

# Denjoy–Wolff theorems, Hilbert metric nonexpansive maps and reproduction–decimation operators

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## Abstract

Let  $K$  be a closed cone with nonempty interior in a Banach space  $X$ . Suppose that  $f : \text{int } K \rightarrow \text{int } K$  is order-preserving and homogeneous of degree one. Let  $q : K \rightarrow [0, \infty)$  be a continuous, homogeneous of degree one map such that  $q(x) > 0$  for all  $x \in K \setminus \{0\}$ . Let  $T(x) = f(x)/q(f(x))$ . We give conditions on the cone  $K$  and the map  $f$  which imply that there is a convex subset of  $\partial K$  which contains the omega limit set  $\omega(x; T)$  for every  $x \in \text{int } K$ . We show that these conditions are satisfied by reproduction–decimation operators. We also prove that  $\omega(x; T) \subset \partial K$  for a class of operator-valued means.

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## 1. Introduction

Let  $D$  be a bounded, open convex set in a finite-dimensional Banach space  $X$  and let  $d$  denote Hilbert’s projective metric on  $D$ . (See Section 2 for a definition of  $d$ .) Let  $f : D \rightarrow D$  be a nonexpansive map in the metric  $d$  (so  $d(f(x), f(y)) \leq d(x, y)$  for all  $x, y \in D$ ) and assume that  $f$  has no fixed points in  $D$ . In this situation, A.F. Beardon [4] has proved the following theorem, which is the starting point for this paper.

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**Theorem 1.1.** (Beardon [4].) *Assume, in addition, that  $D$  is strictly convex (so  $\partial D$  contains no line segments). Then there exists a  $z \in \partial D$  such that  $\lim_{k \rightarrow \infty} f^k(x) = z$  for all  $x \in D$ .*

In Theorem 1.1 and elsewhere in this paper,  $f^k$  will always denote the composition of  $f$  with itself  $k$  times. Note that Theorem 1.1 is a direct analogue of the classical Denjoy–Wolff theorem for analytic maps of the unit disc in  $\mathbb{C}$  into itself. If  $D$  need not be strictly convex, one must consider  $\omega(x; f)$ , the omega limit set of  $x$  under  $f$ , the collection of limit points of the sequence  $\langle f^k(x) \mid k \geq 1 \rangle$ . A result of Nussbaum [16] proves that  $\omega(x; f) \subset \partial D$ , and Karlsson and Noskov [8] have proved

**Theorem 1.2.** (Karlsson and Noskov [8].) *For every  $x \in D$ , there exists  $z \in \omega(x; f) \subset \partial D$  such that  $(1-t)z + ty \in \partial D$  for all  $y \in \omega(x; f)$  and  $0 \leq t \leq 1$ .*

The Karlsson–Noskov result is very far from optimal, but it can be used to derive Beardon’s theorem. Infinite-dimensional generalizations of Theorem 1.2 are given in [19].

If  $K$  is a closed cone with nonempty interior  $\text{int } K$  in a Banach space  $X$ , one can define (see Section 2) Hilbert’s projective metric  $d$  on  $\text{int } K$ . It is well known that if  $f : \text{int } K \rightarrow \text{int } K$  is homogeneous of degree one and order-preserving with respect to  $K$ , it is nonexpansive in the projective metric  $d$ . If  $q : K \rightarrow [0, \infty)$  is a continuous function which is homogeneous of degree one and strictly positive on  $K \setminus \{0\}$ , one can define  $\Sigma = \{x \in \text{int } K \mid q(x) = 1\}$  and define  $T : \Sigma \rightarrow \Sigma$  by  $T(x) = f(x)/q(f(x))$ . The map  $T$  is also nonexpansive with respect to  $d$  and  $f$  has an eigenvector in  $\text{int } K$  if and only if  $T$  has a fixed point in  $\Sigma$ . Many problems in analysis and applications, e.g., DAD-theorems [12] and diffusions on fractals [10], are related to the question of whether  $f$  has an eigenvector in  $\text{int } K$ . However, even if  $X$  is finite-dimensional, this question may be very difficult: see [16] and [17] for some problematic classes of examples (e.g.,  $\mathcal{M}_-$ ) when  $X = \mathbb{R}^n$  and  $K = \mathbb{R}_+^n$ .

If  $T$  has a fixed point  $x_*$  in  $\Sigma$ ,  $X$  is finite-dimensional, and  $K$  is polyhedral,  $\omega(x; T)$  is well understood; it is known (see [18]) that for each  $x \in \Sigma$ ,  $\omega(x; T)$  is a periodic orbit of  $T$  and often (see [16]) equals  $\{x_*\}$ . Compare, also, results in [1]. Note that if  $X$  is finite-dimensional, we can take  $q \in X^*$ ;  $\Sigma$  is then convex and corresponds to  $D$  in Theorems 1.1 and 1.2.

If  $T$  has no fixed points in  $\Sigma$ , what can be said about  $\omega(x; T)$ ? In almost all applications in analysis,  $\Sigma$  is not strictly convex, Theorem 1.1 is not applicable and Theorem 1.2 is somewhat weak. Motivated by these deficiencies, Lins [11] has proved the following.

**Theorem 1.3.** (Lins [11].) *Let  $K$  be a polyhedral cone with nonempty interior in a finite-dimensional Banach space  $X$ . Let  $q \in X^*$  be such that  $q(x) > 0$  for all  $x \in K \setminus \{0\}$  and define  $\Sigma = \{x \in \text{int } K \mid q(x) = 1\}$ . Assume that  $T : \Sigma \rightarrow \Sigma$  is nonexpansive with respect to Hilbert’s projective metric  $d$  and  $T$  has no fixed points in  $\Sigma$ . Then there exists a closed convex set  $U \subset \text{cl } \Sigma \setminus \Sigma$  such that  $\omega(x; T) \subset U$  for all  $x \in \Sigma$ .*

Lins also shows that if  $K = \mathbb{R}_+^n$ ,  $\omega(x; T)$  may comprise an entire face of  $\text{cl } \Sigma \setminus \Sigma$ .

Prior to Lins’ theorem, Nussbaum and Karlsson independently conjectured that Theorem 1.3 is true without the assumption that  $K$  is polyhedral. As of this writing this conjecture remains open. However a resolution is important, for there are problems in analysis where the cone  $K$  is neither polyhedral nor strictly convex.

Here we give a generalization of Theorem 1.3 to include some classes of cones which are neither polyhedral nor strictly convex. Our motivation comes from so-called reproduction–

decimation operators, which arise in the theory of fractal diffusion [10]. General reproduction–decimation operators are order-preserving, homogeneous of degree one maps  $\Lambda : \text{int } K_V \rightarrow \text{int } K_V$ , where  $K_V$  is neither polyhedral nor strictly convex; and it is not in general known when these maps have eigenvectors in  $\text{int } K_V$  (see [13–15,24]). We should emphasize that our results apply to a broad class of order-preserving, homogeneous of degree one maps and there are properties of reproduction–decimation operators (see Remark 4.3) which may not apply to these general maps.

A rough outline of this paper may be in order. In Section 2 we give some definitions and present Theorem 2.1, which generalizes Theorem 1.3 and is our basic abstract result. In Section 3, we discuss cones of positive semi-definite forms and discrete Dirichlet forms and show that the geometric hypotheses of Theorem 2.1 are satisfied in this case. In Section 4 we define a general class of reproduction–decimation operators and apply Theorem 2.1 to obtain Theorem 4.1, which is the Karlsson–Nussbaum conjecture for reproduction–decimation operators. Section 5 uses related ideas to establish Denjoy–Wolff results for maps which arise in the theory of operator means (see [20] and references there). Some of the results in Section 5 are proved in the infinite-dimensional case, which in general poses substantial technical complications (see [19]). For the most part these complications are absent here.

## 2. Preliminaries

Let  $X$  be a Banach space. For any set  $U \subset X$ , we will use the notation  $\text{co } U$  to denote the convex hull of  $U$  and  $\text{cl } U$  to denote the closure of  $U$  in the norm topology on  $X$ . We will also use the notation  $\overline{\text{co}} U$  to denote  $\text{cl}(\text{co } U)$ . A *closed cone* is a convex set  $K \subset X$  such that  $\lambda K \subseteq K$  for all  $\lambda \geq 0$  and  $K \cap (-K) = \{0\}$ . We let  $\text{int } K$  denote the interior of  $K$ . The closed cone  $K$  induces a partial ordering  $\leq_K$  on  $X$  as follows: for any  $x, y \in X$ ,  $x \leq_K y$  if  $y - x \in K$ . If there are positive constants  $\alpha$  and  $\beta$  such that

$$\alpha x \leq_K y \leq_K \beta x \tag{1}$$

then we say that  $x$  and  $y$  are *comparable* and we write  $x \sim_K y$ . The relationship  $\sim_K$  defines an equivalence relation on  $K$  and the equivalence classes of the cone  $K$  under  $\sim_K$  are called the *parts* of  $K$ . Observe that for any points  $x, y \in \text{int } K$ ,  $x \sim_K y$ , thus the interior of  $K$  is a part.

When it is clear which cone  $K$  we refer to, we write  $\leq$  and  $\sim$  instead of  $\leq_K$  and  $\sim_K$ . Following the notation of [16] we define the *Hilbert projective metric* for points  $x \sim y$  as

$$d(x, y) = \log \left( \frac{M(y/x)}{m(y/x)} \right) \tag{2}$$

where  $M(y/x) = \inf\{\beta > 0 \mid y \leq \beta x\}$  and  $m(y/x) = \sup\{\alpha > 0 \mid \alpha x \leq y\}$ . Note that  $d(sx, ty) = d(x, y)$  for  $s > 0, t > 0$ , so  $d$  does not distinguish rays in  $K$ . If  $x, y \in K \setminus \{0\}$  and  $x$  and  $y$  are not comparable, then we will define  $d(x, y) = \infty$ .

If  $D$  is a bounded open convex set in a Banach space  $X$ ,  $K = \{(t, tx) \mid t \geq 0, x \in \text{cl } D\}$  is a closed cone in  $\mathbb{R} \times X$  with interior  $\{(t, tx) \mid t > 0, x \in D\}$ . If we identify  $D$  with  $\{(1, x) \mid x \in D\}$ ,  $d$  gives a metric on  $D$ . In this case, Hilbert’s projective metric on  $D$  is often described using an equivalent definition in terms of cross-ratios. See [7,8,16] for further details.

For any Banach space  $X$ , we let  $X^*$  denote the dual space. If  $K \subset X$  is a closed cone, we let  $K^* = \{\varphi \in X^* \mid \varphi(x) \geq 0 \text{ for all } x \in K\}$ . If  $X$  is finite-dimensional and  $K$  has nonempty interior,

then  $K^*$  is a closed cone in  $X^*$  which we call the *dual cone* of  $K$ . Using the dual cone we may give an alternative formula for the Hilbert projective metric distance between two points  $x, y \in \text{int } K$ :

$$d(x, y) = \sup_{\chi, \psi \in K^* \setminus \{0\}} \log \left( \frac{\chi(x)\psi(y)}{\chi(y)\psi(x)} \right). \tag{3}$$

A *polyhedral cone* is a closed cone  $K$  in a finite-dimensional Banach space  $X$  for which there is a finite collection of linear functionals  $\theta_1, \dots, \theta_N \in X^*$  such that  $K = \{x \in X \mid \theta_i(x) \geq 0, 1 \leq i \leq N\}$ . Note that if  $K$  is a polyhedral cone and  $\text{int } K \neq \emptyset$ , then  $K^*$  is also a polyhedral cone. Suppose that  $K$  is a closed cone with nonempty interior in a Banach space  $X$ . Let  $U$  be a subset of  $X$ , let  $Y$  be a Banach space and let  $K_1$  be a closed cone in  $Y$ . We say that a map  $f : U \rightarrow Y$  is *order-preserving* with respect to  $K$  and  $K_1$  if  $f(x) \leq_{K_1} f(y)$  whenever  $x \leq_K y$ . If  $K$  and  $K_1$  are obvious, then we shall say that  $f$  is order-preserving. A map  $f : U \rightarrow Y$  is a *homogeneous of degree one* if  $f(\lambda x) = \lambda f(x)$  for all  $x \in U$  and  $\lambda > 0$ .

