$ such that (1) $b_{I_*}(t) > 0$ for all $t \in S$ and (2) for all $t \in S$, $w_{I_*}^k(t)$ converges as $k \rightarrow \infty$.*

Then, if $\alpha \in \mathbb{C}$ is an eigenvalue of A and $|\alpha| = r(A) := r$, $\alpha^m = r^m$. If w_{I_} has precisely ν fixed points in S , then we have*

$$N := \{y \in \tilde{X} : r^m y = A^m y\} = \{y \in \tilde{X} : \exists k \geq 1 \text{ with } (r^m I - A^m)^k(y) = 0\} \text{ and} \\ \text{dimension}(N) \leq \nu.$$

If w_{I_} has a unique fixed point in S , then $r(A)$ is an eigenvalue of A of algebraic multiplicity one, and $r(A)$ is the only eigenvalue of A of modulus $r(A)$.*

Proof. If $M = M_u$ is as in the proof of Theorem 5.2, define $\bar{A} = M^{-1}AM$, so, for $x \in C(S)$,

$$(\bar{A}x)(t) = \sum_{i=1}^{\infty} \bar{b}_i(t)x(w_I(t)) \text{ and } \sum_{i=1}^{\infty} \bar{b}_i(t) = 1.$$

Just as in the proof of Theorem 5.2 we have that

$$(\bar{A}^k x)(t) = \sum_{I \in \mathcal{I}_k} \bar{b}_I(t)x(w_I(t)) = \sum_{I \in \mathcal{I}_k} u(t)^{-1} b_I(t) u(w_I(t))x(w_I(t)).$$

It follows that $\bar{b}_{I_*}(t) > 0$ for all $t \in S$. Because the spectrum of \bar{A} equals the spectrum of A , with corresponding eigenvalues having the same algebraic multiplicity, it suffices to prove the theorem for \bar{A} instead of A . Also, by replacing \bar{A} by $r^{-1}\bar{A}$, we can assume that $r = 1$.

Suppose that $\bar{A}(y) = \alpha y$, where $|\alpha| = 1$ and $y \neq 0$ is complex-valued. It follows that $\bar{A}^m(y) = \beta y$, $\beta = \alpha^m$, and we have to prove first that $\beta = 1$. We can assume that $\|y\| = 1$, and we define $S_0 := \{t \in S : |y(t)| = 1\}$. If $t \in S_0$, we have that

$$\sum_{I \in \mathcal{I}_m} \bar{b}_I(t)y(w_I(t)) = \beta y(t).$$

The same argument used in Theorem 5.2 shows that $y(w_I(t)) = \beta y(t)$ for all $I \in \mathcal{I}_m$ such that $b_I(t) > 0$. In particular $y(w_{I_*}(t)) = \beta y(t)$ for all $t \in S_0$. This implies that $w_{I_*}(t) \in S_0$ for all $t \in S_0$. Select some $t \in S_0$. By assumption, $\lim_{k \rightarrow \infty} w_{I_*}^k(t) = \tau$ and necessarily $\tau \in S_0$ and $w_{I_*}(\tau) = \tau$. However, if $\beta \neq 1$, this is a contradiction, because $|y(\tau)| = 1$ and

$$y(w_{I_*}(\tau)) = y(\tau) = \beta y(\tau).$$

Next assume that w_{I_*} has precisely $\nu < \infty$ fixed points in S , and let μ denote the dimension of N (over \mathbb{C}). Because $\overline{A}^m(e) = e$ and $e \in P_o(S)$, our previous results imply that

$$N = \{y \in \tilde{X} : \exists k \geq 1 \text{ with } (I - \overline{A}^m)^k(y) = 0\}.$$

It is known that there exist μ linearly independent, real-valued functions y_1, y_2, \dots, y_μ which form a linear basis of N (over \mathbb{C}). We can also assume that $y_1 = e$. If $y \in N$, our previous arguments show that there exists $\tau \in S$ with $w_{I_*}(\tau) = \tau$ and $|y(\tau)| = \|y\|$. Select a fixed point τ_1 of w_{I_*} and define $\tilde{y}_1 = y_1$ and $\tilde{y}_2 = y_2 - \alpha_1 y_1$, where α_1 is chosen so that $\tilde{y}_2(\tau_1) = 0$. By linear independence, $\tilde{y}_2 \neq 0$, and there exists a fixed point τ_2 of w_{I_*} such that $|\tilde{y}_2(\tau_2)| = \|\tilde{y}_2\| \neq 0$. Because $\tilde{y}_2(\tau_2) \neq 0$ and $\tilde{y}_2(\tau_1) = 0$, we see that $\tau_2 \neq \tau_1$. Arguing by induction, assume that we have defined

$$\tilde{y}_j = y_j - \sum_{s=1}^{j-1} c_{js} y_s, \quad 1 \leq j \leq k$$

and have found fixed points $\tau_1, \tau_2, \dots, \tau_k$ of w_{I_*} in such a way that

$$|\tilde{y}_j(\tau_j)| = \|\tilde{y}_j\| \text{ and } \tilde{y}_j(\tau_s) = 0 \text{ for } 1 \leq s < j.$$

