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INEQUIVALENT MEASURES OF NONCOMPACTNESS AND THE RADIUS OF THE ESSENTIAL SPECTRUM

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Abstract. The Kuratowski measure of noncompactness α on an infinite dimensional Banach space $(X, \|\cdot\|)$ assigns to each bounded set S in X a nonnegative real number $\alpha(S)$ by the formula

$$\alpha(S) = \inf\{\delta > 0 \mid S = \textstyle\bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } \operatorname{diam}(S_i) \le \delta, \text{ for } 1 \le i \le n < \infty\}.$$

In general a map β which assigns to each bounded set S in X a nonnegative real number and which shares most of the properties of α is called a homogeneous measure of noncompactness or homogeneous MNC. Two homogeneous MNC's β and γ on X are called equivalent if there exist positive constants b and c with $b\beta(S) \le \gamma(S) \le c\beta(S)$ for all bounded sets $S \subset X$. There are many results which prove the equivalence of various homogeneous MNC's. Working with $X = \ell^p(\mathbb{N})$ where $1 \le p \le \infty$, we give the first examples of homogeneous MNC's which are not equivalent.

Further, if X is any complex, infinite dimensional Banach space and $L : X \to X$ is a bounded linear map, one can define $\rho(L) = \sup\{|\lambda| \mid \lambda \in \operatorname{ess}(L)\}$, where $\operatorname{ess}(L)$ denotes the essential spectrum of L. One can also define

$$\beta(L) = \inf\{\lambda > 0 \mid \beta(LS) \le \lambda\beta(S) \text{ for every } S \in \mathcal{B}(X)\}.$$

The formula $\rho(L) = \lim_{m\to\infty} \beta(L^m)^{1/m}$ is known to be true if β is equivalent to α, the Kuratowski MNC; however, as we show, it is in general false for MNC's which are not equivalent to α. On the other hand, if B denotes the unit ball in X and β is any homogeneous MNC, we prove that

$$\rho(L) = \limsup_{m\to\infty} \beta(L^m B)^{1/m} = \inf\{\lambda > 0 \mid \lim_{m\to\infty} \lambda^{-m}\beta(L^m B) = 0\}.$$

Our motivation for this study comes from questions concerning eigenvectors of linear and nonlinear cone-preserving maps.

If (X, d) is a complete metric space and S is a bounded subset of X, then K. Kuratowski [10] has defined $\alpha(S)$, the **Kuratowski measure of noncompactness** of S, by

$$\alpha(S) := \inf\{\delta > 0 \mid S = \bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } \operatorname{diam}(S_i) \le \delta, \text{ for } 1 \le i \le n < \infty\}.$$

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Here $\mathrm{diam}(T)$ denotes the diameter of a set $T \subset X$, namely

$$\mathrm{diam}(T) := \sup\{d(x,y) \mid x, y \in T\}.$$

We shall denote by $\mathcal{B}(X)$ the collection of all bounded subsets of X. Kuratowski has shown, and it is straightforward to verify, that α satisfies the following properties:

(K1) $\alpha(S) = 0$ if and only if $\overline{S}$ is compact, for every $S \in \mathcal{B}(X)$;
(K2) $\alpha(S) \leq \alpha(T)$ for every $S, T \in \mathcal{B}(X)$ with $S \subset T$;
(K3) $\alpha(S \cup \{x_0\}) = \alpha(S)$ for every $S \in \mathcal{B}(X)$ and $x_0 \in X$; and
(K4) $\alpha(\overline{S}) = \alpha(S)$ for every $S \in \mathcal{B}(X)$.

If S and T are subsets of a real or complex Banach space $(X, \|\cdot\|)$ and λ is a scalar, we shall let $\mathrm{co}(S)$ denote the convex hull of S, namely the smallest convex set containing S, and we shall write $S + T := \{s + t \mid s \in S \text{ and } t \in T\}$ and $\lambda S := \{\lambda s \mid s \in S\}$. G. Darbo [6] has observed that, assuming the metric on X is the usual one obtained from the norm $\|\cdot\|$, the following properties hold:

(K5) $\alpha(\mathrm{co}(S)) = \alpha(S)$ for every $S \in \mathcal{B}(X)$;
(K6) $\alpha(S + T) \leq \alpha(S) + \alpha(T)$ for every $S, T \in \mathcal{B}(X)$; and
(K7) $\alpha(\lambda S) = |\lambda|\alpha(S)$ for every $S \in \mathcal{B}(X)$ and every scalar λ.

Properties (K5), (K6), and (K7) make the Kuratowski MNC a very useful tool in fixed point theory and functional analysis. Let us also mention the following so-called **set-additivity property**, which holds in any metric space:

(K8) $\alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\}$ for every $S, T \in \mathcal{B}(X)$.

If $(X, \|\cdot\|)$ is a real or complex Banach space, we shall say that a map $\beta : \mathcal{B}(X) \to [0, \infty)$ is a **homogeneous measure of noncompactness on X** or **homogeneous MNC** if β satisfies properties (K1)-(K7), with β replacing α in these conditions. We shall say that β is a **homogeneous, set-additive MNC** if β satisfies properties (K1)-(K8), with β replacing α in these conditions. Our terminology differs from some of the literature [1], [2], [3], [18], where a map satisfying properties (K1)-(K8) is simply called an MNC. Of course these properties are not independent. For example, properties (K2), (K6), and (K7) imply property (K4).

If β and γ are homogeneous MNC's on X, we say that $\boldsymbol{\beta}$ **dominates** $\boldsymbol{\gamma}$ if there exists a number $c > 0$ such that $\gamma(S) \leq c\beta(S)$ for every $S \in \mathcal{B}(X)$. If β and γ are homogeneous MNC's on X such that both β dominates γ and γ dominates β, we say that β and γ are **equivalent**. There are many examples of homogeneous MNC's (see [1], [2], [3], [4], [14], [15], [16], [17], [18]), but up to now all known examples of homogeneous MNC's on a given Banach space X are equivalent. This fact begs the following question.

Question A. Does there exist a Banach space $(X, \|\cdot\|)$ for which there is a homogeneous (possibly set-additive) MNC β on X which is not equivalent to the Kuratowski MNC α on X?

As we shall see below in Theorem 7, where a class of inequivalent MNC's is constructed, Question A is answered in the affirmative.

If $L : X \to X$ is a bounded linear map and β is a homogeneous MNC on X, one can define

$$\begin{aligned} &\beta(L) := \inf\{\lambda \geq 0 \mid \beta(LS) \leq \lambda\beta(S) \text{ for every bounded } S \subset X\}, \\ &\beta^{\#}(L) := \limsup_{m\to\infty} \beta(L^m)^{1/m}, \end{aligned} \tag{1}$$

where we set $\beta(L) = \infty$ if the set in the first line of (1) is empty. If it is in fact the case that $\beta(L) < \infty$, then one easily shows that

$$\beta^{\#}(L) = \lim_{m\to\infty} \beta(L^m)^{1/m} = \inf_{m\geq 1} \beta(L^m)^{1/m}, \tag{2}$$

which follows directly from the fact that $\beta(L^{m+n}) \leq \beta(L^m)\beta(L^n) < \infty$ for every $m \geq 1$ and $n \geq 1$. Lemma 4 below implies that if β is equivalent to the Kuratowski MNC α on X, then there exists a constant $c > 0$, independent of L, with $\beta(L) \leq c\alpha(L) \leq c\|L\| < \infty$. Additionally, if β is equivalent to α, the results of [14] imply that $\beta^{\#}(L) = \rho(L)$, where $\rho(L)$ denotes the radius of the essential spectrum of L. This suggests the following question.

