ON VARIATIONAL PRINCIPLES FOR THE GENERALIZED PRINCIPAL EIGENVALUE OF SECOND ORDER ELLIPTIC OPERATORS AND SOME APPLICATIONS

By

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Dedicated to Professor Shmuel Agmon

1. Introduction

Let P be a second order elliptic operator defined in a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$. Let $W(x)$ be a positive function and denote by λ_0 the principal eigenvalue of the Dirichlet generalized eigenvalue problem

$$\begin{aligned} &Pu = \lambda W(x)u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

If P is a self-adjoint operator it is well known that λ_0 is given by the classical (Rayleigh–Ritz) variational formula, namely

$$\lambda_0 = \inf_{u \in C_0^\infty(\Omega)} \left\{ \frac{(Pu, u)}{(Wu, u)} \right\}, \tag{1.2}$$

where $(\cdot, \cdot)$ denotes the $L_2(\Omega)$ inner product.

In [6,7] Donsker and Varadhan generalized the variational formula (1.2) for general second order elliptic operators with C^∞ coefficients (see also [8,15]). Donsker and Varadhan proved that for $W(x) \equiv 1$, λ_0 is given by

$$\lambda_0 = \inf_{\mu \in \mathcal{M}} \sup_{u \in \mathcal{D}} \int_\Omega \frac{Pu}{u} \mu(dx), \tag{1.3}$$

where $\mathcal{M} \equiv \mathcal{M}(\Omega)$ is the space of all probability measures on $\bar{\Omega}$, and $\mathcal{D}$ denotes the set of all positive functions $u \in C^\infty(\mathbb{R}^n)$ for each of which there exist constants c_1 and c_2 such that $0 < c_1 \leq u(x) \leq c_2 < \infty$ for all $x \in \mathbb{R}^n$. The proof is based on strongly continuous semigroups theory and was motivated by a probability theory

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point of view.

Using the maximum principle Protter and Weinberger [16] proved that for a general second order elliptic operator, λ_0 is given by the variational formula

$$\lambda_0 = \sup_{u>0} \inf_{x\in\Omega} \left\{ \frac{Pu}{W(x)u} \right\}. \tag{1.4}$$

The variational principles (1.2) and (1.4) can be modified to give a formula for a generalized eigenvalues problem with an indefinite weight function (see [2], [10] and Theorem 2.2). Using a test function, the Rayleigh quotient and (1.3) give an upper bound for λ_0 while (1.4) gives a lower bound. On the other hand Donsker and Varadhan [7] showed that the two variational principles are connected via a mini-max theorem.

Suppose now that Ω is an unbounded domain or a domain with a nonsmooth boundary. Then, in general, a principal eigenvalue does not exist. Consider the one-parameter family of operators

$$P_t u = Pu - tW(x)u \quad \text{in } \Omega,$$

where $W \in C^\alpha(\Omega)$, $0 < \alpha \leq 1$ and $t \in \mathbb{R}$. Let

$$\mathcal{C}_P(\Omega) = \{u \in C^2(\Omega) \mid u > 0 \text{ and } Pu = 0 \text{ in } \Omega\},$$

and define

$$\lambda_0(P; W, \Omega) \equiv \lambda_0 = \sup\{t \in \mathbb{R} \mid \mathcal{C}_{P_t}(\Omega) \neq \emptyset\}. \tag{1.5}$$

It is easy to check that for a C^1 bounded domain the two definitions of λ_0 coincide. Moreover, it is well known that with λ_0 defined by (1.5) and under mild regularity conditions the variational principle (1.2) is valid for self-adjoint operators defined on general domains (see for example [1]). Notice that the Protter–Weinberger formula is also valid for elliptic operators in unbounded domains (see Remark 1 in [16], and Corollary 2.3).

Another kind of variational formula for λ_0 was given by M. Schechter [18] for the one-parameter family of Schrödinger operators

$$H_t = -\Delta + \lambda - tW(x) \quad \text{in } \mathbb{R}^n, \tag{1.6}$$

where $W(x)$ is a nonnegative function, $\lambda > 0$ ($\lambda \geq 0$ if $n \geq 3$) is fixed, and $t \in \mathbb{R}$ (see also [19–21]). Schechter proved that

$$(\lambda_0)^{-1} = \inf_{\phi>0} \sup_{x\in\Omega} \frac{\int_\Omega G_{H_0}^{\mathbb{R}^n}(x,y)W(y)\phi(y)dy}{\phi(x)}, \tag{1.7}$$

where $G_P^\Omega(x,y)$ is the minimal positive Green function of the operator P in Ω. Schechter's proof is based on the special properties of $G_{H_0}^{\mathbb{R}^n}(x,y)$. Notice that if $W(x)$ is positive in Ω, one can formally derive (1.7) from the Protter–Weinberger formula (1.4) by substituting $u(x) = \int_\Omega G_P^\Omega(x,y)W(y)\phi(y)dy$.

In this paper we prove that all the above variational principles are valid for a general elliptic operator and an arbitrary domain $\Omega \subseteq \mathbb{R}^n$.

