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SOME NONLINEAR WEAK ERGODIC THEOREMS*

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Abstract. Let C be a cone with nonempty interior in a Banach space and, for $j \geqq 1$, let $f_j: \mathring{C} \to \mathring{C}$ be a sequence of maps. It is frequently assumed that each f_j is homogeneous of degree 1 and order-preserving with respect to the partial ordering induced by C; but it is not assumed that $f_j(C-\{0\}) \subset \mathring{C}$. If $F_m = f_m f_{m-1} \cdots f_1$, the composition of the first m f_j, and if d denotes Hilbert's projective metric, then theorems (usually called weak ergodic theorems in the population biology literature) can be proved ensuring that, for all x and y in $\mathring{C}$, $\lim_{m\to\infty} d(F_m(x), F_m(y)) = 0$ and (if C is normal) $\lim_{m\to\infty} \|(F_m(x)/\|F_m(x)\|) - (F_m(y)/\|F_m(y)\|)\| = 0$. If $u \in \mathring{C}$ is fixed and assumptions on the f_j are strengthened, it can be proved that for every $z \in \mathring{C}$ there exists $\lambda(x) > 0$ such that $\lim_{m\to\infty} \|F_m(x) - \lambda(x) F_m(u)\| = 0$. These theorems are applied to the case where $C = \{x \in \mathbb{R}^n: x_i \geqq 0 \text{ for } 1 \leqq i \leqq n\}$ and where the maps f_j belong to a class M arising in the theory of "means and their iterations" and in certain problems from population biology.

Key words. Hilbert's projective metric, nonlinear weak ergodic theorems, cones, nonlinear cone maps, positive linear operators

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1. Preliminaries. In an effort to make this paper self-contained, we begin by recalling some definitions and theorems from the literature. By a cone C (with vertex at 0) in a Banach space X we mean a closed, convex subset C of X such that (a) $tC \subset C$ for all $t \geqq 0$ and (b) if $x \in C - \{0\}$, then $-x \notin C$. A cone induces a partial ordering on X by

$$x \leqq y \quad \text{if and only if } y - x \in C.$$

Two elements $x, y \in C$ will be called "comparable" if there exist positive reals α and β such that

$$\alpha x \leqq y \leqq \beta x, \qquad \alpha, \beta > 0.$$

If x and y in C are comparable, we follow Bushell [6] and define

$$M(y/x) = \inf\{\beta > 0: y \leqq \beta x\}, \tag{1.1}$$

$$m(y/x) = \sup\{\alpha > 0: \alpha x \leqq y\}. \tag{1.2}$$

If $u \in C - \{0\}$, C_u will denote the set of elements of C that are comparable to u. If u is an element of the interior of C, C_u is the interior of C. In general C_u satisfies all properties of a cone except closedness.

Associated to the set C_u is a natural normed linear space E_u,

$$E_u = \{x \in X: \text{there exists } \alpha > 0 \text{ such that } -\alpha u \leqq x \leqq \alpha u\}.$$

For $x \in E_u$, we define a norm $|x|_u$ by

$$|x|_u = \inf\{\alpha > 0: -\alpha u \leqq x \leqq \alpha u\}.$$

A cone C in a Banach space X is called "normal" if there exists a constant M such that

$$\|x\| \leqq M\|y\|$$

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for all $x, y \in C$ such that $x \leqq y$. A cone C is "total" if the closure of the linear span of C equals X. If C is a cone, C^* will always denote the set of continuous linear functionals $\psi \in X^*$ such that $\psi(x) \geqq 0$ for all $x \in C$. It is not hard to see that if C is a total cone, then C^* is a cone.

The basic technical tool we will use in this paper is the so-called "Hilbert's projective metric" d. If C is a cone in a Banach space X and $u \in C - \{0\}$, then for $x, y \in C_u$, define $\beta = M(y/x)$, $\alpha = m(y/x)$, and

$$d(x, y) = \log\left(\frac{\beta}{\alpha}\right), \tag{1.3}$$

$$\bar{d}(x, y) = \log(\max(\beta, \alpha^{-1})). \tag{1.4}$$

We can easily prove (see [6]) that the projective metric d satisfies all properties of a metric except that $d(y, x) = 0$ if and only if $y = \lambda x$ for some $\lambda > 0$. On the other hand, $\bar{d}$, which was introduced by Thompson [36], is a metric on C_u. If $\Sigma = \{x \in C_u \colon \|x\| = 1\}$, (Σ, d) is a metric space, and it is natural to ask if (Σ, d) is complete. It is proved in [37] that (Σ, d) is complete if and only if

$$\sup\{\|x\| \colon 0 \leqq x \leqq u\} < \infty, \tag{1.5}$$

and (1.5) is satisfied if and only if E_u is a complete normed linear space. Also, one can show that $(C_u, \bar{d})$ is complete if and only if (1.5) is satisfied. It should be noted that the results in [37] are closely related to much earlier theorems of Thompson [36] and Birkhoff (see Theorem 5 in [5] and Remark 1.1 in [25]). Notice also that if $\psi \in C^*$ and $\psi(u) > 0$ and

$$\Sigma_\psi = \{x \in C_u \colon \psi(x) = 1\},$$

then (Σ_ψ, d) is complete if and only if (Σ, d) is complete, because (Σ_ψ, d) and (Σ, d) are isometric.

If $K = \{x \in \mathbb{R}^n \colon x_i \geqq 0 \text{ for } 1 \leqq i \leqq n\}$, K will be called the "standard cone in $\mathbb{R}^n$." Obviously, K is normal, so if $\Sigma_1 = \{x \in \mathring{K} \colon \sum_{i=1}^n x_i = 1\}$ or $\Sigma_2 = \{x \in \mathring{K} \colon x_1 = 1\}$, the remarks above show that (Σ_1, d) and (Σ_2, d) are complete.

We also need to recall some results about positive linear operators. Suppose that C is a cone in a Banach space X and that $L \colon X \to X$ is a bounded linear operator such that $L(C) \subset C$. Assume that Lx and Ly are comparable for all $x, y \in C$ such that $Lx \neq 0$ and $Ly \neq 0$ and define a number $\Delta(L)$, the "projective diameter of $L(C) - \{0\}$" by

$$\Delta(L) = \sup\{d(Lx, Ly) \colon x, y \in C, Lx \neq 0 \text{ and } Ly \neq 0\}. \tag{1.6}$$

If $Lx = 0$ for all $x \in C$, we define $\Delta(L) = 0$. If L is as above and $\Delta(L) < \infty$ we shall say that "L has finite projective diameter."

If $x, y \in C - \{0\}$ are not comparable, define $d(x, y) = \infty$. If $x, y \in C - \{0\}$ and $M(y/x) < \infty$, define

$$\operatorname{osc}(y/x) = M(y/x) - m(y/x). \tag{1.7}$$

If $M(y/x) = \infty$, define $\operatorname{osc}(y/x) = \infty$. If L is a bounded linear operator such that $L(C) \subset C$ and Lx and Ly are nonzero and comparable for all $x, y \in C - \{0\}$, define

$$k(L) = \inf\{k > 0 \colon d(Lx, Ly) \leqq k d(x, y) \text{ for all } x, y \in C - \{0\}\},$$

$$N(L) = \inf\{\lambda > 0 \colon \operatorname{osc}(Ly/Lx) \leqq \lambda \operatorname{osc}(y/x) \text{ for } x, y \in C - \{0\}\}.$$

It is easy to see that $k(L) \leqq 1$ and $N(L) \leqq 1$. However, if $\Delta(L) < \infty$, results of Birkhoff [4], [5] and Hopf [19], with refinements of Ostrowski [27], Bauer [2], Bushell [6], [7], and others [20], [37] imply that

$$N(L) = k(L) = \tanh\left(\frac{\Delta(L)}{4}\right) < 1. \tag{1.8}$$

As a particular example, note that if K is the standard cone in $\mathbb{R}^n$ and L is an $n \times n$ matrix, all of whose entries are positive, then $\Delta(L) < \infty$. In fact, it is not hard to prove that $\Delta(L) = \sup_{i,j} d(Le_i, Le_j)$, where e_i, $1 \leqq i \leqq n$, is the standard basis of $\mathbb{R}^n$. From this observation and (1.8) we derive an explicit formula (see [6], [35]) for $\Delta(L)$ and $k(L)$.

