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Cyclic Differential Equations and Period Three Solutions of Differential-Delay Equations

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The motivation for this paper is the study of nontrivial periodic solutions of differential-delay equations of the type

$$y'(t) = -\alpha g(y(t - N_1), y(t - N_2), \ldots, y(t - N_k)), \tag{0.1}$$

where $g\colon \mathbb{R}^k \to \mathbb{R}$ is a given function, N_i are integers and α is a real parameter. Such equations have been considered in several papers, for instance [2, 5, 6]. It has become apparent that it is important to be able to show that (0.1) has no nontrivial periodic solutions of certain periods. For example, the global Hopf bifurcation theorem, as developed by Alexander and Yorke for ordinary differential equations and later extended to functional differential equations [2, 6], gives relatively little information about (0.1) if one does not know that (0.1) has no nontrivial solutions of certain periods. A well-known result (see Lemma 4.1 in [2]) asserts that Eq. (0.1) can have no nonconstant periodic solutions of period 2. In this paper we show that for *certain* classes of g, (0.1) can have no nonconstant periodic solutions of period 3. The general idea that periodic solutions of (0.1) of period n, n an integer, satisfy an associated system of ordinary differential equations has been used before: see [4] and Lemma 4.1 of [2]. However, in our case we deal with a system of three first order ODEs, instead of two ODEs as in Lemma 4.1 of [2].

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In the first section we consider the cyclic differential equations

$$y_i' = -\alpha f_{i+1}(y_{i+1}) - \beta f_{i+2}(y_{i+2}), \tag{0.2}$$

where the indices i are written mod 3. We prove several theorems which show that (0.2) can have no nonconstant periodic solutions under certain conditions on f_i, α and β. Notice that if $f_0 = f_1 = f_2 = f$, then period 3 solutions of

$$y'(t) = -\alpha f(y(t-1) - \beta f(y(t-2))$$

give rise to period 3 solutions of (0.2) by defining $y_i(t) = y(t-i)$ for $i = 0, 1, 2$. Thus if (0.2) has no nonconstant period 3 solutions, the differential-delay equation also has none.

In the second and third sections of this paper we apply the theorems of the first section in studying

$$x'(t) = \lambda f(x(t-4)) - k\lambda f(x(t-1)). \tag{0.3}$$

This involves a detailed and rather involved study of the imaginary roots of the characteristic equation

$$z - \lambda \exp(-4z) + k\lambda \exp(-z) = 0. \tag{0.4}$$

In the third section we use the results of Sections 1 and 2 to prove the existence of nonconstant periodic solutions of (0.3) of period between 2 and 3 for a wide range of positive λ.

1. Cyclic Differential Equations

In this section we consider the system of three first order differential equations

$$y_i' = -\alpha f_{i+1}(y_{i+1}) - \beta f_{i+2}(y_{i+2}), \tag{1.1}$$

where α and β are real parameters and the indices are written mod 3. We shall always assume that the functions $f_i: \mathbb{R} \to \mathbb{R}$ satisfy

H1. $f_i: \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian, $f_i(0) = 0$ and $uf_i(u) > 0$ for all $u \neq 0$ and for $i = 0$, 1 and 2.

We investigate the trajectories of system (1.1). By scaling the equation it is sufficient to consider the case $\alpha = 1$. Thus we consider

$$y_i' = -f_{i+1}(y_{i+1}) - \beta f_{i+2}(y_{i+2}). \tag{1.2}$$

Denote an arbitrary point of $\mathbb{R}^3$ by $Y = (y_0, y_1, y_2)$ and put $|Y| = \sqrt{(y_0^2 + y_1^2 + y_2^2)}$. Let P be the orthogonal projection of $\mathbb{R}^3$ onto the line $y_0 = y_1 = y_2$. Let $Y(t) = (y_0(t), y_1(t), y_2(t))$ be a solution of (1.2) and put $R(t) = Y(t) - P(Y(t))$, so $R(t) = (1/3)(2y_0 - y_1 - y_2,\ 2y_1 - y_0 - y_2,\ 2y_2 - y_0 - y_1)$. A straightforward calculation gives (letting Z_3 denote the integers mod 3)

$$\frac{3}{2}\frac{d}{dt}|R|^2 = \sum_{i \in Z_3} [(\beta + 1)\, y_i f_i(y_i) - (2 - \beta)\, y_i f_{i+1}(y_{i+1}) - (2\beta - 1)\, y_i f_{i+2}(y_{i+2})]$$

and so

$$\frac{d}{dt}\left(\frac{3|R|^2 - |Y|^2}{2}\right) = \sum_{i \in Z_3} [(\beta + 1)\, y_i f_i(y_i) + (\beta - 1)\, y_i f_{i+1}(y_{i+1}) - (\beta - 1)\, y_i f_{i+2}(y_{i+2})]. \tag{1.3}$$

If $\beta = 1$, then in view of the assumption on the f_i we deduce that

$$\frac{d}{dt}\left(\frac{3|R|^2 - |Y|^2}{2}\right) > 0 \qquad \text{for} \quad Y \neq (0, 0, 0).$$

It follows that if $\beta = 1$, (1.2) has no nontrivial periodic solutions. In fact much more can be said about the solutions of (1.2), as the following theorem shows.

THEOREM 1.1. *Consider the third order system*

$$y_i' = -f_{i+1}(y_{i+1}) - \beta f_{i+2}(y_{i+2}), \tag{1.4}$$

where the indices are written mod 3. Assume that the functions f_i satisfy H1 and that $\beta \neq -1$. For $\beta \neq 1$ assume that there are positive constants a and A such that for all y

$$a\,|y| \leqslant |f_i(y)| \leqslant A\,|y|, \qquad y \in \mathbb{R} \tag{1.5}$$

and

$$1 + \frac{|\beta + 1|}{|\beta - 1|} > \frac{A}{a}. \tag{1.6}$$

Then any solution $Y(t) = (y_0(t), y_1(t), y_2(t))$ defined on a maximal interval $[0, T)$ with $T > 0$ satisfies either

$$\lim_{t \to T^-} |Y(t)| = 0 \qquad \text{or} \qquad \lim_{t \to T} |Y(t)| = +\infty.$$

Proof. We first claim that under our assumptions

$$\frac{d}{dt}\left(\frac{3|R|^2-|Y|^2}{2}\right) \neq 0 \qquad \text{if} \quad Y \neq (0,0,0).$$

We already know this is true for $\beta = 1$. For $\beta \neq 1$, we obtain from (1.3) that

$$\left|\frac{d}{dt}\left(\frac{3}{2}|R|^2 - \frac{1}{2}|Y|^2\right)\right|$$
$$\geqslant |\beta+1| \sum_{i \in Z_3} y_i f_i(y_i) - |\beta - 1| \sum_{i \in Z_3} |y_i f_{i+1}(y_{i+1}) - y_{i+1} f_i(y_i)|.$$

Assumption H1 implies that $y_i f_{i+1}(y_{i+1})$ and $y_{i+1} f_i(y_i)$ have the same sign, and using this fact and (1.5) we obtain

$$|y_i f_{i+1}(y_{i+1}) - y_{i+1} f_i(y_i)| \leqslant (A-a)\,|y_i y_{i+1}|.$$

Using the previous two equations we find

$$\frac{d}{dt}\left(\frac{3|R|^2-|Y|^2}{2}\right) \geqslant a\,|\beta+1| \sum_{i \in Z_3} y_i^2 - (A-a)\,|\beta-1| \sum_{i \in Z_3} |y_i y_{i+1}|.$$

The Cauchy–Schwarz inequality implies

$$\sum_{i \in Z_3} |y_i y_{i+1}| \leqslant \sum_{i \in Z_3} y_i^2$$

so we find

$$\frac{d}{dt}\left(\frac{3|R|^2-|Y|^2}{2}\right) > a\left[|\beta+1| - |\beta-1|\left(\frac{A}{a}-1\right)\right]\left(\sum_{i \in Z_3} y_i^2\right) \qquad (1.7)$$

and Eq. (1.6) implies that this is positive for $Y \neq (0,0,0)$.

Now let $Y(t) = (y_0(t), y_1(t), y_2(t))$ be a solution of (1.4) defined on a maximal interval $[0, T)$. Put

$$m = \liminf_{t \to T^-} |Y(t)| \qquad \text{and} \qquad M = \limsup_{t \to T^-} |Y(t)|.$$

We suppose first that $0 \leqslant m < M \leqslant \infty$ and obtain a contradiction. Choose r_1 and r_2 such that $m < r_1 < r_2 < M$. We can find sequences $\{s_n\}$ and $\{t_n\}$ such that $s_n < t_n < s_{n+1}$ for $n \geqslant 0$, $s_n \to T$ and $t_n \to T$ as $n \to \infty$, $|Y(s_n)| = r_1$, $|Y(t_n)| = r_2$ and $r_1 \leqslant |Y(t)| \leqslant r_2$ for $s_n \leqslant t \leqslant t_n$. (For example, if s_n and t_n have been defined, select numbers σ_{n+1} and τ_{n+1} such that $t_n < \sigma_{n+1} < \tau_{n+1}$, $|Y(\sigma_{n+1})| = r_1$ and $|Y(\tau_{n+1})| = r_2$. Then define s_{n+1} and t_{n+1} by $s_{n+1} =$

$\sup\{s \mid \sigma_{n+1} \leqslant s \leqslant \tau_{n+1}, \quad |Y(s)| = r_1\}$ and $t_{n+1} = \inf\{t \mid s_{n+1} \leqslant t \leqslant \tau_{n+1}, |Y(t)| = r_2\}$.) Note that

$$r_2^2 - r_1^2 = |Y(t_n)|^2 - |Y(s_n)|^2$$
$$= -2 \int_{s_n}^{t_n} \sum_{i \in Z_3} [y_i(s)(f_{i+1}(s)) + \beta f_{i+2}(y_{i+2}(s)))]\, ds.$$

If $B = \sup\{|2 \sum_{i \in Z_3} [y_i(f_{i+1}(y_{i+1}) + \beta f_{i+2}(y_{i+2}))]|: r_1 \leqslant |Y| \leqslant r_2\}$, then $r_2^2 - r_1^2 \leqslant (t_n - s_n)B$. Thus $t_n - s_n \geqslant (r_2^2 - r_1^2)/B$ for all n, and it follows that $T = +\infty$.