These equations imply that the fixed points $\tau_1, \tau_2, \dots, \tau_k$ are all distinct. If $k < \mu$, the reader can easily check that there exist constants $d_{k+1,s}$, $1 \leq s \leq k$, such that if \tilde{y}_{k+1} is defined by

$$\tilde{y}_{k+1} = y_{k+1} - \sum_{s=1}^k d_{k+1,s} \tilde{y}_s,$$

then $\tilde{y}_{k+1}(\tau_j) = 0$ for $1 \leq j \leq k$. By linear independence, $\|\tilde{y}_{k+1}\| \neq 0$, and our inductive hypotheses imply that there exist constants $c_{k+1,s}$ with

$$\tilde{y}_{k+1} = y_{k+1} - \sum_{s=1}^k c_{k+1,s} y_s.$$

It follows that there exists a fixed point τ_{k+1} of w_{I_*} with $|\tilde{y}_{k+1}(\tau_{k+1})| = \|\tilde{y}_{k+1}\| \neq 0$. Since $\tilde{y}_{k+1}(\tau_j) = 0$ for $1 \leq j \leq k$, we see that $\tau_{k+1} \notin \{\tau_j \mid 1 \leq j \leq k\}$. This completes the inductive step. Eventually, we obtain $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_\mu$ and μ distinct fixed points $\tau_1, \tau_2, \dots, \tau_\mu$ of w_{I_*} , so $\mu \leq \nu$.

If w_{I_*} has a unique fixed point, so $\nu = 1$, we have proved that $\dim(N) = 1$; and since

$$N \supset \{y \in \tilde{X} \mid \exists k \geq 1 \text{ with } (I - \overline{A})^k(y) = 0\},$$

1 is an eigenvalue of \bar{A} of algebraic multiplicity one. If $\bar{A}(y) = \alpha y$ for some $y \neq 0$ and some $\alpha \neq 1$ with $|\alpha| = 1$, we have already seen that $\alpha^m = 1$. But this implies that $\bar{A}^m(y) = y$ and y is not a constant function, which contradicts $\dim(N) = 1$. \square

The techniques of proof used in Theorems 5.2 and 5.3 may be applicable even if the exact hypotheses are not satisfied. We illustrate this by considering a previous example.

Corollary 5.3 *For $k > 1$, let $A : X := C(S) \rightarrow X$ be the map given by eq. (31). Then $\alpha = 1$ is the only eigenvalue α of A of modulus 1, and 1 is an eigenvalue of A of algebraic multiplicity one.*

Proof. For any $x \in X$, note that $(Ax)(0) = x(0)$ and $(Ax)(s) = (Ax)(1-s)$ for $0 \leq s \leq 1$. If $A(x) = \alpha x$, where $\|x\| = 1$, $|\alpha| = 1$ and $\alpha \neq 1$, it follows that $x(0) = \alpha x(0)$ and $x(0) = 0$. Since $Ax(0) = Ax(1)$, we find that $x(1) = x(0) = 0$. If $S_0 := \{t \in [0, 1] \mid |x(t)| = \|x\| = 1\}$, it follows that $S_0 \subset (0, 1)$. If $w_2(t) = t^k$, the argument used in the proof of Theorem 5.2 shows that $w_2(S_0) \subset S_0$. If $t \in S_0$, $\lim_{j \rightarrow \infty} w_2^j(t) := \tau$ is an element of S_0 , because S_0 is closed. However, $\tau = 0$ and $0 \notin S_0$, a contradiction.

To complete the proof, it suffices to prove that $N := \{x \in X \mid Ax = x\}$ is one dimensional. Suppose, by way of contradiction, that N contains a nonconstant function x (which can be assumed real-valued). Select $c \in \mathbb{R}$ so that $x_1 := x - ce$ vanishes at $t = 0$. Since $Ax_1 = x_1$, it follows that $x_1(0) = 0 = x_1(1)$. If we now define $S_0 := \{t \in [0, 1] : |x_1(t)| = \|x_1\|\}$, the same argument used in the first part of the proof gives a contradiction. \square

Corollary 5.4 *Let $A : X := C(S) \rightarrow X$ be given by eq. (13) and suppose that $r(A) = 1$. Either assume the hypotheses of Theorem 5.2 or make the following assumptions:*

- (1) *Hypotheses H5.1, H5.2, and H5.3 are satisfied.*
- (2) *There exists $u \in P_0(S)$ with $A(u) = u$.*
- (3) *There exists λ , $0 < \lambda \leq \lambda_0$, with $\bar{\rho}(\lambda) < 1$, where $\bar{\rho}(\lambda)$ is given by eq. (22) and λ_0 is as in H5.3.*
- (4) *There exists $i_* \geq 1$ such that $b_{i_*}(t) > 0$ for all $t \in S$ and $\lim_{k \rightarrow \infty} w_{i_*}^k(t)$ exists for every $t \in S$.*

Then there exists a continuous, finite dimensional linear projection $Q : X \rightarrow X$ of X onto $\{x \in X : Ax = x\}$ such that

$$\lim_{n \rightarrow \infty} \|A^n(x) - Q(x)\| = 0 \text{ for all } x \in X.$$

Furthermore, for every $\mu \in X^$, $(A^*)^n(\mu)$ converges in the weak* topology to $Q^*(\mu)$.*