Let  $d$  denote the Hilbert projective metric on  $\text{int } K$  and suppose that  $f : \text{int } K \rightarrow \text{int } K$  is homogeneous of degree one and order-preserving. It is an immediate consequence of Eqs. (1) and (2) that  $f$  is *nonexpansive* with respect to  $d$ , that is  $d(f(x), f(y)) \leq d(x, y)$  for all  $x, y \in \text{int } K$ . Let  $q : K \rightarrow \mathbb{R}$  be a continuous, homogeneous of degree one map with  $q(x) > 0$  for all  $x \in K \setminus \{0\}$ . We define  $\Sigma = \{x \in \text{int } K \mid q(x) = 1\}$ . For any  $x \in \Sigma$ , let  $T(x) = f(x)/q(f(x))$ , so that  $T : \Sigma \rightarrow \Sigma$ . Note that  $d$  is a metric when restricted to  $\Sigma$  and the map  $T$  is nonexpansive with respect to  $d$ .

For a map  $T : \Sigma \rightarrow \Sigma$  which is nonexpansive with respect to  $d$ , we define the *omega limit set* of  $x$  under  $T$  to be:

$$\omega(x; T) = \bigcap_{n \geq 1} \text{cl} \left( \bigcup_{k \geq n} T^k(x) \right) \tag{4}$$

where  $\text{cl}$  denotes the closure in the norm topology. If the Banach space  $X$  is finite-dimensional, then  $\text{cl } \Sigma$  is compact. This implies that  $\omega(x; T)$  is nonempty (of course, suitable compactness conditions on  $f$  will also give this result). If  $T$  has a fixed point  $x_* \in \Sigma$ , it immediately follows that for any  $x \in \Sigma$  the iterates  $T^k(x)$  are contained in the closed set  $B = \{y \in \Sigma \mid d(x_*, y) \leq d(x_*, x)\}$ , so  $\omega(x; T)$  is contained in  $B$ . If  $\Sigma$  does not contain a fixed point then a result of Nussbaum ([16, Theorem 4.2], also [19, Corollary 3.16]) proves that  $\omega(x; T) \subset \text{cl } \Sigma \setminus \Sigma$ .

A *horofunction*  $h : \Sigma \rightarrow \mathbb{R}$  is a function of the form

$$h(y) = \lim_{k \rightarrow \infty} (d(y, x^k) - d(x^0, x^k)) \tag{5}$$

where  $\{x^k \mid k \geq 0\}$  is a sequence in  $\Sigma$  such that  $x^k \rightarrow \text{cl } \Sigma \setminus \Sigma$  and such that the limit defining  $h$  converges for all  $y \in \Sigma$ . A *horoball* is a sublevel set of a horofunction, that is, a set of the form  $H_R = \{y \in \Sigma \mid h(y) < R\}$  where  $h$  is a horofunction. See [9] for more details about horofunctions and horoballs in the Hilbert geometry.

Suppose that  $K_1 \subset K_2$  are closed cones with nonempty interiors in a Banach space  $X$ . Let  $K_1^*$  and  $K_2^*$  denote the dual cones of  $K_1$  and  $K_2$ , respectively. Note that  $K_2^* \subset K_1^*$ . The following lemma gives conditions under which the Hilbert projective metric on  $\text{int } K_2$  restricted to  $\text{int } K_2 \cap K_1$  is almost equivalent to the projective metric of a polyhedral cone.

**Lemma 2.1.** *Let  $K_1 \subset K_2$  be closed cones with nonempty interiors in a finite-dimensional Banach space  $X$ . Let  $d_2(\cdot, \cdot)$  denote Hilbert’s projective metric induced by  $K_2$ . If there is a polyhedral cone  $K_p$  such that  $K_1 \subset K_p \subset K_2$  and such that every element of  $K_p^*$  is comparable to an element of  $K_2^*$  in the partial ordering induced by  $K_1^*$ , then there is a constant  $c \geq 0$  such that the Hilbert metric distance with respect to  $K_p$ , denoted  $d_p(\cdot, \cdot)$ , satisfies*

$$d_2(x, y) \leq d_p(x, y) \leq d_2(x, y) + c \tag{6}$$

for all  $x, y \in K_1 \cap \text{int } K_2$ .

**Proof.** Since  $K_1 \subset K_p \subset K_2$  it follows immediately that  $d_2(x, y) \leq d_p(x, y)$  for all  $x, y \in K_1 \cap \text{int } K_2$ . Since  $K_p$  is polyhedral, there is a finite collection  $\{\theta_i\}_{i \in I} \subset K_p^* \setminus \{0\}$  such that

$$d_p(x, y) = \max_{i, j \in I} \log \left( \frac{\theta_i(x)\theta_j(y)}{\theta_i(y)\theta_j(x)} \right)$$

whenever  $x$  and  $y$  are comparable in the partial ordering induced by  $K_p$ . For each  $i \in I$  there is a  $\varphi_i \in K_2^*$  such that  $\theta_i$  is comparable to  $\varphi_i$  in the partial ordering induced by  $K_1^*$ . This means that there is an  $\epsilon_i > 0$  such that  $\epsilon_i \varphi_i(x) \leq \theta_i(x) \leq \epsilon_i^{-1} \varphi_i(x)$  for all  $x \in K_1$ . Letting  $\epsilon = \min_{i \in I} \epsilon_i$  we see that for each  $i, j \in I$ ,

$$\begin{aligned} \log \left( \frac{\theta_i(x)\theta_j(y)}{\theta_i(y)\theta_j(x)} \right) &\leq \log \left( \frac{\epsilon^{-2} \varphi_i(x)\varphi_j(y)}{\epsilon^2 \varphi_i(y)\varphi_j(x)} \right) = \log \left( \frac{\varphi_i(x)\varphi_j(y)}{\varphi_i(y)\varphi_j(x)} \right) + \log \left( \frac{1}{\epsilon^4} \right) \\ &\leq d_2(x, y) + \log \left( \frac{1}{\epsilon^4} \right) \end{aligned}$$

since

$$d_2(x, y) = \sup_{\chi, \psi \in K_2^* \setminus \{0\}} \log \left( \frac{\chi(x)\psi(y)}{\chi(y)\psi(x)} \right)$$

and  $\varphi_i, \varphi_j \in K_2^*$ .  $\square$

The following lemma is not difficult, and the proof can be found in several papers. See for example [11].

**Lemma 2.2.** *Let  $K_p$  be a closed polyhedral cone with nonempty interior in a finite-dimensional Banach space  $X$ . Let  $q \in X^*$  satisfy  $q(x) > 0$  for all  $x \in K_p \setminus \{0\}$  and let  $\Sigma_p = \{x \in \text{int } K_p \mid q(x) = 1\}$ . Let  $d_p$  denote the Hilbert metric on  $\Sigma_p$ . Then there is an isometric embedding  $\Phi$  from  $(\Sigma_p, d_p)$  into a subset of a finite-dimensional Banach space  $(Y, \|\cdot\|)$ .*

Suppose that  $Y$  is a finite-dimensional Banach space with norm  $\|\cdot\|$ . Let  $B^* = \{\varphi \in Y^* \mid \varphi(x) \leq 1 \forall x \in Y \text{ with } \|x\| \leq 1\}$ . The following lemma appears in [11]. We include it here for the reader’s convenience.

**Lemma 2.3.** *Let  $y \in Y$  be an element with  $\|y\| = 1$ . Let  $0 < \lambda < 1$ . For any  $R > r > 0$  and any  $z \in Y$  with  $\|z\| \leq R$ , if  $\|z - Ry\| \leq \lambda R$ , then  $\|z - ry\| \leq R - (1 - \lambda)r$ .*

**Proof.** Suppose that  $\|z - ry\| > R - (1 - \lambda)r$ . By the Hahn–Banach theorem there is some  $\varphi \in B^*$  such that  $\|z - ry\| = \varphi(z - ry) > R - (1 - \lambda)r$ . Then,  $\varphi(z) - \varphi(ry) > R - (1 - \lambda)r$  so  $\varphi(ry) < \varphi(z) - R + (1 - \lambda)r$ . Since  $\varphi(z) \leq \|z\| \leq R$  it follows that  $\varphi(ry) < (1 - \lambda)r$  and hence  $\varphi(y) < (1 - \lambda)$ . By scaling,  $(R - r)\varphi(y) = \varphi(Ry - ry) < (1 - \lambda)(R - r)$ . So

$$\varphi(z - Ry) = \varphi(z - ry) - \varphi(Ry - ry) > R - (1 - \lambda)r - (1 - \lambda)(R - r) = \lambda R.$$

Since  $\|z - Ry\| \geq \varphi(z - Ry) > \lambda R$ , we have a contradiction.  $\square$

**Theorem 2.1.** Let  $K_1 \subset K_2$  be closed cones with nonempty interiors in a finite-dimensional Banach space  $X$ . Suppose that  $f : \text{int } K_2 \rightarrow \text{int } K_2$  is order-preserving (in the partial ordering from  $K_2$ ) and homogeneous of degree one. Suppose also that for some  $x^0 \in K_1 \cap \text{int } K_2$ ,  $f^k(x^0) \in K_1$  for all  $k \in \mathbb{N}$ . Let  $q \in X^*$  be a linear functional such that  $q(x) > 0$  for all  $x \in K_2 \setminus \{0\}$ . Define  $\Sigma = \{x \in \text{int } K_2 \mid q(x) = 1\}$  and  $T(x) = f(x)/q(f(x))$  for  $x \in \Sigma$ . If  $T$  has no fixed point in  $\Sigma$  and there is a polyhedral cone  $K_p$ ,  $K_1 \subset K_p \subset K_2$ , satisfying Eq. (6) for all  $x, y \in K_1 \cap \text{int } K_2$ , then there is a convex subset  $U \subset \partial K_2 \cap \text{cl } \Sigma$  such that  $\omega(x; T) \subset U$  for all  $x \in \Sigma$ .

**Remark 2.1.** The conclusion of Theorem 2.1 only depends on having a map  $T : \Sigma \rightarrow \Sigma$  such that: (1)  $T$  is nonexpansive with respect to  $d_2$ , (2)  $T$  has no fixed points in  $\Sigma$  and (3) there exists  $x^0 \in \Sigma \cap K_1$  such that  $T^k(x^0) \in \Sigma \cap K_1$  for all  $k \geq 0$ .

**Remark 2.2.** Suppose that  $T : \Sigma \rightarrow \Sigma$  is nonexpansive with respect to  $d_2$  and  $T(\Sigma \cap K_1) \subset \Sigma \cap K_1$ . Then  $T^k(x^0) \in \Sigma \cap K_1$  for all  $k \geq 0$  and  $x^0 \in \Sigma \cap K_1$ . Also, if  $T$  has no fixed points in  $\Sigma \cap K_1$ , then  $T$  has no fixed points in  $\Sigma$ . After all, suppose that  $x_* \in \Sigma$  is a fixed point of  $T$ . Let  $B_R(x_*) = \{x \in \Sigma \mid d_2(x, x_*) \leq R\}$ . It is known that  $B_R(x_*)$  is convex (see [16, Lemma 4.1]). By choosing  $R$  large enough, we obtain a nonempty, closed, bounded, convex set  $K_1 \cap B_R(x_*)$ . Since  $T$  is nonexpansive with respect to  $d_2$ ,  $T(K_1 \cap B_R(x_*)) \subset K_1 \cap B_R(x_*)$ . Therefore, by the Brouwer fixed point theorem,  $T$  has a fixed point in  $K_1 \cap B_R(x_*)$ .