For notational convenience write $L(t) = (3/2)\,|R(t)|^2 - (1/2)\,|Y(t)|^2$. Equations (1.3) and (1.7) imply that $|dL/dt| > 0$ for $|Y(t)| > 0$ and that there is a positive constant δ such that

$$\frac{dL}{dt} \geqslant \delta \qquad \text{if} \quad r_1 \leqslant |Y(t)| \leqslant r_2.$$

Using the fact that dL/ds is of constant sign for $s > 0$ we find that for $t \geqslant t_n$

$$|L(t) - L(0)| = \int_0^t \left| \frac{dL}{ds} \right| ds$$
$$\geqslant \sum_{j=1}^{n} \int_{s_j}^{t_j} \left| \frac{dL}{ds} \right| ds$$
$$\geqslant n\delta \left(\frac{(r_2^2 - r_1^2)}{B} \right).$$

Choosing t such that $s_{n+1} \leqslant t \leqslant t_{n+1}$ gives

$$r_2^2 \geqslant |Y(t)|^2 \geqslant |L(t)| \geqslant n\delta \frac{(r_2^2 - r_1^2)}{B} - |L(0)|. \tag{1.8}$$

Since Eq. (1.8) is valid for all n, we obtain a contradiction and conclude that $m = M$.

We have proved that $m = M = \lim_{t \to T^-} |Y(t)|$, and it only remains to prove $m = 0$ or $m = \infty$. Suppose not, so that $0 < m = M < \infty$. Then the elementary theory of ordinary differential equations implies that $T = \infty$. Furthermore, there exist positive numbers r_1 and r_2 such that

$$r_1 \leqslant |Y(t)| \leqslant r_2 \qquad \text{for} \quad 0 \leqslant t < \infty.$$

Just as before, there exist a positive constant δ such that

$$\frac{dL}{dt} \geqslant \delta \qquad \text{for} \quad t \geqslant 0$$

and it follows that

$$|L(t)| \geqslant \delta t - |L(0)| \qquad \text{for} \quad t \geqslant 0. \tag{1.9}$$

Letting t approach ∞ in (1.9) contradicts the assumption that $|Y(t)|$ is bounded. It follows that $m = 0$ or $m = \infty$, and the theorem is proved. ∎

Remark 1.1. It can be seen that both possibilities of the theorem can occur. For instance, in the case $f_0 = f_1 = f_2$ and $\beta = 1$, let $y(t)$ be a solution of the differential equation $y'(t) = -2f_0(y(t))$. Then by the assumptions on f_0, $y(t) \to 0$ as $t \to \infty$ and $Y(t) = (y(t), y(t), y(t))$ is a solution of (1.4). Note also that if $Y(t)$ is another solution such that $L(0) = (3/2)|R(0)|^2 - (1/2)|Y(0)|^2 > 0$, then $L(t) > L(0)$ for $t > 0$ and $Y(t)$ does not approach 0 as $t \to T^-$ and so $|Y(t)| \to +\infty$ as $t \to T^-$.

Remark 1.2. If $\beta = -1$, Theorem 1.1 provides no information about Eq. (1.4). In fact, if $\beta = -1$ and $f_0 = f_1 = f_2 = f$, $Y(t) = (c, c, c)$ is a solution of (1.4) for any constant c, so the theorem's conclusion is false in this case. In addition, for certain functions f, Eq. (1.4) possesses nonconstant periodic solutions. For example, suppose $f(x)$ is an odd, C^1 function such that $xf(x) > 0$ for $x \neq 0$ and

$$f'(0) > \left(\frac{\pi}{3\sqrt{3}}\right) > \lim_{x \to \infty} \frac{f(x)}{x}.$$

Then a theorem of Kaplan and Yorke [4] asserts that the equation

$$x'(t) = -f(x(t-1) - f(x(t-2))$$

has a solution $x(t)$ such that $x(0) = 0$, $x(t) > 0$ for $0 < t < 3$ and $x(t+3) = -x(t)$ for all t. If we define $y_i(t) = x(t - 2i)$ for $i = 0, 1, 2$, one can easily see that

$$y_i'(t) = -f(y_{i+1}) + f(y_{i+2}),$$

where the indices are written mod 3.

Remark 1.3. If the sign in Eq. (1.3) is changed, so one considers

$$y_1' = f_{i+1}(y_{i+1}) + \beta f_{i+2}(y_{i+2}),$$

then the conclusion of Theorem 1.1 remains valid if $[0, T)$ is taken to be the maximum negative interval on which the solution $Y(t)$ is defined.

COROLLARY 1.1. *Consider the differential-delay equation*

$$x'(t) = -g(x(t-1), x(t-4),..., x(t-3k-1)) - g(x(t-n), x(t-n-3),..., x(t-n-3k). \quad (1.10)$$

where $g: \mathbb{R}^{k+1} \to \mathbb{R}$ is a function and n is an integer which is congruent to 2 mod 3. Define $f(u) = g(u, u,..., u)$ and assume that $f: \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian and $uf(u) > 0$ for $u \neq 0$. Then Eq. (1.10) has no solutions of period 3 which are not identically zero.

Remark 1.4. We emphasize that by a solution of period 3 we simply mean a solution such that $x(t+3) = x(t)$ for all t; 3 is *not* assumed to be the minimal period. We shall see at the end of this section that if attention is restricted to special classes of periodic solutions of differential-delay equations much stronger results can be obtained.

Proof. If $x(t)$ is a solution of Eq. (1.10) of period 3, then $x(t)$ satisfies

$$x'(t) = -f(x(t-1)) - f(x(t-2)).$$

If we define $y_i(t) = x(t-i)$ and $f_i = f$ for $i = 0, 1$ and 2 then $y_i(t)$ satisfies Eq. (1.2) with $\beta = 1$ (assuming the indices are written mod 3). Theorem 1.1 implies that Eq. (1.2) has no nonconstant periodic solutions.

The question of exactly what assumptions will insure that Eq. (1.1) has no nonconstant periodic solutions appears to be quite delicate (we will mention some examples later). A case of particular interest for applications to differential-delay equations is when $f_0 = f_1 = f_2 = f$. Even in this case it is unclear what are the best assumptions of f to eliminate nonconstant periodic solutions of (1.1). Indeed our next theorem shows that the conclusions of Theorem 1.1 may hold under assumptions quite different from those of Theorem 1.1.

THEOREM 1.2. *Consider the system of three first order equations*

$$y_i' = -f(y_{i+1}) - \beta f(y_{i+2}), \quad (1.11)$$

where the indices i are written mod 3. *Assume that* $f: \mathbb{R} \to \mathbb{R}$ *is locally Lipschitzian,* f *is monotonically increasing and* f *is odd* ($f(-u) = -f(u)$). *Assume that* $\frac{1}{3} < \beta < 3$. *Then any solution* $Y(t) = (y_0(t), y_1(t), y_2(t))$ *of* (1.11) *defined on a maximal interval* $[0, T)$ $(T > 0)$ *satisfies either* $\lim_{t \to T} |Y(t)| = 0$ or $\lim_{t \to T^-} |Y(t)| = +\infty$.

Proof. Equation (1.3) reduces in our situation to

$$\frac{d}{dt}\frac{3|R|^2-|Y|^2}{2}$$
$$=(\beta+1)\sum_{i\in Z_3} y_i f(y_i)+(\beta-1)\sum_{i\in Z_3}[y_i f(y_{i+1})-y_{i+1}f(y_i)].$$

Noting that $y_i f(y_{i+1})$ and $y_{i+1}f(y_i)$ are always of the same sign and writing $z_i=|y_i|$ we see

$$\begin{aligned}|y_i f(y_{i+1})-y_{i+1}f(y_i)| &\leqslant \max\{|y_i f(y_{i+1})|, |y_{i+1}f(y_i)|\}\\ &=\max\{z_i f(z_{i+1}), z_{i+1}f(z_i)\}\\ &\leqslant z_j f(z_j),\end{aligned}$$

where $j=i+1$ if $z_{i+1}\geqslant z_i$ and $j=i$ if $z_{i+1}<z_i$. Thus

$$\left|\sum_{i\in Z_3}[y_i f(y_{i+1})-y_{i+1}f(y_i)]\right|\leqslant z_{i(1)}f(z_{i(1)})+2z_{i(2)}f(z_{i(2)}),$$

where $i(0)$, $i(1)$ and $i(2)$ are chosen so that $z_{i(0)}\leqslant z_{i(1)}\leqslant z_{i(2)}$. It follows that

$$\begin{aligned}\frac{d}{dt}\left(\frac{3|R|^2-|Y|^2}{2}\right) &\geqslant [(\beta+1)-2|\beta-1|]\sum_{i\in Z_3} z_i f(z_i)\\ &\geqslant [(\beta+1)-2|\beta-1|]\sum_{i\in Z_3} y_i f(y_i).\end{aligned}$$

The assumptions on β insure that $(\beta+1)-2|\beta-1|>0$, so

$$\frac{d}{dt}\left(\frac{3}{2}|R|^2-\frac{1}{2}|Y|^2\right)>0 \qquad \text{for} \quad Y(t)\neq(0,0,0).$$