Proof. By assumption, there exists $u \in P_0(S)$ with $Au = u$. Theorems 5.2 and 5.3 imply that if α is an eigenvalue of A of modulus one, then $\alpha = 1$. We assume that $\bar{\rho}(\lambda) < 1$ or $\bar{s}(\lambda) < 1$ for some λ with $0 < \lambda \leq \lambda_0$. Since $\bar{\rho}(\lambda) \leq \bar{s}(\lambda)$, Theorem 5.1 implies that there exists λ , $0 < \lambda \leq \lambda_0$, with $\rho(A_\lambda) < 1$. The existence of Q and the convergence properties of

$A^n(x)$ as $n \rightarrow \infty$ now follow from Theorem 3.3 and Remark 3.5, and the final statement of the corollary is immediate from the definition of weak* convergence. \square

Remark 5.3. If $S = \{1, 2, \dots, n\}$ with the metric inherited from \mathbb{R} , then one can identify $C(S)$ with \mathbb{R}^n by identifying $f \in C(S)$ with $(f(1), f(2), \dots, f(n))$. If $B = (b_{jk})$ is an $n \times n$ nonnegative matrix, then, writing elements of \mathbb{R}^n as column vectors, B induces a linear map $x \rightarrow Bx$ on \mathbb{R}^n and a corresponding map $A : C(S) \rightarrow C(S)$ by

$$(Af)(s) = \sum_{j \in S} b_j(s) f(w_j(s)),$$

where $b_j(s) := b_{sj}$ and $w_j(s) := j$ for all $s, j \in S$. Clearly, A is a Perron-Frobenius operator of the form in eq. (13). Recall that A is irreducible (in the sense of Section 3) if and only if, for every ordered pair (i, j) with $1 \leq i, j \leq n$, there exists a positive integer $m = m(i, j)$ such that the (i, j) entry of B^m is positive. It is immediate from eq. (33) in this case that $s_1(\lambda) = 0$ for $0 < \lambda \leq 1$, so $\bar{s}(\lambda) = 0$ for $0 < \lambda \leq 1$.

Now suppose that (S, d) is a general compact metric space, that $A : X := C(S) \rightarrow X$ is given by eq. (13) and that H5.1-H5.3 are satisfied. Assume also that A is irreducible and that $\bar{s}(1) < r(A)$, where $\bar{s}(1)$ is as in eq. (33). Because $\bar{\rho}(1) \leq \bar{s}(1)$, Theorem 5.1 implies that there exists $u \in P(S)$, $u \neq 0$, with $Au = ru$ and $r = r(A)$. Irreducibility of A implies that $u \in P_o(S)$. In this situation, Hennion asserts in Theorem 2 of [6] that $r(A)$ is the only eigenvalue of A of modulus $r(A)$. However, even in the case that $S = \{1, 2, \dots, n\}$, this assertion is false: a nonnegative matrix B may have eigenvalues α with $|\alpha| = r(B)$ and $\alpha \neq r(B)$. The 2×2 matrix B defined by $b_{ii} = 0$ and $b_{ij} = 1$ for $i \neq j$ provides the simplest example.

Remark 5.4. Theorems 5.1, 5.2, and 5.3 are slight generalizations of unpublished notes written by the author in 1989 in response to a question from Jeff Geronimo. The theorems generalize results in [1], [6] and [9, 11].

Theorem 5.1 provides little information about the eigenvector $u \in P(S)$. By strengthening the hypotheses in Theorem 5.1, one can obtain much more information about u , in particular proving that $u \in P_o(S)$. We begin with a lemma.

Lemma 5.4 *If $M > 0$, $0 < \lambda \leq 1$, and $\theta > 0$, define*

$$K(M, \theta, \lambda) := \{x \in C(S) : 0 \leq x(s) \leq x(t) \exp(Md(s, t)^\lambda) \forall s, t \in S \text{ with } d(s, t) \leq \theta\}.$$

Then $K := K(M, \theta, \lambda)$ is a closed cone in X (see Section 3 for definitions), and the set $\{x \in K : \|x\| \leq 1\} := B_1 \cap K$ is compact.

Proof. We leave to the reader the exercise of proving that K is a closed cone. To prove that $B_1 \cap K$ is compact, it suffices to prove that $B_1 \cap K$ is equicontinuous. If $x \in B_1 \cap K$, let $S_0 = \{t \in S : x(t) = 0\}$. Clearly S_0 is compact (possibly empty), and the definition of K implies that if $s \in S_0$ and $d(t, s) \leq \theta$, then $t \in S_0$. Thus, if $s, t \in S$ and $d(s, t) \leq \theta$, either (a) $s \in S_0$ and $t \in S_0$ or (b) $s \in S \setminus S_0$ and $t \in S \setminus S_0$. In either case we claim that if $d(s, t) \leq \theta$ and $x \in B_1 \cap K$ we have

$$|x(s) - x(t)| \leq Md(s, t)^\lambda,$$

which implies equicontinuity. The inequality is obvious in case (a). In case (b) the definition of K implies that for $d(s, t) \leq \theta$ we have

$$|\log(x(s)) - \log(x(t))| \leq Md(s, t)^\lambda.$$

We can assume that $0 < x(s) \leq x(t) \leq 1$, and the mean value theorem implies that for some ξ with $\log(x(s)) \leq \xi \leq \log(x(t)) \leq 0$ we have

$$\begin{aligned} |x(s) - x(t)| &= \exp(\log(x(t))) - \exp(\log(x(s))) \\ &= \exp(\xi)[\log(x(t)) - \log(x(s))] \leq Md(s, t)^\lambda. \end{aligned}$$