**Proof of Theorem 2.1.** Let  $x^k = T^k(x^0)$  for  $k \geq 1$ . Since  $\lim_{k \rightarrow \infty} d_2(x^0, x^k) = \infty$ , and  $d_2(x^0, x^k) \leq d_p(x^0, x^k)$  we may choose a subsequence of integers  $k_i$  so that

$$d_p(x^0, x^{k_i}) > d_p(x^0, x^m) \quad \text{for all } m < k_i. \tag{7}$$

We claim that there is a refinement of  $k_i$  such that

$$d_p(x^{k_i}, x^{k_j - m}) \leq d_p(x^0, x^{k_j}) - \frac{1}{4}d_p(x^0, x^{k_i}) \tag{8}$$

whenever  $k_i$  and  $m$  are fixed and  $k_j$  is sufficiently large. Assume this claim for now.

Since  $\Sigma$  is locally compact, the Ascoli–Arzela theorem implies that by replacing  $k_j$  with a further subsequence of itself we may assume the horofunction  $h$  defined below exists for all  $y \in \Sigma$ :

$$h(y) = \lim_{j \rightarrow \infty} d_2(y, x^{k_j}) - d_2(x^0, x^{k_j}).$$

By the  $d_2$ -nonexpansiveness of  $T$  and Eq. (6), we observe that

$$\begin{aligned} h(x^{k_i+m}) &= \lim_{j \rightarrow \infty} d_2(x^{k_i+m}, x^{k_j}) - d_2(x^0, x^{k_j}) \\ &\leq \liminf_{j \rightarrow \infty} d_2(x^{k_i}, x^{k_j-m}) - d_2(x^0, x^{k_j}) \\ &\leq \liminf_{j \rightarrow \infty} d_p(x^{k_i}, x^{k_j-m}) - d_p(x^0, x^{k_j}) + c. \end{aligned}$$

Now, by Eq. (8),  $d_p(x^{k_i}, x^{k_j-m}) \leq d_p(x^{k_j}, x^0) - \frac{1}{4}d_p(x^{k_i}, x^0)$  for  $j$  sufficiently large. Therefore we obtain

$$h(x^{k_i+m}) \leq -\frac{1}{4}d_p(x^{k_i}, x) + c.$$

This implies that  $\lim_{m \rightarrow \infty} h(x^m) = -\infty$ .

For any constant  $M \geq 0$  let  $H_M$  denote the horoball  $H_M = \{y \in \Sigma \mid h(y) \leq -M\}$ . Since Hilbert metric balls are convex (see e.g., [16, Lemma 4.1]), it follows that each  $H_M$  is convex. Let  $\text{cl } H_M$  denote the norm closure of  $H_M$  in  $X$ . Since  $h(x^m) \rightarrow -\infty$  it follows that  $\omega(x^0; T) \subset \text{cl } H_M$  for all  $M \geq 0$ . Since  $h(y)$  is finite for all  $y \in \Sigma$ , we must have  $\bigcap_{M \geq 0} \text{cl } H_M \subset \text{cl } \Sigma \setminus \Sigma$ . Thus  $\omega(x^0; T) \subset \bigcap_{M \geq 0} \text{cl } H_M$  which is a convex subset of  $\text{cl } \Sigma \setminus \Sigma = \partial K_2 \cap \text{cl } \Sigma$ .

We now consider the case where  $x \in \text{int } K_2$  but  $x \neq x^0$ . Observe that for any  $x \in \Sigma$ ,  $d_2(T^k(x), T^k(x^0)) \leq d_2(x, x^0)$  for all  $k > 0$ . Thus,  $h(T^k(x)) \leq h(T^k(x^0)) + d_2(x, x^0)$ ,  $\forall k > 0$ . Therefore  $\lim_{k \rightarrow \infty} h(T^k(x)) = -\infty$  for all  $x \in \Sigma$ , and thus  $\omega(x; T) \subset \bigcap_{m \geq 0} \text{cl } H_m$  for all  $x \in \Sigma$ .

It remains to prove Eq. (8). By Lemma 2.2 there is an isometric embedding  $\Phi$  from  $(K_p \cap \Sigma, d_p)$  into a subset of a finite-dimensional normed space  $(Y, \|\cdot\|)$ . For each  $x^k$  let  $\hat{x}^k = \Phi(x^k)$ . We may assume without loss of generality that  $\Phi(x^0) = 0$ . Therefore,  $d(x^0, x^k) = \|\hat{x}^k\|$  for all  $k > 0$ . Since  $x^{k_j}$  is assumed to satisfy Eq. (7),  $\|\hat{x}^{k_j}\| > \|\hat{x}^m\|$  for all  $m < k_j$ .

Since the unit ball in  $Y$  is compact, there is a point  $\bar{y} \in Y$  with  $\|\bar{y}\| = 1$  which is an accumulation point of the sequence  $\hat{x}^{k_i} / \|\hat{x}^{k_i}\|$ ,  $i \geq 1$ . By replacing  $k_i$  with a refinement we may assume that

$$\left\| \frac{\hat{x}^{k_i}}{\|\hat{x}^{k_i}\|} - \bar{y} \right\| \leq 2^{-i} \quad \text{for all } i \geq 1.$$

Thus,

$$\|\hat{x}^{k_i} - (\|\hat{x}^{k_i}\| \bar{y})\| \leq 2^{-i} \|\hat{x}^{k_i}\|, \quad \forall i \geq 1.$$

For each  $i \geq 1$  we denote  $\|\hat{x}^{k_i}\| \bar{y}$  by  $y^i$ . Then the equation above becomes:

$$\|\hat{x}^{k_i} - y^i\| \leq 2^{-i} \|\hat{x}^{k_i}\|, \quad \forall i \geq 1. \tag{9}$$

Fix some  $i \geq 1$ . Note that  $\|\hat{x}^{k_j-m}\| < \|\hat{x}^{k_j}\|$  by Eq. (7). Also

$$\begin{aligned} \|\hat{x}^{k_j-m} - y^j\| &\leq \|\hat{x}^{k_j-m} - \hat{x}^{k_j}\| + \|\hat{x}^{k_j} - y^j\| \\ &\leq d_2(x^{k_j-m}, x^{k_j}) + c + 2^{-j} \|\hat{x}^{k_j}\| \leq md_2(x^0, T(x^0)) + c + 2^{-j} \|\hat{x}^{k_j}\| \end{aligned}$$

by the nonexpansiveness of  $T$  under the metric  $d_2$ . Note that

$$c_m = md_2(x^0, T(x^0)) + c$$

is a constant which depends only on  $m$ . Thus

$$\|\hat{x}^{k_j-m} - y^j\| \leq c_m + 2^{-j} \|\hat{x}^{k_j}\|. \tag{10}$$

Eq. (10) implies that for  $j$  large enough  $\|\hat{x}^{k_j-m} - y^j\| \leq \frac{1}{4} \|\hat{x}^{k_j}\| = \frac{1}{4} \|y^j\|$ . Using Lemma 2.3 with  $\lambda = \frac{1}{4}$ ,  $r = \|\hat{x}^{k_i}\|$ ,  $R = \|\hat{x}^{k_j}\|$ ,  $y = \bar{y}$  and  $z = \hat{x}^{k_j-m}$  we obtain

$$\|\hat{x}^{k_j-m} - y^i\| \leq \|\hat{x}^{k_j}\| - \frac{3}{4} \|\hat{x}^{k_i}\|. \tag{11}$$

Hence,

$$\begin{aligned} d_p(x^{k_i}, x^{k_j-m}) &= \|\hat{x}^{k_i} - \hat{x}^{k_j-m}\| \leq \|\hat{x}^{k_i} - y^i\| + \|y^i - \hat{x}^{k_j-m}\| \\ &\leq 2^{-i} \|\hat{x}^{k_i}\| + \|\hat{x}^{k_j}\| - \frac{3}{4} \|\hat{x}^{k_i}\| \end{aligned}$$

by Eqs. (9) and (11). Thus, for  $i \geq 1$  and  $j$  large enough,

$$d_p(x^{k_i}, x^{k_j-m}) \leq \|\hat{x}^{k_j}\| - \frac{1}{4} \|\hat{x}^{k_i}\| = d_p(x^0, x^{k_j}) - \frac{1}{4} d_p(x^0, x^{k_i})$$

which proves Eq. (8).  $\square$

### 3. Positive semi-definite forms and discrete Dirichlet forms

A class of nonlinear order-preserving homogeneous of degree one maps appears in the study of diffusion on fractals. These “reproduction–decimation operators” are defined on the interior of a certain cone of positive semi-definite forms. In this section we will introduce this cone of positive semi-definite forms, as well as the cone of discrete Dirichlet forms. In the following section we will define a general class of reproduction–decimation operators and show how the results of the previous chapter allow us to establish a Denjoy–Wolff type result for these operators even though the cone of positive semi-definite forms is neither polyhedral nor strictly convex.

Let  $S$  be a finite set. If we think of  $S$  as a measure space with the counting measure, then  $L^2(S)$  is a finite-dimensional Hilbert space consisting of the functions  $x : S \rightarrow \mathbb{R}$ . The inner product on  $L^2(S)$  is  $\langle x, y \rangle = \sum_{i \in S} x(i)y(i)$ . On  $L^2(S)$  we have a *standard basis* consisting of the functions  $e_i$ ,  $i \in S$ , where  $e_i(j) = \delta_{ij}$ , the Kronecker delta. We let  $\mathbb{1}_S$  be the function  $\mathbb{1}_S(i) = 1$  for all  $i \in S$ . For  $x, y \in L^2(S)$  we let  $x \wedge y$  denote the minimum of  $x$  and  $y$  in  $L^2(S)$ , that is,  $(x \wedge y)(i) = \min\{x(i), y(i)\}$ .

We are interested in the set  $X_S$  of quadratic forms  $E : L^2(S) \rightarrow \mathbb{R}$  given by an expression of the form

$$E(x) = \frac{1}{2} \sum_{i \neq j \in S} c_{ij} (x(i) - x(j))^2 \tag{12}$$



where each  $c_{ij} \in \mathbb{R}$  and  $c_{ji} = c_{ij}$  for all  $i \neq j$ . The set  $X_S$  is a finite-dimensional vector space with dimension equal to  $n(n - 1)/2$  where  $n = \text{card } S$ . We can therefore assume that  $X_S$  is a Banach space. Note that the constants  $c_{ij}$  are uniquely determined for each  $E \in X_S$  by the formula  $c_{ij} = \varphi_{ij}(E)$  where

$$\varphi_{ij}(E) = \frac{1}{4}(E(e_i - e_j) - E(e_i + e_j)). \tag{13}$$

Recall that a quadratic form  $E \in X_S$  is positive semi-definite if  $E(x) \geq 0$  for all  $x \in L^2(S)$ . In the space  $X_S$  we let  $K_S$  denote the cone of positive semi-definite forms, that is

$$K_S = \{E \in X_S \mid E(x) \geq 0 \forall x \in L^2(S)\}.$$

A *discrete Dirichlet form* is a quadratic form  $E \in X_S$  such that  $E(x \wedge \mathbb{1}_S) \leq E(x)$  for all  $x \in L^2(S)$ . It is known (see [10]) that any  $E$  given by Eq. (12) is a discrete Dirichlet form if and only if  $c_{ij} \geq 0$  for every  $i \neq j$ . From this characterization we see that the set of discrete Dirichlet forms is a closed cone in  $X_S$  which we denote  $D_S$ . Furthermore,

$$D_S = \{E \in X_S \mid \varphi_{ij}(E) \geq 0 \forall i, j \in S \text{ with } i \neq j\}$$

where each  $\varphi_{ij}$  is given by Eq. (13). Since there are only finitely many  $\varphi_{ij}$  and each one is a linear functional on  $X_S$ , we see that  $D_S$  is a polyhedral cone. It is also clear that  $D_S \subset K_S$ .