The remainder of the proof is the same as that of Theorem 1.1 and we leave it to the reader. ∎

Remark 1.5. If $Y\in\mathbb{R}^3$, $P(Y)$ denotes the orthogonal projection of Y on the line $y_0=y_1=y_2$ and $R=Y-P(Y)$, then the quantity

$$(3/2)|R|^2-(1/2)|Y|^2=|R|^2-(1/2)|PY|^2$$

plays a central role in the proofs of Theorems 1.1 and 1.2. If $Q_{+++}=\{Y\in\mathbb{R}^3 \mid y_i\geqslant 0 \text{ for } i=0, 1, \text{ and } 2\}$ and $Q_{---}=\{Y\in\mathbb{R}^3 \mid y_i\leqslant 0$ for $i=0, 1,$ and $2\}$ and $Q=\mathbb{R}^3-(Q_{+++}\cup Q_{---})$, one can show that

$((3|R|^2 - |Y|^2)/2) > 0$ for $Y \in Q$. If $\beta > -1$ and $Y(t)$ denotes a solution of (1.2), we know that

$$\frac{d}{dt}\left(\frac{3|R|^2 - |Y|^2}{2}\right) > 0$$

if the hypotheses of Theorem 1.1 or Theorem 1.2 are satisfied. We conclude that if $Y(0) \in \bar{Q}$ and $Y(0) \neq (0, 0, 0)$ (and the hypotheses of Theorem 1.1 or Theorem 1.2 are satisfied), then $\lim_{t\to T^-} |Y(t)| = \infty$. It is also worth noting that if $\beta \geqslant 0$ and the f_i satisfy H1 and $Y(0) \in \bar{Q}$, then $Y(t) \in \bar{Q}$ for all positive t in the domain of Y.

The crudeness of our estimates in Theorems 1.1 and 1.2 suggests that sharper results should be true. One is encouraged in this hope by the following theorem, which deals with the case $\beta = 0$, this being a case of particular interest and indeed one of the motivations of our study.

THEOREM 1.3. *Consider the third order system*

$$y_i' = -f_{i+1}(y_{i+1}), \tag{1.12}$$

where the indices are written mod 3. *Assume that* $f_i : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable*, $f_i(0) = 0$ *and* $f_i'(u) > 0$ *for all* u. *Then for any solution* $Y(t)$ *of* (1.12) *defined on a maximal positive interval* $[0, T)$ *either* $\lim_{t\to T^-} |Y(t)| = 0$ *or* $\lim_{t\to T^-} |Y(t)| = +\infty$.

Proof. Let F_i be an anti-derivative of f_i for $i = 0, 1, 2$. Define $W: \mathscr{R}^3 \to \mathbb{R}$ by

$$W(Y) = \sum_{i\in Z_3} [F_i(y_i) - y_i f_{i+2}(y_{i+2})],$$

where $Y = (y_0, y_1, y_2)$. Let $V(t) = W(Y(t))$ and note that

$$\begin{aligned}\frac{dV}{dt} &= \sum_{i\in Z_3} [-f_i(y_i) f_{i+1}(y_{i+1}) + f_{i+1}(y_{i+1}) f_{i+2}(y_{i+2}) \\ &\quad + y_i f_i(y_i) f'_{i+2}(y_{i+2})] \\ &= \sum_{i\in Z_3} y_i f_i(y_i) f'_{i+2}(y_{i+2}). \end{aligned} \tag{1.12}$$

By our assumptions, the right hand side of (1.12) is positive for $Y(t) \neq (0, 0, 0)$, so we have already shown that (1.12) can have no nonconstant periodic solutions.

Define $G(Y) = \sum_{i\in Z_3} y_i f_i(y_i) f'_{i+2}(y_{i+2})$, so $G(Y) > 0$ if $Y \neq (0, 0, 0)$. As in Theorem 1.1, let $m = \lim_{t\to T^-} \inf |Y(t)|$ and $M = \lim_{t\to T^-} \sup |Y(t)|$. We suppose first that $0 \leqslant m < M \leqslant \infty$, choose r_1 and r_2 with $m < r_1 < r_2 < M$ and obtain a contradiction. Choose sequences $\{s_n\}$ and $\{t_n\}$ as in Theorem 1.1. Just as in the proof of Theorem 1.1, there exists $\varepsilon > 0$ such that

$$V(t) \geqslant V(0) + \varepsilon n \qquad \text{for} \quad s_n \leqslant t \leqslant t_n.$$

If $B = \sup\{W(Y): r_1 \leqslant |Y| \leqslant r_2\}$ we see that

$$B \geqslant V(0) + \varepsilon n$$

for all n. This is impossible, so $m = M$. The argument that $0 < m = M < \infty$ is impossible is essentially the same as in Theorem 1.1, and we leave it to the reader. We conclude that $m = M = 0$ or $m = M = \infty$. ∎

Remark 1.6. The conclusions of Theorem 1.3 remain the same for the system

$$y_i' = f_{i+1}(y_{i+1})$$

(f_i as in Theorem 1.3 and indices written mod 3) except that the interval $[0, T)$ must be taken to be the maximum negative interval of definition for $Y(t)$.

COROLLARY 1.2. *Consider the differential-delay equation*

$$y'(t) = -g(u_1(t), u_2(t), \ldots, u_n(t)), \tag{1.13}$$

where $u_j(t) = y(t - 3j + 2)$. If $f(u) =^{\text{def}} g(u, u, \ldots, u)$ is continuously differentiable, $f(0) = 0$ and $f'(u) > 0$ for all u, then (1.13) has no nonconstant periodic solution of period 3. The same conclusion holds if, instead of the above assumptions, f is locally Lipschitzian, $uf(u) > 0$ for $u \neq 0$ and

$$a\,|u| \leqslant |f(u)| \leqslant A\,|u|$$

for all u and for positive constants a and A such that $A/a < 2$.

Proof. Suppose $y(t)$ is a nonconstant periodic solution of (1.13) of period 3. Put $y_i(t) = y(t - i)$ for $i = 0, 1, 2$. Clearly $Y(t) = (y_0(y), y_i(t), y_2(t))$ is a periodic solution of period 3 of the system $y_i' = -f(y_{i+1})$, $i = 0, 1, 2$. Theorem 1.1 or Theorem 1.3 shows this is not possible. ∎

COROLLARY 1.3. *Consider the differential-delay equation*

$$x'(t) = -f(x(t-1)). \tag{1.14}$$

Assume that $f(u)$ *is continuously differentiable,* $f(0) = 0$, $f'(u) > 0$ *for all* u

and f is odd. Then Eq. (1.14) *has no nonconstant periodic solution such that* $x(t+3) = -x(t)$ *for all* t.

Proof. If $x(t)$ is a nonconstant periodic solution of (1.14) such that $x(t+3) = -x(t)$ for all t, define $y_0(t) = x(t)$, $y_1(t) = x(t-4)$ and $y_2(t) = x(t-2)$. One easily checks that

$$y_i' = f(y_{i+1}),$$

where the indices are written mod 3, and this contradicts Theorem 1.3 and Remark 1.6. ∎

Remark 1.7. It is natural to ask how close Theorems 1.1 and 1.3 are to best-possible when they are applied to the system

$$y_i' = -f(y_{i+1}) \tag{1.15}$$

(where the indices are written mod 3). Angelstorf [1] has proved the following theorem, which shows that one must assume more than f locally Lipschitzian and $uf(u) > 0$ for all $u \neq 0$ in order to eliminate nonconstant periodic solutions.

THEOREM (Angelstorf [1, p. 49]). *Let* $f_\lambda : \mathbb{R} \to \mathbb{R}$, $\lambda \geqslant 1$, *be a parametrized family of odd continuous functions which satisfy the following conditions*:

(1) *There exists a constant* $k < 2$ *such that for all* $\lambda \geqslant 1$ *and all* $x \geqslant \frac{1}{2}$ *one has* $1 \leqslant f_\lambda(x) \leqslant k$.

(2) *For* $1 \leqslant \lambda_1 < \lambda_2$ *one has* $0 \leqslant f_{\lambda_1}(x) < f_{\lambda_2}(x)$ *for all* $x \in (0, k/2)$.

(3) *For all* $\lambda \geqslant 1$ *one has* $0 \leqslant f_\lambda(x) \leqslant \lambda$ *for all* $x \geqslant 0$.

(4) *There exists a constant* $c > k/4$ *such that for all* $\lambda \geqslant 1$

$$\int_0^{1/2} f_\lambda(x)\, dx \geqslant c\lambda.$$

Then there exists $\lambda \geqslant 1$ *such that the diferential-delay*

$$u'(t) = -f_\lambda(u(t-1))$$

has a periodic solution $u(t)$ *such that* $u(t+3/2) = -u(t)$ *for all* t *and* $u(t) > 0$ *for* $0 < t < 3/2$.

If $y_i(t) = u(t-i)$ *for* $i = 0, 1, 2$ *and* $f = f_\lambda$, *then* $Y(t) = (y_0(t), y_1(t), y_2(t))$ *is a nonconstant periodic solution of* (1.15).