The outline of this paper is as follows. In Section 2 we give some basic definitions, fix notations and recall some results. In Section 3 we prove that the Donsker–Varadhan formula is valid for the general case. In Section 4 we show that the Schechter formula (1.7) holds for the general case provided the operator P is subcritical in Ω, namely P admits a positive minimal Green function in Ω. As an application we show in Section 5 that for an operator which is formally self-adjoint, $\lambda_0 > 0$, if and only if the Birman–Schwinger integral operator is a bounded L_2 operator.

It turns out that if ϕ is a positive supersolution of the equation $(P - \lambda_0 W(x))u = 0$ in Ω, then $\phi(x)$ is a minimizer of (1.7). It is natural to ask if there exists a minimizer for (1.7) that satisfies the equation

$$(1.8) \qquad \int_\Omega G_P^\Omega(x,y)W(y)\phi(y)dy = (\lambda_0)^{-1}\phi(x) \quad \text{for all } x \in \Omega.$$

It turns out that if P_{λ_0} is critical in Ω, the answer is yes (for the definition of criticality see Section 2). We conjecture that this is also true in the subcritical case. The conjecture turns out to be true for the special case of the radial Schrödinger operators in $\mathbb{R}^n$ (Theorem 4.6).

We conclude the paper in Section 6 where we prove a generalized Riemann–Lebesgue lemma. The lemma was proved first under more restrictive assumptions by Cantrell and Cosner [4,5]. This lemma has applications in population dynamics and refuge theory.

For the sake of brevity we shall omit some of the proofs. In [12] we intend to give the details of the proofs. Moreover, we shall present a functional analytic approach, give analogous results for general boundary conditions and discuss some more related results.

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2. Preliminaries

We consider a second order elliptic operator P acting on functions u in a domain $\Omega \subseteq \mathbb{R}^n$. We shall deal with an elliptic operator of the form

$$Pu = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u, \tag{2.1}$$

where $\partial_i = \partial/\partial x_i$ and $x = (x_1, \ldots, x_n) \in \Omega$. We assume that the coefficients of P are real and Hölder continuous and that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma(x)\sum_{i=1}^{n}\xi_i^2 \tag{2.2}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, where $\gamma(x)$ is a positive continuous function.

Let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of smooth bounded domains which exhaust Ω. Suppose that for all $1 \leq k$ the Dirichlet Green function $G_P^{\Omega_k}(x,y)$ of P exists and is positive. It is easy to see that $\{G_P^{\Omega_k}(x,y)\}_{k=1}^{\infty}$ is an increasing sequence of functions which by the Harnack inequality converges for every $x, y \in \Omega$ either to $G_P^{\Omega}(x,y)$, the positive *minimal Green function* of P in Ω, or to infinity. Recall [13], that P is said to be *a subcritical operator* in Ω in the first case and *critical* in the second case. P is said to be *supercritical* in Ω if $\mathcal{C}_P(\Omega) = \emptyset$. It turns out [13, Corollary 4.3] that P is critical in Ω if and only if P admits a ground state (in the sense of Agmon [1, Definition 5.1]) with an eigenvalue zero.

In the sequel we shall always assume that λ_0 is defined by (1.5). Let $x_0 \in \Omega$ be fixed, we shall denote by $\mathcal{K}$ the set of all positive C^2 functions u in $\bar{\Omega}$ such that $u(x_0) = 1$. Consider the one-parameter family of operators $P_t = P - tW(x)$, where $t \in \mathbb{R}$ and $W(x) \in C^{\alpha}(\Omega)$. Denote by S the set of all $t \in \mathbb{R}$ such that $\mathcal{C}_{P_t}(\Omega) \neq \emptyset$, and by S_+, S_-, S_0 the set of $t \in \mathbb{R}$ such that P_t is subcritical, supercritical and critical in Ω respectively. We have

Theorem 2.1 ([14]). *S is a closed convex set and $S_0 \subseteq \partial S$. Moreover, if $S \neq \emptyset$ then S is bounded if and only if W changes its sign in Ω and $S = \mathbb{R}$ if and only if $W \equiv 0$.*

Suppose that the sign of $W(x)$ in Ω is not restricted, and denote

$$W_+(W_-) = \{x \in \Omega \mid W(x) \geq 0(\leq 0)\}. \tag{2.3}$$

Suppose that $S \neq \emptyset$, we shall write $S = [\lambda_-, \lambda_+]$. Recall (see for example [3] p. 92) that $t \in S$ if and only if there exists a positive supersolution of the equation $P_t u = 0$ in Ω. Thus one can use Theorem 2.1 to generalize the Protter–Weinberger variational principle (see [10] and the references therein).

Theorem 2.2 *Suppose that* $u(x) \in C^2(\Omega), u(x) > 0$ *in* Ω . *Define*

(2.4) $$\rho_+(u, W) \equiv \rho_+(u) = \sup\{t \in \mathbb{R} \mid P_t u \geq 0 \text{ on } W_+\},$$

(2.5) $$\rho_-(u, W) \equiv \rho_-(u) = \inf\{t \in \mathbb{R} \mid P_t u \geq 0 \text{ on } W_-\}.$$

Then

(2.6) $$\lambda_+ = \sup\left\{\rho_+(u) \mid u \in C^2(\Omega),\ u > 0,\ \rho_+(u) \geq \rho_-(u)\right\},$$

(2.7) $$\lambda_- = \inf\left\{\rho_-(u) \mid u \in C^2(\Omega),\ u > 0,\ \rho_+(u) \geq \rho_-(u)\right\}.$$