Both  $K_S$  and  $D_S$  have nonempty interior in  $X_S$ . Let  $\langle \mathbb{1}_S \rangle^\perp$  denote the subspace  $\{x \in L^2(S) \mid \langle x, \mathbb{1}_S \rangle = 0\}$ . It is a straightforward exercise, which we leave to the reader, to prove that

$$\begin{aligned} \text{int } K_S &= \{E \in K_S \mid \exists c > 0 \text{ with } E(x) \geq c \langle x, x \rangle \forall x \in \langle \mathbb{1}_S \rangle^\perp\}, \\ \text{int } D_S &= \{E \in D_S \mid \varphi_{ij}(E) > 0 \text{ for all } i, j \in S \text{ with } i \neq j\}. \end{aligned} \tag{14}$$

The class of operators we are interested in is defined on  $D_S \cap \text{int } K_S$ . We would like to use Theorem 2.1 to establish a Denjoy–Wolff type theorem for this class of maps. In order to do this, we must first prove the following proposition.

**Proposition 3.1.** *If  $S$  is a finite set with  $\text{card } S \geq 3$ , and  $D_S$  and  $K_S$  are defined as above, then there is a closed polyhedral cone  $C_p \subset X_S$  such that  $D_S \subset C_p \subset K_S$  and every element in  $C_p^*$  is comparable to an element of  $K_S^*$  in the partial ordering induced by  $D_S^*$ .*

In order to prove this proposition, we need to consider the dual cones of  $D_S$  and  $K_S$ . One can easily show that

$$D_S^* = \left\{ \sum_{i \neq j \in S} r_{ij} \varphi_{ij} \mid r_{ij} \geq 0, r_{ij} = r_{ji} \text{ for all } i \neq j \right\} \tag{15}$$

where  $\varphi_{ij}$  is the linear functional on  $X_S$  given by Eq. (13). Finding a nice characterization of  $K_S^*$  takes a little more work. In what follows, for any  $x \in L^2(S)$ , let  $|x|$  denote the “variation norm” of  $x$ , that is,

$$|x| = \max_{i, j \in S} |x(i) - x(j)|.$$

Although  $|\cdot|$  is not norm on  $L^2(S)$ , it is a norm on the subspace  $\langle \mathbb{1}_S \rangle^\perp$ . We recall that a *sufficient set* for a closed cone  $K$  in a Banach space  $X$  is a subset  $U \subset K^*$  such that  $K = \{x \in X \mid \varphi(x) \geq 0 \forall \varphi \in U\}$ .

**Lemma 3.1.** *Let  $n = \text{card } S$ . The dual cone of  $K_S$  is*

$$K_S^* = \left\{ \sum_{k=1}^{n(n-1)/2+1} t_k \hat{x}_k \mid t_k \geq 0, x_k \in \langle \mathbb{1}_S \rangle^\perp \text{ with } |x_k| = 1 \right\}$$

where, for any  $x \in L^2(S)$ ,  $\hat{x} \in X_S^*$  is the linear functional such that  $\hat{x}(E) = E(x)$  for  $E \in X_S$ .

**Proof.** Since  $E(x + \lambda \mathbb{1}_S) = E(x)$  for all  $E \in X_S$  and  $\lambda \in \mathbb{R}$ , it follows that  $E$  is positive semi-definite if and only if  $E(x) \geq 0$  for all  $x \in \langle \mathbb{1}_S \rangle^\perp$  with  $|x| = 1$ . Thus the set of linear functionals  $\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\}$  is a sufficient set for  $K_S$ . Therefore,

$$K_S^* = \text{cl} \left\{ \sum_{k=1}^N t_k \hat{x}_k \mid N > 0, t_k \geq 0, x_k \in \langle \mathbb{1}_S \rangle^\perp \text{ with } |x_k| = 1 \right\}.$$

We will now show that the set  $\{\sum_{k=1}^N t_k \hat{x}_k \mid N > 0, t_k \geq 0, |x_k| = 1\}$  is closed. Observe that the set  $\{x \in \langle \mathbb{1}_S \rangle^\perp: |x| = 1\}$  is compact. We leave it to the reader to verify that the map  $x \mapsto \hat{x}$  is continuous, and this implies that  $\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\}$  is compact. An application of Carathéodory’s theorem proves that  $\text{co}\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\}$  is compact [23, Theorem 17.2]. Observe that if  $E \in \text{int } K_S$  and  $|x| = 1$ , then  $\hat{x}(E) > 0$ . This implies that  $0 \notin \text{co}\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\}$ .

Since  $\text{co}\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\}$  is compact and does not contain zero, the set

$$\bigcup_{\lambda \geq 0} \lambda(\text{co}\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\})$$

is closed. To see this, suppose that  $\theta_k$  is a sequence in  $\text{co}\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\}$  and  $b_k \geq 0$  is a sequence of real numbers such that  $b_k \theta_k \rightarrow \zeta$ . Then, since  $\text{co}\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\}$  is compact, a subsequence  $\theta_{k_i}$  converges to some  $\theta_\infty \in \text{co}\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\}$ . Since  $\theta_\infty \neq 0$ , the corresponding subsequence  $b_{k_i}$  must also converge to some  $b_\infty \geq 0$  as  $i \rightarrow \infty$ . Then  $\zeta = b_\infty \theta_\infty$ , so  $\zeta \in \bigcup_{\lambda \geq 0} \lambda(\text{co}\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\})$ , and therefore  $\bigcup_{\lambda \geq 0} \lambda(\text{co}\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\})$  is closed. Now observe that

$$\begin{aligned} & \bigcup_{\lambda \geq 0} \lambda(\text{co}\{\hat{x}: x \in \langle \mathbb{1}_S \rangle^\perp, |x| = 1\}) \\ &= \left\{ \sum_{k=1}^N t_k \hat{x}_k \mid N > 0, t_k \geq 0, x_k \in L^2(S) \text{ with } |x_k| = 1 \right\}, \end{aligned}$$

and by Carathéodory’s theorem for convex sets (see [23, Theorem 17.1]) we may assume that  $N = \dim X_S^* + 1 = n(n - 1)/2 + 1$ .  $\square$

Since  $D_S \subset K_S$  it follows that  $K_S^* \subset D_S^*$ . Also note that  $D_S^*$  is a polyhedral cone, since  $D_S$  is polyhedral. Let  $\varphi \in D_S^*$ . By Eq. (15),  $\varphi \in D_S^*$  has the form  $\varphi = \sum_{i \neq j \in S} s_{ij} \varphi_{ij}$  where each  $s_{ij} = s_{ji}$  and  $s_{ij} \geq 0$ . Let  $I = I(\varphi)$  be the set of pairs  $(i, j) \in S \times S$  such that  $s_{ij} = 0$ . Then the part of the cone  $D_S^*$  containing  $\varphi$  is the set

$$U_I = \left\{ \sum_{i \neq j \in S} r_{ij} \varphi_{ij} \mid r_{ij} = 0 \text{ for } (i, j) \in I, r_{ij} > 0 \text{ otherwise} \right\}. \tag{16}$$

For a collection of pairs  $I \subset S \times S$  with the property that  $(i, j) \in I$  implies  $(j, i) \in I$ , we will say that two elements  $i, j \in S$  are *I-connected* if there is a path  $i_k \in S, k = 1, \dots, p$ , such that  $i_1 = i, i_p = j$  and  $(i_k, i_{k+1}) \in I$  for  $1 \leq k < p$ . Note that *I-connectedness* is an equivalence relation on  $S$ .

**Lemma 3.2.** *Suppose  $U_I$  is a part of  $D_S^*$  given by Eq. (16) and every *I-connected* pair  $i, j \in S$  satisfies  $(i, j) \in I$ . Then  $U_I \cap K_S^* \neq \emptyset$ .*

**Proof.** For any  $x \in L^2(S)$ ,

$$\hat{x}(E) = E(x) = \frac{1}{2} \sum_{i \neq j \in S} \varphi_{ij}(E) (x(i) - x(j))^2,$$

or

$$\hat{x} = \frac{1}{2} \sum_{i \neq j \in S} (x(i) - x(j))^2 \varphi_{ij}.$$

Therefore,  $\hat{x} \in U_I$  if and only if  $x(i) = x(j)$  exactly when  $(i, j) \in I$ .

By assumption, every *I-connected* pair  $i, j \in S$  satisfies  $(i, j) \in I$ . Furthermore, we know that *I-connectedness* is an equivalence relation. Therefore,  $S$  can be partitioned into the equivalence classes under this relation, and we see that  $i, j \in S$  will be contained in the same equivalence class if and only if  $(i, j) \in I$ . We may now construct an  $x \in L^2(S)$  such that  $x(i) = x(j)$  if and only if  $(i, j) \in I$ . Then  $\hat{x} \in U_I$ , and it is clear that  $\hat{x}$  is also in  $K_S^*$ .  $\square$

**Lemma 3.3.** *If  $U_I$  is a part of  $D_S^*$  given by Eq. (16) and  $U_I \cap K_S^* = \emptyset$ , then there is an  $E \in X_S$  such that  $\theta(E) \geq 0$  for all  $\theta \in K_S^*$  but  $\varphi(E) < 0$  for all  $\varphi \in U_I$ .*

**Proof.** By Lemma 3.2, if  $U_I \cap K_S^* = \emptyset$ , then there is an *I-connected* pair  $i, j \in S$  such that  $(i, j) \notin I$ . Let  $J$  be the set of pairs  $(i, j) \in S \times S$  such that  $i \neq j$  and  $i, j$  are *I-connected*. We see immediately that,  $J \neq I$ .

Let  $E$  be given by Eq. (12) with  $c_{ij} = 0$  if  $(i, j) \notin J, c_{ij} = -1$  if  $(i, j) \in J \setminus I$  and for  $(i, j) \in I$  let  $c_{ij} = M$  where  $M > 0$  is a large constant which we will specify later. Then for any  $x \in L^2(S)$ ,

$$\begin{aligned} \hat{x}(E) &= \frac{1}{2} \sum_{i \neq j \in S} c_{ij} (x(i) - x(j))^2 \\ &= \frac{1}{2} \sum_{(i,j) \in I} M (x(i) - x(j))^2 - \frac{1}{2} \sum_{(i,j) \in J \setminus I} (x(i) - x(j))^2. \end{aligned}$$

Choose  $(p, q) \in I$  such that  $(x(p) - x(q))^2 = \max_{(i,j) \in I} (x(i) - x(j))^2$ . Note that for any pair  $(i, j) \in J$ , there is a path  $i_k \in S, k = 1, \dots, N, i_1 = i, i_N = j$ , and  $(i_k, i_{k+1}) \in I$  for all  $1 \leq k < N$ . Furthermore, we may choose  $N$  to be less than or equal to  $n = \text{card } S$ . Therefore, for all  $(i, j) \in J$ ,

$$\begin{aligned} (x(i) - x(j))^2 &= |x(i) - x(j)|^2 \\ &\leq (|x(i_1) - x(i_2)| + |x(i_2) - x(i_3)| + \dots + |x(i_{N-1}) - x(i_N)|)^2, \\ &\leq (n|x(p) - x(q)|)^2 = n^2(x(p) - x(q))^2. \end{aligned}$$

By letting  $M > \text{card}(J \setminus I)n^2$  we can see that

$$\begin{aligned} \hat{x}(E) &= \frac{1}{2} \sum_{(i,j) \in I} M(x(i) - x(j))^2 - \frac{1}{2} \sum_{(i,j) \in J \setminus I} (x(i) - x(j))^2 \\ &\geq \frac{1}{2} \text{card}(J \setminus I)n^2(x(p) - x(q))^2 - \frac{1}{2} \sum_{(i,j) \in J \setminus I} (x(i) - x(j))^2 \\ &\geq \frac{1}{2} \text{card}(J \setminus I)n^2(x(p) - x(q))^2 - \frac{1}{2} \text{card}(J \setminus I) \max_{(i,j) \in J} (x(i) - x(j))^2 \geq 0. \end{aligned}$$

Since the constant  $M$  did not depend on  $x$ , it follows that  $\hat{x}(E) \geq 0$  for all  $x \in L^2(S)$ . Therefore, by Lemma 3.1,  $\theta(E) \geq 0$  for all  $\theta \in K_S^*$ .