Remark 1.8. The results we have obtained depend very strongly on the fact that we are dealing with a system of *three* first order ODEs. One could, for example, consider the system of n first order ODEs

$$y_i' = -f(y_{i+1}), \tag{1.16}$$

where the indices are now written mod n and $n > 3$, and one could seek conditions on f which insure that (1.16) has no nonconstant periodic solutions. However, results in [7, 8] show that any conditions must necessarily be very restrictive. Specifically, suppose $g: \mathbb{R} \to \mathbb{R}$ is a continuous function such that $xg(x) > 0$ for all $x \neq 0$, $g'(0) > 0$, and g is bounded below. In addition, assume that $-b = \lim_{x \to -\infty} g(x)$ and $a = \lim_{x \to +\infty} g(x)$ exist (allowing $a = +\infty$ if g is monotonic increasing). Then it is proved in [7, 8] that for every number p with $4 < p < 2 + a/b + b/a$ there exists $\alpha = \alpha_p > 0$ such that the equation

$$x'(t) = -\alpha g(x(t-1))$$

has a nonconstant periodic solution of period p. In particular, if n is an integer such that $4 < n < 2 + a/b + b/a$ and $f(x) = \alpha_n g(x)$, then (1.16) has a nonconstant periodic solution of period n.

Until now we have discussed general periodic solutions of, for example,

$$x'(t) = -g(x(t-1)). \tag{1.17}$$

However, if one restricts attention to a subclass of the periodic solutions of (1.17), namely, the "slowly oscillating periodic solutions," one can sharpen Theorem 1.3. Recall that a periodic solution $x(t)$ of (1.17) is called "slowly oscillating" if $x(0) = 0$, $x(t) > 0$ for $0 < t < z_1$, where $z_1 > 1$, $x(t) < 0$ for $z_1 < t < z_2$, where $z_1 - z_1 > 1$ and $x(t + z_2) = x(t)$ for all t.

PROPOSITION 1.1. *Assume that $g: \mathbb{R} \to \mathbb{R}$ is a continuous, monotonic increasing function such that $xg(x) > 0$ for all $x \neq 0$. Then for any number p such that $2 < p \leqslant 3$ Eq.* (1.17) *has no slowly oscillating periodic solution of period p.*

Proof. Suppose $x(t)$ is a slowly oscillating periodic solution of (1.17) of period p, where $2 < p \leqslant 3$. Let z_1 and z_2 denote the first and second zeros of $x(t)$, and for notational convenience write $z_1 = 1 + b_1$ and $z_2 - z_1 = 1 + b_2$, so $b_1 + b_2 \leqslant 1$. The properties of g imply that $x(t)$ is increasing on $[0, 1]$, decreasing on $[1, z_1 + 1]$ and increasing on $[z_1 + 1, z_2 + 1]$. If one writes $h = x(1)$ and $h_1 = -x(z_1 + 1)$, one can easily see that h and h_1 are positive; and using (1.17) and the fact that g is monotonic one finds

$$h = \int_0^{b_1} g(x(s))\, ds < b_1 g(x(b_1)). \tag{1.18}$$

Integrating $x'(t)$ from z_1 to z_1+1 gives

$$\begin{aligned} h_1 &= \int_{b_1}^{1} g(x(s))\,ds + \int_{1}^{1+b_1} g(x(s))\,ds \\ &> (1-b_1)\, g(x(b_1)) \\ &> \left(\frac{1-b_1}{b_1}\right) h. \end{aligned} \tag{1.19}$$

If we define $h_2 = x(z_2+1)$, then essentially the same arguments show that

$$h_2 > \left(\frac{1-b_2}{b_2}\right) h_1. \tag{1.20}$$

Since we are assuming that $x(t)$ is periodic, $h_2 = h$ and (1.19) and (1.20) yield

$$h > \left(\frac{1-b_2}{b_2}\right)\left(\frac{1-b_1}{b_1}\right) h. \tag{1.21}$$

Inequality (1.21) implies that

$$b_1 b_2 > (1-b_2)(1-b_1). \tag{1.22}$$

Inequality (1.22) gives a contradiction, since $b_1 + b_2 \leqslant 1$. ∎

Remark 1.9. Numerical studies of Jurgens, Peitgen and Saupe [9] suggest that for every p with $2 < p < 3$ there is a number $\alpha = \alpha_p$ such that the equation

$$\begin{aligned} x'(t) &= -\alpha f(x(t-1)), \\ f(x) &\stackrel{\text{def}}{=} x(1+x^8)^{-1} \end{aligned} \tag{1.23}$$

has a slowly oscillating periodic solution $x(t)$ such that $x(t) > 0$ for $0 < t < p/2$ and $x(t+p/2) = -x(t)$ for all t. However, nothing rigorous has been proved.

2. The Characteristic Equation $z + \lambda\alpha_0 e^{-z} + \lambda\beta_0 e^{-4z} = 0$

We are interested in showing how the global Hopf bifurcation theorem for functional differential equations [2, 6] can be used in conjunction with the results of Section 1 to obtain new results about the equation

$$x'(t) = -\lambda\alpha_0 f(x(t-1) - \lambda\beta_0 f(x(t-4)). \tag{2.1}$$

In particular we shall prove existence of solutions of period p with $2 < p < 3$. The main remaining technical obstable is to give a detailed discussion of the pure imaginary roots of

$$\psi(z, \lambda) \stackrel{\text{def}}{=} z + \lambda\alpha_0 e^{-z} + \lambda\beta_0 e^{-4z} = 0, \tag{2.2}$$

where α_0 and β_0 are fixed reals $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$. Once we give such an analysis, our discussion of (2.1) will closely parallel that of Chow and Mallet-Paret in [2] for the equation

$$x'(t) = -\lambda\alpha_0 f(x(t-1)) - \lambda\beta_0 f(x(t-2)). \tag{2.3}$$

Unfortunately, the analysis of (2.2) seems more difficult than the analysis of the characteristic equation

$$z + \lambda\alpha_0 e^{-z} + \lambda\beta_0 e^{-2z} = 0 \tag{2.4}$$

studied in [2]. The next theorem is our starting point.

THEOREM 2.1. *Assume that α_0 and β_0 are real numbers such that $\alpha_0 + \beta_0 \neq 0$ and consider the characteristic equation*

$$\psi(z, \lambda) \stackrel{\text{def}}{=} z + \lambda\alpha_0 e^{-z} + \lambda\beta_0 e^{-4z} = 0,$$

where $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$. If $\psi(iv, \mu) = 0$ for some pure imaginary number iv and some real number μ, then

$$\frac{\partial\psi}{\partial z} \neq 0,$$

where the partial derivative is evaluated at $z = iv$ and $\lambda = \mu$. Furthermore, there exist positive numbers ε and δ such that for each real number λ with $|\lambda - \mu| < \delta$ there exists a unique complex number $z = z(\lambda)$ such that $|z - iv| < \varepsilon$ and $\psi(z, \lambda) = 0$. The map $\lambda \to z(\lambda)$ is continuously differentiable.

Proof. If $\partial\psi/\partial z \neq 0$, the latter part of the theorem is simply the implicit function theorem, so it suffices to prove $\partial\psi/\partial z \neq 0$. Suppose not. Then for some pure imaginary number $z = iv$ and a real number μ one obtains

$$z + \mu\alpha_0 e^{-z} + \mu\beta_0 e^{-4z} = 0,$$

$$1 - \mu\alpha_0 e^{-z} - 4\mu\beta_0 e^{-4z} = 0.$$

Taking the real and imaginary part of each equation gives

$$\mu\alpha_0 \cos \nu + \mu\beta_0 \cos 4\nu = 0, \tag{2.5}$$

$$\mu\alpha_0 \sin \nu + \mu\beta_0 \sin 4\nu = \nu, \tag{2.6}$$

$$\mu\alpha_0 \cos \nu + 4\mu\beta_0 \cos 4\nu = 1, \tag{2.7}$$

$$\mu\alpha_0 \sin \nu + 4\mu\beta_0 \sin 4\nu = 0. \tag{2.8}$$

(Notice that if $\alpha_0 + \beta_0 = 0$, $\nu = 0$ and $\mu = (3\beta_0)^{-1}$ provides a solution of (2.5)–(2.8), so the condition $\alpha_0 + \beta_0 \neq 0$ is necessary). Equation (2.7) implies $\mu \neq 0$, and subtracting (2.5) from (2.7) gives

$$3\beta_0 \cos 4\nu = \mu^{-1}$$

so $\beta_0 \neq 0$ and $\cos 4\nu \neq 0$. Equation (2.5) then implies that $\alpha_0 \neq 0$ and $\cos \nu \neq 0$. If $\sin \nu = 0$, then $\sin 4\nu = 0$ and (2.6) implies $\nu = 0$. But then Eq. (2.5) would give $\alpha_0 + \beta_0 = 0$, which we have assumed is not true. Thus we can assume $\sin \nu \neq 0$ and (from (2.8)), $\sin 4\nu \neq 0$.

Equation (2.5) gives

$$\beta_0 = -\left(\frac{\alpha_0 \cos \nu}{\cos 4\nu}\right).$$

Substituting the latter expression for β_0 in (2.8) and simplifying gives

$$(\alpha_0 \sin \nu)[1 - 16(\cos \nu)^2(\cos 2\nu)(\cos 4\nu)^{-1}] = 0$$

and since we have shown $\alpha_0 \sin \nu \neq 0$,

$$(\cos 4\nu) - 16(\cos \nu)^2(\cos 2\nu) = 0. \tag{2.9}$$

If $\cos \nu =^{\mathrm{def}} x$, recall that $\cos(4\nu) = 8x^4 - 8x^2 + 1$ and $\cos(2\nu) = 2x^2 - 1$ so (2.9) becomes

$$-24x^4 + 8x^2 + 1 = 0, \quad x = \cos \nu. \tag{2.10}$$

Since x^2 is positive, $x^2 = (2 + \sqrt{10})/12$, and (2.10) has only two real solutions, namely,

$$x = \pm\left(\frac{2 + \sqrt{10}}{12}\right)^{1/2} \stackrel{\mathrm{def}}{=} \pm\delta. \tag{2.11}$$

The equation

$$\cos \nu = +\delta$$

has a unique solution $\nu = \nu_0$ such that $0 < \nu < \pi$ (in fact $\pi/4 < \nu_0 < \pi/2$). Similarly, the equation

$$\cos \nu = -\delta.$$

has a unique solution $\nu = \nu_1$ such that $0 < \nu < \pi$, namely, $\nu_1 = \pi - \nu_0$. It is clear that the general solution of the equation

$$\cos \nu = \pm\delta \tag{2.12}$$

is given by $\nu = \pm\nu_0 + n\pi$, where n is any integer.

