

Periodic points of nonexpansive maps and nonlinear generalizations of the Perron-Frobenius theory

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Abstract. Let $K^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, 1 \leq i \leq n\}$ and suppose that $f : K^n \rightarrow K^n$ is nonexpansive with respect to the l_1 -norm, $\|x\|_1 = \sum_{i=1}^n |x_i|$, and satisfies $f(0) = 0$. Let $P_3(n)$ denote the (finite) set of positive integers p such that there exists f as above and a periodic point $\xi \in K^n$ of f of minimal period p . For each $n \geq 1$ we use the concept of “admissible arrays on n symbols” to define a set of positive integers $Q(n)$ which is determined solely by number theoretical and combinatorial constraints and whose computation reduces to a finite problem. In a separate paper the sets $Q(n)$ have been explicitly determined for $1 \leq n \leq 50$, and we provide this information in an appendix. In our main theorem (Theorem 3.1) we prove that $P_3(n) = Q(n)$ for all $n \geq 1$. We also prove that the set $Q(n)$ and the concept of admissible arrays are intimately connected to the set of periodic points of other classes of nonlinear maps, in particular to periodic points of maps $g : D_g \rightarrow D_g$, where $D_g \subset \mathbb{R}^n$ is a lattice (or lower semilattice) and g is a lattice (or lower semilattice) homomorphism.

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1. Introduction

If D is a set and $g : D \rightarrow D$ a map, g^j will denote the j -fold composition of g with itself. If $\xi \in D$ and $g^p(\xi) = \xi$ for some $p \geq 1$, ξ will be called a *periodic point of g of period p* and p will be called the *minimal period* if $g^j(\xi) \neq \xi$ for $1 \leq j < p$. If D is a subset of a vector space V and $\|\cdot\|$ is a norm on V , a map $f : D \rightarrow V$ is called *nonexpansive with respect to $\|\cdot\|$* if

$$\|f(x) - f(y)\| \leq \|x - y\| \quad \text{for all } x, y \in D. \quad (1.1)$$

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We define $\mathcal{H}(D, \|\cdot\|)$ to be the set of maps $h : D \rightarrow D$ such that h is nonexpansive with respect to $\|\cdot\|$. If V is finite dimensional, a norm $\|\cdot\|$ on V is polyhedral if there exist finitely many continuous linear functionals θ_i , $1 \leq i \leq N$, such that

$$\|x\| = \max \{|\theta_i(x)| : 1 \leq i \leq N\}.$$

Important examples of polyhedral norms on \mathbb{R}^n are the l_∞ -norm or sup-norm, $\|\cdot\|_\infty$, and the l_1 -norm, $\|\cdot\|_1$:

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \quad \text{and} \quad \|x\|_1 = \sum_{i=1}^n |x_i|. \quad (1.2)$$

If D is a closed subset of a vector space V , $h \in \mathcal{H}(D, \|\cdot\|)$ and $x \in D$, it is of interest in many applications to understand the behaviour of $h^j(x)$ as $j \rightarrow \infty$. If V is finite dimensional and $\|\cdot\|$ is polyhedral, it is known (see [1], [2], [5], [6], [8], [9], [19], [20]) that if $h \in \mathcal{H}(D, \|\cdot\|)$ and $\{\|h^j(x)\| : j \geq 1\}$ is bounded, then $h^j(x)$ approaches a periodic orbit of h , i.e., there exists a periodic point $\xi = \xi_x \in D$ of minimal period $p = p_x$ and $\lim_{j \rightarrow \infty} h^{jp}(x) = \xi$. Furthermore, there is an integer $N = N(\dim V, \|\cdot\|)$, independent of $h \in \mathcal{H}(D, \|\cdot\|)$ such that the minimal period p of any periodic point of any $h \in \mathcal{H}(D, \|\cdot\|)$ satisfies $p \leq N$. Thus

$$\mathcal{P}(D, \|\cdot\|) = \{p \mid \exists h \in \mathcal{H}(D, \|\cdot\|) \text{ and a periodic point of } h \text{ of minimal period } p\} \quad (1.3)$$

is a finite set, and it is reasonable to ask whether one can explicitly determine $\mathcal{P}(D, \|\cdot\|)$ or some naturally defined subset of $\mathcal{P}(D, \|\cdot\|)$.

In this paper we take $V = \mathbb{R}^n$, $\|\cdot\| = \|\cdot\|_1$, the l_1 -norm, and $D = K^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, 1 \leq i \leq n\}$. We define $\mathcal{F}_3(n)$ by

$$\mathcal{F}_3(n) = \{h \in \mathcal{H}(D, \|\cdot\|) \mid h(0) = 0\}, \quad (1.4)$$

and we define a natural subset of $\mathcal{P}(K^n, \|\cdot\|_1)$ by

$$P_3(n) = \{p \geq 1 \mid \exists h \in \mathcal{F}_3(n) \text{ and a periodic point of } h \text{ of minimal period } p\}. \quad (1.5)$$

We shall determine $P_3(n)$ precisely in terms of combinatorial and number theoretical constraints. In an appendix we shall explicitly list the sets $P_3(n)$ for $1 \leq n \leq 50$.

To describe a special case of our main theorem, it is necessary to define an admissible array on n symbols.

Suppose that L is a finite, totally ordered set with total ordering \prec , that Σ is a set with n elements, and that for each $i \in L$, $\theta_i : \mathbb{Z} \rightarrow \Sigma$ is a map. (\mathbb{Z} will always denote the integers and \mathbb{N} the natural numbers.) We shall say that $\{\theta_i \mid i \in L\}$ is an *admissible array on n symbols* if the maps θ_i , $i \in L$, satisfy the following conditions:

- (1) For each $i \in L$, the map $\theta_i : \mathbb{Z} \rightarrow \Sigma$ is periodic of minimal period p_i , where $1 \leq p_i \leq n$. Furthermore, for $1 \leq j < k \leq p_i$ we have $\theta_i(j) \neq \theta_i(k)$.
- (2) If $m_1 < m_2 < \dots < m_{r+1}$ is any increasing sequence of $(r + 1)$ elements of L ($r \geq 1$) and if

$$\theta_{m_i}(s_i) = \theta_{m_{i+1}}(t_i)$$

for $1 \leq i \leq r$, then

$$\sum_{i=1}^r (t_i - s_i) \not\equiv 0 \pmod{\rho},$$

where ρ is the greatest common divisor of $\{p_{m_i} \mid 1 \leq i \leq r + 1\}$.

If $\{\theta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L\}$ is an admissible array on n symbols as above, we define the period of the admissible array to be the least common multiple of $\{p_\lambda \mid \lambda \in L\}$. Generally, if S is a finite set of positive integers, $\text{gcd}(S)$ will denote the greatest common divisor of the elements of S and $\text{lcm}(S)$ will denote the least common multiple of the elements of S . We sometimes denote an admissible array by θ instead of $\{\theta_\lambda \mid \lambda \in L\}$, where $\theta : \mathbb{Z} \times L \rightarrow \Sigma$ and $\theta(j, \lambda) = \theta_\lambda(j)$. We define a set $Q(n)$ by

$$Q(n) = \{p \in \mathbb{N} \mid \exists \text{ an admissible array on } n \text{ symbols which has period } p\}. \tag{1.6}$$

For purposes of computing $Q(n)$, one can always assume (see [16]) that L is a subset of \mathbb{N} with the usual ordering and $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$.

Readers may reasonable wonder whether $Q(n)$ admits a simple, inductive description. However, results described at the end of Section 4 of this paper suggest that such a hope is too optimistic.

Our principal result is a more detailed and precise version of the following theorem.

Theorem A. $P_3(n) = Q(n)$ for all $n \geq 1$.

Theorem A is a precise generalization of an aspect of the classical Perron-Frobenius theory [7] of matrices with nonnegative entries. If M is a nonnegative $n \times n$ matrix whose columns all sum to one (a column stochastic matrix) then $x \mapsto Mx =: f(x)$ (x a column vector in \mathbb{R}^n) determines an l_1 -norm nonexpansive map of K^n to K^n with $f(0) = 0$. Perron-Frobenius theory (see [11] or Section 9 of [16] for details) implies that, for each $x \in K^n$, there exists a periodic point ξ_x of f of minimal period $p_x = p$ with $\lim_{k \rightarrow \infty} f^{kp}(x) = \xi_x$. Furthermore, the number p_x is the order of some element of the symmetric group on n letters; and if p is the order of an element of the symmetric group on n letters, then there exists a column stochastic matrix M and a periodic point ξ of M of minimal period p .

We conclude this introduction with an outline of the paper. In Section 2 some theorems and definitions from earlier papers are recalled and a few new results

are given. In particular we define several different classes of maps in \mathbb{R}^n : $\mathcal{F}_1(n)$, $\mathcal{F}_2(n)$, $\mathcal{F}_3(n)$, $\mathcal{G}_1(n)$ and $\mathcal{G}_2(n)$; and for each class of maps, we consider the integers p for which there exists a map f in the class and a periodic point of the map of minimal period p . Next we recall various elementary definitions concerning lattices and lower semilattices in \mathbb{R}^n . In Propositions 2.1 and 2.2 we begin the description of an intimate connection between lattices and lower semilattices in \mathbb{R}^n on the one hand and admissible arrays and periodic points of maps $f \in \mathcal{F}_3(n)$ on the other.

In Section 3 we prove Theorem 3.1, which is the basic result in this paper and a generalization of Theorem A. The proof takes all of Section 3 and is carried out in a sequence of eleven lemmas. We observe that for each $p \in Q(n)$ there is a “minimal” admissible array θ of period p . For each such array θ , we associate an integral-preserving and order-preserving map $M(\theta) := f : K^n \rightarrow K^n$ and a periodic point $y = y(\theta)$ of f of minimal period p . This is accomplished in Lemma 3.6 and proves Theorem A. If $\theta = \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\}$ and θ_i has minimal period p_i , the remainder of the section is devoted to relating the structure of the lower semilattice V generated by $\{f^j(y) \mid j \geq 0, y = y(\theta), f = M(\theta)\}$ to the structure of θ . For example, if p_i is as above, we prove that there are irreducible elements $z^i \in V$, $i \in L$, $z^i \leq y = y(\theta)$, such that z^i is a periodic point of f of minimal period p_i .

In Section 4 we derive some consequences of Theorem 3.1 and list some open questions.

An appendix lists the elements of $Q(n)$ for $1 \leq n \leq 50$.

2. Background material: admissible arrays and lower semi-lattices

The cone $K^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, 1 \leq i \leq n\}$ induces a partial ordering by $x \leq y$ if and only if $y - x \in K^n$. We shall write $x < y$ if $x \leq y$ and $x \neq y$. If $y - x \notin K^n$ we write $x \not\leq y$; and if $x \not\leq y$ and $y \not\leq x$, we shall say that x and y are *incomparable* or *not comparable*. A map $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *order-preserving* if $f(x) \leq f(y)$ for all $x, y \in D$ with $x \leq y$. If $f_i(x)$ denotes the i^{th} coordinate of $f(x)$, then f is called *integral-preserving* if

$$\sum_{i=1}^n f_i(x) = \sum_{i=1}^n x_i \quad \text{for all } x \in D;$$

and f will be termed *sup-norm-decreasing* if

$$\|f(x)\|_\infty \leq \|x\|_\infty \quad \text{for all } x \in D.$$

We wish to define some refinements of $\mathcal{F}_3(n)$.

Definition 2.1. Define $u = (1, 1, \dots, 1) \in \mathbb{R}^n$ and consider the following conditions on maps $f : K^n \rightarrow K^n$:

- (1) $f(0) = 0$,
- (2) f is order-preserving,
- (3) f is integral-preserving,
- (4) f is nonexpansive with respect to the l_1 -norm,
- (5) $f(\lambda u) = \lambda u$ for all $\lambda > 0$,
- (6) f is sup-norm-decreasing.

We define sets of maps $\mathcal{F}_j(n)$, $1 \leq j \leq 3$, by

$$\begin{aligned} \mathcal{F}_1(n) &= \{f : K^n \rightarrow K^n \mid f \text{ satisfies (1), (2), (3) and (5)}\}, \\ \mathcal{F}_2(n) &= \{f : K^n \rightarrow K^n \mid f \text{ satisfies (1), (2) and (3)}\}, \end{aligned}$$

and

$$\mathcal{F}_3(n) = \{f : K^n \rightarrow K^n \mid f \text{ satisfies (1) and (4)}\}.$$

A proposition of Crandall and Tartar [3] implies that if $f : K^n \rightarrow K^n$ is integral-preserving, then it is order-preserving if and only if it is l_1 -norm nonexpansive. Thus we see that

$$\mathcal{F}_1(n) \subset \mathcal{F}_2(n) \subset \mathcal{F}_3(n). \tag{2.1}$$

If $f : K^n \rightarrow K^n$ is integral-preserving and order-preserving, one can easily check that f satisfies (5) if and only if f is sup-norm-decreasing. Thus we have

$$\mathcal{F}_1(n) = \{f : K^n \rightarrow K^n \mid f \text{ satisfies (1), (2), (3) and (6)}\}.$$

Using this characterization of $\mathcal{F}_1(n)$ and a result of Lin and Krengel [4], we see that if $f \in \mathcal{F}_1(n)$ and $y \in K^n$ is a periodic point of f , then there is a permutation σ , depending on f and y , such that $f(y) = (y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)})$.

Definition 2.2. We define sets of positive integers $P_j(n)$, $1 \leq j \leq 3$, by

$$P_j(n) = \{p \geq 1 \mid \exists f \in \mathcal{F}_j(n) \text{ and a periodic point of } f \text{ of minimal period } p\}. \tag{2.2}$$

Our theorems will describe the sets $P_j(n)$, $2 \leq j \leq 3$, precisely and provide considerable information about $P_1(n)$.

For $x, y \in \mathbb{R}^n$, we shall denote by $x \wedge y$ and $x \vee y$ the standard lattice operation on \mathbb{R}^n :

$$\begin{aligned} x \wedge y &:= z, & z_i &= \min\{x_i, y_i\} & \text{for } 1 \leq i \leq n \\ x \vee y &:= w, & w_i &= \max\{x_i, y_i\} & \text{for } 1 \leq i \leq n. \end{aligned}$$

If $B = \{x^j \mid 1 \leq j \leq k\}$ is a finite set of points in \mathbb{R}^n , we shall write in the obvious notation $x^1 \wedge x^2 \wedge \cdots \wedge x^k = \bigwedge_{j=1}^k x^j = \bigwedge_{x \in B} x$ and $x^1 \vee x^2 \vee \cdots \vee x^k = \bigvee_{j=1}^k x^j = \bigvee_{x \in B} x$. As usual, we define $x^+ = x \vee 0$; and for a set A , $|A|$ denotes the cardinality of A . A subset V of \mathbb{R}^n will be called a *lower semilattice* if $x \wedge y \in V$ whenever $x \in V$ and $y \in V$; V will be called a *lattice* if $x \wedge y \in V$ and $x \vee y \in V$ whenever $x \in V$ and $y \in V$.

A set $V \subset \mathbb{R}^n$ will be called *finite lower semilattice* (respectively, *finite lattice*) if V is a lower semilattice (lattice) and $|V| < \infty$. If $A \subset \mathbb{R}^n$, there is a minimal semilattice $V \supset A$ and a minimal lattice $W \supset A$ (minimal in the sense of set inclusion); V will be called the lower semilattice generated by A and W the lattice generated by A . Furthermore, (see [11, p. 954])

$$V = \left\{ \bigwedge_{z \in T} z \mid T \subset A, 1 \leq |T| < \infty \right\} \quad \text{and}$$

$$W = \left\{ \bigvee_{w \in T} w \mid T \subset V, 1 \leq |T| < \infty \right\},$$

so V and W are finite if A is finite.

If V is a lattice (respectively, lower semilattice), a map $h : V \rightarrow V$ will be called a lattice homomorphism (respectively, a lower semilattice homomorphism) if

$$h(x \wedge y) = h(x) \wedge h(y) \quad \text{and} \quad h(x \vee y) = h(x) \vee h(y) \quad \text{for all } x, y \in V.$$

(respectively, $h(x \wedge y) = h(x) \wedge h(y)$ for all $x, y \in V$).

Definition 2.3. If $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, we shall write $f \in \mathcal{G}_1(n)$ (respectively, $f \in \mathcal{G}_2(n)$) if and only if D is a lower semilattice (respectively, lattice), $f(D) \subset D$ and f is a lower semilattice homomorphism (respectively, a lattice homomorphism).

Definition 2.4. If p is a positive integer, we shall write $p \in Q_1(n)$ (respectively $p \in Q_2(n)$) if there exists a map $f \in \mathcal{G}_1(n)$ (respectively $f \in \mathcal{G}_2(n)$) and a periodic point ξ of f of minimal period p .

One of the authors [17] first observed that there is an intimate connection between periodic points of maps $f \in \mathcal{F}_3(n)$ and periodic points of lower semilattice homomorphisms. The following proposition, which is essentially a special case of Proposition 2.1 in [11] gives a refinement of the observation in [17]. In the result below note that a norm $\|\cdot\|$ on \mathbb{R}^n is called *monotonic* if $\|x\| \leq \|y\|$ whenever $0 \leq x \leq y$; the norm is *strictly monotonic* if $\|x\| < \|y\|$ whenever $0 \leq x < y$.

