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A DIFFERENTIAL-DELAY EQUATION ARISING IN OPTICS AND PHYSIOLOGY*

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Abstract. In recent papers the authors have studied differential-delay equations E_ε of the form $\varepsilon\dot{x}(t) = -x(t)+f(x(t-1))$. For functions like $f(x) = \mu_1 + \mu_2 \sin(\mu_3 x + \mu_4)$, such equations arise in optics, while for choices like $f(x) = \mu x^\nu e^{-x}$ and $f(x) = \mu x^\nu(1+x^\lambda)^{-1}$ and for $x \geqq 0$, the equation has been suggested in physiological models. Under varying hypotheses on f (labeled (I), (II), and (III) below), previous work has given theorems concerning existence and asymptotic properties as $\varepsilon \to 0^+$ of periodic solutions of E_ε which oscillate about a value α such that $f(\alpha) = \alpha$. However, verifying (I), (II), or (III) for specific examples can be difficult. This paper gives general principles that help in verifying (I), (II), or (III), and then applies these results to specific classes of functions of interest.

Key words. singularly perturbed differential-delay equation, slowly oscillating periodic solution, Schwarzian derivative

AMS(MOS) subject classifications. 26A18, 34K15, 34K25

1. Introduction. The singularly perturbed differential-delay equation

$$\varepsilon\dot{x}(t) = -x(t)+f(x(t-1)), \tag{1.1}$$

which arises in various models in optics, biology, and physiology, has been studied by many authors. See, for example, [2], [4], [5], [7]-[14], [17]-[22], [24], [25], and the references in [20]-[22]. Recently, Mallet-Paret and Nussbaum [20]-[22] have explored the relation between (1.1) and the discrete system

$$x_n = f(x_{n-1}) \tag{1.2}$$

obtained by formally setting $\varepsilon = 0$ in (1.1). Some of the main results of [20], [21] concern the existence and asymptotic behavior of square-wavelike periodic solutions of (1.1) for small ε. However, these results require that f satisfy various hypotheses, which will be given in § 2 below and which may be nontrivial to verify. Typical nonlinearities of interest are

$$f(x) = \mu_1 + \mu_2 \sin(\mu_3 x + \mu_4), \tag{1.3}$$

which arise in optics, and

$$f(x) = \mu x^\nu e^{-x}, \quad x \geqq 0, \tag{1.4}$$

$$f(x) = \mu x^\nu (1+x^\lambda)^{-1}, \quad x \geqq 0, \tag{1.5}$$

which arise in biological and physiological models. See, for example, [16], where the function in (1.5) is used in (1.1) (for $\nu = 0$ or $\nu = 1$, $\lambda > 0$ and $\mu > 0$) to model blood diseases. (Note that various constants appear in the equations in [16], but that by

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change of variables the equations (4a) and (4b) in [16] are subsumed by our equation (1.1) with f as in (1.5).)

Unfortunately, verifying the hypotheses of § 2 even for the above simple-looking functions is not trivial and was not carried out in [21] (primarily for reasons of space). For example, one of our hypotheses involves global conditions expressed as qualitative properties of the dynamical system (1.2) and may be hard to check. It seems a significant body of theory is needed, even for the functions in (1.3)–(1.5), to determine exactly when our hypotheses are satisfied; routine calculations are insufficient. Our purpose here is to develop such a theory and then to apply it to determine parameter values for which the above nonlinearities satisfy various hypotheses. Although we have not given actual numerical ranges of parameters where our hypotheses are satisfied, we can, with a simple computer program easily obtain most of them from our results. Thus, this paper may be viewed as a companion to [21], for here we show how to apply the general results of [21] to specific systems of scientific interest.

Our interest naturally extends beyond the nonlinearities in (1.3)–(1.5); however, because so many of the basic difficulties are already apparent for these nonlinearities, we will view them as models and work out their theory in as much detail as possible. Even so, we will leave open questions for these examples.

2. Hypotheses on f and their implications. The following hypotheses were shown in [20], [21] to imply various results about the differential equation (1.1). Note that these hypotheses are arranged in increasing order of strength, and that all assume the condition $f(0)=0$. This assumption is merely a normalization; more generally the functions of interest will have a nonzero fixed point $f(x_0)=x_0$, and it will be necessary to translate this point to the origin before analyzing the function.

We say a function f is *monotone decreasing* in an interval I in case $f(x_1)\geqq f(x_2)$ whenever $x_1<x_2$ and $x_1, x_2\in I$. We say f is *strictly decreasing* in I in case $f(x_1)>f(x_2)$ for all such x_1 and x_2. We make analogous definitions of *monotone increasing* and *strictly increasing.*

We let $f^n:\mathbb{R}\to\mathbb{R}$ denote the n-fold composition of the function f with itself.

We now present four hypotheses a function f can satisfy. These were introduced in [20], [21].

(0) The function $f:\mathbb{R}\to\mathbb{R}$ is continuous, satisfies $f(0)=0$, is differentiable at $x=0$ satisfying $f'(0)<-1$, and is monotone decreasing in some neighborhood of $x=0$.

(I) The function f satisfies hypothesis (0). In addition there exist numbers $A>0$ and $B>0$ such that

$$f([-B, A])\subseteq[-B, A],$$
$$xf(x)<0 \quad \text{if } x\in[-B, A]-\{0\}.$$

(II) There exist A and B such that (I) holds. In addition there exist positive numbers $a\leqq A$ and $b\leqq B$ such that if $x_0\in[-B, A]-\{0\}$ and x_n is given by (1.2), then

$$f^n(x)=x_n\to\{-b, a\} \quad \text{as } n\to\infty.$$

(III) There exist A and B such that (I) holds. In addition f is monotone decreasing in $[-B, A]$ and (II) holds with $a=A$ and $b=B$.

Note that $f(a)=-b$ and $f(-b)=a$ must hold in hypothesis (II). If f only satisfies (I), then there still must exist a and b satisfying $f(a)=-b$ and $f(-b)=a$, respectively. However, the orbit $\{-b, a\}$ of (1.2) need not be stable and attract iterates x_n, and a and b need not be unique.

If (I) holds and f is monotone decreasing on $[-B, A]$ and if f^2 has a unique positive fixed point $a \in (0, A]$, then it is easy to show that (III) is satisfied. To see this, first observe that f^2 has a unique negative fixed point $-b \in [-B, 0)$, $-b = f(a)$. If $-b_1$ and $-b_2$ were negative fixed points of f^2, then $f(-b_1)$ and $f(-b_2)$ would be positive fixed points of f^2, so

$$a = f(-b_1) = f(-b_2),$$

and we could conclude that

$$f(a) = f^2(-b_1) = -b_1 = f^2(-b_2) = -b_2.$$

Next, note that f^2 is monotone increasing on $[-B, A]$ (because f is monotone decreasing) and that there exists $\varepsilon > 0$ such that $|f^2(x)| > x$ for $0 < |x| < \varepsilon$ (because f is monotone decreasing and $f'(0) < -1$). It follows that $f^2(x) > x$ for $0 < x < a$; otherwise, the intermediate value theorem would imply that f^2 has a positive fixed point x_1, with $0 < x_1 < a$. If $0 < y_0 < a$ and $y_n = f^{2n}(y_0)$ we conclude that

$$y_0 < f^2(y_0) = y_1 < f^2(a) = a,$$

and generally that

$$y_n < y_{n+1} < a \quad \forall n \geqq 1.$$

It follows that y_n converges to a limit y, and since $f^2(y) = y$, it must be true that $y = a$. A similar argument shows that if $-b < z_0 < 0$, then

$$\lim_{n\to\infty} f^{2n}(z_0) = -b.$$

Finally, we can deduce that if $-b \leqq x \leqq a$ and $x \neq 0$, then

$$\lim_{n\to\infty} f^n(x) = \{-b, a\}.$$

In fact we can conclude slightly more. If $A > a$, the uniqueness of the positive fixed point of f implies that $f^2(A) < A$ (we know $f^2(A) \leqq A$). Thus the intermediate value theorem implies that if $a < x \leqq A$, $f^2(x) < x$. Using this fact and the fact that f^2 is monotone increasing, we see that if $a < y_0 \leqq A$ and $y_n = f^{2n}(y_0)$, then

$$a < y_{n+1} < y_n \quad \text{for all } n.$$

As before this implies $y_n \to a$. A similar argument shows that if $-B \leqq z_0 < -b$, then

$$\lim_{n\to\infty} f^{2n}(z_0) = -b.$$

Finally, we can conclude that if $-B \leqq x \leqq A$ and $x \neq 0$, then

$$\lim_{n\to\infty} f^n(x) = \{-b, a\}.$$

If, however, f is not monotone decreasing on $[-B, A]$, then verifying (II) directly may be quite difficult, as it involves examining all iterates $x_n = f^n(x_0)$ of an arbitrary initial condition x_0. Furthermore, even if $f'(x) < 0$ for $x \in [-B, A]$, a direct proof that f^2 has exactly one fixed point in $(0, A]$ may not be easy. Fortunately, our theorems will eliminate the need for such an approach, at least in the cases of interest. Instead, checking (II) will involve only local calculations, with no need to iterate f. The main property of f that allows for such a simplification is that it possess a negative Schwarzian derivative. This property was first used in the study of interval maps by Allwright [1] and Singer [27]. If $f'(x) < 0$ for $-B < x < A$ and f has negative Schwarzian derivative on $(-B, A)$, the results of § 7 will imply f^2 has a unique fixed point in $(0, A]$.

In [21], we showed that (I), (II), and (III) each imply results about solutions of (1.1). The solutions of interest are *slowly oscillating periodic solutions*, or SOP-*solutions*. A solution $x(t)$ of (1) is called an SOP-solution if there exist quantities

$$q>1 \quad \text{and} \quad \bar{q}>q+1$$

such that

$$\begin{aligned}&x(0)=x(q)=x(\bar{q})=0,\\&x(t)>0 \quad \text{in } (0, q),\\&x(t)<0 \quad \text{in } (q, \bar{q}),\\&x(t+\bar{q})=x(t) \quad \forall t.\end{aligned}$$

For the functions of interest it will always be the case that $xf(x)<0$ whenever $x\neq 0$ is in the range of such a solution. In particular this will imply that the zeros of $x(t)$ are all simple.

An SOP solution $x(t)$ is called an *S-solution* if it satisfies

$$x(t+q)=-x(t) \quad \forall t,$$

in addition to the above conditions. Necessarily f is an odd function throughout the range of an S-solution. Also, $\bar{q}=2q$ for any S-solution.

The following results, which are proved in [21], describe the existence and asymptotic properties for small ε of SOP-solutions and S-solutions when (I), (II), or (III) holds.

THEOREM 2.1. *Assume f satisfies* (I). *Then there exists $\varepsilon_0>0$ such that for each positive $\varepsilon<\varepsilon_0$* (1) *possesses an* SOP-*solution satisfying*

$$x(t)\in(-B, A) \quad \forall t. \tag{2.1}$$

In addition, there exist positive numbers ε_1, γ, K_1, K_2, r_1, and r_2 such that if $x(t)$ is any SOP-*solution of* (1.1) *satisfying* (2.1), *and if $0<\varepsilon<\varepsilon_1$, then*

$$\begin{aligned}&x(t)>\gamma \quad \textit{for } K_2\varepsilon\leqq t\leqq q-K_2\varepsilon,\\&x(t)<-\gamma \quad \textit{for } q+K_2\varepsilon\leqq t\leqq \bar{q}-K_2\varepsilon,\\&|\dot{x}(t)|\geqq K_1/\varepsilon \quad \textit{whenever } |x(t)|\leqq\gamma,\\&1+\varepsilon r_1\leqq q\leqq 1+\varepsilon r_2,\\&1+\varepsilon r_1\leqq \bar{q}-q\leqq 1+\varepsilon r_2.\end{aligned}$$

THEOREM 2.2. *Assume f satisfies* (II). *Then given $\delta>0$ there exist $\varepsilon_2>0$ and $K_2>0$ such that if $x(t)$ is any* SOP-*solution of* (1) *satisfying* (2.1), *and if $0<\varepsilon<\varepsilon_2$, then*

$$|x(t)-\text{sqw}\,(t)|\leqq\delta \quad \textit{in } [\varepsilon K_2, q-\varepsilon K_2]\cup[q+\varepsilon K_2, \bar{q}-\varepsilon K_2]$$

where sqw (t) *is the two-periodic square-wavefunction defined by*

$$\text{sqw}(t)=\begin{cases} a & \textit{in } [0,1),\\ -b & \textit{in } [1,2),\end{cases}$$

$$\text{sqw}\,(t+2)=\text{sqw}\,(t) \quad \forall t.$$

THEOREM 2.3. *Assume f satisfies* (III). *Let $x(t)$ be any* SOP-*solution of* (1) *satisfying* (2.1) *for some $\varepsilon>0$ (with $a=A$ and $b=B$), and let $p\in(0, q)$ and $\bar{p}\in(q, \bar{q})$ be such that*

$$x(p)=\max x(t) \quad \textit{and} \quad x(\bar{p})=\min x(t).$$

Then $x(t)$ is monotone increasing in $(0, p)$, monotone decreasing in $(p, \bar{p})$, and monotone increasing in $(\bar{p}, \bar{q})$.

THEOREM 2.4. *Assume f satisfies* (I) *and that in addition $A = B$ and $f(-x) = -f(x)$ for all $x \in [-A, A]$. Then there exists $\varepsilon_0 > 0$ such that for each positive $\varepsilon < \varepsilon_0$,* (2.1) *possesses an S-solution satisfying*

$$x(t) \in (-A, A) \quad \forall t.$$

In the case of Theorem 2.2 we easily see that

$$x(t) \to \mathrm{sqw}\,(t) \quad \text{as } \varepsilon \to 0$$

uniformly on compact subsets of $\mathbb{R} - \mathbb{Z}$, for SOP-solutions $x(t)$. Also, when f is odd the S-solutions obtained in Theorem 2.4 are of course SOP-solutions, and hence satisfy the conclusions of Theorems 2.1, 2.2, and 2.3 when the appropriate hypotheses hold.

3. Some specific functions f. We consider $f_k: \mathbb{R} \to \mathbb{R}$, for $1 \leqq k \leqq 5$, defined as follows:

$$f_1(x) = \mu - x^2,$$

$$f_2(x) = x^3 - \mu x,$$

$$f_3(x) = -\mu[\sin(x+\theta) - \sin\theta],$$

$$f_4(x) = \mu x^\nu e^{-x}, \qquad x \geqq 0,$$

$$f_5(x) = \frac{\mu x^\nu}{x^\lambda + 1}, \qquad x \geqq 0.$$

The values of f_4 and f_5 for $x < 0$ are immaterial, so for definiteness we set

$$f_k(x) = f_k(0) \quad \text{if } x < 0 \text{ and } k = 4 \text{ or } 5,$$

always assuming $\nu \geqq 0$ and $\lambda \geqq 0$. The functions f_1 and f_2 give model problems with the simplest possible nonlinearities; in particular f_1 is the much-studied quadratic map of the interval [6], [15]. The function f_2 is an odd function, so by Theorem 2.4 there is the possibility of obtaining S-solutions of (1.1). The function f_3 seems at first to be a special case of the general trigonometric nonlinearity (1.3) arising in optical models; we will show, however, that f_3 can always be obtained from (1.3) by means of a linear transformation of the differential equation (1.1). The function f_4 occurs in biological and physiological models as noted earlier, as does f_5 when $\nu = 1$ or $\nu = 0$.

Our object is to determine ranges of the parameters μ, θ, ν, and λ for which the hypotheses (0), (I), (II), and (III) hold for a suitable translate of each f_k. By "suitable translate" we mean that a transformation taking a fixed point x_0 of f_k to the origin must generally be made before verifying the hypothesis in question. Indeed, for f_4 and f_5 it is not the fixed point $x = 0$ that is of interest, but rather some nontrivial fixed point $x_0 > 0$ about which we do our analysis. If $f: \mathbb{R} \to \mathbb{R}$ possesses a fixed point x_0, then letting $y = x - x_0$ in (1) yields

$$\varepsilon \dot{y}(t) = -y(t) + g(y(t-1)) \tag{3.1}$$

where

$$g(y) = f(y + x_0) - f(x_0) \tag{3.2}$$

satisfies $g(0) = 0$. When we say *a hypothesis* (such as (0), (I), (II), or (III)) *holds for a function f at a fixed point x_0*, we mean that the hypothesis holds for the transformed function g as stated.

We complete this section by showing how the function f in (1.3) can be reduced to the normal form f_3. In fact, we will show that the parameters μ and θ can always be chosen to satisfy

$$\mu \geqq 0 \quad \text{and} \quad 0 \leqq \theta \leqq \pi. \tag{3.3}$$

First note that the function f in (1.3) is bounded and so must have at least one fixed point, and possibly more than one. Let x_0 denote such a point. Then the function g in (3.2) has the same form as in (1.3) but possibly with a different value of μ_4; we continue to denote the new value by μ_4. The fact that $g(0)=0$ implies from the form (1.3) that $\mu_1 = -\mu_2 \sin \mu_4$, and so

$$g(y) = \mu_2[\sin(\mu_3 y + \mu_4) - \sin \mu_4].$$

Now assuming $\mu_2 \neq 0$ and $\mu_3 \neq 0$ (otherwise g is identically zero), we set

$$z = \pm \mu_3 y \tag{3.4}$$

with the sign $\pm$ to be determined later. The differential equation (3.1) now becomes

$$\varepsilon \dot{z}(t) = -z(t) + h(z(t-1))$$

where

$$h(z) = \mu_2\mu_3[\sin(z \pm \mu_4) - \sin(\pm\mu_4)].$$

Upon setting

$$\mu = |\mu_2\mu_3|,$$
$$\theta = \begin{cases} \pm\mu_4 \pmod{2\pi} & \text{if } \mu_2\mu_3 < 0, \\ \pm\mu_4 + \pi \pmod{2\pi} & \text{if } \mu_2\mu_3 > 0, \end{cases}$$

we see that the function h has precisely the form of f_3. In addition, an appropriate choice of sign in (3.4) ensures that μ and θ satisfy (3.3).

In our subsequent analysis we will usually assume that the function $f(x)$ in (1.3) has been written in the normal form:

$$f_3(x) = -\mu[\sin(x+\theta) - \sin\theta] \tag{3.5}$$

with $\mu > 0$ and $0 \leqq \theta \leqq \pi$. However, the reader should remember that writing the function in normal form conceals certain difficulties. First, as previously noted, the function f in (1.3) may have several fixed points. For each such fixed point of f, different parameters μ and θ in the normal form f_3 will, in general, be obtained. Second, we usually want to know for what ranges of the *original* parameters μ_1, μ_2, μ_3, and μ_4 in (1.3) does the function $f(x)$ satisfy hypotheses (0), (I), (II), or (III). The parameters in the normal form are written in terms of a fixed point of f in (1.3), and this fixed point is typically not explicitly known. Thus transferring information about the normal form back to the original function may present some nontrivial calculus problems.

4. The local condition (0). Here we discuss the existence of a fixed point of f_k at which condition (0) holds; clearly this is the case at a fixed point x_0 if and only if $f_k'(x_0) < -1$. For each function f_k, with parameters θ, ν, and λ in appropriate ranges, we will show that a critical value μ_0 of the parameter μ exists such that (0) holds at an appropriate fixed point x_0 if and only if $\mu > \mu_0$.

We begin with the model functions f_1 and f_2. If $\mu > -1/4$ then f_1 has two fixed points; the larger one,

$$x_0 = \frac{-1+\sqrt{4\mu+1}}{2},$$

interests us here. We see that $f_1'(x_0) = 1-\sqrt{4\mu+1}$, and a short calculation reveals that (0) holds there if and only if $\mu > \mu_0 = \frac{3}{4}$. For the nonlinearity f_2, with the fixed point $x_0 = 0$, we have $f_2'(0) = -\mu$; thus (0) holds there if and only if $\mu > \mu_0 = 1$.

At the fixed point $x_0 = 0$ of f_3, we have $f_3'(0) = -\mu \cos\theta$, so a necessary and sufficient condition for (0) to hold here is that $\mu \cos\theta > 1$. In particular this condition and the restrictions (3.3) imply that $\mu > 0$ and $0 \leqq \theta < \pi/2$. Thus we obtain $\mu_0 = 1/\cos\theta$.

Before discussing the functions f_4 and f_5 it is convenient to prove a simple theorem.