Proof It is easy to check that if $\rho_+(u) \geq \rho_-(u)$ then u is a positive supersolution of the equations $P_{\rho_+(u)}v = 0$ and $P_{\rho_-(u)}v = 0$ in Ω. Hence $\lambda_- \leq \rho_-(u) \leq \rho_+(u) \leq \lambda_+$. Moreover, λ_+ (λ_-) is attained by any $u \in \mathcal{C}_{P_{\lambda_+}}(\Omega)$ $(\mathcal{C}_{P_{\lambda_-}}(\Omega))$. $\square$

If $W(x)$ is positive in Ω, then for any positive function $u(x) \in C^2(\Omega)$, $\rho_-(u) = -\infty$, while

$$\rho_+(u) = \inf_{x\in\Omega}\left\{\frac{Pu}{Wu}\right\}.$$

Moveover, in this case we have $\lambda_- = -\infty$ and $\lambda_+ = \lambda_0$, where λ_0 is defined by (1.5). Hence Theorem 2.2 implies the Protter–Weinberger variational principle:

Corollary 2.3 *Suppose* $W(x) > 0$ *in* Ω. *Then*

(2.8) $$\lambda_0 = \sup_{u\in\mathcal{K}}\inf_{x\in\Omega}\left\{\frac{Pu}{W(x)u}\right\}.$$

In Section 3 we shall use (2.8) and a mini-max theorem to derive a generalization of the Donsker–Varadhan formula.

3. The Donsker–Varadhan principle

In this section we shall prove that the Donsker–Varadhan variational principle holds for λ_0 (λ_0 is defined by (1.5)). We shall prove it for an elliptic operator P of the form considered in Section 2 which is defined on a general domain $\Omega \subseteq \mathbb{R}^n$ and for a positive function $W \in C^\alpha(\Omega)$.

Recall that $\mathcal{M}$ is the space of all probability measures on $\bar{\Omega}$. Denote by $\mathcal{M}_0(\Omega)$ the subset of $\mathcal{M}$ which consists of all measures with a compact support in $\bar{\Omega}$. We have

Theorem 3.1 *Let P be an elliptic operator defined in a domain $\Omega \subseteq \mathbb{R}^n$, and let $W(x) \in C^\alpha(\Omega)$ be a positive function. Then*

$$\lambda_0 = \inf_{\mu\in\mathcal{M}} \sup_{u\in\mathcal{K}} \int_\Omega \frac{Pu}{W(x)u}\mu(dx) = \inf_{\mu\in\mathcal{M}_0(\Omega)} \sup_{u\in\mathcal{K}} \int_\Omega \frac{Pu}{W(x)u}\mu(dx). \tag{3.1}$$

We shall first prove the theorem for a bounded domain. Our proof is a modification which simplifies the proof in [7]. Moreover, it applies to a much wider class of operators. We have

Lemma 3.2 (i) *Let P be an elliptic operator defined in a C^1 bounded domain. Then*

$$\lambda_0 = \inf_{\mu\in\mathcal{M}} \sup_{u\in\mathcal{K}} \int_\Omega \frac{Pu}{W(x)u}\mu(dx) \equiv \inf_{\mu\in\mathcal{M}} I(\mu). \tag{3.2}$$

(ii) *Assume further that P is an operator with $C^{1,\alpha}(\bar\Omega)$ coefficients. Let $\phi(x)$ and $\vartheta(x)$ be the ground states with an eigenvalue zero of P_{λ_0} and $P^*_{\lambda_0}$ respectively (P^* is the formal adjoint of P). Assume also that $\int_\Omega \vartheta(x)W(x)\phi(x)dx = 1$. Then $\mu_0 \equiv \vartheta(x)W(x)\phi(x)dx$ is the unique minimizer among all the smooth probability measures for the variational formula (3.2).*

Proof of the Lemma (i) Denote by $\mathcal{V}$ the subspace

$$\mathcal{V} = \{v \in C^2(\bar\Omega) | v(x_0) = 0\}.$$

Then the right hand side of (3.2) is equivalent to

$$\inf_{\mu\in\mathcal{M}} \sup_{v\in\mathcal{V}} \int_\Omega \frac{P(e^v)}{W(x)e^v}\mu(dx). \tag{3.3}$$

Consider the function

$$f(v,\mu) = \int_\Omega \frac{P(e^v)}{W(x)e^v}\mu(dx) \tag{3.4}$$

defined on $C^2(\bar\Omega) \times \mathcal{M}$. Using the observation of H. Berestyki and P. L. Lions (see also [7,11,14]), we see that for any $0 \le t \le 1$ and $v_1, v_2 \in C^2(\Omega)$

$$f(tv_1 + (1-t)v_2, \mu) = tf(v_1,\mu) + (1-t)f(v_2,\mu) \tag{3.5}$$

$$+t(1-t)\int_\Omega \frac{\exp 2(v_1 - v_2)}{W(x)} \sum_{i,j=1}^n a_{ij}(x)\partial_i(\exp(v_2 - v_1))\partial_j(\exp(v_2 - v_1))\mu(dx).$$