It remains to show that  $\varphi(E) < 0$  for all  $\varphi \in U_I$ . However, if  $\varphi \in U_I$  then  $\varphi = \sum_{i \neq j \in S} r_{ij} \varphi_{ij}$  with  $r_{ij} \geq 0$  for all pairs  $i \neq j \in S$ , and  $r_{ij} = 0$  if and only if  $(i, j) \in I$ . Therefore

$$\varphi(E) = \sum_{(i,j) \in J \setminus I} -r_{ij} < 0. \quad \square$$

**Proof of Proposition 3.1.** We will construct a polyhedral cone  $C_p^*$  such that  $K_S^* \subset C_p^* \subset D_S^*$  and such that every element  $\varphi \in C_p^*$  is comparable to an element  $\theta \in K_S^*$  under the partial ordering induced by  $D_S^*$ .

To construct  $C_p^*$ , we will intersect the polyhedral cone  $D_S^*$  with finitely many closed half-spaces of the form  $H_E = \{\theta \in X_S^* \mid \theta(E) \geq 0\}$  where  $E \in X_S$ . This will ensure that  $C_p^*$  is also polyhedral. For any part  $U_I$  of  $D_S^*$ , if  $U_I \cap K_S^* = \emptyset$ , then Lemma 3.3 implies that there is an  $E \in X_S$  such that  $\varphi(E) < 0$  for all  $\varphi \in U_I$  while  $\theta(E) \geq 0$  for all  $\theta \in K_S^*$ . Therefore, there is a closed half-space  $H_E$  such that  $H_E$  contains  $K_S^*$  but is disjoint from  $U_I$ . Since  $D_S^*$  is a polyhedral cone, it only has finitely many parts and therefore, by intersecting the cone  $D_S^*$  with finitely many half-spaces  $H_E$  we may obtain a polyhedral cone  $C_p^*$  such that  $K_S^* \subset C_p^* \subset D_S^*$ , and every element of  $C_p^*$  lies in a part  $U_I$  of  $D_S^*$  such that  $U_I \cap K_S^* \neq \emptyset$ . Thus every element of  $C_p^*$  is comparable to an element of  $K_S^*$  in the partial ordering of  $D_S^*$ .  $\square$

#### 4. Denjoy–Wolff theorems for reproduction–decimation operators

One motivation for this paper arises from so-called “reproduction–decimation operators.” Such operators arise in the study of diffusion on fractals: see [10, Chapter 3]. Here we shall

describe a general class of reproduction–decimation operators and show that our general Denjoy–Wolff theorem is applicable. Our operators will be defined on the interior of an appropriate cone, and will be order-preserving and homogeneous of degree one. If one knows that a given operator has an eigenvector in the interior of the cone, our Denjoy–Wolff theorem is irrelevant. The point is that proving a given reproduction–decimation operator has such an eigenvector may be a difficult problem, and there are examples where no such eigenvector exists (see [6] and [24, pp. 650, 651]).

Let  $V$  and  $W$  be finite sets with  $V \subset W$ . We view  $V$  and  $W$  as measure spaces with the counting measure, so we can form the real Hilbert spaces  $L^2(V)$  and  $L^2(W)$ . Since  $V$  and  $W$  will be fixed, we shall write  $H = L^2(W)$ . We define an orthogonal projection  $P : H \rightarrow H$  by  $(Px)(i) = x(i)$  for  $i \in V$  and  $(Px)(i) = 0$  for  $i \in W \setminus V$ . It is easy to see that  $H_1 := P(H)$  and  $L^2(V)$  are naturally isometric as Hilbert spaces by the map  $x \in H_1$  goes to  $x|_V$ . We shall write  $Q = I - P$ , where  $I$  denotes the identity operator on  $H$ ; and  $H_2 := Q(H)$  is naturally isometric to  $L^2(W \setminus V)$  and  $H_2 = H_1^\perp$ .

For  $S = V$  or  $S = W$ , we will continue use the definitions of the previous sections for  $X_S$ ,  $K_S$ , and  $D_S$ . The following lemma is an easy consequence of Eq. (14) and the fact that  $c_{ij} = \varphi_{ij}(E) \geq 0$  for  $i \neq j$  when  $E \in D_S$ .

**Lemma 4.1.** (See [10, Chapter 3].) *If  $E \in D_S$  is given by Eq. (12), then  $E \in D_S \cap \text{int } K_S$  if and only if whenever  $i \in S$  and  $j \in S$  with  $i \neq j$ , there exist  $i_0 = i, i_1, \dots, i_p = j$  in  $S$  with  $i_{k-1} \neq i_k$  for  $1 \leq k \leq p$  and  $c_{i_{k-1}i_k} > 0$  for  $1 \leq k \leq p$ .*

We now need to define a general reproduction operator, which will be a linear map from  $X_V$  to  $X_W$ . For  $1 \leq k \leq N$  let  $\psi_k : V \rightarrow W$  be a map. Define  $\Psi_k : L^2(W) \rightarrow L^2(V)$  by  $(\Psi_k x)(i) = x(\psi_k(i))$ . Note that

$$\Psi_k(\mathbb{1}_W) = \mathbb{1}_V \quad \text{and} \quad \Psi_k(x \wedge \mathbb{1}_W) = \Psi_k(x) \wedge \mathbb{1}_V \quad \forall x \in L^2(W). \tag{17}$$

If  $\eta_k, 1 \leq k \leq N$ , are given positive reals, we define a reproduction operator  $R : X_V \rightarrow X_W$  by

$$R(E) = \sum_{k=1}^N \eta_k E \circ \Psi_k. \tag{18}$$

Reproduction operators satisfy the following properties.

**Proposition 4.1.** *If  $R : X_V \rightarrow X_W$  is defined by Eq. (18), then:*

- (a)  $R$  is linear.
- (b)  $R(K_V) \subset K_W$ .
- (c)  $R(D_V) \subset D_W$ .
- (d) If  $E_1, E_2 \in X_V$  satisfy  $E_1 \leq_{K_V} E_2$ , then  $R(E_1) \leq_{K_W} R(E_2)$ .

**Proof.** Both (a) and (b) follow immediately from Eq. (18). If  $E \in D_V$  and  $x \in L^2(W)$ , we obtain from Eqs. (17) and (18) that

$$R(E)(x \wedge \mathbb{1}_W) \leq R(E)(x).$$

Therefore  $R(E) \in D_W$ , which proves (c). From (a) and (b) it follows that if  $E_1 \leq_{K_V} E_2$ , then  $R(E_2 - E_1) \in K_W$  so  $R(E_1) \leq_{K_W} R(E_2)$ .  $\square$

**Definition 4.1.** For  $\psi_k : V \rightarrow W$ ,  $1 \leq k \leq N$ , we shall say that  $\{\psi_k \mid 1 \leq k \leq N\}$  satisfies *condition I* if, for all  $j, j' \in W$ , there exist  $k_0, k_1, \dots, k_p$  with  $j \in \psi_{k_0}(V)$ ,  $j' \in \psi_{k_p}(V)$  and  $\psi_{k_s}(V) \cap \psi_{k_{s-1}}(V) \neq \emptyset$  for  $1 \leq s \leq p$ .

Note that condition I is satisfied by reproduction operators corresponding to fractals satisfying certain connectivity conditions. See [10, Section 1.6], for more details.

**Lemma 4.2.** If  $\{\psi_k \mid 1 \leq k \leq N\}$  satisfies condition I,  $\eta_k > 0$  for  $1 \leq k \leq N$ , and  $R$  is defined by Eq. (18), then

$$R(\text{int } K_V) \subset \text{int } K_W. \tag{19}$$

**Proof.** By Eq. (12), it suffices to prove that if  $x \in L^2(W)$ ,  $E \in \text{int } K_V$  and  $R(E)(x) = 0$ , then  $x = \lambda \mathbb{1}_W$  for some  $\lambda \in \mathbb{R}$ . However,

$$R(E)(x) = \sum_{k=1}^N \eta_k E(\psi_k(x)),$$

so if  $R(E)(x) = 0$ ,  $E(\psi_k(x)) = 0$  for  $1 \leq k \leq N$ . Because  $E \in \text{int } K_V$ , it follows that there exists  $\lambda_k \in \mathbb{R}$  with

$$\psi_k(x) = x \circ \psi_k = \lambda_k \mathbb{1}_k \quad \text{for } 1 \leq k \leq N.$$

It follows that  $x(i) = \lambda_k$  for all  $i \in \varphi_k(V)$ ,  $1 \leq k \leq N$ . If  $j, j' \in W$ , select  $k_0, k_1, \dots, k_p$  as in the definition of condition I. It follows that  $\lambda_{k_s} = \lambda_{k_{s+1}}$  for  $0 \leq s < p$ , so  $\lambda_{k_0} = x(j) = \lambda_{k_p} = x(j')$ , and  $x$  is a scalar multiple of  $\mathbb{1}_W$ .  $\square$

It remains to define the “decimation operator”  $\Phi : \text{int } K_W \rightarrow \text{int } K_V$ . This operator is sometimes called the “trace operator” or (see [2]) the “shorted operator.” If  $x \in L^2(V)$ , we identify  $x$  with an element of  $H_1 \subset L^2(W)$  by defining  $x(i) = 0$  for  $i \in W \setminus V$ . Then the *decimation operator*  $\Phi : K_W \rightarrow K_W$  is defined by

$$\Phi(E)(x) = \inf\{E(x + y) \mid y \in H_2\}. \tag{20}$$

Note that  $\Phi(E)$  can also be considered a map of  $H \rightarrow H$  with  $\Phi(E)(H) \subset H_1$  and  $\Phi(E)(H_2) = \{0\}$ . Although the map  $\Phi$  is defined on  $K_W$ , it may not be continuous on  $K_W$ , see the remarks after Corollary 3 to Theorem 1 in [2]. Corollary 1 to Theorem 1 of [2] implies that if  $E_1, E_2 \in K_W$  and  $E_1 \leq_{K_W} E_2$ , then  $\Phi(E_1) \leq_{K_V} \Phi(E_2)$ . In fact, by [2, Theorem 4],  $\Phi$  is superadditive (that is,  $\Phi(E_1 + E_2) \geq_{K_V} \Phi(E_1) + \Phi(E_2)$  for all  $E_1, E_2 \in K_W$ ). It is clear that  $\Phi(\lambda E) = \lambda \Phi(E)$  for  $\lambda > 0$  and  $E \in \text{int } K_W$ . If  $E \in \text{int } K_W$ , then by Eq. (14) there is a constant  $c > 0$  such that  $E(z) \geq c\langle z, z \rangle$  for all  $z \in L^2(W)$ . Thus, for  $x \in H_1$ ,

$$\begin{aligned} \Phi(E)(x) &= \inf\{E(x + y) \mid y \in H_2\} \\ &\geq \inf\{c\langle x + y, x + y \rangle \mid y \in H_2\} \geq c\langle x, x \rangle. \end{aligned}$$

This implies that  $\Phi(E) \in \text{int } K_V$  for all  $E \in \text{int } K_W$ .