If C is a cone and $D \subset C$, a map $f: D \to C$ will be called nonexpansive with respect to d if

$$d(f(x), f(y)) \leqq d(x, y) \quad \text{for all } x, y \in D. \tag{1.9}$$

We have the obvious modification for $\bar{d}$. A map $f: D \to C$ will be called "order-preserving" if $f(x) \leqq f(y)$ for all $x, y \in D$ such that $x \leqq y$. The map f will be called "homogeneous of degree 1" on D if

$$f(tx) = tf(x) \quad \text{for all } t > 0 \quad \text{and} \quad x \in D \quad \text{such that } tx \in D,$$

and will be called "subhomogeneous" on D if $f(tx) \geqq tf(x)$ for all t, $0 < t \leqq 1$, and $x \in D$ such that $tx \in D$. It is an easy exercise that if $u \in C - \{0\}$, $D = C_u$ and $f: D \to D$ is order preserving and homogeneous of degree 1, then f is nonexpansive with respect to d: see [6], [25], [29]. Thompson [36] observed that, if $f: C_u \to C_u$ is subhomogeneous and order-preserving, then f is nonexpansive with respect to $\bar{d}$. Potter [29] observed that, for $\psi \in C^*$ with $\psi(u) > 0$, the restriction of f to $\Sigma_\psi = \{x \in C_u : \psi(x) = 1\}$ is nonexpansive with respect to d.

Now suppose that C is a cone in a Banach space X, $u \in C - \{0\}$, and S is a collection of maps $f: C_u \to C_u$. In most of this paper we will assume that f is order preserving and homogeneous of degree 1 for every $f \in S$. Suppose that $f_j \in S$, $1 \leqq j < \infty$, is a sequence of functions in S and define

$$F_n = f_n f_{n-1} f_{n-2} \cdots f_1 \tag{1.10}$$

for $n \geqq 1$. We are interested in finding further conditions ensuring that for all $x, y \in C_u$,

$$\lim_{n \to \infty} d(F_n(x), F_n(y)) = 0. \tag{1.11}$$

Such results are called "weak ergodic theorems" in the population biology literature [11], [17]. The linear theory is well understood: see the excellent survey article [11] by Cohen. If (1.5) is satisfied, it is known (see eq. (1.20a) in [25]) that there exists a constant M such that

$$\begin{aligned} &\|x - y\| \leqq M[\exp(d(x, y)) - 1] \quad \text{for all } x, y \in \Sigma, \\ &\Sigma = \{x \in C_u : \|x\| = 1\}. \end{aligned} \tag{1.12}$$

Using (1.12) we can see that if (1.5) and (1.11) hold and the functions f_j are homogeneous of degree 1, then

$$\lim_{n \to \infty} \|F_n(x)\|F_n(x)\|^{-1} - F_n(y)\|F_n(y)\|^{-1}\| = 0. \tag{1.13}$$

Note that if (1.5) is satisfied, $C \cap E_u$ is a normal cone with nonempty interior C_u in the Banach space E_u, so by working in E_u we can assume that C is normal with nonempty interior.

In fact, the question we are asking is motivated by a particular class of maps $\mathcal{M}$ defined on the interior of the standard cone K in $\mathbb{R}^n$, so we recall the definition of $\mathcal{M}$ (see [24], the Introduction to [25], and § 4 of [23]). Recall that a probability vector σ is a vector $\sigma \in K$ such that $\sum_{i=1}^n \sigma_i = 1$. If r is a real number and σ a probability vector, define $M_{r\sigma} : \mathring{K} \to \mathbb{R}$ by

$$M_{r\sigma}(x) = \left(\sum_{i=1} \sigma_i x_i^r \right)^{1/r}.$$

If $r = 0$, define

$$M_{0\sigma}(x) = \prod_{i=1}^{n} x_i^{\sigma_i}.$$

Such maps, of course, have an extensive classical theory, described in [18]. For $1 \leqq i \leqq n$, let Γ_i be a finite collection of ordered pairs (r, σ), with r a real number and σ a probability vector. For $(r, \sigma) \in \Gamma_i$ let $c_{ir\sigma}$ be a positive real number and define $f_i : \mathring{K} \to (0, \infty)$ by

$$f_i(x) = \sum_{(r,\sigma)\in\Gamma_i} c_{ir\sigma} M_{r\sigma}(x). \tag{1.14}$$

Define f_i to be the ith component of a map $f : \mathring{K} \to \mathring{K}$. If $f : \mathring{K} \to \mathring{K}$ can be written in this form, we will write $f \in M$. If f can be written as in (1.14) in such a way that $r \geqq 0$ for all $(r, \sigma) \in \Gamma_i$ and $1 \leqq i \leqq n$, we say that $f \in M_+$; if f can be written as in (1.14) such that $r < 0$ for all $(r, \sigma) \in \Gamma_i$ and $1 \leqq i \leqq n$, we say that $f \in M_-$. Note that if f is a linear map such that $f(\mathring{K}) \subset \mathring{K}$, then $f \in M_+ \cap M_-$. We define $\mathcal{M}(\mathcal{M}_+, \mathcal{M}_-)$ to be the smallest set of maps $f : \mathring{K} \to \mathring{K}$ such that $\mathcal{M} \supset M(\mathcal{M}_+ \supset M_+, \mathcal{M}_- \supset M_-)$ and $\mathcal{M}(\mathcal{M}_+, \mathcal{M}_-)$ is closed under addition of functions, composition of functions, and multiplication by positive scalars. The class $\mathcal{M}$ arises in the theory of "means and their iterations"; see [1], [12], [15], [23]–[26]. It is proved in [26] (this is not hard) that if $f \in \mathcal{M}$, then f extends continuously to K and $f | \mathring{K}$ is C^∞. We will see that establishing weak ergodic theorems may already be nontrivial when $S = \mathcal{M}$ and $C = K$. Existing nonlinear weak ergodic theorems as, for example, in the work of Fujimoto and Krause [16], are frequently inapplicable. On the other hand, we have not attempted to give an all-inclusive abstract framework: there are examples to which our general theorems are not directly applicable but which can be handled with theorems from [16].

2. Some nonlinear weak ergodic theorems. The following definition will play a crucial role in our subsequent work.

DEFINITION 2.1. Let C be a cone in a Banach space X and D a subset of C such that all elements of D are comparable. Suppose that $f_j : D \to D$, $j \geqq 1$, is a sequence of maps and define $F_m = f_m f_{m-1} \cdots f_1$ to be the composition of the first m functions f_j, $1 \leqq j \leqq m$. We say that $\langle f_j \rangle$ has "the bounded orbit property" (with respect to Hilbert's projective metric) if for every $x \in D$, there exist $y \in D$ and $R > 0$ (possibly depending on x) such that

$$d(F_m(x), y) \leqq R \quad \text{for all } m \geqq 1.$$

If each of the functions f_j is nonexpansive with respect to d, it is an easy exercise (left to the reader) to prove that $\langle f_j \rangle$ has the bounded orbit property if and only if there exist $x_0, y_0 \in D$ and $R_0 > 0$ such that

$$d(F_m(x_0), y_0) \leqq R_0 \quad \text{for all } m \geqq 1.$$

If $D = C_u$ for some $u \in C - \{0\}$ and each of the functions $f_j: D \to D$ in Definition 2.1 is homogeneous of degree 1 and order-preserving, select $\psi \in C^*$ such that $\psi(u) > 0$ and define $\Sigma_\psi = \{x \in C_u: \psi(x) = 1\}$ and $h_j: D \to D$ by

$$h_j(x) = f_j(x)/\psi(f_j(x)).$$

It is another easy exercise (again left to the reader) to prove that $\langle h_j \rangle$ satisfies the bounded orbit property if and only if $\langle f_j \rangle$ does. The maps h_j can also be considered maps of $D_1 = \Sigma_\psi$ to itself and $\langle h_j \rangle$ satisfies the bounded orbit property on D_1 if and only if $\langle h_j \rangle$ satisfies the bounded orbit property on D.