Question B. Is it the case that $\beta^{\#}(L) = \rho(L)$ for any homogeneous MNC β on X, where $\rho(L)$ denotes the radius of the essential spectrum of L? If this is not the case, is there an analogous formula for $\rho(L)$ which holds for any homogeneous MNC β?

For a general homogeneous MNC β which is not equivalent to α, we shall establish in Theorem 8 below that it may happen that $\beta^{\#}(L) \neq \rho(L)$, and in fact it may happen that $\beta(L^m) = \infty$ for all $m \geq 1$. Elsewhere [13], we shall construct an example for which

$$\liminf_{m\to\infty} \beta(L^m)^{1/m} < \limsup_{m\to\infty} \beta(L^m)^{1/m} = \infty.$$

In such cases $\beta^{\#}(L) = \infty$ while $\rho(L) < \infty$. As will be shown in Theorem 10 below, in place of the quantity $\beta^{\#}(L)$ the appropriate quantity to consider is

$$\beta^*(L) := \limsup_{m\to\infty} \beta(L^m B_1(0))^{1/m} = \inf\{\lambda > 0 \mid \lim_{m\to\infty} \lambda^{-m}\beta(L^m B_1(0)) = 0\}, \tag{3}$$

as it is the case that $\beta^*(L) = \rho(L)$ for every homogeneous MNC β and every bounded linear operator L on X. We denote

$$B_r(x) := \{y \in X \mid \|y - x\| < r\} \tag{4}$$

both here and below.

Remark. In order for $\rho(L)$ to be defined above, one needs to have a linear operator on a complex Banach space. Suppose instead that X is a real Banach space, β is a homogeneous MNC on X, and $L : X \to X$ is a bounded linear map. The complexification $\widehat{X}$ of X equals $\{(u, v)|u, v \in X\}$. If one identifies (u, v) with $u+iv$ where $i^2 = -1$, and defines

$$\|u + iv\| := \sup_{0\leq\theta\leq 2\pi} \|(\cos\theta)u + (\sin\theta)v\|,$$

then $\widehat{X}$ becomes a complex Banach space. The linear map L then extends to a complex linear map $\widehat{L}$ on $\widehat{X}$ by $\widehat{L}(u + iv) = Lu + iLv$. It is also the case that β extends to a homogeneous MNC $\widehat{\beta}$ on $\widehat{X}$ as follows. For $x = u + iv \in \widehat{X}$ define $\mathrm{Re}(x) := u$, and for $\widehat{S} \in \mathcal{B}(\widehat{X})$ define $\mathrm{Re}(\widehat{S}) := \{\mathrm{Re}(x) \mid x \in \widehat{S}\}$ and set

$$\widehat{\beta}(\widehat{S}) := \sup_{0\leq\theta\leq 2\pi} \beta(\mathrm{Re}(e^{i\theta}\widehat{S})). \tag{5}$$

One can prove that $\widehat{\beta}$ is a homogeneous MNC on the complex Banach space $\widehat{X}$, that $\widehat{\beta}(\widehat{L}^m) = \beta(L^m)$, and that $\widehat{\beta}(\widehat{L}^m\widehat{B}_1(0)) = \beta(L^m B_1(0))$, where $\widehat{B}_1(0)$ (respectively,

$B_1(0)$) denotes the unit ball in $\widehat{X}$ (respectively, X). It follows that

$$\widehat{\beta}^{\#}(\widehat{L}) = \beta^{\#}(L), \qquad \widehat{\beta}^{*}(\widehat{L}) = \beta^{*}(L) \tag{6}$$

both hold. We remark also that if α denotes the Kuratowski MNC on a real Banach space X and $\widehat{\alpha}$ denotes its complexification as above, then $\widehat{\alpha}$ is in fact the Kuratowski MNC on $\widehat{X}$. We omit the proofs of these results, which are straightforward for the most part, except for the proof that $\widehat{\alpha}$ is the Kuratowski MNC on $\widehat{X}$; this is given as Proposition 11.

Our interest in Questions A and B and the related issues above arises from the question of the "correct" definition of the "cone essential spectral radius," denoted $\rho_C(f)$, for a map $f : C \to C$. Here C is a closed cone in a Banach space and f is a continuous, homogeneous, order-preserving map. This question is, in turn, related to the problem of existence of an eigenvector of f in C with eigenvalue equal to $r_C(f)$, the "cone spectral radius of f," and to showing that $\rho_C(f) \le r_C(f)$; see [11] and [17]. In future work, related to this paper, we shall discuss deficiencies in the definition of $\rho_C(f)$ in [11], [17], and theorems about existence of eigenvectors of f.

Theorems 7, 8, and 10 are the main results of this paper. In Theorem 7 we shall present the first known example of an infinite dimensional Banach space $Y = \ell^p(\mathbb{N})$ and a homogeneous, set-additive MNC γ_Y on Y which is not equivalent to the Kuratowski MNC, thereby answering Question A in the affirmative. In fact, we provide a large class of such inequivalent MNC's γ_Y. Much more general results for other spaces are given in [12], but it seems worthwhile to illustrate our approach here in this relatively simple case with a self-contained proof. (In fact we use some ideas from [12] in the example considered in Theorem 8.) In Theorem 8 we study the quantities $\gamma_Z(\Lambda^m)$ and $\gamma_Z^{\#}(\Lambda)$ for homogeneous, set-additive MNC's γ_Z on $Z = \ell^p(\mathbb{N} \times \mathbb{N})$ related to the MNC's γ_Y of Theorem 7, for a particular shift operator Λ on the space Z. We demonstrate the pathological features of these quantities noted above, in particular that in general $\gamma_Z^{\#}(\Lambda) \ne \rho(\Lambda)$, which thereby gives a negative answer to the first part of Question B. In Theorem 10 we prove for a general homogeneous MNC β on a Banach space X that $\beta^*(L)$ rather than $\beta^{\#}(L)$ is the "correct" quantity to consider in studying $\rho(L)$. In particular we show that $\beta^*(L) = \rho(L)$ always holds for all bounded linear operators on X, thus providing an affirmative answer to the second part of Question B.

Due to the following result proved in [12], the issue of whether or not a homogeneous MNC satisfies the set-additivity property (K8) is often unimportant.

Proposition 1 (see [12])**.** *Let $(X, \|\cdot\|)$ be a Banach space and β a homogeneous MNC on X. For $S \in \mathcal{B}(X)$, define $\gamma(S)$ by*

$$\gamma(S) := \inf\{\max_{1\le i\le n} \beta(S_i) \mid S = \bigcup_{i=1}^{n} S_i \textit{ for some } S_i \textit{ with } 1 \le i \le n < \infty\}. \tag{7}$$

Then γ is a homogeneous, set-additive MNC on X with $\gamma(S) \le \beta(S)$ for all bounded $S \subset X$. Moreover, $\gamma = \beta$ if β itself is a homogeneous, set-additive MNC.

Before presenting our main results we make some fundamental observations.

Proposition 2. *Let $(X, \|\cdot\|)$ be a Banach space and β a homogeneous MNC on X. Then the Kuratowski MNC α dominates β.*

Proof. Let $c := \beta(B_1(0))$, recalling the notation (4). Then homogeneity implies that $\beta(B_r(0)) = rc$. If $S \in \mathcal{B}(X)$ and $d := \alpha(S)$, then given $\varepsilon > 0$, there exists a finite collection of sets $S_1, S_2, \ldots, S_n$ with $S = \bigcup_{i=1}^n S_i$, and with $\mathrm{diam}(S_i) \leq d + \varepsilon$ for $1 \leq i \leq n$. For each i select $x_i \in S_i$ and define $T := \{x_i \mid 1 \leq i \leq n\}$. Note that $S \subset T + B_{d+\varepsilon}(0)$, so property (K6), along with (K1) and (K2), implies that

$$\beta(S) \leq \beta(T) + \beta(B_{d+\varepsilon}(0)) = \beta(B_{d+\varepsilon}(0)) = (d+\varepsilon)c.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\beta(S) \leq cd = c\alpha(S)$. □

The next result was obtained independently by Furi and Vignoli in [7] and by Nussbaum in Section A of [16].