Therefore, as in [7] if we impose the C^2-topology then $f(v,\mu)$ is a continuous strictly concave function of v for each fixed μ, and for each fixed v it is a continuous linear function of μ. Since μ varies on a compact set (in the weak* topology), the mini-max theorem of Sion [22] applies and we have

$$\begin{aligned}(3.6)\qquad &\inf_{\mu\in\mathcal{M}}\sup_{v\in\mathcal{V}} f(v,\mu) = \sup_{v\in\mathcal{V}}\inf_{\mu\in\mathcal{M}} f(v,\mu)\\ &=\sup_{u\in\mathcal{K}}\inf_{\mu\in\mathcal{M}}\int_\Omega \frac{Pu}{W(x)u}\mu(dx) = \sup_{u\in\mathcal{K}}\inf_{x\in\Omega}\left\{\frac{Pu}{W(x)u}\right\} = \lambda_0,\end{aligned}$$

where the last equality follows from Corollary 2.3.

(ii) Define $g(v) = f(v,\mu_0)$, where f is defined by (3.4) and v belongs to the convex hull of $C^2(\bar\Omega)\cup\{\log\phi\}$. A direct computation shows that $v_0\equiv\log\phi$ is a critical point of g and $g(v_0)=\lambda_0$. Since $g(v)$ is strictly concave it follows that $I(\mu_0)=\lambda_0$.

Define $\mu_t=(1-t)\mu_0+t\mu_1$ where $\mu_1\in\mathcal{M}$ is a probability measure with a smooth density and $0\le t\le 1$. It follows from the explicit formula of $I(\mu)$ (see [7, Lemma 3.3]) that $I(\mu_t)$ is strictly convex at $t=0$. Since $\lambda_0\le I(\mu)$ (see (3.9) below), it follows that μ_0 is the unique smooth minimizer. □

Remark 3.3 (i) The second part of the lemma was proved also by Y. Kifer using probabilistic methods [9, Proposition 3.1]. Kifer also shows that μ_0 is the invariant measure of a certain Markov process.

(ii) Since the second part of the lemma implies $\lambda_0\ge\inf_{\mu\in\mathcal{M}} I(\mu)$ and it is clear that $\lambda_0\le I(\mu)$ for all $\mu\in\mathcal{M}$ (see (3.9) below), we see that the second part of the lemma provides us also an alternative proof of the Donsker–Varadhan principle.

Proof of Theorem 3.1 Denoting $\lambda_0^{(k)}=\lambda_0(\Omega_k)$, we see that

$$\begin{aligned}(3.7)\qquad &\inf_{\mu\in\mathcal{M}}\sup_{u\in\mathcal{K}}\int_\Omega\frac{Pu}{W(x)u}\mu(dx)\le\inf_{\mu\in\mathcal{M}_0(\Omega)}\sup_{u\in\mathcal{K}}\int_\Omega\frac{Pu}{W(x)u}\mu(dx)\\ &\le\inf_{\mu\in\mathcal{M}_0(\Omega_k)}\sup_{u\in\mathcal{K}}\int_\Omega\frac{Pu}{W(x)u}\mu(dx)\le\lambda_0^{(k)}.\end{aligned}$$

Since $\lambda_0^{(k)}\to\lambda_0$ as $k\to\infty$ it follows that

$$(3.8)\qquad \inf_{\mu\in\mathcal{M}}\sup_{u\in\mathcal{K}}\int_\Omega\frac{Pu}{W(x)u}\mu(dx)\le\inf_{\mu\in\mathcal{M}_0(\Omega)}\sup_{u\in\mathcal{K}}\int_\Omega\frac{Pu}{W(x)u}\mu(dx)\le\lambda_0.$$

On the other hand, by Corollary 2.3 we have

$$(3.9)\qquad \lambda_0=\sup_{u\in\mathcal{K}}\inf_{x\in\Omega}\left\{\frac{Pu}{W(x)u}\right\}\le\sup_{u\in\mathcal{K}}\int_\Omega\frac{Pu}{W(x)u}\mu(dx)$$

for any $\mu \in \mathcal{M}$. Therefore,

$$\lambda_0 \leq \inf_{\mu \in \mathcal{M}} \sup_{u \in \mathcal{K}} \int_{\Omega} \frac{Pu}{W(x)u} \mu(dx). \tag{3.10}$$

Combining (3.8) and (3.10) we obtain (3.1). □

4. The Schechter principle

Suppose now that P is subcritical in Ω, and $W(x)$ is a nonnegative Hölder continuous function in Ω (W may vanish in Ω). In this section we show that the variational principle (1.7) holds in the general case. First we need to prove some lemmas.

Lemma 4.1 *Suppose that P is a subcritical operator in Ω and $W(x) \geq 0, W \in C^{\alpha}(\Omega)$. Assume that $t \in S_+$ and denote by $G_t^{\Omega}(x,y)$ the Green function of the operator*

$$P_t = P - tW(x) \quad in\ \Omega. \tag{4.1}$$

Then $G_t^{\Omega}(x,y)$ satisfies the resolvent equation

$$G_t^{\Omega}(x,y) = G_P^{\Omega}(x,y) + t\int_{\Omega} G_P^{\Omega}(x,z)W(z)G_t^{\Omega}(z,y)dz. \tag{4.2}$$

Proof See [14, the first part of the proof of Theorem 2.1 (ii)]. □

We also have (see Theorem 2.1 and Equation (2.12) in [14])

Lemma 4.2 *Under the hypotheses of Lemma 4.1, assume that $u \in \mathcal{C}_{P_t}(\Omega)$ where $t > 0$ and $t \in S$. Then for every $x \in \Omega$*

$$t\int_{\Omega} G_P^{\Omega}(x,z)W(z)u(z)dz \leq u(x). \tag{4.3}$$

Moreover, if $t = \lambda_0$ and $\lambda_0 \in S_0$ then

$$\lambda_0 \int_{\Omega} G_P^{\Omega}(x,z)W(z)u(z)dz = u(x), \quad x \in \Omega, \tag{4.4}$$

where u is a ground state of the equation $(P - \lambda_0 W(x)u = 0$ in Ω.