THEOREM 4.1. *Let $\bar{f}:[0,\infty) \to [0,\infty)$ be a continuous function that is C^1 on $(0,\infty)$. Assume there exists $\theta \geqq 0$ such that $\bar{f}'(x) > 0$ for $0 < x < \theta$ and $\bar{f}'(x) < 0$ for $x > \theta$, and there exists $s_0 > 0$ such that $(d/dx)(x\bar{f}(x))$ is positive for $0 < x < s_0$ and negative for $x > s_0$. Then for $\mu \geqq \theta(\bar{f}(\theta))^{-1}$, the equation $\mu\bar{f}(x) = x$ has a unique solution $x = x_0(\mu)$ such that $x_0(\mu) \geqq \theta$ and $\mu\bar{f}$ satisfies* (0) *at $x_0(\mu)$ if and only if $\mu > s_0(\bar{f}(s_0))^{-1}$.*

Proof. The existence and uniqueness of $x_0(\mu)$ is trivial. Since $s(\bar{f}(s))^{-1}$ is strictly increasing for $s > \theta$, we can define $\mu(s) = s(\bar{f}(s))^{-1}$ and parameterize by $s \geqq \theta$, so $\mu(s)\bar{f}(s)$ has fixed point $s \geqq \theta$. Thus the set of μ such that $\mu f'(x_0(\mu)) < -1$ is the same as $\{\mu(s): \mu(s)\bar{f}'(s) < -1\}$. A calculation shows that $\mu(s)\bar{f}'(s) < -1$ if and only if $(d/ds)(s\bar{f}(s)) < 0$, i.e., if and only if $s > s_0$. □

For the function $\bar{f}_4(x) = x^\nu e^{-x}$ we easily compute that the conditions of Theorem 4.1 are satisfied for $\nu \geqq 0$ and that $s_0 = \nu+1$ and $\mu\bar{f}_4(x)$ satisfies (0) at x_0 if and only if $\mu > (\nu+1)(\bar{f}_4(\nu+1))^{-1}$. For the function $\bar{f}_5(x) = x^\nu(1+x^\lambda)^{-1}$, we easily compute that the hypotheses of Theorem 4.1 are satisfied if $\nu \geqq 0$ and $\lambda > \nu+1$ and that $s_0^\lambda = (\nu+1)(\lambda-\nu-1)^{-1}$. Thus $\mu\bar{f}_5(x)$ satisfies condition (0) at x_0 if and only if $\mu > s_0(\bar{f}_5(s_0))^{-1}$, where $s_0^\lambda = (\nu+1)(\lambda-\nu-1)^{-1}$.

Table 1 summarizes the previous results by giving the range of parameters for which f_k satisfies (0) at a fixed point x_0. Note again that in the case of f_3 only the point $x_0 = 0$ is considered, even though there may be other fixed points at which (0) holds.

TABLE 1

Fixed points x_0 of f_k and critical parameter values μ_0. We have (0) *holding at x_0 if and only if $\mu > \mu_0$, provided the parameters θ, ν, and λ satisfy the given restrictions.*

k	x_0	μ_0	Restrictions
1	$\dfrac{-1+\sqrt{4\mu+1}}{2}$	$\dfrac{3}{4}$	
2	0	1	
3	0	$\dfrac{1}{\cos\theta}$	$0 \leqq \theta < \dfrac{\pi}{2}$
4	unique fixed point $x_0 > \nu$	$(\nu+1)^{1-\nu} e^{\nu+1}$	$\nu \geqq 0$
5	unique fixed point $x_0 > \left(\dfrac{\nu}{\lambda-\nu}\right)^{1/\lambda}$	$\log\mu_0 = \log\lambda - \left(\dfrac{\nu-1}{\lambda}\right)\log(\nu+1) - \left(\dfrac{\lambda-\nu+1}{\lambda}\right)\log(\lambda-\nu-1)$	$\nu \geqq 0,\quad \lambda > \nu+1$

When $\nu=1$ in the function $f_5(x)$, we can explicitly compute $x_0=(\mu-1)^{1/\lambda}$, and the conditions on the parameters become $\lambda<2$ and $\mu>\lambda(\lambda-2)^{-1}$. More generally, it is of interest to locate the fixed points of f_4 and f_5 more precisely. The following result gives the asymptotic behavior of $x_0(\mu)$ for large μ; we omit the proof because we will not actually use the result here and because the proof involves only standard arguments from asymptotic analysis. Note that we use the standard "big O" and "little o" notation: If $h(\mu)$ and $g(\mu)$ are complex-valued functions defined for large positive μ and if $g(\mu)$ is nonzero for large μ, then we write

$$h(\mu)=O(g(\mu)) \quad \text{if and only if } \limsup_{\mu\to+\infty}\frac{|h(\mu)|}{|g(\mu)|}<\infty,$$

$$h(\mu)=o(g(\mu)) \quad \text{if and only if } \limsup_{\mu\to+\infty}\frac{|h(\mu)|}{|g(\mu)|}=0.$$

THEOREM 4.2. *For $\nu\geqq 0$ and $\lambda>\nu+1$ define numbers $\theta_4=\nu$ and $\theta_5^\lambda=\nu(\lambda-\nu)^{-1}$. The functions $f_4(x)=\mu\bar f_4(x)$ and $f_5(x)=\mu\bar f_5(x)$ have (for sufficiently large μ) a unique fixed point $x_0(\mu)$ such that $x_0(\mu)>\theta_j$ and $x_0(\mu)$ satisfies*

$$x_0(\mu)=\log(\mu)-(1-\nu)\log(\log\mu)+O\left(\frac{\log(\log\mu)}{\log\mu}\right) \quad \textit{for } f_4,$$

$$x_0(\mu)=\mu^{(1/(\lambda+1-\nu))}-\left(\frac{1}{\lambda+1-\nu}\right)\mu^{((1-\lambda)/(\lambda+1-\nu))} + O(\mu^{((1-2\lambda)/(\lambda+1-\nu))}) \quad \textit{for } f_5,$$

where log *denotes natural logarithm.*

5. General results on piecewise monotone functions. We wish to determine when a hypothesis (I), (II), or (III) holds for f_k at a point x_0 given in Table 1. As noted earlier these three conditions are global, and verifying them for specific functions may be difficult. To aid us in this task we will first obtain some general criteria for these hypotheses to hold; we will then apply these criteria to the functions f_k of interest.

To begin, we introduce condition (PM) (piecewise monotone) on a function f; observe that each f_k satisfies (PM) at the fixed point x_0 and parameter ranges of Table 1:

(PM) *The function $f:\mathbb{R}\to\mathbb{R}$ satisfies hypothesis* (0). *In addition, there exist (possibly infinite) quantities $0<\xi\leqq\alpha\leqq\infty$ and $0<\eta\leqq\beta\leqq\infty$ such that*
 (i) *$f(-\beta)=0$ if $\beta<\infty$;*
 (ii) *f is monotone increasing and strictly positive in $(-\beta,-\eta)$ if $\eta<\infty$;*
 (iii) *f is monotone decreasing in $(-\eta,\xi)$ but not in any larger open interval;*
 (iv) *f is monotone increasing and strictly negative in (ξ,α) if $\xi<\infty$;*
 (v) *$f(\alpha)=0$ if $\alpha<\infty$.*
 Furthermore, if $\xi=\eta=\infty$ (so f is monotone decreasing in all of $\mathbb{R}$), then $|f^2(x)|<|x|$ for some x.

Figure 1 depicts a function satisfying (PM). (Note that $f(\alpha)=0$ is *not* required for this function as $\alpha=\infty$. That is, $\lim_{x\to\infty}f(x)$ may either be zero or strictly negative.) For any function satisfying (PM) we have $xf(x)<0$ if $|x|\neq 0$ is sufficiently small, since $f(0)=0$ and $f'(0)<-1$ by (0). A first question we consider for such a function is when it also satisfies (I).

Suppose both (PM) and (I) hold for f. Then the quantities A and B in (I) clearly satisfy

$$A<\alpha \quad\text{and}\quad B<\beta. \tag{5.1}$$

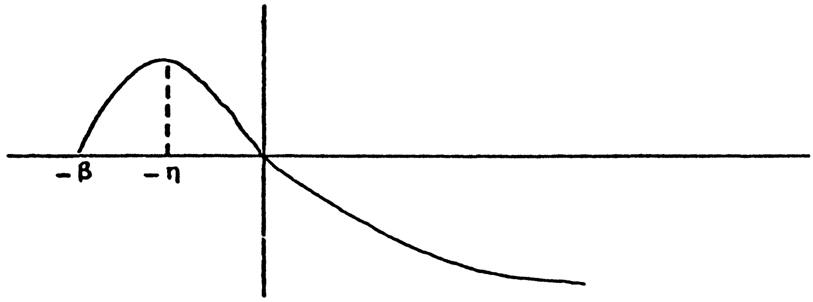

FIG. 1

On the other hand, if f satisfies (PM) and A and B are positive numbers satisfying (5.1), then f also satisfies (I) if and only if

$$f([-B, A]) \subseteq [-B, A]. \tag{5.2}$$

Finally, if both (5.1) and (5.2) hold for a function satisfying (PM), then there exists a minimal interval $[-B_*, A_*] \subseteq (-\beta, \alpha)$, with both A_* and B_* strictly positive, for which (5.2) is an equality:

$$f([-B_*, A_*]) = [-B_*, A_*].$$

This is so because $f'(0) < -1$.

Now assume f satisfies (PM) and define a continuous monotone decreasing function $f_*: R \to R$ by

$$f_*(x) = \begin{cases} f(-\eta) & \text{in } (-\infty, -\eta] \quad \text{if } \eta < \infty, \\ f(x) & \text{in } (-\eta, \xi), \\ f(\xi) & \text{in } [\xi, \infty) \quad \text{if } \xi < \infty. \end{cases} \tag{5.3}$$

If A and B are positive numbers satisfying (5.1), then clearly

$$f([-B, A]) = [f_*(A), f_*(-B)], \tag{5.4}$$

so that the equality in (5.2) holds at $A = A_*$ and $B = B_*$ if and only if

$$f_*(A_*) = -B_* \quad \text{and} \quad f_*(-B_*) = A_*. \tag{5.5}$$

Thus, (I) holds if and only if there exist two distinct points in the interval $(-\beta, \alpha)$ that are mapped into one another by f_*. From this basic fact we conclude the following result.

PROPOSITION 5.1. *Assume f satisfies* (PM), *define the function f_* by* (5.3), *and define quantities*

$$\begin{aligned} A_* &= \inf\{A > 0 \mid f_*^2(A) = A\}, \\ B_* &= \inf\{B > 0 \mid f_*^2(-B) = -B\}. \end{aligned} \tag{5.6}$$

Then A_* and B_* are well-defined positive numbers satisfying (5.5). *Hypothesis* (I) *holds for f if and only if*

$$A_* < \alpha \quad \textit{and} \quad B_* < \beta$$

And in such a case we can take $A = A_*$ and $B = B_*$ in the statement of (I). *A sufficient condition for* (I) *to hold is that both*

$$f_*^2(\alpha) < \alpha \quad \textit{if} \quad \alpha < \infty, \qquad f_*^2(-\beta) > -\beta \quad \textit{if } \beta < \infty. \tag{5.7}$$

Another sufficient condition for (I) *to hold is that*

$$(5.8) \qquad f_*(\alpha) > -\beta, \quad f_*^2(\alpha) < \alpha, \quad \alpha < \infty,$$

while a third sufficient condition for (I) *to hold is that*

$$(5.9) \qquad f_*(-\beta) < \alpha, \quad f_*^2(-\beta) > -\beta, \quad \beta < \infty.$$

Proof. The existence and positivity of A_* and B_* follow from the fact that $|f_*^2(x)| > |x|$ for small $|x| \neq 0$ (since $(f_*^2)'(0) > 1$) and from the fact that $|f_*^2(x)| < |x|$ for some x. The latter inequality holds because f_*^2 is a bounded function if either ξ or η is finite; if $\xi = \eta = \infty$, the inequality is assumed in the definition of (PM). The first part of the proposition now follows easily from the monotonicity of f_* and the discussion above.

The assumptions on f imply that f_*^2 always has a positive fixed point and a negative fixed point. If $\alpha = \beta = \infty$, f satisfies (I) by what we have already proved. In all other cases it suffices to prove that f_*^2 has a fixed point in $(0, \alpha)$ and a fixed point in $(-\beta, 0)$. The reader can easily verify that (5.7), (5.8), or (5.9) are all sufficient to ensure this. For example, if $\alpha < \infty$ and (5.8) is satisfied, f_*^2 has a fixed point x_0 in $(0, \alpha)$ because $f_*^2(\alpha) < \alpha$ and $(f_*^2)'(0) > 1$, and then $f_*(x_0)$ is a fixed point of f_*^2 in $(-\beta, 0)$ (because f_* is monotone and $f_*(\alpha) > -\beta$). □

The following related result tells when the monotonicity condition (III) holds for a function that satisfies (PM).

PROPOSITION 5.2. *Assume f satisfies* (PM), *and let f_*, A_*, and B_* be as in Proposition* 5.1. *Then* (III) *holds for f if and only if*

$$(5.10) \qquad A_* \leqq \xi \quad \text{and} \quad B_* \leqq \eta.$$

As before, we have $A = A_*$ and $B = B_*$ in the statement of (III). *A sufficient condition for* (5.10) *to hold is that both*

$$(5.11) \qquad f_*^2(\xi) \leqq \xi \quad \text{if } \xi < \infty, \qquad f_*^2(-\eta) \geqq -\eta \quad \text{if } \eta < \infty$$

should hold.

Another sufficient condition for (5.10) *to be satisfied is that*

$$(5.12) \qquad f_*(\xi) \geqq -\eta, \quad f_*^2(\xi) \leqq \xi, \quad \xi < \infty$$

while a third sufficient condition for (5.10) *to hold is that*

$$(5.13) \qquad f_*(-\eta) \leqq \xi, \quad f_*^2(-\eta) \geqq -\eta, \quad \eta < \infty.$$

Proof. This follows the proof of Proposition 5.1 once we recall f is monotone in $(-\eta, \xi)$, but in no larger open interval. □

Suppose the function f_*^2 has exactly one fixed point in $(0, \infty)$. Then an easy argument implies that f_*^2 has a unique fixed point in $(-\infty, 0)$. If $A_* < \alpha$ and $B_* < \infty$ (A_* and B_* as in (5.6)), we must have $f_*^2(\alpha) < \alpha$ (if $\alpha < \infty$) and $f_*^2(-\beta) > -\beta$ (if $\beta < \infty$): otherwise the intermediate value theorem would imply that f_*^2 has a fixed point in $[\alpha, \infty)$ or $(-\infty, -\beta]$, contradicting uniqueness. Thus if f_*^2 has a unique positive fixed point, the sufficient condition in (5.7) that f satisfy (I) is also necessary. Furthermore, a little additional thought shows that (5.7) is satisfied if and only if (5.8) is satisfied, or (5.9) is satisfied, or $\alpha = \beta = \infty$ (assuming f_*^2 has a unique positive fixed point).

Similarly, if f is as in Proposition 5.2 and f_*^2 has a unique positive fixed point, then f satisfies (III) if and only if f satisfies (5.11). Also f satisfies (III) if and only if f satisfies (5.12), or (5.13), or $\xi=\eta=\infty$.

Inequalities (5.7) and (5.11) are, in general, easier to verify than (5.6) and (5.10), so it is useful to have theorems that ensure f_*^2 has a unique positive fixed point. Furthermore, we have already seen in the discussion in § 2 the importance of knowing that $f^2:[-B, A]\to[-B, A]$ (A and B as in § 2) has a unique positive fixed point. For example, if f is monotone decreasing and $f([-B, A])\subset[-B, A]$, we saw in § 2 that f satisfies (III) on $[-B, A]$ if f^2 has a unique positive fixed point in $[-B, A]$.

If the function f_*^2 were convex downward in (τ, ∞) and convex upward in $(-\infty, \tau)$ for some real τ, then f_*^2 would have a unique fixed point in $(0, \infty)$. This type of convexity assumption is clumsy to deal with, but a related concept, that of negative Schwarzian derivative, is readily verifiable for many functions of interest and can be used to prove a variety of results, including the uniqueness of the positive fixed point of f_*^2. Remarkably, each of the five functions f_k has a negative Schwarzian derivative for most of the parameter values of interest, so it will be natural for us to make the assumption of negative Schwarzian derivative in most of our subsequent theorems.

6. The Schwarzian derivative. The Schwarzian derivative Sf of a function $f: I\to R$ in an interval I is defined to be the function

$$Sf(x)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

at those points $x\in I$ where f is three times differentiable and $f'(x)\neq 0$. At all other points of I we consider Sf to be undefined. The Schwarzian derivative originated in the theory of conformal mappings and was first used in the study of interval maps by Allwright [1] and Singer [27].

In this section we present several basic properties of Schwarzian derivatives and of functions whose Schwarzian derivative is negative. An important sufficient condition for the Schwarzian derivative of a function to be negative is given in Proposition 6.2.

Our first proposition collects some well-known results about the Schwarzian derivative (see [6], [27]). Statement (v) in Proposition 6.1, although elementary, is quite useful and does not appear to have been explicitly stated in the literature.

PROPOSITION 6.1. *Let $f: I\to\mathbb{R}$ and $y: J\to\mathbb{R}$ be functions defined on intervals I and J. Then*

(i) $S(g\circ f)(x)=Sf(x)+[f'(x)]^2 Sg(f(x))$ *and*

(ii) $S(g\circ f)(x)<0$ *if* $Sf(x)<0$ *and* $Sg(f(x))<0$

hold whenever Sf is defined at $x\in I$ and Sg is defined at $f(x)\in J$. Also, if m is the Möbius transformation

$$m(x)=\frac{c_1x+c_2}{c_3x+c_4},\qquad c_1c_4-c_2c_3\neq 0,$$

then

(iii) $Sm(x)=0$ *where defined, and hence*

$$S(m\circ f)(x)=Sf(x)$$

if Sf is defined at $x\in I$ and $c_3f(x)+c_4\neq 0$. Finally,

(iv) $(d^2/dx^2)(|f'(x)|^{-1/2})>0$ *if and only if* $Sf(x)<0$, *and*

(v) *if* $(d^2/dx^2)\log|f'(x)|<0$ *then* $Sf(x)<0$

hold whenever Sf is defined.

Proof. These are straightforward but tedious calculations, which we omit. We do note that (ii) and (iii) follow easily from (i). Also,

$$h''(x) > 0 \quad \text{implies } \frac{d^2}{dx^2}(e^{h(x)}) > 0;$$

so with $h(x) = -\frac{1}{2}\log|f'(x)|$ we obtain (v) from (iv). □

COROLLARY 6.1 [27, Prop. 2.4]. *Let $f: I \to \mathbb{R}$ be three times differentiable in an open interval I, and assume*

$$f'(x) \neq 0 \quad \textit{and} \quad Sf(x) < 0 \quad \textit{for all } x \in I.$$

Then the function $|f'(x)|$ does not attain a minimum in I. That is, there does not exist $w \in I$ such that $|f'(w)| \leqq |f'(x)|$ for all $x \in I$.

Proof. This follows immediately from (iv) of Proposition 6.1. □

LEMMA 6.1. *Let $f: I \to I$ be three times differentiable in an interval I (not necessarily open), with range in I, and assume that $Sf(x) < 0$ at each $x \in I$ for which $f'(x) \neq 0$. Assume that for some $w \in I$ (possibly an endpoint) we have*

$$f(w) = w \quad \textit{and} \quad |f'(w)| \leqq 1.$$

In addition, assume

$$f''(w) = 0 \quad \textit{if } f'(w) = 1$$

at this fixed point w. Then there exists a relatively open neighborhood $U \subseteq I$, with $w \in U$, such that

$$f(U) \subseteq U,$$
$$f^n(x) \to w \quad \textit{as } n \to \infty, \quad \textit{for each } x \in U.$$

Proof. In the case of a strict inequality $|f'(w)| < 1$ it is an elementary exercise to show that the set

$$U = (w - \delta, w + \delta) \cap I$$

satisfies the conclusions of the lemma if $\delta > 0$ is small enough. The same set also works if $f'(w) = 1$: we have $f''(w) = 0$ (by assumption) and $f'''(w) < 0$ (since $Sf(w) < 0$). It follows that if $g(x) = f(x) - x$, $g^{(j)}(w) = 0$ for $0 \leqq j \leqq 2$ and $g^{(3)}(w) < 0$. Thus Taylor's theorem implies that there exists $\delta > 0$ such that $f(x) < x$ for $x \in (w, w + \delta) \cap I$ and $f(x) > x$ for $x \in (w - \delta, w) \cap I$. Since $f'(w) = 1$ we also have $f(x) \geqq w$ for $x \in (w, w + \delta) \cap I$ and $f(x) \leqq w$ for $x \in (w - \delta, w) \cap I$. It follows that if $x \in (w - \delta, w + \delta) \cap I$, $x_n = f^n(x)$ is a monotonic sequence bounded above by w (if $x \leqq w$) or below by w (if $w \leqq x$). Thus the sequence (x_n) converges to ξ and necessarily $f(\xi) = \xi$. By construction w is the only fixed point of f in $(w - \delta, w + \delta) \cap I$, so $\xi = w$ and x_n converges to ξ.