Proposition 3 (see [7] and Section A of [16]). *Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space. If $Q := \{x \in X \mid \|x\| \leq 1\}$ and if α denotes the Kuratowski MNC on X, then $\alpha(Q) = 2$.*

Lemma 4 below is an easy result; see [14] or Section A of [16]. However, as we shall see later, Lemma 4 may fail for general homogeneous MNC's.

Lemma 4 (see [14] or Section A of [16]). *Let $(X_i, \|\cdot\|_i)$, for $i = 1, 2$, be Banach spaces, let α_i denote the Kuratowski MNC on X_i, and let $L : X_1 \to X_2$ be a bounded linear map. Define*

$$\alpha(L) := \inf\{\lambda \geq 0 \mid \alpha_2(LS) \leq \lambda\alpha_1(S) \text{ for every bounded } S \subset X_1\}.$$

Then we have $\alpha(L) \leq \|L\|$. Further, if β_i is a homogeneous MNC on X_i, with β_i equivalent to α_i for $i = 1, 2$, then there exists a constant $c > 0$, independent of L, such that

$$\beta_2(LS) \leq c\alpha(L)\beta_1(S) \leq c\|L\|\beta_1(S)$$

for every $S \in \mathcal{B}(X_1)$.

Our next lemma is true in greater generality (see [12]), but the following version will suffice for our purposes.

Lemma 5. *Let $(X_i, \|\cdot\|_i)$, for $i = 1, 2$, be Banach spaces, and let $L : X_1 \to X_2$ be a one-one, continuous linear map of X_1 onto X_2. If β_2 is a homogeneous MNC on X_2, define, for $S \in \mathcal{B}(X_1)$,*

$$\widetilde{\beta}_2(S) := \beta_2(LS).$$

Then $\widetilde{\beta}_2$ is a homogeneous MNC on X_1, and $\widetilde{\beta}_2$ is set-additive if β_2 is set-additive. If α_i denotes the Kuratowski MNC on X_i and if β_2 is equivalent to α_2, then $\widetilde{\beta}_2$ is equivalent to α_1.

Proof. The fact that $\widetilde{\beta}_2$ is a homogeneous (set-additive) MNC on X_1 follows easily from the fact that L is a linear homeomorphism of X_1 onto X_2. Details are left to the reader.

To see that $\widetilde{\beta}_2$ is equivalent to α_1 if β_2 is equivalent to α_2, observe that $\widetilde{\beta}_2$ is equivalent to $\widetilde{\alpha}_2$, where $\widetilde{\alpha}_2(S) := \alpha_2(LS)$. Thus it suffices to prove that $\widetilde{\alpha}_2$ is equivalent to α_1. However, if S is a bounded subset of X_1, then Lemma 4 implies that $\alpha_2(LS) \leq \|L\|\alpha_1(S)$ and $\alpha_1(S) = \alpha_1(L^{-1}LS) \leq \|L^{-1}\|\alpha_2(LS)$. This proves that $\widetilde{\alpha}_2$ and α_1 are equivalent. □

The following lemma will be convenient in establishing Theorem 7.

Lemma 6. *Let $(X_i, \|\cdot\|_i)$, for $i = 1, 2$, be Banach spaces, let α_i denote the Kuratowski MNC on X_i, and let $L : X_1 \to X_2$ be a one-one, continuous linear map of X_1 onto X_2. Suppose there exists a homogeneous MNC β_2 on X_2 which is inequivalent to α_2. Then there exists a homogeneous, set-additive MNC γ_2 on X_2 which is inequivalent to α_2. Further, there exists a homogeneous, set-additive MNC γ_1 on X_1 which is inequivalent to α_1.*

Proof. Proposition 2 implies that α_2 dominates β_2, so there must exist a sequence of bounded sets $S_n \subset X_2$ with $\alpha_2(S_n) > 0$ and $\lim_{n\to\infty} \frac{\beta_2(S_n)}{\alpha_2(S_n)} = 0$. Let γ_2 be the homogeneous, set-additive MNC derived from β_2 as in Proposition 1. Then it is immediate that $\gamma_2(S) \le \beta_2(S)$ for all $S \in \mathcal{B}(X_2)$, and so $\lim_{n\to\infty} \frac{\gamma_2(S_n)}{\alpha_2(S_n)} = 0$. Define $\widetilde{\gamma}_2$ and $\widetilde{\alpha}_2$ as in Lemma 5, so $\widetilde{\gamma}_2(T) := \gamma_2(LT)$ and $\widetilde{\alpha}_2(T) := \alpha_2(LT)$ for $T \in \mathcal{B}(X_1)$. Then Lemma 5 implies that $\widetilde{\gamma}_2$ and $\widetilde{\alpha}_2$ are homogeneous, set-additive MNC's on X_1 and that $\widetilde{\alpha}_2$ is equivalent to α_1, so in particular there exists $c > 0$ such that $\widetilde{\alpha}_2(T) \le c\alpha_1(T)$ for every $T \in \mathcal{B}(X_1)$. If we define $T_n := L^{-1}S_n$, it follows that

$$\lim_{n\to\infty} \left(\frac{\widetilde{\gamma}_2(T_n)}{\alpha_1(T_n)} \right) \le c \lim_{n\to\infty} \left(\frac{\widetilde{\gamma}_2(T_n)}{\widetilde{\alpha}_2(T_n)} \right) = c \lim_{n\to\infty} \left(\frac{\gamma_2(S_n)}{\alpha_2(S_n)} \right) = 0,$$

so $\widetilde{\gamma}_2$ and α_1 are inequivalent. If we define $\gamma_1 := \widetilde{\gamma}_2$, the proof is complete. □

Let $1 \le p \le \infty$ and let $\mathbb{N}$ denote the natural numbers. We define the Banach space $Y := \ell^p(\mathbb{N})$ in the usual way: Elements $y \in Y$ are maps $y : \mathbb{N} \to \mathbb{C}$ such that $\|y\|_Y := (\sum_{i=1}^\infty |y(i)|^p)^{1/p} < \infty$. As usual, we interpret $\|y\|_Y := \sup_{i\in\mathbb{N}} |y(i)|$ if $p = \infty$. (We remark that if we instead take the corresponding real Banach space of maps $y : \mathbb{N} \to \mathbb{R}$, then the construction below is still valid with the obvious changes.) Similarly, the Banach space $Z := \ell^p(\mathbb{N} \times \mathbb{N})$ is the set of maps $z : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ such that $\|z\|_Z := (\sum_{i=1}^\infty \sum_{j=1}^\infty |z(i,j)|^p)^{1/p} < \infty$, and again with the corresponding supremum norm if $p = \infty$. It is well-known that there is a one-one map $\sigma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ of $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$, and that σ induces a linear isometry $L_\sigma : Y \to Z$ by composition, namely $L_\sigma y := y \circ \sigma$. We want to prove that there is a homogeneous, set-additive MNC γ_Y on Y which is inequivalent to the Kuratowski MNC α_Y on Y. By Lemma 6 it suffices to prove that there exists a homogeneous MNC β_Z on Z which is inequivalent to the Kuratowski MNC α_Z on Z.