Using the Riesz decomposition theorem, Lemma 4.1 and Lemma 4.2 we have

Corollary 4.3 *Let u be a positive supersolution of the equation $P_t v = 0$ in Ω, where t is a positive number. Then u satisfies inequality (4.3). Moreover, either we have a strict inequality in (4.3) for every $x \in \Omega$, or we have an equality in (4.3) for all $x \in \Omega$.*

Proof There remains to prove the last statement of the corollary. One can easily check that if u is a positive supersolution of the equation $P_t v = 0$ in Ω, then $w(x) = u(x) - t\int G_P^\Omega(x,z)w(z)u(z)dz$ is a nonnegative supersolution of the equation $Pv = 0$ in Ω. Hence, either $w(x)$ is strictly positive or identically zero. Notice that if u satisfies an equality in (4.3) then $u \in C_{P_t}(\Omega)$. □

Denote $C_+(\Omega) = \{\phi \in C(\Omega) | \phi > 0\}$. The next theorem is the generalization of the Schechter variational formula.

Theorem 4.4 *Suppose that P is subcritical in $\Omega \subseteq \mathbb{R}^n$, and let $W(x) \in C^\alpha(\Omega)$, $W \not\equiv 0$ be a nonnegative function. Then*
(i) $\lambda_0 = \lambda_0(P, W, \Omega)$ *is given by*

$$(4.5)\qquad (\lambda_0)^{-1} = \inf_{\phi \in C_+(\Omega)} \sup_{x\in\Omega} \left\{ (\phi(x))^{-1} \int_\Omega G_P^\Omega(x,y)W(y)\phi(y)dy \right\}.$$

(ii) *If ϕ is a positive supersolution of the equation $P_{\lambda_0}u = 0$ in Ω, then $\phi(x)$ is a minimizer of* (4.5).
(iii) *If ϕ is a minimizer of* (4.5) *then $v(x) = \int_\Omega G_P^\Omega(x,y)W(y)\phi(y)dy$ is a positive supersolution of the equation $P_{\lambda_0}u = 0$ in Ω. Moreover, the minimizer is unique if and only if $\lambda_0 \in S_0$.*

Proof (i) Denote the right hand side of (4.5) by ν and notice that $\lambda_0 \geq 0$. It follows from Lemma 4.2 that $(\lambda_0)^{-1} \geq \nu$. Suppose that $\nu < (\lambda_0)^{-1}$, then for $\varepsilon > 0$ small enough there exists a continuous positive function ϕ_ε such that

$$(4.6)\qquad \sup_{x\in\Omega} \left\{ (\phi_\varepsilon(x))^{-1} \int_\Omega G_P^\Omega(x,y)W(y)\phi_\varepsilon(y)dy \right\} \equiv \lambda^{-1} = (\lambda_0 + \varepsilon)^{-1} < (\lambda_0)^{-1}.$$

Consider the function $v_\varepsilon(x) = \int_\Omega G_P^\Omega(x,y)W(y)\phi_\varepsilon(y)dy$. Then v_ε is a positive supersolution of the equation $(P - \lambda W(x))u = 0$ in Ω, which is a contradiction of the definition of λ_0.
(ii) Follows directly from Corollary 4.3 and (4.5).
(iii) It is easy to check that $v(x)$ is a positive supersolution of $P_{\lambda_0}u = 0$ in Ω. Recall that $\lambda_0 \in S_0$ if and only if, up to a positive factor, there is only one positive supersolution of $P_{\lambda_0}u = 0$ in Ω (this supersolution is in fact the ground state). Therefore, the last statement of the theorem follows from part (ii). □

It follows from Lemma 4.2 that if $\lambda_0 \in S_0$ then the minimizer satisfies equality (4.4). It is natural to raise the following conjecture:

Conjecture 4.5 *There exists $u \in \mathcal{C}_{P_{\lambda_0}}(\Omega)$ that satisfies*

$$\lambda_0 \int_\Omega G_P^\Omega(x,z)W(z)u(z)dz = u(x), \quad x \in \Omega. \tag{4.4}$$

In other words, we ask whether there exists $u \in \mathcal{C}_{P_{\lambda_0}}(\Omega)$ which is a potential, in the sense of potential theory, with respect to the operator P in Ω. Notice that equality (4.4) can hold also for $\lambda \neq \lambda_0$. For example, consider the operator $P = -\Delta + 1$ in $\mathbb{R}^n$ and let $W \equiv 1$, then equality (4.4) holds for every $0 < \lambda \leq 1 = \lambda_0$ and $u \in \mathcal{C}_{P_\lambda}(\Omega)$.