In the remaining case, when $f'(w) = -1$, we have $f^2(w) = w$ and $(f^2)'(w) = 1$, and a simple calculation (see [27, p. 261]) gives us that $(f^2)''(w) = 0$. By (ii) of Proposition 6.1 the Schwarzian derivative of f^2 satisfies $Sf^2(x) < 0$ whenever $(f^2)'(x) \neq 0$. Thus f^2 satisfies the conditions on f already considered in the preceding paragraph, so there exists $U_0 \subseteq I$ satisfying the conclusions of the lemma for the function f^2 instead of f. From this we see that the set $U = U_0 \cap f(U_0)$ satisfies the conclusions of the lemma for the function f. □

The next result gives an easily verified condition for the Schwarzian derivative of a function to be negative. Recall first that the order ω of an entire function $f: \mathbb{C} \to \mathbb{C}$ is the infimum of all numbers $\kappa > 0$ such that $|f(z)|\, e^{-|z|^\kappa}$ is bounded on $\mathbb{C}$. (If no such

κ exists then f is said to have infinite order.) Nontrivial entire functions of finite order possess a product representation

$$f(z)=e^{\Omega(z)}z^k\prod_n\left(1-\frac{z}{z_n}\right)E_M\left(\frac{z}{z_n}\right)$$

with at most countably many factors, where Ω is a polynomial of degree at most $[\omega]$ (the greatest integer less than or equal to the order), $k\geqq 0$ is the multiplicity of $z=0$ as a root of f, the numbers z_n are the other roots of f listed according to multiplicity, E_M is the function

$$E_M(z)=\exp\left(\sum_{n=1}^{M}\frac{z^n}{n}\right)$$

with $E_0(z)=1$, and $M\geqq 0$ is an integer satisfying $M\leqq\omega\leqq M+1$. In addition, it is the case that

$$\sum_n\frac{1}{|z_n|^{M+1}}<\infty, \tag{6.1}$$

and the infinite product converges uniformly on compact subsets of $\mathbb{C}$. We also recall that the order of the derivative f' equals the order ω of f.

PROPOSITION 6.2. *Let f be an entire function of order $\omega<2$ such that $f(x)\in\mathbb{R}$ whenever $x\in\mathbb{R}$, and such that all zeros of the derivative f' are real. Then either*

$$Sf(x)<0\quad \text{whenever } f'(x)\neq 0 \text{ and } x\in\mathbb{R},$$

or f is a linear function $f(x)=c_0x+c_1$.

Proof. We note that $M=0$ or 1 in the infinite product representation for the derivative f'. Denoting the zeros of f' by $x_n\in\mathbb{R}$ we have for this function either

$$f'(x)=e^{\Omega(x)}x^k\prod_n\left(1-\frac{x}{x_n}\right),\quad\text{or} \tag{6.2}$$

$$f'(x)=e^{\Omega(x)}x^k\prod_n\left(1-\frac{x}{x_n}\right)e^{x/x_n}, \tag{6.3}$$

where we restrict our attention to real values x of the argument. In either case (6.2) or (6.3) we have $\Omega''(x)=0$ for all x, and

$$\sum_n\frac{1}{|x_n|^2}<\infty. \tag{6.4}$$

Now assume that f' does possess a root; otherwise $f'(x)=e^{\Omega(x)}$ and the result is easily checked. In the first case, (6.2), we have

$$\log|f'(x)|=\Omega(x)+k\log|x|+\sum_n\log\left|1-\frac{x}{x_n}\right|,$$

so term-by-term differentiation (*justified by* (6.1)) *gives*

$$\frac{d^2}{dx^2}\log|f'(x)|=-\frac{k}{x^2}-\sum_n\frac{1}{(x-x_n)^2}<0 \tag{6.5}$$

for $x\neq x_n$ and $x\neq 0$ if $k>0$. In the second case, (6.3), the same formula, (6.5), holds. In either case the result $Sf(x)<0$ follows from (v) of Proposition 6.1. □

7. Verifying (I), (II), and (III). The following hypotheses are strengthened versions of (PM) involving a negative Schwarzian derivative condition. Under these conditions, verifying (I), (II), and (III) reduces to essentially local calculations.

(NS_1) *The function* $f:\mathbb{R}\to\mathbb{R}$ *satisfies hypothesis* (PM). *In addition f is three times differentiable in* $(-\eta, \xi)$ *and satisfies* $f'(x)<0$ *and* $Sf(x)<0$ *in* $(-\eta, \xi)$.

(NS_2) *The function* $f:\mathbb{R}\to\mathbb{R}$ *satisfies* (PM). *In addition, f is three times differentiable in* $(-\beta, \alpha)$ *and satisfies* $f'(x)\neq 0$ *and* $Sf(x)<0$ *if* $x\in(-\beta,\alpha)$ *and* $x\neq \xi, -\eta$.

Under hypothesis (NS_1), Theorem 7.1 gives necessary and sufficient condition for (I) and (III) to hold, thereby extending Propositions 5.1 and 5.2. Under hypothesis (NS_2), Theorem 7.3 gives an easily verified necessary and sufficient condition for (II) to hold provided (I) also holds. Theorem 7.2 will be useful in verifying (I), (II), and (III) for specific functions.

THEOREM 7.1. *Assume f satisfies* (NS_1) *and define f_* by* (5.3) *as before. Then f satisfies* (I) *if and only if*

$$f_*^2(\alpha)<\alpha \quad \text{if } \alpha<\infty, \qquad f_*^2(\beta)>-\beta \quad \text{if } \beta>\infty. \tag{7.1}$$

Also, f satisfies (I) *if and only if at least one of the following three conditions holds*:
(1) $\alpha=\beta=\infty$; *or* (2) $\alpha<\infty$ *and* $f_*(\alpha)>-\beta$ *and* $f_*^2(\alpha)<\alpha$; *or* (3) $\beta<\infty$ *and* $f_*(-\beta)<\alpha$ *and* $f_*^2(-\beta)>-\beta$.

The function f satisfies (III) *if and only if*

$$f_*^2(\xi)\leqq\xi \quad \text{if } \xi<\infty, \qquad f_*^2(-\eta)\geqq-\eta \quad \text{if } \eta<\infty. \tag{7.2}$$

Also, f satisfies (III) *if and only if at least one of the following three conditions is satisfied*:
(1) $\xi=\eta=\infty$; *or* (2) $\xi<\infty$ *and* $f_*(\xi)\geqq-\eta$ *and* $f_*^2(\xi)\leqq\xi$; or (3) $\eta<\infty$ *and* $f_*(-\eta)\leqq\xi$ *and* $f_*^2(-\eta)\geqq-\eta$.

Recall that we have

$$f_*^2(\alpha)=f_*^2(\xi) \quad \text{and} \quad f_*^2(-\beta)=f_*^2(-\eta) \tag{7.3}$$

for the quantities in Theorem 7.1.

THEOREM 7.2. *Assume f satisfies* (NS_1), *except that*

$$f'(0)=-k, \qquad 0<k\leqq 1,$$

holds instead of $f'(0)<-1$. *If* $0<k<1$, *assume that* $f''(0)\geqq 0$. *Then with $f_*(x)$ given by* (5.3) *we have*

$$|f_*^2(x)|<|x| \quad \forall x\neq 0.$$

Note that (7.1) and (7.2) hold under the hypotheses of Theorem 7.2.

THEOREM 7.3. *Assume f satisfies both* (NS_2) *and* (I). *Then f satisfies* (II) *if and only if both*
(i) $|(f^2)'(x)|\leqq 1$, *and*
(ii) $(f^2)''(x)=0$ *if* $(f^2)'(x)=1$
hold whenever

$$f^2(x)=x, \qquad x\in(0,\alpha), \quad f(x)\in(-\beta,0). \tag{7.4}$$

Equivalently, f satisfies (II) *if and only if both* (i) *and* (ii) *hold whenever*

$$f^2(x)=x, \qquad x\in(-\beta,0), \quad f(x)\in(0,\alpha). \tag{7.5}$$

LEMMA 7.1. *Assume f satisfies* (NS_1), *and define f_*, A_*, and B_* by* (5.3) *and* (5.6). *Then A_* and $-B_*$ are the unique nonzero fixed points of f_*^2. Moreover,* $f_*^2(x)-x$

changes sign at these points, with $|f_*^2(x)|>|x|$ *in* $(-B_*, A_*)-\{0\}$, *and* $|f_*^2(x)|<|x|$ *in* $(-\infty, -B_*)\cup(A_*, \infty)$.

Proof. First observe that a (possibly infinite) quantity $\xi_*>0$ exists such that

$$(f_*^2)'(x)>0 \quad \text{in } [0, \xi_*),$$

$$f_*^2(x)=f_*^2(\xi_*) \quad \text{in } [\xi_*, \infty) \quad \text{if } \xi_*<\infty.$$

This follows easily from the definition of f_*, and we see that $\xi_*=\xi$ if $f_*(\xi)\geqq-\eta$ and $\xi_*=f_*^{-1}(-\eta)<\xi$ if $f_*(\xi)<-\eta$ (these formulas hold even if ξ or η is infinite). Also,

$$Sf_*^2(x)<0 \quad \text{in } [0, \xi_*)$$

by (ii) of Proposition 6.1. By assumption, we have

$$(f_*^2)'(0)>1. \tag{7.6}$$

Now A_* is the smallest positive fixed point of f_*^2 in $(0, \infty)$. If $A_*\geqq\xi_*$, then clearly A_* is the only such fixed point; so suppose that $A_*<\xi_*$. Because A_* is the smallest positive fixed point of f_*^2, we have $f_*^2(x)>x$ for $0<x<A_*$, which implies

$$(f_*^2)'(A_*)\leqq 1. \tag{7.7}$$

By Lemma 6.1 the derivative $(f_*^2)'$ does not attain a minimum in any open subinterval of $(0, \xi_*)$. Using this and the fact that $(f_*^2)'(0)>1$, we conclude from (7.7) that

$$(f_*^2)'(x)<1 \quad \text{for } A_*<x<\xi_*. \tag{7.8}$$

Integrating (7.8) from A_* to u for $A_*<u\leqq\xi_*$, we obtain

$$f_*^2(u)-f_*^2(A_*)=f_*^2(u)-A_*<\int_{A_*}^{u} 1\,dx=u-A_*, \tag{7.9}$$

and (7.9) implies $f_*^2(x)<x$ for $A_*<x\leqq\xi_*$, and hence for all $x>A_*$.

The analysis for $-B_*$ is similar and is left to the reader. □

Proof of Theorem 7.1. As was noted in § 5, the sufficient conditions (5.7) and (5.11) in Propositions 5.1 and 5.2 are also necessary if f_*^2 has a unique positive fixed point. Lemma 7.1 shows that f_*^2 has a unique positive fixed point.

Proof of Theorem 7.2. Obviously f^2 and f_*^2 agree on $[0, \xi_*]$. A simple calculation (see [27, p. 261]) shows that

$$(f_*^2)''(0)=f''(0)[k^2-k]. \tag{7.10}$$

Since we assume that $f''(0)\geqq 0$ if $0<k<1$, (7.10) implies

$$(f_*^2)''(0)\leqq 0 \tag{7.11}$$

for $0<k\leqq 1$. If strict inequality holds in (7.11), the mean value theorem implies that there exists $\delta>0$ such that

$$0<(f_*^2)'(x)<(f_*^2)'(0)=k^2 \quad \text{for } 0<x<\delta. \tag{7.12}$$

If $(f_*^2)''(0)=0$, the negative Schwarzian condition implies

$$(f_*^2)'''(0)<0,$$

and by using Taylor's formula we again see that there exists $\delta>0$ such that (7.12) is satisfied. Lemma 6.1 now implies

$$(f_*^2)'(x)<k^2 \quad \text{for } 0<x<\xi_*, \tag{7.13}$$

for if (7.13) failed for some x, $(f_*^2)'$ would achieve its minimum at an interior point of $(0, x)$. By integrating inequality (7.13) from zero to x for $x \leqq \xi_*$, we easily obtain

$$f_*^2(x) < x \quad \text{for } 0 < x \leqq \xi_*,$$

and hence

$$f_*^2(x) < x \quad \text{for } 0 < x. \tag{7.14}$$

Inequality (7.14) implies that f_*^2 has no negative fixed points y (otherwise $f_*(y)$ would be a positive fixed point of f_*^2), and since $f_*^2(x) > x$ for small negative x, we conclude that $f_*^2(x) > x$ for all $x < 0$. □

LEMMA 7.2. *Assume f satisfies both* (NS_1) *and* (I). *If $a \in (-\beta, \alpha)$ is such that $f(a) \in (-\beta, \alpha)$ and $f^2(a) = a$, then we have in fact $a, f(a) \in [-B, A] \subseteq (-\beta, \alpha)$ where A and B are as in* (I).

Proof. Assume without loss of generality that $a > 0$. The monotonicity of f_* and the fact that $|f(x)| \leqq |f_*(x)|$ in $(-\beta, \alpha)$ imply

$$a = f^2(a) \leqq f_*(f(a)) \leqq f_*^2(a),$$

and from this we have

$$a \leqq A_* \tag{7.15}$$

by Lemma 7.1. On the other hand, the inclusion $f^2([-B, A]) \subset [-B, A]$ (which follows from (I)) and (5.3) imply that $f_*^2(A) \leqq A$, so

$$A_* \leqq A \tag{7.16}$$

by Lemma 7.1. From (7.15) and (7.16) we have $a \in [-B, A]$. The proof that $f(a) \in [-B, A]$ is analogous. □

LEMMA 7.3. *Assume f satisfies both* (PM) *and* (I), *and is differentiable in $(-\beta, \alpha)$ with $f'(x) \neq 0$ there, except at $x = \xi$ and $x = -\eta$ (this is true in particular if f satisfies* (NS_2) *and* (I)). *Then with A as in* (I), *the following hold*:

(i) *The critical points of f^2 in $(0, A)$ are isolated, and $(f^2)'$ changes sign at each such point.*

(ii) *If a point w in the open interval $(0, A)$ is a local maximum of f^2, then it is a global maximum in $[0, A]$*:

$$f^2(w) = \max_{[0,A]} f^2(x). \tag{7.17}$$

(iii) *If f^2 possesses a critical point in the closed interval $[0, A]$ and if w in the closed interval $[0, A]$ is as in* (7.17), *then w is a critical point*:

$$(f^2)'(w) = 0.$$

Proof. This lemma follows directly from several elementary observations based on the shape of the graph of f as in (PM), and the fact that f maps the interval $[-B, A] \subseteq (-\beta, \alpha)$ into itself.

At a critical point $x \in (0, A)$ of f^2 we have

$$(f^2)'(x) = f'(f(x))f'(x) = 0$$

and so either $x = \xi$ or $f(x) = -\eta$. As $f(x) = -\eta$ for at most two points in $(0, \alpha)$, we conclude that f^2 has at most three critical points in $(0, A)$. Of course, these points are isolated. If either $A \leqq \xi$ or $f(\xi) \geqq -\eta$, then f^2 has at most one critical point in $(0, A)$, and this point, if it exists, is a local maximum. In this case (i), (ii), and (iii) clearly hold, so the lemma is proved.

On the other hand, suppose $A>\xi$ and $f(\xi)<-\eta$. Then we see that $x=\xi$ is a local minimum of f^2, that $f(x)=-\eta$ has either one or two solutions ζ in $(0, A]$ and that they are local maxima for f^2 with the (common, if there are two solutions ζ) value

$$f^2(\zeta)=f(-\eta)=\max_{[-B,0]} f(x)=\max_{[0,A]} f^2(x). \tag{7.18}$$

In particular, (i) and (ii) hold. To prove (iii), we note that if $w\in[0, A]$ satisfies (7.17), then from (7.18) we have $f^2(w)=f(-\eta)$, and hence $f(w)=-\eta$. Thus $(f^2)'(w)=f'(-\eta)f'(w)=0$ as claimed. □

LEMMA 7.4. *Assume f satisfies the hypotheses of Lemma 7.3. Suppose there exists an interval $J=[r, s]$ with $0\leqq r<s\leqq A$ such that*

$$f^2(J)=J,\ f^2(\partial J)=\partial J, \text{ where } \partial J=\{r, s\}, \text{ and } J-\partial J \text{ contains a critical point of } f^2. \tag{7.19}$$

Furthermore, assume that it is not true that $f^2(r)=s=f^2(s)$. Then there exists $v\in J-\partial J$ such that $f^2(v)=s$. Also, if $w\in\partial J$ is such that $f^2(w)=s$, then $(f^2)'(w)=0$.

Proof. Our assumptions imply that (a) $f^2(r)=s$ and $f^2(s)=r$, or (b) $f^2(r)=r$ and $f^2(s)=s$, or (c) $f^2(r)=r=f^2(s)$. In case (a), let $v=\sup\{x<s: (f^2)'(x)=0\}$. By using (i) of Lemma 7.3, the fact that f^2 achieves its minimum on J at s and the assumption that f^2 has a critical point in (r, s), we see that $r<v<s$, and $(f^2)'(x)<0$ for $v<x<s$. Lemma 7.3 implies that $(f^2)'(x)$ changes sign at v, so f^2 has a local maximum at v. A similar argument applies in cases (b) and (c) and shows that f^2 always has a critical point v in (r, s) at which f^2 has a local maximum. Note, however, that this argument fails if $f^2(r)=s=f^2(s)$.

Because $f^2(J)\subset J$, we have $f^2(v)\leqq s$; but part (ii) of Lemma 7.3 implies

$$f^2(v)=\max_{[0,A]} f^2(x)\geqq s, \tag{7.20}$$

so we conclude that

$$f^2(v)=s=\max_{[0,A]} f^2(x). \tag{7.21}$$

If there exists $w\in\partial J$ such that $f^2(w)=s$, (7.21) and part (iii) of Lemma 7.3 imply that $(f^2)'(w)=0$. □

Proof of Theorem 7.3. We establish the last statement of the theorem first, by showing (i) and (ii) both hold for all fixed points of f^2 satisfying (7.4) if and only if they hold for all fixed points of f^2 satisfying (7.5). Indeed, this is an easy consequence of the following two observations. First, if $f^2(x)=x$ then $f^2(y)=y$, where $y=f(x)$, and we have $(f^2)'(x)=(f^2)'(y)$. Second, if $(f^2)'(x)=1$ for this point, then $(f^2)''(x)=(f^2)''(y)$. Thus, we need only consider fixed points of f^2 satisfying (7.5).

By Lemma 7.2 we may further restrict our attention to fixed points $x\in(0, A]$ (with $f(x)\in[-B, 0)$ holding automatically), where A and B are as in (I). We will therefore prove that for f to satisfy (II) it is necessary and sufficient that each fixed point of f^2 in $(0, A]$ should satisfy both (i) and (ii).

Necessity. Assume that f satisfies (II) for some a and b, but that either (i) or (ii) fails for $x=a$. From (II) we see that $x=a$ is the only fixed point of both f^2 and of f^4 in $(0, A]$, and so we have

$$\begin{aligned} &f^n(x)>x \quad \text{in } (0, a),\\ &f^n(x)<x \quad \text{in } (a, A] \quad \text{if } a<A \end{aligned} \tag{7.22}$$

for $n=2$ and 4, because $(f^n)'(0)>1$ and $f^n(A)\leqq A$. Observing that $f^2(a)=a$ implies $(f^4)'(a)=[(f^2)'(a)]^2\geqq 0$, we conclude from (7.22) that $(f^4)'(a)\leqq 1$; hence $|(f^2)'(a)|\leqq 1$. Thus (i) holds for $x=a$.

We therefore assume that (ii) fails for $x=a$, and so

$$(f^2)'(a)=1 \quad \text{and} \quad (f^2)''(a)\neq 0. \tag{7.23}$$

We now see that in order for (7.22) to hold with $n=2$ it is necessary that

$$(f^2)''(a)>0. \tag{7.24}$$

Furthermore, the fixed point a must be located at the endpoint A of the interval, that is,

$$f^2(A)=A \quad \text{as } a=A.$$

Now observe that the interval $(0,A)$ contains a critical point of f^2. Indeed if this were not so, then $(f^2)'$ would attain a positive minimum in $(0,A)$ because of (7.23) and the fact that $(f^2)'(0)>1$; however, this would contradict Corollary 6.1.

Thus we see that the interval $J=[0,A]$ satisfies the hypotheses of Lemma 7.4. But then we conclude from this result, with $w=A$, that $(f^2)'(A)=0$. This contradicts (7.23), completing the proof of necessity.