Using Eq. (20) we can show that  $\Phi(D_W \cap \text{int } K_W) \subset D_V \cap \text{int } K_V$ . If  $x \in L^2(V)$ ,  $E \in D_W \cap \text{int } K_W$  and we identify  $L^2(V)$  with  $H_1$  as above, Eq. (20) gives  $\Phi(E)(x \wedge \mathbb{1}_V) = \inf\{E(x \wedge \mathbb{1}_W + y) \mid y \in H_2\}$ . Because  $y \wedge \mathbb{1}_W \in H_2$  for  $y \in H_2$  and because  $(x + y) \wedge \mathbb{1}_W = x \wedge \mathbb{1}_W + y \wedge \mathbb{1}_W$ , we see that

$$\begin{aligned} & \inf\{E(x \wedge \mathbb{1}_W + y) \mid y \in H_2\} \\ & \leq \inf\{E(x \wedge \mathbb{1}_W + y \wedge \mathbb{1}_W) \mid y \in H_2\} \\ & = \inf\{E((x + y) \wedge \mathbb{1}_W) \mid y \in H_2\} \\ & \leq \inf\{E(x + y) \mid y \in H_2\} = \Phi(E)(x). \end{aligned}$$

This proves that  $\Phi(D_W \cap \text{int } K_W) \subset D_V \cap \text{int } K_V$ . We collect these results in the following proposition.

**Proposition 4.2.** *If  $\Phi : K_W \rightarrow K_V$  is defined by Eq. (20), then*

- (a)  $\Phi(\text{int } K_W) \subset \text{int } K_V$ .
- (b)  $\Phi$  is homogeneous of degree one.
- (c) If  $E_1, E_2 \in \text{int } K_W$  satisfy  $E_1 \leq_{K_W} E_2$ , then  $\Phi(E_1) \leq_{K_V} \Phi(E_2)$ .
- (d)  $\Phi(D_W \cap \text{int } K_W) \subset D_V \cap \text{int } K_V$ .

A reproduction–decimation operator is a map  $\Lambda : K_V \rightarrow K_V$  given by  $\Lambda = \Phi \circ R$  where  $\Phi$  is a decimation operator and  $R$  is a reproduction operator. If  $R$  is defined by Eq. (18) and  $\{\psi_k \mid 1 \leq k \leq N\}$  satisfies condition I, then  $\Lambda(\text{int } K_V) \subset \text{int } K_V$ . If condition I is not satisfied, then  $\Lambda = \Phi \circ R$  is still defined as a map from  $K_V$  to  $K_V$ , but it may not map  $\text{int } K_V$  into itself. Furthermore,  $\Lambda$  may not be continuous on all of  $K_V$ , but it will be continuous on  $D_V$  (see [5, Theorem 7.2]). Note that  $\Lambda$  is also referred to as a “renormalization” operator by some authors [22,24].

By using Theorem 2.1 and Proposition 3.1 we immediately obtain the following result.

**Theorem 4.1.** *Assume that  $\text{card } V \geq 3$  and that  $f : \text{int } K_V \rightarrow \text{int } K_V$  is order-preserving (in the partial ordering from  $K_V$ ), homogeneous of degree one, and satisfies  $f(D_V \cap \text{int } K_V) \subset D_V$ . Let  $q \in X_V^*$  be a linear functional such that  $q(E) > 0$  all  $E \in K_V \setminus \{0\}$ . Define  $\Sigma := \{E \in \text{int } K_V \mid q(E) = 1\}$  and  $T(E) := f(E)/q(f(E))$  for  $E \in \Sigma$ , and assume that  $T$  has no fixed points in  $\Sigma$ . Then there is a convex set  $U \subset \partial K_V \cap \text{cl } \Sigma$  such that  $\omega(E; T) \subset U$  for all  $E \in \Sigma$ .*

**Remark 4.1.** For any two elements  $E_1, E_2 \in \Sigma$ , elementary arguments (see [19]) imply that any element  $\bar{E}_1 \in \omega(E_1; T)$  is comparable in the partial ordering on  $K_V$  to some element  $\bar{E}_2 \in \omega(E_2; T)$  and vice-versa.

**Remark 4.2.** Theorem 7.2 of [5] implies that  $f|_{D_V \cap \text{int } K_V}$  has a continuous extension  $\tilde{f} : D_V \rightarrow D_V$ , and a standard argument using the fixed point index then shows that  $\tilde{f}$  has an eigenvector  $E \in D_V$  with a corresponding eigenvalue  $\lambda > 0$ . This does not show that there is an eigenvector in  $D_V \cap \text{int } K_V$ , and Theorem 4.1 applies to the case where such an eigenvector fails to exist.

We can apply Theorem 4.1 to reproduction–decimation operators.

**Corollary 4.1.** *Let notation be as above, and let  $R$  and  $\Phi$  be as defined in Eqs. (18) and (20). Assume (see Definition 4.1) that  $\{\psi_k \mid 1 \leq k \leq N\}$  satisfies condition I. Let  $q$  be a linear functional which is positive on  $K_V \setminus \{0\}$  and define  $\Sigma = \{E \in \text{int } K_V \mid q(E) = 1\}$ . For  $E \in \text{int } K_V$ , define  $\Lambda(E) = \Phi(R(E))$  and  $T(E) = \Lambda(E)/q(\Lambda(E))$ . Then  $\Lambda(\text{int } K_V) \subset \text{int } K_V$ ,  $\Lambda(D_V \cap \text{int } K_V) \subset D_V \cap \text{int } K_V$ ,  $\Lambda$  is homogeneous of degree one and  $\Lambda$  is order-preserving in the partial ordering from  $K_V$ . If  $E \in \Sigma$ , let  $\omega(E; T)$  denote the omega limit set of  $E$  under the map  $T$ . If  $T$  has no fixed points in  $\Sigma$  (or equivalently,  $\Lambda$  has no eigenvectors in  $\text{int } K_V$ ), then we have*

$$\text{co}\left(\bigcup_{E \in \Sigma} \omega(E; T)\right) \subset \partial K_V \cap \text{cl } \Sigma.$$

Furthermore, for  $E_1, E_2 \in \Sigma$ , every element of  $\omega(E_1; T)$  is comparable to an element of  $\omega(E_2; T)$  in the partial ordering from  $K_V$ .

**Proof.** Under the given assumption we have proved that  $R(\text{int } K_V) \subset \text{int } K_W$  and  $R(D_V \cap \text{int } K_V) \subset D_W \cap \text{int } K_W$  in Proposition 4.1 and Lemma 4.2. Because  $R$  is linear it follows that  $R$  is order-preserving as a map from  $K_V$  to  $K_W$  and homogeneous of degree one. We have also seen that  $\Phi : \text{int } K_W \rightarrow \text{int } K_V$  is order-preserving and homogeneous of degree one and that  $\Phi(D_W \cap \text{int } K_W) \subset D_V \cap \text{int } K_V$ . It follows that  $\Lambda$  satisfies the conditions of Theorem 4.1, so Corollary 4.1 follows from Theorem 4.1 and from the remarks immediately following Theorem 4.1.  $\square$

**Remark 4.3.** In Corollary 4.1 we only use the fact that  $\Lambda$  is order-preserving, homogeneous of degree one and  $\Lambda(D_V \cap \text{int } K_V) \subset D_V \cap \text{int } K_V$ . We do not use any other special properties of reproduction–decimation operators. Some interesting results that only apply to reproduction–decimation operators are known. Peirone has shown [21, Theorem 4.22] that if a reproduction–decimation operator  $\Lambda$  has an eigenvector  $\bar{E} \in \text{int } K_V$  then for all  $E \in \text{int } K_V$ , the omega limit set  $\omega(E; T)$  is a single point in  $\text{int } K_V$ , and that point is an eigenvector of  $\Lambda$ . In [21], Peirone also gives conditions which imply that there is a unique eigenvector in the interior (up to multiplication by a scalar). In [24, Lemma 5.13] and the comments which immediately follow, Sabot shows that if a reproduction–decimation operator has two linearly independent eigenvectors in  $D_V \cap \text{int } K_V$ , then the set of eigenvectors in  $D_V \cap \text{int } K_V$  is unbounded in the Hilbert metric on  $\text{int } K_V$ . We would like to thank Roberto Peirone for bringing these results to our attention.

We can say more about the omega limit sets of normalized reproduction–decimation operators in the special case when  $\text{card } V = 3$ . In this case, the set  $\Sigma = \{E \in \text{int } K_V \mid q(E) = 1\}$  is strictly convex. After all, if  $E_1$  and  $E_2$  are distinct elements of  $\text{cl } \Sigma \setminus \Sigma$ , then there are nonzero elements  $x_1, x_2 \in \langle \mathbb{1}_V \rangle^\perp$  such that  $E_1(x_1) = 0$ ,  $E_2(x_2) = 0$ . Since  $q(E_1) = q(E_2) = 1$  and  $E_1 \neq E_2$ , it follows that  $E_1$  is not a scalar multiple of  $E_2$  and therefore  $x_1$  is not a scalar multiple of  $x_2$ . This implies that  $E_1(x_2) > 0$ , otherwise  $E_1 = 0$  which cannot be the case since  $q(E_1) = 1$ . Similarly,  $E_2(x_1) > 0$ . We see from this that  $\lambda E_1 + (1 - \lambda)E_2 \in \Sigma$  for  $0 < \lambda < 1$ . Therefore  $\Sigma$  is strictly convex.

We can now apply the Denjoy–Wolff type theorem established by Beardon (see [3], and also [7]) for Hilbert metric nonexpansive maps on strictly convex domains to conclude that there exists  $\bar{E} \in \partial K_V$  such that  $T^k(E) \rightarrow \bar{E}$  as  $k \rightarrow \infty$ , for all  $E \in \Sigma$ . This is a stronger result than we are able to prove when  $n > 3$ . Moreover, since  $T(\Sigma \cap D_V) \subset \Sigma \cap D_V$ , it follows that



$\bar{E} \in \Sigma \cap D_V$ . If we assume that  $V = \{1, 2, 3\}$ , an easy argument then implies that  $\bar{E}$  is one of the following three discrete Dirichlet forms:

- (a)  $E_1(x) = \beta(x(2) - x(3))^2$ ,
- (b)  $E_2(x) = \beta(x(1) - x(3))^2$  or
- (c)  $E_3 = \beta(x(1) - x(2))^2$

where  $\beta$  is determined by the condition that  $q(B) = 1$ . Note that Peirone has studied the case when  $\text{card } V = 3$  in [22], and this observation is crucial in that paper, although Peirone establishes it by a different method.

### 5. Denjoy–Wolff theorems for operator-valued maps

In this section we consider another setting, possibly infinite-dimensional, in which the ideas exploited in Theorem 2.1 are useful. Our motivation comes from the kinds of operator-valued maps which arise in the study of operator-valued means: see [20] and references in [20]. It seems natural to begin more generally, however.