Proposition 2.1 (Compare Proposition 2.1 in [11]). *Let $\|\cdot\|$ be a strictly monotonic norm on \mathbb{R}^n and suppose that $f : K^n \rightarrow K^n$ is nonexpansive with respect to $\|\cdot\|$ and $f(0) = 0$. Let $\{p_i\}_{i=1}^\infty$ be a given, strictly increasing sequence of positive integers and define V by*

$$V = \left\{ x \in K^n \mid \lim_{i \rightarrow \infty} f^{p_i}(x) = x \right\}.$$

Assume either (a) f is order-preserving or (b) $\|\cdot\|$ is the l_1 -norm. In case (a), V is a closed set and a lattice, $f(V) \subset V$, f is a lattice homomorphism on V and $f|V$ is an isometry. In case (b), V is a closed set and a lower semilattice, f is a lower semilattice homomorphism on V and $f|V$ is an isometry.

Proof. The continuity of f immediately implies that $f(V) \subset V$. If $x, y \in V$, the nonexpansiveness of f implies that

$$\|f(x) - f(y)\| \leq \|x - y\|.$$

If strict inequality holds in the above inequality, the nonexpansiveness of f implies that

$$\|f^{p_i}(x) - f^{p_i}(y)\| \leq \|f(x) - f(y)\| < \|x - y\|,$$

which contradicts $\lim_{i \rightarrow \infty} f^{p_i}(x) = x$ and $\lim_{i \rightarrow \infty} f^{p_i}(y) = y$. Thus $f|V$ is an isometry. In case (a), Proposition 2.1 of [11] implies that if $x, y \in V$, then $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$; and it follows easily that $x \wedge y \in V$ and $x \vee y \in V$ and $f : V \rightarrow V$ is a lattice homomorphism. If $y \in \text{clo } V$ and $\epsilon > 0$, there exists $x \in V$ such that $\|x - y\| < \epsilon/2$. The nonexpansiveness of f implies that $\|f^{p_i}(y) - f^{p_i}(x)\| < \epsilon/2$ for all i ; and since $\lim_{i \rightarrow \infty} \|f^{p_i}(x) - x\| = 0$, we conclude that there exists i_0 such that $\|f^{p_i}(y) - y\| < \epsilon$ for $i \geq i_0$. This shows that $y \in V$ and V is closed.

In case (b), the proofs that $f(V) \subset V$, $f|V$ is an isometry and V is closed remain unchanged. To complete the proof, it suffices to show that $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in V$. An examination of the proof of Proposition 2.1 in [11] shows that the same proof applies and proves that $f(x \wedge y) = f(x) \wedge f(y)$ if we can prove that for all $j \geq 1$,

$$f^j(x \wedge y) \leq f^j(x) \quad \text{and} \quad f^j(x \wedge y) \leq f^j(y).$$

Because $x, y \in K^n$, the definition of the l_1 -norm gives

$$\|x - (x \wedge y)\| = \|x\| - \|x \wedge y\|.$$

The nonexpansiveness of f^j implies that

$$\|f^j(x) - f^j(x \wedge y)\| \leq \|x - (x \wedge y)\| = \|x\| - \|x \wedge y\|. \tag{2.4}$$

Because f^j is nonexpansive and $f^j(0) = 0$, we see that $\|f^j(x \wedge y)\| \leq \|x \wedge y\|$; and because $f|V$ is an isometry and $f(0) = 0$, we have $\|f^j(x)\| = \|x\|$ for all $j \geq 1$. Suppose, by way of contradiction, that $f^j(x \wedge y) \not\leq f^j(x)$ for some j . Then there exists i , $1 \leq i \leq n$, with

$$(f^j(x \wedge y))_i > (f^j(x))_i.$$

It follows easily (using (2.4) also) that

$$\|f^j(x)\| - \|f^j(x \wedge y)\| < \|f^j(x) - f^j(x \wedge y)\| \leq \|x\| - \|x \wedge y\|. \tag{2.5}$$

Since $\|f^j(x)\| = \|x\|$, we conclude from (2.5) that

$$\|x \wedge y\| < \|f^j(x \wedge y)\|,$$

which contradicts $\|f^j(x \wedge y)\| \leq \|x \wedge y\|$. Thus we find that $f^j(x \wedge y) \leq f^j(x)$ for all $j \geq 1$, and the same proof shows that $f^j(x \wedge y) \leq f^j(y)$ for all $j \geq 1$. \square

Corollary 2.1. *Let $\|\cdot\|$ and f be as in Proposition 2.1 and assume that f satisfies either condition (a) or (b) of Proposition 2.1. For a fixed positive integer p , let $W = \{x \in K^n \mid f^p(x) = x\}$. In case (a) or case (b), W is a closed set, $f(W) \subset W$ and $f|_W$ is an isometry. In case (a), W is a lattice and $f : W \rightarrow W$ is a lattice homomorphism. In case (b), W is a lower semilattice and $f : W \rightarrow W$ is a lower semilattice homomorphism. If $y \in W$ and $A_y = \{f^j(y) \mid 0 \leq j < p\}$, let W_y be the lattice generated by A_y and let V_y be the lower semilattice generated by A_y . In case (a), $f(W_y) \subset W_y \subset W$ and $f|_{W_y}$ is a lattice homomorphism. In case (b), $f(V_y) \subset V_y$ and $f|_{V_y}$ is a lower semilattice homomorphism.*

Proof. In the notation of Proposition 2.1, let $p_i := ip$ and let V be as defined in Proposition 2.1, so $W \subset V$. Continuity of f implies that W is closed, and it is immediate that $f(W) \subset W$. Proposition 2.1 implies that $f|_V$ is an isometry, so $f|_W$ is an isometry. In case (a), Proposition 2.1 implies that f preserves the lattice operations on V and hence on W . In particular, if $x, z \in W$,

$$f^p(x \vee z) = f^p(x) \vee f^p(z) = x \vee z \quad \text{and} \quad f^p(x \wedge z) = f^p(x) \wedge f^p(z) = x \wedge z,$$

so W is closed under the lattice operations and W is a lattice. The same argument shows that W is a lower semilattice in case (b) and $f|_W$ is a lower semilattice homomorphism. Since $A_y \subset W$ and W is a lattice in case (a), $W_y \subset W$ in case (a) and $f|_{W_y}$ preserves the lattice operations. Using this fact we see in case (a) that $f(W_y)$ is a lattice and that $A_y = f(A_y) \subset f(W_y)$. Thus $f(W_y) \cap W_y$ is a lattice which contains A_y , and this contradicts the minimality of W_y unless $f(W_y) = W_y$. Thus we conclude that $f(W_y) = W_y$. In case (b), essentially the same argument shows that V_y is a lower semilattice, $f(V_y) = V_y \subset W$ and $f|_{V_y}$ is a lower semilattice homomorphism. \square

Proposition 2.1 and Corollary 2.1 show that $P_3(n) \subset Q_1(n)$ and that lower semilattices in \mathbb{R}^n and periodic points of lower semilattice homomorphisms should play an important role in our analysis. To describe the connection precisely, we need to recall some standard definitions concerning lower semilattices.

If $V \subset \mathbb{R}^n$ is a finite lower semilattice and $A \subset V$, we say that A is “bounded above in V ” if there exists $b \in V$ such that $a \leq b$ for all $a \in A$, b is called an “upper bound for A in V ”. If A is bounded above in V , then because V is a finite lower semilattice, there exists $\beta \in V$ such that β is an upper bound for A in V , and $\beta \leq b$ for all $b \in V$ for which b is an upper bound for A in V . We shall write

$$\beta = \sup_V(A). \tag{2.6}$$

If V is a finite lower semilattice, $A \subset V$ is bounded above and $f : V \rightarrow V$ is a lower semilattice homomorphism such that $f^p(x) = x$ for all $x \in V$ and some fixed $p \geq 1$, then it is easy to show (see [11], [15], [17]) that

$$f(\sup_V(A)) = \sup_V(f(A)). \tag{2.7}$$

If $V \subset \mathbb{R}^n$ is a finite lower semilattice, $x \in V$ and $A_x := \{y \in V : y < x\}$, we shall say that x is *irreducible in V* or x is *join-irreducible in V* if A_x is empty or if

$$\sup_V(A_x) := z < x. \tag{2.8}$$

If x is irreducible in a finite lower semilattice V in \mathbb{R}^n and $z := \sup_V(A_x)$, we define

$$I_V(x) = \{j \mid 1 \leq j \leq n, z_j < x_j\} \tag{2.9}$$

and we define $I_V(x) = \{j \mid 1 \leq j \leq n\}$ if A_x is empty. If V is a finite lower semilattice in \mathbb{R}^n and $f : V \rightarrow V$ is a lower semilattice homomorphism and there exists $p \geq 1$ with $f^p(y) = y$ for all $y \in V$, then it follows easily from (2.7) that $f(x)$ is an irreducible element of V whenever $x \in V$ is an irreducible element of V . Furthermore (see [15], [17]) every element of V is a periodic point of f and the minimal period p_x of any irreducible $x \in V$ satisfies $p_x \leq n$.

If W is a finite lower semilattice and $x \in W$, we define $h_W(x)$, the *height of x in W* by

$$h_W(x) = \sup \{k \geq 0 \mid \exists y^0, y^1, \dots, y^k \in W \text{ with } y^k = x \text{ and } y^j < y^{j+1} \text{ for } 0 \leq j < k\}. \tag{2.10}$$

We define $h_W(x) = 0$ if there does not exist $u \in W$ with $u < x$.

With these definitions we can now recall a basic result from [15]. In the statement of the following proposition, recall that we have already defined admissible arrays on n symbols in Section 1.

Proposition 2.2 (See Proposition 1.1 and 1.2 in [15]). *Let W be a lower semilattice in \mathbb{R}^n and $g : W \rightarrow W$ a lower semilattice homomorphism. Assume that $\xi \in W$ is a periodic point of g of minimal period p , let $A = \{g^j(\xi) \mid j \geq 0\}$ and let V be the lower semilattice generated by A . Define $f = g|_V$ considered as a map from V to V , so $f^p(x) = x$ for all $x \in V$. Then there exist elements $y^i \in V$, $1 \leq i \leq m$, with the following properties:*

- (1) $y^i \leq \xi$ for $1 \leq i \leq m$.
- (2) y^i is an irreducible element of V , so y^i is a periodic point of f of minimal period p_i , where $1 \leq p_i \leq n$.
- (3) $p = \text{lcm}(\{p_i \mid 1 \leq i \leq m\})$, the least common multiple of p_1, p_2, \dots, p_m .

- (4) $h_V(y^i) \leq h_V(y^{i+1})$ for $1 \leq i < m$, where $h_V(\cdot)$ is the height function on V defined by eq. (2.10).
 (5) For $1 \leq i < j \leq m$, the sets $\{f^k(y^i) : k \geq 0\}$ and $\{f^k(y^j) : k \geq 0\}$ are disjoint.
 (6) For $1 \leq i < j \leq m$, the elements y^i and y^j are not comparable.

If y^i , $1 \leq i \leq m$, are any elements of V which satisfy properties (1)–(6) above, define $L = \{i \in \mathbb{N} \mid 1 \leq i \leq m\}$ with the standard ordering and $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$. For $0 \leq j < p_i$ select $a_{ij} \in I_V(f^j(y^i))$, where $I_V(\cdot)$ is defined by eq. (2.9). Define $\theta_i : \mathbb{Z} \rightarrow \Sigma$ by $\theta_i(j) = a_{ij}$ for $0 \leq j < p_i$ and θ_i is periodic of period p_i . Then it follows that $\{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\} = \theta$ is an admissible array on n symbols, θ_i has minimal period p_i and θ has period p .

By using Proposition 2.1 and the remarks preceding it and recalling Definition 2.2 and 2.4, it is easy to prove (see [15]) that

$$P_1(n) \subset P_2(n) \subset P_3(n) \subset Q_1(n) \subset Q(n) \quad (2.11)$$

and

$$P_2(n) \subset Q_2(n) \subset Q_1(n) \subset Q(n). \quad (2.12)$$

Now suppose that $\theta = \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\}$ is an admissible array on n symbols and that θ_i , $i \in L$, is periodic of minimal period p_i . Our goal in the next section is to associate to θ a map $f = M(\theta) \in \mathcal{F}_2(n)$ and a periodic point $\xi = \xi(\theta)$ of f of minimal period $p = \text{lcm}(\{p_i \mid i \in L\})$. In conjunction with (2.11) and (2.12), this will show that

$$P_2(n) = P_3(n) = Q_2(n) = Q_1(n) = Q(n), \quad (2.13)$$

which is a sharpening of Theorem A. Furthermore, if $A = \{f^j(\xi) \mid j \geq 0\}$ and V is the lower semilattice generated by A , we shall prove that, under mild further assumptions on θ , there exist elements $y^i \in V$, $1 \leq i \leq m := |L|$, which satisfy properties (1), (2), (3), (5) and (6) of Proposition 2.2 and “in essence” also satisfy property (4).

3. $P_2(\mathbf{n})=P_3(\mathbf{n})=Q(\mathbf{n})$: the fundamental theorem

Suppose that L is a finite, totally ordered set with ordering \prec , that Σ is a set with n elements, that $\{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\} = \theta$ is an admissible array on n symbols, and that θ_i has period p_i . Note that the definition of admissible array depends on the ordering of L . If $|L| = l$ and the elements of L are ordered by $\lambda_1 \prec \lambda_2 \prec \cdots \prec \lambda_l$, we shall say that $\alpha, \beta \in L$ are adjacent if $\alpha = \lambda_i$ and $\beta = \lambda_{i+1}$ or $\alpha = \lambda_{i+1}$ and $\beta = \lambda_i$. If $\alpha \in L$, we shall always denote by $R(\theta_\alpha)$ the range of θ_α :

$$R(\theta_\alpha) = \{j \mid j = \theta_\alpha(k), k \in \mathbb{Z}\} \subset \Sigma. \quad (3.1)$$

An admissible array can be visualized as a semi-infinite array of elements of Σ , the α^{th} -row being the elements $\dots, \theta_\alpha(0), \theta_\alpha(1), \dots, \theta_\alpha(k), \dots$.

If $m \in R(\theta_\alpha)$ and $m = \theta_\alpha(j)$, we shall say that $m_+ := \theta_\alpha(j+1)$ is the immediate successor of m in row α and $m_- := \theta_\alpha(j-1)$ is the immediate predecessor of m in row α . This notion is well-defined, for if $m = \theta_\alpha(j_1)$, then the properties of arrays imply that $j_1 - j$ is a multiple of p_α and $\theta_\alpha(j_1 - 1) = m_-$ and $\theta_\alpha(j_1 + 1) = m_+$.

If L_1 is a finite, totally ordered set with $|L_1| = |L|$, $\varphi : L_1 \rightarrow L$ is a one-one order-preserving map of L_1 onto L , Σ_1 is a finite set with $|\Sigma_1| = |\Sigma|$, $\psi : \Sigma \rightarrow \Sigma_1$ is a one-one map and $\theta'_\beta := \psi \circ \theta_{\varphi(\beta)}$ for $\beta \in L_1$, then one easily checks that $\{\theta'_\beta : \mathbb{Z} \rightarrow \Sigma_1 \mid \beta \in L_1\}$ is an admissible array on n symbols and that θ'_β has minimal period $p_{\varphi(\beta)} := p'_\beta$. By virtue of this observation we can assume if desired that $L = \{i \in \mathbb{N} \mid 1 \leq i \leq l\}$ with the usual ordering and that $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$.

If $\{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\} =: \theta$ is an admissible array on n symbols and $L_1 \subset L$ is a proper subset of L with the ordering inherited from L then $\{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L_1\}$ is an admissible array on n symbols and is called a proper subarray of θ . If θ_i has minimal period p_i for $i \in L$, we shall say that $\theta = \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\}$ is a “minimal admissible array on n symbols” if, for every proper subset $L_1 \subset L$,

$$\text{lcm}(\{p_i \mid i \in L_1\}) < \text{lcm}(\{p_i \mid i \in L\}). \tag{3.2}$$

If θ is not a minimal admissible array on n symbols, it is clear that there exists a proper subset L_1 of L such that

$$\text{lcm}(\{p_i \mid i \in L_1\}) = \text{lcm}(\{p_i \mid i \in L\}) \tag{3.3}$$

and for every proper subset L_2 of L_1

$$\text{lcm}(\{p_i \mid i \in L_2\}) < \text{lcm}(\{p_i \mid i \in L_1\}). \tag{3.4}$$

Equations (3.3) and (3.4) imply that $\{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L_1\}$ is a subarray of θ and a minimal admissible array on n symbols whose period is the same as the period of θ . Thus we see that

$$Q(n) = \{p \mid p \text{ is the period of a minimal admissible array on } n \text{ symbols}\}. \tag{3.5}$$

If $\{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\}$ is a minimal admissible array on n symbols and if $\pi(x)$ denotes the number of primes q such that $q \leq x$, one can prove that

$$|L| \leq \pi(n/2 - 1) + 1 \quad \text{for } n \neq 5 \quad \text{and} \quad |L| \leq 2 \quad \text{for } n = 5.$$

Suppose that $(L, <)$ is a finite totally ordered set and $\{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\}$ is an admissible array on n symbols. If we denote by L' the set L given a different total ordering $<'$, it is not necessarily true that $\{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L'\}$ is an admissible

array. Suppose, however, that $|L| = m$ and that the ordering on L is $\lambda_i \prec \lambda_{i+1}$ for $1 \leq i \leq m$. Suppose that there exists j , $1 \leq j \leq m$, such that $R(\theta_{\lambda_j}) \cap R(\theta_{\lambda_{j+1}})$ is empty. Define a one-one map $\sigma : \{i \in \mathbb{N} \mid 1 \leq i \leq m\} \rightarrow \{i \in \mathbb{N} \mid 1 \leq i \leq m\}$ by $\sigma(i) = i$ for $i \notin \{j, j+1\}$ and $\sigma(j) = j+1$ and $\sigma(j+1) = j$ (σ is a row transposition) and define a total ordering \prec' on L by $\lambda_{\sigma(i)} \prec' \lambda_{\sigma(i+1)}$ for $1 \leq i \leq m$. If L' denotes the set L with the total ordering \prec' , we claim that $\theta' := \{\theta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L'\}$ is an admissible array on n symbols. If $m_1 \prec' m_2 \prec' \dots \prec' m_{r+1}$ and $\theta_{m_i}(s_i) = \theta_{m_{i+1}}(t_i)$ for $1 \leq i \leq r$, we have to prove that

$$\sum_{i=1}^r (s_i - t_i) \not\equiv 0 \pmod{\rho}, \quad \rho := \gcd(p_{m_1}, p_{m_2}, \dots, p_{m_{r+1}}). \tag{3.6}$$

Because λ_j and λ_{j+1} are adjacent with respect to \prec and \prec' and because the intersection of $R(\theta_{\lambda_j})$ and $R(\theta_{\lambda_{j+1}})$ is empty, at most one element of $\{\lambda_j, \lambda_{j+1}\}$ is an element of $\{m_i \mid 1 \leq i \leq r+1\}$. Using this fact one can see that $m_1 \prec m_2 \prec \dots \prec m_{r+1}$, and eq. (3.6) follows because θ is an admissible array.