Theorem 7. *Let $1 \le p \le \infty$ and let Y denote the Banach space $\ell^p(\mathbb{N})$ with the usual norm. Let α_Y denote the Kuratowski MNC on Y. Then there exists a homogeneous, set-additive MNC γ_Y on Y which is inequivalent to α_Y.*

Proof. With $Z = \ell^p(\mathbb{N} \times \mathbb{N})$ and with the norm $\|\cdot\|_Z$ as above, let α_Z denote the Kuratowski MNC on Z. By the remarks above, it suffices to prove that there exists a homogeneous MNC β_Z on Z which is inequivalent to α_Z.

For simplicity, we shall denote α_Z and β_Z simply by α and β, respectively, and we denote $\mathcal{B} := \mathcal{B}(Z)$, the set of bounded subsets of Z. Also for simplicity, in what follows we shall assume that $p < \infty$, as the arguments for $p = \infty$ are similar.

Let a_n, for $n \ge 1$, be a nonincreasing sequence of positive reals with $a_1 \le 1$ and $\lim_{n\to\infty} a_n = 0$. Define a Banach space $(\widetilde{Z}, \|\cdot\|_{\widetilde{Z}})$ to be the set of maps $z : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ such that

$$\|z\|_{\widetilde{Z}} := \left(\sum_{i=1}^\infty a_i^p \sum_{j=1}^\infty |z(i,j)|^p \right)^{1/p} < \infty,$$

and let $\widetilde{\alpha}$ denote the Kuratowski MNC on $\widetilde{Z}$. Note that $Z \subset \widetilde{Z}$ and that

$$\|z\|_{\widetilde{Z}} \le \|z\|_Z \tag{8}$$

for all $z \in Z$. For each integer $n \ge 1$ define the linear projection $P_n : Z \to Z$ by setting $P_n z = x$, where

$$x(i,j) = \begin{cases} z(i,j), & \text{for } 1 \le i \le n, \\ 0, & \text{for } i > n. \end{cases}$$

Note also that $P_n : \widetilde{Z} \to \widetilde{Z}$ is a projection and that $P_n\widetilde{Z} = P_n Z$. It is easy to see that, for all $z \in Z$,

$$\|P_n z\|_Z \le \|z\|_Z, \qquad \|P_n z\|_{\widetilde{Z}} \le \|z\|_{\widetilde{Z}}, \qquad \|P_n z\|_Z \le a_n^{-1}\|P_n z\|_{\widetilde{Z}}, \tag{9}$$

and in fact the second and third inequalities in (9) are valid for every $z \in \widetilde{Z}$. Thus by Lemma 4, using (8) and (9), we have that

$$\begin{aligned} &\widetilde{\alpha}(S) \le \alpha(S), \qquad \alpha(P_n S) \le \alpha(S), \qquad \widetilde{\alpha}(P_n S) \le \widetilde{\alpha}(S), \\ &\alpha(P_n S) \le a_n^{-1}\widetilde{\alpha}(P_n S), \end{aligned} \tag{10}$$

for every $S \in \mathcal{B}$. We now define $\mathcal{A} \subset \mathcal{B}$ by

$$\mathcal{A} := \{S \in \mathcal{B} \mid \lim_{n\to\infty} \alpha((I - P_n)S) = 0\}. \tag{11}$$

The reader can easily verify that if $S, T \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, then the sets $\operatorname{co}(S)$, λS, $\overline{S}$, and $S + T$ are all elements of $\mathcal{A}$. Furthermore, if $S \in \mathcal{B}$, then $P_n S \in \mathcal{A}$ for every integer $n \ge 1$.

With these preliminaries we define $\beta : \mathcal{B} \to [0, \infty)$ by

$$\beta(S) := \inf\{\widetilde{\alpha}(A) + \alpha(B) \mid S \subset A + B, \text{ for some } A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}. \tag{12}$$

We claim that β is a homogeneous MNC on Z, that β is inequivalent to α, and that $\beta(S) = \widetilde{\alpha}(S)$ for all $S \in \mathcal{A}$.

Observe first that for any $S \in \mathcal{B}$, if we take $A := \{0\}$ and $B := S$ in equation (12), we see that $\beta(S) \le \alpha(S)$.

If $S \in \mathcal{A}$ and we take $A := S$ and $B := \{0\}$ in (12), we see that $\beta(S) \le \widetilde{\alpha}(S)$. On the other hand, if $S \in \mathcal{A}$ and $S \subset A + B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have, using the first inequality in (10), that

$$\widetilde{\alpha}(S) \le \widetilde{\alpha}(A) + \widetilde{\alpha}(B) \le \widetilde{\alpha}(A) + \alpha(B),$$

so we obtain from (12) that $\widetilde{\alpha}(S) \le \beta(S)$. We conclude that $\widetilde{\alpha}(S) = \beta(S)$ for $S \in \mathcal{A}$, as claimed.

The fact that β satisfies property (K2) (with β replacing α) is obvious. It follows that if $S \in \mathcal{B}$, then $\beta(S) \le \beta(\operatorname{co}(S))$. On the other hand, given $\varepsilon > 0$, select $A \in \mathcal{A}$ and $B \in \mathcal{B}$ so that $S \subset A + B$ and $\beta(S) \le \widetilde{\alpha}(A) + \alpha(B) < \beta(S) + \varepsilon$. Note that $\operatorname{co}(A) + \operatorname{co}(B)$ is a convex set containing S, so $\operatorname{co}(S) \subset \operatorname{co}(A) + \operatorname{co}(B)$. Since $\operatorname{co}(A) \in \mathcal{A}$, we conclude that

$$\beta(\operatorname{co}(S)) \le \widetilde{\alpha}(\operatorname{co}(A)) + \alpha(\operatorname{co}(B)) = \widetilde{\alpha}(A) + \alpha(B) < \beta(S) + \varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, $\beta(\operatorname{co}(S)) = \beta(S)$. Thus β satisfies property (K5).

If $S, T \in \mathcal{B}$ and $\varepsilon > 0$, select $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$ such that $S \subset A_1 + B_1$ and $T \subset A_2 + B_2$, with $\widetilde{\alpha}(A_1) + \alpha(B_1) \le \beta(S) + \varepsilon$ and $\widetilde{\alpha}(A_2) + \alpha(B_2) \le \beta(T) + \varepsilon$.

Note that $A := A_1 + A_2 \in \mathcal{A}$ and $B := B_1 + B_2 \in \mathcal{B}$, and also that $S + T \subset A + B$. It follows that

$$\beta(S+T) \leq \widetilde{\alpha}(A) + \alpha(B) = \widetilde{\alpha}(A_1 + A_2) + \alpha(B_1 + B_2)$$
$$\leq \widetilde{\alpha}(A_1) + \alpha(B_1) + \widetilde{\alpha}(A_2) + \alpha(B_2) \leq \beta(S) + \beta(T) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we see that $\beta(S+T) \leq \beta(S) + \beta(T)$, so β satisfies property (K6).

The fact that β satisfies property (K7), namely $\beta(\lambda S) = |\lambda|\beta(S)$ for all $S \in \mathcal{B}$ and $\lambda \in \mathbb{C}$, follows easily from the definition (12) of β and the fact that $\widetilde{\alpha}$ and α satisfy property (K7). Details are left to the reader.

If $S \in \mathcal{B}$, property (K2) implies that $\beta(S) \leq \beta(\overline{S})$. On the other hand, we have for any $\varepsilon > 0$ that $\overline{S} \subset S + B_\varepsilon(0)$. Thus from the homogeneity of β and from properties (K2) and (K6), we have that

$$\beta(\overline{S}) \leq \beta(S) + \beta(B_\varepsilon(0)) = \beta(S) + \varepsilon\beta(B_1(0)).$$

This shows that $\beta(\overline{S}) \leq \beta(S)$ and proves property (K4).