Suppose that C is a cone with nonempty interior in a finite-dimensional Banach space X and $f: \mathring{C} = D \to \mathring{C}$ is homogeneous of degree 1 and order-preserving. We can define $f_j = f$ for all $j \geqq 1$ and ask whether $\langle f_j \rangle$ satisfies the bounded orbit property with respect to d. It is a special case of results in § 4 of [25] that $\langle f_j \rangle$ satisfies the bounded orbit property if and only if f has an eigenvector $v \in \mathring{C}$ (so $f(v) = \lambda v$). Note that if f extends continuously to C, it certainly has an eigenvector in $C - \{0\}$, but the question of whether f has an eigenvector in $\mathring{C}$ may be quite subtle, even if $f \in \mathcal{M}_-$. The reader is referred to [25], [26] for further details. Even the simple-looking four-dimensional map $f \in M_-$ in [34] requires some care. For a complete, rigorous analysis see [26].

We will need the following simple geometrical lemma to prove our first weak ergodic theorem.

LEMMA 2.1. *Let C be a cone in a Banach space X and let S be a subset of C. Assume that all elements of S are comparable and that there exists $\rho > 0$ such that $d(x, y) \leqq \rho$ for all $x, y \in S$. If $w \in C$ and $w = x_1 + x_2$ for points $x_1, x_2 \in S$, we have that $x_1 \geqq \lambda w$ or $x_2 \geqq \lambda w$, where $\lambda = \frac{1}{2}\exp(-\rho)$.*

Proof. Suppose that $w = x_1 + x_2$ as above and that $d(x_1, x_2) = \tau \leqq \rho$. It suffices to prove that

$$x_1 \geqq \mu w \quad \text{or} \quad x_2 \geqq \mu w \quad \text{where } \mu = \tfrac{1}{2}\exp(-\tau).$$

It follows easily from the definition of Hilbert's projective metric that

$$d(x_1, w) \leqq \tau \quad \text{and} \quad d(x_2, w) \leqq \tau. \tag{2.1}$$

In fact, we have strict inequality in (2.1) if $\tau > 0$. Formulae (2.1) imply that there exist positive constants α_j and β_j for $j = 1, 2$ such that

$$\alpha_j w \leqq x_j \leqq \beta_j w \quad \text{and} \quad (\beta_j/\alpha_j) \leqq \exp(\tau). \tag{2.2}$$

Assume that $\alpha_j < \mu$ for $j = 1$ and $j = 2$. Then we obtain

$$x_j \leqq \beta_j w = (\alpha_j/\mu)\mu(\beta_j/\alpha_j) w \leqq c_j w, \tag{2.3}$$

where

$$c_j = (\tfrac{1}{2})(\alpha_j/\mu) < \tfrac{1}{2}. \tag{2.4}$$

By adding (2.3) for $j = 1$ and $j = 2$ we obtain

$$w = x_1 + x_2 \leqq (c_1 + c_2) w = cw. \tag{2.5}$$

Since the constant c in (2.5) is less than 1, we have a contradiction, and therefore it must be true that $\alpha_1 \geqq \mu$ or $\alpha_2 \geqq \mu$. □

We will actually use Lemma 2.1 in the following less general version.

LEMMA 2.2. *Let C be a cone in a Banach space X and let $A: X \to X$ be a bounded linear operator such that $A(C) \subset C$ and A has finite projective diameter (so $\Delta(A) < \infty$*

for $\Delta(A)$ as in (1.6)). Then if $z \in C$ and $z = x + y$ for $x, y \in C$, we have $Ax \geqq \lambda Az$ or $Ay \geqq \lambda Az$, where

$$\lambda = (\tfrac{1}{2}) \exp(-\Delta(A)).$$

Proof. In the notation of Lemma 2.1, define S by

$$S = \{Ax: x \in C \text{ and } Ax \neq 0\}, \tag{2.6}$$

so the diameter ρ of S with respect to Hilbert's projective metric is $\Delta(A)$. If $Ax = 0$ or $Ay = 0$ for x and y as in the statement of the lemma, the result is obvious. Otherwise, if we define $w = Az$, $x_1 = Ax$, and $x_2 = Ay$, Lemma 2.2 follows immediately from Lemma 2.1. □

With these preliminaries we can establish our first weak ergodic theorem. In reading the statement of Theorem 2.1 below, recall that if A and B are linear maps and C is a cone, we say that $A \leqq B$ (in the partial ordering induced by C) if $A(x) \leqq B(x)$ for all $x \in C$.

THEOREM 2.1. *Let C be a cone with nonempty interior in a Banach space X and for each $j \geqq 1$ let $f_j: \mathring{C} \to \mathring{C}$ be a map that is order-preserving and homogeneous of degree 1. For $m \geqq 1$ let $F_m = f_m f_{m-1} \cdots f_1$ denote the composition of the first m maps f_j and let F_0 denote the identity. Assume that there exist $u \in \mathring{C}$, an integer $p \geqq 1$, a real number $R > 0$, and a sequence of bounded linear operators $A_i: X \to X$, $i \geqq 1$, with the following properties:*

(a) *For every $j \geqq 1$, f_j is continuously Fréchet differentiable on $B_R(F_{j-1}(u))$, where $B_R(y) = \{x \in \mathring{C}: d(x, y) < R\}$.*

(b) *The operator A_i satisfies $A_i(\mathring{C}) \subset (\mathring{C})$ for all $i \geqq 1$.*

(c) *If $g_j = f_{jp} f_{jp-1} \cdots f_{jp-p+1}$ and $G_m = g_m g_{m-1} \cdots g_1 = F_{mp}$, then $g_j'(x) \geqq A_j$ for all $x \in B_R(G_{j-1}(u))$ and all $j \geqq 1$.*

(d) *There exists a positive constant k such that $A_j(G_{j-1}(u)) \geqq k g_j(G_{j-1}(u))$ for $j \geqq 1$.*

If A_j has finite projective diameter, let $\Delta(A_j)$ be the projective diameter as in (1.6), and otherwise define $\Delta(A_j) = \infty$ and $\exp(-\Delta(A_j)) = 0$. Then if we have that

$$\lim_{N\to\infty} \sum_{j=1}^{N} \exp(-\Delta(A_j)) = \infty, \tag{2.7}$$

it follows that

$$\lim_{m\to\infty} d(F_m(x), F_m(y)) = 0 \quad \textit{for all } x, y \in \mathring{C}. \tag{2.8}$$

Also, if C is normal, (1.13) is satisfied for all $x, y \in \mathring{C}$. In particular, if $A_j = A$ for all $j \geqq 1$ and $\Delta(A) < \infty$, then (2.8) is satisfied.

Proof. By the triangle inequality it suffices to prove (2.8) for all $x \in \mathring{C}$ and for $y = u$. As has already been noted, each map f_j is nonexpansive with respect to d, so for any $x \in \mathring{C}$, $d(F_m(x), F_m(u))$ is a monotonic decreasing sequence of reals. Thus to prove (2.8) it suffices to prove that

$$\lim_{j\to\infty} d(G_j(x), G_j(u)) = 0 \quad \text{for all } x \in \mathring{C}. \tag{2.9}$$

Fix a number R_1, $0 < R_1 < R$, and suppose we can prove that there exists a sequence of numbers λ_j with $0 < \lambda_j \leqq 1$, such that if $d(y, G_j(u)) \leqq R_1$, $j \geqq 0$, then

$$d(g_{j+1}(y), g_{j+1}(G_j(u))) \leqq \lambda_{j+1} d(y, G_j(u)), \tag{2.10}$$

$$\lim_{N\to\infty} \prod_{j=m}^{N} \lambda_j = 0 \quad \text{for any } m \geqq 1. \tag{2.11}$$

Repeated application of (2.10) and (2.11) then implies that if $d(G_{m-1}(x), G_{m-1}(u)) \leqq R_1$ for some $m \geqq 1$, then

$$d(G_N(x), G_N(u)) \leqq \left(\prod_{j=m}^{N} \lambda_j \right) \quad \text{for all } N \geqq m,$$

which establishes (2.9) in the case $d(x, u) \leqq R_1$.