Equations (2.5) and (2.7) give

$$\mu = (3\beta_0 \cos 4\nu)^{-1} \tag{2.13}$$

and (2.6) and (2.8) then give

$$-3\mu\beta_0 \sin 4\nu = -\tan(4\nu) = \nu. \tag{2.14}$$

Equation (2.14) yields

$$\tan(4\nu_0) = -\nu_0 + n\pi, \tag{2.15}$$

where n is any integer.

The equation $\cos 4\nu = 8x^4 - 8x^2 + 1$ uniquely determines $\cos 4\nu$, and since $\pi < 4\nu_0 < 2\pi$, $\sin 4\nu_0$ is also uniquely determined. An unpleasant calculation gives

$$\tan(4\nu_0) = \frac{1}{8}\left[\frac{(10 + 8\sqrt{10})^{1/2}(4 - \sqrt{10})}{(\sqrt{10} - 1)}\right] = 0.29 \tag{2.16}$$

Since $\pi/4 < \nu_0 < \pi/2$ it follows easily from (2.16) that (2.15) cannot be satisfied for any integer n. ∎

Remark 2.1. One can seek to locate the comples solutions of

$$g(z) = z + \alpha e^{-z} + \beta e^{-4z} = 0 \tag{2.17}$$

instead of just the pure imaginary solutions. Assume that α and β are real parameters with $\beta \neq 0$. For each integer $n \geqslant 1$, define $G_n = \{z: n\pi \leqslant \operatorname{Im}(z) \leqslant (n+1)\pi\}$ and define $G_0 = \{z: |\operatorname{Im}(z)| \leqslant \pi\}$. Define $h(z)$ by

$$h(z) = z + \beta e^{-4z} = 0.$$

It is easy to see that $\operatorname{Im}(tg(z) + (1-t)\,h(z)) \neq 0$ for $z \in \partial(G_n)$, $n \geqslant 0$ and $0 \leqslant t \leqslant 1$ and that the set $\{z \in G_n \mid tg(z) + (1-t)h(z) = 0$ for some t, $0 \leqslant t \leqslant 1\}$ is compact. It follows from Rouche's theorem that $g(z)$ has the same number of zeros in G_n (counting multiplicities) as $h(z)$. It is a classical result (which one can also prove simply, using Rouche's theorem) that $h(z)$ has precisely two zeros in G_n for $n \geqslant 1$ and precisely four zeros in G_0.

If one uses the same kind of reasoning to study

$$z + \alpha e^{-z} + \beta e^{-2z} = 0 \tag{2.18}$$

one finds that (2.18) has a unique zero $z_n = z_n(\alpha, \beta)$ in G_n for each $n \geqslant 1$ and exactly two zeros in G_0 (assuming $\beta \neq 0$). It follows that $z_n(\alpha, \beta)$ is a continuous function of (α, β) for $\beta \neq 0$.

If $z(\lambda)$ is as in Theorem 2.1, we shall actually need to find conditions on α_0 and β_0 which insure

$$\operatorname{Re}\left(\frac{dz}{d\lambda}\right) > 0.$$

The following elementary lemma is a first step in this direction.

LEMMA 2.1. *Define* $\gamma(v) = \cos(4v)/\cos(v)$ *for* $\cos(v) \neq 0$. *Then one has* $\gamma(-v) = \gamma(v)$ *and* $\gamma(v) = -\gamma(\pi - v)$. *There exists a number* v_* *with*

$$(2\pi/3) < v_* < (3\pi/4)$$

and

$$(\cos v_*)^2 = \frac{2 + \sqrt{10}}{12}$$

such that $\gamma'(v) > 0$ *for* $v \in (\pi/2, v_*)$ *and* $\gamma'(v) < 0$ *for* $v \in (v_*, \pi)$. *One also has*

$$\max_{\pi/2 < v < \pi} \gamma(v) = \gamma(v_*) = 8\sqrt{3}(\sqrt{10} - 1)[9(\sqrt{10} + 2)^{1/2}]^{-1} > \sqrt{2}.$$

Proof. The first statement of the lemma is obvious. A calculation gives

$$\begin{aligned} \gamma'(v) &= [-4 \sin(4v) \cos v + \sin v \cos(4v)](\cos v)^{-2} \\ &= (\sin v)(\cos v)^{-2}[-16(\cos v)^2(\cos 2v) + (\cos 4v)]. \end{aligned}$$

If we define $x = \cos(v)$, it follows that for $0 < v < \pi$

$$\operatorname{sgn}(\gamma'(v)) = \operatorname{sgn}[-24x^4 + 8x^2 + 1], \tag{2.19}$$

where $\operatorname{sgn}(\beta) = +1$ for $\beta > 0$ and $\operatorname{sgn}(\beta) = -1$ for $\beta < 0$. It follows from (2.19) that for $0 < \nu < \pi$

$$\operatorname{sgn}(\gamma'(\nu)) = \operatorname{sgn}\left(\left(\frac{2+\sqrt{10}}{12}\right) - x^2\right), \qquad x = \cos(\nu). \tag{2.20}$$

If ν_* is the unique number ν such that

$$\pi/2 < \nu < \pi$$

and

$$(\cos \nu)^2 = \frac{2+\sqrt{10}}{12}$$

it follows that γ is increasing on $(\pi/2, \nu_*]$ and decreasing on $[\nu_*, \pi]$. We also have

$$\left(\cos\left(\frac{2\pi}{3}\right)\right)^2 = \frac{1}{4} < \frac{2+\sqrt{10}}{12} = (\cos \nu^*)^2$$

and

$$\left(\cos\left(\frac{3\pi}{4}\right)\right)^2 = \frac{1}{2} > \frac{2+\sqrt{10}}{12}$$

so

$$2\pi/3 < \nu_* < 3\pi/4. \tag{2.21}$$

The argument above shows that γ has its maximum on $(\pi/2, \pi]$ at $\nu = \nu_*$. As in the proof of Theorem 2.1, we can compute $\cos(4\nu)$ in terms of $x = \cos(\nu)$, and the resulting calculation yields the formula for $\gamma(\nu_*)$ in our lemma. Because γ is decreasing on $[\nu_*, \pi]$,

$$\gamma(\nu_*) > \gamma(3\pi/4) = \sqrt{2}.$$

which completes the proof. ∎

Remark 2.2. It is easy to see that $\gamma(\nu) = 1$ for $\nu \in [0, \pi]$ if and only if $\nu = 0$, $\nu = 2\pi/5$, $\nu = 2\pi/3$, or $\nu = 4\pi/5$. Similarly, one finds $\gamma(\nu) = -1$ for $\nu \in [0, \pi]$ if and only if ν is one of the values $\pi/5, \pi/3, 3\pi/5$, or π. Finally, $\gamma(\nu) = 0$ for $\nu \in [0, \pi]$ if and only if $\nu = \pi/8 + j(\pi/4)$, $j = 0, 1, 2, 3$.

We shall also need the following simple consequence of Lemma 2.1 and Remark 2.2.

LEMMA 2.2. *Let* I_0, I_1, I_2 *and* I_3 *denote the open intervals* $(0, \pi/5)$, $(\pi/3, 2\pi/5)$, $(3\pi/5, 2\pi/3)$ and $(4\pi/5, \pi)$, *respectively. If k is a real number with* $|k| < 1$, *there is a unique number* $\nu_j = \nu_j(k) \in I_j$ *such that* $\gamma(\nu_j) = k$, *and* $\nu_j(k)$ *is a monotonic, real analytic function of k for* $|k| < 1$. *Furthermore, if* $\gamma(\nu) = k$ *for some* $\nu \in [0, \pi]$, *then* $\nu = \nu_j(k)$ *for some j.*

Proof. Lemma 2.1 and Remark 2.2 show that $\gamma'(\nu) \neq 0$ for $\nu \in I_j$ and $\gamma(I_j) = (-1, 1)$. It follows that if $\gamma_j =^{\text{def}} \gamma \mid I_j$, then $\nu_j(k) = \gamma_j^{-1}(k)$. Since γ_j is real analytic on I_j and $\gamma_j'(\nu) \neq 0$ for $\nu \in I_j$, the inverse function theorem implies that $\nu_j(k)$ is real analytic and monotonic. The fact that there are no other solutions of $\gamma(\nu) = k$ follows from Lemma 2.1 and the fact that $\pi/5 < \pi - \nu^* < \pi/3$ and $2\pi/3 < \nu^* < 4\pi/5$. ∎

In the notation of Theorem 2.1, if $\psi(i\nu, \mu) = 0$ for a pure imaginary number $i\nu$ and a real number μ, there is a C^1, complex-valued function $z(\lambda)$ defined for λ near μ such that $z(\mu) = i\nu$ and $\psi(z(\lambda), \lambda) = 0$. Our next theorem determines $\operatorname{sgn}(\operatorname{Re}(z'(\lambda)))$ (evaluated at $\lambda = \mu$) and will be crucial for our application in Section 3.