Conjecture 4.5 turns out to be true for the special case where P is a radial Schrödinger operator in $\mathbb{R}^n, n \geq 3$ and $W(x) = W(r)$, where $r = |x|$. More precisely, we consider the one-parameter family of operators

$$P_t = -\Delta + V(r) - tW(r) \quad \text{in } \mathbb{R}^n, \tag{4.7}$$

where $W, V \in C^\alpha(\mathbb{R}_+), W \geq 0$ and $t \in \mathbb{R}$. We have

Theorem 4.6 *Let P_t be the operator in (4.7). Assume that P_0 is subcritical in $\mathbb{R}^n$, and $W \geq 0$. Denote $G(x,y) = G_{P_0}^{\mathbb{R}^n}(x,y)$. Suppose that $\lambda_0 > 0$ and let $u \in \mathcal{C}_{P_{\lambda_0}}(\mathbb{R}^n)$. Then*

$$\lambda_0 \int_\Omega G(x,z)W(z)u(z)dz = u(x), \quad x \in \mathbb{R}^n.$$

Remark 4.7 (i) If $\lambda_0 \in S_+$ (P and Ω are general) then one can find a minimizer of (4.5) which is not a supersolution of $P_{\lambda_0}u = 0$ in Ω. For example, one can take any function $\phi(x)$ that satisfies

$$G_{P_{\lambda_0}}(x,y_0) - G_P(x,y_0) \leq \phi(x) \leq G_{P_{\lambda_0}}(x,y_0) \quad \text{in } \Omega.$$

(ii) Comparing the Protter–Weinberger variational principle (2.8) with (4.5), we have to remember that if we want to estimate λ_0 using (2.8) we need to find a positive supersolution of $P_t u = 0$ in Ω ($t \approx \lambda_0$), which in general is not easy to find. On the other hand, in order to use (4.5) we need to know the minimal Green function of P in Ω which is also, in general, not known. Notice that we could derive (4.5) from (2.8) for $W(x) > 0$, provided we know that for positive supersolutions the integral in (4.5) is finite.

We can now derive the analog of the Donsker–Varadhan principle using (4.5) instead of (2.8).

Theorem 4.8 *Let P be a subcritical operator in Ω, and let $W(x) \in C^{\alpha}(\Omega)$ be a nonnegative function. Then*

$$(4.8)\qquad \begin{aligned}(\lambda_0)^{-1} &= \sup_{\mu\in\mathcal{M}}\ \inf_{\phi\in C_+(\Omega)}\left\{\int_\Omega(\phi(x))^{-1}\left\{\int_\Omega G_P^\Omega(x,y)W(y)\phi(y)dy\right\}\mu(dx)\right\}\\ &= \sup_{\mu\in\mathcal{M}_0(\Omega)}\ \inf_{\phi\in C_+(\Omega)}\left\{\int_\Omega(\phi(x))^{-1}\left\{\int_\Omega G_P^\Omega(x,y)W(y)\phi(y)dy\right\}\mu(dx)\right\}.\end{aligned}$$

The proof of the theorem is almost the same as the proofs of Theorem 3.1 and Lemma 3.2, hence we shall only outline the proof.

Proof As in the proof of Theorem 3.1 it is enough to prove the theorem for a smooth bounded domain. Using the Hölder inequality we see that the functional

$$(4.9)\qquad f_x(v) = \log\int_\Omega G_P^\Omega(x,y)W(y)\exp(v(y))dy - v(x)$$

is convex for every $x \in \Omega$. Therefore, $\exp(f_x(v))$ is also a convex functional. It follows that the function

$$(4.10)\qquad f(v,\mu) = \int_\Omega\left\{\exp(-v(x))\int_\Omega G_P^\Omega(x,y)W(y)\exp(v(y))dy\right\}\mu(dx)$$

defined on $C(\Omega)\times\mathcal{M}$ is a lower semicontinuous (see [17], p. 356, Example 1) and convex function of v and for each fixed μ, and for each fixed v it is a continuous linear function of μ. Since μ varies on a compact set (in the weak* topology), the mini-max theorem of Sion ([22], see also [11]) applies and we have

$$(4.11)\qquad \begin{aligned}&\sup_{\mu\in\mathcal{M}}\ \inf_{\phi\in C_+(\bar\Omega)}\left\{\int_\Omega\left\{(\phi(x))^{-1}\int_\Omega G_P^\Omega(x,y)W(y)\phi(y)dy\right\}\mu(dx)\right\}\\ &=\sup_{\mu\in C}\ \inf_{v\in C(\Omega)} f(v,\mu) = \inf_{v\in C(\Omega)}\ \sup_{\mu\in\mathcal{M}} f(v,\mu)\\ &= \inf_{\phi\in C_+(\bar\Omega)}\left\{\sup_{\mu\in\mathcal{M}}\int_\Omega\left\{(\phi(x))^{-1}\int_\Omega G_P^\Omega(x,y)W(y)\phi(y)dy\right\}\mu(dx)\right\}\\ &= \inf_{\phi\in C_+(\bar\Omega)}\left\{\sup_{x\in\Omega}\left\{(\phi(x))^{-1}\int_\Omega G_P^\Omega(x,y)W(y)\phi(y)dy\right\}\right\} = (\lambda_0)^{-1},\end{aligned}$$

where the last equality follows from Theorem 4.4. □

Remark 4.9 (i) Let $\beta(x)$ be a continuous nondecreasing convex function defined on $\mathbb{R}$, then by replacing $\exp \circ \log$ with $\beta \circ \log$ in (4.10) and using the same technique as in the proof of the theorem we obtain