If $d(y, G_{j-1}(u)) = \rho > R_1$, Proposition 1.9 in [25] implies that there exists y_1 on the line segment connecting y to $G_{j-1}(u)$ such that

$$d(y, y_1) = \rho - R_1 \quad \text{and} \quad d(y_1, G_{j-1}(u)) = R_1. \tag{2.12}$$

Using the nonexpansivity of g_j and (2.10) we obtain from (2.12) that

$$d(g_j(y), G_j(u)) \leqq \rho - R_1 + \lambda_j R_1. \tag{2.13}$$

If $\rho \leqq R_2$, $R_2 > R_1$, we obtain from (2.13) that

$$d(g_j(y), G_j(u)) \leqq \mu_j d(y, G_{j-1}(u)), \tag{2.14}$$

where

$$\mu_j = [R_2 - (1 - \lambda_j) R_1] R_2^{-1}. \tag{2.15}$$

Formula (2.14) was proved under the assumption that $R_1 < \rho \leqq R_2$, but because $\mu_j \geqq \lambda_j$, the equation holds for $\rho \leqq R_2$.

If $d(x, u) \leqq R_2$, the nonexpansive property of G_j implies that

$$d(G_j(x), G_j(u)) \leqq R_2 \quad \text{for all } j \geqq 0.$$

Thus by repeated applications of (2.14) we obtain

$$d(G_N(x), G_N(u)) \leqq \left(\prod_{j=1}^{N} \mu_j \right) d(x, u),$$

so (2.9) will follow if we can prove that

$$\lim_{N \to \infty} \left(\prod_{j=1}^{N} \mu_j \right) = 0. \tag{2.16}$$

If $\lambda_j \leqq \frac{1}{2}$ for infinitely many indices j, we can easily see that there exists a constant $c < 1$ such that $0 \leqq \mu_j \leqq c$ for infinitely many indices c and (2.16) will be satisfied. Thus we can assume that $\frac{1}{2} < \lambda_j \leqq 1$ for $j \geqq m$, so $\frac{1}{2} < \mu_j \leqq 1$ for $j \geqq m$. Under these conditions it is well known and easily checked that

$$\lim_{N \to \infty} \prod_{j=m}^{N} \lambda_j = 0 \Leftrightarrow \sum_{j=m}^{\infty} -\log(\lambda_j) = \infty \Leftrightarrow \sum_{j=m}^{\infty} (1 - \lambda_j) = \infty,$$

$$\lim_{N \to \infty} \prod_{j=m}^{N} \mu_j = 0 \Leftrightarrow \sum_{j=m}^{\infty} -\log(\mu_j) = \infty \Leftrightarrow \sum_{j=m}^{\infty} (1 - \mu_j) = \infty.$$

Because $(1 - \mu_j) = (R_1/R_2)(1 - \lambda_j)$, the equations above imply that (2.16) is satisfied.

Thus it suffices to prove that (2.10) and (2.11) can be satisfied. For a fixed $\psi \in C^* - \{0\}$ it suffices (by homogeneity) to define $u_j = G_j(u)/\psi(G_j(u))$ and to prove that

$$d(g_{j+1}(u_j), g_{j+1}(y)) \leqq \lambda_{j+1} d(y, u_j)$$

for all y such that $\psi(y) = 1$ and $d(y, u_j) \leqq R_1 < R$. Recall (see Lemma 4.1 in [25] or argue directly) that

$$V_{R_1}(u_j) = \{y : d(y, u_j) \leqq R_1\}$$

is convex. For notational convenience, define $G_{j+1}=g$, $A_{j+1}=A$, and $u_j=v$ and let α and β be positive numbers such that

$$\alpha v \leqq y \leqq \beta v \quad \text{and} \quad \log(\beta/\alpha)=\rho=d(y,v).$$

The normalization $\psi(y)=\psi(v)=1$ implies that $\alpha \leqq 1$ and $\beta \geqq 1$. If we define

$$w_t=(1-t)(\alpha v)+ty \quad \text{and} \quad z_t=(1-t)y+t(\beta v), \quad 0\leqq t\leqq 1,$$

we obtain

$$g(y)-g(\alpha v)=\int_0^1 g'(w_t)(y-\alpha v)\,dt,$$

$$g(\beta v)-g(y)=\int_0^1 g'(z_t)(\beta v-y)\,dt.$$

If we recall that $g'(z_t)\geqq A$, we obtain from the preceding two equations that

$$\alpha g(v)+A(y-\alpha v)\leqq g(y)\leqq \beta g(v)-A(\beta v-y). \tag{2.17}$$

Note that

$$A(y-\alpha v)+A(\beta v-y)=(\beta-\alpha)Av,$$

so Lemma 2.2 implies that there exists a positive constant $\gamma=\gamma_{j+1}=(\frac{1}{2})\exp(-\Delta(A_{j+1}))$ such that

$$A(y-\alpha v)\geqq \gamma(\beta-\alpha)Av \quad \text{or} \quad A(\beta v-y)\geqq \gamma(\beta-\alpha)Av.$$

For definiteness we assume that

$$A(y-\alpha v)\geqq \gamma(\beta-\alpha)Av. \tag{2.18}$$

The proof in the other case is essentially the same.

Next, remember that we assume the existence of a positive constant k, independent of $j\geqq 0$, such that

$$A(v)=A_{j+1}(u_j)\geqq kg(v)=kg_{j+1}(u_j). \tag{2.19}$$

If we use (2.17)-(2.19) we obtain

$$\alpha g(v)+k\gamma(\beta-\alpha)g(v)\leqq g(y)\leqq \beta g(v), \tag{2.20}$$

where $\log(\beta/\alpha)=d(y,v)$ and $\gamma=(\frac{1}{2})\exp(-\Delta(A_{j+1}))$. Formula (2.20) implies that $k\gamma\leqq 1$ and

$$d(g(v),g(y))\leqq \log\left(\frac{\beta}{\alpha+k\gamma(\beta-\alpha)}\right). \tag{2.21}$$

If we define $s=\beta/\alpha$, with $1\leqq s\leqq \exp(R_1)$, and recall that $d(v,y)=\log(s)$, we obtain from (2.21) that $d(g(v),g(y))=d(g_{j+1}(u_j),g_{j+1}(y))\leqq \lambda_{j+1}d(u_j,y)$, where

$$\lambda_{j+1}=\sup_{1<s\leqq \exp(R_1)}\frac{\varphi_1(s)}{\varphi_2(s)}, \tag{2.22}$$

with $\varphi_1(s)=\log[s(1+k\gamma(s-1))^{-1}]$ and $\varphi_2(s)=\log(s)$.

Because $\varphi_j(1)=0$, the generalized mean value theorem implies that for each s, $1<s\leqq \exp(R_1)$, there exists σ with $1<\sigma<s$ such that

$$\varphi_1(s)/\varphi_2(s)=\varphi_1'(\sigma)/\varphi_2'(\sigma)=(1-k\gamma)(1+k\gamma(\sigma-1))^{-1}, \tag{2.23}$$

so

$$(2.24) \qquad \lambda_{j+1} = 1 - \left(\frac{k}{2}\right) \exp(-\Delta(A_{j+1})).$$

The same sort of argument that we have used already proves that

$$\lim_{N\to\infty} \prod_{j=m}^{N} \lambda_j = 0 \quad \text{for every } m \geqq 1$$

if and only if (2.7) is satisfied.

This completes the proof of (2.8); the remaining assertions of the theorem are straightforward and left to the reader. □

Remark 2.1. Let hypotheses and notation be as in Theorem 2.1. However, do not assume condition (d), and suppose that $\langle g_j \rangle$ satisfies the bounded orbit property. Then condition (d) is equivalent to the following condition (d′):

(d′) There exists a positive constant k_1 such that

$$A_j(u) \geqq k_1 g_j(u) \quad \text{for all } j \geqq 1.$$

We will prove that (d′) implies (d); the opposite implication is proved similarly. If u_j is as defined in the proof of Theorem 2.1, the bounded orbit hypothesis implies that there exist positive constants c_j and d_j, $j \geqq 1$, such that

$$c_j u_0 \leqq u_j \leqq d_j u_0 \quad \text{and} \quad d_j / c_j \leqq M,$$

where M is a constant independent of j. It follows that

$$c_j g_{j+1}(u_0) \leqq g_{j+1}(u_j) \leqq d_j g_{j+1}(u_0),$$

$$c_j A_{j+1}(u_0) \leqq A_{j+1}(u_j) \leqq d_j A_{j+1}(u_0).$$

If we use these inequalities and hypothesis (d′) we find that

$$g_{j+1}(u_j) \leqq d_j g_{j+1}(u_0) \leqq k_1^{-1} d_j A_{j+1}(u_0) \leqq k_1^{-1}\left(\frac{d_j}{c_j}\right) A_{j+1}(u_j)$$

$$\leqq k_1^{-1} M A_{j+1}(u_j),$$

which is equivalent to hypothesis (d).