This proves equicontinuity. \square

Suppose that C is a closed cone in a real Banach space Y and $L : Y \rightarrow Y$ is a bounded linear map with $L(C) \subset C$. Define

$$\begin{aligned} \|L\|_C &= \sup\{\|L(y)\| : y \in C \text{ and } \|y\| \leq 1\} \text{ and} \\ \alpha_C(L) &= \inf\{k \geq 0 : \alpha(L(B)) \leq k\alpha(B) \forall B \subset C, B \text{ bounded}\}. \end{aligned}$$

Define (see [15]) $r_C(L)$, the cone spectral radius of L , and $\rho_C(L)$, the cone essential spectral radius of L , by

$$\begin{aligned} r_C(L) &:= \inf_{n \geq 1} \|L^n\|_C^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|L^n\|_C^{\frac{1}{n}} \text{ and} \\ \rho_C(L) &:= \inf_{n \geq 1} (\alpha_C(L^n))^{\frac{1}{n}} = \lim_{n \rightarrow \infty} (\alpha_C(L^n))^{\frac{1}{n}}. \end{aligned}$$

It is proved in [15] that if $\rho_C(L) < r_C(L)$, then there exists $y \in C \setminus \{0\}$ with $L(y) = ry$, $r = r_C(L)$. In the case that $L|_C$ is compact, this was proved by Bonsall [3].

Theorem 5.4 *Let (S, d) be a compact metric space and suppose that $A : X := C(S) \rightarrow X$ is given by eq. (13). Assume the following:*

- (1) *H5.1 and H5.2 are satisfied.*
- (2) *There exist $M_0 > 0$, $\theta > 0$, λ with $0 < \lambda < 1$ and an integer n such that for all $I \in \mathcal{I}_n$, we have $b_I \in K(M_0, \theta, \lambda)$, where $K(M_0, \theta, \lambda)$ is as in Lemma 5.4.*
- (3) *For n and θ as in (2), there exists c with $0 < c < 1$ such that*

$$d(w_I(t), w_I(s)) \leq c^n d(t, s)$$

for all $t, s \in S$ with $d(t, s) \leq \theta$ and for all $I \in \mathcal{I}_n$.

- (4) *For all $t \in S$, $\sum_{I \in \mathcal{I}_n} b_I(t) > 0$.*

Then there exists $M > 0$ such that $A^n(K(M, \theta, \lambda)) \subset K(M, \theta, \lambda)$, and A^n has an eigenvector $u \in K(M, \theta, \lambda)$ with eigenvalue $(r(A))^n := r^n > 0$. If S is connected, $u(s) > 0$ for all $s \in S$; and in general $S_0 := \{s \in S : u(s) = 0\}$ is open and closed. If $\theta \geq \text{diameter}(S)$, $u(s) > 0$ for all $s \in S$. The operator A has an eigenvector

$$v = \sum_{j=0}^{n-1} r^{-j} A^j(u) \in P(S)$$

with eigenvalue $r = r(A)$.

Proof. Select M so that $M_0 + Mc^{n\lambda} \leq M$. We claim that $A^n(K) \subset K$, where $K := K(M, \theta, \lambda)$. If $I \in \mathcal{I}_n$ and $s, t \in S$ satisfy $d(s, t) \leq \theta$, then

$$d(w_I(t), w_I(s)) \leq c^n d(s, t) < \theta.$$

It follows that if $d(s, t) \leq \theta$ and $x \in K$ and $I \in \mathcal{I}_n$, then we have

$$\begin{aligned} x(w_I(s)) &\leq x(w_I(t)) \exp(Md(w_I(s), w_I(t))^\lambda) \leq x(w_I(t)) \exp(Mc^{n\lambda} d(s, t)^\lambda) \text{ and} \\ b_I(s) &\leq b_I(t) \exp(M_0 d(s, t)^\lambda). \end{aligned}$$

Using these inequalities, we find that

$$\begin{aligned} (A^n x)(s) &= \sum_{I \in \mathcal{I}_n} b_I(s) x(w_I(s)) = \sum_{I \in \mathcal{I}_n} b_I(t) x(w_I(t)) \exp((M_0 + c^{n\lambda} M) d(s, t)^\lambda) \\ &\leq ((A^n x)(t)) \exp(Md(s, t)^\lambda). \end{aligned}$$

The above inequality proves that $A^n(K) \subset K$.