Definition 3.1. Let θ and θ' be defined as above. We shall say that θ and θ' are equivalent under an allowable row transposition. If $\theta = \{\theta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L\}$ and $\tilde{\theta} = \{\theta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in \tilde{L}\}$ are admissible arrays and $L = \tilde{L}$ as sets but L and \tilde{L} possibly have different total orderings, we shall say that θ and $\tilde{\theta}$ are *equivalent* if L and \tilde{L} have the same ordering or if one can go from θ to $\tilde{\theta}$ by a finite sequence of allowable row transpositions.

If L is a finite totally ordered set, $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$ and $\theta = \{\theta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L\}$ is an admissible array on n symbols, we next define certain natural quantities associated to θ . If $m \in \Sigma$, we define $L(m; \theta) \subset L$ and $\rho(m; \theta) \in \mathbb{N} \cup \{0\}$ by

$$L(m; \theta) = \{\alpha \in L \mid m \in R(\theta_\alpha)\} \quad \text{and} \quad \rho(m; \theta) = |L(m; \theta)|,$$

where $R(\theta_\alpha)$ is given by eq. (3.1). We define $\rho(m; \theta) = 0$ if $L(m; \theta)$ is empty. If $\rho(m; \theta) \geq 1$, we can write

$$L(m; \theta) := \{\alpha(m, \nu) \in L : 1 \leq \nu \leq \rho(m; \theta)\}, \tag{3.8}$$

where

$$\alpha(m, 1) \prec \alpha(m, 2) \prec \dots \prec \alpha(m, \rho), \quad \rho := \rho(m; \theta). \tag{3.9}$$

Thus $\alpha(m, \nu)$ is the index in L of the ν^{th} -row of θ in which m appears; and if $\alpha \notin L(m; \theta)$, $m \notin R(\theta_\alpha)$. We define $\Gamma(\theta) \subset \Sigma \times \mathbb{N}$ and $D(\theta) \subset K^n$ by

$$\Gamma(\theta) = \{(m, \nu) \in \Sigma \times \mathbb{N} : \rho(m) \geq 1 \quad \text{and} \quad 1 \leq \nu \leq \rho(m)\} \tag{3.10}$$

and

$$D(\theta) = \{x \in K^n \mid 0 \leq x_i \leq \rho(i) \quad \text{for} \quad 1 \leq i \leq n\}. \tag{3.11}$$

If $(m, \nu) \in \Gamma(\theta)$, we define $\gamma(m, \nu) \in \Sigma$ to be the immediate successor of m in row $\alpha(m, \nu)$, i.e.,

$$\gamma(m, \nu) = k = \theta_\alpha(j + 1), \quad \text{where } m = \theta_\alpha(j) \quad \text{and} \quad \alpha = \alpha(m, \nu). \tag{3.12}$$

We extend γ to a map of $\Sigma \times \mathbb{N}$ into Σ by defining

$$\gamma(m, \nu) = m \quad \text{for } (m, \nu) \notin \Gamma(\theta). \tag{3.13}$$

If $(m, \nu) \in \Sigma \times \mathbb{N}$, we define $M_{m,\nu} : K^n \rightarrow \mathbb{R}$ by

$$M_{m,\nu}(x) := (x_m - (\nu - 1))^+ \wedge 1, \quad x = (x_1, x_2, \dots, x_n), \tag{3.14}$$

so $M_{m,\nu}(x)$ is that fraction of x_m which lies between $\nu - 1$ and ν . For γ defined by (3.12) and (3.13), we define $f = M(\theta)$, $f : K^n \rightarrow K^n$, by $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$, where

$$f_k(x) = \sum_{(m,\nu), \gamma(m,\nu)=k} M_{m,\nu}(x). \tag{3.15}$$

It is easy to prove that $f = M(\theta) \in \mathcal{F}_2(n)$. We prove a slightly more general fact.

Lemma 3.1. *Let $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$ and suppose that $\gamma : \Sigma \times \mathbb{N} \rightarrow \Sigma$ is a given map. For each $(m, \nu) \in \Sigma \times \mathbb{N}$ let $a_{m\nu}$ be a nonnegative real number such that $\sum_{\nu=1}^\infty a_{m\nu} = \infty$. For $(m, \nu) \in \Sigma \times \mathbb{N}$ and $x \in K^n$, define $W_{m,\nu} : K^n \rightarrow [0, \infty)$ by*

$$W_{m,\nu}(x) = \left(x_m - \sum_{j=1}^{\nu-1} a_{mj} \right)^+ \wedge a_{m\nu}.$$

Define $f : K^n \rightarrow K^n$ by $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$, where

$$f_k(x) := \sum_{\gamma(m,\nu)=k} W_{m,\nu}(x).$$

Then f is integral-preserving and order-preserving, so $f \in \mathcal{F}_2(n)$.

Proof. The map $x \mapsto W_{m,\nu}(x)$ is an order-preserving map, so $x \mapsto f_k(x)$ is order-preserving for $1 \leq k \leq n$. This implies that f is order-preserving.

Notice that for any $x = (x_1, x_2, \dots, x_n)$ and $m \in \Sigma$

$$x_m = \sum_{\nu=1}^\infty W_{m,\nu}(x) \quad \text{and} \quad \sum_{m=1}^n x_m = \sum_{(m,\nu) \in \Sigma \times \mathbb{N}} W_{m,\nu}(x).$$

It follows that

$$\begin{aligned} \sum_{k=1}^n f_k(x) &= \sum_{k=1}^n \sum_{\gamma(m,\nu)=k} W_{m,\nu}(x) \\ &= \sum_{(m,\nu) \in \Sigma \times \mathbb{N}} W_{m,\nu}(x) = \sum_{m=1}^n x_m, \end{aligned}$$

so f is integral-preserving. □

If $x = (x_1, x_2, \dots, x_n)$ and x_i denotes the volume of sand in a container of infinite volume C_i , one can describe a simple physical sand-shifting procedure which moves the sand around and leaves volume $f_k(x)$ in C_k , f as in Lemma 3.1: see Example 2 in [14] for details. The map $f = M(\theta)$ corresponds to the case $a_{m\nu} = 1$ for all (m, ν) and γ is given by (3.12) and (3.13).

Lemma 3.2. *Let L be a finite, totally ordered set with ordering $\lambda_1 \prec \lambda_2 \prec \dots \prec \lambda_l$, let $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$ and suppose that $\theta := \{\theta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L\}$ is an admissible array on n symbols. If $\sigma : \Sigma \rightarrow \Sigma$ is a one-one map, let $\tilde{\theta}$ denote the admissible array $\{\sigma \circ \theta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L\}$ and define $\hat{\sigma} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\hat{\sigma}(x) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$. If $f = M(\theta)$ and $\tilde{f} = M(\tilde{\theta})$, we have*

$$\hat{\sigma} \circ \tilde{f} \circ (\hat{\sigma})^{-1} = f. \tag{3.16}$$

If θ and θ' are equivalent admissible arrays on n symbols (see Definition 3.1), then

$$M(\theta') = M(\theta). \tag{3.17}$$

Proof. Let $\gamma : \Sigma \times \mathbb{N} \rightarrow \Sigma$ be a map determined by eq. (3.12) and eq. (3.13) for the admissible array θ , and let $\tilde{\gamma}$ be the corresponding map for $\tilde{\theta}$. One can easily check that

$$\tilde{\gamma}(\sigma(m), \nu) = \sigma(\gamma(m, \nu)).$$

If $M_{m,\nu}(y)$ is defined by eq. (3.14), one also easily sees that

$$M_{\sigma(m),\nu}(\hat{\sigma}^{-1}(y)) = M_{m,\nu}(y).$$

Using these two equations we see that

$$\begin{aligned} \tilde{f}_{\sigma(k)}(\hat{\sigma}^{-1}(y)) &= \sum_{\tilde{\gamma}(\sigma(m),\nu)=\sigma(k)} M_{\sigma(m),\nu}(\hat{\sigma}^{-1}(y)) \\ &= \sum_{\gamma(m,\nu)=k} M_{m,\nu}(y) = f_k(y), \end{aligned} \tag{3.18}$$

and eq. (3.18) is equivalent to the equation $\hat{\sigma} \tilde{f} (\hat{\sigma})^{-1} = f$.

Next suppose that θ and θ' are equivalent admissible arrays on n symbols. To prove that $M(\theta) = M(\theta')$, it suffices to assume that θ and θ' are equivalent under an allowable row transposition, since the general case follows by repeated application of this special case. Thus let L' equal L as a set but with a different total ordering \prec' ; the ordering \prec' is the same as \prec except that for some j , $1 \leq j \leq l$, $\lambda_{j+1} \prec' \lambda_j$, and $R(\theta_{\lambda_j}) \cap R(\theta_{\lambda_{j+1}})$ is empty. It is easy to check (see (3.8), (3.10) and (3.11)) that $L(m; \theta) = L(m; \theta')$, $\Gamma(\theta) = \Gamma(\theta')$ and $D(\theta) = D(\theta')$. Because, for $m \in \Sigma$, m is an element of at most one of $R(\theta_{\lambda_j})$ and $R(\theta_{\lambda_{j+1}})$, one sees that (in the notation of eq. (3.9))

$$\alpha(m, 1) \prec' \alpha(m, 2) \prec' \dots \prec' \alpha(m, \rho), \quad \rho = \rho(m; \theta). \tag{3.19}$$

If $\gamma'(m, \nu)$ is defined by (3.12) and (3.13) for the admissible array θ' , it follows from (3.19) that $\gamma'(m, \nu) = \gamma(m, \nu)$ for all $(m, \nu) \in \Sigma \times \mathbb{N}$, so $M(\theta) = M(\theta')$. \square

With these preliminary results and definitions, we state our basic theorem, which is a refinement of Theorem A.

Theorem 3.1. *For $n \geq 1$, let $Q(n)$ be defined by eq. (1.6), $P_j(n)$, $1 \leq j \leq 3$, be defined by Definition 2.2 and $Q_j(n)$, $1 \leq j \leq 2$, be defined by Definition 2.4. Then for all $n \geq 1$ we have $P_1(n) \subset P_2(n)$ and*

$$P_2(n) = P_3(n) = Q_2(n) = Q_1(n) = Q(n). \tag{3.20}$$

If $L = \{i \in \mathbb{N} \mid 1 \leq i \leq l\}$ with the usual total ordering and $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$, suppose that $\theta := \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\}$ is a minimal admissible array on n symbols and that θ_i is periodic of minimal period p_i for $i \in L$. If $f = M(\theta)$ and $D = D(\theta)$ are as in equations (3.11) and (3.15), then $f : K^n \rightarrow K^n$ is integral preserving and order-preserving and $f(D) \subset D$. Furthermore, f has a periodic point $y \in D$ with the following properties:

- (1) y has minimal period $p = \text{lcm}(\{p_i : i \in L\})$.
- (2) If $A = \{f^j(y) \mid j \geq 0\}$ and V is the lower semilattice generated by A , then $f(V) \subset V$, $f : V \rightarrow V$ is a lower semilattice homomorphism and $f^p(x) = x$ for all $x \in V$.
- (3) For each $i \in L$ there exists an irreducible element $z^i \in V$ of V such that $z^i \leq y$ and z^i is a periodic point of f of minimal period p_i . Furthermore, $y = \bigvee_{i \in L} z^i$ and the elements z^i and z^k are not comparable for $1 \leq i < k \leq l$.
- (4) If $I_V(\cdot)$ is defined by eq. (2.9), $I_V(f^j(z^i)) = \{\theta_i(j)\}$.
- (5) If $i, k \in L$ and $i < k$ and $R(\theta_i) \cap R(\theta_k)$ is nonempty, then $h_V(z^i) < h_V(z^k)$, where $h_V(\cdot)$ is given by eq. (2.10) and $R(\theta_\lambda)$ by eq. (3.1).
- (6) There exists a one-one map $\varphi : L \rightarrow L$ such that if L' denotes L with the new total ordering $\varphi(1) \prec' \varphi(2) \prec' \dots \prec' \varphi(l)$, then $\theta' = \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L'\}$ is an admissible array on n symbols which is equivalent to θ and $h_V(z^{\varphi(i)}) \leq h_V(z^{\varphi(i+1)})$ for $1 \leq i < l$.

Lemma 3.1 implies that $f \in \mathcal{F}_2(n)$, so if we prove Property (1) in Theorem 3.1, then eq. (3.5) and the remarks at the end of Section 2 imply that eq. (3.20) holds. We already know that $P_1(n) \subset P_2(n)$. Thus Property (1) already gives a refinement of Theorem A. The remaining properties listed in Theorem 3.1 provide an approximate converse to Proposition 2.2. Notice the assumption that θ is minimal automatically implies that property (5) in Proposition 2.2 will be satisfied.

Property (6) of Theorem 3.1 implies that if we initially replace the minimal admissible array θ with a total ordering \prec on L by an appropriate equivalent admissible array θ' with total ordering \prec' on L , then $h_V(z^i) \leq h_V(z^k)$ whenever $i \prec' k$. After relabelling, this corresponds to property (4) in Proposition 2.2. Property (5) of Theorem 3.1 shows that if $i < k$ then it is often true that $h_V(z^i) < h_V(z^k)$, but in general this inequality may fail if $R(\theta_i) \cap R(\theta_k)$ is empty.

We shall now establish a sequence of lemmas which will cumulatively establish Properties (1)–(6) of Theorem 3.1.

Lemma 3.3. *Let L be a finite, totally ordered set with a total ordering \prec , let $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$, let $\theta = \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\}$ be an admissible array on n symbols, and write $f = M(\theta)$ and $D = D(\theta)$ (see eq. (3.11) and eq. (3.15)). Then it follows that $f(D) \subset D$.*

Proof. For a fixed $k \in \Sigma$ and $x \in D$, we have to prove that $0 \leq f_k(x) \leq \rho(k)$, where $\rho(k) := \rho(k; \theta)$ is as in eq. (3.7). Because $x \in D$, we have that $M_{m,\nu}(x) = 0$ for $\nu > \rho(m; \theta)$, so, for $\Gamma(\theta)$ as in eq. (3.10),

$$f_k(x) = \sum_{(m,\nu) \in \Gamma(\theta), \gamma(m,\nu)=k} M_{m,\nu}(x).$$

Let $L(k; \theta) = \{\alpha(k, r) \mid 1 \leq r \leq \rho(k)\}$ (see eq. (3.8) and eq. (3.9)). If $(m, \nu) \in \Gamma(\theta)$ and $\gamma(m, \nu) = k$, there must exist r such that k is the immediate successor of m in row $\alpha(k, r)$. If we write $k = \theta_\alpha(j_r)$ for $\alpha = \alpha(k, r)$, then $m = \theta_\alpha(j_r - 1)$ is uniquely determined, and ν is also uniquely determined (row $\alpha(k, r)$ is the ν^{th} row of the admissible array θ in which m appears). It follows that for each r , $1 \leq r \leq \rho(k)$, there is at most one $(m, \nu) \in \Gamma(\theta)$ such that k is the immediate successor of m in row $\alpha(k, r)$. If we write $\Gamma_k(\theta) = \{(m, \nu) \in \Gamma(\theta) \mid \gamma(m, \nu) = k\}$, it follows that $|\Gamma_k(\theta)| \leq \rho(k)$ and

$$f_k(x) = \sum_{(m,\nu) \in \Gamma_k(\theta)} M_{m,\nu}(x) \leq \sum_{(m,\nu) \in \Gamma_k(\theta)} 1 \leq \rho(k),$$

so $f(D) \subset D$. □

The next lemma is not strictly necessary for the proof of Theorem 3.1. We include the lemma because it sheds light on the nature of periodic points of $f = M(\theta)$ and the dynamics of iterates of f .

Lemma 3.4. *Let $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$ and let $\gamma : \Sigma \times \mathbb{N} \rightarrow \Sigma$ be a given map. Define a map $f : K^n \rightarrow K^n$ by*

$$f_k(x) = \sum_{\gamma(m,\nu)=k} M_{m,\nu}(x) \quad \text{for } k = 1, 2, \dots, n,$$

where $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$ and $M_{m,\nu}(x)$ is given by eq. (3.14). Then f is integral-preserving and order-preserving, and for every $x \in K^n$ there exists $j = j_x \in \mathbb{N}$ such that $f^j(x) := \xi$ is a periodic point of f .