It may be unclear how we can expect to find operators A_j such as those in Theorem 2.1. The next corollary shows that under mild assumptions a scalar multiple of $g_j'(G_{j-1}(u))$ can serve as A_j.

COROLLARY 2.1. *Let C be a cone with nonempty interior in a Banach space X, and let $f_j : \mathring{C} \to \mathring{C}$, $j \geqq 1$, be a sequence of maps that are order-preserving and homogeneous of degree 1. For a fixed $p \geqq 1$, let g_j and G_m be as defined in Theorem 2.1, and assume that there exist $r > 0$ and $u \in \mathring{C}$ such that f_j is C^1 on $B_r(F_{j-1}(u))$ for all $j \geqq 1$, where $B_r(x) = \{y \in \mathring{C}: d(y, x) < r\}$. Define $u_j = G_j(u)/\|G_j(u)\|$ and assume that there exist positive constants c and $\rho \leqq r$ such that, if $d(x, u_{j-1}) < \rho$ and $j \geqq 1$, then*

$$(2.25) \qquad g_j'(x) \geqq c g_j'(u_{j-1}).$$

Finally, assume that the operators $B_j = g_j'(G_{j-1}(u)) = g_j'(u_{j-1})$ satisfy

$$(2.26) \qquad \sum_{j=1}^{\infty} \exp(-\Delta(B_j)) = \infty,$$

where $\Delta(B_j)$ *is given by* (1.6) *if* B_j *has finite projective diameter, and that they satisfy* $\exp(-\Delta(B_j)) = 0$ *otherwise. Then it follows that*

$$\lim_{n\to\infty} d(F_n(x), F_n(y)) = 0 \quad \textit{for all } x, y \in \mathring{C}.$$

If C *is normal,* (1.13) *is also satisfied for all* $x, y \in \mathring{C}$.

Proof. Define $A_j = cB_j$ for c as in (2.25). It suffices to prove that the hypotheses of Theorem 2.1 are satisfied. Formula (2.25) implies (taking $\rho = R$) that hypothesis (c) of Theorem 2.1 is satisfied.

Because g_i is order-preserving, B_i and $A_i = cB_i$ are also order-preserving, so $A_i(C) \subset C$. The homogeneity of g_i implies that

$$g_i(u_{i-1}) = B_i(u_{i-1}) \in \mathring{C}, \tag{2.27}$$

and using (2.27) and the order-preserving property of A_i we conclude that $A_i(\mathring{C}) \subset \mathring{C}$. Thus hypothesis (b) of Theorem 2.1 is satisfied. Hypothesis (d) of Theorem 2.1 also follows directly from (2.27). Finally, because $\Delta(cB_i) = \Delta(B_i)$, (2.7) is equivalent to (2.26). □

The homogeneity of the functions f_j in Theorem 2.1 plays less of a role than it might at first seem to. We illustrate this by stating a result that follows by essentially the same argument as Theorem 2.1. First, we need a lemma proved by Potter (see [29]) in the case where the function g is defined, order-preserving, and subhomogeneous on *all* of $\mathring{C}$.

LEMMA 2.3. (Compare [29].) *Let* C *be a cone with nonempty interior in a Banach space* X, *and for* $\psi \in C^* - \{0\}$ *and* $\lambda > 1$ *define* $\Sigma = \{x \in \mathring{C} : \psi(x) = 1\}$ *and* $D = \{x \in \mathring{C} : \lambda^{-1} < \psi(x) < \lambda\}$. *Assume that* $f: D \to \mathring{C}$ *is order-preserving and subhomogeneous on* D. *Then* $f|\Sigma$ *is nonexpansive with respect to Hilbert's projective metric* d, *so* $d(f(x), f(y)) \leqq d(x, y)$ *for all* $x, y \in \Sigma$.

Proof. If x_0 and x_1 are points in Σ with $d(x_0, x_1) = \rho$, define $x_s = (1-s)x_0 + sx_1$. For a given integer $n > 1$, it follows from Proposition 1.9 in [25] that there are real numbers s_j, with $s_j < s_{j+1}$ for $0 \leqq j < n$ and $s_0 = 0$ and $s_n = 1$ and (writing $y_j = x_{s_j}$)

$$d(y_j, y_{j+1}) = \rho n^{-1} \quad \text{for } 0 \leqq j < n.$$

Note that in general $s_j \neq jn^{-1}$.

Choose n so large that $\rho n^{-1} < \lambda$. It suffices to prove that

$$d(f(y_j), f(y_{j+1})) \leqq d(y_j, y_{j+1}) = \rho n^{-1}, \tag{2.28}$$

for then the triangle inequality gives

$$d(f(x_0), f(x_1)) \leqq \sum_{j=0}^{n-1} d(f(y_j), f(y_{j+1})) \leqq n(\rho n^{-1}) = d(x_0, x_1).$$

For a fixed j select numbers α and β so that

$$\alpha y_j \leqq y_{j+1} \leqq \beta y_j \quad \text{and} \quad \log\left(\frac{\beta}{\alpha}\right) = \rho n^{-1}. \tag{2.29}$$

Because $\psi(y_j) = \psi(y_{j+1}) = 1$, we easily obtain from (2.29) that $0 < \alpha \leqq 1 \leqq \beta$ and $\alpha^{-1} \leqq \rho n^{-1} < \lambda$ and $\beta < \rho n^{-1} < \lambda$. It follows that the points αy_j, $\beta^{-1} y_{j+1}$, y_j, and y_{j+1} all lie in D. By using the subhomogeneity and the order-preserving property of f on D we find that

$$\alpha f(y_j) \leqq f(\alpha y_j) \leqq f(y_{j+1}) \quad \text{and} \quad \beta^{-1} f(y_{j+1}) \leqq f(\beta^{-1} y_{j+1}) \leqq f(y_j),$$

so

$$d(f(y_j), f(y_{j+1})) \leqq \log\left(\frac{\beta}{\alpha}\right).$$

□

We can now give a version of Theorem 2.1 for order-preserving, subhomogeneous operators. Once we know Lemma 2.3, the proof of the next theorem follows by essentially the same argument as in Theorem 2.1 and is left to the reader.

THEOREM 2.2. *Let C be a cone with nonempty interior in a Banach space X and for fixed $\psi \in C^* - \{0\}$ and $\lambda > 1$ define $D = \{x \in \mathring{C} : \lambda^{-1} < \psi(x) < \lambda\}$ and $\Sigma = \{x \in \mathring{C} : \psi(x) = 1\}$. Suppose that $g_j : D \to \mathring{C}$, $j \geqq 1$, is a sequence of order-preserving, subhomogeneous maps and define*

$$h_j(x) = g_j(x)/\psi(g_j(x)),$$

with $H_m = h_m h_{m-1} \cdots h_1$ and H_0 equal to the identity. For some $u \in \Sigma$ define $u_j = H_j(u)$ and assume that there exists $R > 0$ and a sequence of bounded linear operators A_j, $j \geqq 1$, such that:

(a) *g_j is continuously Fréchet differentiable on $B_R(u_{j-1}) \cap D$ for all $j \geqq 1$, where $B_R(y) = \{x \in \mathring{C} : d(x, y) < R\}$.*

(b) *$A_i(\mathring{C}) \subset \mathring{C}$ for all $i \geqq 1$ and $g_i'(x) \geqq A_i$ for all $x \in B_R(u_{i-1}) \cap D$ and $i \geqq 1$.*

(c) *There exists a positive constant k such that $A_j(u_{j-1}) \geqq k g_j(u_{j-1})$ for all $j \geqq 1$.*

If we have

$$\lim_{N \to \infty} \sum_{j=1}^{N} \exp(-\Delta(A_j)) = \infty,$$

where $\Delta(A_j)$ is given by (1.6), then it follows that

$$\lim_{n \to \infty} d(H_n(x), H_n(y)) = 0 \quad \text{for all } x, y \in \Sigma,$$

and

$$\lim_{n \to \infty} \|H_n(x) - H_n(y)\| = 0 \quad \text{for all } x, y \in \Sigma$$

if C is normal.