Let  $K$  be a closed cone with nonempty interior  $\text{int } K$  in a Banach space  $X$ . For a fixed integer  $m \geq 2$ , let  $K^m = \prod_{i=1}^m K = K \times K \times \dots \times K$  denote the Cartesian product of  $K$  with itself  $m$  times and let  $Y = \prod_{i=1}^m X = X \times X \times \dots \times X$  denote the Cartesian product of  $X$  with itself  $m$  times. Elements of  $Y$  are ordered  $m$ -tuples  $(A_1, A_2, \dots, A_m)$ , with  $A_i \in X$  for  $1 \leq i \leq m$ , and  $Y$  is a Banach space with norm

$$\|(A_1, A_2, \dots, A_m)\| = \sum_{i=1}^m \|A_i\|.$$

$K^m$  is a closed cone with interior  $\text{int } K^m = \prod_{i=1}^m \text{int } K$ . We assume that  $f : \text{int } K^m \rightarrow \text{int } K^m$  is an order-preserving map (in the partial ordering induced by  $K^m$ ) and that  $f$  is homogeneous of degree one. We also assume that  $q : K^m \setminus \{0\} \rightarrow (0, \infty)$  is a continuous map which is homogeneous of degree one, and we shall write  $\Sigma = \{x \in \text{int } K^m \mid q(x) = 1\}$ . If we take  $q(x) = \|x\|$ ,  $\Sigma$  is bounded in norm; but in infinite dimensions there may not exist a linear  $q$  for which  $\Sigma$  is bounded in norm.

We shall maintain the above notation in this section, and if the above assumptions on  $f$ ,  $q$ , and  $\Sigma$  are met, we shall say that hypothesis 4.1 (or H4.1) is satisfied.

As usual  $\mathbb{R}_+^m = \{x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m \mid x_i \geq 0 \text{ for } 1 \leq i \leq m\}$  and  $\text{int } \mathbb{R}_+^m$  denotes the interior of  $\mathbb{R}_+^m$ . If  $\alpha \in \mathbb{R}_+^m$  and  $\sum_{i=1}^m \alpha_i = 1$ , we shall call  $\alpha$  a *probability vector*.

Under the assumptions of H4.1, we shall define  $T : \Sigma \rightarrow \Sigma$  by

$$T(x) = \frac{f(x)}{q(f(x))}. \tag{21}$$

Our crucial hypothesis is that  $f$  leaves invariant a certain finite-dimensional cone. We shall say that H4.2 is satisfied if H4.1 is satisfied and there exists  $E = (E_1, E_2, \dots, E_m) \in \text{int } K^m$  such that for all  $x = (x_1, \dots, x_m) \in \text{int } \mathbb{R}_+^m$  there exists  $y = (y_1, \dots, y_m) \in \text{int } \mathbb{R}_+^m$  with

$$f(x_1 E_1, x_2 E_2, \dots, x_m E_m) = (y_1 E_1, y_2 E_2, \dots, y_m E_m). \tag{22}$$

We now describe an important special case in which H4.1 is satisfied. Let  $H$  be a real Hilbert space and denote by  $\mathcal{L}(H)$  the Banach space of linear maps  $B : H \rightarrow H$ . Let  $X \subset \mathcal{L}(H)$  denote the Banach space of self-adjoint linear maps  $A \in \mathcal{L}(H)$ . Recall that  $A \in X$  is called *positive semi-definite* if  $\langle Ah, h \rangle \geq 0$  for all  $h \in H$  and  $A \in X$  is called *positive definite* if there exists  $c > 0$  such that  $\langle Ah, h \rangle \geq c\langle h, h \rangle$  for all  $h \in H$ . We define

$$K = \{A \in X \mid A \text{ is positive semi-definite}\}, \tag{23}$$

and we note that  $K$  is a closed cone in  $X$  and that

$$\text{int } K = \{A \in X \mid A \text{ is positive definite}\}. \tag{24}$$

For a fixed integer  $m \geq 2$ ,  $Y = \prod_{i=1}^m X$  and  $K^m = \prod_{i=1}^m K$  are defined as before, so elements of  $\text{int } K^m$  are ordered  $m$ -tuples  $A = (A_1, A_2, \dots, A_m)$ , where  $A_i \in X$  is positive definite for  $1 \leq i \leq m$ . To define an order-preserving, homogeneous of degree one map  $f : \text{int } K^m \rightarrow \text{int } K^m$ , for each  $\alpha \in \mathbb{R}_+^m \setminus \{0\}$  and  $A = (A_1, A_2, \dots, A_m) \in \text{int } K^m$ , let

$$M_\alpha(A) = \left( \sum_{i=1}^m \alpha_i A_i^{-1} \right)^{-1}. \tag{25}$$

It is well known that  $A \mapsto A^{-1}$  is an order-reversing map of  $\text{int } K$  into  $\text{int } K$ , so  $A \mapsto M_\alpha(A)$  is an order-preserving map of  $\text{int } K^m$  into  $\text{int } K$ ; and it is clear that  $A \mapsto M_\alpha(A)$  is homogeneous of degree one. More generally if  $U_i : H \rightarrow H$  is an orthogonal map (so  $U_i^* U_i = U_i U_i^* = I$ ) for  $1 \leq i \leq m$ , and we write  $U = (U_1, U_2, \dots, U_m)$ , define

$$M_{\alpha,U}(A) = \left( \sum_{i=1}^m \alpha_i U_i^* A_i^{-1} U_i \right)^{-1}, \tag{26}$$

and note that  $A \mapsto M_{\alpha,U}(A)$  is an order-preserving map from  $\text{int } K^m$  to  $\text{int } K$ .

For each  $k$ ,  $1 \leq k \leq m$ , let  $\Gamma_k$  be a finite, nonempty set of ordered pairs  $(\alpha, U)$ , where  $\alpha \in \mathbb{R}_+^m \setminus \{0\}$  and  $U = (U_1, U_2, \dots, U_m)$  is an ordered  $m$ -tuple of orthogonal operators  $U_j : H \rightarrow H$ . Define  $f_k : \text{int } K^m \rightarrow \text{int } K$  by

$$f_k(A) = \sum_{(\alpha,U) \in \Gamma_k} M_{\alpha,U}(A) \tag{27}$$

where  $A = (A_1, A_2, \dots, A_m) \in \text{int } K^m$ , and define  $f : \text{int } K^m \rightarrow \text{int } K^m$  by

$$f(A) = (f_1(A), f_2(A), \dots, f_m(A)). \tag{28}$$

If  $I$  denotes the identity operator on  $H$  and if we take  $E = (I, I, \dots, I)$  in the statement of H4.2, we see that  $f$  satisfies H4.2. We also note that, in general, if  $f : \text{int } K^m \rightarrow \text{int } K^m$  and  $g : \text{int } K^m \rightarrow \text{int } K^m$  both satisfy H4.2 for the same  $E$ , then  $g \circ f$  satisfies H4.2 for that  $E$ . If  $\Gamma'_k$  is a finite subset of  $\mathbb{R}_+^m \setminus \{0\}$  for  $1 \leq k \leq m$ , and  $f'_k : \text{int } K^m \rightarrow \text{int } K$  is defined by

$$f'_k(A) = \sum_{\alpha \in \Gamma'_k} M_\alpha(A) \tag{29}$$

and  $f$  is defined by Eq. (29), then we can take  $E = (E_1, E_1, \dots, E_1)$  for any  $E_1 \in \text{int } K$  and  $f$  will satisfy H4.2 for this  $E$ .

We now return to the more general setting of H4.1. Recall that a closed cone  $C$  in a Banach space  $X$  is called finite-dimensional if the linear span of  $C$  is a finite-dimensional linear subspace of  $X$ . If  $C$  is a closed, finite-dimensional cone in a Banach space  $X$ , it is a standard consequence of the Hahn–Banach theorem that there exists a continuous linear functional  $\theta \in X^*$  with  $\theta(A) > 0$  for all  $A \in C \setminus \{0\}$ . If, in the context of H4.1, we take  $E = (E_1, E_2, \dots, E_m) \in \text{int } K^m$ , we can define a finite-dimensional closed cone  $C_E \subset K^m$  by

$$C_E = \{(x_1 E_1, x_2 E_2, \dots, x_m E_m) \mid x = (x_1, x_2, \dots, x_m) \in \mathbb{R}_+^m\}. \tag{30}$$

The reader can verify that  $A = (x_1 E_1, x_2 E_2, \dots, x_m E_m)$  is contained in  $\text{int } K^m \cap C_E$  if and only if  $x = (x_1, x_2, \dots, x_m) \in \text{int } \mathbb{R}_+^m$ . One can also verify that if  $A = (x_1 E_1, x_2 E_2, \dots, x_m E_m) \in \text{int } K^m \cap C_E$  and  $B = (y_1 E_1, y_2 E_2, \dots, y_m E_m) \in \text{int } K^m \cap C_E$ , then

$$d_{C_E}(A, B) = d_{K^m}(A, B) = d(x, y). \tag{31}$$

Here  $d_{C_E}$  (respectively,  $d_{K^m}$ ) denotes Hilbert’s projective metric in  $C_E$  (respectively,  $K^m$ ), and  $d$  denotes Hilbert’s projective metric on  $\mathbb{R}_+^m$ . The linear map

$$L : (x_1, x_2, \dots, x_m) \rightarrow (x_1 E_1, x_2 E_2, \dots, x_m E_m)$$

gives a linear isomorphism between  $\mathbb{R}_+^m$  and  $C_E$ , so  $C_E$  is a polyhedral cone.

**Lemma 5.1.** *Assume H4.2 and for every  $x = (x_1, \dots, x_m) \in \text{int } \mathbb{R}_+^m$ , let  $y = (y_1, \dots, y_m) = g(x)$  be given by Eq. (22). Then  $g : \text{int } \mathbb{R}_+^m \rightarrow \text{int } \mathbb{R}_+^m$  is order-preserving (in the partial ordering induced by  $\mathbb{R}_+^m$ ) and homogeneous of degree one. If  $k \geq 1$ ,  $E = (E_1, E_2, \dots, E_m)$  is as in Eq. (22) and  $z = (z_1, z_2, \dots, z_m) = g^k(x)$ , we also have*

$$f^k(x_1 E_1, x_2 E_2, \dots, x_m E_m) = (z_1 E_1, z_2 E_2, \dots, z_m E_m). \tag{32}$$

**Proof.** For any  $u, v \in \text{int } \mathbb{R}_+^m$ , one can easily verify that

$$(u_1 E_1, \dots, u_m E_m) \leq (v_1 E_1, \dots, v_m E_m)$$

in the partial ordering on  $K^m$  if and only if  $u \leq v$  in the partial ordering on  $\mathbb{R}_+^m$ . Since we assume that  $f$  is order-preserving it follows that  $g$  is order-preserving. The final statement of Lemma 5.1 follows easily by induction.  $\square$

**Proposition 5.1.** *Let  $C$  be a closed cone with nonempty interior in a Banach space  $X$ . Let  $q : C \rightarrow [0, \infty)$  be a continuous map which is homogeneous of degree one and satisfies  $q(x) > 0$  for all  $x \in C \setminus \{0\}$ . Define  $\Sigma = \{x \in \text{int } C \mid q(x) = 1\}$  and let  $T : \Sigma \rightarrow \Sigma$  be a map which is nonexpansive with respect the Hilbert’s projective metric  $d$  on  $C$ . Let  $D \subset C$  be a closed finite-dimensional cone such that  $\Sigma \cap D$  is nonempty and  $T(\Sigma \cap D) \subset \Sigma \cap D$ . Then  $T$  has a fixed point in  $\Sigma$  if and only if  $T$  has a fixed point in  $\Sigma \cap D$ .*

**Proof.** Since  $\Sigma \cap D$  is nonempty, the closed cone  $C_1 = C \cap \text{span}\{D\}$  (where  $\text{span}\{D\}$  denotes the linear span of  $D$ ) is nonempty. If  $d_{C_1}$  denotes Hilbert’s projective metric on  $C_1$ , and  $d_C$  denote Hilbert’s projective metric on  $C$ , then for any  $x, y \in \text{int } C \cap C_1$ ,  $d_{C_1}(x, y) = d_C(x, y)$ . This follows from the definition of Hilbert’s projective metric, since for  $x, y \in \text{span}\{D\}$  and  $\beta > \alpha > 0$ ,  $\alpha x \leq_{C_1} y \leq_{C_1} \beta x$  if and only if  $\alpha x \leq_C y \leq_C \beta x$ .