If $T \in \mathcal{B}$ and $\overline{T}$ is compact, then $\beta(T) = 0$ because $\beta(T) \leq \alpha(T) = 0$. If $\overline{T}$ is compact and $S \in \mathcal{B}$, we claim that $\beta(S \cup T) = \beta(S)$, which certainly implies that property (K3) is satisfied. Property (K2) implies that $\beta(S) \leq \beta(S \cup T)$. To see the opposite inequality, select $x_0 \in S$, define $\Gamma := (T \cup \{x_0\}) + \{-x_0\}$, and note that $\overline{\Gamma}$ is compact and that $S \cup T \subset S + \Gamma$. Therefore

$$\beta(S \cup T) \leq \beta(S + \Gamma) \leq \beta(S) + \beta(\Gamma) \leq \beta(S) + \alpha(\Gamma) = \beta(S),$$

and so property (K3) holds.

Note that we do not claim that β necessarily satisfies property (K8).

We now establish property (K1), which, along with the inequivalence of β and α, is the main point of our construction. First, as noted above, if $S \in \mathcal{B}$ and $\overline{S}$ is compact, then $\beta(S) = 0$. Now suppose, conversely, that $S \in \mathcal{B}$ and $\beta(S) = 0$. We have to show that $\alpha(S) = 0$, which implies that $\overline{S}$ is compact. Given $\varepsilon > 0$, equation (12) implies that there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $S \subset A + B$ and $\widetilde{\alpha}(A) + \alpha(B) < \varepsilon$. Equation (11) implies that there exists an integer N with $\alpha((I - P_N)A) < \varepsilon$. It follows that $(I - P_N)S \subset (I - P_N)A + (I - P_N)B$ and so

$$\alpha((I - P_N)S) \leq \alpha((I - P_N)A) + \alpha((I - P_N)B)$$
$$\leq \alpha((I - P_N)A) + \alpha(B) + \alpha(P_N B)$$
$$\leq \alpha((I - P_N)A) + 2\alpha(B) < 3\varepsilon,$$

where the second inequality in (10) has been used. Next, for N as above, define $\kappa := a_N\varepsilon \leq \varepsilon$ and select $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ with $S \subset A' + B'$ such that $\widetilde{\alpha}(A') + \alpha(B') < \kappa$. The inequalities in (10) imply that $\widetilde{\alpha}(P_N A') < \kappa$ and $\alpha(P_N B') < \kappa$, and also $\alpha(P_N A') \leq a_N^{-1}\widetilde{\alpha}(P_N A')$. It follows that

$$\alpha(P_N S) \leq \alpha(P_N A') + \alpha(P_N B')$$
$$\leq \frac{\widetilde{\alpha}(P_N A')}{a_N} + \alpha(P_N B') < \left(\frac{1}{a_N} + 1\right)\kappa \leq 2\varepsilon.$$

Thus

$$\alpha(S) \leq \alpha((I - P_N)S) + \alpha(P_N S) < 3\varepsilon + 2\varepsilon = 5\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, $\alpha(S) = 0$.

Finally, we show that β is inequivalent to α. For any $n \geq 1$ define

$$(13) \qquad Z_n := \{z \in Z \mid z(i,j) = 0 \text{ for } i \neq n\}, \qquad S_n := \{z \in Z_n \mid \|z\|_Z \leq 1\}.$$

Note that $(Z_n, \|\cdot\|_Z)$ and $(Z_n, \|\cdot\|_{\widetilde{Z}})$ are infinite dimensional Banach spaces, and in fact $\|z\|_Z = a_n^{-1}\|z\|_{\widetilde{Z}}$ for every $z \in Z_n$. Thus Proposition 3 implies that $\alpha(S_n) = 2$, and also, since S_n is also the closed ball of radius a_n in the space $(Z_n, \|\cdot\|_{\widetilde{Z}})$, Proposition 3 implies that $\widetilde{\alpha}(S_n) = 2a_n$. Further, $S_n \in \mathcal{A}$ and so we have that $\widetilde{\alpha}(S_n) = \beta(S_n)$, as noted earlier in this proof. Thus

$$\lim_{n\to\infty} \left(\frac{\beta(S_n)}{\alpha(S_n)}\right) = \lim_{n\to\infty} a_n = 0,$$

and it follows that β and α are inequivalent. □

The above theorem suggests the following general question.

Open Question. Is it the case that for any infinite dimensional Banach space $(X, \|\cdot\|)$ there exists a homogeneous (possibly set-additive) MNC β which is not equivalent to the Kuratowski MNC α on X?

In [12], we provide a partial answer to the above Open Question, by showing that for a large class of Banach spaces of interest in analysis, there does exist a homogeneous, set-additive MNC which is not equivalent to the Kuratowski MNC. In particular, this is verified for general Hilbert spaces; for the Banach spaces $L^p(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a general measure space and $1 \leq p \leq \infty$; for $C(K)$, where K is a compact Hausdorff space; and for the Sobolev space $W^{m,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set. We believe, however, that an answer (positive or negative) to the Open Question is probably difficult and probably will involve techniques beyond those used in [12].

Our next main result studies $\beta(\Lambda^m)$ and $\beta^{\#}(\Lambda)$ and the corresponding quantities for γ, for the MNC $\beta = \beta_Z$ in the proof of Theorem 7 and the homogeneous, set-additive MNC $\gamma = \gamma_Z$ derived from β by Proposition 1. Recall the definitions and properties (1), (2), of $\beta(\Lambda^m)$ and $\beta^{\#}(\Lambda)$. We shall take Λ to be a particular shift operator.

Theorem 8. *With $Z = \ell^p(\mathbb{N}\times\mathbb{N})$, where $1 \leq p \leq \infty$, define $\Lambda : Z \to Z$ by $\Lambda z = x$, where $x(i,j) = z(i+1,j)$ for every $(i,j) \in \mathbb{N}\times\mathbb{N}$. Also fix a nonincreasing sequence $\{a_n\}_{n=1}^{\infty}$ as in the proof of Theorem 7, with β the homogeneous MNC on Z given by equation (12), and γ the homogeneous, set-additive MNC derived from β as in Proposition 1. Then for every $m \geq 1$,*

$$\beta(\Lambda^m) = \gamma(\Lambda^m) = \mu_m := \sup_{n\geq 1}\left(\frac{a_n}{a_{n+m}}\right),$$

with the above formula serving as the definition of $\mu_m \in (1, \infty]$.

Remark. It is easily seen that $\|\Lambda^m\|_{\mathcal{L}(Z)} = 1$ for every $m \geq 1$, so $\alpha(\Lambda^m) \leq 1$ by Lemma 4, where α is the Kuratowski MNC on Z. (Here and below we let $\|\cdot\|_{\mathcal{L}(X)}$ denote the operator norm associated to a space X.) In fact one easily sees that $\alpha(\Lambda^m) = 1$ for every m, and so by earlier remarks we have that $\alpha^{\#}(\Lambda) = \rho(\Lambda) = 1$.

Proof of Theorem 8. Let $m \geq 1$ be an integer which will be fixed for the remainder of the proof. Generally, we shall use the notation and constructions from the proof of Theorem 7, assuming as well that $p < \infty$.

Let $S \in \mathcal{B}$ with $S = \bigcup_{i=1}^n S_i$ for some S_i where $n < \infty$. Then $\Lambda^m S = \bigcup_{i=1}^n \Lambda^m S_i$ and so

$$\gamma(\Lambda^m S) \le \max_{1\le i\le n} \beta(\Lambda^m S_i) \le \beta(\Lambda^m) \max_{1\le i\le n} \beta(S_i)$$

from the definition (7) of γ and from Lemma 4. As the above inequalities are valid for every S_i, it follows that $\gamma(\Lambda^m S) \le \beta(\Lambda^m)\gamma(S)$ and thus $\gamma(\Lambda^m) \le \beta(\Lambda^m)$.