Sufficiency. Assume that f satisfies (NS_2) and (I) for some A and B, and that (i) and (ii) in the statement of the theorem hold at each fixed point of f^2 in $(0,A]$. First note that f^2 has at least one fixed point in $(0,A]$; this follows from the inclusion $f^2([\delta,A])\subseteq[\delta,A]$, which is true for sufficiently small $\delta>0$ because $(f^2)'(0)>1$. Choose any such fixed point a, that is,

$$f^2(a)=a\in(0,A],$$

and consider its domain of attraction W in $[0,A]$ defined by

$$W=\{x\in[0,A]\,|\,f^{2n}(x)\to a \text{ as } n\to\infty\}.$$

Clearly $a\in W$ and $0\notin W$. By using Lemma 6.1 with f^2 in place of f we see that the set W is relatively open in $[0,A]$. Let $I\subseteq W$ denote the maximal connected component of W containing $x=a$; thus I is an interval of the form

$$I=(r,s), \quad \text{or else} \quad I=(r,A]$$

where in either case the quantities r and s satisfy

$$0\leqq r<a, \quad a<s\leqq A \quad \text{if } a<A.$$

Because $f(W)\subseteq W$, $f(I)$ is a connected subset of W containing a, the maximality of the connected component implies

$$f(I)\subseteq I. \tag{7.25}$$

Continuity implies that $f^2(\bar{I})\subset\bar{I}$. However, if $I=(r,s)$, we must have that

$$f^2(r),f^2(s)\in\{r,s\}; \tag{7.26}$$

otherwise r or s would be in I. If $I=(r,A]$, the same reasoning implies

$$f^2(r)=r. \tag{7.27}$$

Now observe that neither the point r nor s (if $I=(r,s)$) can be a nonzero fixed point of f^2. By Lemma 6.1 we know that each fixed point of f^2 in $(0,A]$ attracts iterates

$f^{2n}(x)$ of all nearby points x. However, we know that those points $x \in I$ near r or s satisfy $f^{2n}(x) \to a$ instead. Therefore, if $I = (r, s)$, we must have

$$f^2(s) = r,$$

and we must have $f^2(r) = s$ unless $r = 0$. If $I = (r, A]$, we must have $f^2(r) = r$, and we have just seen that this implies $r = 0$. Thus, if $I = (r, A]$, the domain of attraction of the fixed point a of f^2 is the entire interval $(0, A]$, so condition (II) holds. Therefore for the remainder of the proof we assume $I = (r, s)$.

Because $f^2(s) = r$ and $s > 0$ we have $r > 0$, so the previous remarks imply that $f^2(r) = s$. Define $g = f^4$, so g maps $[r, s]$ into itself, g has negative Schwarzian derivative on $[r, s]$, and r, a, and s are fixed points of g. Note that

$$f'(x) = (f^2)'(f^2(x))(f^2)'(x), \tag{7.28}$$

so

$$0 \leqq g'(a) = ((f^2)'(a))^2 \leqq 1. \tag{7.29}$$

Lemma 2.6 of [27] proves that if g is any continuous function on an interval $[r, s]$ and if $g(r) = r$, $g(s) = s$, g is C^3 on (r, s) and $g'(x) \neq 0$ and $(Sg)(x) < 0$ for $x \in (r, s)$, then $g'(a) < 1$ if $a \in (r, s)$ and $g(a) = a$. (Note that the proof of Lemma 2.6 in [27] requires only that g' not vanish on (r, s), although the result is stated slightly less generally.) Thus by Lemma 2.6 of [27] and (7.29) we obtain a contradiction unless $g'(x_0) = 0$ for some $x_0 \in (r, s)$. Because $f^2(I) \subset I$, (7.28) implies f^2 has a critical point in I. Lemma 7.4 now implies there exists $v \in I$ such that $f^2(v) = s$. Since $\lim_{n\to\infty} f^{2n}(x) = a$ for any $x \in I$ and $f^{2n}(v) = r$ or s, we have a contradiction, and the proof is complete. □

If $g(x, \theta)$ is defined for (x, θ) near $(0, \theta_*)$, and if $g(0, \theta) = 0$ and $\partial g(0, \theta_*)/\partial x = -1$, it is natural to ask whether the map $x \to g(x, \theta)$ satisfies (III) at zero for some interval $(\theta_* - \delta, \theta_*)$ or $(\theta_*, \theta_* + \delta)$, $\delta > 0$. This question was answered by Allwright in [1]. He assumed that $x \to g(x, \theta)$ has negative Schwarzian derivative for all x, but his proof only requires that $x \to g(x, \theta)$ have negative Schwarzian derivative for x near zero and θ near θ_*. Thus we obtain the following result, which may also be obtained as a simple consequence of Theorem 7.2.

COROLLARY 7.1 (see Allwright [1]). *Assume $g(x, \theta)$ is defined and continuous for $|x| < \delta_1$ and $|\theta - \theta_*| < \delta_2$. In addition suppose g is C^3 in the x-variable and g has a negative Schwarzian derivative (with respect to the x-variable) at $x = 0$ and $\theta = \theta_*$. Assume $g(0, \theta) = 0$ for $|\theta - \theta_*| < \delta_2$ and $g'(0, \theta_*) = \partial g(0, \theta_*)/\partial x = -1$. Finally, assume $\partial^2 g/\partial\theta\partial x$ is defined and continuous on the domain of g and $\partial^2 g(0, \theta_*)/\partial\theta\partial x \neq 0$. Then there exists $\delta > 0$ such that the map $x \to g(x, \theta)$ satisfies* (III) *for $\theta_* < \theta < \theta_* + \delta$ if $\partial^2 g(0, \theta_*)/\partial\theta\partial x < 0$, while the map $x \to g(x, \theta)$ satisfies* (III) *for $\theta_* - \delta < \theta < \theta_*$ if $\partial^2 g(0, \theta_*)/\partial\theta\partial x > 0$.*

Of course what Allwright really shows is that fixed points of period 2 of the map $x \to g(x, \theta)$ are bifurcating at $\theta = \theta_*$ from the trivial fixed point $x = 0$. The negative Schwarzian condition at $x = 0$ ensures that the bifurcation is such that (III) is satisfied. In fact, the negative Schwarzian condition is also essentially necessary for (III) to be satisfied locally. The following proposition indicates the sense in which necessity is meant. Since the proposition follows by standard arguments in local bifurcation theory, the proof is omitted.

PROPOSITION 7.1. *Assume $g(x, \theta)$ is a C^4 function defined on an open neighborhood of $(0, \theta_*)$. Assume $g(0, \theta) = 0$ for $(0, \theta)$ in the domain of g, $\partial g(0, \theta_*)/\partial x = -1$, and $\partial^2 g(0, \theta_*)/\partial\theta\partial x = \eta_1 \neq 0$. Suppose the Schwarzian derivative of g (with respect to the x-variable) at $x = 0$ and $\theta = \theta_*$ is nonzero and denote this Schwarzian derivative by η_2.*

Then there exist positive numbers ε and δ and continuous functions $\xi_+(\theta)$ and $\xi_-(\theta)$ that are defined for $\theta_* \leqq \theta \leqq \theta_* + \delta$ if $\eta_1\eta_2 > 0$ and for $\theta_* - \delta \leqq \theta \leqq \theta_*$ if $\eta_1\eta_2 < 0$ and that have the following properties:

(1) *$g(\xi_+(\theta), \theta) = \xi_-(\theta)$ and $g(\xi_-(\theta), \theta) = \xi_+(\theta)$,*

(2) *$\xi_+(\theta_*) = 0 = \xi_-(\theta_*)$ and the range of ξ_+ is in $[0, \varepsilon]$ and the range of ξ_- is in $[-\varepsilon, 0]$, and*

(3) *if $g(g(x, \theta), \theta) = x$ for some (x, θ) with $0 < |x| < \varepsilon$ and $|\theta - \theta_*| \leqq \delta$, then θ must be in the domain of ξ_+ and $x = \xi_+(\theta)$ or $x = \xi_-(\theta)$.*

Note that if the Schwarzian derivative is positive, the map $x \to g(x, \theta)$ takes $[\xi_-(\theta), \xi_+(\theta)]$ into itself, but $\partial g(0, \theta)/\partial x > -1$ for θ in the domain of ξ_+.

We need one more theorem for our applications in § 9. Roughly speaking, our next result asserts that for functions with negative Schwarzian derivative, (II) fails before (I).

THEOREM 7.4. *Assume f satisfies* (PM) *and α or β is finite (α, β, ξ, and η are as in the definition of* (PM)). *Suppose f is C^3 on $[-\beta, \alpha]$, $f'(x) \neq 0$ for $x \neq -\eta$ and $x \neq \xi$, and the Schwarzian derivative $Sf(x)$ is negative on $(-\beta, \alpha)$ for $x \neq -\eta, \xi$. If $f([-\beta, \alpha]) \subset [-\beta, \alpha]$ and $f_*^2(\alpha) = \alpha$ or $f_*^2(-\beta) = -\beta$ (f_* as in (5.3)), then there exists $\gamma \in (-\beta, \alpha)$, $\gamma \neq 0$, such that $(f^2)(\gamma) = \gamma$ and $(f^2)'(\gamma) < -1$.*

Proof. Assume for definiteness that $f_*^2(\alpha) = \alpha$. We assume that the theorem is false, so $(f^2)'(\gamma) \geqq -1$ for every nonzero $\gamma \in [-\beta, \alpha]$ such that $f^2(\gamma) = \gamma$, and we try to obtain a contradiction. Recall that Lemma 6.1 implies that if $f^2(\gamma) = \gamma$ and $-1 \leqq (f^2)'(\gamma) < 1$, then γ is a "locally stable fixed point of f^2" in the sense that there exists $\delta > 0$ such that $\lim_{n\to\infty} f^{2n}(x) = \gamma$ for all x such that $|x - \gamma| < \delta$.

There are two cases to consider, each corresponding to a different qualitative appearance of f^2.

Case 1. Assume that $-\eta < f(\xi) = f_*(\alpha)$. In this case we have $f_*^2(\alpha) = f^2(\xi) = \alpha$, and we can easily verify that $(f^2)'(x) > 0$ for $0 \leqq x \leqq \xi$ and $(f^2)'(x) < 0$ for $\xi < x < \alpha$. For notational convenience, define $\xi = \xi_2$ in Case 1.

Case 2. Assume that $f(\xi) = f_*(\alpha) \leqq -\eta$. In this case we have $f_*^2(\alpha) = f(-\eta) = \alpha$. Furthermore, there exist a unique number ξ_1, $0 < \xi_1 \leqq \xi$ and a unique number ξ_2, $\xi \leqq \xi_2 < \alpha$ such that $f(\xi_1) = f(\xi_2) = -\eta$. Using this information, we can easily check that $(f^2)'(x) > 0$ for $0 \leqq x < \xi_1$, $(f^2)'(x) < 0$ for $\xi_1 < x < \xi$, $(f^2)'(x) > 0$ for $\xi < x < \xi_2$ and $(f^2)'(x) < 0$ for $\xi_2 < x < \alpha$.

It follows that (in Case 1 or Case 2), $f^2(\xi_2) = \alpha$ and $(f^2)'(x) < 0$ for $\xi_2 < x < \alpha$. Because $f^2(\alpha) = \alpha$, the intermediate value theorem implies that there is a unique number γ, $\xi_2 < \gamma < \alpha$, such that $f^2(\gamma) = \gamma$. Our assumptions imply

$$-1 \leqq (f^2)'(\gamma) \leqq 0,$$

so our previous remarks imply that γ is a locally stable fixed point of f^2.

Just as in the proof of Theorem 7.3, let $U = \{x \in [0, \alpha] : f^{2n}(x) \to \gamma\}$, so U is an open set, and let U_1 be the maximal connected component of U containing γ, so U_1 is also an open set and $f^2(U_1) \subset U_1$. (Note that $0 \notin U$ and $\alpha \notin U$.) Since U_1 is connected, we can write $U_1 = (r, s)$. If $r < \xi_2$, we have $\xi_2 \in U_1$, so $f^4(\xi_2) = 0 \in U_1$, a contradiction. Thus we must have $\xi_2 \leqq r$. If $s = \alpha$, we obtain $0 = f^2(\alpha) \in U_1$ (because $f^2(U_1) \subset U_1$), and this is impossible because $r \geqq \xi_2$. Thus we must have $s < \alpha$. Just as in the proof of Theorem 7.3, we must have $f^2(r) \in \{r, s\}$ and $f^2(s) \in \{r, s\}$.

There are several possibilities to consider. If $f^2(r) = r$ or $f^2(s) = s$, we contradict the fact that γ is the unique fixed point of f^2 in the interval $[\xi_2, \alpha]$. The only other possibility is that $f^2(r) = s$ and $f^2(s) = r$. If we write $g = f^4$, we know that g has a negative Schwarzian derivative on $[r, s]$ and that r, s and γ are fixed points of g. If

$g'(x) \neq 0$ for $r < x < s$, Lemma 2.6 of [27] implies that $g'(\gamma) > 1$, which is a contradiction. Thus there must exist $x_0 \in (r, s)$ such that $g'(x_0) = 0$. By using the chain rule we see that

$$x_0 = \xi \quad \text{or } f^2(x_0) = \xi \quad \text{or } f(x_0) = -\eta \quad \text{or } f^3(x_0) = -\eta.$$

Because $f^2(x_0) \in (r, s)$, $x_0 \in (r, s)$, and $r \geqq \xi_2$, the only possibility is that $f(x_0) = -\eta$ or $f^3(x_0) = -\eta$. In particular, we must be in Case 2 and have $f(\xi) \leqq -\eta$ and $f(-\eta) = \alpha$. But then we again obtain a contradiction: either $\alpha = f^2(x_0) \in (r, s)$ or $\alpha = f^4(x_0) \in (r, s)$. Since we have obtained a contradiction in all cases, the theorem is proved. □

Remark 7.1. Note that we have actually proved somewhat more than is claimed. If f is as in Theorem 7.4 and $f_*^2(\alpha) = \alpha$ and ξ_2 is as defined in the proof, an examination of the previous argument shows that there exists γ with $\xi_2 < \gamma < \alpha$ such that $f^2(\gamma) = \gamma$ and $(f^2)'(\gamma) < -1$. An analogous statement is true if $f_*^2(-\beta) = -\beta$.

In fact, the same kinds of arguments used in Theorem 7.4 allow a much more detailed picture of the fixed points of $g = f^2$. If f is as in Theorem 7.4 and $f_*^2(\alpha) = \alpha$, and ξ_1 and ξ_2 are as in the proof of Theorem 7.4 (so that $\xi_1 = \xi_2 = \xi$ if $f(\xi) \geqq -\eta$), then we can prove g has no nonzero fixed points on $[0, \xi_1]$. If $g(\xi) \leqq \xi$, then g has unique fixed points γ_1 in $[\xi_1, \xi]$ and γ_2 in $(\xi, \xi_2]$, and $g'(\gamma_2) > 1$. If $g(\xi) > \xi$, then g has no fixed points in $[\xi_1, \xi]$ and g either has zero, 1, or 2 fixed points in $[\xi, \xi_2]$. If g has exactly one fixed point γ_1 in $[\xi, \xi_2]$, then $g'(\gamma_1) = 1$ and $g''(\gamma_1) > 0$. If g has exactly two fixed points $\gamma_1 < \gamma_2$ in $[\xi, \xi_2]$, then $0 < g'(\gamma_1) < 1$ and $g'(\gamma_2) > 1$. Because it is very long, we omit the proof.

COROLLARY 7.2. *Suppose f is as in Theorem 7.4 and $f_n : \mathbb{R} \to \mathbb{R}$, $n \geqq 1$, is a sequence of C^1 functions such that $f_n(x) \to f(x)$ and $f_n'(x) \to f'(x)$ uniformly on compact intervals. Assume f_n satisfies condition* (0) *and positive numbers A_n and B_n exist such that $f_n([-B_n, A_n]) \subset [-B_n, A_n]$, $xf_n(x) < 0$ for all $x \in [-B_n, A_n] - \{0\}$, and $A_n \to \alpha$ and $B_n \to \beta$ as $n \to \infty$ (α and β are as in the definition of* (PM) *for f). If $f_*^2(\alpha) = \alpha$ or $f_*^2(\beta) = \beta$, then there exists $\gamma \in (-\beta, \alpha)$ such that $f^2(\gamma) = \gamma$ and $(f^2)'(\gamma) < -1$, and there exists a sequence $(\gamma_n) \to \gamma$, defined for n sufficiently large, such that $\gamma_n \in (-B_n, A_n)$, $f_n^2(\gamma_n) = \gamma_n$ and $(f_n^2)'(\gamma_n) < -1$.*

Proof. This follows immediately from Theorem 7.4 and elementary calculus arguments. □

8. The Schwarzian derivative of f_k. To apply the theory of § 7 to the functions f_k, we must first show $Sf_k(x) < 0$ for the appropriate ranges of x. As the Schwarzian derivative is invariant under translation, it is sufficient to work directly with the functions f_k rather than with the corresponding normalized functions

$$g_k(x) = f_k(x + x_0) - f_k(x_0). \tag{8.1}$$

PROPOSITION 8.1. *For the functions f_k we have*

$$Sf_k(x) < 0 \quad \text{whenever } f_k'(x) \neq 0 \tag{8.2}$$

for the ranges of parameters and values of x in Table 2. Also, hypothesis (NS_1) *or* (NS_2) *holds at the fixed point x_0 for all $\mu > \mu_0$ as indicated in Table 2, where x_0 and μ_0 are as in Table* 1.

The data of Table 2 are sufficient but not necessary for the Schwarzian derivative of f_k to be negative, or for (NS_1) or (NS_2) to hold. For example, we have not ruled out the possibility that in at least part of the range $0 < \nu < 1$ the condition (NS_2) might hold for f_4 or f_5, rather than the weaker condition (NS_1).

Proof. Rather than prove this result by direct (but lengthy) calculation of the Schwarzian derivatives, we use the results of § 6 to simplify our work.

TABLE 2

Ranges where the Schwarzian derivative of f_k is negative (provided $f_k'(x) \neq 0$), and where (NS_1) *or* (NS_2) *holds at the fixed point x_0 for all $\mu > \mu_0$. The values of x_0 and μ_0 are as in Table* 1.

k	Range where the Schwarzian derivative is negative (where (8.2) holds)	Hypothesis (NS_1) or (NS_2) holding at x_0 for all $\mu > \mu_0$
1	all $x \in \mathbb{R}$, $\mu \in \mathbb{R}$	(NS_2)
2	all $x \in \mathbb{R}$ when $\mu \geqq 0$	(NS_2)
3	all $x \in \mathbb{R}$, $\mu, \theta \in \mathbb{R}$	(NS_2) when $0 \leqq \theta < \pi/2$
4	$x > 0$ when $\nu \geqq 1$	(NS_2) when $\nu \geqq 1$ or $\nu = 0$
	$x > \nu$ when $0 \leqq \nu < 1$	(NS_1) when $0 < \nu < 1$
5	$x > 0$ when $\nu \geqq 1$ and $\lambda > \nu + 1$	(NS_2) when $\nu \geqq 1$ and $\lambda > \nu + 1$, or when $\nu = 0$ and $\lambda > 1$
	$x > \left(\frac{\nu}{\lambda - \nu}\right)^{1/\lambda}$ when $0 \leqq \nu < 1$ and $\lambda > \nu + 1$	(NS_1) when $0 < \nu < 1$ and $\lambda > \nu + 1$

The cases $k = 1$, 2, and 3 follow immediately from Proposition 6.2. (An easy alternate proof in the case of f_3 is to note that $Sf_3(x) \leqq f_3'''(x)/f_3'(x) = -1$ wherever $f_3'(x) \neq 0$.) Essentially the same argument as in the proof of Proposition 6.2 also works for f_4, even though this is not an entire function when ν is not an integer. Assuming $\mu \neq 0$, for $x > 0$ and $x \neq \nu$ we have

$$\frac{d^2}{dx^2} \log |f_4'(x)| = -\frac{\nu - 1}{x^2} - \frac{1}{(x - \nu)^2}, \tag{8.3}$$

which is negative if $\nu \geqq 1$. If $0 < \nu < 1$ and $x > \nu$ then (8.3) is bounded above by $-(\nu - 1)/x^2 - 1/x^2 = -\nu/x^2$, and hence is again negative. When $\nu = 0$, a direct calculation of the Schwarzian derivative shows that $Sf_4(x) = -\frac{1}{2}$ for $x > 0$. The claims of the proposition now follow directly. In particular, we see that (NS_1) holds for f_4 when $0 < \nu < 1$ and $\mu > \mu_0$, when we note that f_4 achieves its maximum at $x = \nu$ and this critical point lies to the left of the fixed point x_0.

The calculations for f_5 are somewhat more involved, but some simplification can be achieved by considering the reciprocal of this function. Setting $m(x) = \mu/x$ where $\mu \neq 0$, from (iii) of Proposition 6.1 we have that for $x > 0$ and $f_5'(x) \neq 0$

$$Sf_5(x) = Sh(x)$$

where

$$h(x) = m(f_5(x)) = x^{-\nu} + x^{\lambda - \nu}.$$

Further calculation yields

$$Sh(x) = -\frac{Py^2 + Qy + R}{x^2[(\lambda - \nu)y - \nu]^2}$$

where

$$P = \tfrac{1}{2}(\lambda - \nu)^2[(\lambda - \nu)^2 - 1],$$

$$Q = \nu(\lambda - \nu)[(\lambda - \nu)^2 + 3\nu(\lambda - \nu) + \nu^2 + 1],$$

$$R = \tfrac{1}{2}\nu^2(\nu^2 - 1),$$

$$y = x^\lambda, \qquad y \neq \frac{\nu}{\lambda - \nu}.$$

If $\nu \geqq 0$ and $\lambda > \nu+1$ as in Table 1 then $P>0$ and $Q>0$. If in addition either $\nu \geqq 1$ or $\nu = 0$, then $R \geqq 0$, and so $Sf_5(x)<0$ for $x>0$. In this case f_5 satisfies (NS_2) when $\mu > \mu_0$, as claimed. On the other hand, if $\lambda > \nu+1$ but $0<\nu<1$, then a calculation reveals $Py^2+Qy+R>0$ at $y=\nu/(\lambda-\nu)$. Thus,

$$Py^2+Qy+R>0 \quad \text{for } y \geqq \frac{\nu}{\lambda-\nu}$$

because $P>0$ and $Q>0$; hence $Sf_5(x)<0$ for $x>(\nu/(\lambda-\nu))^{1/\lambda}$. Again, as f_5 achieves its maximum at $x=(\nu/(\lambda-\nu))^{1/\lambda}$ and this point lies to the left of x_0 when $\mu>\mu_0$, it follows that f_5 satisfies (NS_1), as claimed. □

9. Applications of the general theory to the function f_k. We will now use the results of § 7 to determine ranges of parameters for which hypothesis (I), (II), or (III) holds for f_k at a fixed point x_0 of Table 1. In this connection it will first be useful to make a few general remarks.