THEOREM 2.2. *Let notation be as in Theorem* 2.1 *and assume that* $\beta_0 < 0$ *and* $|\alpha_0| < |\beta_0|$. *If* $\psi(i\nu, \mu) = 0$ *for some pure imaginary* $i\nu$ *and some* $\mu > 0$, *then*

$$\operatorname{Re}(z'(\lambda))|_{\lambda=\mu} > 0.$$

There exist precisely four numbers θ_j *with* $0 < \theta_j < 2\pi$ $(0 \leqslant j \leqslant 3)$ *such that if* $\psi(i\nu, \mu) = 0$ *for positive real numbers* μ *and* ν, *then*

$$\nu = \theta_j + 2m\pi \tag{2.22}$$

for some integer $m \geqslant 0$ *and some j*, $0 \leqslant j \leqslant 3$, *and*

$$\mu = \nu(\alpha_0 \sin \theta_j + \beta_0 \sin 4\theta_j)^{-1}. \tag{2.23}$$

The numbers θ_j *satisfy* $\theta_0 \in (9\pi/5, 2\pi)$, $\theta_1 \in (\pi/3, 2\pi/5)$, $\theta_2 \in (4\pi/3, 7\pi/5)$ *and* $\theta_3 \in (4\pi/5, \pi)$.

Proof. If μ and ν are positive numbers such that $\psi(i\nu, \mu) = 0$, then μ and ν satisfy Eqs. (2.5) and (2.6). In the notation of Lemma 2.1, Eq. (2.5) is equivalent to

$$\gamma(\nu) = -(\alpha_0/\beta_0) \tag{2.24}$$

and in the notation of Lemma 2.2, (2.24) has precisely four solutions $\nu \in (0, \pi)$. These four solutions are ν_j $(0 \leqslant j \leqslant 3)$, where $\nu_j \in I_j$ and ν_j and I_j

are as in Lemma 2.2. It follows that the general positive solution ν of (2.24) is

$$\nu = \nu_j + 2m\pi$$

or (2.25)

$$\nu = (2\pi - \nu_j) + 2m\pi,$$

where m is a nonnegative integer and $0 \leqslant j \leqslant 3$.

Equation (2.6) implies that

$$\mu = \nu(\alpha_0 \sin \nu + \beta_0 \sin 4\nu)^{-1}$$

and since we assume $\mu > 0$, ν in Eq. (2.25) must be chosen so that

$$\alpha_0 \sin \nu + \beta_0 \sin 4\nu > 0.$$

If one uses Eq. (2.5) and the expressions for $\cos 4\nu$ and $\cos 2\nu$ in terms of $\cos \nu$ one obtains

$$\begin{aligned} &\alpha_0 \sin \nu + \beta_0 \sin 4\nu \\ &\quad = (\alpha_{0\ \sin\ \nu})[1 + (4\beta_0/\alpha_0) \cos \nu \cos 2\nu] \\ &\quad = (\alpha_0 \sin \nu)(\cos 4\nu)^{-1}[\cos 4\nu - 4(\cos \nu)^2 \cos 2\nu] \\ &\quad = -\beta_0(\tan \nu)[1 - 4(\cos \nu)^2]. \end{aligned} \tag{2.26}$$

It follows from (2.26) and the location of ν_j given in Lemma 2.2 that

$$\begin{aligned} &\alpha_0 \sin \nu_j + \beta_0 \sin 4\nu_j < 0 \qquad \text{for } j = 0 \text{ or } j = 2, \\ &\alpha_0 \sin \nu_j + \beta_0 \sin 4\nu_j > 0 \qquad \text{for } j = 1 \text{ or } j = 3. \end{aligned}$$

It follows that the general positive solution ν of (2.5) such that $\alpha_0 \sin \nu + \beta_0 \sin 4\nu$ is also positive is given by

$$\nu = \theta_j + 2m\pi, \qquad m \geqslant 0, \tag{2.27}$$

where $\theta_j = 2\pi - \nu_j$ for $j = 0$ or 2 and $\theta_i = \nu_i$ for $j = 1$ or 3.

It remains to prove that $\operatorname{Re}(z'(\lambda)) > 0$. If $\psi(z, \lambda)$ is as in Theorem 2.1 and $\psi(i\nu, \mu) = 0$, a calculation using the implicit function theorem gives

$$\operatorname{Re}(z'(\lambda))|_{\lambda=\mu} = (|\partial\psi/\partial z|^{-2})[\nu(\alpha_0 \sin \nu + 4\beta_0 \sin 4\nu)]$$

so (for $\nu > 0$)

$$\operatorname{sgn}(\operatorname{Re}(z'(\lambda)))|_{\lambda=\mu} = \operatorname{sgn}(\alpha_0 \sin \nu + 4\beta_0 \sin 4\nu). \tag{2.28}$$

If one uses the facts that ν satisfies (2.5) and that $\cos(4\nu)$ and $\cos(2\nu)$ can be written in terms of $\cos\nu = x$, one finds

$$\alpha_0 \sin\nu + 4\beta_0 \sin 4\nu = (-\beta_0)\tan(\nu)(-24x^4 + 8x^2 + 1) \qquad (2.29)$$

so

$$\operatorname{sgn}(\operatorname{Re}(z'(\lambda))) = \operatorname{sgn}((\tan\nu)(-24x^4 + 8x^2 + 1)). \qquad (2.30)$$

Recalling Eq. (2.10) in the proof of Theorem 2.1, we know that if $p(x) =^{\text{def}} -24x^4 + 8x^2 + 1$

$$\begin{aligned} p(x) > 0 &\qquad \text{for } x^2 < \frac{2+\sqrt{10}}{12}, \\ p(x) < 0 &\qquad \text{for } x^2 > \frac{2+\sqrt{10}}{12}. \end{aligned} \qquad (2.31)$$

If $x = \cos(\nu_j) = \cos(\theta_j)$ $(0 \leqslant j \leqslant 3)$ and one uses the fact that $\nu_j \in I_j$ (I_j as in Lemma 2.2), one obtains from (2.31) that

$$\begin{aligned} p(\cos(\theta_j)) < 0 &\qquad \text{for } j = 0 \text{ or } j = 3, \\ p(\cos(\theta_j)) > 0 &\qquad \text{for } j = 1 \text{ or } j = 2 \end{aligned} \qquad (2.32)$$

and

$$\begin{aligned} \tan(\theta_j) < 0 &\qquad \text{for } j = 0 \text{ or } 3, \\ \tan(\theta_j) > 0 &\qquad \text{for } j = 1 \text{ or } 2. \end{aligned} \qquad (2.33)$$

It follows from Eqs. (2.31)–(2.33) that for $\nu = \theta_j + 2m\pi$, $0 \leqslant j \leqslant 3$ and $m \geqslant 0$,

$$\operatorname{sgn}(\tan(\nu)(-24x^4 + 8x^2 + 1)) > 0, \qquad x = \cos\nu.$$

We have already seen that all solutions $\nu > 0$ of $\psi(i\nu, \mu) = 0$ for which $\mu > 0$ are of the form $\theta_j + 2m\pi$, so (2.31) implies that

$$\operatorname{Re}(z'(\lambda)) > 0$$

and the proof is complete. ∎

Remark 2.3. It is also possible to carry out the same analysis (assuming $\beta_0 < 0$) in the case $1 < -(\alpha_0/\beta_0) < \max_{\pi/2 < \nu < \pi} \gamma(\nu)$, where $\gamma(\nu)$ is as in Lemma 2.1. In this case the equation

$$\gamma(\nu) = -\frac{\alpha_0}{\beta_0}$$

has precisely three solutions $\nu \in (0, \pi)$, say, ν_0, ν_1 and ν_2 (in order of size). The general solution (ν, μ) of Eqs. (2.5) and (2.6) such that $\nu > 0$ and $\mu > 0$ is given by

$$\nu = \nu_j + 2m\pi, \qquad m \geqslant 0,$$

$$\mu = \nu(\alpha_0 \sin \nu + \beta_0 \sin 4\nu)^{-1}.$$

If $\psi(i\nu, \mu) = 0$ and $z(\lambda)$ is as in Theorem 2.1, then

$$\operatorname{Re} z'(\lambda)|_{\lambda=\mu} > 0 \qquad \text{for } \nu = \nu_j + 2m\pi, j = 0 \text{ or } 2,$$

$$\operatorname{Re} z'(\lambda)|_{\lambda=\mu} < 0 \qquad \text{for } \nu = \nu_1 + 2m\pi.$$

Since we shall not need this result, we shall give no further details.

We shall also need a simple proposition concerning positive roots of the characteristic equations.

PROPOSITION 2.1. *Assume that* $\beta < 0$ *and* α *is real and define* $\psi(z) = z + \alpha e^{-z} + \beta e^{-4z}$. *If* $\alpha < |\beta|$, *the equation* $\psi(z) = 0$ *has precisely one positive, real solution* z. *If* $\alpha > |\beta|$, *the equation* $\psi(z) = 0$ *has no positive real solution.*

Proof. Suppose that $\psi(z) = 0$ for some $z > 0$. Then one has

$$\psi'(z) = 1 - \alpha e^{-z} - 4\beta e^{-4z}$$
$$= 1 + z - 3\beta e^{-4z}.$$

Since $z > 0$ and $\beta < 0$, it follows that $\psi'(z) > 1$ at any positive value of z such that $\psi(z) = 0$. This fact implies immediately that $\psi(z) = 0$ has at most one positive solution and can have no positive solution if $\psi(0) > 0$. If $\alpha > |\beta|$, then $\psi(0) > 0$, and the above remarks show that there are no positive roots. If $|\beta| > \alpha$, then $\psi(0) < 0$; and since $\lim_{z \to +\infty} \psi(z) = +\infty$, the equation $\psi(z) = 0$ has at least one root (and therefore exactly one positive root). ∎

For our work in Section 3 we would like to know that if $i\nu$ is a solution of (2.2), then $ni\nu$ is not a solution for any integer $n > 1$, but this seems to be subtle question. Instead, we shall have to make do with the following proposition.