Next suppose that $S \in \mathcal{A}$, again with $S = \bigcup_{i=1}^n S_i$ for some S_i. Then $S_i \in \mathcal{A}$ for each i, and $\beta(S) = \widetilde{\alpha}(S)$ and $\beta(S_i) = \widetilde{\alpha}(S_i)$, as noted in the proof of Theorem 7. Thus

$$\beta(S) = \widetilde{\alpha}(S) = \max_{1\le i\le n} \widetilde{\alpha}(S_i) = \max_{1\le i\le n} \beta(S_i)$$

from the set-additivity of $\widetilde{\alpha}$, and this implies that $\gamma(S) = \beta(S)$.

Now recall the set $S_n \subset Z$ as in (13) and the fact, noted in the proof of Theorem 7, that $\beta(S_n) = 2a_n$. Certainly $S_n \in \mathcal{A}$, and so also $\gamma(S_n) = 2a_n$. Observing that $\Lambda^m S_{n+m} = S_n$ for every $n \ge 1$, we have that $\gamma(\Lambda^m S_{n+m}) = (\frac{a_n}{a_{n+m}})\gamma(S_{n+m})$ and therefore $\gamma(\Lambda^m) \ge \frac{a_n}{a_{n+m}}$. Taking the supremum over $n \ge 1$, we conclude that $\gamma(\Lambda^m) \ge \mu_m$. It remains to prove that $\beta(\Lambda^m) \le \mu_m$. If $\mu_m = \infty$ we are done, so assume for the remainder of the proof that $\mu_m < \infty$.

Recall the Banach space $(\widetilde{Z}, \|\cdot\|_{\widetilde{Z}})$ in the proof of Theorem 7. For any $z \in \widetilde{Z}$ we have that

$$\begin{aligned}\|\Lambda^m z\|_{\widetilde{Z}} &= \left(\sum_{i=m+1}^{\infty} a_{i-m}^p \sum_{j=1}^{\infty} |z(i,j)|^p \right)^{1/p} \\ &\le \left(\sum_{i=m+1}^{\infty} \mu_m^p a_i^p \sum_{j=1}^{\infty} |z(i,j)|^p \right)^{1/p} \le \mu_m \|z\|_{\widetilde{Z}},\end{aligned}$$

and it follows that $\Lambda^m \widetilde{Z} \subset \widetilde{Z}$ and $\|\Lambda^m\|_{\mathcal{L}(\widetilde{Z})} \le \mu_m$. On the other hand, let $n > m$ and take any $z \in Z_n$, with Z_n as in (13). Then $\Lambda^m z \in Z_{n-m}$ and so $z, \Lambda^m z \in \widetilde{Z}$ with

$$\|\Lambda^m z\|_{\widetilde{Z}} = a_{n-m} \left(\sum_{j=1}^{\infty} |z(n,j)|^p \right)^{1/p} = \left(\frac{a_{n-m}}{a_n} \right) \|z\|_{\widetilde{Z}}.$$

It follows that $\|\Lambda^m\|_{\mathcal{L}(\widetilde{Z})} \ge \mu_m$ and thus

$$\|\Lambda^m\|_{\mathcal{L}(\widetilde{Z})} = \mu_m. \tag{14}$$

Now take any $S \in \mathcal{B}$ and $\varepsilon > 0$. Then there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ so that $S \subset A + B$ and

$$\beta(S) \le \widetilde{\alpha}(A) + \alpha(B) < \beta(S) + \varepsilon,$$

by the definition (12) of β. The reader can verify that $\alpha((I - P_n)\Lambda^m A) = \alpha((I - P_{n+m})A)$, which implies that $\Lambda^m A \in \mathcal{A}$. We have $\Lambda^m S \subset \Lambda^m A + \Lambda^m B$ and also $\mu_m \ge 1$, so it follows from Lemma 4, from (14), and because $\|\Lambda^m\|_{\mathcal{L}(Z)} = 1$ that

$$\begin{aligned}\beta(\Lambda^m S) &\le \widetilde{\alpha}(\Lambda^m A) + \alpha(\Lambda^m B) \\ &\le \|\Lambda^m\|_{\mathcal{L}(\widetilde{Z})}\widetilde{\alpha}(A) + \|\Lambda^m\|_{\mathcal{L}(Z)}\alpha(B) \\ &= \mu_m \widetilde{\alpha}(A) + \alpha(B) \le \mu_m(\beta(S) + \varepsilon).\end{aligned}$$

We conclude that $\beta(\Lambda^m) \le \mu_m$, as desired; hence $\beta(\Lambda^m) = \gamma(\Lambda^m) = \mu_m$, as claimed. □

Remark. Any value $s \in (1, \infty]$ for the quantity $\beta^{\#}(\Lambda)$ can be obtained by a suitable choice of the sequence $\{a_n\}_{n=1}^{\infty}$ in the above construction. If $s \in (1, \infty)$, then taking $a_n = s^{-n}$ gives $\beta(\Lambda^m) = \mu_m = s^m$, and hence $\beta^{\#}(\Lambda) = s$. If $s = \infty$, then taking, for example, $a_n = n^{-n}$ gives $\beta(\Lambda^m) = \mu_m = \infty$ for every m, and hence $\beta^{\#}(\Lambda) = \infty$.

While the above construction has been carried out for the space $\ell^p(\mathbb{N} \times \mathbb{N})$, where $1 \le p \le \infty$, with the aid of results in [12] analogs of Theorem 8 can be proved for a variety of infinite dimensional Banach spaces which arise naturally in analysis.

We return again to the general case. Let $(X, \|\cdot\|)$ be any complex, infinite dimensional Banach space, β an arbitrary homogeneous MNC on X, and $L : X \to X$ any bounded linear map. There are several inequivalent definitions of $\text{ess}(L)$, the essential spectrum of L, and all definitions actually apply when $L : \mathcal{D}(L) \subset X \to X$ is closed and densely defined. For example, F.E. Browder [5] defines $\text{ess}(L)$ to be the set of $\lambda \in \mathbb{C}$ such that (a) λ is an accumulation point of $\sigma(L)$, the spectrum of L, or that (b) $\mathcal{R}(\lambda I - L)$, the range of $\lambda I - L$, is not closed, or that (c) $\bigcup_{i=1}^{\infty} \mathcal{N}((\lambda I - L)^i)$ is infinite dimensional, where $\mathcal{N}(B)$ denotes the null space of a linear map B. Another possible definition is $\text{ess}(L) = \{\lambda \in \mathbb{C} \mid \lambda I - L \text{ is not Fredholm of index } 0\}$. F. Wolf [19] defines $\text{ess}(L) = \{\lambda \in \mathbb{C} \mid \lambda I - L \text{ is not Fredholm}\}$, and T. Kato [9] defines $\text{ess}(L) = \{\lambda \in \mathbb{C} \mid \lambda I - L \text{ is not semi-Fredholm}\}$. Simple examples involving shift operators on $\ell^2(\mathbb{N})$ show that these definitions are not equivalent. However, by using classical results of Gohberg and Krein [8] and index theory for semi-Fredholm operators (see [9]), one can prove that for all definitions, $\text{ess}(L)$ is nonempty and that

$$\rho(L) := \sup\{|\lambda| \mid \lambda \in \text{ess}(L)\} \tag{15}$$

is the same for all definitions of $\text{ess}(L)$. If $|\lambda| > \rho(L)$ and $\lambda \in \sigma(L)$, then λ is an eigenvalue of L of finite algebraic multiplicity, λ is an isolated point of $\sigma(L)$, and $\lambda I - L$ is Fredholm of index 0.