By Lemma 5.4, the set $\{x \in K : \|x\| \leq 1\}$ is compact, so $A^n|_K$ is compact and $\rho_K(A^n) = 0$. The constant function e is in K , so

$$r_K(A^n) \geq \lim_{j \rightarrow \infty} \|A^{nj}(e)\| = r(A^n) = (r(A))^n.$$

The opposite inequality is obvious, so we conclude that $r_K(A^n) = (r(A))^n$. By assumption (4) and H5.1, there exists $\delta > 0$ with

$$\sum_{I \in \mathcal{I}_n} b_I(t) \geq \delta,$$

and it follows that $A^n(e) \geq \delta e$ and $r(A^n) \geq \delta$. Using the remarks which preceded Theorem 5.4, we conclude that there exists $u \in K$, $\|u\| = 1$, with $A^n(u) = r^n u$, and $r = r(A)$. The proof of Lemma 5.4 showed that $K \subset X_\lambda$, so $u \in X_\lambda$. The proof of Lemma 5.4 also showed that $S_0 = \{t \in S \mid u(t) = 0\}$ is open and closed, so if S is connected, S_0 must be empty. Similarly, we saw that $S_0 = \{t \in S \mid d(t, S_0) \leq \theta\}$, so if $\theta \geq \text{diam}(S)$, S_0 must be empty. Finally, the fact that v is an eigenvector of A with eigenvalue r follows by a straightforward calculation. \square

6 Perron-Frobenius Operators: Existence of Periodic Points

We now wish to give a version of Theorem 3.1 for Perron-Frobenius operators. In order to do this properly, we need a slight generalization of Lemma 3.2 in which we arrange that the functions f_i of Lemma 3.2 are Lipschitzian.

Lemma 6.1 *Let $m \geq 2$ be an integer and suppose that E_i , $0 \leq i \leq m$, are closed, nonempty subsets of a compact metric space (S, d) and that $E_m = E_0$. Assume also that $\bigcap_{i=0}^{m-1} E_i = \emptyset$ and let r be a positive real number. Then there exist positive, Lipschitzian functions $f_i : S \rightarrow (0, \infty)$, $0 \leq i \leq m$, with $f_m = f_0$ and $f_i(s) = r f_{i-1}(s)$ for all $s \in E_i$, $1 \leq i \leq m$.*

Proof. A compactness argument shows that there exists $\delta > 0$ such that $\cap_{i=1}^{m-1} V_\delta(E_i) = \emptyset$, where $V_\delta(E_i) := \{s \in S : d(s, E_i) \leq \delta\}$. Define $h_i(t) = \log(r)$ for all $t \in V_\delta(E_i)$. Because $\cap_{i=0}^{m-1} V_\delta(E_i) = \emptyset$, Lemma 3.1 implies that there exist continuous extensions $\hat{h}_i : S \rightarrow \mathbb{R}$ with $\hat{h}_i(t) = \log(r)$ for all $t \in V_\delta(E_i)$ and $\sum_{i=0}^{m-1} \hat{h}_i(t) = 0$ for all $t \in S$.

By compactness, there exists a finite open covering $\{B_{\frac{\delta}{2}}(s_k) : 1 \leq k \leq n\}$ of S , where $B_{\frac{\delta}{2}}(s_k)$ denotes an open ball of radius $\frac{\delta}{2}$ and center s_k . As is well-known, there exists a Lipschitz partition of unity subordinate to this covering, so there exist nonnegative, Lipschitz functions $\phi_k : S \rightarrow [0, \infty)$, $1 \leq k \leq n$, with $\text{support}(\phi_k) \subset B_{\frac{\delta}{2}}(s_k)$ and $\sum_{k=1}^n \phi_k(t) = 1$ for all $t \in S$. We now define functions \tilde{h}_i , $0 \leq i \leq m$, by

$$\tilde{h}_i(s) = \sum_{k=1}^n \phi_k(s) \hat{h}_i(s_k).$$

Because ϕ_k , $1 \leq k \leq n$, is Lipschitzian, the functions \tilde{h}_i , $1 \leq i \leq m$, are Lipschitzian. We claim that (a) $\sum_{i=1}^m \tilde{h}_i(t) = 0$ for all $t \in S$ and (b) $\tilde{h}_i(t) = \log(r)$ for all $t \in E_i$, $1 \leq i \leq m$. Claim (a) follows from the corresponding fact for the functions \hat{h}_i :

$$\sum_{i=0}^{m-1} \tilde{h}_i(s) = \sum_{i=0}^{m-1} \sum_{k=1}^n \phi_k(s) \hat{h}_i(s_k) = \sum_{k=1}^n \phi_k(s) \left(\sum_{i=0}^{m-1} \hat{h}_i(s_k) \right) = 0.$$

For claim (b), notice that if $t \in E_i$ and $\phi_k(t) \neq 0$, then $d(t, s_k) < \frac{\delta}{2}$ and $s_k \in V_\delta(E_i)$ and $\hat{h}_i(s_k) = \log(r)$. It follows that, for $t \in E_i$,

$$\tilde{h}_i(t) = \sum_{k=1}^n \phi_k(t) \hat{h}_i(s_k) = \sum_{k=1}^n \phi_k(t) (\log(r)) = \log(r).$$

We define $g_i(t) = \exp(\tilde{h}_i(t))$ and note that g_i is Lipschitz. If we now argue as in the proof of Lemma 3.2, the conclusion of Lemma 6.1 follows. \square

We can now state a version of Theorem 3.1 for Perron-Frobenius operators. As usual, A_λ is the map induced by A on the space X_λ of Hölder continuous functions on S .