Suppose that $f(x)$ is a continuous function with fixed point x_0 and assume that the function $g(x)$ defined by

$$f(x+x_0)-x_0=g(x)$$

satisfies (PM). Recall that f satisfies (I), (II), or (III) at x_0 if and only if g satisfies the corresponding hypothesis at zero. If α, β, ξ, and η are the quantities in (PM) for g, then define $x_1=x_0-\beta$, $x_2=x_0-\eta$, $x_3=x_0+\xi$, and $x_4=x_0+\alpha$. We are considering the function $f(x)$ on the interval $[x_1,x_4]$, and $f(x)>x_0$ for $x_1<x<x_0$, $f(x_1)=x_0$ if $x_1>-\infty$, $f(x)<x_0$ for $x_0<x<x_4$, and $f(x_4)=x_0$ if $x_4<\infty$. Furthermore, $[x_2,x_3]$ is the maximum interval containing x_0 on which f is monotone decreasing and f is monotone increasing on $[x_1,x_2]$ and $[x_3,x_4]$. Upon defining the function $f_*:\mathbb{R}\to\mathbb{R}$ by

$$f_*(x)=\begin{cases} f(x_2) & \text{in } (-\infty,x_2] \quad \text{if } x_2>-\infty,\\ f(x) & \text{in } [x_2,x_3],\\ f(x_3) & \text{in } [x_3,\infty) \quad \text{if } x_3<\infty,\end{cases}$$

we see that (7.1) and (7.2) of Theorem 7.2 for g become

$$(f_*^2)(x_4)<x_4 \quad \text{if } x_4<\infty,$$
$$(f_*^2)(x_1)>x_1 \quad \text{if } x_1>-\infty, \tag{9.1}$$

and

$$f^2(x_3)\leqq x_3 \quad \text{if } x_3<\infty,$$
$$f_*^2(x_2)\geqq x_2 \quad \text{if } x_2>-\infty. \tag{9.2}$$

Thus if g satisfies (PM) and f is C^3 with negative Schwarzian derivative on (x_2,x_3), then f satisfies hypothesis (I) at x_0 if and only if the inequalities (9.1) hold, and f satisfies (III) if and only if (9.2) holds.

Now suppose that f is a continuous function with fixed point x_0, that g satisfies (PM), and x_1, x_2, x_3, and x_4 are as defined above. Suppose that φ is a C^1 function defined on an interval (y_1,y_4), that $\varphi'(y)>0$ for $y_1<y<y_4$, and that $\varphi(y_1)=x_1$ and $\varphi(y_4)=x_4$. If $x_1=-\infty$, then assume for convenience that $y_1=-\infty$, and similarly if $x_4=\infty$, then assume $y_4=\infty$. It is easy to check that f satisfies hypothesis (I), (II), or (III) at x_0 if and only if $\varphi^{-1}f\varphi$ satisfies the corresponding hypothesis at $y_0=\varphi^{-1}(x_0)$. Furthermore, writing $h=\varphi^{-1}f\varphi$, we easily verify that

$$h_*=\varphi^{-1}f_*\varphi. \tag{9.3}$$

If f is C^3 on $[x_2, x_3]$ and $Sf(x)<0$ for $x_2<x<x_3$, it follows from (9.1)-(9.3) and the above remarks that $h=\varphi^{-1}f\varphi$ satisfies (I) at y_0 if and only if

$$(h_*^2)(y_4)<y_4 \quad\text{and}\quad (h_*^2)(y_1)>y_1, \tag{9.4}$$

where $y_j=\phi^{-1}(x_j)$ for $0\leqq j\leqq 4$. Similarly, h satisfies (III) at y_0 if and only if

$$(h_*^2)(y_3)\leqq y_3 \quad\text{and}\quad (h_*^2)(y_1)\geqq y_1. \tag{9.5}$$

Note that h need not have negative Schwarzian derivative on (y_2, y_3).

The above observation sometimes simplifies calculations, since it may be easier to work with h and h_* than with f and f_*.

We now begin the analyses of the functions f_k. Consider first the function f_1. We easily see that with x_0 as in Table 1 and $\mu>\mu_0=\frac{3}{4}$ we have $\alpha+x_0=\xi+x_0=\infty$, $-\beta+x_0=-x_0$, and $-\eta+x_0=0$, so

$$f_{1^*}(x)=\begin{cases}\mu & \text{in } (-\infty, 0],\\ \mu-x^2 & \text{in } (0,\infty).\end{cases}$$

We have $f_{1^*}^2(-\eta+x_0)\geqq-\eta+x_0$ if and only if $\mu-\mu^2\geqq 0$, and $f_{1^*}^2(-\beta+x_0)\geqq-\beta+x_0$ if and only if $\mu-\mu^2\geqq -x_0$, that is,

$$2\mu^2-2\mu+1<\sqrt{4\mu+1}. \tag{9.6}$$

Inequality (9.6) is equivalent to

$$\mu^3-2\mu^2+2\mu-2<0,$$

as a short calculation shows; and this in turn is equivalent to

$$\mu<\mu_*\cong 1.5437$$

where μ_* is the unique real root of $\mu^3-2\mu^2+2\mu-2=0$. By Theorem 7.1 we conclude from these calculations, and the data of Table 1, that (I) holds at x_0 if and only if $\frac{3}{4}<\mu<\mu_*$, and that (III) holds if and only if $\frac{3}{4}<\mu\leqq 1$.

To determine those values of μ between $\mu=\frac{3}{4}$ and $\mu=\mu_*$ at which (II) holds, we must consider points of period 2 for the map f_1 in the interval $(-\beta+x_0, \alpha+x_0)=(-x_0,\infty)$. Assuming that $\frac{3}{4}<\mu<\mu_*$, we consider points x_1 and x_2 satisfying

$$f_1(x_1)=x_2 \quad\text{and}\quad f_1(x_2)=x_1 \tag{9.7}$$

and lying on either side of x_0 in the above interval:

$$-x_0<x_1<x_0<x_2. \tag{9.8}$$

As noted earlier such points do exist; in fact, in the closed interval $[-B+x_0, A+x_0]\subseteq(-\beta+x_0, \alpha+x_0)$. Writing out the equations in (9.7) gives, after some manipulation, that $x_1+x_2=1$ and

$$\mu=1-x_1x_2.$$

Further, the derivative of f_1^2 at either of these points is

$$(f_1^2)'(x_1)=(f_1^2)'(x_2)=f_1'(x_1)f_1'(x_2)=4x_1x_2=4(1-\mu).$$

In the range $\frac{3}{4}<\mu\leqq\frac{5}{4}$ we therefore have $-1\leqq(f_1^2)'(x_1)<1$, so (II) holds by Theorem 7.3. Theorem 7.3 also implies the uniqueness of solutions of (9.7) and (9.8) for $\frac{3}{4}<\mu\leqq\frac{5}{4}$. Of course, since $x_1+x_2=1$, we can also solve for x_1 and x_2 and directly obtain uniqueness. If, on the other hand, $\frac{5}{4}<\mu<\mu_*$, the same calculations show $|(f_1^2)'(x_j)|>1$, and (II) does not hold. Table 3 summarizes our results for f_1 by indicating the parameter ranges where (I), (II), or (III) holds.

The situation for f_2 is very similar to that for f_1. In particular the same sort of analysis as above yields intervals of the parameter μ in which various hypotheses hold. Thus we easily find that f_2 satisfies (I) for $1<\mu<3\sqrt{3}/2$ and (III) for $1<\mu<\frac{3}{2}$. If $f_2(x_1)=x_2$ and $f_2(x_2)=x_1$, where $-\sqrt{\mu}<x_2<0<x_1<\sqrt{\mu}$, then $x_1f_2(x_1)-x_2f_2(x_2)=0$, from which we derive $\mu=x_1^2+x_2^2$, if $x_1\neq -x_2$, or $x_1=-x_2$. The equation

$$f_2(x_1)-x_2-f_2(x_2)+x_1=0$$

implies

$$x_1^2+x_1x_2+x_2^2=\mu-1,$$

so if $x_1\neq -x_2$ we have $x_1^2+x_2^2=\mu$ and

$$x_1x_2=-1. \tag{9.9}$$

Now if $x_2=-x_1^{-1}$, a simple calculation shows that the defining equations for x_2 and x_1 are equivalent to the single equation

$$x_1^4-\mu x_1^2=-1. \tag{9.10}$$

The quadratic equation (9.10) has a real, positive solution if and only if $\mu\geqq 2$. Therefore, if $1<\mu<2$, we must have $x_2=-x_1$. Since f_2 is odd, the defining equations for x_1 and x_2 reduce to the single equation

$$f_2(x_1)=x_1^3-\mu x_1=-x_1,$$

so we obtain

$$x_1=\sqrt{\mu-1} \quad\text{and}\quad x_2=-\sqrt{\mu-1}. \tag{9.11}$$

Substituting (9.11) into the following equation, we obtain

$$\begin{aligned}(f_2^2)'(x_1)=f_2'(x_2)f_2'(x_1)&=(3x_2^2-\mu)(3x_1^2-\mu)\\ &=(2\mu-3)^2.\end{aligned} \tag{9.12}$$

Equation (9.12) implies that $|(f_2^2)'(x_j)|<1$ if $1<\mu<2$, so (II) holds for $1<\mu<2$. On the other hand, if $\mu>2$ and we take $x_2=-x_1$, Theorem 7.3 and (9.12) imply that (II) is not satisfied.

Finally, if $\mu=2$, a direct calculation using the above information shows that $x_1=1$ and $x_2=-1$ are the only nonzero fixed points of f_2^2 and

$$(f_2^3)''(x_1)=0,$$

so Theorem 7.3 again implies that (II) is satisfied. All of this information is summarized in Table 3.

TABLE 3

$f_k(x)$ satisfies (I) at x_0 if and only if $\mu_0<\mu<\mu_1$, $f_k(x)$ satisfies (II) if and only if $\mu_0<\mu\leqq\mu_2$, and $f_k(x)$ satisfies (III) at x_0 if and only if $\mu_0<\mu\leqq\mu_3$.

k	x_0	μ_0	μ_1	μ_2	μ_3
1	$(-1+\sqrt{4\mu+1})/2$	$\frac{3}{4}$	$\mu_1^3-2\mu_1^2+2\mu_1-2=0$ $\mu_1\cong 1.5437$	$\frac{5}{4}$	1
2	0	1	$3\sqrt{3}/2$	2	$\frac{3}{2}$

For the map f_3 with fixed point $x_0=0$ and parameter θ chosen in the range $0 \leqq \theta < \pi/2$ established earlier, we again find intervals of μ in which (I) and (III) hold. The range where (II) holds, however, is still not clear. Recall the critical value $\mu_0 = \mu_0(\theta) = 1/\cos\theta$ from Table 1.

THEOREM 9.1. *There exist continuous functions*

$$\rho_3, \tau_3 : \left[0, \frac{\pi}{2}\right) \to (0, \infty)$$

satisfying

$$\mu_0(\theta) = \frac{1}{\cos\theta} < \rho_3(\theta) < \tau_3(\theta)$$

such that if $0 \leqq \theta \leqq \pi/2$, *then* (I) *holds for* f_3 *at* $x_0 = 0$ *if and only if* $\mu_0(\theta) < \mu < \tau_3(\theta)$, *and* (III) *holds if and only if* $\mu_0(\theta) < \mu \leqq \rho_3(\theta)$.

Motivated by the results for f_1 and f_2, we might expect the existence of a third function σ_3 satisfying $\rho_3(\theta) < \sigma_3(\theta) < \tau_3(\theta)$ and such that (II) holds if and only if $\mu_0(\theta) < \mu \leqq \sigma_3(\theta)$. We believe this to be the case, but we have not pursued this question here. However, we can easily prove by an implicit function theorem argument that condition (II) holds for $\mu_0(\theta) < \mu < \rho_3(\theta) + \delta_3(\theta)$ for some sufficiently small $\delta_3(\theta) > 0$: the period 2 points $\{x_1 x_2\}$ of f_3 for $\mu = \rho_3(\theta)$ are "super-stable" (that is, $(f_3^2)'(x_1) = (f_3^2)'(x_2) = 0$) and so must persist for μ slightly larger than $\rho_3(\theta)$. On the other hand, Corollary 7.2 implies that there exists $\delta_4(\theta) > 0$ such that for $\tau_3(\theta) - \delta_4(\theta) < \mu < \tau_3(\theta)$ the function f_3^2 has a fixed point γ (in the relevant interval) such that $(f_3^2)'(\gamma) < -1$, and this implies (II) fails for $\tau_3(\theta) - \delta_4(\theta) < \mu < \tau_3(\theta)$. By using Remark 7.1 we can also show that, for θ near zero and μ near $\tau_3(\theta)$, f_3^2 has a second fixed point $\bar{\gamma}$ for which $(f_3^2)'(\bar{\gamma}) > 1$.

Proof of Theorem 9.1. Assuming $\mu > 0$ and $0 \leqq \theta < \pi/2$, we note the following values:

$$\alpha = \pi - \theta \leqq \beta = \pi + \theta, \qquad \xi = \frac{\pi}{2} - \theta \leqq \eta = \frac{\pi}{2} + \theta$$

for f_3 in condition (PM). We also note the following formulas:

$$f_{3*}(\alpha) = f_{3*}(\xi) = -\mu(1 - \sin\theta),$$
$$f_{3*}(-\beta) = f_{3*}(-\eta) = \mu(1 + \sin\theta).$$

In order to use Theorem 7.1 for determining when (I) or (III) holds, we must calculate both $(f_{3*}^2)(\alpha) = (f_{3*}^2)(\xi)$ and $(f_{3*}^2)(-\beta) = (f_{3*}^2)(-\eta)$, and examine (7.1) or (7.2). In fact, we claim

$$\begin{aligned} &f_{3*}^2(\alpha) < \alpha \quad \text{implies } f_{3*}^2(-\beta) > -\beta, \\ &f_{3*}^2(\xi) \leqq \xi \quad \text{implies } f_{3*}^2(-\eta) \geqq -\eta. \end{aligned} \tag{9.13}$$

Thus, we need only verify the inequality $f_{3*}^2(\alpha) < \alpha$ to conclude (I), and $f_{3*}^2(\xi) \leqq \xi$ to conclude (III).

We prove only the first implication (9.13), as the proof of the other is similar. Suppose

$$f_{3*}^2(-\beta) \leqq -\beta. \tag{9.14}$$

Then as f_{3*} achieves its minimum at $x = \xi$, we have

$$f_{3*}(\xi) \leqq -\beta \tag{9.15}$$

from (9.14), and so

$$f_{3*}^2(\xi)=f_{3*}(-\beta) \tag{9.16}$$

as f_{3*} is constant to the left of $-\beta$. Thus, from (9.14)-(9.16) we obtain

$$\begin{aligned} f_{3*}^2(\alpha)&=f_{3*}^2(\xi)=f_{3*}(-\beta)=\mu(1+\sin\theta)\\ &\geqq \mu(1-\sin\theta)=-f_{3*}(\xi)\geqq\beta\geqq\alpha. \end{aligned}$$

The required inequality $f_{3*}^2(\alpha)\geqq\alpha$, from which the implication (9.13) follows, is now proved.

We now calculate the quantity

$$(f_{3*}^2)(\xi)=(f_{3*}^2)(\alpha)=f_{3*}(-\mu(1-\sin\theta)) \tag{9.17}$$

and compare it with either α or ξ, as described above. Let $h_3(\mu,\theta)$ denote $(f_{3*}^2)(\alpha)$; then we easily see

$$h_3(\mu,\theta)=\begin{cases}\mu[\sin\theta-\sin(\theta-\mu(1-\sin\theta))] & \text{if } \mu\leqq\dfrac{\pi/2+\theta}{1-\sin\theta},\\[2ex] \mu(1+\sin\theta) \quad \text{if } \mu\geqq\dfrac{\pi/2+\theta}{1-\sin\theta}. & \end{cases}$$

A simple calculation shows that $\partial h_3(\mu,\theta)/\partial\mu>0$ for all $\mu>0$ in the two ranges of μ for which h_3 is defined. Thus $h_3(\mu,\theta)$ is strictly increasing in μ, for each fixed $\theta\in[0,\pi/2)$, and assumes every positive value exactly once for $\mu>0$. Thus there exist continuous functions $\rho_3,\tau_3:[0,\pi/2)\to(0,\infty)$ satisfying

$$h_3(\rho_3(\theta),\theta)=\alpha=\pi-\theta, \qquad h_3(\tau_3(\theta),\theta)=\xi=\pi/2-\theta.$$

By Theorem 7.2 we also have

$$h_3(1/\cos,\theta)<\xi,$$

since $f_3'(0)=-1$ when $\mu=1/\cos\theta$. Thus,

$$1/\cos\theta<\rho_3(\theta)<\tau_3(\theta)$$

and the result follows immediately. □

As has already been noted in § 3, if the function f_3 is not in our normal form, there may be some difficulties in determining the ranges of the original parameters for which (I) or (III) is satisfied. To illustrate this point, we consider

$$\varepsilon\dot{x}(t)=-x(t)+\mu(1-\sin(x(t-1))), \tag{9.18}$$

which has been studied numerically by Chow and Green [4]. For each $\mu>0$, we can easily see that

$$\mu(1-\sin x)=x, \qquad 0<x<\pi/2, \tag{9.19}$$

has a unique fixed point $\theta=\theta(\mu)\in(0,\pi/2)$, and using the implicit function theorem we can see that $\theta'(\mu)>0$ for $\mu>0$. The question is does $\mu(1-\sin x)$ satisfy condition (I) or (III) at $x=\theta(\mu)$.

PROPOSITION 9.1. *For each $\mu>0$, the function $\mu f(x)=\mu(1-\sin x)$ has a unique fixed point $\theta=\theta(\mu)\in(0,\pi/2)$. There exist numbers μ_0 (μ_0 is approximately equal to* 1.1773) *and μ_1 (μ_1 is approximately equal to* 2.3879) *such that $\mu f(x)$ satisfies* (I) *at*

$\theta(\mu)$ if and only if $\mu_0<\mu<\mu_1$ and $\mu f(x)$ satisfies (III) *at $\theta(\mu)$ if and only if $\mu_0<\mu\leqq\pi/2$. The equation*

$$\frac{\theta\cos\theta}{1-\sin\theta}=1,\qquad 0<\theta<\frac{\pi}{2},$$

has a unique solution $\theta_0\in(0,\pi/2)$ and $\mu_0=\theta_0/(1-\sin\theta_0)$. The equation

$$\theta+\left(\frac{\theta}{1-\sin\theta}\right)=\pi,\qquad 0<\theta<\frac{\pi}{2},$$

has a unique solution $\theta_1\in(0,\pi/2)$ and $\mu_1=\theta_1/(1-\sin\theta_1)$.