Now let α denote the Kuratowski MNC on X and define η, the **ball measure of noncompactness on $\mathbf{X}$**, by

$$\eta(S) := \inf\{r > 0 \mid S \subset \bigcup_{i=1}^{n} B_r(x_i) \text{ for some } x_i \in X, \text{ for } 1 \le n < \infty\},$$

with $B_r(x)$ as in (4). It is well-known that η is a homogeneous, set-additive MNC and that

$$\frac{\alpha(S)}{2} \le \eta(S) \le \alpha(S) \tag{16}$$

for every $S \in \mathcal{B}(X)$. If $L : X \to X$ is a bounded linear map, it is also known (see Lemma 1 in [14]) that

$$\eta(L^m) = \eta(L^m B_1(0)). \tag{17}$$

It follows from equations (16) and (17) and earlier remarks that

$$\rho(L) = \eta^{\#}(L) = \lim_{m\to\infty} \eta(L^m B_1(0))^{1/m} = \lim_{m\to\infty} \alpha(L^m B_1(0))^{1/m} = \alpha^*(L), \tag{18}$$

where $\rho(L)$ is as in (15) and where we recall that $\beta^*(L)$, for any homogeneous MNC β, is given by (3). As any such β is dominated by α by Proposition 2, it follows from (18) that

$$\beta^*(L) \le \alpha^*(L) = \rho(L). \tag{19}$$

We claim that $\beta^*(L) = \rho(L)$. To prove this we shall use an old result of Yood [20] and some facts about semi-Fredholm operators (see [9]). In the following lemma, recall that a map f from a topological space U to a topological space V is called **proper** if $f^{-1}(K)$ is compact (possibly empty) for every compact $K \subset V$.

Lemma 9 (Yood [20]). *Let X and Y be Banach spaces (real or complex) and $L : X \to Y$ a bounded linear map. Then the map $L|S : S \to Y$ is proper for every closed, bounded $S \subset X$ if and only if $\mathcal{N}(L)$, the null space of L, is finite dimensional, and $\mathcal{R}(L)$, the range of L, is closed.*

Theorem 10. *Let X be a complex Banach space, $L : X \to X$ a bounded linear map, and β any homogeneous MNC on X. Then*

$$\beta^*(L) = \rho(L),$$

where $\beta^*(L)$ is given by equation (3) *and $\rho(L)$ by equation* (15). *If instead X is a real Banach space, then*

$$\beta^*(L) = \rho(\widehat{L}), \tag{20}$$

where $\widehat{L} : \widehat{X} \to \widehat{X}$ is the complexification of L and $\widehat{X}$ is the complexification of X.

Proof. First suppose that X is a complex Banach space. Let $r > 0$ and $|\lambda| > \beta^*(L)$, and denote $L_\lambda := \lambda^{-1}L$. Then by equation (3),

$$\lim_{m\to\infty} \beta(L_\lambda^m B_r(0)) = \lim_{m\to\infty} r\beta(L_{|\lambda|}^m B_1(0)) = 0. \tag{21}$$

Let $Q_r := \overline{B_r(0)}$. We claim that $(I - L_\lambda)|Q_r$ is proper, equivalently, that $(\lambda I - L)|Q_r$ is proper. As $r > 0$ is arbitrary, this implies that $(\lambda I - L)|S$ is proper for every closed, bounded $S \subset X$. To prove our claim, let $K \subset X$ be compact and let $T := \{x \in Q_r \mid (I - L_\lambda)x \in K\}$. The set T is closed, by continuity. If $x \in T$, then $x = L_\lambda x + y$ for some $y \in K$, and it follows for all $m \geq 1$ that $x = L_\lambda^m x + \sum_{i=0}^{m-1} L_\lambda^i y$. This implies that

$$T \subset L_\lambda^m T + \left(\sum_{i=0}^{m-1} L_\lambda^i\right) K \subset L_\lambda^m Q_r + K_m, \tag{22}$$

where $K_m := (\sum_{i=0}^{m-1} L_\lambda^i)K$ is compact. It follows from (22) that

$$\beta(T) \leq \beta(L_\lambda^m Q_r) + \beta(K_m) = \beta(L_\lambda^m Q_r) \leq \beta(\overline{L_\lambda^m B_r(0)}) = \beta(L_\lambda^m B_r(0)),$$

and with (21) it follows that $\beta(T) = 0$. Thus T is compact. Yood's lemma now implies that $\mathcal{N}(\lambda I - L)$ is finite dimensional and $\mathcal{R}(\lambda I - L)$ is closed, that is, $\lambda I - L$ is a semi-Fredholm operator with index $i(\lambda I - L) := \dim(\mathcal{N}(\lambda I - L)) - \operatorname{codim}(\mathcal{R}(\lambda I - L)) < \infty$. Moreover, the value of $i(\lambda I - L)$ is independent of such a λ due to the continuity of the index of semi-Fredholm operators. As $\lambda I - L$ is invertible for $|\lambda| > \|L\|$, this value is $i(\lambda I - L) = 0$. Thus $\lambda I - L$ is Fredholm of index 0 for all λ with $|\lambda| > \beta^*(L)$. Using Wolf's definition of $\operatorname{ess}(L)$ we have that $\rho(L) \leq \beta^*(L)$; thus $\rho(L) = \beta^*(L)$ from (19).

If X is a real Banach space, then (20) follows from (6) and the surrounding remark. □

Lastly, we prove the following result, which was discussed in a remark above.

Proposition 11. *Let X be a real Banach space, let α denote the Kuratowski MNC on X, and let $\widehat{\alpha}$ denote its complexification, as in* (5)*. Then $\widehat{\alpha}$ is also the Kuratowski MNC on $\widehat{X}$.*

Proof. With $\widehat{\alpha}$ denoting the complexification of α as in the statement of the proposition, let $\overline{\alpha}$ denote the Kuratowski MNC on $\widehat{X}$. We must show that $\widehat{\alpha} = \overline{\alpha}$. First observe that if $\widehat{S} \subset \widehat{X}$ is any bounded set, then $\operatorname{diam}(e^{i\theta}\widehat{S}) = \operatorname{diam}(\widehat{S})$ and $\operatorname{diam}(\operatorname{Re}(\widehat{S})) \leq \operatorname{diam}(\widehat{S})$; hence

$$\operatorname{diam}(\operatorname{Re}(e^{i\theta}\widehat{S})) \leq \operatorname{diam}(\widehat{S}),$$

for any $\theta \in \mathbb{R}$. Now with such an $\widehat{S}$ fixed, denote $\overline{a} = \overline{\alpha}(\widehat{S})$ and let $\varepsilon > 0$. Then $\widehat{S} = \bigcup_{j=1}^{n} \widehat{S}_j$ for some sets $\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n \subset \widehat{X}$, each with $\operatorname{diam}(\widehat{S}_j) \leq \overline{a} + \varepsilon$. For any $\theta \in \mathbb{R}$ we have that $\operatorname{Re}(e^{i\theta}\widehat{S}) = \bigcup_{j=1}^{n} \operatorname{Re}(e^{i\theta}\widehat{S}_j)$, and as $\operatorname{diam}(\operatorname{Re}(e^{i\theta}\widehat{S}_j)) \leq \overline{a} + \varepsilon$, it follows that $\alpha(\operatorname{Re}(e^{i\theta}\widehat{S})) \leq \overline{a} + \varepsilon$. Taking the supremum over θ and letting $\varepsilon \to 0$ now gives $\widehat{\alpha}(\widehat{S}) \leq \overline{a} = \overline{\alpha}(\widehat{S})$.