Proof. We already know (Lemma 3.1) that f is integral-preserving and order-preserving. For each $r \in \mathbb{R}$, let $[r]$ denote the greatest integer $m \leq r$; and for $x \in K^n$, $x = (x_1, x_2, \dots, x_n)$, define

$$\text{Fract}(x) = \{x_i - [x_i] \mid x_i - [x_i] > 0\} \quad \text{and} \quad N(x) = |\{i \mid x_i - [x_i] > 0\}|.$$

We claim that $N(f(x)) \leq N(x)$. To see this, let $\mu = N(x)$ and let

$$S_k = \{(m, \nu) \mid \gamma(m, \nu) = k\}.$$

Exactly μ of the ordered pairs (m, ν) satisfy $0 < M_{m,\nu}(x) < 1$, and $M_{m,\nu}(x) = 0$ or $M_{m,\nu}(x) = 1$ for all other (m, ν) . It follows that if any set S_k , $1 \leq k \leq n$, contains more than one ordered pair (m, ν) with $0 < M_{m,\nu}(x) < 1$, then there are at most $(\mu - 2)$ sets S_k which contain at least one ordered pair (m, ν) with $0 < M_{m,\nu}(x) < 1$. Thus $N(f(x)) < N(x)$ unless each set S_k contains at most one ordered pair (m, ν) with $0 < M_{m,\nu}(x) < 1$, in which case $N(f(x)) = N(x)$. Since $0 \leq N(f^j(x)) \leq n$ for all $j \geq 0$, there must exist $j_1 \geq 0$, dependent on x , such that $N(f^j(x))$ is constant for all $j \geq j_1$. By the remarks above,

$$\text{Fract}(f^j(x)) = \text{Fract}(f^k(x)) \quad \text{for all } j, k \geq j_1.$$

Because $\|f^j(x)\|_1 = \|x\|_1$ for all $j \geq 1$ and $\text{Fract}(f^j(x))$ is fixed for $j \geq j_1$, $f^j(x) \in \Gamma$ for all $j \geq j_1$, where Γ is a finite set. Thus there must exist $j_1 \leq j < k$ with $f^k(x) = f^j(x)$, so $f^j(x)$ is a periodic point of period $k - j$. \square

In the generality of Lemma 3.4 it is unclear how to determine in terms of γ whether one has a periodic point ξ of f of some specified minimal period p . To treat this problem we shall have to exploit the fact that our γ arises from an admissible array θ by means of eq. (3.12).

If $x \in D(\theta)$, we have $0 \leq x_m \leq \rho(m; \theta)$ for $m \in \Sigma$. In general if r is a nonnegative real and $0 \leq r \leq \rho(m; \theta)$ (where $\theta = \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\}$ is an admissible array on n symbols), we can write r uniquely as

$$r = \sum_{\lambda \in L} r_\lambda, \tag{3.21}$$

where $0 \leq r_\lambda \leq 1$ for $\lambda \in L$, and the numbers r_λ are uniquely determined by (3.21) and the conditions

- (a) $r_\lambda = 0$ if $m \notin R(\theta_\lambda)$;
- (b) $r_\lambda = 1$ if there exists $\lambda' \in L$, $\lambda \prec \lambda'$, with $r_{\lambda'} > 0$.

If $L = \{j \in \mathbb{N} \mid 1 \leq j \leq l\}$ with a total ordering \prec (not necessarily the standard ordering) and r_1, r_2, \dots, r_l are obtained as above from r , we shall write

$$C_m(r) = (r_1, r_2, \dots, r_l). \tag{3.22}$$

Note that $C_m(r)$ depends on m, r and θ and in particular depends on the ordering on L . In this notation, given $x = (x_1, x_2, \dots, x_n) \in D(\theta)$, we shall write

$$(x_{m,1}, x_{m,2}, \dots, x_{m,l}) = C_m(x_m); \tag{3.23}$$

and if $f = M(\theta)$, $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$, $x \in D(\theta)$ and $x_{m,\lambda}$ are as in eq. (3.23) one can see that

$$f_k(x) = \sum_* x_{m,\lambda}, \tag{3.24}$$

where the summation in (3.24) is taken over all $(m, \lambda) \in \Sigma \times L$ such that there exists j , depending on (m, λ) , with $m = \theta_\lambda(j)$ and $k = \theta_\lambda(j + 1)$.

The following lemma provides a crucial step in finding an appropriate periodic point $y \in D(\theta)$ of $f = M(\theta)$.

Lemma 3.5. *Let $L = \{i \in \mathbb{N} \mid 1 \leq i \leq l\}$ with a total ordering \prec (not necessarily the standard ordering) and $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$ and suppose that $\theta = \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L\}$ is an admissible array on n symbols and that θ_λ is periodic of minimal period p_λ , $\lambda \in L$. For each $\lambda \in L$ let c_λ be a fixed real number with $0 < c_\lambda < 1$. Then, for $\mu \in \mathbb{Z}$, $m \in \Sigma$ and $\lambda \in L$ there exist real numbers $y_{m,\lambda}^\mu$ which are uniquely defined by the following properties:*

- (1) $y_{m,\lambda}^\mu = c_\lambda$ if $\theta_\lambda(\mu) = m$.
- (2) $y_{m,\lambda}^\mu = 1$ if $m = \theta_\lambda(j)$ for some j and there exist $\nu \in \mathbb{Z}$ and $\lambda' \in L$, $\lambda \prec \lambda'$, such that

$$y_{\theta_\lambda(j+\nu),\lambda'}^{\mu+\nu} > 0.$$

- (3) $y_{m,\lambda}^\mu = 0$ if neither the conditions of (1) nor of (2) are satisfied.

Proof. Without loss of generality we can assume that the ordering \prec on L is the usual ordering $<$. We construct numbers $y_{m,\lambda}^\mu$ which satisfy (1), (2) and (3) by backward induction on $\lambda \in L$, starting with $\lambda = l$. If $\lambda = l$, condition (2) cannot hold and we define

$$y_{m,l}^\mu = c_l \text{ if } \theta_l(\mu) = m \quad \text{and} \quad y_{m,l}^\mu = 0 \text{ otherwise.}$$

Assume, by way of induction, that numbers $y_{m,\lambda}^\mu$ have been defined for all $\mu \in \mathbb{Z}$, $m \in \Sigma$ and $\lambda > \lambda_1$, $\lambda \in L$, in such a way that conditions (1), (2) and (3) above are satisfied. In order to define $y_{m_1,\lambda_1}^{\mu_1}$, it suffices to prove that if

$$\theta_{\lambda_1}(\mu_1) = m_1, \tag{3.25}$$

then condition (2) cannot hold. We shall assume that condition (2) and eq. (3.25) both hold (for $\mu = \mu_1$, $m = m_1$ and $\lambda = \lambda_1$), and we shall obtain a contradiction. If condition (2) holds, there exist $\nu_1 \in \mathbb{Z}$, $\lambda_2 > \lambda_1$, $\lambda_2 \in L$ and j_1 with

$$\theta_{\lambda_1}(j_1) = m_1 = \theta_{\lambda_1}(\mu_1) \tag{3.26}$$

and

$$y_{\theta_{\lambda_1}(j_1+\nu_1),\lambda_2}^{\mu_1+\nu_1} > 0. \tag{3.27}$$

We define $\mu_2 = \mu_1 + \nu_1$ and $m_2 = \theta_{\lambda_1}(j_1 + \nu_1)$. Equation (3.27) and the induction hypothesis imply that either

$$\theta_{\lambda_2}(\mu_1 + \nu_1) = \theta_{\lambda_1}(j_1 + \nu_1) = m_2$$

or (the condition (2) case) there exist j_2 and ν_2 in \mathbb{Z} and $\lambda_3 > \lambda_2$, $\lambda_3 \in L$, with

$$\theta_{\lambda_2}(j_2 + \nu_1) = \theta_{\lambda_1}(j_1 + \nu_1) \tag{3.28}$$

and

$$y_{\theta_{\lambda_2}(j_2+\nu_1+\nu_2),\lambda_3}^{\mu_2+\nu_2} = y_{\theta_{\lambda_2}(j_2+\nu_1+\nu_2),\lambda_3}^{\mu_1+\nu_1+\nu_2} > 0. \tag{3.29}$$

We define $\mu_3 = \mu_2 + \nu_2 = \mu_1 + \nu_1 + \nu_2$ and $m_3 = \theta_{\lambda_2}(j_2 + \nu_1 + \nu_2)$. By the inductive hypothesis, either

$$\theta_{\lambda_2}(j_2 + \nu_1 + \nu_2) = \theta_{\lambda_3}(\mu_1 + \nu_1 + \nu_2) \tag{3.30}$$

or there exist j_3 and ν_3 in \mathbb{Z} and $\lambda_4 > \lambda_3$, $\lambda_4 \in L$, with

$$\theta_{\lambda_3}(j_3 + \nu_1 + \nu_2) = \theta_{\lambda_2}(j_2 + \nu_1 + \nu_2) \tag{3.31}$$

and

$$y_{\theta_{\lambda_3}(j_3+\nu_1+\nu_2+\nu_3),\lambda_4}^{\mu_3+\nu_3} = y_{\theta_{\lambda_3}(j_3+\nu_1+\nu_2+\nu_3),\lambda_4}^{\mu_1+\nu_1+\nu_2+\nu_3} > 0. \tag{3.32}$$

Continuing in this way we obtain $\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_k \leq l$, and so we must eventually fall into the case of condition (1) of the lemma. Thus, for some $k \geq 1$

and sequence $\lambda_1 < \lambda_2 < \dots < \lambda_{k+1} \leq l$, we eventually obtain the following set of $(k+1)$ equations:

$$\theta_{\lambda_1}(j_1) = \theta_{\lambda_1}(\mu_1) := m_1 \quad (3.33)$$

$$\theta_{\lambda_s}\left(j_s + \sum_{i=1}^{s-1} \nu_i\right) = \theta_{\lambda_{s-1}}\left(j_{s-1} + \sum_{i=1}^{s-1} \nu_i\right) := m_s, \quad 2 \leq s \leq k \quad (3.34)$$

$$\theta_{\lambda_{k+1}}\left(\mu_1 + \sum_{i=1}^k \nu_i\right) = \theta_{\lambda_k}\left(j_k + \sum_{i=1}^k \nu_i\right) := m_{k+1}. \quad (3.35)$$

We define $\rho = \gcd(p_{\lambda_1}, \dots, p_{\lambda_{k+1}})$ and use equations (3.34) and (3.35) and the definition of admissible arrays to conclude that

$$\sum_{s=2}^k \left[\left(j_s + \sum_{i=1}^{s-1} \nu_i \right) - \left(j_{s-1} + \sum_{i=1}^{s-1} \nu_i \right) \right] + \left[\left(\mu_1 + \sum_{i=1}^k \nu_i \right) - \left(j_k + \sum_{i=1}^k \nu_i \right) \right] = \mu_1 - j_1$$

and

$$\mu_1 - j_1 \not\equiv 0 \pmod{\rho}.$$

However, eq. (3.33) implies that $\mu_1 - j_1 \equiv 0 \pmod{p_{\lambda_1}}$, and this contradicts the last equation and completes the proof. \square

Remark 3.1. An examination of the proof of Lemma 3.5 shows that (for $y_{m,\lambda}^\mu$ as in Lemma 3.5) $y_{m,\lambda}^\mu = 1$ if and only if there exist $k \geq 1$, elements $\lambda := \lambda_1, \lambda_2, \dots, \lambda_{k+1}$ of L with $\lambda_1 < \lambda_2 < \dots < \lambda_{k+1}$ and integers j_1, j_2, \dots, j_k and $\nu_1, \nu_2, \dots, \nu_k$ such that

$$m = \theta_\lambda(j_1) \quad (3.36)$$

$$\theta_{\lambda_s}\left(j_s + \sum_{i=1}^{s-1} \nu_i\right) = \theta_{\lambda_{s-1}}\left(j_{s-1} + \sum_{i=1}^{s-1} \nu_i\right) := m_s, \quad 2 \leq s \leq k \quad (3.37)$$

$$\theta_{\lambda_{k+1}}\left(\mu + \sum_{i=1}^k \nu_i\right) = \theta_{\lambda_k}\left(j_k + \sum_{i=1}^k \nu_i\right) := m_{k+1}. \quad (3.38)$$

Alternatively, if we define

$$\tilde{\nu}_s := \sum_{i=1}^{s-1} \nu_i \quad \text{for } 2 \leq s \leq k+1,$$

we see that $y_{m,\lambda}^\mu = 1$ if and only if there exist $k \geq 1$, elements $\lambda_1 < \lambda_2 < \dots < \lambda_{k+1}$ of L and integers j_1, j_2, \dots, j_k and $\tilde{\nu}_2, \tilde{\nu}_3, \dots, \tilde{\nu}_{k+1}$ such that

$$m = \theta_{\lambda_1}(j_1) \tag{3.39}$$

$$\theta_{\lambda_s}(j_s + \tilde{\nu}_s) = \theta_{\lambda_{s-1}}(j_{s-1} + \tilde{\nu}_s) := m_s, \quad 2 \leq s \leq k \tag{3.40}$$

$$\theta_{\lambda_{k+1}}(\mu + \tilde{\nu}_{k+1}) = \theta_{\lambda_k}(j_k + \tilde{\nu}_{k+1}) := m_{k+1}. \tag{3.41}$$

Remark 3.2. In the notation of Lemma 3.5, suppose that L' denotes L with a different total ordering \prec' and that $\theta' = \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L'\}$ is an admissible array on n symbols and θ' is equivalent to θ . One can construct numbers $\eta_{m,\lambda}^\mu$ which satisfy the analogues of properties (1), (2) and (3) of Lemma 3.5:

(1)' $\eta_{m,\lambda}^\mu = c_\lambda$ if $\theta_\lambda(\mu) = m$.

(2)' $\eta_{m,\lambda}^\mu = 1$ if $m = \theta_\lambda(j)$ for some j and there exist $\nu \in \mathbb{Z}$ and $\lambda' \in L', \lambda \prec' \lambda'$, such that

$$\eta_{\theta_\lambda(j+\nu),\lambda'}^{\mu+\nu} > 0.$$

(3)' $\eta_{m,\lambda}^\mu = 0$ if neither the conditions of (1)' nor of (2)' are satisfied.

Notice that these conditions are almost identical to conditions (1)–(3), except that we have $\lambda \prec' \lambda'$ in condition (2)' instead of $\lambda \prec \lambda'$. It is a simple fact that for all $\mu \in \mathbb{Z}, m \in \Sigma$ and $\lambda \in L$

$$\eta_{m,\lambda}^\mu = y_{m,\lambda}^\mu.$$

We shall need this fact and the fact that $M(\theta) = M(\theta')$ later.

For the remainder of this section we shall denote by $y_{m,\lambda}^\mu$ ($\mu \in \mathbb{Z}, m \in \Sigma$ and $\lambda \in L$) numbers which are defined as in Lemma 3.5.

Lemma 3.6. *Let assumptions and notation be as in Lemma 3.5 and let $y_{m,\lambda}^\mu$ ($\mu \in \mathbb{Z}, m \in \Sigma$ and $\lambda \in L$) be numbers as defined in Lemma 3.5. Define*

$$y_m^\mu := \sum_{\lambda \in L} y_{m,\lambda}^\mu \quad \text{and} \quad y^\mu = (y_1^\mu, y_2^\mu, \dots, y_n^\mu).$$

Then we have

- (1) $0 \leq y_{m,\lambda}^\mu \leq 1$ for all $\mu \in \mathbb{Z}, m \in \Sigma$ and $\lambda \in L$.
- (2) $y_{m,\lambda}^\mu = 0$ if $m \notin R(\theta_\lambda)$, the range of θ_λ .
- (3) If $y_{m,\lambda'}^\mu > 0$ for some $\lambda' \in L$, then $y_{m,\lambda}^\mu = 1$ for all $\lambda \in L$ such that $\lambda \prec \lambda'$ and $m \in R(\theta_\lambda)$.
- (4) In the notation of eq. (3.23), $(y_{m,1}^\mu, y_{m,2}^\mu, \dots, y_{m,l}^\mu) = C_m(y_m^\mu)$ for all $\mu \in \mathbb{Z}, m \in \Sigma$.
- (5) For all μ, ν and $j \in \mathbb{Z}$ and $\lambda \in L$ we have $y_{\theta_\lambda(j),\lambda}^\mu = y_{\theta_\lambda(j+\nu),\lambda}^{\mu+\nu}$.
- (6) $f(y^\mu) = y^{\mu+1}$ for all $\mu \in \mathbb{Z}$ and y^0 is a periodic point of f of minimal period $p = \text{lcm}(\{p_\lambda : \lambda \in L\})$.

Proof. Property (1) is immediate from Lemma 3.5. If $m \notin R(\theta_\lambda)$, then neither condition (1) nor condition (2) of Lemma 3.5 can be satisfied, so condition (3) of Lemma 3.5 applies and $y_{m,\lambda}^\mu = 0$. This proves property (2). Property (3) is a direct consequence of condition (2) of Lemma 3.5: take $\nu = 0$. Properties (1),(2) and (3) and the equation $y_m^\mu = \sum_{\lambda \in L} y_{m,\lambda}^\mu$ imply property (4).