(4.12)
$$\beta \circ (-\log \lambda_0) = \sup_{\mu \in \mathcal{M}} \left\{ \inf_{\phi \in C_+(\Omega)} \int_\Omega \beta \circ \log\{\frac{1}{\phi(x)} \int_\Omega G_P^\Omega(x,y)W(y)\phi(y)dy\}\mu(dx) \right\}.$$

(ii) If $\Omega \subset\subset \mathbb{R}^n$ one can derive Theorem 4.8 directly from the Donsker–Varadhan principle [12].

The next Theorem shows that $\mu_0 = \vartheta(x)W(x)\phi(x)dx$ is again an extremal measure.

Theorem 4.10 *Let P be a subcritical operator with $C^{1,\alpha}(\bar{\Omega})$ coefficients defined on a C^1 bounded domain and let $W(x) \in C^\alpha(\bar{\Omega})$ be a nonnegative function. Let $\phi(x)$ and $\vartheta(x)$ be the ground states with an eigenvalue zero of P_{λ_0} and $(P_{\lambda_0})^*$ respectively. Assume also that*

$$\int_\Omega \vartheta(x)W(x)\phi(x)dx = 1. \tag{4.13}$$

Then

$$(\lambda_0)^{-1} = \inf_{u \in C_+(\Omega)} \left\{ \int_\Omega \vartheta(x)W(x)\frac{\phi(x)}{u(x)} \left\{ \int_\Omega G_P^\Omega(x,y)W(y)u(y)dy \right\} dx \right\}. \tag{4.14}$$

Moreover, if u is a minimizer of (4.14) then $u(x) = \phi(x)$ for all $x \in$ supp W.

Remark 4.11 (i) In the general case the integral in (4.13) may be infinite.

(ii) The theorem is the analog of Lemma 3.2 (ii) (for the Donsker–Varadhan principle) and of Theorem 2.3 in [11] (for matrices). The proof is basically the same as the proof of Lemma 3.2 (ii).

(iii) The measure $\mu_0 = \vartheta(x)W(x)\phi(x)dx$ appears naturally in the context of positive solutions. For other examples see Lemma 3.2 and [9,14,15].

(iv) If P is a subcritical operator with $C^{1,\alpha}(\bar{\Omega})$ coefficients in a smooth bounded domain Ω and if $W \in C^\alpha(\bar{\Omega})$ is a nonzero nonnegative function, then all the other assumptions of Theorem 4.10 are satisfied and the theorem holds. Therefore, as in Remark 3.3, and under the regularity assumption of the coefficients one can use Theorem 4.10 to obtain an alternative proof for Theorem 4.8. We leave the details to the reader.

5. The symmetric case

In this section we shall assume that P is formally self-adjoint. Therefore, the positive minimal Green function $G_P^\Omega(x,y)$, if it exists, is a symmetric function in the two variables x and y. Moreover, we can consider P as a symmetric operator in $L_2(\Omega)$ with a domain $\mathcal{D}(P) = C_0^1(\Omega)$ and let $\tilde{P}$ be its Friedrichs extension. Then it is well known that

$$\lambda_0 = \lambda_0(P, 1, \Omega) = \inf \sigma(\tilde{P}) \tag{5.1}$$

and

$$\lambda_\infty = \lambda_\infty(P, 1, \Omega) \equiv \sup_{k>0}\{\lambda_0(P, 1, \Omega \setminus \Omega_k)\} = \inf \sigma_{\mathrm{ess}}(\tilde{P}), \tag{5.2}$$

where $\sigma(\tilde{P})$ and $\sigma_{\mathrm{ess}}(\tilde{P})$ are the spectrum and the essential spectrum of $\tilde{P}$ (see for example [1]).

In this section we shall discuss some further properties of $\lambda_0(P, W, \Omega)$, where P is formal self-adjoint, W is a nonnegative function and $\Omega \subseteq \mathbb{R}^n$. First we shall show the connection between the boundedness of the Birman–Schwinger integral operator, as an operator in L_2, and the positiveness of λ_0.

Theorem 5.1 *Let P be a formal self-adjoint subcritical operator in Ω, and let $W \in C^\alpha(\Omega)$ be a nonzero nonnegative function.*

Then $\lambda_0 = \lambda_0(P, W, \Omega) > 0$ if and only if the integral operator

$$Tf(x) = \int_\Omega K(x,y) f(y) dy \tag{5.3}$$

is a bounded operator on $L_2(\Omega)$, where

$$K(x,y) = W^{1/2}(x) G_P^\Omega(x,y) W^{1/2}(y) \tag{5.4}$$

is the Birman–Schwinger kernel. Moreover,

$$||T||^2 = (\lambda_0)^{-2} = \inf_{u \in C_+(\Omega)} \sup_{x \in \Omega}\{(u(x))^{-1} T^2 u(x)\}. \tag{5.5}$$

The proof is based on Theorem 4.4 and the Schechter generalization of the Schur test [19, Theorem 3 and Equation (12) therein].