Now denote $\widehat{a} = \widehat{\alpha}(\widehat{S})$. Then $\alpha(\operatorname{Re}(e^{i\theta}\widehat{S})) \leq \widehat{a}$ for every θ. Fix $m > 0$ and let $\theta_k = \frac{2\pi k}{m}$ for $1 \leq k \leq m$. Also fix $\varepsilon > 0$. Then for each k in the above range there exist sets $S_{k,j} \subset X$ for $1 \leq j \leq n_k < \infty$ such that $\operatorname{Re}(e^{i\theta_k}\widehat{S}) = \bigcup_{j=1}^{n_k} S_{k,j}$ with

$$\operatorname{diam}(S_{k,j}) \leq \alpha(\operatorname{Re}(e^{i\theta_k}\widehat{S})) + \varepsilon \leq \widehat{a} + \varepsilon.$$

Now define $\widehat{S}_{k,j} = \{x \in \widehat{S} \mid \operatorname{Re}(e^{i\theta_k}x) \in S_{k,j}\}$, so clearly $\widehat{S} = \bigcup_{j=1}^{n_k} \widehat{S}_{k,j}$ for every k. Now consider all sequences $\sigma = (j_1, j_2, \ldots, j_m)$ where $1 \leq j_k \leq n_k$, and for each such σ let $\widehat{T}_\sigma = \bigcap_{k=1}^{m} \widehat{S}_{k,j_k}$. Then $\widehat{S} = \bigcup \widehat{T}_\sigma$, where the union is taken over all possible such sequences σ, of which there are finitely many. We wish to obtain an upper bound for the diameter of $\widehat{T}_\sigma$ for each σ. Fixing $\sigma = (j_1, j_2, \ldots, j_m)$, let $x, y \in \widehat{T}_\sigma$. For any k with $1 \leq k \leq m$ we have that $x, y \in \widehat{S}_{k,j_k}$, and therefore $\operatorname{Re}(e^{i\theta_k}x), \operatorname{Re}(e^{i\theta_k}y) \in S_{k,j_k}$. Thus

$$\|\operatorname{Re}(e^{i\theta_k}(x-y))\| = \|\operatorname{Re}(e^{i\theta_k}x) - \operatorname{Re}(e^{i\theta_k}y)\| \leq \operatorname{diam}(S_{k,j_k}) \leq \widehat{a} + \varepsilon.$$

Denoting $x - y = u + iv$, where $u, v \in X$, this can be written as

$$\|(\cos\theta_k)u - (\sin\theta_k)v\| \leq \widehat{a} + \varepsilon.$$

Now for any $\theta \in [0, 2\pi]$, there exists k such that $|\theta - \theta_k| \leq \frac{2\pi}{m}$. Then

$$\begin{aligned}
&\|(\cos\theta)u - (\sin\theta)v\| \\
&\quad \leq \|(\cos\theta_k)u - (\sin\theta_k)v\| + \|(\cos\theta - \cos\theta_k)u - (\sin\theta - \sin\theta_k)v\| \\
&\quad \leq \|(\cos\theta_k)u - (\sin\theta_k)v\| + \frac{2\pi}{m}\|u\| + \frac{2\pi}{m}\|v\| \leq \widehat{a} + \varepsilon + \frac{4\pi}{m}\|x-y\|.
\end{aligned}$$

Taking the supremum over θ in the first term above gives $\|u - iv\|$, and upon noting that $\|u - iv\| = \|u + iv\| = \|x - y\|$ we obtain

$$\|x-y\| \leq \widehat{a} + \varepsilon + \frac{4\pi}{m}\|x-y\| \leq \widehat{a} + \varepsilon + \frac{4\pi}{m}\operatorname{diam}(\widehat{S}).$$

As $x, y \in T_\sigma$ are arbitrary, this gives an upper bound for $\operatorname{diam}(T_\sigma)$ and thus an upper bound

$$\overline{\alpha}(\widehat{S}) \leq \widehat{a} + \varepsilon + \frac{4\pi}{m}\operatorname{diam}(\widehat{S})$$

for the Kuratowski MNC of $\widehat{S}$. As ε and m are arbitrary, it follows that $\overline{\alpha}(\widehat{S}) \leq \widehat{a} = \widehat{\alpha}(\widehat{S})$. With this, the proposition is proved. $\square$

References

[1] R.R. Akhmerov, M.I. Kamenskij, A.S. Potapov, A.E. Rodkina, and B.N. Sadovskij, *Measures of Noncompactness and Condensing Operators* (in Russian), Nauka, Novosibirsk, 1986; English translation: Birkhäuser Verlag, Basel, 1992. MR1153247 (92k:47104)

[2] J. Appell, Measures of noncompactness, condensing operators and fixed points: An application-oriented survey, *Fixed Point Theory* **6** (2005), pp. 157-229. MR2196709 (2006h:47121)

[3] J.M. Ayerbe Toledano, T. Dominguez Benavides, and G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser Verlag, Basel, 1997. MR1483889 (99e:47070)

[4] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980. MR591679 (82f:47066)

[5] F.E. Browder, On the spectral theory of elliptic differential operators, *Math. Ann.* **142** (1961), pp. 22-130. MR0209909 (35:804)

[6] G. Darbo, Punti uniti in trasformazioni a condominio non compatto, *Rend. Sem. Mat. Univ. Padova* **24** (1955), pp. 84-92. MR0070164 (16:1140f)

[7] M. Furi and A. Vignoli, On a property of the unit sphere in a linear normed space, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **18** (1970), pp. 333-334. MR0264373 (41:8969)

[8] I. Gohberg and M.G. Krein, The basic propositions on defect numbers, root numbers and indices of linear operators, *Amer. Math. Soc. Translations, Series 2*, vol. 13 (1960), pp. 185-264. MR0113146 (22:3984)

[9] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966. MR0203473 (34:3324)

[10] K. Kuratowski, Sur les espaces complets, *Fund. Math.* **15** (1930), pp. 301-309.

[11] J. Mallet-Paret and R.D. Nussbaum, Eigenvalues for a class of homogeneous cone maps arising from max-plus operators, *Discrete and Continuous Dynamical Systems* **8** (2002), pp. 519-562. MR1897866 (2003c:47088)

[12] J. Mallet-Paret and R.D. Nussbaum, Inequivalent measures of noncompactness, *Ann. Mat. Pura Appl.*, to appear.

[13] J. Mallet-Paret and R.D. Nussbaum, Generalizing the Krein-Rutman theorem, measures of noncompactness and the fixed point index, *J. Fixed Point Theory and Appl.* **7** (2010), pp. 103-143.

[14] R.D. Nussbaum, The radius of the essential spectrum, *Duke Math. J.* **37** (1970), pp. 473-478. MR0264434 (41:9028)

[15] R.D. Nussbaum, A generalization of the Ascoli theorem and an application to functional differential equations, *J. Math. Anal. Appl.* **35** (1971), pp. 600-610. MR0289898 (44:7085)

[16] R.D. Nussbaum, The fixed point index for local condensing maps, *Ann. Mat. Pura Appl.* **89** (1971), pp. 217-258. MR0312341 (47:903)

[17] R.D. Nussbaum, Eigenvalues of nonlinear operators and the linear Krein-Rutman theorem, in *Fixed Point Theory*, Springer Lecture Notes in Math., vol. 886, Springer-Verlag, Berlin, 1981, pp. 309-331. MR643014 (83b:47068)

[18] B.N. Sadovskij, Limit-compact and condensing operators, Uspekhi Mat. Nauk **27** (1972), pp. 81-146 (in Russian). MR0428132 (55:1161)

[19] F. Wolf, On the essential spectrum of partial differential boundary problems, *Comm. Pure Appl. Math.* **12** (1959), pp. 211-228. MR0107750 (21:6472)

[20] B. Yood, Properties of linear transformations preserved under addition of a completely continuous transformation, *Duke Math. J.* **18** (1951), pp. 599-612. MR0044020 (13:355f)

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