It suffices to prove property (5) for the case $\nu = 1$, since the general case follows by repeated applications of this case. If $\theta_\lambda(\mu) = \theta_\lambda(j)$, then $\mu - j \equiv 0 \pmod{p_\lambda}$. It follows that $(\mu + 1) - (j + 1) \equiv 0 \pmod{p_\lambda}$, so $\theta_\lambda(\mu + 1) = \theta_\lambda(j + 1)$. Thus if $y_{\theta_\lambda(j),\lambda}^\mu = c_\lambda$, it follows that $y_{\theta_\lambda(j+1),\lambda}^{\mu+1} = c_\lambda$. The same argument shows that if $y_{\theta_\lambda(j+1),\lambda}^{\mu+1} = c_\lambda$, then $y_{\theta_\lambda(j),\lambda}^\mu = c_\lambda$, so $y_{\theta_\lambda(j+1),\lambda}^{\mu+1} = c_\lambda$ if and only if $y_{\theta_\lambda(j),\lambda}^\mu = c_\lambda$. We know that $y_{\theta_\lambda(j),\lambda}^\mu = 1$ if and only if there exists $\nu \in \mathbb{Z}$ and $\lambda' > \lambda$ with $y_{\theta_\lambda(j+\nu),\lambda'}^{\mu+\nu} > 0$. But then we have

$$y_{\theta_\lambda(j+1+\nu_1),\lambda'}^{\mu+1+\nu_1} > 0,$$

where $\nu_1 = \nu - 1$, so $y_{\theta_\lambda(j+1),\lambda}^{\mu+1} = 1$. The converse follows similarly, so $y_{\theta_\lambda(j),\lambda}^\mu = 1$ if and only if $y_{\theta_\lambda(j+1),\lambda}^{\mu+1} = 1$. Since 0 is the only other possible value of $y_{\theta_\lambda(j),\lambda}^\mu$ and $y_{\theta_\lambda(j+1),\lambda}^{\mu+1}$, we see that

$$y_{\theta_\lambda(j),\lambda}^\mu = y_{\theta_\lambda(j+1),\lambda}^{\mu+1}.$$

Properties (4) and (5) and eq. (3.24) now imply that

$$\begin{aligned} (f(y^\mu))_k &= \sum_{*} y_{m,\lambda}^\mu = \sum_{*} y_{\theta_\lambda(j+1),\lambda}^{\mu+1} \\ &= \sum_{\lambda \in L, k \in R(\theta_\lambda)} y_{k,\lambda}^{\mu+1} = y_k^{\mu+1}, \end{aligned} \tag{3.42}$$

where it is understood that the summation \sum_* is taken over $(m, \lambda) \in \Sigma \times L$ such that there exists $j = j(m, \lambda)$ with $\theta_\lambda(j) = m$ and $\theta_\lambda(j+1) = k$. Thus $f(y^\mu) = y^{\mu+1}$.

If $m = \theta_\lambda(j)$ for some j , property (5) implies that

$$y_{\theta_\lambda(j+p_\lambda),\lambda}^{\mu+p_\lambda} = y_{m,\lambda}^{\mu+p_\lambda} = y_{m,\lambda}^\mu.$$

If $m \notin R(\theta_\lambda)$, $0 = y_{m,\lambda}^{\mu+p_\lambda} = y_{m,\lambda}^\mu$; so in all cases, $y_{m,\lambda}^{\mu+p_\lambda} = y_{m,\lambda}^\mu$. If $p = \text{lcm}(\{p_\lambda \mid \lambda \in L\})$, it follows that $y_{m,\lambda}^{\mu+p} = y_{m,\lambda}^\mu$ for all $\mu \in \mathbb{Z}$, $\lambda \in L$, $m \in \Sigma$. It follows that $y_m^{\mu+p} = y_m^\mu$ and $y^{\mu+p} = y^\mu$ for all $\mu \in \mathbb{Z}$, $m \in \Sigma$. Since $f(y^\mu) = y^{\mu+1}$, we see that $f^p(y^0) = y^p = y^0$ and y^0 has period p .

It remains to show that p is the minimal period of y^0 , i.e., $f^j(y^0) \neq y^0$ for $0 < j < p$. Let μ be the minimal positive integer such that $y^\mu = y^0$. For any given $\lambda \in L$ it suffices to show that $p_\lambda \mid \mu$. Thus take a fixed $\lambda \in L$ and define $m = \theta_\lambda(0)$.

For any $\nu \in \mathbb{Z}$, properties (2) and (3) of our lemma imply that there is at most one $\lambda' \in L$ with $0 < y_{m,\lambda'}^\nu < 1$. It follows that y_m^0 equals c_λ plus an integer, so $y_m^0 = y_m^\mu$ is not an integer. Because we have

$$y_m^\mu = \sum_{\lambda' \in L} y_{m,\lambda'}^\mu = y_m^0 = \sum_{\lambda' \in L} y_{m,\lambda'}^0,$$

our previous remarks imply that there exists exactly one $\lambda' := \lambda_1 \in L$ with $\theta_{\lambda_1}(\mu) = m$ and $0 < y_{m,\lambda_1}^\mu < 1$. By using property (2) and (3), we easily see that $y_m^\mu < y_m^0$ if $\lambda_1 < \lambda$ and $y_m^\mu > y_m^0$ if $\lambda_1 > \lambda$, so the only possibility is that $\lambda_1 = \lambda$. However, if $\lambda_1 = \lambda$, $\theta_\lambda(\mu) = \theta_\lambda(0)$; and since θ_λ has minimal period p_λ , we conclude that $p_\lambda | \mu$, which completes the proof. \square

Remark 3.3. Lemma 3.6 proves that if θ is an admissible array on n symbols and $f = M(\theta)$, then f has a periodic point $y \in D(\theta)$ of minimal period equal to the period of θ . As already remarked, this proves eq. (3.20) and condition (1) of Theorem 3.1. Note that the assumption that θ is minimal is not actually needed to prove condition (1) of Theorem 3.1. Because $f = M(\theta) \in \mathcal{F}_2(n)$, condition (2) of Theorem 3.1 follows from general remarks in Section 2.

Remark 3.4. The points y^μ , $\mu \in \mathbb{Z}$, constructed in Lemma 3.6 depend on the particular admissible array θ . If θ' is an equivalent admissible array, we obtain corresponding points w^μ , $\mu \in \mathbb{Z}$. However, by using Remark 3.2 we see that $w^\mu = y^\mu$ for all μ , and we have already noted (Lemma 3.2) that $M(\theta) = M(\theta')$.

For the remainder of this section y_m^μ and y^μ will be as defined in Lemma 3.6. We shall write $A = \{y^\mu : \mu \in \mathbb{Z}\}$, the periodic orbit of y^0 under f , and shall denote by V the lower semilattice generated by A .

To proceed further, we shall need a technical lemma in which, for the first time, the assumption that the admissible array θ is minimal plays a role.

Lemma 3.7. *Let assumptions and notation be as in Lemma 3.5 and suppose in addition that θ is minimal and $p := \text{lcm}(\{p_\lambda \mid \lambda \in L\}) > 1$. Minimality implies that for each $i \in L$, there exists a prime number m_i and a positive integer α_i such that $m_i^{\alpha_i} | p_i$ but $m_i^{\alpha_i} \nmid p_j$ for $j \in L, j \neq i$. With this choice of m_i , we define q_i by*

$$q_i = \text{lcm} \left(\{p_j \mid j \in L, j \neq i\} \cup \left\{ \frac{p_i}{m_i} \right\} \right) \tag{3.43}$$

and note that $q_i < p$. For a fixed $i \in L$ and $m \in \Sigma$, we define $\tilde{S} = \{\mu \in \mathbb{Z} \mid y_{m,i}^\mu = 1\}$. Then it follows that

$$\tilde{S} + \tau q_i = \tilde{S} \quad \text{for all } \tau \in \mathbb{Z}.$$

Proof. If $m \notin R(\theta_i)$, we know that $y_{m,i}^\mu = 0$ for all $\mu \in \mathbb{Z}$, \tilde{S} is empty, and the lemma is trivially true. Thus we assume that there exists $j_1 \in \mathbb{Z}$ with $\theta_i(j_1) = m$,

and we define $\lambda_1 = i$. If $\mu \in \tilde{S}$, then by Remark 3.1 there exist $k \geq 1$, elements $\lambda_1 < \lambda_2 < \dots < \lambda_{k+1}$ of L and integers j_1, j_2, \dots, j_k and $\tilde{\nu}_2, \tilde{\nu}_3, \dots, \tilde{\nu}_{k+1}$ such that equations (3.39)–(3.40) are satisfied. These numbers depend on μ . Define $q = \gcd(p_{\lambda_1}, p_{\lambda_2}, \dots, p_{\lambda_{k+1}})$, so $q|q_i$. If $\tau \in \mathbb{Z}$, there exist integers A_1, A_2, \dots, A_{k+1} with

$$\tau q = A_1 p_{\lambda_1} + A_2 p_{\lambda_2} + \dots + A_{k+1} p_{\lambda_{k+1}}.$$

Define $\mu' = \mu + \tau q$, $j'_s = j_s + \sum_{t=1}^s A_t p_{\lambda_t}$ for $1 \leq s \leq k$ and $\tilde{\nu}'_s = \tilde{\nu}_s - \sum_{t=1}^{s-1} A_t p_{\lambda_t}$ for $2 \leq s \leq k+1$. Observe that $m = \theta_{\lambda_1}(j'_1)$; and if we substitute μ' for μ , j'_s for j_s ($1 \leq s \leq k$) and $\tilde{\nu}'_s$ for $\tilde{\nu}_s$ for $2 \leq s \leq k+1$, we see that equation (3.40) and (3.41) are satisfied. It follows from Remark 3.1 that $y_{m,\lambda}^{\mu'} = 1$, i.e., $\mu + \tau q \in \tilde{S}$ for all $\tau \in \mathbb{Z}$. Since $q|q_i$, we conclude that $\mu + tq_i \in \tilde{S}$ for all $t \in \mathbb{Z}$, so $\tilde{S} + tq_i \subset \tilde{S}$ for all $t \in \mathbb{Z}$. If $\mu \in \tilde{S}$ and $t \in \mathbb{Z}$, it follows that $(\mu - tq_i) \in \tilde{S}$, so $\mu = (\mu - tq_i) + tq_i \in \tilde{S} + tq_i$. This shows that $\tilde{S} \subset \tilde{S} + tq_i$, so $\tilde{S} + tq_i = \tilde{S}$ for all $t \in \mathbb{Z}$. \square

Remark 3.2 implies that if we replace θ in Lemma 3.7 by an equivalent admissible array θ' and define a corresponding set \tilde{S}' , then $\tilde{S}' = \tilde{S}$.

We now construct the elements z^i , $i \in L$, of Theorem 3.1 and begin to establish their properties.

Lemma 3.8. *Let $L = \{1, 2, \dots, l\}$ with a total ordering \prec , let $\Sigma = \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$ and suppose that $\theta := \{\theta_\lambda : \mathbb{Z} \rightarrow \Sigma \mid \lambda \in L\}$ is a minimal admissible array on n symbols and that θ_λ has minimal period p_λ for $\lambda \in L$. Define $p = \text{lcm}(\{p_\lambda \mid \lambda \in L\})$ and let m_i , α_i and q_i be as in Lemma 3.7. Let $y_{m,\lambda}^\mu$, y_m^μ and y^μ be as in Lemma 3.5 and 3.6 and define V to be the lower semilattice generated by $\{y^\mu \mid 0 \leq \mu < p\}$. For $i \in L$ define S_i and \tilde{S}_i by*

$$S_i = \{\mu \in \mathbb{Z} \mid y_{m,i}^\mu \geq y_{m,i}^0, m = \theta_i(0)\}$$

$$\tilde{S}_i = \{\mu \in \mathbb{Z} \mid y_{m,i}^\mu = 1, m = \theta_i(0)\}.$$

Define $z^i \in V$ by

$$z^i = \bigwedge_{\mu \in S_i} y^\mu.$$

Then z^i is an irreducible element of V and a periodic point of $f := M(\theta)$ of minimal period p_i . Furthermore, for all $j \in \mathbb{Z}$ we have

$$I_V(f^j(z^i)) = \{\theta_i(j)\}.$$

Proof. By applying property (5) of Lemma 3.6 we see that if $\mu \in S_i$, then $\mu + tp_i \in S_i$ for all $t \in \mathbb{Z}$ and hence that $S_i + tp_i = S_i$ for all $t \in \mathbb{Z}$. Because $y_{m,i}^0 = c_i$,

$m = \theta_i(0)$, $y_{m,i}^\mu \geq y_{m,i}^0$ if and only if either (a) $y_{m,i}^\mu = c_i$ or (b) $y_{m,i}^\mu = 1$; and we know that $y_{m,i}^\mu = c_i$ if and only if $\mu = tp_i$ for some $t \in \mathbb{Z}$. Thus we can write

$$S_i = \tilde{S}_i \cup \{tp_i \mid t \in \mathbb{Z}\},$$

and no element of \tilde{S}_i is an integral multiple of p_i . Because we have, for all $\mu \in \mathbb{Z}$, that

$$y_m^\mu = \sum_{\lambda \in L} y_{m,\lambda}^\mu, \quad m = \theta_i(0),$$

we derive from Lemma 3.6 that $\mu \in S_i$ if and only if $y_m^\mu \geq y_m^0$.

We know (see Section 2) that $f = M(\theta)$ is a lower semilattice homomorphism on V and in fact a lattice homomorphism on the lattice generated by V . Thus we obtain

$$\begin{aligned} f^{p_i}(z^i) &= \bigwedge_{\mu \in S_i} f^{p_i}(y^\mu) = \bigwedge_{\mu \in S_i} y^{\mu+p_i} \\ &= \bigwedge_{\nu \in S_i+p_i} y^\nu = \bigwedge_{\nu \in S_i} y^\nu \\ &= z^i. \end{aligned}$$

Thus z^i is a periodic point of f of minimal period γ , where $\gamma|p_i$. Assume, by way of contradiction, that $\gamma \neq p_i$. For $t \in \mathbb{Z}$ we have

$$f^{t\gamma}(z^i) = \bigwedge_{\mu \in S_i} f^{t\gamma}(y^\mu) = \bigwedge_{\nu \in S_i+t\gamma} y^\nu = z^i.$$

It follows from this equation that $y_m^\nu \geq z_m^i = y_m^0$, $m = \theta_i(0)$, for all $\nu \in S_i+t\gamma$, i.e., $S_i+t\gamma \subset S_i$. Because $S_i+t\gamma \subset S_i$ for all $t \in \mathbb{Z}$, we derive that $S_i+t\gamma = S_i$ for all $t \in \mathbb{Z}$. In particular, $0+\gamma = \gamma \in S_i$; and since we assume that γ is a proper divisor of p_i , $\gamma \in \tilde{S}_i$. Lemma 3.7 implies that $\gamma+ tq_i \in \tilde{S}_i$ for all $t \in \mathbb{Z}$, so $\gamma+ q_i \in S_i$ and $(\gamma+ q_i) + (-\gamma) \in S_i - \gamma \subset S_i$, i.e., $q_i \in S_i$. In the notation of Lemma 3.7, we know that $m_i^{\alpha_i} | p_i$ but $m_i^{\alpha_i} \nmid q_i$, so $q_i \notin \{tp_i \mid t \in \mathbb{Z}\}$. It follows that $q_i \in \tilde{S}_i$. However, Lemma 3.7 now implies that $q_i - q_i = 0 \in \tilde{S}_i$, a contradiction; and we conclude that z^i has minimal period p_i .