Proof. The idea of the proof is to parameterize by the fixed point $\theta\in(0,\pi/2)$ instead of by μ. If $\theta\in(0,\pi/2)$ is the fixed point of $\mu f(x)$, then

$$\mu=\frac{\theta}{1-\sin\theta}.$$

Thus, for $0<\theta<\pi/2$, define $g(x,\theta)=g_\theta(x)$ by

$$g(x,\theta)=g_\theta(x)=\left(\frac{\theta}{1-\sin\theta}\right)(1-\sin(x+\theta))-\theta.$$

We can easily check that the map $\theta\to(\theta/1-\sin\theta))$ is strictly increasing for $0<\theta<\pi/2$ and

$$\begin{aligned}&\{\mu:\mu f(x)\text{ satisfies (III) at }\theta(\mu)\}\\&\quad=\{\theta/(1-\sin\theta): 0<\theta<\pi/2\text{ and }g_\theta(x)\text{ satisfies (III) at zero}\},\end{aligned}$$

with an analogous equation concerning (I). If the numbers $\alpha=\alpha(\theta)$, $\beta=\beta(\theta)$, $\xi=\xi(\theta)$, and $\eta=\eta(\theta)$ are as in the definition of condition (PM) for g_θ, then $\alpha=\pi-2\theta$, $\beta=\pi+2\theta$, $\xi=\pi/2-\theta$, and $\eta=\pi/2+\theta$. The same argument as in Theorem 9.1 shows that g_θ satisfies (III) if and only if

$$(g^2_{\theta^*})\left(\frac{\pi}{2}-\theta\right)\leqq\frac{\pi}{2}-\theta,\tag{9.20}$$

$$g'_\theta(0)=-\left(\frac{\theta\cos\theta}{1-\sin\theta}\right)<-1.\tag{9.21}$$

Since $g_{\theta^*}(\pi/2-\theta)=-\theta>-\eta$, we easily compute that (9.20) holds if and only if

$$\left(\frac{\theta}{1-\sin\theta}\right)\leqq\frac{\pi}{2}.\tag{9.22}$$

Thus g_θ, $0<\theta<\pi/2$, satisfies (III) if and only if (9.21) and (9.22) hold. Similarly, we see that g_θ satisfies (I) if and only if (9.21) is valid and

$$\begin{aligned}&(g^2_{\theta^*})(\pi-2\theta)=\left(\frac{\theta}{1-\sin\theta}\right)-\theta<\pi-2\theta,\quad\text{that is,}\\&\left(\frac{\theta}{1-\sin\theta}\right)+\theta<\pi,\qquad 0<\theta<\tfrac{\pi}{2}.\end{aligned}\tag{9.23}$$

It remains to study where (9.21)-(9.23) hold. Obviously the function $(\theta/(1-\sin\theta))+\theta=h(\theta)$ satisfies $h(0)=0$, $\lim_{\theta\to\pi/2}h(\theta)=\infty$ and $h'(\theta)>0$ for $0<\theta<\pi/2$, so there is a unique number θ_1 such that

$h(\theta_1)=\pi$ and $h(\theta)<\pi$ if and only if $0<\theta_1<\pi$.

In order to study where $k(\theta)=(\theta\cos\theta/(1-\sin\theta))>1$ for $\theta\in(0,\pi/2)$, first observe that the mean value theorem gives

$$1-\sin\theta=\left(\frac{\pi}{2}-\theta\right)\cos(\psi),\qquad \theta<\psi<\frac{\pi}{2}.$$

If $\pi/4\leqq\theta\leqq\pi/2$, this implies

$$k(\theta)=\left(\frac{\theta}{\pi/2-\theta}\right)\left(\frac{\cos\theta}{\cos\psi}\right)>1.$$

On the other hand, if $0<\theta<\pi/4$ we have

$$k'(\theta)=[(\cos\theta-\theta\sin\theta)(1-\sin\theta)+\theta\cos^2\theta](1-\sin\theta)^{-2}.$$

Because we also have

$$\cos\theta-\theta\sin\theta>\cos\theta-\sin\theta\geqq 0\quad\text{for } 0<\theta\leqq\pi/4,$$

we conclude that $k'(\theta)>0$ for $0<\theta\leqq\pi/4$. From the above facts it follows that $k(\theta)=1$ has a unique solution $\theta_0\in(0,\pi/2)$, that $0<\theta_0<\pi/4$, and that $k(\theta)>1$ for $\theta\in(0,\pi/2)$ if and only if $\theta_0<\theta<\pi/2$. This completes the proof of the proposition. (Approximate values of θ_0 and θ_1, and hence of μ_0 and μ_1, can easily be computed using Newton's method.) □

Proposition 9.1 and the results summarized in § 2 provide some explanation of the numerical results in [4]. For example, if $\mu_0<\mu\leqq\pi/2$ we see for small ε the regular SOP-solutions predicted by Theorem 2.3. Theorem 2.1 asserts that SOP-solutions persist for $\mu_0<\mu<\mu_1$ and $\varepsilon>0$ sufficiently small. However, these solutions apparently lose stability for μ near μ_1 and $\varepsilon>0$ small: for μ near μ_1 Chow and Green appear to have found, numerically, periodic solutions that are *not* SOP-solutions.

We now want to examine when (I), (II), or (III) is satisfied by the functions f_4 or f_5. With $k=4$ or $k=5$ and fixed parameters $\nu\geqq 0$ and $\lambda>\nu+1$ (when $k=5$), we will write

$$f_k(x)=\mu\bar{f}_k(x), \tag{9.24}$$

where the function $\bar{f}_k(x)$ does not depend on μ. In our next several theorems we will discuss the range of μ for which the functions f_4 and f_5 satisfy (I) or (III), and we will return later to (II). The next theorem provides a reasonably sharp and general answer concerning when (III) holds for functions $\mu\bar{f}(x)$; the question of (I) seems more difficult.

THEOREM 9.2. *Assume $\bar{f}:(0,\infty)\to(0,\infty)$ is a C^3 function and there exists a number $\theta\geqq 0$ such that $\bar{f}'(s)>0$ for $0<s<\theta$ and $\bar{f}'(s)<0$ for $s>\theta$. Assume there exists a unique number $s_0>\theta$ such that*

$$\frac{d}{ds}(s\bar{f}(s))|_{s=s_0}=0,$$

and assume also

$$\frac{d}{ds}(s\bar{f}(s))|_{s=s_0}<0\quad\forall s>s_0.$$

Finally, assume $(S\bar{f})(x)<0$ for $x>\theta$. Define functions $\mu(s)$, $x_1(s)$, $x_2(s)$, and $x_3(s)$ by $\mu(s)=s(\bar{f}(s))^{-1}$, $x_1(s)=\mu(s)\bar{f}(\theta)$, and

$$x_j(s)=\mu(s)\bar{f}(x_{j-1}(s))\quad\textit{for } j\geqq 2.$$

If $x_0(\mu)$ denotes the unique fixed x of $\mu f(x)$ such that $x \geqq \theta$ (for $\mu \geqq \theta(\bar{f}(\theta))^{-1}$), then

$$\{\mu\colon \mu\bar{f}(x) \text{ satisfies (I) at } x_0(\mu)\} = \{\mu(s)\colon s > s_0 \text{ and } x_3(s) > s\}$$

and

$$\{\mu\colon \mu f(x) \text{ satisfies (III) at } x_0(\mu)\} = \{\mu(s)\colon s > s_0 \text{ and } x_2(s) \geqq \theta\}.$$

There exists a number $\rho > \mu(s_0) \equiv \mu_0$ such that

$$\{\mu\colon \mu f(x) \text{ satisfies (III) at } x_0(\mu)\} = (\mu_0, \rho],$$

and $\rho < \infty$ if $\lim_{x\to\infty} x\bar{f}(x) = 0$. The number ρ is $z(\bar{f}(\theta))^{-1}$ where z is the unique solution of $z\bar{f}(z) = \theta\bar{f}(\theta)$ such that $z > s_0$ (if such a solution exists).

If D is an open subset of $\mathbb{R}^m$ and $\bar{f}\colon(0,\infty)\times D \to (0,\infty)$ is a C^3 map such that $x \to \bar{f}(x, \gamma)$ satisfies the conditions of our theorem for each $\gamma \in D$ (so $\theta = \theta(\gamma)$ and $s_0 = s_0(\gamma)$ exist and are easily proven to be continuous), then the number $\rho = \rho(\gamma)$ is also a continuous function of γ. (If $\rho(\gamma) = \infty$ for some γ, continuity is interpreted in the obvious way.)

Proof. First assume $\bar{f}$ is independent of $\gamma \in D$. For a given $\mu > 0$ let $x_0 = x_0(\mu) \geqq \theta$ and define $\delta = \delta(\mu) \leqq \theta$ by

$$\bar{f}(\delta) = \bar{f}(x_0).$$

Theorem 4.1 implies that $\mu\bar{f}(x)$ satisfies condition (0) at x_0 if and only if $\mu > \mu_0 = s_0(\bar{f}(s_0))^{-1} = \mu(s_0)$, and the results of § 7 imply that $\mu\bar{f}(x)$ satisfies (III) at x_0 if and only if $\mu > \mu_0$ and

$$(\mu\bar{f})^2(\theta) \geqq \theta \tag{9.25}$$

where $(\mu\bar{f})^j$ is the composition of $\mu\bar{f}$ with itself j times. Similarly, $\mu\bar{f}(x)$ satisfies (I) at x_0 if and only if $\mu > \mu_0$ and

$$(\mu\bar{f})^2(\theta) > \delta = \delta(\mu),$$

or equivalently $\mu > \mu_0$ and

$$(\mu\bar{f})^3(\theta) > x_0(\mu). \tag{9.26}$$

(Note that $\mu\bar{f}(\theta) > \mu\bar{f}(x_0) = x_0$, so $(\mu\bar{f})^2(\theta) < x_0$.)

As in Theorem 4.1 the key idea is to parameterize by $s = x_0(\mu)$ instead of μ. Since $\mu\bar{f}(x_0) = x_0$, this gives

$$\mu = s(\bar{f}(s))^{-1} = \mu(s),$$

which is an increasing function of s for $s \geqq \theta$. In terms of this parameterization we find that $\mu(s)\bar{f}(x)$ satisfies (III) at $s > \theta$ if and only if $s > s_0$ and

$$x_2(s) \geqq \theta. \tag{9.27}$$

Similarly, we see that $\mu(s)\bar{f}(x)$ satisfies (I) at s if and only if $s > s_0$ and

$$x_3(s) > s. \tag{9.28}$$

Since Theorem 7.2 and Corollary 7.1 imply that there exists $\delta > 0$ such that $\mu(s)\bar{f}(x)$ satisfies (III) for $s_0 < s \leqq s_0 + \delta$, we have proved the first part of the theorem.

It remains to prove that (III) is satisfied on an interval $(\mu_0, \rho]$. Equivalently, it suffices to prove that

$$\{s\colon s > s_0, x_2(s) \geqq \theta\}$$

is an interval. We know that $x_2(s) > \theta$ for s near s_0, so it suffices to prove $x_2'(s) < 0$ for $s > s_0$. A calculation gives

$$x_1'(s) = \bar{f}(\theta)(\bar{f}(s))^{-2}[\bar{f}(s) - s\bar{f}'(s)],$$

and since $\bar{f}'(s) < 0$ for $s > s_0$, $x_1'(s) > 0$ for all $s > s_0$. Note that we have

$$\mu(s)\bar{f}(\theta) = x_1(s) > \mu(s)\bar{f}(s) = s > s_0,$$

so we find that

$$\lim_{s\to\infty} x_1(s) = \infty.$$

We can write

$$\mu(s) = x_1(s)(\bar{f}(\theta))^{-1},$$

so if we define $g(x) = (\bar{f}(\theta))^{-1}x\bar{f}(x)$,

$$x_2(s) = g(x_1(s)), \qquad x_2'(s) = g'(x_1(s))x_1'(s).$$

However, $x_1(s) > s_0$ and we have assumed that $g'(x) < 0$ for $x > s_0$, so $x_2'(s) < 0$.

If $\sigma > s_0$ is such that $x_2(\sigma) = \theta$, and if $z = x_1(\sigma)$ the above calculation shows that $z\bar{f}(z) = \theta\bar{f}(\theta)$ and $\rho = \mu(\sigma) = z(\bar{f}(\theta))^{-1}$.

If we assume $\lim_{x\to\infty} x\bar{f}(x) = 0$, we obtain (because $x_1(s) \to \infty$ as $s \to \infty$) the following

$$\lim_{s\to\infty} x_2(s) = \lim_{s\to\infty} g(x_1(s)) = 0.$$

This implies that for all large s, $x_2(s) < \theta$, so ρ is finite in this case.

If $\bar{f}$ depends on a parameter $\gamma \in D$, the continuity of $\rho(\gamma)$ follows easily from the implicit function theorem at points where $\rho(\gamma) < \infty$, and continuity at points γ where $\rho(\gamma) = \infty$ is also easy. Details are left to the reader. □

Remark (9.1). *If $\bar{f}: (0, \infty) \to (0, \infty)$ is as Theorem 9.2 and $\varphi: (\alpha, \beta) \to (0, \infty)$ is a C^1 map onto $(0, \infty)$ with positive derivative, the remarks at the beginning of this section show that $\varphi^{-1}(\mu\bar{f})\varphi$ satisfies (I) or (III) if and only if $\mu\bar{f}$ satisfies (I) or (III).*

As an immediate consequence of Theorem 9.2 we obtain Theorem 9.3.

THEOREM 9.3. *Let $f_k(x)$, $k = 4$ or 5, be as usual and assume $\nu > 0$ and $\lambda > \nu + 1$ if $k = 5$. Define $\mu_k(s) = s(\bar{f}_k(s))^{-1}$ and define $\theta_k = \nu$ for $k = 4$ and $\theta_k^\lambda = \nu(\lambda - \nu)^{-1}$ for $k = 5$. Define $\sigma_k = \nu + 1$ for $k = 4$ and $\sigma_k^\lambda = (\nu + 1)(\lambda - v - 1)^{-1}$ for $k = 5$. Define z_k to be the unique solution $z > \sigma_k$ of*

$$z\bar{f}_k(z) = \theta_k\bar{f}_k(\theta_k).$$

If x_0 denotes generically the unique fixed point greater than θ_k of $\mu\bar{f}_k(x)$, then

$$\{\mu: \mu\bar{f}_k(x) \text{ satisfies (III) at } x_0\} = (\mu_k(\sigma_k), \rho_k],$$

where $\rho_k = z_k(\bar{f}_k(\theta_k))^{-1}$ is a continuous function of ν and λ. If $x_1(s) = \mu_k(s)\bar{f}_k(\theta_k)$ and $x_j(s) = \mu_k(s)\bar{f}_k(x_{j-1}(s))$ for $j \geqq 2$, then

$$\{\mu: \mu\bar{f}_k(x) \text{ satisfies (I) at } x_0\} = \{\mu_k(s): s > \sigma_k \text{ and } x_3(s) > s\}.$$

If $\nu = 0$, then $\mu\bar{f}_k(x)$ satisfies (III) at x_0 for all $\mu > \mu_0$.

Proof. The number θ_k plays the role of θ in Theorem 9.2, and σ_k the role of s_0. We have already verified the negative Schwarzian condition on $\bar{f}_k$ and the other hypotheses of Theorem 9.2 are easily verified, so Theorem 9.3 follows directly from Theorem 9.2. □

We now want to study more precisely when $\mu \bar{f}_4(x)$ satisfies (I). It is convenient to give a calculus lemma first.

LEMMA 9.1. *If* $\nu > 0$, *then*

$$\left(\frac{\nu+2}{\nu}\right)^{\nu+1} > e^2, \tag{9.29}$$

and if $\nu \geqq 1$

$$e < \left(\frac{\nu+2}{\nu+1}\right)\left(\frac{\nu+1}{\nu}\right)^{\nu}. \tag{9.30}$$

If $0 \leqq c < \sqrt{5}-2$, *then there exists* $\nu(c) \geqq 1$ *such that the following inequality is valid for* $\nu \geqq \nu(c)$:

$$e^{1+c} \leqq \left(\frac{\nu+1+c}{\nu}\right)^{\nu}\left(\frac{\nu+2}{\nu+1+c}\right). \tag{9.31}$$

Proof. By taking natural logarithms we see that (9.29) is equivalent to proving

$$(\nu+1)\log\left(1+\frac{2}{\nu}\right) > 2,$$

and the above inequality is equivalent to

$$(\nu+1)\int_0^{2/\nu}\left(\frac{1}{1+t}\right)dt > \int_0^{2/\nu} \nu\, dt.$$

The above inequality is equivalent to

$$\int_0^{1/\nu}\left(\frac{1-\nu t}{1+t}\right)dt = I_1 > \int_{1/\nu}^{2/\nu}\left(\frac{\nu t-1}{1+t}\right)dt = I_2.$$

Change variables in I_1 by setting $t = 1/\nu - \rho$ to obtain

$$I_1 = \int_0^{1/\nu}\left(\frac{\nu\rho}{1+(1/\nu)-\rho}\right)d\rho$$

and change variables in I_2 by setting $t = 1/\nu+\rho$ to obtain

$$I_2 = \int_0^{1/\nu}\left(\frac{\nu\rho}{1+(1/\nu)+\rho}\right)d\rho.$$

Since $\nu > 0$, we have

$$\nu\rho(1+\nu^{-1}-\rho)^{-1} > \nu\rho(1+\nu^{-1}+\rho)^{-1} \quad \text{for } 0 < \rho < \nu^{-1},$$

so $I_1 > I_2$.

The proofs of (9.30) and (9.31) are like the proof of (9.29). If $\nu \geqq 1$, (9.30) is equivalent to

$$1 < \log\left(1+\frac{1}{\nu+1}\right) + \nu\log\left(1+\frac{1}{\nu}\right).$$

Expressing both sides as integrals, the above inequality is equivalent to proving

$$\int_0^{1/\nu} \nu\, dt < \int_0^{1/\nu}\left(\frac{\nu+1}{1+t}\right)dt - \int_{1/(\nu+1)}^{1/\nu}\left(\frac{1}{1+t}\right)dt.$$

Simplification shows that the above inequality is equivalent to

$$0 < \int_0^{1/(\nu+1)}\left(\frac{1-\nu t}{1+t}\right)dt - \int_{1/(\nu+1)}^{1/\nu}\left(\frac{\nu t}{1+t}\right)dt = J_1 - J_2. \tag{9.32}$$

Making the change of variables $t=1/\nu-\rho$ in J_2, we have

$$J_2=\int_0^{1/\nu-1/(\nu+1)}\left(\frac{1-\nu t}{1+\nu^{-1}-t}\right)dt. \tag{9.33}$$

Since $\nu\geqq 1$ we can easily verify that

$$(\nu+1)^{-1}\geqq \nu^{-1}-(\nu+1)^{-1},$$

and using (9.32) and (9.33) we see that $J_1>J_2$ if

$$(1-\nu t)(1+t)^{-1}>(1-\nu t)(1+\nu^{-1}-t)^{-1}\quad \text{for } 0<t<\nu^{-1}-(\nu+1)^{-1}. \tag{9.34}$$

However, again using that $\nu\geqq 1$, we can check that inequality (9.34) holds, so $J_1>J_2$.

By taking logarithms we see that inequality (9.31) is equivalent to

$$1+c\leqq \nu\log\left(1+\frac{1+c}{\nu}\right)+\log\left(1+\frac{1-c}{\nu+1+c}\right),$$

or, by expressing both sides as integrals,

$$\int_0^{(1+c)/\nu}\nu\,dt\leqq\int_0^{(1+c)/\nu}\left(\frac{\nu}{1+t}\right)dt+\int_0^{(1-c)/(\nu+1+c)}\left(\frac{1}{1+t}\right)dt.$$

By simplifying we see that inequality (9.31) is equivalent to

$$K_1=\int_0^{(1+c)/\nu}\left(\frac{\nu t}{1+t}\right)dt\leqq\int_0^{(1-c)/(\nu+1+c)}\left(\frac{1}{1+t}\right)dt=K_2.$$

Since $(1+t)^{-1}<1$ for $t>0$ we see that

$$K_1<\int_0^{(1+c)/\nu}(\nu t)\,dt=\left(\frac{1}{2}\right)\frac{(1+c)^2}{\nu}. \tag{9.35}$$

On the other hand, $(1+t)^{-1}>1-t$, so

$$K_2>\int_0^{(1-c)/(\nu+1+c)}(1-t)\,dt=\left(\frac{1-c}{\nu+1+c}\right)-\frac{1}{2}\left(\frac{1-c}{\nu+1+c}\right)^2. \tag{9.36}$$

For a given c, $0<c<1$, it follows from (9.35) and (9.36) that $K_1<K_2$ if

$$\left(\frac{1}{2}\right)\frac{(1+c)^2}{\nu}<\left(\frac{1-c}{\nu+1+c}\right)-\frac{1}{2}\left(\frac{1-c}{\nu+1+c}\right)^2.$$

Multiplying by $2\nu(\nu+1+c)$, we see that the above inequality is equivalent to

$$0<(1-4c-c^2)\nu-(1+c)^3-(1-c)^2\left(\frac{\nu}{\nu+1+c}\right). \tag{9.37}$$

If $1-4c-c^2>0$, i.e., $c<\sqrt{5}-2$, then inequality (9.37) will be satisfied for all $\nu\geqq\nu(c)$, where

$$\nu(c)=(1-4c-c^2)^{-1}[(1+c)^3+(1-c)^2]=(1-4c-c^2)^{-1}[c^3+4c^2+c+2]. \tag{9.38}$$ □

THEOREM 9.4. *There exists a continuous function*

$$\tau_4:(0,\infty)\to(0,\infty)$$

satisfying $\mu_0(\nu)=(\nu+1)^{1-\nu}e^{\nu+1}<\rho_4(\nu)<\tau_4(\nu)$, *where* $\rho_4=\rho_4(\nu)$ *is as in Theorem* 9.3, *such that* (I) *holds for* $\mu\bar{f}_4(x)$ (*at its unique fixed point in* (ν,∞)) *if and only if* $\mu_0(\nu)<\mu<\tau_4(\nu)$. *Furthermore, we have that*

$$\begin{gathered}\lim_{\nu\to 0^+}\tau_4(\nu)=\infty \text{ and } \lim_{\nu\to\infty}[\tau_4(\nu)-\mu_0(\nu)]=0, \text{ and for } \nu\geqq 2\\ \tau_4(\nu)-\mu_0(\nu)\leqq\mu(2\nu)-\mu(\nu+1)\leqq(e^2/2\nu)^\nu(2\nu),\end{gathered} \tag{9.39}$$

where we have defined $\mu(s)=s^{1-\nu}e^s$. *For* $0<c<\sqrt{5}-2$ *and for* $\nu\geqq\nu(c)$ ($\nu(c)$ *as in* (9.38)) *we have*

$$[\mu_0(\nu),\rho_4(\nu)]\supset\{s^{1-\nu}e^s:\nu+1<s\leqq\nu+1+c\}. \tag{9.40}$$

Proof. For notational convenience we write $\bar{f}_4(x)=\bar{f}(x)$, $\theta=\nu$, and $s_0=\nu+1$, and define $x_j(s)$ as in Theorem 9.2. Theorems 9.2 and 9.3 imply

$$[\mu_0(\nu),\rho_4(\nu)]=\{\mu(s):x_2(s)\geqq\theta\},$$

$$\{\mu:\mu\bar{f}(x)\text{ satisfies (I)}\}=\{\mu(s):s>s_0\text{ and }x_3(s)>s\}.$$

Furthermore, we have already shown that

$$x_1'(s)>0,\quad x_2'(s)<0\quad\forall s>s_0.$$

We fix $\nu>0$ and first show $x_3(s)<s$ for all large s or, equivalently, that

$$\log(s^{-1}x_3(s))<0\quad\forall\text{ large }s.$$

Using the definition of $x_j(s)$, we find

$$\begin{aligned}\log(s^{-1}x_3(s))&<-\nu\log s+s+\nu\log x_2\\&=-\nu^3\log s+(1+\nu+\nu^2)s+\nu^3\log\nu-\nu^3-\nu^{\nu+1}e^{-\nu}s^{1-\nu}e^s.\end{aligned} \tag{9.41}$$

Because the e^s term is dominant for large s, the right-hand side of (9.41) is negative for all large s, so (for fixed $\nu>0$) for every sufficiently large μ, $\mu\bar{f}(x)$ does not satisfy (I).