Theorem 6.1 *Let (S, d) be a compact metric space and assume that the following conditions hold:*

- (a) *Hypotheses 5.1, 5.2 and 5.3 are satisfied and $A : X := C(S) \rightarrow X$ is defined by eq. (13).*
- (b) *For λ_0 as in H5.3 and $\bar{\rho}(\lambda)$ as in eq. (22), there exists λ , $0 < \lambda \leq \lambda_0$, such that either $\bar{\rho}(\lambda) < r(A)$ or, more generally, $\rho(A_\lambda) < r(A)$.*
- (c) *There exist closed, nonempty sets $E_i \subset S$, $0 \leq i \leq m$, m a prime number, such that $E_m = E_0$, $\cap_{i=1}^m E_i = \emptyset$, and whenever $f \in C(S)$ and $f|_{E_i} = 0$ for some i with $0 \leq i \leq m-1$, it follows that $Af|_{E_{i+1}} = 0$.*
- (d) *The spectral radius $r(A)$ of A equals one, and there exists $\theta \in P(S) \cap X_\lambda$ with $A(\theta) = \theta$ and $\theta(s_0) > 0$ for some $s_0 \in E := \cup_{i=1}^m E_i$.*

Then there exist $a > 0$ and $b > 0$ and a periodic point $g_0 \in X_\lambda$ of A of minimal period m with $a\theta \leq g_0 \leq b\theta$.

Note that Theorem 3.3 implies that if $r(A) = 1$ and conditions (a) and (b) of Theorem 6.1 are satisfied, then there exists $\theta \in (P(S) \setminus \{0\}) \cap X_\lambda$ with $A(\theta) = \theta$. Theorems 3.3 and 3.4 provide conditions under which one can also guarantee that $\theta(s_0) > 0$ for some $s_0 \in E$.

Proof of Theorem 6.1. By Lemma 6.1, there exist Lipschitzian functions $f_i \in P_c(S)$, $0 \leq i \leq m$, with $f_m = f_0$ and $f_i|E_i = r f_{i-1}|E_i$ for $1 \leq i \leq m$, where r is a positive real number, $r \neq 1$. Replacing the functions f_i by $\tilde{f}_i := \theta f_i$, we see that for $1 \leq i \leq m$, $\tilde{f}_i \in X_\lambda$, $\tilde{f}_i|E_i = r \tilde{f}_{i-1}|E_i$, and there exist positive reals a and b with $a\theta \leq \tilde{f}_i \leq b\theta$.

We now proceed as in the proof of Theorem 3.1. Define $Y_\lambda = \prod_{i=0}^{m-1} X_\lambda$ to be the Banach space of ordered m -tuples $y = (x_0, x_1, \dots, x_{m-1})$ of elements of X_λ , and let $\Phi_\lambda : Y_\lambda \rightarrow Y_\lambda$ be defined by

$$\Phi_\lambda((x_0, x_1, \dots, x_{m-1})) = (A_\lambda x_{m-1}, A_\lambda x_0, A_\lambda x_1, \dots, A_\lambda x_{m-2}).$$

Define a set $C_\lambda \subset Y_\lambda$ by

$$C_\lambda := \{(g_0, g_1, \dots, g_{m-1}) \in Y_\lambda : a\theta \leq g_j \leq b\theta \text{ and } g_j|E_j = r g_{j-1}|E_j \text{ for } 1 \leq j \leq m\}.$$

As usual, in the previous equation indices are written mod m , so $g_m = g_0$. Our previous remarks show that $(\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{m-1}) \in C_\lambda$, so $C_\lambda \neq \emptyset$. One easily sees that C_λ is closed and convex, and the same argument as in Theorem 3.1 shows that $\Phi_\lambda(C_\lambda) \subset C_\lambda$.

If $g = (g_0, g_1, \dots, g_{m-1}) \in C_\lambda$, note that

$$\Phi_\lambda^{km}(g) = (A_\lambda^{km} g_0, A_\lambda^{km} g_1, \dots, A_\lambda^{km} g_{m-1}).$$