We must show that z^i is irreducible in V . If $\zeta \in V$ and $\zeta < z^i$, we first claim that ζ_m , the m^{th} -coordinate of ζ (where $m = \theta_i(0)$), satisfies $\zeta_m < z_m^i$. Recall that there exists a set $J \subset \{\mu \mid 0 \leq \mu < p\}$ such that $\zeta = \bigwedge_{\mu \in J} y^\mu$. If $\zeta_m \geq z_m^i$, then $y_m^\mu \geq z_m^i = y_m^0$ for all $\mu \in J$. This implies that $J \subset S_i$ and that $\zeta \geq z^i = \bigwedge_{\mu \in S_i} y^\mu$, a contradiction. Since $\zeta_m < z_m^i$, there exists $\mu \in J$ with $\mu \notin S_i$. For this μ we have $y_{m,i}^\mu = 0$, which implies, using Lemma 3.6, that

$$\zeta_m \leq y_m^\mu \leq y_m^0 - c_i = z_m^i - c_i, \quad m = \theta_i(0). \tag{3.44}$$

Unfortunately, (3.44) is insufficient to prove that z^i is irreducible in V . It remains to find $w \in V$ such that $\zeta \leq w \wedge z^i < z^i$ for all $\zeta \in V$ with $\zeta < z^i$. We define

$$w := y^{q_i} = f^{q_i}(y^0).$$

In the notation of Lemma 3.5 and 3.6 we have for $1 \leq j \leq n$

$$w_j := y_j^{q_i} = \sum_{\lambda=1}^l y_{j,\lambda}^{q_i}.$$

If $\lambda \in L$ and $\lambda \neq i$, we know that $p_\lambda | q_i$, so property (5) of Lemma 3.6 implies that, for $\lambda \neq i$, $y_{j,\lambda}^{q_i} = y_{j,\lambda}^0$. It follows that

$$\begin{aligned} w_j &= \sum_{\lambda=1}^l y_{j,\lambda}^{q_i} = \sum_{\lambda \neq i} y_{j,\lambda}^0 + y_{j,i}^{q_i} \quad \text{and} \\ y_j^0 &= \sum_{\lambda \neq i} y_{j,\lambda}^0 + y_{j,i}^0. \end{aligned} \tag{3.45}$$

Lemma 3.7 implies that $y_{j,i}^{q_i} = 1$ if and only if $y_{j,i}^0 = 1$. If $\theta_i(0) = m = j$, so $0 < y_{m,i}^0 := c_i < 1$, this implies that $y_{m,i}^{q_i} \neq 1$. Since q_i is not an integral multiple of p_i , it follows that $y_{m,i}^{q_i} = 0$ and eq. (3.45) implies that

$$w_m < y_m^0. \tag{3.46}$$

If $j = \theta_i(q_i)$, the same argument shows that

$$y_{\theta_i(q_i)}^0 < w_{\theta_i(q_i)}. \tag{3.47}$$

Finally, if $j \neq \theta_i(q_i)$ and $j \neq \theta_i(0)$, then neither $y_{j,i}^{q_i}$ nor $y_{j,i}^0$ can equal c_i and hence each equals 0 or 1. Because $y_{j,i}^{q_i} = 1$ if and only if $y_{j,i}^0 = 1$, we conclude that $y_{j,i}^{q_i} = y_{j,i}^0$ if $j \neq \theta_i(q_i)$ and $j \neq \theta_i(0)$. Using eq. (3.45) it follows that, for $j \neq \theta_i(q_i)$ and $j \neq \theta_i(0)$,

$$w_j = y_j^0. \tag{3.48}$$

In the case of eq. (3.46), eq. (3.45) actually implies that

$$w_m = y_m^0 - c_i. \tag{3.49}$$

Combining these equations, we see that $(w \wedge z^i)_j = w_j \wedge z_j^i = z_j^i$ for $j \neq i$ (because $z^i \leq y^0$) and $(w \wedge z^i)_m = w_m = y_m^0 - c_i$. If $\zeta \in V$ and $\zeta < z^i$, we have already seen that $\zeta_m \leq y_m^0 - c_i$, so $\zeta \leq w \wedge z^i < z^i$, which proves that z^i is irreducible

in V . Since $m = \theta_i(0)$ is the only coordinate j for which $(w \wedge z^i)_j < z_j^i$, we have also proved that

$$I_V(z^i) = \{\theta_i(0)\}.$$

Since z^i is irreducible in V we know (see Section 2) that for $\nu \in \mathbb{Z}$, $f^\nu(z^i)$ is irreducible in V . Essentially the same reasoning as above shows that

$$I_V(f^\nu(z^i)) = \{\theta_i(\nu)\}.$$

To see this, fix $\nu \in \mathbb{Z}$ and write $u = f^{q_i}(f^\nu(y^0)) = y^{\nu+q_i}$. The same reasoning as above shows that $u_j = y_j^\nu$ for $j \neq \theta_i(\nu)$ and $j \neq \theta_i(\nu + q_i)$,

$$u_{\theta_i(\nu)} + c_i = y_{\theta_i(\nu)}^\nu \quad \text{and} \quad u_{\theta_i(\nu+q_i)} = y_{\theta_i(\nu+q_i)}^\nu + c_i.$$

It follows as before that if $\zeta \in V$, $\zeta < f^\nu(z^i)$, then $\zeta \leq u \wedge f^\nu(z^i) < f^\nu(z^i)$. Since $\theta_i(\nu)$ is the only coordinate in which $u \wedge f^\nu(z^i)$ is less than $f^\nu(z^i)$, we also see that

$$I_V(f^\nu(z^i)) = \{\theta_i(\nu)\},$$

which completes the proof. □

Remark 3.5. If the admissible array θ in Lemma 3.8 is replaced by an equivalent admissible array θ' , we see by virtue of Remarks 3.2 and 3.4 that the sets S_i and \tilde{S}_i and the irreducible elements z^i remain unchanged. We shall need this observation later in the proof of Lemma 3.11.

It remains to prove, under assumptions of Lemma 3.8 that $\bigvee_{i \in L} z^i = y$ and that z^i and z^k are not comparable for $i, k \in L$ with $i \neq k$.

Lemma 3.9. *Let notation and assumptions be as in Lemma 3.8. If i and k are unequal elements of L , then z^i and z^k are not comparable; and we have*

$$y = \bigvee_{i \in L} z^i. \tag{3.50}$$

Proof. We begin by proving eq. (3.50). By construction, we know that $z^i \leq y$ for $i \in L$ and that $z^i = \bigwedge_{\mu \in S_i} y^\mu$, where S_i is defined in Lemma 3.8. Thus, to prove eq. (3.50), we must prove that for each m with $1 \leq m \leq n$, there exists $t \in L$ such that $y_m^\mu \geq y_m^0 := y_m$ for all $\mu \in S_t$, so $z_m^t \geq y_m$. If $y_m^0 - [y_m^0]$, the fractional part of y_m^0 , is strictly positive, we know that there exists $i \in L$ with $m = \theta_i(0)$; and if we take $t := i$, the definition of S_i implies that $y_m^\mu \geq y_m^0$ for all $\mu \in S_t$.

Thus we take a fixed m , $1 \leq m \leq n$, and assume that $y_m^0 - [y_m^0] = 0$. If $y_m^0 = 0$, eq. (3.50) is satisfied in the m^{th} -coordinate, and there is nothing to prove. Otherwise, we define $i \in L$ such that $y_{m,i}^0 = 1$ and $y_{m,j}^0 = 0$ for $j \in L$ and $j > i$. For notational convenience, write $i = i_1$ and select $j_1 \in \mathbb{Z}$ with $\theta_{i_1}(j_1) = m$. Using

equations (3.39)–(3.41) in Remark 3.1, we see that there exist $k \geq 1$, elements $i := i_1 < i_2 < \dots < i_{k+1}$ of L , and integers j_1, j_2, \dots, j_k and $\nu_2, \nu_3, \dots, \nu_{k+1}$ such that

$$m = \theta_{i_1}(j_1), \tag{3.51}$$

$$\theta_{i_s}(j_s + \nu_s) = \theta_{i_{s-1}}(j_{s-1} + \nu_s), \quad 2 \leq s \leq k, \tag{3.52}$$

$$\theta_{i_{k+1}}(\nu_{k+1}) = \theta_{i_k}(j_k + \nu_{k+1}). \tag{3.53}$$

We define $t = i_{k+1}$, and we must prove that if $\mu \in S_t$, i.e., if $y_{\theta_t(0),t}^\mu > 0$, then $y_{m,i_1}^\mu = 1$. Thus select a fixed $\mu \in S_t$, define $\nu'_s = \nu_s + \mu$ for $2 \leq s \leq k + 1$ and $j'_s = j_s - \mu$ for $1 \leq s \leq k$. Equations (3.51)–(3.53) become

$$m'_1 = \theta_{i_1}(j'_1) = \theta_{i_1}(j_1 - \mu), \tag{3.54}$$

$$\theta_{i_s}(j'_s + \nu'_s) = \theta_{i_{s-1}}(j'_{s-1} + \nu'_s) := m'_s, \quad 2 \leq s \leq k, \tag{3.55}$$

$$\theta_{i_{k+1}}(-\mu + \nu'_{k+1}) = \theta_{i_k}(j'_k + \nu'_{k+1}) := m'_{k+1}. \tag{3.56}$$

The assumption that $\mu \in S_k$, $t := i_{k+1}$, and eq. (3.56) imply that

$$0 < y_{\theta_t(-\mu),t}^0 = y_{\theta_t(-\mu+\nu'_{k+1}),t}^{\nu'_{k+1}} = y_{\theta_{i_k}(j'_k+\nu'_{k+1}),i_{k+1}}^{\nu'_{k+1}}.$$

By Lemma 3.5 we conclude that

$$1 = y_{\theta_{i_k}(j'_k+\nu'_{k+1}),i_k}^{\nu'_{k+1}} = y_{\theta_{i_k}(j'_k+\nu'_k),i_k}^{\nu'_k} = y_{m'_k,i_k}^{\nu'_k}.$$

Assume that for some s , $2 \leq s \leq k$, we have proved that

$$1 = y_{m'_s,i_s}^{\nu'_s} = y_{\theta_{i_s}(j'_s+\nu'_s),i_s}^{\nu'_s}.$$

Equation (3.55) then implies that

$$1 = y_{\theta_{i_{s-1}}(j'_{s-1}+\nu'_s),i_s}^{\nu'_s},$$

so Lemma 3.5 implies that (defining $\nu_1 = -\mu$ and $\nu'_1 = 0$)

$$1 = y_{\theta_{i_{s-1}}(j'_{s-1}+\nu'_s),i_{s-1}}^{\nu'_s} = y_{\theta_{i_{s-1}}(j'_{s-1}+\nu'_{s-1}),i_{s-1}}^{\nu'_{s-1}} = y_{m'_{s-1},i_{s-1}}^{\nu'_{s-1}}. \tag{3.57}$$

It follows by induction that (3.57) is satisfied for $2 \leq s \leq k$ and that

$$1 = y_{m'_1,i_1}^{\nu'_1} := y_{\theta_{i_1}(j_1+\nu_1),i_1}^{\nu_1+\mu} = y_{\theta_{i_1}(j_1),i_1}^\mu = y_{m,i_1}^\mu. \tag{3.58}$$

Equation (3.58) is the desired result and completes the proof of eq. (3.50).

It remains to prove that z^i and z^k are not comparable for $1 \leq i < k \leq l$. Suppose not. Then eq. (3.50) implies that there is a proper subset L_1 of L such that

$$y = \bigvee_{i \in L_1} z^i. \tag{3.59}$$

If q is a positive integer, the results of Section 2 imply that

$$f^q(y) = \bigvee_{i \in L_1} f^q(z^i). \tag{3.60}$$

If we define $q = \text{lcm}(\{p_i \mid i \in L_1\})$, eq. (3.60) implies that y is a periodic point of f of period q . Because the admissible array is minimal, $q < p := \text{lcm}(\{p_i \mid i \in L\})$, and we have already proved that y is a periodic point of f of minimal period p . We have obtained a contradiction, so the proof of the lemma is complete. \square

Note that Lemma 3.8 and 3.9 together establish properties (3) and (4) of Theorem 3.1, so it only remains to prove properties (5) and (6).

Lemma 3.10. *Let hypotheses and notation be as in Lemma 3.8. If $i, k \in L$ and $i < k$ and $R(\theta_i) \cap R(\theta_k)$ is nonempty, then $h_V(z^i) < h_V(z^k)$.*

Proof. We first claim that there exists $\gamma \in \mathbb{Z}$ such that

$$S_k \subset \tilde{S}_i + \gamma. \tag{3.61}$$

Because we assume that $R(\theta_i) \cap R(\theta_k)$ is nonempty, there exist $s, t \in \mathbb{Z}$ with $\theta_k(t) = \theta_i(s)$. If $\nu \in S_k$, then by definition we have $y_{\theta_k(0),k}^\nu > 0$. Using Lemmas 3.5 and 3.6 we deduce that $y_{\theta_k(t),k}^{\nu+t} > 0$, so $y_{\theta_k(s),k}^{\nu+t} > 0$ and $y_{\theta_k(s),i}^{\nu+t} = 1$ and $y_{\theta_k(0),i}^{\nu+t-s} = 1$. The final inequality shows that $\nu + (t - s) \in \tilde{S}_i$, which implies that eq. (3.61) holds with $\gamma = s - t$.

Using eq. (3.61) and the definitions of z^k and z^i we see that

$$f^{-\gamma}(z^k) = \bigwedge_{\nu \in S_k - \gamma} y^\nu \geq \bigwedge_{\mu \in \tilde{S}_i} y^\mu.$$

By definition we have that $y_{\theta_i(0),i}^\mu = 1$ for all $\mu \in \tilde{S}_i$, so $y_{\theta_i(0)}^\mu > y_{\theta_i(0)}^0$ for all $\mu \in \tilde{S}_i$ and

$$\bigwedge_{\mu \in \tilde{S}_i} y^\mu > \bigwedge_{\mu \in S_i} y^\mu = z^i.$$

We conclude that

$$f^{-\gamma}(z^k) > z^i,$$

from which we easily derive that

$$h_V(f^{-\gamma}(z^k)) > h_V(z^i). \tag{3.62}$$

Finally recall that for any $x \in V$ and $m \in \mathbb{Z}$, we have

$$h_V(f^m(x)) = h_V(x),$$

so we derive from (3.62) that

$$h_V(z^k) > h_V(z^i),$$

which completes the proof. □

We can now complete the proof of Theorem 3.1 by proving property (6) of Theorem 3.1.

Lemma 3.11. *Let notation and assumptions be as in Lemma 3.8. There exists a total ordering \prec' on L such that if L' denotes L with the total ordering \prec' , then $\theta' = \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L'\}$ is an admissible array equivalent to θ and all $i, k \in L'$ with $i \prec' k$ we have $h_V(z^i) \leq h_V(z^k)$.*

Proof. Suppose that \prec' is any total ordering of L such that if L' denotes the set L with this total ordering, then $\theta' = \{\theta_i : \mathbb{Z} \rightarrow \Sigma \mid i \in L'\}$ is an admissible array equivalent to θ . (Call such a total ordering “allowable”.) By Remark 3.5, the irreducible elements ζ^i which one obtains from the construction in Lemma 3.8 when one starts with θ' satisfy $\zeta^i = z^i$. Once we know this, we can apply Lemma 3.10 to the array θ' and conclude that if $i, k \in L'$, $i \prec' k$ and $R(\theta_i) \cap R(\theta_k)$ is nonempty, then $h_V(z^i) < h_V(z^k)$.

For each allowable ordering \prec' on L , we define an integer $N(L, \prec')$ by

$$N(L, \prec') = \left| \{(i, j) \in L \times L : i \prec' j \text{ and } h_V(z^i) \leq h_V(z^j)\} \right|. \tag{3.63}$$

If there exists an allowable total ordering \prec' for which $h_V(z^i) \leq h_V(z^k)$ for all $(i, j) \in L \times L$ with $i \prec' j$, we are done. Thus we assume, by way of contradiction, that such a total ordering does not exist; and we select a particular total ordering \prec' as above such that

$$N(L, \prec') = \max \{N(L, \prec'') \mid \prec'' \text{ is an allowable total ordering on } L\}.$$

By assumption, there exist $i, j \in L$ with $i \prec' j$ and $h_V(z^i) > h_V(z^j)$. There exist elements $i := i_1 \prec' i_2 \prec' i_3 \prec' \dots \prec' i_k := j$ in L such that all other elements λ of L satisfy $\lambda \prec' i_1$ or $i_k \prec' \lambda$. There must exist s , $1 \leq s \leq k$, with $h_V(z^{i_s}) > h_V(z^{i_{s+1}})$, or we contradict $h_V(z^i) > h_V(z^j)$. Lemma 3.10 and the remarks at the beginning of the proof imply that $R(\theta_{i_s}) \cap R(\theta_{i_{s+1}})$ is empty. We define a new total ordering

\prec'' on L by defining $i_{s+1} \prec'' i_s$ and $\lambda \prec'' \mu$ if and only if $\lambda \prec' \mu$ unless $\lambda = i_s$ and $\mu = i_{s+1}$. The definition of equivalent admissible arrays implies that \prec'' is allowable, and eq. (3.63) implies

$$N(L, \prec'') = N(L, \prec') + 1.$$

This contradicts the maximality of $N(L, \prec')$ and completes the proof. □

Lemmas 3.1–3.11 provide a complete proof of Theorem 3.1 in the case that $p > 1$, where p is the period of the minimal admissible array θ . However, if $p = 1$, $y := y^0$ is a fixed point of $f = M(\theta)$, $V = \{y^0\}$, $z^1 = y$, and the theorem is trivial.

4. Consequences of the main theorem and open questions

Instead of considering l_1 -norm nonexpansive maps $f : K^n \rightarrow K^n$ with $f(0) = 0$ and defining $P_3(n)$, we can consider l_1 -norm nonexpansive maps $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and define $R(n)$, an analogue of $P_3(n)$:

$$R(n) := \{p \in \mathbb{N} \mid \exists f : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ such that } f \text{ is } l_1\text{-norm nonexpansive} \\ \text{and } f \text{ has a periodic point of minimal period } p\}.$$

The following theorem is essentially proved in Example 3, Section 1 of [14] and is closely related to an observation in [18].

Theorem 4.1. *If $R(n)$ is defined as above and $Q(n)$ by eq. (1.6), we have*

$$R(n) \subset Q(2n).$$

Proof. It is proved in [14] that $R(n) \subset P_3(2n)$, and Theorem 3.1 implies that $P_3(2n) = Q(2n)$. □

It seems unlikely that $R(n) = Q(2n)$ in general. For example, if $n = 3$, it is proved in [16] that

$$Q(6) = \{j \in \mathbb{N} \mid 1 \leq j \leq 6\} \cup \{12\},$$

and one can prove that $\{j \in \mathbb{N} \mid 1 \leq j \leq 6\} \subset R(3)$. However, we conjecture that $12 \notin R(3)$.

Question 4.1. Can one characterize $R(n)$ precisely by number theoretic and combinatorial constraints analogous to Theorem 3.1?