If $\nu\geqq1$, it is a calculus exercise (which we leave to the reader) to prove that the derivative of the right-hand side of (9.41) with respect to s is negative for $s\geqq2\nu$. Another calculus exercise left to the reader is to verify that

$$\frac{d}{d\nu}(\log(s^{-1}x_3(s))|_{s=2\nu})<0\quad\text{for }\nu\geqq2.$$

A direct calculation shows that the right-hand side of (9.41) is negative for $\nu=2$ and $s=4=2\nu$, and combining the above information we conclude that

$$\log(s^{-1}x_3(s))<0\quad\text{for }s\geqq2\nu\text{ and }\nu\geqq2. \tag{9.42}$$

It follows from inequality (9.42) that, for $\nu\geqq2$,

$$\{\mu:\mu\bar{f}(x)\text{ satisfies (I)}\}\subset\{\mu(s):\nu+1<s<2\nu\}=(\mu(\nu+1),\mu(2\nu)).$$

Because $\mu(s)$ is increasing for $s>\nu-1$ we have

$$\begin{aligned}\mu(2\nu)-\mu(\nu+1)<\mu(2\nu)-\mu(\nu)&=\nu\left(\frac{e}{\nu}\right)^{\nu}\left[2\left(\frac{e}{2}\right)^{\nu}-1\right]\\&\leqq(2\nu)\left(\frac{e^2}{2\nu}\right)^{\nu},\end{aligned}$$

which immediately gives (9.39) (although we have not yet proved that $\tau_4(\nu)$ exists).

To prove that $\tau_4(\nu)\to\infty$ as $\nu\to0^+$ (assuming the existence of $\tau_4(\nu)$), it suffices to prove that given any $M>0$, there exists $\delta=\delta(M)$ such that for $0<\nu<\delta(M)$ and $\nu+1<s\leqq M$,

$$\log(s^{-1}x_3(s))=-\nu^3\log s+(1+\nu+\nu^2)s+\nu^3\log\nu-\nu^3-\nu^{\nu+1}s^{1-\nu}e^{s-\nu}-x_2>0. \tag{9.43}$$

Because $\nu^\nu e^{-\nu}$ converges to one as $\nu\to 0^+$, we see that $x_1(s)=s^{1-\nu}e^s\nu^\nu e^{-\nu}$ converges uniformly on $[1,M]$ to $s\,e^s$ as $\nu\to 0^+$, and using this we see that

$$\lim_{\nu\to 0^+}x_2(s)=\lim_{\nu\to 0^+}s^{1-\nu}e^s x_1^\nu e^{-x_1}=s\exp(s-s\,e^s),$$

and that the convergence is uniform in $s\in[1,M]$. Using this information, we see that

$$\lim_{\nu\to 0^+}(\log(s^{-1}x_3(s)))=s-s\exp(s-s\,e^s), \tag{9.44}$$

and that the convergence is uniform in $s\in[1,M]$. Since the right-hand side of (9.44) is positive on $(0,\infty)$, there exists $\delta=\delta(M)>0$ such that

$$\log(s^{-1}x_3(s))>0 \quad \text{for } 1\leqq s\leqq M,\quad 0<\nu<\delta.$$

We need only prove the existence of $\tau_4(\nu)$, or equivalently that

$$\{s>\nu+1:\log(s^{-1}x_3(s))<0\} \quad \text{is an interval.}$$

It is convenient to make the following observation first.

We claim that if $x_1(s)\leqq\nu+2$ and $s>\nu+1$, then $x_2(s)>\nu$. Because $x_1'(s)>0$ and $x_2'(s)<0$ for $s>\nu+1$, it suffices to prove that if $x_1(s)=\nu+2$, then $x_2(s)>\nu$. However, if $x_1(s)=\nu+2$, we find as in the proof of Theorem 9.2, that

$$x_2=x_1\bar f(x_1)(\bar f(\theta))^{-1}=\nu^{-\nu}e^\nu(\nu+2)(\nu+2)^\nu e^{-(\nu+2)},$$

so $x_2(s)>\nu$ if

$$\left(\frac{\nu+2}{\nu}\right)^{\nu+1}>e^2,$$

which is (9.25). We conclude that if $x_1(s)\leqq\nu+2$, then $\mu(t)\bar f(x)$ satisfies (III) for $\nu+1<t\leqq\nu+2$. A calculation shows that $x_1(\nu+1+c)\leqq\nu+2$ if and only if

$$e^{1+c}\leqq\left(\frac{\nu+1+c}{\nu}\right)^\nu\left(\frac{\nu+2}{\nu+1+c}\right), \tag{9.45}$$

and Lemma 9.1 implies that if $0<c<\sqrt5-2$ and $\nu\geqq\nu(c)$, inequality (9.45) holds. This proves the inclusion (9.40).

Logarithmic differentiation easily yields the following formulas:

$$\begin{aligned}
&\frac{dx_1}{ds}=x_1\left[\frac{s+1-\nu}{s}\right],\\
&\frac{dx_2}{ds}=x_2\left[\frac{s+1-\nu}{s}\right][\nu+1-x_1],\\
&\frac{d}{ds}\left(\log\left(\frac{x_3(s)}{s}\right)\right)=\left(\frac{s+1-\nu}{s}\right)[1+(\nu-x_2)(\nu+1-x_1)]-\left(\frac{1}{s}\right).
\end{aligned} \tag{9.46}$$

Define s_* to be the first $s>\nu+1$ such that $\log(s^{-1}x_3(s))=0$, so we know

$$\frac{d}{ds}\log(s^{-1}x_3(s))|_{s=s_*}=s_*^{-1}[(s_*+1-\nu)(1-(\nu-x_2)(x_1-\nu-1))-1]\leqq 0. \tag{9.47}$$

Theorem 9.3 implies $x_2(s_*) < \nu$, and the remarks above show $x_1(s_*) > \nu+2$. To complete the proof we need only show

$$(s+1-\nu)[1-(\nu-x_2(s))(x_1(s)-\nu-1)]-1 \equiv \phi(s) < 0$$

for $s > s_*$, and because $\phi(s_*) \leqq 0$, it suffices to prove

$$\begin{aligned} \phi'(s) &= (\nu - x_2)(x_1 - \nu - 1) - (s+1-\nu)^2 s^{-1}(x_1 - \nu - 1)^2 x_2 \\ &\quad - (s+1-\nu)^2 x^{-1} x_1(\nu - x_2) < 0 \end{aligned} \tag{9.48}$$

for $s > s_*$.

Case 1. Assume $\nu \geqq 1$. Using the estimates $x_1 - \nu - 1 > 1$ and $\nu - x_2 > 0$ in the formula for $\phi'(s)$ for $s > s_*$, we obtain

$$\begin{aligned} \phi'(s) &< 1 - (\nu - x_2) - x_2(s+1-\nu)^2 s^{-1} - (s+1-\nu)^2 s^{-1}(\nu+2)(\nu - x_2) \\ &= 1 - (\nu - x_2)[1 + (\nu+1)(s+1-\nu)^2 s^{-1}] - (s+1-\nu)^2 s^{-1}\nu. \end{aligned}$$

The previous inequality shows

$$\phi'(s) < 1 - (s+1-\nu)^2 s^{-1}\nu. \tag{9.49}$$

The function on the right-hand side of (9.49) is decreasing for $s > \nu+1$, so inequality (9.49) implies that, for $s \geqq s_*$,

$$\phi'(s) < 1 - \frac{4\nu}{\nu+1} < 0.$$

Case 2. Assume $0 < \nu < 1$. Because x_2 is decreasing and less than ν for $s \geqq s_*$ and x_1 is increasing and greater than $\nu+1$ for $s > \nu+1$, $(\nu - x_2)(x_1 - \nu - 1)$ is an increasing function of s for $s \geqq s_*$. At $s = s_*$, inequality (9.47) implies

$$\begin{aligned} &\tfrac{1}{2} < 1 - (s_* + 1 - \nu)^{-1} \leqq (\nu - x_2(s_*))(x_1(s_*) - \nu - 1), \quad \text{so} \\ &\tfrac{1}{2} < (\nu - x_2(s))(x_1(s) - \nu - 1) \quad \text{for } s \geqq s_*. \end{aligned} \tag{9.50}$$

Using the equation for $\phi'(s)$ in (9.48), we see

$$\phi'(s) < 1 - (\nu - x_2)(x_1 - 1 - \nu) - (s+1-\nu)^2 s^{-1}(\nu - x_2)x_1. \tag{9.51}$$

Because $0 < \nu < 1$, we have $(s+1-\nu)^2 s^{-1} > s > 1$, so from (9.50) and (9.51) we derive

$$\phi'(s) < 1 - (\nu - x_2)(x_1 - 1 - \nu) - (\nu - x_2)(x_1 - 1 - \nu) < 0$$

for $s > s_*$. The proof is now complete. □

Next we want to analyze when $\mu \bar{f}_5(x)$ satisfies (I). Unfortunately, our results are incomplete. We conjecture that there exists a continuous function $\tau_5(\nu, \lambda)$ (allowing $\tau_5(\nu, \lambda) = \infty$) defined for $\nu > 0$ and $\lambda > \nu+1$ such that $\mu \bar{f}_5(x)$ satisfies (I) if and only if $\mu_0(\nu, \lambda) < \mu < \tau_5(\nu, \lambda)$ (where $\mu_0(\nu, \lambda)$ is as in Table 1). By using Theorem 9.3 and Theorem 9.5, we have given a computer-assisted proof of this conjecture for various specific ν and λ, but we have not proved it in general.

THEOREM 9.5. *Assume $\nu > 0$ and $\lambda > \nu+1$, and let $x_0 = x_0(\mu, \nu, \lambda)$ and $\mu_0 = \mu_0(\nu, \lambda)$ be as in Table 1 for the function $f_5(x) = \mu \bar{f}_5(x)$. The function $\mu \bar{f}_5(x)$ satisfies* (I) *at x_0 for all large μ if*

$$\nu + 1 < \lambda \leqq \nu + \left(\frac{1}{2\nu}\right) + \left(\frac{1}{2\nu}\right)\sqrt{4\nu^2+1} \equiv \phi(\nu), \tag{9.52}$$

while if $\lambda > \phi(\nu)$, there exists a number $\gamma = \gamma(\nu, \lambda) < \infty$ such that $\mu \bar{f}_5(x)$ does not satisfy (I) *at x_0 for any $\mu > \gamma(\nu, \lambda)$. If $\nu + 1 < \lambda \leqq \nu + 1 + (1/2\nu)$, $\mu \bar{f}_5(x)$ satisfies condition* (I) *at x_0 for all $\mu > \mu_0(\nu, \lambda)$.*

Proof. While not essential, a change of variables will simplify our calculations. For $x>0$, define $\psi(x)=x^{1/\lambda}$, so

$$(\psi^{-1}(\mu\bar{f}_5)\psi)(x)=\mu^\lambda x^\nu(1+x)^{-\lambda}\equiv\mu^\lambda\bar{h}_5(x).$$

The remarks at the beginning of this section show that $\mu\bar{f}_5(x)$ satisfies (I) at x_0 if and only if $\mu^\lambda\bar{h}_5(x)$ satisfies (I) at x_0^λ. As in Theorem 9.2 we see that $\mu^\lambda\bar{h}_5(x)$ satisfies (I) at x_0^λ if and only if $\mu>\mu_0$ and

$$(\mu^\lambda\bar{h}_5)^3(\theta)>x_0^\lambda, \tag{9.53}$$

where $\theta=\nu(\lambda-\nu)^{-1}$, the point where $\bar{h}_5$ achieves its maximum.

As before, it is convenient to parameterize by $s=x_0^\lambda$, the fixed point of $\mu^\lambda\bar{h}_5$. Define a function $\mu(s)$ by

$$\mu(s)^\lambda=s^{1-\nu}(1+s)^\lambda=s(\bar{h}_5(s))^{-1},$$

so s is a fixed point of $\mu(s)^\lambda\bar{h}_5(x)$. Define $x_1(s)=\mu(s)\bar{h}_5(\theta)$ and $x_j(s)=\mu(s)\bar{h}_5(x_{j-1}(s))$ for $j>1$. Just as in Theorem 9.2 we obtain from (9.53) that

$$\{\mu\colon \mu\bar{f}_5(x) \text{ satisfies (I) at } x_0\}=\{\mu(s)\colon s>s_0 \text{ and } x_3(s)>s_0\}, \tag{9.54}$$

where $s_0=(\nu+1)(\lambda-\nu-1)^{-1}$ as in (9.51). The proof that $x_1'(s)>0$ and $x_2'(s)<0$ for all $s>s_0$ is as in Theorem 9.2 and is left to the reader.

A calculation shows

$$\begin{aligned} x_1(s)&=x_1=s^{\lambda+1-\nu}(1+s^{-1})^\lambda\theta_1, \quad \text{where}\\ \theta_1&=\left(\frac{\nu}{\lambda}\right)^\nu\left(\frac{\lambda-\nu}{\lambda}\right)^{\lambda-\nu}<1. \end{aligned} \tag{9.55}$$

A further calculation yields

$$x_2(s)=x_2=s^{1-(\lambda-\nu)^2}\theta_1^{\nu-\lambda}(1+s^{-1})^{\lambda+\nu\lambda-\lambda^2}(1+x_1^{-1})^{-\lambda}. \tag{9.56}$$

Equation (9.55) shows $\lim_{s\to\infty}x_1(s)=\infty$, and (9.56) gives

$$\lim_{s\to\infty}x_2(s)=\lim_{s\to\infty}s^{1-(\lambda-\nu)^2}\theta_1^{\nu-\lambda}=0,$$

because $\lambda>\nu+1$. Substituting (9.56) in the formula for $x_3(s)$, we obtain

$$\begin{aligned} s^{-1}x_3(s)&=s^{\lambda-\nu(\lambda-\nu)^2}\theta_1^{\nu(\nu-\lambda)}(1+s^{-1})^\delta(1+x_1^{-1})^{-\nu\lambda}(1+x_2)^{-\lambda} \quad \text{where}\\ \delta&=\lambda+\nu\lambda+\nu^2\lambda-\nu\lambda^2. \end{aligned} \tag{9.57}$$

Equation (9.56) implies

$$\lim_{s\to\infty}s^{-1}x_3(s)=\lim_{s\to\infty}s^{\lambda-\nu(\lambda-\nu)^2}\theta_1^{\nu(\nu-\lambda)}.$$

Because $\lambda-\nu(\lambda-\nu)^2=0$ for $\lambda=\phi(\nu)$, $\lambda-\nu(\lambda-\nu)^2>0$ for $\nu+1<\lambda<\phi(\nu)$, and $\lambda-\nu(\lambda-\nu)^2<0$ for $\lambda>\phi(\nu)$, we obtain

$$\lim_{s\to\infty}s^{-1}x_3(s)=\begin{cases}\infty & \text{for } \nu+1<\lambda<\phi(\nu),\\ \theta_1^{\nu(\nu-\lambda)} & \text{for } \lambda=\phi(\nu),\\ 0 & \text{for } \lambda>\phi(\nu).\end{cases} \tag{9.58}$$

Using (9.58) and (9.54) and recalling that $0<\theta_1<1$, we obtain the first part of the theorem.

It remains to prove the final part of the theorem. We know (from Theorem 7.2 and the remarks at the beginning of this section) that $s^{-1}x_3(s) > 1$ for $s = s_0$. Therefore, to prove that $\mu\bar{f}_5(x)$ satisfies (I) for all $\mu > \mu_0$ (when $\nu+1 < \lambda \leqq \nu+1+(1/2\nu)$) we need only prove that $s^{-1}x_3(s)$ is an increasing function for $s \geqq s_0$. Because $x_1'(s) > 0$ and $x_2'(s) < 0$ for $s \geqq s_0$, it is clear that $(1+x_1^{-1})^{-\nu\lambda}(1+x_2)^{-\lambda}$ is an increasing function of s. Thus, by using (9.57) we see that $s^{-1}x_3(s)$ is increasing if

$$\frac{d}{ds} s^{\lambda-\nu(\lambda-\nu)^2}(1+s^{-1})^{\lambda+\nu\lambda+\nu^2\lambda-\nu\lambda^2} \geqq 0 \quad \text{for } s \geqq s_0. \tag{9.59}$$

By differentiating logarithmically, we see that inequality (9.59) will hold if

$$s^{-1}(s+1)^{-1}[(\lambda-\nu(\lambda-\nu)^2)(s+1) - \lambda(1+\nu+\nu^2-\nu\lambda)] \geqq 0 \quad \text{for } s \geqq s_0. \tag{9.60}$$

Since $s+1 \geqq \lambda(\lambda-\nu-1)^{-1}$ for $s \geqq s_0$ we see that (9.60) will be satisfied if

$$[\lambda-\nu(\lambda-\nu)^2]\left(\frac{\lambda}{\lambda-\nu-1}\right) - \lambda[1+\nu(1+\nu-\lambda)] \geqq 0. \tag{9.61}$$

Recalling that $\lambda > \nu+1$ and simplifying, we see that (9.61) will be satisfied if

$$\lambda \leqq 1+\nu+(1/2\nu),$$

and this completes the proof. □

If $\nu = 1$, Theorem 9.5 ensures that $\mu\bar{f}_5(x)$ satisfies (I) at x_0 for all $\mu > \mu_0$ if $2 < \lambda \leqq 2.5$, while the number $\phi(1)$ equals $(\frac{1}{2})(3+\sqrt{5})$ or approximately 2.618. We can, however, give an ad hoc argument (which we omit) and prove that, for $\nu = 1$ and $2 < \lambda \leqq (\frac{1}{2})(3+\sqrt{5})$, $\mu\bar{f}_5(x)$ satisfies (I) for all $\mu > \mu_0$.

We now want to study when a function $\mu\bar{f}(x)$ satisfies (II) at a fixed point x_0; our particular interest, of course, is $\bar{f} = \bar{f}_4$ or $\bar{f} = \bar{f}_5$. We first make some preliminary calculations concerning local stability of period 2 points of $\mu\bar{f}(x) = f(x)$.

Suppose, for some μ, there exist numbers $0 < x_1 < x_2$ satisfying

$$\mu\bar{f}(x_1) = x_2 \quad \text{and} \quad \mu\bar{f}(x_2) = x_1. \tag{9.62}$$

Then we have

$$x_1\bar{f}(x_1) = x_2\bar{f}(x_2) = c, \tag{9.63}$$

$$\mu = \frac{x_1x_2}{c}. \tag{9.64}$$

Conversely, if $0 < x_1 < x_2$, with x_1 and x_2 satisfying (9.60), and μ is defined by (9.64), then x_1 and x_2 also satisfy (9.62).