PROPOSITION 2.2. *Assume that* $\beta_0 < 0$ *and consider the equation*

$$z + \lambda\alpha e^{-z} + \lambda\beta_0 e^{-4z} = 0, \tag{2.34}$$

where α *is real,* $|\alpha/\beta_0| < 1$ *and* z *is complex. Then there exists a countable* *set of* α, $S(\beta_0)$, *such that if* $\alpha \notin S(\beta_0)$ *and* $|\alpha| < |\beta_0|$ *and* $\lambda > 0$ *Eq.* (2.34) *has at most one pair of pure imaginary solutions.*

Proof. For $|\alpha| < |\beta_0|$ let $\nu_j(\alpha)$, $0 \leqslant j \leqslant 3$, be the four solutions ν (written in order of increasing size) such that $0 \leqslant \nu \leqslant \pi$ and $\gamma(\nu) =^{\text{def}} (\cos(4\nu))(\cos\nu)^{-1} = -\alpha\beta_0^{-1}$ (see Lemma 2.1). Recall (Lemma 2.2) that the functions $\nu_j(\alpha)$ are real analytic. As before, define $\theta_j(\alpha) = 2\pi - \nu_j(\alpha)$ are real analytic. As before, define $\theta_j(\alpha) = 2\pi - \nu_j(\alpha)$ for $j = 0$ or 1 and $\theta_j(\alpha) = \nu_j(\alpha)$ otherwise. For a given α, the positive values λ for which (2.34) has pure imaginary solutions are precisely the numbers

$$\lambda_{j,m}(\alpha) = (\theta_j(\alpha) + 2m\pi)(\alpha \sin\theta_j(\alpha) + \beta_0 \sin 4\theta_j(\alpha))^{-1}$$

for $0 \leqslant j \leqslant 3$ and $m \geqslant 0$. If we can prove that for $(j_1, m_1) \neq (j_2, m_2)$, $\lambda_{j_1,m_1}(\alpha) \neq \lambda_{j_2,m_2}(\alpha)$, then for any given $\lambda > 0$, (2.34) has at most one pair of pure imaginary solutions. It is clear that

$$\lambda_{j,m}(\alpha) < \lambda_{j,m+1}(\alpha).$$

Suppose we can prove that for any $\eta > 0$ and any two pairs of integers (j_1, m_1) and (j_2, m_2) with $j_1 \neq j_2$

$$\{\alpha: |\alpha| \leqslant |\beta_0| - \eta \text{ and } \lambda_{j_1,m_1}(\alpha) = \lambda_{j_2,m_2}(\alpha)\}$$

is finite. Then we define $S(\beta_0)$ by

$$\begin{aligned} S(\beta_0) \overset{\text{def}}{=} \{\alpha: |\alpha| < |\beta_0| \text{ and } \\ &\lambda_{j_1,m_1}(\alpha) = \lambda_{j_2,m_2}(\alpha) \text{ for some pairs} \\ &(j_1, m_1) \text{ and } (j_2, m_2) \text{ with } j_1 \neq j_2\}. \end{aligned}$$

This set will be a countable union of finite sets and hence countable.

Thus it suffices to prove that $\{\alpha: |\alpha| \leqslant |\beta_0| - \eta,\ \lambda_{j_1,m_1}(\alpha) = \lambda_{j_2,m_2}(\alpha)\}$ is finite if $j_1 \neq j_2$. If the set is empty, we are done. It α_* lies in the set, it suffices to prove that α_* is an isolated point in the set. If α_* is not isolated, then, because the functions $\lambda_{j_1,m_1}(\alpha)$ and $\lambda_{j_2,m_2}(\alpha)$ are real analytic, they must be identically equal. In particular we have

$$\lambda_{j_1,m_1}(0) = \lambda_{j_2,m_2}(0).$$

However, the numbers $\theta_j(0)$, $0 \leqslant j \leqslant 3$, are easily computed to be of the form

$$\frac{3\pi + 4p\pi}{8}, \qquad 0 \leqslant p \leqslant 3.$$

Therefore there must be integers k and p, $k \neq p$, $0 \leqslant k, p \leqslant 3$, such that

$$\theta_{j_1}(0) = \frac{3\pi + 4p\pi}{8},$$

$$\theta_{j_2}(0) = \frac{3\pi + 4k\pi}{8}.$$

If one substitutes in the formulas for $\lambda_{j_1, m_1}(0)$ and $\lambda_{j_2, m_2}(0)$ and simplifies one obtains

$$p - k = 4(m_2 - m_1)$$

which is impossible. ∎

3. Periodic Solutions of $x'(t) = \lambda f(x(t-4) + k\lambda f(x(t-1))$

By scaling one can assume that $\beta_0 = -1$ in (2.1); we write $\alpha_0 = k$ and consider the equation

$$x'(t) = \lambda f(x(t-4) - k\lambda f(x(t-1)) \tag{3.1}$$

for $\lambda > 0$. We shall eventually prove under further assumptions on λ, k and f that (3.1) has a periodic solution with period between 2 and 3. We begin with a simple proposition.

Lemma 3.1. *Assume that $k \neq 1$ in* (3.1) *and that $f: \mathbb{R} \to \mathbb{R}$ is a continuous function such that $xf(x) > 0$ for all $x \neq 0$ and $f(x) > -B > -\infty$ for all x. If $x(t)$ is a periodic solution of period T of* (3.1) *one has*

$$\sup_t |x(t)| \leqslant |\lambda| BT(2 + 2|k|). \tag{3.2}$$

If f is Lipschitzian on a neighborhood of 0 *and $T_0 > 0$, there exists a positive number ε such that* (3.1) *has no nonconstant periodic solutions of period $T \leqslant T_0$ for $0 < \lambda \leqslant \varepsilon$.*

Proof. Assume that $x(t)$ is periodic solution of (3.1) of period T. Integrating (3.1) over an interval of lenth T gives

$$0 = \lambda(1 - k) \int_0^T f(x(s))\, ds$$

and since $\lambda > 0$ and $k \neq 1$

$$0 = \int_0^T f(x(s))\, ds. \tag{3.3}$$

Since we assume that $xf(x) > 0$ for $x \neq 0$, (3.3) shows that if $x(t)$ is not identically zero, it must achieve both positive and negative values (if $x(t)$ were always nonnegative and positive at a point, the integral in (3.3) would be positive). In particular, there must be a time t_0, $0 \leqslant t_0 \leqslant T$, such that $x(t_0) = 0$. Let $J_+ = \{t \in [0, T]\colon\ x(t) \geqslant 0\}$ and $J_- = \{t \in [0, T]\colon\ x(t) < 0\}$. Equation (3.3) gives

$$\int_{J_+} f(x(t))\, dt = -\int_{J_-} f(x(t))\, dt < TB$$

and we conclude that

$$\int_0^T |f(x(s))|\, ds < 2TB. \tag{3.4}$$

If t_0 is as above, select $t_1 \in [0, T]$ such that

$$|x(t_1)| = \sup_{0 \leqslant t \leqslant T} |x(t)|.$$

Then we have

$$\begin{aligned} |x(t_1)| &= |x(t_1) - x(t_0)| \\ &= |\lambda| \left| \int_{t_0 - 4}^{t_1 - 4} f(x(s))\, ds + k \int_{t_0 - 1}^{t_1 - 1} f(x(s))\, ds \right| \\ &\leqslant |\lambda|\, (2TB + |k|\, (2TB)) \\ &= |\lambda|\, BT(2 + 2\,|k|). \end{aligned} \tag{3.5}$$

Suppose that $|f(x)| \leqslant C\,|x|$ for $|x| \leqslant \delta_0$. We want to show that (3.1) has no nonconstant periodic solutions of period less than T_0 for λ small. If $x(t)$ is a periodic solution of (3.1) of period $T \leqslant T_0$, (3.2) shows that

$$\sup_t |x(t)| \leqslant \delta_0$$

if

$$0 < \lambda < B^{-1} T_0^{-1} (2 + 2\,|k|)^{-1} \delta_0. \tag{3.6}$$

Thus if λ satisfies (3.6) and $\sup_t |x(t)| = \varepsilon > 0$, we obtain from (3.1) that

$$|x'(t)| \leqslant C\lambda(\varepsilon + |k|\, \varepsilon). \tag{3.7}$$

Assume that λ is so small that

$$C\lambda(1 + |k|)\, T_0 < 1. \tag{3.8}$$

If t_0 and t_1 are defined as before,

$$\varepsilon = |x(t_1)| \leqslant \int_{t_0}^{t_1} |x'(s)|\, ds \leqslant C\lambda(1 + |k|)\, T_0 \varepsilon \tag{3.9}$$

and (3.9) contradicts (3.8) for $\varepsilon \neq 0$. It follows that for λ satisfying (3.6) and (3.8), Eq. (3.1) has no periodic solution which is not identically zero. ∎

Remark 3.1. Lemma 3.1 is false if $k = 1$, since then (3.1) is satisfied by any constant solution.

We also need to recall some notation from Section 2.