Remark 4.1. One can ask a question which is superficially related to Question 4.1 but is actually very different. Consider maps $f : D_f \rightarrow D_f \subset \mathbb{R}^n$ such that f is l_1 -norm nonexpansive. Note that even if D_f is finite, such a map may not have an

l_1 -norm nonexpansive extension $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Define $\tilde{R}(n)$ to be the set of positive integers p such that there exists an l_1 -norm nonexpansive map $f : D_f \rightarrow D_f$ which has a periodic point of minimal period p . For $n \geq 3$, one expects $\tilde{R}(n)$ to be strictly larger than $R(n)$. Can one characterize $\tilde{R}(n)$ precisely by number theoretical and combinatorial constraints? An upper bound of $n!2^{2^n}$ for elements of $\tilde{R}(n)$ has been obtained by Misiurewicz [8], but this estimate is probably far from sharp.

Our next theorem is a combination of Theorem 1.1 of [11] and Theorem 3.1.

Theorem 4.2. *Let $\|\cdot\|$ be a strictly monotonic norm on \mathbb{R}^n and suppose that $f : K^n \rightarrow K^n$ is order-preserving, nonexpansive with respect to $\|\cdot\|$ and satisfies $f(0) = 0$. For every $x \in K^n$ there exists a periodic point $\eta_x \in K^n$ of f of minimal period p_x and*

$$\lim_{k \rightarrow \infty} f^{kp_x}(x) = \eta_x.$$

The integer p_x satisfies $p_x \in Q(n)$.

Proof. The existence of η_x and p_x is proved in Theorem 1.1 of [11]. Writing $y = \eta_x$ and $p = p_x$, y is a periodic point of f of minimal period p ; and if $A = \{f^j(y) : 0 \leq j < p\}$ and V denotes the lattice generated by A , then Proposition 2.1 implies that $f|_V$ is a lattice homomorphism of V onto V and $f^p(x) = x$ for all $x \in V$. It follows that $p \in Q_2(n) = Q(n)$. \square

For a fixed strictly monotonic norm $\|\cdot\|$, it is unknown exactly what integers p_x can arise for maps f satisfying the conditions of Theorem 4.2.

Question 4.2. For a fixed strictly monotonic norm $\|\cdot\|$ on \mathbb{R}^n , let $\mathcal{H}(K^n, \|\cdot\|)$ denote the set of maps $f : K^n \rightarrow K^n$ such that $f(0) = 0$, f is order-preserving and f is nonexpansive with respect to $\|\cdot\|$. Let $T_1(n) = \{p \in \mathbb{N} \mid \exists f \in \mathcal{H}(K^n, \|\cdot\|) \text{ and a periodic point } y \text{ of } f \text{ of minimal period } p\}$, so $T_1(n) \subset Q(n)$. Can one characterize $T_1(n)$ in terms of number theoretic and combinatorial constraints?

For each $n \geq 1$, let $S(n)$ be a set of positive integers and assume that $1 \in S(1)$. An important role in [16] is played by collections $\{S(n) : n \geq 1\}$ which satisfy so-called ‘‘rule A’’ and ‘‘rule B’’, as defined in Section 8 of [16].

Definition 4.1. We shall say that $\{S(n) \mid n \geq 1\}$ satisfies rule A if for all positive integers m_1 and m_2 and all integers $p_j \in S(m_j)$, $j = 1, 2$, it is true that $\text{lcm}(p_1, p_2) \in S(m_1 + m_2)$.

Because we assume that $1 \in S(1)$, rule A implies that $S(n) \subset S(n + 1)$ for all $n \geq 1$.

Definition 4.2. We shall say that $\{S(n) \mid n \geq 1\}$ satisfies rule B if whenever m and r are positive integers and $p_j \in S(m)$ for $1 \leq j \leq r$, then $\text{rlcm}(p_1, p_2, \dots, p_r) \in S(rm)$.

One can define a smallest possible collection of positive integers $\{P(n) \mid n \geq 1\}$ such that $P(1) = \{1\}$ and $\{P(n) \mid n \geq 1\}$ satisfies rule A and rule B.

Definition 4.3. We define inductively a collection of positive integers $P(n)$ for $n \geq 1$ by $P(1) = \{1\}$ and, for $n > 1$, $p \in P(n)$ if and only if either

- (A) $p = \text{lcm}(p_1, p_2)$, where $p_j \in P(n_j)$ for $j = 1, 2$, and $n = n_1 + n_2$ or
- (B) $p = r\text{lcm}(p_1, p_2, \dots, p_r)$, where $r > 1$, $m \geq 1$, $p_i \in P(m)$ for $1 \leq i \leq r$ and $n = rm$.

One easily checks that $\{P(n) \mid n \geq 1\}$ as defined in Definition 4.3 does satisfy rules A and B. One can also easily prove that $P(n)$ contains the set of all positive orders of the elements of the symmetric group on n letters, i.e.,

$$P(n) \supset \left\{ \text{lcm}(m_1, m_2, \dots, m_t) \mid m_i \geq 1, \sum_{i=1}^t m_i = n, t \geq 1 \right\}, \quad (4.1)$$

but in general $P(n)$ is significantly larger than the right hand side of (4.1), eg. $12 \in P(6)$.

The significance of rules A and B for our questions is indicated by the following result, which is essentially proved in Section 3 of [10] and Section 8 of [16].

Corollary 4.1. *The collection $\{P_1(n) \mid n \geq 1\}$ satisfies rule A and rule B. The collection $\{Q(n) \mid n \geq 1\}$ satisfies rule A and rule B.*

Proof. The fact that $\{P_1(n) \mid n \geq 1\}$ satisfies rule A and rule B is proved in Section 3 of [10] (though stated in different terminology). The fact that $\{P_2(n) \mid n \geq 1\}$ satisfies rule A and rule B is proved in Section 8 of [16], and it follows from Theorem 3.1 that $\{Q(n) \mid n \geq 1\}$ satisfies rule A and rule B. \square

One can prove directly, without the aid of Theorem 3.1, that $\{Q(n) \mid n \geq 1\}$ satisfies rule A and rule B, but the proof is not trivial. It is not obvious how to prove geometrically, without the use of Theorem 3.1, that $\{P_3(n) \mid n \geq 1\}$ satisfies rule A and rule B.

It follows from Corollary 4.1 that $P(n) \subset P_1(n) \subset Q(n)$ for all $n \geq 1$, and indeed it is proved in Theorem 6.3 of [16] that $P(n) = Q(n)$ for $1 \leq n \leq 50$. Thus we have

$$P(n) = P_1(n) = P_2(n) = P_3(n) = Q_1(n) = Q_2(n) = Q(n), \quad 1 \leq n \leq 50. \quad (4.2)$$

One might conjecture from eq. (4.2) that $P(n) = Q(n)$ for all $n \geq 1$, but it is proved in Theorem 7.10 of [16] that this is false and in fact almost certainly $P(n) \neq Q(n)$ for infinitely many n . A special case of Theorem 7.10 of [16] is the following result.

Theorem 4.3. *Suppose that λ_4 and $\lambda_1 := \lambda_4 + 2$ are prime numbers with $\lambda_4 \geq 11$ and $\lambda_4 \neq 41$. Then it follows that*

$$q := 2^3 \times 7^2 \times \lambda_1 \times \lambda_4 \in Q(56 + 2\lambda_4) \quad \text{and} \quad q \in P(57 + 2\lambda_4)$$

but $q \notin P(56 + 2\lambda_4)$. In particular, $q = 56056 \in Q(78)$ and $q \in P(79)$, but $q \notin P(78)$.

The results of Section 7 in [16] leave open the question whether $P_1(n) = Q(n)$ for all $n \geq 1$.

Question 4.3. Is it true that $P_1(n) = Q(n)$ for all $n \geq 1$? If not, can one describe $P_1(n)$ precisely in terms of number theoretical and combinatorial constraints?

Appendix Numerical data

If $p \in P_2(n)$ and $q|p$ it is observed in [16] that $q \in P_2(n)$, and since $P_2(n) = Q(n)$, the same property holds for $Q(n)$. We define an element $p \in Q(n)$ to be “maximal” if $mp \notin Q(n)$ for all $m > 1$, $m \in \mathbb{N}$. It follows easily that $Q(n)$ is the set of all divisors of maximal elements of $Q(n)$.

The results in the following tables are all obtained in [16], and the reader is directed to [16] for further details. Note that the following table shows that $Q(n)$ is unequal to the set of orders of elements of the symmetric group on n letters for $n = 6$ and $8 \leq n \leq 50$. A later paper by S.V.L and R.D.N. will prove that $Q(n)$ is unequal to this set of orders for all $n \geq 8$.

n	maximal elements of $Q(n)$
1	[1]
2	[2]
3	[2, 3]
4	[3, 4]
5	[4, 5, 6]
6	[5, 12]
7	[7, 10, 12]
8	[7, 10, 15, 24]
9	[14, 15, 18, 20, 24]
10	[14, 18, 21, 24, 40, 60]
11	[11, 18, 21, 24, 28, 40, 60]
12	[11, 28, 35, 36, 42, 120]
13	[13, 22, 35, 36, 84, 120]
14	[13, 22, 33, 36, 90, 120, 140, 168]
15	[26, 33, 44, 105, 120, 140, 168, 180]
16	[26, 39, 44, 55, 66, 126, 140, 180, 210, 240, 336]
17	[17, 39, 52, 55, 72, 126, 132, 180, 240, 280, 336, 420]
18	[17, 52, 65, 77, 78, 110, 132, 144, 240, 252, 280, 336, 360, 420]
19	[19, 34, 65, 77, 110, 144, 156, 165, 240, 252, 264, 336, 360, 840]
20	[19, 34, 51, 91, 130, 154, 156, 165, 198, 220, 252, 264, 720, 1680]
21	[38, 51, 68, 91, 130, 154, 195, 198, 231, 264, 312, 440, 660, 720, 1260, 1680]

- 22 [38, 57, 68, 85, 102, 182, 195, 234, 260, 312, 396, 462, 504, 528, 616, 720, 880, 1260, 1320, 1680]
- 23 [23, 57, 76, 85, 182, 204, 234, 273, 312, 385, 396, 462, 504, 520, 528, 616, 720, 780, 880, 1260, 1320, 1680]
- 24 [23, 76, 95, 114, 119, 143, 170, 204, 234, 273, 312, 364, 520, 616, 720, 770, 780, 792, 924, 1680, 2520, 2640]
- 25 [46, 95, 119, 143, 170, 228, 255, 300, 364, 408, 455, 468, 546, 720, 792, 990, 1008, 1540, 1560, 1680, 1848, 2520, 2640]
- 26 [46, 69, 133, 190, 228, 238, 255, 300, 306, 340, 408, 572, 720, 792, 910, 936, 1008, 1155, 1540, 1680, 1848, 1980, 2184, 2520, 2640, 3120]
- 27 [69, 92, 133, 190, 238, 285, 300, 306, 357, 408, 429, 456, 572, 680, 792, 936, 1020, 1080, 1170, 1386, 1512, 1820, 1980, 2184, 2310, 2640, 3080, 3120, 3696, 3780, 5040]
- 28 [92, 115, 138, 187, 266, 285, 300, 306, 342, 357, 380, 408, 456, 476, 572, 680, 792, 858, 936, 1020, 1080, 1365, 1386, 1512, 1980, 2340, 2640, 3120, 3640, 3696, 3780, 4368, 5040, 9240]
- 29 [29, 115, 187, 266, 276, 300, 342, 399, 456, 476, 595, 612, 714, 715, 760, 936, 1080, 1140, 1512, 1584, 1638, 1716, 2040, 2340, 2640, 2730, 2772, 3120, 3640, 3696, 3780, 3960, 4368, 5040, 6160, 9240]
- 30 [29, 161, 209, 221, 230, 276, 300, 342, 374, 399, 456, 532, 595, 612, 715, 760, 1140, 1144, 1428, 1584, 1638, 1716, 2040, 2640, 2772, 3120, 3640, 3696, 3960, 4368, 4680, 5040, 5460, 6160, 7560, 9240]
- 31 [31, 58, 161, 209, 221, 230, 300, 345, 374, 532, 552, 561, 612, 665, 684, 798, 1001, 1144, 1530, 1716, 1872, 2040, 2280, 2380, 2856, 2860, 3120, 3276, 4368, 4680, 5040, 5544, 7560, 7920, 10920, 18480]
- 32 [31, 58, 87, 247, 322, 345, 414, 418, 442, 460, 552, 561, 665, 684, 748, 1001, 1596, 1785, 1872, 2040, 2100, 2280, 2380, 2856, 2860, 3060, 3276, 3432, 4290, 4680, 5544, 6240, 7560, 7920, 8736, 10080, 10920, 13860, 36960]
- 33 [62, 87, 116, 247, 322, 414, 418, 442, 483, 552, 600, 627, 663, 684, 748, 920, 935, 1122, 1380, 1710, 2100, 2142, 2280, 2380, 2574, 2660, 2860, 3060, 3192, 3432, 3570, 4004, 4080, 4290, 5712, 6240, 6552, 7560, 7920, 8316, 8736, 9360, 10080, 11880, 21840, 27720, 36960]
- 34 [62, 93, 116, 145, 174, 253, 414, 483, 494, 552, 600, 627, 644, 836, 900, 920, 1326, 1380, 1768, 1870, 1995, 2100, 2280, 2448, 2574, 2660, 3003, 3192, 3420, 3432, 4004, 4284, 4488, 5720, 6120, 6240, 6552, 7560, 7920, 8160, 8316, 8580, 8736, 9360, 9520, 10080, 11424, 11880, 14280, 16380, 21840, 27720, 36960]
- 35 [93, 124, 145, 253, 348, 494, 600, 644, 741, 805, 828, 836, 900, 966, 1045, 1105, 1254, 1309, 1326, 1768, 1870, 2100, 2394, 2448, 2660, 2760, 3420, 3990, 4284, 4488, 4560, 5148, 6006, 6120, 6240, 6384, 6552, 6864, 7560, 7920, 8008, 8160, 8316, 8736, 9360, 9520, 10080, 11424, 11440, 11880, 14280, 16380, 17160, 21840, 27720, 36960]

- 36 [124, 155, 186, 203, 290, 299, 323, 348, 506, 600, 741, 805, 828, 900, 988, 1045, 1368, 1768, 1932, 2100, 2210, 2394, 2508, 2618, 2652, 2760, 2805, 3300, 3420, 3740, 4488, 4560, 4896, 5148, 5320, 6384, 6864, 7560, 7980, 8008, 8160, 8568, 9520, 10010, 10080, 11424, 11440, 11880, 12012, 12240, 14280, 15840, 17160, 18720, 22176, 32760, 36960, 41580, 43680, 55440]
- 37 [37, 155, 203, 290, 299, 323, 372, 435, 506, 600, 696, 759, 828, 900, 988, 1235, 1463, 1482, 1547, 2070, 2090, 2100, 2210, 2508, 2618, 2736, 2760, 3220, 3300, 3366, 3740, 3864, 4488, 4560, 4788, 4896, 5304, 5320, 5610, 6384, 6840, 7560, 7980, 8160, 8568, 10010, 10080, 10296, 11424, 11880, 12240, 13104, 15840, 18720, 22176, 24024, 28560, 32760, 34320, 36960, 41580, 43680, 55440]
- 38 [37, 217, 310, 372, 406, 435, 522, 580, 598, 600, 696, 759, 900, 1012, 1292, 1547, 2100, 2415, 2470, 2760, 2926, 3220, 3300, 3315, 3366, 3864, 3900, 3927, 4140, 4180, 4420, 4896, 5236, 5304, 5472, 5928, 6270, 7480, 7560, 8160, 8568, 8976, 9120, 9576, 10032, 10080, 10296, 11220, 11424, 11880, 12240, 12768, 12870, 13104, 13680, 15840, 16632, 18720, 20020, 21420, 22176, 24024, 28560, 31920, 32760, 34320, 36960, 41580, 43680, 55440]
- 39 [74, 217, 310, 406, 465, 522, 598, 609, 696, 744, 897, 900, 969, 1012, 1160, 1265, 1292, 1518, 1729, 1740, 2470, 2898, 2926, 3094, 3220, 3300, 3762, 3900, 3978, 4140, 4180, 4200, 4830, 4896, 5304, 5472, 5520, 5928, 6270, 6630, 6732, 7560, 7728, 7854, 8568, 8840, 8976, 9120, 9576, 10032, 10080, 10472, 11880, 12240, 12768, 13104, 13680, 14040, 14960, 15444, 15840, 16632, 18720, 21420, 22176, 22440, 30030, 31920, 32760, 34320, 36036, 36960, 40040, 41580, 43680, 48048, 49140, 51480, 55440, 57120]
- 40 [74, 111, 319, 391, 434, 465, 522, 558, 609, 620, 696, 744, 812, 897, 900, 1160, 1196, 1265, 1292, 1656, 1740, 1938, 2898, 3036, 3300, 3458, 3705, 3900, 3978, 4140, 4200, 4389, 4940, 5304, 5520, 5852, 5928, 6188, 6440, 6545, 6732, 7524, 7728, 8360, 8568, 8840, 8976, 9282, 9576, 9660, 10032, 10472, 12540, 13260, 14040, 14960, 15444, 15708, 19656, 21420, 22440, 23940, 24480, 27360, 30030, 34320, 36036, 40040, 48048, 49140, 51480, 57120, 63840, 83160, 110880, 131040]
- 41 [41, 111, 148, 319, 391, 434, 558, 651, 744, 812, 1015, 1044, 1196, 1218, 1240, 1495, 1615, 1771, 1794, 1860, 2431, 2530, 3036, 3300, 3312, 3458, 3480, 3876, 3900, 3978, 4446, 4940, 5304, 5520, 5796, 5928, 6300, 6440, 7410, 7524, 7728, 8280, 8400, 8778, 8840, 9282, 9576, 9660, 10032, 11704, 12376, 13090, 13260, 13464, 14040, 15444, 16720, 17136, 19656, 23940, 24480, 25080, 27360, 31416, 34320, 36036, 42840, 44880, 48048, 49140, 51480, 57120, 63840, 83160, 110880, 120120, 131040]