Remark 2.2. Note that in the statement of Theorem 2.2 there is no integer p analogous to the one in the statement of Theorem 2.1. If we assume that the g_j satisfy a "ray-preserving property" as in [16], then

$$\{g_j(tx) : t > 0\} \subset \{s g_j(x) : s > 0\} \quad \text{for } j \geqq 1 \quad \text{and} \quad x \in \Sigma.$$

Then if $G_m = g_m g_{m-1} \cdots g_1$, we can verify (see [16]) that

$$G_m(x)/\psi(G_m(x)) = H_m(x).$$

Using this fact we can then give a version of Theorem 2.2 that directly generalizes Theorem 2.1. Without the ray-preserving property there seems to be no necessary connection between G_m and H_m.

However, the assumption of the ray-preserving property for order-preserving, subhomogeneous operators can be restrictive. To see this, suppose that $f_j : \mathring{C} \to \mathring{C}$, $j = 1, 2$, is order-preserving and $f_j(tx) = t^{\lambda_j} f_j(x)$ for $0 < t \leqq 1$ and $x \in \mathring{C}$, where $0 < \lambda_j \leqq 1$. Then f_j is subhomogeneous, order-preserving, and ray-preserving. However, $f = f_1 + f_2$ is subhomogeneous and order-preserving, but not ray-preserving unless $\lambda_1 = \lambda_2$.

Remark 2.3. In [16], Fujimoto and Krause have obtained weak ergodic theorems for ray-preserving maps of the standard cone $K = \{x \in \mathbb{R}^n : x_i \geqq 0 \text{ for } 1 \leqq i \leqq n\}$ into itself. If $\Gamma = \{x \in K : \psi(x) = 1\}$ ($\psi \in K^* - \{0\}$) and $f_j : K \to K$, $j \geqq 1$, is a sequence of maps for which we want to establish a weak ergodic theorem, then the assumption of "uniform pointwise boundedness" in Theorem 4 of [16] (assuming, for simplicity that $r = 1$ in the statement of that theorem) implies that there exist $a, b \in \mathring{K}$ such that

$$a \leqq f_j(x)/\psi(f_j(x)) = h_j(x) \leqq b \quad \text{for all } x \in \Gamma. \tag{2.30}$$

Formula (2.30) (or its analogue for $r \geqq 1$) implies that $h_j : \Gamma \cap \mathring{K} \to \Gamma \cap \mathring{K}$, $j \geqq 1$, satisfies the bounded orbit property with respect to d, but uniform pointwise boundedness represents a less general condition than the bounded orbit property does. In particular, if $f_j \in \mathcal{M}$ for $j \geqq 1$, many examples of interest do not satisfy (2.30) or even the condition that $f_j(K - \{0\}) \subset \mathring{K}$, but they may satisfy the bounded orbit property. Of course (see § 4 of [16]), there are also many examples where the uniform pointwise boundedness assumption is satisfied.

Remark 2.4. The reader will note that the bounded orbit property is not assumed in Theorem 2.1, Corollary 2.1, or Theorem 2.2. Nevertheless, the bounded orbit property plays a crucial role: to verify the hypotheses of Corollary 2.1 or Theorem 2.2 for examples of interest, we will typically have to verify the bounded orbit property.

In the framework of Theorem 2.1 it is natural to ask if a stronger conclusion can be obtained. Does there exist $v \in \mathring{C}$ such that

$$\lim_{n\to\infty} d(F_n(x), v) = 0 \quad \text{and} \quad \lim_{n\to\infty} \|F_n(x)/\|F_n(x)\| - v\| = 0 \quad \text{for all } x \in \mathring{C}?$$

A similar question can be asked for Theorem 2.2. Such results are called "strong ergodic theorems" in [11]. As we now show, such a theorem can be derived easily from Theorem 2.1.

THEOREM 2.3. *Let C be a cone with nonempty interior in a Banach space X and $f_j : \mathring{C} \to \mathring{C}$, $j \geqq 1$, a sequence of maps that are homogeneous of degree 1 and order-preserving. Assume that there exists $v \in \mathring{C}$, $\|v\| = 1$, such that*

$$\lim_{j\to\infty} d(f_j(v), v) = 0, \tag{2.31}$$

where d denotes Hilbert's projective metric. Assume there exists $R > 0$, an integer $p \geqq 1$, and a sequence of bounded linear operators A_j, $j \geqq 1$, with the following properties:

(a) *If g_j is defined as in Theorem 2.1, g_j is continuously Fréchet differentiable on $B_R(v) = \{x \in \mathring{C} : d(x, v) < R\}$ for all $j \geqq 1$.*

(b) *For all $x \in B_R(v)$ and all $j \geqq 1$ we have $g_j'(x) \geqq A_j$.*

(c) *The operators A_j satisfy $A_j(\mathring{C}) \subset \mathring{C}$, A_j has finite projective diameter $\Delta(A_j)$, and*

$$\sup \{\Delta(A_j) : j \geqq 1\} < \infty. \tag{2.32}$$

(d) *There exists a positive constant k such that $A_j(v) \geqq k g_j(v)$ for all $j \geqq 1$. Then it follows that* $\lim_{n\to\infty} d(F_n(x), v) = 0$ *for all $x \in \mathring{C}$, and* $\lim_{n\to\infty} \|F_n(x)/\|F_n(x)\| - v\| = 0$ *if C is normal.*

Proof. By using the triangle inequality and the nonexpansiveness of f_j we can verify that

$$d(g_j(v), v) \leqq \sum_{i=jp-p+1}^{jp} d(f_i(v), v),$$

so

$$\lim_{j\to\infty} d(g_j(v), v) = 0.$$

We claim that to prove the theorem it suffices to prove that

$$\lim_{n\to\infty} d(G_n x, v) = \lim_{n\to\infty} d(F_{np} x, v) = 0 \quad \text{for all } x \in \mathring{C}. \tag{2.33}$$

To see this, note that the triangle inequality and the nonexpansiveness of the f_j imply that for $np < m < np + p$,

$$d(F_m x, v) \leqq d(F_{np} x, v) + \sum_{j=0}^{m-np-1} d(f_{m-j} v, v). \tag{2.34}$$

Formulae (2.31), (2.33), and (2.34) imply that

$$\lim_{m\to\infty} d(F_m x, v) = 0 \quad \text{for all } x \in \mathring{C}.$$

Thus it suffices to work with g_j and G_j.