From the proof of Theorem 5.1 follows

Corollary 5.2 *Suppose* $\lambda_0 = 0$. *Then* T *is an unbounded operator on* $L_p(\Omega), 1 \leq p \leq \infty$, *where* T *is defined by (5.3) and (5.4).*

The next theorem provided us with a sufficient condition for the invariance of the infimum of $\sigma_{\mathrm{ess}}(\tilde{P})$ (see also Theorem 5.1 in [20]). Since the theorem holds also for nonsymmetric operators, we shall state the theorem in terms of λ_∞ (λ_∞ is defined by (5.2)).

Theorem 5.3 *Suppose that* P *is an elliptic operator of the form (2.1) and that* P *is defined on* Ω. *Assume also that* $\lambda_\infty = 0$. *Consider the two-parameter family of operators*

$$P_{t,s}u = (P - tW + s)u, \quad t, s \in \mathbb{R} \tag{5.6}$$

where $W \in C^\alpha(\Omega)$ *is a nonnegative function. Suppose that for some* $s_0 \geq 0$

$$\lim_{k\to\infty} \lambda_0(P + s_0, W_k, \Omega) = \infty, \tag{5.7}$$

where

$$W_k(x) = \begin{cases} 0, & x \in \Omega_k, \\ W(x), & x \in \Omega \setminus \Omega_k. \end{cases} \tag{5.8}$$

Then

$$\lambda_\infty(P - W, 1, \Omega) = \lambda_\infty(P, 1, \Omega) = 0. \tag{5.9}$$

Remark 5.4 Suppose that $\lambda_0(P, 1, \Omega) = 0$, and let $W \geq 0$. Theorem 2.1 implies that the function

$$\lambda_0(s) \equiv \lambda_0(P + s, W, \Omega), \quad s \geq 0, \tag{5.10}$$

is a nondecreasing concave function. It follows that Theorems 4.1 and 4.2 in [20] hold also in the general case and can be derived directly from these properties of λ_0.

6. The Riemann–Lebesgue lemma

In this section we derive from the Donsker–Varadhan principle a generalized Riemannian–Lebesgue lemma which has applications in population dynamics and refuge theory [4,5]. We have

Theorem 6.1 *Let P be a subcritical elliptic operator with Hölder continuous coefficients which is defined in $\Omega \subset\subset \mathbb{R}^n$. Let $\{m_j\}_{j=1}^{\infty} \subseteq L_\infty(\Omega) \cap C^\alpha(\bar{\Omega})$ be a sequence of functions such that $\|m_j\|_\infty \leq M$. Denote*

$$\lambda_0^j \equiv \lambda_0(m_j) = \sup\{t \in \mathbb{R} | \mathcal{C}_{P-tm_j}(\Omega) \neq \emptyset\}. \tag{6.1}$$

Then $\lambda_0(m_j) \to \infty$ as $j \to \infty$ if and only if

$$\limsup_{j\to\infty} \int_\Omega m_j(x)\mu(dx) \leq 0 \tag{6.2}$$

for all probability measures μ in $\bar{\Omega}$ with density in $L_1(\Omega)$.

Proof We shall give here the proof of only one part of the theorem. Suppose that $\lambda_0(m_j) \to \infty$ as $j \to \infty$. By the Donsker–Varadhan formula we have

$$\inf_{\mu\in\mathcal{M}} \left\{ \sup_{u\in\mathcal{K}} \left\{ \int_\Omega \frac{Pu}{u}\mu(dx) \right\} - \lambda_0^j \int_\Omega m_j(x)\mu(dx) \right\} \geq 0. \tag{6.3}$$

Let μ be fixed, then

$$I(\mu) \equiv \sup_{u\in\mathcal{K}} \int_\Omega \frac{Pu}{u}\mu(dx) \geq \lambda_0^j \int_\Omega m_j(x)\mu(dx). \tag{6.4}$$

From the explicit formula for $I(\mu)$ [7], Lemma 3.3] it follows that $I(\mu)$ is finite if μ is sufficiently smooth. Therefore

$$I(\mu)/\lambda_0^j \geq \int_\Omega m_j(x)\mu(dx), \tag{6.5}$$

and as $j \to \infty$ we obtain (6.2). Note that in the proof of this part we have not used the boundedness of Ω and $\{m_j\}$. □

Example 6.2 (i) Let P be an elliptic operator in $\mathbb{R}^n$ with constant coefficients and define $m_j(x) = m_0(x + je_1)$, where $m_0 \in C_0^\infty(\mathbb{R}^n), e_1 = (1, 0, \dots, 0)$ and $j > 0$. Then $\lambda_0(m_j) = \lambda_0(m_0)$, but $\lim_{j\to\infty} \int_\Omega m_j(x)\mu(dx) = 0$ for all probability measures μ in $\mathbb{R}^n$ with density in $L_1(\mathbb{R}^n)$. Therefore, one part of Theorem 6.1 is not valid if Ω is an unbounded domain.

(ii) Let $Py = -y'', \Omega = (0, \pi)$ and $m_j(x) = \sin jx$. It follows from the classical Riemann–Lebesgue lemma that $\lambda_0(m_j) \to \infty$ as $j \to \infty$.

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