Note also that A^m is a Perron-Frobenius operator and that $\rho(A_\lambda^m) = (\rho(A_\lambda))^m < 1$. If i is the inclusion map of X_λ into X , we obviously have that

$$\|i A_\lambda^{km}(g_j)\| \leq b \|\theta\| \text{ for } k \geq 1,$$

so Theorem 5.1 implies that $\{A_\lambda^{km}(g_j) : k \geq 0\} := S_j$ is bounded in the X_λ norm for each j , $0 \leq j \leq m-1$. Since $\rho(A_\lambda^m) < 1$, select $\nu \geq 1$ and $c < 1$ such that $\alpha(A_\lambda^\nu) = c < 1$. (Here α denotes the measure of noncompactness in X_λ .) Since $A_\lambda^{\nu m}(S_j)$ differs from S_j by only a finite set, we have $\alpha(A_\lambda^{\nu m}(S_j)) = \alpha(S_j)$, so we conclude that

$$\alpha(S_j) = \alpha(A_\lambda^{\nu m}(S_j)) \leq c\alpha(S_j),$$

and $\alpha(S_j) = 0$, $0 \leq j \leq m-1$. It follows that $\{\Phi_\lambda^{km}(g) | k \geq 0\}$ has compact closure in X_λ for every $g \in C_\lambda$. Since $\Phi_\lambda^\mu(g) \in C_\lambda$ if $0 \leq \mu \leq m-1$ and $g \in C_\lambda$, it follows that $\{\Phi_\lambda^{km+\mu}(g) : k \geq 0\}$ has compact closure in X_λ for $g \in C_\lambda$ and $0 \leq \mu \leq m-1$. One derives from this that $\{\Phi_\lambda^j(g) : j \geq 0\}$ has compact closure in X_λ for every $g \in C_\lambda$.

Now take a fixed $h \in C_\lambda$ and define D_λ by

$$D_\lambda := \overline{\text{co}}(\{\Phi_\lambda^j(h) | j \geq 0\}) \subset C_\lambda.$$

Mazur's theorem implies that D_λ is a compact, convex set in X_λ , and one easily checks that $\Phi_\lambda(D_\lambda) \subset D_\lambda$, so Schauder's fixed point theorem implies that Φ_λ has a fixed point $g_0 \in D_\lambda$. The remainder of the proof now follows as in Theorem 3.1. \square

Our next corollary gives a simple situation in which condition (c) of Theorem 6.1 is satisfied.

Corollary 6.1 *Let hypotheses be as in Theorem 6.1, except replace condition (c) by the following assumption:*

(γ) *For a prime number m , there exist closed, nonempty sets $E_j \subset S$, $0 \leq j \leq m$, with $E_m = E_0$ and $\bigcap_{j=1}^m E_j = \emptyset$ such that $w_i(E_j) \subset E_{j-1}$ for $1 \leq j \leq m$ and $i \geq 0$.*

Then the conclusion of Theorem 6.1 remains valid.

Proof. The reader can verify that condition (γ) of Corollary 6.1 implies condition (c) of Theorem 6.1. \square

Corollary 6.1 shows why the assumptions of Corollary 5.1 are much too restrictive if one is interested in periodic points. For suppose condition (γ) of Corollary 6.1 is satisfied. Then there cannot exist a positive integer k and $I \in \mathcal{I}_k$ such that w_I is a Lipschitz map with Lipschitz constant $c < 1$. For suppose, by way of contradiction, that such an I exists. Condition (γ) implies that $w_I^m(E_k) \subset E_k$ for $1 \leq k \leq m$. Because w_I^m is a contraction, w_I^m has a fixed point $z_k \in E_k$ for $1 \leq k \leq m$; and the condition that $\bigcap_{k=1}^m E_k = \emptyset$ implies that there must exist $1 \leq k < l \leq m$ with $z_k \neq z_l$. Thus the contraction mapping w_I^m has two distinct fixed points, a contradiction.

To illustrate the use of Theorem 6.1 and Corollary 6.1 we consider a slight variant of the operator studied in Corollary 5.2. Let $S = [0, 1]$ and let $k \geq 1$ denote a fixed real number. Define $A : X := C(S) \rightarrow X$ by

$$(Ax)(t) = tx((1-t)^k) + (1-t)x(1-t^k) \quad (35)$$

A calculation yields

$$\begin{aligned} (A^2x)(t) &= t(1-t)^k x((1-(1-t)^k)^k) + t(1-(1-t)^k)x(1-(1-t)^{k^2}) \\ &\quad + (1-t)(1-t^k)x(t^{k^2}) + (1-t)t^k x(1-(1-t^k)^k). \end{aligned}$$

Corollary 6.2 *Let $A : X := C([0, 1]) \rightarrow X$ be the map given by eq. (35). Then for all $k \geq 1$ we have $r(A) = 1$. There exists δ , $0 < \delta < 1$, such that if $k > 2 - \delta$ and $0 < \lambda \leq 1$, then $\rho(A_\lambda) < 1 = r(A)$ and A satisfies conditions (a), (b), (c) and (d) of Theorem 5.1. If $k > 2 - \delta$, the map A has a periodic point $g_0 \in P_o(S)$ of minimal period 2. If $k > 1$ and $\alpha \in \mathbb{C}$ is an eigenvalue of A and $|\alpha| = 1$, then $\alpha = 1$ or $\alpha = -1$. If $k > 1$, 1 is an eigenvalue of A of algebraic multiplicity one; and if -1 is an eigenvalue of A , it is of algebraic multiplicity 1. If $k > 2 - \delta$, -1 is an eigenvalue of A .*

Proof. We argue as in Corollary 5.2. If e denotes the function identically equal to one, then $A(e) = e$, so $r(A) = 1$. By Theorem 5.1, we will obtain that $\rho(A_\lambda) < 1$ for $0 < \lambda \leq 1$ if we can prove that $\rho_m(1) < 1$ for some $m \geq 1$ (see equations (19) and (20)). Using Remark