Select a fixed number R_1, $0 < R_1 < R$, and define $V_{R_1}(v) = \{z\colon d(z, v) \leqq R_1\}$. Essentially the same argument used in the proof of Theorem 2.1 actually shows that for all $x, y \in V_{R_1}(v)$,

$$d(g_j(x), g_j(y)) \leqq c_j d(x, y),$$

where $c_j = [1 - (k/2)\, e^{-R_1} \exp(-\Delta(A_j))]$. (Note the extra factor e^{-R_1} in c_j; this is unnecessary if x or y equals v.) Because we assume that $\Delta(A_j)$ is bounded above, we have

$$\sup\{c_j\colon j \geqq 1\} = c < 1.$$

Now select any number ε, $0 < \varepsilon < R_1$, and suppose that $x \in V_\varepsilon(v)$, so $d(x, v) \leqq \varepsilon$. Then we obtain

$$\begin{aligned} d(g_j(x), v) &\leqq d(g_j(x), g_j(v)) + d(g_j(v), v) \\ &\leqq c\varepsilon + d(g_j(v), v). \end{aligned}$$

It follows that there exists an integer $m = m(\varepsilon)$ such that

$$g_j(V_\varepsilon(v)) \subset V_\varepsilon(v) \quad \text{for } j \geqq m(\varepsilon).$$

We now apply Theorem 2.1. For $m = m(\varepsilon)$ as above, define $\varphi_i(x) = g_{m+i}(x)$ and $\Phi_j = \varphi_j \varphi_{j-1} \cdots \varphi_1$. The sequence φ_j, $j \geqq 1$, satisfies the bounded orbit property; in fact, $\Phi_j(v) \in V_\varepsilon(v)$ for all $j \geqq 1$. It is also easy to check that all the hypotheses of Theorem 2.1 are satisfied, so

$$\lim_{j\to\infty} d(\Phi_j(y), \Phi_j(v)) = 0 \quad \text{for all } y \in \mathring{C}.$$

Taking $y = G_m(x)$ for $x \in \mathring{C}$, we conclude that

$$\lim_{j\to\infty} d(G_{j+m}(x), \Phi_j(v)) = 0 \quad \text{for all } x \in \mathring{C}.$$

Since $\Phi_j(v) \in V_\varepsilon(v)$ for all $j \geqq 1$, we conclude that for any fixed $x \in \mathring{C}$ there exists n such that

$$d(G_j(x), v) < 2\varepsilon \quad \text{for all } j \geqq n.$$

Since $\varepsilon > 0$ can be taken as small as desired, the proof is complete. □

Remark 2.5. Suppose that $f\colon \mathring{C} \to \mathring{C}$ is a map and $f(v) = \lambda v$ for some $v \in \mathring{C}$. Assume that $f_j\colon \mathring{C} \to \mathring{C}$, $j \geqq 1$, is a sequence of maps as in Theorem 2.3 and that for every $x \in \mathring{C}$ we have

$$\lim_{j\to\infty} d(f_j(x), f(x)) = 0.$$

Then it is certain that $\lim_{j\to\infty} d(f_j(v), v) = 0$, which provides a situation where we can find a vector v as in Theorem 2.3. However, such a vector v may well exist even if the functions f_j do not converge to a function f.

Remark 2.6. Essentially the same argument as in Theorem 2.3 can be extended, as in Theorem 2.2, to the case of order-preserving maps that are subhomogeneous. Details are left to the reader.

Under the hypotheses of Theorem 2.1, it is natural to ask if, for every given $x \in \mathring{C}$, there exists a positive number $\gamma = \gamma(x)$ such that

$$\lim_{n\to\infty} \|F_n(x) - \gamma F_n(u)\| = 0.$$

If $f_j = f$ for all $j \geqq 1$ and $f(u) = u$, this question is considered at length in § 3 of [25]. Before considering the general case it is convenient to prove a simple lemma.

LEMMA 2.4. *Let C be a normal cone with nonempty interior in a Banach space X. If $u \in \mathring{C}$ and R, c_1, and c_2 are positive reals, define V by*

$$V = \{x \in \mathring{C}: d(x, u) \leqq R \text{ and } c_1 \leqq \|x\| \leqq c_2\},$$

where d denotes Hilbert's projective metric. Then there exists $\rho > 0$ so that, for every $x \in V$,

$$\{z \in X: \|z - x\| < \rho\} \subset \mathring{C}.$$

Proof. If $x \in V$, there exist positive numbers α and β such that

$$\alpha u \leqq x \leqq \beta u \quad \text{and} \quad \beta\alpha^{-1} \leqq e^R. \tag{2.35}$$

By the definition of normality there exists a constant M so that if $0 \leqq x \leqq y$ then

$$\|x\| \leqq M\|y\|. \tag{2.36}$$

Combining (2.35) and (2.36) gives

$$\alpha\|u\| \leqq M\|x\| \leqq Mc_2 \quad \text{and} \quad c_1 \leqq \|x\| \leqq M\beta\|u\|. \tag{2.37}$$

Formulae (2.35) and (2.37) imply that

$$(c_1/M\|u\|)\, e^{-R} \leqq \alpha \quad \text{and} \quad \beta \leqq (Mc_2/\|u\|)\, e^R.$$

Thus there exists a number $\gamma \geqq 1$, independent of $x \in V$, such that

$$\gamma^{-1}u \leqq x \leqq \gamma u. \tag{2.38}$$

Because $\gamma^{-1}u \in \mathring{C}$, there exists $\rho > 0$ such that $\gamma^{-1}u + z \in \mathring{C}$ for all $z \in X$ with $\|z\| < \rho$, and (2.38) then implies that $x + z \in \mathring{C}$ for all $x \in V$ and all z with $\|z\| < \rho$. □

If C is a cone with nonempty interior in a Banach space X, $x \in \mathring{C}$ and $\{y: \|y - x\| < \rho\} \subset \mathring{C}$, then we can easily prove that for $\|y - x\| < \rho$ we have

$$(\rho - \|y - x\|)\rho^{-1}x \leqq y \leqq (\rho + \|y - x\|)\rho^{-1}x. \tag{2.39}$$

THEOREM 2.4. *Let notation and assumptions be as in Theorem 2.1. Assume also that $\langle f_j \rangle$ satisfies the bounded orbit property, that C is normal, and that*

$$\sup\{\|F_m(u)\|: m \geqq 0\} < \infty \quad \textit{and} \quad \inf\{\|F_m(u)\|: m \geqq 0\} > 0. \tag{2.40}$$

Then for every $x \in \mathring{C}$ there exists a positive number $\gamma = \gamma(x)$ such that

$$\lim_{m\to\infty} \|F_m(x) - \gamma F_m(u)\| = 0. \tag{2.41}$$

The map $x \to \gamma(x)$ is homogeneous of degree 1, order-preserving, and continuous.

Proof. It is known (see [32]) that there exists an equivalent norm on X whose restriction to C is order-preserving. Thus we can assume that if $0 \leqq x \leqq y$, then $\|x\| \leqq \|y\|$.

For a fixed $x \in \mathring{C}$, define numbers α_n and β_n by

$$\alpha_n = m(F_n(x)/F_n(u)) \quad \text{and} \quad \beta_n = M(F_n(x)/F_n(u)),$$

so

$$\alpha_n F_n(u) \leqq F_n(x) \leqq \beta_n F_n(u). \tag{2.42}$$

Applying f_{n+1} to (2.42) we find that

$$\alpha_n F_{n+1}(u) \leqq F_{n+1}(x) \leqq \beta_n F_{n+1}(u),$$

and we conclude from this inequality that

$$\alpha_n \leqq \alpha_{n+1}, \quad \beta_{n+1} \leqq \beta_n, \quad \alpha_n \leqq \beta_n \quad \text{for all } n, \tag{2.43}$$

so $\lim_{n\to\infty} \alpha_n = \alpha$ and $\lim_{n\to\infty} \beta_n = \beta$.

If we can prove that $\alpha = \beta$, we can define $\gamma(x) = \beta$ and (by normality of the cone) it is evident from (2.42) that (2.41) is satisfied. Furthermore, the homogeneity and order-preserving properties of $\gamma(x)$ follow immediately from the formula

$$\gamma(x) = \lim_{n\to\infty} M(F_n(x)/F_n(u)).$$

Because γ is homogeneous of degree 1 and order-preserving, we can easily derive from (2.39) that γ is continuous.