Define κ to be the derivative

$$\kappa = (f^2)'(x_1) = f'(x_1)f'(x_2) = \mu^2\bar{f}'(x_1)\bar{f}'(x_2)$$

occurring in Theorem 7.3. A short calculation gives

$$\kappa = u(x_1)u(x_2), \tag{9.65}$$

where $u(x)$ is the function

$$u(x) = \frac{x\bar{f}'(x)}{\bar{f}(x)}. \tag{9.66}$$

Our basic idea is to use c as a parameter and to express x_1, x_2, μ, and κ as functions of c. To make this rigorous, assume that $\bar{f}:[0,\infty)\to[0,\infty)$ is continuous and C^2 on $(0,\infty)$ and that, if $g(x)=x\bar{f}(x)$, then there exists a number $s_0>0$ such that

$$g'(x)>0 \quad \text{for } 0<x<s_0, \qquad g'(x)<0 \quad \text{for } x>s_0. \tag{9.67}$$

For simplicity in the statement of our theorems, further assume that

$$\lim_{x\to\infty} g(x)=0. \tag{9.68}$$

Define c_* by

$$c_*=g(s_0)=\max_{x>0} g(x), \tag{9.69}$$

and $g_1=g\,|[0,c_*]$ and $g_2=g\,|[c_*,\infty)$. Then for $0<c\leqq c_*$ (9.63) and the condition $0<x_1<x_2$ determine x_1 and x_2 as functions of c:

$$x_1=x_1(c)=g_1^{-1}(c)\in(0,s_0] \quad\text{and}\quad x_2=x_2(c)=g_2^{-1}(c)\in[s_0,\infty), \tag{9.70}$$

where g_1^{-1} and g_2^{-1} are the inverse functions of g_1 and g_2, respectively. Because $g_1'(x)>0$ for $0<x<s_0$ and g_1 is continuous on $[0,s_0]$, we obtain that x_1 is continuous on $[0,c_*]$ and C^1 on $(0,c_*)$, $x_1(0)=0$, $x_1(c_*)=s_0$ and $x_1'(c)>0$ for $0<c<c_*$. Similarly, we find that x_2 is continuous on $(0,c_*]$ and C^1 on $(0,c_*)$, $x_2(c_*)=s_0$ and $x_2'(c)<0$ for $0<c<c_*$. Note that (9.68) ensures that the domain of x_2 is $(0,c_*]$ and $\lim_{c\to0^+}x_2(c)=\infty$. Having defined $x_1(c)$ and $x_2(c)$, we then have that $\mu=\mu(c)$ and $\kappa=\kappa(c)$, given by (9.64) and (9.65), respectively, are functions of c. To make further progress we must establish some of the properties of $\mu(c)$ and $\kappa(c)$.

LEMMA 9.2. *Assume $\bar{f}:[0,\infty)\to[0,\infty)$ is continuous and C^2 on $(0,\infty)$. If $g(x)=x\bar{f}(x)$ assume there exists $s_0>0$ such that $g'(x)>0$ for $0<x<s_0$ and $g'(x)<0$ for $x>s_0$ and $\lim_{x\to\infty}g(x)=0$. Let $x_1(c)$ and $x_2(c)$ be as defined before for $0<c\leqq c_*=g(s_0)$, and let $\kappa(c)$ and $\mu(c)$ be defined by (9.64) and (9.65). Then $\mu(c)$ and $\kappa(c)$ have the following properties*:

(i) *$\mu(c)\to s_0^2/c_*>0$ and $\kappa(c)\to1$ as $c\to c_*$.*

(ii) *$\mu(c)\to\infty$ as $c\to0^+$. If $u(x)$ is defined by equation (9.66), assume that $\lim_{x\to0+}u(x)=L_1$, where L_1 is finite, and that $\lim_{x\to\infty}u(x)=L_2$, where we allow $L_2=-\infty$. Then we have $\lim_{c\to0^+}\kappa(c)=L_1L_2$.*

(iii) *Suppose that there exists $\theta\geqq0$ such that $\bar{f}'(x)>0$ for $0<x<\theta$ and $\bar{f}'(x)<0$ for $x>\theta$ (so $\theta<s_0$), and that $u'(x)<0$ for $x>\theta$. Define $v(x)=-u(x)$ for $x\geqq\theta$ and let $v^{-1}(\gamma)$ denote the inverse map. Note that $\lim_{x\to\infty}v(x)=-L_2>1$ under our assumptions, define $\delta=\max(-L_2^{-1},-L_1)$, and assume that*

$$g(v^{-1}(\gamma))>g(v^{-1}(1/\gamma)) \quad \textit{for } \delta<\gamma<1. \tag{9.71}$$

Then we have $\kappa(c)<1$ for $0<c<c_*$. If we define $\Phi(t)=\log(g(v^{-1}(t)))$ for $\delta<t<1$, inequality (9.71) will be satisfied if

$$\Phi'(t)<-\left(\frac{1}{t^2}\right)\Phi'\left(\frac{1}{t}\right) \quad \textit{for } \delta<t<1. \tag{9.72}$$

(iv) *If $\kappa(c)<1$ for $0<c<c_*$, then $\mu'(c)<0$ for $0<c<c_*$.*

(v) *If $u'(x)<0$ for all $x>0$ and if $c\in(0,c_*)$ is such that $\kappa(c)\leqq0$, then $\kappa'(c)>0$.*

Proof. (i) By definition of s_0, $g'(s_0)=f(s_0)+s_0f'(s_0)=0$, so we obtain $u(s_0)=-1$. Since we have already noted that $x_1(c)$ and $x_2(c)$ approach s_0 as $c\to c_*$, we conclude that

$$\lim_{c\to c_*}\kappa(c)=\lim_{c\to c_*}u(x_1)u(x_2)=(-1)^2=1.$$

Using the same sort of reasoning, we obtain

$$\lim_{c\to c_*} \mu(c) = \lim_{c\to c_*} \frac{x_1 x_2}{c} = \frac{s_0^2}{c_*}.$$

(ii) Because $\bar{f}$ is continuous at zero, there exists a constant M such that

$$Mx \geqq g(x) \quad \text{for } x \text{ small and positive.}$$

For $c>0$ and small it follows that

$$Mx_1(c) \geqq g(x_1(c)) = c \quad \text{or} \quad x_1 \geqq c/M.$$

Using this estimate, we see that for $c>0$ small,

$$\mu(c) \geqq (1/M)x_2(c),$$

and because $x_2(c)\to\infty$ as $c\to 0^+$, $\mu(c)\to\infty$ as $c\to 0^+$. Because $\lim_{c\to 0^+} x_1(c)=0$ and $\lim_{c\to 0^+} x_2(c)=\infty$, $\lim_{c\to 0^+}\kappa(c)=\lim_{x\to 0} u(x)\lim_{x\to\infty} u(x)=L_1L_2$.

(iii) Assume that u, f, and g satisfy the given assumptions but that there exists c, $0<c<c_*$, such that $\kappa(c)\geqq 1$. Our assumptions imply $L_1\geqq 0$ and $L_2\leqq 0$, so $\lim_{c\to 0^+}\kappa(c)\leqq 0$. Thus, by choosing a different number c, $0<c<c_*$, we can assume that $\kappa(c)=1$. Because $u(x_2)<\mu(\sigma)=-1$ (since $x_2(c)>\sigma$) and because $u(x)>0$ for $x<\theta$, we must have $x_1(c)\geqq\theta$. If $\theta>0$, so $u(\theta)=0$, then we must in fact have $x_1(c)>0$ and of course $x_1(c)>\theta$ if $\theta=0$. If we write $\gamma=v(x_1(c))$, then we must have $0=v(\theta)<\gamma<1=v(s_0)$ if $\theta>0$ and $-L_1<\gamma<1$ if $\theta=0$. Furthermore, we must have $\gamma^{-1}=v(x_2(c))$, so $\gamma^{-1}\in(1,-L_2)$, and we obtain the estimates $\delta<\gamma<1$. Now (9.63) gives

$$g(v^{-1}(\gamma)) = g(v^{-1}(1/\gamma)),$$

which contradicts (9.71).

Note that (9.71) is equivalent to

$$\Phi(1)-\Phi(\gamma) < \Phi(1)-\Phi(\gamma^{-1}) \quad \text{for } \delta<\gamma<1. \tag{9.73}$$

Using the fundamental theorem of calculus and changing variables in the integral for the right-hand side, we see that (9.73) is equivalent to

$$\int_\gamma^1 \Phi'(t)\,dt < \int_\gamma^1 -\left(\frac{1}{t^2}\right)\Phi'\left(\frac{1}{t}\right)dt,$$

so (9.72) implies (9.73) and (9.71).

(iv) A calculation shows

$$(\mu(c))^2 = \frac{x_1x_2}{\bar{f}(x_1)\bar{f}(x_2)} \quad \text{where } x_j = x_j(c), \tag{9.74}$$

so it suffices to show that if $\kappa(c)<1$, then

$$\frac{d}{dc}\log\left(\frac{x_1x_2}{\bar{f}(x_1)\bar{f}(x_2)}\right) < 0. \tag{9.75}$$

Using the formula $x_j'(c)=(\bar{f}(x_j)^{-1}(1+u(x_j)))^{-1}$ (which we obtain by differentiating $x_j\bar{f}(x_j)=c$), we find that (9.75) is equivalent to

$$\sum_{j=1}^2 (x_j\bar{f}(x_j)(1+u(x_j)))^{-1} - \sum_{j=1}^2 \bar{f}'(x_j)(\bar{f}(x_j)^2(1+u(x_j)))^{-1} < 0.$$

Multiplying the above inequality by $x_1x_2\bar{f}(x_1)\bar{f}(x_2)=c^2$ and simplifying, we find that the above inequality is equivalent to

$$2c(1+u(x_1))^{-1}(1+u(x_2))^{-1}(1-\kappa(c))<0. \tag{9.76}$$

Recall that $g'(x)>0$ for $0<x<s_0$ and $g'(x)<0$ for $x>s_0$, which implies $u(x)>-1$ for $0<x<s_0$ and $u(x)<-1$ for $x>s_0$. From this we conclude that $(1+u(x_1))(1+u(x_2))$ is negative and that $\mu'(c)<0$ if and only if $\kappa(c)<1$.

(v) If $\kappa(c)\leqq 0$, we must have $u(x_1(c))\geqq 0$, since $u(x_2(c))<-1$ for $0<c<c_*$. It follows that

$$\kappa'(c)=u'(x_1)x_1'u(x_2)+u(x_1)u'(x_2)x_2'>0,$$

because we are assuming that $u'(x)<0$ for all $x>0$, and that $x_1'(c)>0$ and $x_2'(c)<0$ for $0<c<c_*$. □

With $j=4$ or 5 and parameters $\nu>0$ and $\lambda>\nu+1$, let $\bar{f}_j$ be defined as in (9.24) and let functions $\mu_j(c)$ and $\kappa_j(c)$ be defined by substituting $\bar{f}_j$ for $\bar{f}$ in (9.63)-(9.65). Define $g_j(x)=x\bar{f}_j(x)$, so that $g_j'(x)>0$ for $0<x<\sigma_j$ (where σ_j is given in Theorem 9.3), and $g_j'(x)<0$ for $x>\sigma_j$. Also recall that $\bar{f}_j'(x)>0$ for $0<x<\theta_j$ and $\bar{f}_j'(x)<0$ for $x>\theta_j$, where θ_j is an in Theorem 9.3. From Lemma 9.2 we obtain Lemma 9.3.

LEMMA 9.3. *With $j=4$ or 5 and parameters $\nu>0$ and $\lambda>\nu+1$, the functions $\mu_j(c)$ and $\kappa_j(c)$ (defined for $0<c\leqq c_{*j}=g_j(\sigma_j)$) have the following properties*:

(i) $u_j(c)\to\sigma_j^2/c_{*j}>0$ *and* $\kappa_j(c)\to 1$ *as* $c\to c_{*j}$.

(ii) $\mu_j(c)\to\infty$, $\kappa_4(c)\to-\infty$ *and* $\kappa_5(c)\to-\nu(\lambda-\nu)<0$ *as* $c\to 0^+$.

(iii) $\kappa_j(c)<1$, *if* $0<c<c_{*j}$.

(iv) $\mu_j'(c)<0$, *if* $0<c<c_{*j}$.

(v) $\kappa_j'(c)>0$, *if* $0<c<c_{*j}$.

Proof. Define $u_j(x)=x\bar{f}_j'(x)(\bar{f}_j(x))^{-1}$ and $v_j(x)=-u_j(x)$. A calculation gives

$$u_4(x)=\nu-x,\qquad u_5(x)=[\nu-(\lambda-\nu)x^\lambda](1+x^\lambda)^{-1},$$

so $u_j'(x)<0$ for all $x>0$. A further calculation gives

$$v_4^{-1}(t)=\nu+t,\qquad (v_5^{-1}(t))^\lambda=(\nu+t)(\lambda-\nu-t)^{-1}.$$

Lemma 9.2 will imply Lemma 9.3 if we can prove that (9.71) is satisfied with g_j and v_j replacing g and v, and Lemma 9.2 implies that this will be the case if

$$\Phi_j'(t)<-\left(\frac{1}{t^2}\right)\Phi_j'\left(\frac{1}{t}\right)\quad\text{for } \delta_j<t<1, \tag{9.77}$$

where $\Phi_j(t)=\log g_j(v_j^{-1}(t))$ and $\delta_4=0$ and $\delta_5=(\lambda-\nu)^{-1}$. We can easily check that

$$\Phi_4'(t)=(1-t)(\nu+t)^{-1},\qquad \Phi_5'(t)=(1-t)(\nu+t)^{-1}(\lambda-\nu-t)^{-1},$$

so for $j=4$ inequality (9.77) is equivalent to

$$(1-t)(\nu+t)^{-1}<(1-t)(\nu t^3+t^2)^{-1}\quad\text{for } 0<t<1 \tag{9.78}$$

and for $j=5$ inequality (9.77) is equivalent to

$$\begin{aligned}&(1-t)(\nu+t)^{-1}(\lambda-\nu-t)^{-1}<(1-t)t^{-1}(\nu t+1)^{-1}(\lambda t-\nu t-1)^{-1}\\&\text{for } (\lambda-\nu)^{-1}<t<1.\end{aligned} \tag{9.79}$$

Since $0<t<1$, (9.78) is obviously true. Because $0<t<1$ we have $\nu+t>\nu t^2+t>0$;

and by using the fact that $\lambda > \nu$ we can see

$$\lambda - \nu - t > \lambda t - \nu t - 1 > 0 \quad \text{for } (\lambda - \nu)^{-1} < t < 1,$$

so inequality (9.79) is valid. □

With the aid of Lemma 9.2 we can give conditions under which a function $\mu\bar{f}(x)$ satisfies (II) precisely for $\mu \in (\mu_0, \sigma]$, where $\sigma > \mu_0$.

THEOREM 9.6. *Assume $\bar{f}:[0,\infty)\to[0,\infty)$ is a continuous map that is C^3 on $(0,\infty)$. Assume there exists $\theta \geqq 0$ such that $\bar{f}'(x)>0$ for $0<x<\theta$ and $\bar{f}'(x)<0$ for $x>\theta$. If $g(x)=x\bar{f}(x)$, assume there exists $s_0>0$ such that $g'(x)>0$ for $0<x<s_0$ and $g'(x)<0$ for $x>s_0$. Define $u(x)=x\bar{f}'(x)(\bar{f}(x))^{-1}$ and $v(x)=-u(x)$ and assume $u'(x)<0$ for all $x>0$ and*

$$g(v^{-1}(\gamma)) > g(v^{-1}(1/\gamma)) \quad \textit{for } \delta < \gamma < 1, \tag{9.80}$$

where δ is defined as in Lemma 9.2 and v^{-1} is the inverse map of v. (Recall that inequality (9.80) is satisfied if inequality (9.72) holds.) Finally, suppose there exists a C^3 map ψ of an interval (a,b) onto $(0,\infty)$ such that $\psi'(x)>0$ for $x\in(a,b)$ and $\psi^{-1}(\mu\bar{f})\psi$ has negative Schwarzian derivative for all $x\in(a,b)$. Then there exists $\sigma>\mu_0=\bar{f}(s_0)s_0^{-1}$ such that $\mu\bar{f}(x)$ satisfies (II) at $x_0(\mu)=x_0$, the unique fixed point of $\mu f(x)$ in the interval (θ,∞), if and only if $\mu_0<\mu\leqq\sigma$. If $\mu(c)$ and $\kappa(c)$ are defined as in Lemma 9.2 and $\kappa(c_2)=-1$, then $\mu(c_2)=\sigma$.

Proof. If $f=\mu\bar{f}$ is as in Theorem 7.3 or 7.4, but we assume that $\psi^{-1}f\psi$ (ψ as in Theorem 9.6) has negative Schwarzian derivative everywhere instead of supposing that f has, we can still easily see (using the remarks at the beginning of this section) that the conclusions of Theorem 7.3 and 7.4 are satisfied.

Now let $\mu(c)$ and $\kappa(c)$ be as in Lemma 9.2. Theorem 9.3 implies that $\mu\bar{f}(x)$ satisfies condition (0) at $x_0(\mu)$ if and only if $\mu>\mu_0=\bar{f}(s_0)s_0^{-1}$, and Lemma 9.2 implies that $\mu_0=\mu(c_*)$ and that $\mu(c)>\mu_0$ for $0<c<c_*$. Since (Lemma 9.2) $\mu'(c)<0$ for $0<c<c_*$ and $\lim_{c\to 0^+}\mu(c)=\infty$, we will work with the parameter c instead of $\mu>\mu_0$. Define a number c_1 by

$$c_1 = \inf\{c>0\colon \mu(\gamma)\bar{f}(x) \text{ satisfies (I) at } x_0 \text{ for } c\leqq\gamma<c_*\}.$$

Corollary 7.1 implies that $c_1<c_*$. Define c_2 as in the statement of the theorem if $\kappa(c)=-1$ has a solution $c>0$; otherwise define $c_2=0$. Lemma 9.2 implies that $\kappa(c)<1$ for $0<c<c_*$ and $\kappa(c)\geqq-1$ if and only if $c\geqq c_2$. Thus Theorem 7.3 will imply that $\mu(c)\bar{f}(x)$ satisfies (II) if and only if $c_2\leqq c<c_*$ if we can prove that $c_2>c_1$ when $c_1>0$. However, if $c_1>0$, Theorem 7.4 and Corollary 7.2 imply that there exists $\delta<0$ such that $\mu(c_1)\leqq\mu<\mu(c_1)+\delta$, $(\mu\bar{f})^2$ has a fixed point x such that $(d/dx)(\mu\bar{f})^2(x)<-1$. If $c_2\leqq c_1$, this contradicts Lemma 9.2, so we must have $c_2>c_1$. □

As an immediate consequence of Theorem 9.6 and Lemma 9.3 we obtain Corollary 9.1.

COROLLARY 9.1. *For parameters $\nu\geqq1$ and $\lambda>\nu+1$ let $\bar{f}_4(x)$ and $\bar{f}_5(x)$ be as defined before and let $\mu_0(\nu)$ and $\mu_0(\nu,\lambda)$ be as defined in Table 1 for the functions $\bar{f}_4$ and $\bar{f}_5$, respectively. If $\bar{f}_j$ has its maximum on $(0,\infty)$ at θ_j, there exist continuous functions $\sigma_4(\nu)$ and $\sigma_5(\nu,\lambda)$ such that μf_j satisfies condition (II) at the unique fixed point of μf_j in (θ_j,∞) if and only if $\mu_0(\nu)<\mu\leqq\sigma_4(\nu)$ for $j=4$ or $\mu_0(\nu,\lambda)<\mu\leqq\sigma_5(\nu,\lambda)$ for $j=5$.*

Proof. We need prove the continuity and finiteness of σ_j, and this follows easily from the results of Lemma 9.3. □

If $\psi:(a,b)\to(0,\infty)$ is as in Theorem 9.6, the results of this section also apply to the functions $\psi^{-1}(\mu\bar{f}_j)\psi$ for $\nu\geqq1$ and $\lambda>\nu+1$. Taking $\psi(x)=ax$, $a>0$, we obtain, for example, the conclusion of Corollary 9.1 for $\mu_1x^\nu e^{-ax}$ and $\mu_1x^\nu(1+bx^\lambda)^{-1}$, where

$\mu_1 > 0$ and $b = a^\lambda$ is an arbitrary positive number. Taking $\psi(x) = x^p$ for $p > 0$, we obtain the same results for $\mu_2 x^\nu e^{-bx^p}$ and $\mu_2 x^\nu (1 + bx^{\lambda p})^{-1/p}$, where $\mu_2 > 0$, $b > 0$, and $p > 0$.

Although we will not pursue this here, we can establish the conclusions of Corollary 9.1 for other classes of functions, e.g., $\mu \bar{f}_6(x)$, where

$$\bar{f}_6(x) = x^\nu \exp(-x(1+ax)),$$

where $\nu \geqq 1$ and $a > 0$. The major problem is verifying (9.71) or (9.72).

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