DEFINITION 3.1. If $-1 < k < 1$, then $v_j = v_j(k)$ (for $0 \leqslant j \leqslant 3$) denote the four solutions $v \in [0, \pi]$, written in order of increasing size, of the equation

$$\gamma(v) = \cos(4v)[\cos v]^{-1} = k.$$

Define $\theta_j = \theta_j(k)$ by $\theta_j(k) = v_j(k)$ for $j = 1$ or $j = 3$ and $\theta_j(k) = 2\pi - v_j(k)$ for $j = 1$ or 2. For each integer j, $0 \leqslant j \leqslant 3$, and each nonnegative integer m, $m \geqslant 0$, define

$$\mu_{j,m} = \mu_{j,m}(k) = (\theta_j + 2\pi m)(k \sin \theta_j + \sin 4\theta_j)^{-1}, \tag{3.10}$$

where $\theta_j = \theta_j(k)$ in (3.10).

The results of Section 2 show that

$$k \sin \theta_j + \sin 4\theta_j > 0$$

and that the characteristic equation

$$z - \lambda e^{-4z} + k\lambda e^{-z} = 0 \tag{3.11}$$

has a pure imaginary root z for some $\lambda > 0$ if and only if

$$\lambda = \mu_{j,m}(k)$$

for some j and m.

The following lemma follows by standard local analysis near the zero solution, and we omit the proof.

LEMMA 3.2. *Assume that $f: \mathbb{R} \to \mathbb{R}$ is continuously differentiable on a neighborhood of* 0, *that $f'(0) = 1$, and that k in* (3.1) *satisfies $|k| < 1$. There exists a continuous function $\delta(T, \lambda)$ such that*

$$\delta(T,\lambda) \geqslant 0 \qquad \textit{for } (T,\lambda) \in (2,3) \times (0,\infty),$$
$$\delta(2\pi n/(\theta_j + 2m\pi), \mu_{j,m}) = 0 \qquad \textit{for integers } n \geqslant 1, 0 \leqslant j \leqslant 3 \textit{ and } m \geqslant 0,$$
$$\delta(T,\lambda) > 0 \qquad \textit{otherwise,}$$

and such that any nonconstant periodic solution of (3.1) *of period* $T \in (2,3)$ *satisfies*

$$\sup_{-4 \leqslant \theta \leqslant 0} |x(t+\theta)| > \delta(T,\lambda) \tag{3.12}$$

for all t.

The proof of the main theorem of this section now follows (with the aid of our previous results) by essentially the same argument used in Section 4 of [2] so we shall assume familiarity with [2] and give a sketchy treatment. Define $X = C([-4,0])$ = continuous, real-valued functions on $[-4,0]$ in the sup norm. Following the standard terminology for functional differential equations, if x is a continuous function defined on $[t-4,t]$, define $x_t \varepsilon X$ by $x_t(\theta) = x(t+\theta)$. Define $f^\lambda\colon X \to R$ by

$$f^\lambda(\phi) = \lambda f(\phi(-4)) - k\lambda f(\phi(-1)),$$

where f is as in (3.1). Then (3.1) can be written as

$$x'(t) = f^\lambda(x_t).$$

Define $M(T,\lambda)$ to be the right hand side of (3.2), where B is a lower bound for $f(x)$ in (3.1) and k is as in (3.1). Define open sets-$\Omega \subset (0,\infty) \times X \times (0,\infty)$ and $\Omega(\lambda)$ by

$$\Omega = \{(T,\phi,\lambda)\colon 2 < T < 3, \delta(T,\lambda) < \|\phi\| < M(T,\lambda)\} \tag{3.13}$$

and

$$\Omega(\lambda) = \{(T,\phi)\colon (T,\phi,\lambda) \in \Omega\}.$$

For each $\lambda > 0$ define a set $S(\lambda)$ by

$$S(\lambda) = \{\nu > 0\colon i\nu \text{ is a solution of } (3.11)\} \tag{3.14}$$

so $S(\lambda)$ has between 0 and 4 elements.

THEOREM 3.1. *Assume that* $f\colon \mathbb{R} \to \mathbb{R}$ *is a continuously differentiable function such that* $f(0) = 0$, $f'(0) = 1$, f *is bounded below by a constant*

$-B > -\infty$ and $f'(x) > 0$ for all x. Assume that the constant k in (3.1) satisfies $|k| < 1$. Define λ_0 by

$$\lambda_0 = \min\{\mu_{0,0}(k), \mu_{1,1}(k), \mu_{2,0}(k), \mu_{3,0}(k)\}.$$

If $\lambda > \lambda_0$ and $\lambda \neq \mu_{j,m}(k)$ for any j, $0 \leqslant j \leqslant 3$, or $m \geqslant 0$, then $d(f, \Omega(\lambda)$, the Fuller index of f^λ on $\Omega(\lambda)$, is defined and nonzero. In particular, for $\lambda > \lambda_0$ and $\lambda \neq \mu_{j,m}(k)$, (3.1) has a nonconstant periodic solution of period p with $2 < p < 3$. If $\mu \geqslant \lambda_0$ and $\mu = \mu_{j,m}(k)$ for some j and m, then the change in the Fuller index of f^λ on $\Omega(\lambda)$ as λ passes through μ is

$$\lim_{\varepsilon \to 0+} d(f^{\lambda+\varepsilon}, \Omega(\lambda+\varepsilon)) - d(f^{\lambda-\varepsilon}, \Omega(\lambda-\varepsilon)) = -\sum_{\nu \in S(\mu)} \sum_{(\nu/\pi) < n < (3\nu/2\pi)} \frac{1}{n}. \tag{3.15}$$

(n in Eq. (3.15) denotes an integer and $S(\mu)$ is as in (3.14)).

Proof. If $x(t)$ is a solution of (3.1) of period 3, then $x(t)$ satisfies

$$x'(t) = \lambda(1-k) f(x(t-1)).$$

Corollary 1.3 shows that this is impossible unless $x(t)$ is identically zero. Lemma 4.1 in [2] shows that 0 is the only solution of (3.1) of period 2. If $\lambda \neq \mu_{j,m}$ for and j, $0 \leqslant j \leqslant 3$, or $m \geqslant 0$, then the definition of $\delta(T, \lambda)$ and $M(T, \lambda)$ implies that

$$\delta(T, \lambda) < \|x_t\| < M(T, \lambda)$$

for any nonzero solution of (3.1) of period T and any t. It follows that the Fuller index, $d(f^\lambda, \Omega(\lambda))$, is defined for $\lambda > 0$, $\lambda \neq \mu_{j,m}$. Lemma 2.1 implies that there exists $\varepsilon > 0$ such that (3.1) has no nontrivial periodic solutions of period less than or equal to 3 for $0 < \lambda < \varepsilon$. It follows that

$$d(f^\lambda, \Omega(\lambda)) = 0 \qquad \text{for} \quad 0 < \lambda < \varepsilon. \tag{3.16}$$

One can easily check (using Lemma 2.2) that λ_0 is the smallest positive number λ such that Eq. (3.11) has a pure imaginary solution $i\nu$ with $\nu > 0$ and

$$2 < n(2\pi/\nu) < 3$$

for some integer n. Suppose that we can prove that (3.15) holds for every $\mu = \mu_{j,m}$. Then (3.15) and (3.16) together imply that $d(f^\lambda, \Omega(\lambda))$ is negative for $\lambda > \lambda_0$ and $\lambda \neq \mu_{j,m}$ and consequently (3.1) will have, for such λ, a nonconstant periodic solution with period T satisfying $2 < T < 3$.

Thus it suffices to prove (3.15). First suppose that the characteristic equation (3.11) has at most one pure imaginary root $i\nu$, $\nu > 0$, for each $\lambda > 0$. Then (3.15) follows directly from a formula in [2], for we have proved (Theorem 2.2) that pure imaginary solutions of (3.11) pass through the imaginary axis with positive velocity as λ increases and (Proposition 2.1) (3.11) has exactly one positive root.

In the general case select $\varepsilon > 0$ small enough that the change in Fuller index at μ is

$$d(f^{\mu+\varepsilon}, \Omega(\mu+\varepsilon)) - d(f^{\mu-\varepsilon}, \Omega(\mu-\varepsilon)).$$

Select (by Proposition 2.2) a number k' so close to k that if $g^{\lambda}: X \to R$ is defined by

$$g^{\lambda}(\phi) = \lambda f(\phi(-4)) - k'\lambda f(\phi(-1))$$

then

$$d(f^{\mu\pm\varepsilon}, \Omega(\mu \pm \varepsilon)) = d(g^{\mu\pm\varepsilon}, \Omega(\mu + \varepsilon))$$

and also

$$z + \lambda e^{-4z} - k'\lambda e^{-z} = 0 \tag{3.17}$$

has at most one pure imaginary solution $i\nu$ with $\nu > 0$ for every $\lambda > 0$. We can also assume (3.17) has no pure imaginary solutions for $\lambda = \mu \pm \varepsilon$. Applying (3.15) to g^{λ} gives

$$\begin{aligned} & d(g^{\mu+\varepsilon}, \Omega(\mu+\varepsilon)) - d(g^{\mu-\varepsilon}, \Omega(\mu-\varepsilon)) \\ & = \sum_{\nu \in S'} \sum_{(\nu/\pi) < n < (3\nu/2\pi)} \frac{1}{2}, \end{aligned} \tag{3.18}$$

where S' is the set of positive reals ν such that ν is a solution of (3.17) for some λ with $|\lambda - \mu| \leqslant \varepsilon$. Taking the limit as $k' \to k$ gives (3.15) for f^{λ}. ∎

Remark 3.2. Note that the period in Theorem 3.1 need *not* be the minimal period.

Remark 3.3. If, for a specific f and k, one can prove that the Hopf bifurcation at the points $\mu_{j,m}(k)$ is always generic, i.e., always takes place either to the right or to the left of $\mu_{j,m}(k)$, then one can easily sharpen Theorem 3.1 by proving that Eq. (3.1) has a nonconstant periodic solution of period between 2 and 3 for every $\lambda > \lambda_0$. For reasons of length we have not pursued this point here.

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