- 42 [41, 148, 185, 222, 341, 377, 437, 558, 638, 651, 744, 782, 868, 1015, 1044, 1240, 1495, 1615, 1771, 1800, 1860, 2431, 2436, 2530, 2584, 3300, 3312, 3480, 3588, 3795, 3876, 3900, 4446, 5100, 5187, 5520, 5796, 6072, 6300, 6916, 7315, 7524, 7728, 7735, 7956, 8280, 8400, 9282, 9880, 10032, 10608, 11704, 11856, 12376, 13464, 14820, 15444, 16720, 16830, 17136, 17556, 19320, 20592, 24480, 25080, 26180, 26520, 27360, 31416, 34320, 38610, 42840, 44880, 47880, 48048, 51480, 57120, 63840, 72072, 80080, 83160, 98280, 110880, 120120, 131040, 180180]
- 43 [43, 82, 185, 341, 377, 437, 444, 638, 782, 868, 957, 1044, 1085, 1116, 1173, 1302, 2093, 2261, 2584, 2610, 2717, 2990, 3480, 3542, 3588, 3600, 3720, 3795, 3876, 3900, 4060, 4446, 4554, 4872, 5060, 5100, 5796, 6072, 6300, 6460, 6916, 8400, 9724, 10374, 11704, 11856, 13464, 14630, 15048, 15120, 15444, 15470, 15912, 16560, 17556, 19635, 20592, 23100, 24480, 26180, 27360, 29640, 31416, 33660, 34272, 37128, 38610, 38640, 44880, 47880, 50160, 51480, 53040, 57120, 63840, 68640, 72072, 80080, 83160, 85680, 96096, 98280, 110880, 120120, 131040, 180180]
- 44 [43, 82, 123, 259, 370, 403, 444, 682, 754, 874, 957, 1085, 1116, 1173, 1276, 1564, 2093, 2261, 2604, 2717, 2990, 3045, 3480, 3542, 3600, 3720, 3900, 4060, 4485, 4554, 4872, 5100, 5220, 5313, 5700, 6072, 6300, 6460, 6600, 7176, 7293, 7752, 8400, 8645, 8892, 9690, 9724, 10120, 11856, 15048, 15120, 15180, 15912, 16560, 18360, 18810, 19152, 20748, 23100, 23562, 24480, 25704, 27360, 28980, 29260, 29640, 30940, 34272, 37128, 38640, 39780, 47880, 53040, 57120, 62832, 63840, 64260, 67320, 70224, 77220, 78540, 83160, 85680, 89760, 96096, 98280, 100320, 104720, 110880, 131040, 144144, 160160, 180180, 205920, 240240]
- 45 [86, 123, 164, 259, 370, 403, 555, 682, 754, 874, 888, 1023, 1116, 1131, 1276, 1311, 1564, 1595, 1914, 1955, 2346, 2790, 3600, 3654, 3720, 4060, 4186, 4340, 4485, 5100, 5208, 5220, 5382, 5700, 5814, 5980, 6090, 6460, 6600, 6960, 7176, 7752, 8400, 9044, 9108, 9690, 9744, 9900, 10626, 10868, 11592, 12144, 12600, 14168, 14586, 15048, 15120, 15912, 16560, 17290, 17784, 18360, 19448, 20240, 23100, 24480, 25704, 26928, 27300, 27360, 28980, 29260, 30360, 34272, 37620, 38304, 38640, 39780, 41496, 43890, 46410, 47124, 53040, 57120, 59280, 61880, 63840, 64260, 67320, 70224, 74256, 77220, 83160, 85680, 89760, 95760, 98280, 100320, 104720, 110880, 125664, 131040, 144144, 157080, 180180, 205920, 480480]
- 46 [86, 129, 164, 205, 246, 493, 518, 555, 666, 740, 806, 888, 1023, 1131, 1364, 1508, 1595, 2088, 2622, 3255, 3496, 3600, 3654, 3720, 3828, 3910, 4340, 5100, 5208, 5220, 5580, 5700, 5814, 6600, 6783, 6960, 7752, 7800, 8120, 8151, 8372, 8400, 9044, 9384, 9744, 9900, 10764, 10868, 12155, 12180, 12558, 12600, 12920, 14352, 15048, 15120, 15912, 17710, 17784, 18216, 18360, 19380, 19448, 20520, 21252, 22230, 23100, 23184, 23920, 24288, 24480, 25704, 26334, 26928, 27300, 27360, 27846,

- 28336, 28728, 29172, 33120, 34272, 34580, 35880, 37620, 38304, 39780, 40480, 41496, 43890, 46410, 47124, 53040, 57120, 57960, 58520, 59280, 60720, 61880, 63840, 64260, 67320, 70224, 71820, 74256, 77280, 83160, 85680, 89760, 95760, 98280, 100320, 104720, 108108, 110880, 125664, 131040, 144144, 154440, 157080, 205920, 360360, 480480]
- 47 [47, 129, 172, 205, 492, 493, 518, 666, 777, 806, 888, 1209, 1364, 1480, 1508, 1705, 1885, 2046, 2185, 2220, 2233, 2262, 2622, 2737, 3190, 3289, 3496, 3553, 3600, 3828, 3906, 3910, 4176, 4340, 5100, 5580, 5700, 6510, 6960, 7308, 7440, 7752, 7800, 8120, 8372, 8400, 9044, 9384, 9744, 9900, 10416, 10440, 10764, 10868, 11628, 11700, 12180, 12558, 12600, 12920, 13200, 13566, 14352, 15120, 16302, 17710, 17784, 18216, 19380, 20520, 21252, 23184, 23920, 24288, 24310, 25935, 27300, 28336, 28728, 30096, 31824, 33120, 35880, 37620, 40480, 44460, 46200, 52668, 53856, 55692, 57960, 58344, 59280, 60720, 69160, 70224, 71820, 77280, 79560, 82992, 83160, 89760, 94248, 98280, 100320, 106080, 108108, 110880, 123760, 125664, 128520, 131040, 134640, 144144, 148512, 154440, 171360, 175560, 185640, 191520, 205920, 314160, 360360, 480480]
- 48 [47, 172, 215, 258, 287, 407, 410, 492, 527, 551, 666, 777, 888, 986, 1036, 1209, 1480, 1612, 1705, 1885, 2220, 2232, 2233, 3190, 3496, 3553, 3600, 3906, 4092, 4176, 4370, 4524, 4785, 5100, 5244, 5474, 5580, 5700, 5865, 6578, 6900, 6960, 7308, 7440, 7656, 7800, 7820, 8400, 8680, 9384, 9744, 9900, 10416, 10440, 10465, 10764, 11628, 11700, 12558, 12600, 13020, 13200, 14352, 16744, 17784, 18088, 20520, 21736, 22610, 22770, 23920, 24360, 27132, 27170, 27300, 28336, 28728, 31122, 31824, 32604, 33120, 34034, 35420, 35880, 36432, 38760, 42504, 44460, 46200, 48620, 51870, 52668, 53856, 55692, 58344, 59280, 69160, 70224, 71820, 77280, 79560, 82992, 100320, 106080, 108108, 110880, 115920, 117040, 121440, 123760, 128520, 131040, 134640, 148512, 150480, 154440, 166320, 171360, 175560, 185640, 188496, 191520, 196560, 205920, 270270, 480480, 628320, 720720]
- 49 [94, 215, 287, 407, 410, 516, 527, 551, 615, 984, 986, 1036, 1295, 1332, 1479, 1554, 1612, 2015, 2387, 2418, 2639, 2940, 3059, 3410, 3600, 3770, 4092, 4199, 4370, 4440, 4464, 4466, 4524, 4785, 5474, 5700, 5742, 6380, 6900, 7038, 7308, 7440, 7656, 7800, 7812, 7820, 8400, 8680, 9384, 10416, 10488, 11160, 11628, 11700, 11730, 12600, 13020, 13156, 13200, 14212, 18088, 19800, 20880, 20930, 21528, 22610, 27132, 27170, 27300, 31122, 33120, 34034, 35420, 35700, 36432, 38760, 42900, 45540, 46200, 48620, 48720, 50232, 53130, 53856, 58344, 59280, 60192, 63648, 65208, 69160, 71760, 72930, 77280, 82992, 85008, 88920, 100320, 103740, 105336, 106080, 110880, 111384, 115920, 121440, 123760, 128520, 131040, 134640, 140448, 143640, 148512, 150480, 154440,

159120, 166320, 171360, 185640, 188496, 191520, 196560, 205920,
 235620, 288288, 351120, 480480, 540540, 628320, 720720]
 50 [94, 141, 301, 430, 481, 516, 574, 589, 615, 738, 814, 820, 984, 1054,
 1102, 1295, 1332, 1479, 1972, 2015, 2387, 2639, 2940, 3059, 3108,
 3410, 3770, 4199, 4440, 4464, 4466, 4836, 5115, 5655, 5742, 6555,
 6699, 7038, 7440, 7656, 7812, 8184, 8211, 8740, 9048, 9867, 10416,
 10488, 10659, 10948, 11160, 11400, 11628, 12760, 13156, 13800, 14212,
 15600, 15640, 18768, 19019, 19140, 20400, 20880, 21528, 23400, 23460,
 24840, 26040, 26400, 26910, 29070, 31878, 33120, 34776, 35568, 36432,
 36540, 39600, 41860, 43758, 45220, 45540, 46368, 48720, 50232, 50400,
 53130, 53856, 54264, 54340, 54600, 59280, 60192, 62244, 63648, 65208,
 68068, 70840, 71400, 71760, 72930, 77280, 77520, 82992, 85008, 85800,
 86940, 88920, 92400, 97240, 100320, 105336, 106080, 110880, 111384,
 115920, 116688, 121440, 128520, 131040, 134640, 140448, 141372,
 143640, 148512, 150480, 154440, 159120, 166320, 171360, 188496,
 191520, 196560, 201960, 205920, 207480, 288288, 351120, 371280,
 471240, 480480, 540540, 628320, 720720]

The following list contains the factorizations of the maximal elements for $Q(42)$.

$Q[42] = [41, 2^2 \times 37, 5 \times 37, 2 \times 3 \times 37, 11 \times 31, 13 \times 29, 19 \times 23, 2 \times 3^2 \times 31,$
 $2 \times 11 \times 29, 3 \times 7 \times 31, 2^3 \times 3 \times 31, 2 \times 17 \times 23, 2^2 \times 7 \times 31, 5 \times 7 \times 29,$
 $2^2 \times 3^2 \times 29, 2^3 \times 5 \times 31, 5 \times 13 \times 23, 5 \times 17 \times 19, 7 \times 11 \times 23,$
 $2^3 \times 3^2 \times 5^2, 2^2 \times 3 \times 5 \times 31, 11 \times 13 \times 17, 2^2 \times 3 \times 7 \times 29,$
 $2 \times 5 \times 11 \times 23, 2^3 \times 17 \times 19, 2^2 \times 3 \times 5^2 \times 11, 2^4 \times 3^2 \times 23,$
 $2^3 \times 3 \times 5 \times 29, 2^2 \times 3 \times 13 \times 23, 3 \times 5 \times 11 \times 23, 2^2 \times 3 \times 17 \times 19,$
 $2^2 \times 3 \times 5^2 \times 13, 2 \times 3^2 \times 13 \times 19, 2^2 \times 3 \times 5^2 \times 17, 3 \times 7 \times 13 \times 19,$
 $2^4 \times 3 \times 5 \times 23, 2^2 \times 3^2 \times 7 \times 23, 2^3 \times 3 \times 11 \times 23, 2^2 \times 3^2 \times 5^2 \times 7,$
 $2^2 \times 7 \times 13 \times 19, 5 \times 7 \times 11 \times 19, 2^2 \times 3^2 \times 11 \times 19, 2^4 \times 3 \times 7 \times 23,$
 $5 \times 7 \times 13 \times 17, 2^2 \times 3^2 \times 13 \times 17, 2^3 \times 3^2 \times 5 \times 23, 2^4 \times 3 \times 5^2 \times 7,$
 $2 \times 3 \times 7 \times 13 \times 17, 2^3 \times 5 \times 13 \times 19, 2^4 \times 3 \times 11 \times 19, 2^4 \times 3 \times 13 \times 17,$
 $2^3 \times 7 \times 11 \times 19, 2^4 \times 3 \times 13 \times 19, 2^3 \times 7 \times 13 \times 17, 2^3 \times 3^2 \times 11 \times 17,$
 $2^2 \times 3 \times 5 \times 13 \times 19, 2^2 \times 3^3 \times 11 \times 13, 2^4 \times 5 \times 11 \times 19,$
 $2 \times 3^2 \times 5 \times 11 \times 17, 2^4 \times 3^2 \times 7 \times 17, 2^2 \times 3 \times 7 \times 11 \times 19,$
 $2^3 \times 3 \times 5 \times 7 \times 23, 2^4 \times 3^2 \times 11 \times 13, 2^5 \times 3^2 \times 5 \times 17,$
 $2^3 \times 3 \times 5 \times 11 \times 19, 2^2 \times 5 \times 7 \times 11 \times 17, 2^3 \times 3 \times 5 \times 13 \times 17,$
 $2^5 \times 3^2 \times 5 \times 19, 2^3 \times 3 \times 7 \times 11 \times 17, 2^4 \times 3 \times 5 \times 11 \times 13,$
 $2 \times 3^3 \times 5 \times 11 \times 13, 2^3 \times 3^2 \times 5 \times 7 \times 17, 2^4 \times 3 \times 5 \times 11 \times 17,$
 $2^3 \times 3^2 \times 5 \times 7 \times 19, 2^4 \times 3 \times 7 \times 11 \times 13, 2^3 \times 3^2 \times 5 \times 11 \times 13,$
 $2^5 \times 3 \times 5 \times 7 \times 17, 2^5 \times 3 \times 5 \times 7 \times 19, 2^3 \times 3^2 \times 7 \times 11 \times 13,$
 $2^4 \times 5 \times 7 \times 11 \times 13, 2^3 \times 3^3 \times 5 \times 7 \times 11, 2^3 \times 3^3 \times 5 \times 713,$
 $2^5 \times 3^2 \times 5 \times 7 \times 11, 2^3 \times 3 \times 5 \times 7 \times 11 \times 13, 2^5 \times 3^2 \times 5 \times 7 \times 13,$
 $2^2 \times 3^2 \times 5 \times 7 \times 11 \times 13]$

We conclude with a table containing the factorization of the largest element of $Q(n)$ for $1 \leq n \leq 50$.

n	largest element of $Q(n)$	n	largest element of $Q(n)$
1	1	26	$2^4 \times 3 \times 5 \times 13$
2	2	27	$2^4 \times 3^2 \times 5 \times 7$
3	3	28	$2^3 \times 3 \times 5 \times 7 \times 11$
4	2^2	29	$2^3 \times 3 \times 5 \times 7 \times 11$
5	2×3	30	$2^3 \times 3 \times 5 \times 7 \times 11$
6	$2^2 \times 3$	31	$2^4 \times 3 \times 5 \times 7 \times 11$
7	$2^2 \times 3$	32	$2^5 \times 3 \times 5 \times 7 \times 11$
8	$2^3 \times 3$	33	$2^5 \times 3 \times 5 \times 7 \times 11$
9	$2^3 \times 3$	34	$2^5 \times 3 \times 5 \times 7 \times 11$
10	$2^2 \times 3 \times 5$	35	$2^5 \times 3 \times 5 \times 7 \times 11$
11	$2^2 \times 3 \times 5$	36	$2^4 \times 3^2 \times 5 \times 7 \times 11$
12	$2^3 \times 3 \times 5$	37	$2^4 \times 3^2 \times 5 \times 7 \times 11$
13	$2^3 \times 3 \times 5$	38	$2^4 \times 3^2 \times 5 \times 7 \times 11$
14	$2^3 \times 3 \times 7$	39	$2^5 \times 3 \times 5 \times 7 \times 17$
15	$2^2 \times 3^2 \times 5$	40	$2^5 \times 3^2 \times 5 \times 7 \times 13$
16	$2^4 \times 3 \times 7$	41	$2^5 \times 3^2 \times 5 \times 7 \times 13$
17	$2^2 \times 3 \times 5 \times 7$	42	$2^2 \times 3^2 \times 5 \times 7 \times 11 \times 13$
18	$2^2 \times 3 \times 5 \times 7$	43	$2^2 \times 3^2 \times 5 \times 7 \times 11 \times 13$
19	$2^3 \times 3 \times 5 \times 7$	44	$2^4 \times 3 \times 5 \times 7 \times 11 \times 13$
20	$2^4 \times 3 \times 5 \times 7$	45	$2^5 \times 3 \times 5 \times 7 \times 11 \times 13$
21	$2^4 \times 3 \times 5 \times 7$	46	$2^5 \times 3 \times 5 \times 7 \times 11 \times 13$
22	$2^4 \times 3 \times 5 \times 7$	47	$2^5 \times 3 \times 5 \times 7 \times 11 \times 13$
23	$2^4 \times 3 \times 5 \times 7$	48	$2^4 \times 3^2 \times 5 \times 7 \times 11 \times 13$
24	$2^4 \times 3 \times 5 \times 11$	49	$2^4 \times 3^2 \times 5 \times 7 \times 11 \times 13$
25	$2^4 \times 3 \times 5 \times 11$	50	$2^4 \times 3^2 \times 5 \times 7 \times 11 \times 13$

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