Since  $C_1$  is finite-dimensional, there exists a linear functional  $\theta \in X^*$  such that  $\theta(x) > 0$  for all  $x \in C_1 \setminus \{0\}$ . Let  $S = \{x \in \text{int } C \cap C_1 \mid \theta(x) = 1\}$ . For any  $x \in S \cap D$ , define the map  $\hat{T}(x)$  by

$$\hat{T}(x) = \frac{T(x/q(x))}{\theta(T(x/q(x)))}.$$

Suppose that  $z \in \Sigma$  is a fixed point of  $T$ . Let  $y \in S \cap D$ , define  $R = d(y, z)$  and let  $\Gamma = \{x \in S \cap D \mid d(x, y) \leq R\}$ . Since  $\{x \in \text{int } C \mid d(x, z) \leq R\}$  is convex (see Lemma 4.1 in [16]), it follows that  $\Gamma$  is convex. Suppose that  $x \in \Gamma$ . Then clearly,  $\hat{T}(x) \in S \cap D$ , and furthermore,

$$d_{C_1}(\hat{T}(x), z) = d_{C_1}(T(x/q(x)), z) \leq d_{C_1}(x/q(x), z) = d_{C_1}(x, z) \leq R.$$

Therefore  $\hat{T}(\Gamma) \subset \Gamma$ . We claim that  $\Gamma$  is compact. It is clear that  $\Gamma$  is closed in Hilbert’s projective metric as a subset of  $S$ . Since  $C_1$  is a finite-dimensional cone, it is well known that the metric space  $(S, d_{C_1})$  is proper, that is every closed bounded subset of  $(S, d_{C_1})$  is compact. Thus  $\Gamma$  is compact. Since  $\hat{T} : \Gamma \rightarrow \Gamma$  is nonexpansive with respect to  $d_{C_1}$ , it is continuous, and we may apply the Brouwer fixed point theorem to conclude that  $\hat{T}$  has a fixed point  $\hat{z} \in \Gamma$ . This concludes the proof, since  $\hat{z}/q(\hat{z}) \in \Sigma \cap D$  is a fixed point of  $T$ .  $\square$

**Remark 5.1.** In our particular application, we will not need Proposition 5.1, but we believe that Proposition 5.1 has independent interest and may prove useful in more general settings.

If H4.1 holds and  $A \in \Sigma$ , recall that  $\omega(A; T)$ , the omega limit set of  $A$  under  $T$ , is given by Eq. (4). If  $\text{cl}(\bigcup_{k \geq 1} T^k(A))$  is compact,  $\omega(A; T)$  is compact and nonempty; but in general  $\omega(A; T)$  may be empty.

In the following, recall that a closed cone  $C$  in a Banach space  $X$  is *normal* if there exists a constant  $M$  such that whenever  $A, B \in C$  and  $A \leq B$  it follows that  $\|A\| \leq M\|B\|$ .

**Theorem 5.1.** Assume H4.2, let  $C_E$  be given by Eq. (30) and assume that  $T|_{\Sigma \cap C_E}$  has no fixed points. Assume that the cone  $K^m$  in H4.2 is normal and that there exists a constant  $M_1$  such that for all  $A, B \in K^m$  with  $A \leq B$  we have

$$q(A) \leq M_1 q(B). \tag{33}$$

Then there exists a nonempty, proper subset  $J \subset \{1, 2, \dots, m\}$  with the following property: if  $A \in \Sigma$  and  $\epsilon > 0$ , there exists a constant  $n = n(\epsilon, A)$  such that for all  $B = (B_1, \dots, B_m) \in \overline{\text{co}}(\bigcup_{k \geq n} T^k(A))$  we have  $\|B_j\| < \epsilon$  for all  $j \in J$ . In particular, if  $B \in \bigcap_{n \geq 1} \overline{\text{co}}(\bigcup_{k \geq n} T^k(A))$ , then  $B_j = 0$  for all  $j \in J$ . If we replace the assumptions that  $K^m$  is normal and that Eq. (33) holds by the assumption that for every  $A \in \Sigma$ ,  $\text{cl}(\bigcup_{k \geq n} T^k(A))$  is compact, then the conclusions of Theorem 5.1 still hold.

**Proof.** By using Theorem 2.2 in [11] directly or by using Theorem 2.1 with  $K_1 = K_2 = K_p$ , we see that  $\overline{\text{co}}(\omega(E; T)) \subset \partial C_E$ . It follows that there exists a nonempty, proper subset  $J \subset \{1, 2, \dots, m\}$  such that for every  $B = (B_1, B_2, \dots, B_m) \in \overline{\text{co}}(\omega(E; T))$ ,  $B_j = 0$  for all  $i \in J$ .

If  $A \in \Sigma$ , there exist  $\alpha > 0$  and  $\beta > 0$  with  $\alpha E \leq A$  and  $A \leq \beta E$ . Because  $f$  is order-preserving it follows that  $\alpha f^k(E) \leq f^k(A)$  and  $f^k(A) \leq \beta f^k(E)$  for all  $k \geq 1$ . We now assume that Eq. (33) holds, so  $q(f^k(A)) \leq \beta M_1 q(f^k(E))$  and  $\alpha M_1^{-1} q(f^k(E)) \leq q(f^k(A))$ . Because  $\text{cl}(\bigcup_{k \geq n} T^k(E))$  is compact for each  $n \geq 1$  and  $\omega(E; T) = \bigcap_{n \geq 1} (\text{cl}(\bigcup_{k \geq n} T^k(E)))$ , point set topology implies that for every open neighborhood  $U$  of  $\omega(E; T)$ , there exists an integer  $n(U) \geq 1$  such that  $\text{cl}(\bigcup_{k \geq n} T^k(E)) \subset U$  for all  $n \geq n(U)$ . It follows that for every  $\epsilon > 0$ , there exists an integer  $n(\epsilon)$  such that if  $B = T^k(E) = (B_1, \dots, B_m)$  and  $k \geq n(\epsilon)$ , then  $\|B_j\| \leq \epsilon$  for all  $j \in J$ . Furthermore, if  $B \in \overline{\text{co}}(\bigcup_{k \geq n} T^k(E))$  and  $n \geq n(\epsilon)$ , then  $\|B_j\| \leq \epsilon$  for all  $j \in J$ . If  $B = (B_1, \dots, B_m) \in \bigcup_{k \geq n} T^k(A)$  and  $n \geq n(\epsilon)$ , we conclude that

$$B = T^k(A) = \frac{f^k(A)}{q(f^k(E))} \frac{q(f^k(E))}{q(f^k(A))} \leq \beta T^k(E) \left( \frac{M_1}{\alpha} \right).$$

Because we assume that  $K^m$  (or, equivalently,  $K$ ) is normal, we conclude that for all  $j \in J$ ,  $\|B_j\| \leq (\frac{\beta}{\alpha}) M_1 M \epsilon$ , where  $M$  is the constant in the definition of normality. It follows that for  $n \geq n(\epsilon)$  and for all  $B = (B_1, \dots, B_m) \in \overline{\text{co}}(\bigcup_{k \geq n} T^k(A))$ ,  $\|B_j\| \leq (\frac{\beta}{\alpha}) M_1 M \epsilon$  for  $j \in J$ . This proves the first part of Theorem 5.1.

Now suppose that  $K^m$  may not be normal and that Eq. (33) may fail, but that  $\text{cl}(\{T^k(A) \mid k \geq 1\})$  is compact for some  $A \in \Sigma$ . We first claim that

$$\sup \left\{ \frac{q(f^k(E))}{q(f^k(A))} : k \geq 1 \right\} \leq \infty.$$

If not, there exists a sequence  $k_i \rightarrow \infty$  with  $q(f^{k_i}(A))/q(f^{k_i}(E)) \rightarrow 0$ . By taking a further subsequence, we can assume also that  $T^{k_i}(A) \rightarrow \eta$  and  $T^{k_i}(E) \rightarrow \zeta$  where  $\eta$  and  $\zeta$  are nonzero elements of  $K^m$ . However, we have that

$$\alpha T^{k_i}(E) \leq T^{k_i}(A) \left( \frac{q(f^{k_i}(A))}{q(f^{k_i}(E))} \right),$$

and letting  $i \rightarrow \infty$  we conclude that  $-\alpha \zeta \in K^m$ , which contradicts the definition of a cone. Thus there is a constant  $\gamma$  such that  $q(f^k(E))/q(f^k(A)) \leq \gamma$  for all  $k \geq 1$ , and that implies that

$$T^k(A) \leq \beta T^k(E) \left( \frac{q(f^k(E))}{q(f^k(A))} \right) \leq \beta \gamma T^k(E). \tag{34}$$

Let  $(T^k(A))_j$  denote the  $j$ th component of  $T^k(A)$  for  $1 \leq j \leq m$ . We claim that for each  $\epsilon > 0$ , there exists  $N(\epsilon)$  with  $\|(T^k(A))_j\| \leq \epsilon$  for all  $j \in J$  and for all  $k \geq N(\epsilon)$ . If not, there exist  $j \in J$  and a sequence  $k_i \rightarrow \infty$  such that  $\|(T^{k_i}(A))_j\| \geq \epsilon$  for all  $i$ . By taking a subsequence, we can assume that  $(T^{k_i}(A))_j \rightarrow \eta_j$  as  $i \rightarrow \infty$ , where  $\eta_j \in K$ , and  $\|\eta_j\| \geq \epsilon$ . On the other hand, we already know that  $(T^{k_i}(E))_j \rightarrow 0$ , so taking limits in Eq. (34), we find that  $-\eta_j \in K$ , a contradiction. This completes the proof.  $\square$

Theorem 5.1 is directly applicable to the case of operator-valued means.

**Corollary 5.1.** *Let  $H$  be a real Hilbert space, let  $X$  denote the Banach space of bounded, self-adjoint linear maps  $B : H \rightarrow H$  and let  $K$  denote the cone of positive semi-definite operators in  $X$ . For a given integer  $m \geq 2$ , let  $K^m = \prod_{i=1}^m K$  and let  $F : \text{int } K^m \rightarrow \text{int } K^m$  denote the composition of a finite number of maps of the form given by Eqs. (27) and (28). Assume that  $F$  has no eigenvectors in  $C_E \cap \text{int } K^m$ , where  $C_E = \{(x_1 I, x_2 I, \dots, x_m I) \mid x = (x_1, x_2, \dots, x_m) \in \mathbb{R}_+^m\}$ . For  $A = (A_1, \dots, A_m) \in K^m$ , define  $q(A) = \|A\| = \sum_{i=1}^m \|A_i\|$ , define  $\Sigma = \{A \in \text{int } K^m \mid q(A) = 1\}$  and define  $T : \Sigma \rightarrow \Sigma$  by  $T(A) = F(A)/q(F(A))$ . Then there exists a nonempty, proper subset  $J$  of  $\{1, 2, \dots, m\}$  with the following property: if  $A \in \Sigma$  and  $\epsilon > 0$ , there exists an integer  $n = n(\epsilon, A)$  such that for all  $B = (B_1, B_2, \dots, B_m) \in \overline{\text{co}}(\bigcup_{k \geq n} T^k(A))$ ,  $\|B_j\| < \epsilon$  for all  $j \in J$ .*

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