5.2 and eq. (30) for $m = 1$ or $m = 2$ and $\lambda = 1$, we find that it suffices to prove that $\max\{k\sigma_1(t; k) : 0 \leq t \leq 1\} < 1$ or $\max\{k^2\sigma_2(t; k) : 0 \leq t \leq 1\} < 1$, where $\sigma_1(t; k)$ and $\sigma_2(t; k)$ are as in the proof of Corollary 5.2. Thus it follows from the proof of Corollary 5.2 that there exists $\delta, 0 < \delta < 1$, such that if $k > 2 - \delta$, then $\rho(A_\lambda) < 1$ for $0 < \lambda \leq 1$. Theorem 5.1 now implies that (for $k > 2 - \delta$) A satisfies (a), (b), (c) and (d) of Theorem 5.1.

Now define $E_0 = \{0\}$ and $E_1 = \{1\}$. For any $x \in X$, $(Ax)(1) = x(0)$ and $(Ax)(0) = x(1)$, so one easily sees that E_0 and E_1 satisfy condition (c) of Theorem 6.1 with $m = 2$. We have already checked the other hypotheses of Theorem 6.1, so (for $k > 2 - \delta$) A has a periodic point $g_0 \in P_\circ(S)$ of minimal period 2.

If $k > 2 - \delta$ and $g_1 := A(g_0)$, then $g_0 - g_1$ is an eigenvector of A with eigenvalue -1 . If $k > 1$ and $\alpha \in \mathbb{C}$ is an eigenvalue of \tilde{A} of modulus one and if $u \in \tilde{X}$ is a corresponding eigenvector, we obtain from our formula for A^2 that

$$(\tilde{A}^2 u)(0) = u(0) = \alpha^2 u(0) \text{ and } (\tilde{A}^2 u)(1) = u(1) = \alpha^2 u(1).$$

If $u(0) \neq 0$ or $u(1) \neq 0$, it follows that $\alpha^2 = 1$. If $u(0) = 0$ and $u(1) = 0$, we argue as in Theorem 5.3 and Corollary 5.3. Let $M = \max\{|u(t)| : t \in S\}$ and $\Sigma_0 = \{t \in [0, 1] : |u(t)| = M\}$. Our assumptions imply that $\Sigma_0 \subset (0, 1)$. Because $|(A^2 u)(t)| = |u(t)|$, arguing as in Theorem 5.3 and using the formula for A^2 we see that if $t \in \Sigma_0$, then $t^{k^2} \in \Sigma_0$. Iterating, we find that $t^{k^{2n}} \in \Sigma_0$; and since $k > 1$, we obtain by letting $n \rightarrow \infty$ that $0 \in \Sigma_0$, a contradiction.

Thus we have proved that if $k > 1$ and u is an eigenvector of \tilde{A}^2 with eigenvalue $\beta \in \mathbb{C}$ of modulus one, then $\beta = 1$ and $u(0) \neq 0$ or $u(1) \neq 0$.

Suppose now that we have two linearly independent fixed points e and g of \tilde{A}^2 . If $h_0 := g - g(0)e$, we must have that $h_0(1) \neq 0$; otherwise, h_0 will be a fixed point of \tilde{A}^2 which vanishes at 0 and 1. By multiplying h_0 by a constant, we obtain a fixed point h of \tilde{A}^2 with $h(0) = 0$ and $h(1) = 1$. If v is any fixed point of \tilde{A}^2 , there exist constants c and d such that $v - ce - dh$ vanishes at 0 and 1. Since $v - ce - dh$ is a fixed point of \tilde{A}^2 , it follows from our previous remarks that $v = ce + dh$, so $\{x \in \tilde{X} \mid \tilde{A}^2(x) = x\}$ is two dimensional. Because $\tilde{A}^2(e) = e$ and $e \in P_\circ(S)$, we know that $\{\|A^k\| : k \geq 1\}$ is bounded, and our previous theorems imply that

$$\{x \in \tilde{X} \mid \tilde{A}^2(x) = x\} = \{x \in \tilde{X} \mid (I - \tilde{A}^2)^j(x) = x \text{ for some } j \geq 0\}.$$

If A has an eigenvalue -1 , g must correspond to an eigenvalue -1 , and the corollary follows easily. If A does not have an eigenvalue -1 , there cannot exist an element g as above. For if g exists and -1 is not an eigenvalue, one easily argues (work in the linear space spanned by g and $A(g)$) that $A(g) = g$. However, this implies that $g(0) = g(1)$, which gives the contradiction that $h_0(1) = 0$. It follows that if -1 is not an eigenvalue of A , 1 is still an eigenvalue of \tilde{A} of algebraic multiplicity one. \square

As in the case of Corollary 5.2, we conjecture that if $k > 1$, A has a periodic point of minimal period 2. If $k = 1$, $Ax(t) = x(1 - t)$ and 1 and -1 are eigenvalues of infinite multiplicity.

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Department of Mathematics, Hill Center
Rutgers University
110 Frelinghuysen Road
Piscataway, NJ 08854-8019

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