Theorem 2.1 implies that

$$\lim_{n\to\infty} \|F_n(x)\|F_n(x)\|^{-1} - F_n(u)\|F_n(u)\|^{-1}\| = 0, \tag{2.44}$$

and (2.42) gives

$$\alpha_n \leqq \gamma_n = \|F_n(x)\|/\|F_n(u)\| \leqq \beta_n. \tag{2.45}$$

Formulae (2.40) and (2.45) imply that $\|F_n(x)\|$ is bounded above and below by positive reals. Using the fact that $\|F_n(x)\|$ is bounded above, we obtain from (2.44)

$$\lim_{n\to\infty} \|F_n(x) - \gamma_n F_n(u)\| = \lim_{n\to\infty} \varepsilon_n = 0. \tag{2.46}$$

We now use Lemma 2.4. The bounded orbit property implies that there exists $R > 0$ such that

$$d(\gamma_n F_n(u), u) = d(F_n(u), u) \leqq R \quad \text{for all } n \geqq 1.$$

Furthermore, as already noted, there exist positive reals c_1 and c_2 such that

$$c_1 \leqq \gamma_n \|F_n(u)\| = \|F_n(x)\| \leqq c_2.$$

Thus Lemma 2.3 implies that there exists $\rho > 0$ such that if $\|z - \gamma_n F_n(u)\| < \rho$ for some $n \geqq 0$, then $z \in \mathring{C}$. If ε_n is defined as in (2.46), then (2.39) implies that if $\varepsilon_n < \rho$ for $n \geqq N$, then for $n \geqq N$,

$$[(\rho - \varepsilon_n)\rho^{-1}]\gamma_n F_n(u) \leqq F_n(x) \leqq [(\rho + \varepsilon_n)\rho^{-1}]\gamma_n F_n(u). \tag{2.47}$$

It follows that

$$0 \leqq \beta_n - \alpha_n \leqq (\rho + \varepsilon_n)\rho^{-1}\gamma_n - (\rho - \varepsilon_n)\rho^{-1}\gamma_n = 2\varepsilon_n \rho^{-1}\gamma_n,$$

and we conclude (using (2.46) and the fact γ_n is bounded) that

$$\lim_{n\to\infty} (\beta_n - \alpha_n) = 0. \qquad \square$$

As an easy corollary of Theorem 2.4 we mention the following result, a slightly weaker version of which has been proved by Cohen in [12]. Of course, in this simple situation it is also possible to give an elementary, direct proof, so the following corollary is meant only as an illustration of Theorem 2.4.

COROLLARY 2.2. (Compare [12].) *Let* $K=\{(x_1,x_2)\in\mathbb{R}^2: x_i\geqq 0 \text{ for } i=1,2\}$. *Let* λ_j, $j\geqq 1$, *be a sequence of real numbers such that* $0\leqq\lambda_j\leqq 1$, *and for each* $j\geqq 1$ *define*

$$f_j:\mathring{K}\to\mathring{K} \quad by \quad f_j(x_1,x_2)=((1-\lambda_j)x_1+\lambda_j x_2,\, x_1^{1-\lambda_j}x_2^{\lambda_j})$$

and define $F_n=f_nf_{n-1}\cdots f_1$. *Then for each* $x=(x_1,x_2)\in\mathring{K}$, *there exists* $\gamma=\gamma(x)>0$ *such that*

$$\lim_{n\to\infty}F_n(x)=(\gamma,\gamma).$$

Proof. Let $u=(1,1)$ so $f_j(u)=u$ for all $j\geqq 1$. Thus f_j, $j\geqq 1$, satisfies the bounded orbit property in Theorem 2.1 and (2.40) in Theorem 2.4. The functions f_j are clearly all order-preserving, homogeneous of degree 1, and C^1 on $\mathring{K}$. For a fixed $R>0$, define

$$A_j=e^{-R}\begin{pmatrix}1-\lambda_j & \lambda_j\\ 1-\lambda_j & \lambda_j\end{pmatrix}.$$

It is easy to check that $f_j'(x)\geqq A_j$ for all x with $d(x,u)\leqq R$ and

$$A_j(u)\geqq e^{-R}f_j(u)=e^{-R}u.$$

Also, A_j has one-dimensional range, so $\Delta(A_j)=0$. If we take $p=1$ (in the notation of Theorem 2.1), we thus see that all hypotheses of Theorems 2.1 and 2.4 are satisfied, so the conclusion of the corollary follows from Theorem 2.4. □

The preceding theorems typically make some assumption of differentiability. These assumptions are motivated by the applications we have in mind and can certainly be weakened. The hypotheses of Theorem 2.1 represent only a convenient way to obtain the estimates in (2.10) and (2.11). To illustrate this point we mention the following theorem, whose proof is essentially the first part of the proof of Theorem 2.1. Details are left to the reader.

THEOREM 2.5. *Let* C *be a cone with nonempty interior in a Banach space* X. *Let* Σ *be a subset of* $\mathring{C}$ *and* $S=\{y\in\mathring{C}: \|y\|=1\}$ *and assume that for each* $y\in S$ *there is a unique positive number* $\lambda=\lambda(y)$ *such that* $\lambda y\in\Sigma$. *Suppose that the* $h_j:\Sigma\to\Sigma$, $j\geqq 1$, *is a sequence of maps and that there exists* $u\in\Sigma$ *such that if* $H_m=h_mh_{m-1}\cdots h_1$ (*the composition of the first* m *functions* h_j) *and* H_0 *denotes the identity, then*

$$d(h_m(x),H_m(u))\leqq d(x,H_{m-1}(u)) \quad \text{for all } x\in\Sigma \quad \text{and} \quad m\geqq 1.$$

(*Here* d *denotes Hilbert's projective metric.*) *In addition, assume that there exist* $R>0$ *and a sequence of reals* λ_j, $j\geqq 1$, *such that* $0\leqq\lambda_j\leqq 1$ *for all* j,

$$d(h_m(x),H_m(u))\leqq\lambda_m d(x,H_{m-1}(u)) \quad \text{for all } x\in\Sigma \text{ with } d(x,H_{m-1}(u))\leqq R, \text{ and}$$

$$\lim_{N\to\infty}\prod_{j=m}^{N}\lambda_j=0 \quad \text{for all } m\geqq 1.$$

It then follows that for every $x\in\Sigma$,

$$\lim_{n\to\infty}d(H_n(x),H_n(u))=0.$$

3. Applications: verifying the bounded orbit property. In this section we show how the results of § 2 can be applied in the case where $f_j\in M$ for all $j\geqq 1$, M being the class defined in § 1. We shall show that the main difficulty lies in verifying the bounded orbit property with respect to d. If $f_j\in M_+$ for all $j\geqq 1$, we will give reasonably general conditions ensuring that the bounded orbit property is satisfied. As we have already noted in § 2, if $f_j\in M_-$ for all $j\geqq 1$, verifying the bounded orbit property can be a difficult problem even when $f_j=f$ for all $j\geqq 1$.

We begin with some needed notation and definitions. We will always denote by K the standard cone in $\mathbb{R}^n$,

$$K = \{x \in \mathbb{R}^n : x_i \geqq 0 \text{ for } 1 \leqq i \leqq n\}. \tag{3.1}$$

If $f_j \in M$ is a sequence of functions for $j \geqq 1$, we will denote by f_{ji}, $1 \leqq i \leqq n$, the ith component of the function $f_j : \mathring{K} \to \mathring{K}$. By definition of M, there is a finite collection Γ_{ji} of ordered pairs (r, σ), r a real number and σ a probability vector, and positive real numbers $c_{jir\sigma}$ for $(r, \sigma) \in \Gamma_{ji}$, such that

$$f_{ji}(x) = \sum_{(r,\sigma)\in\Gamma_{ji}} c_{jir\sigma} M_{r\sigma}(x). \tag{3.2}$$

Because $(x^r)^{1/r} = x$, the set Γ_{ji} described above may not be uniquely determined by the function f_{ji}. However, we will need some control of the size of the numbers r, which appear in (3.2). Thus we make the following definitions. Suppose that $\phi \in M$ so ϕ_i, the ith component of ϕ, can be written

$$\phi_i(x) = \sum_{(r,\sigma)\in G_i} c_{ir\sigma} M_{r\sigma}(x), \tag{3.3}$$

where G_i is a finite collection of ordered pairs (r, σ), r a real number and σ a probability vector, and $c_{ir\sigma} > 0$. If $\phi \in M_+$, the sets G_i can be chosen so that $r \geqq 0$ for all $(r, \sigma) \in G_i$.

DEFINITION 3.1. If $\phi \in M$, we define $\mu(\phi)$ by

$$\begin{aligned}\mu(\phi) = \inf\{\mu > 0 : \ & \phi_i(x) \text{ can be expressed as in (3.3)},\\ & \text{and } |r| \leqq \mu \text{ for all } (r,\sigma) \in G_i \text{ and for } 1 \leqq i \leqq n\}.\end{aligned} \tag{3.4}$$

If $\phi \in M_+$, we define $\nu(\phi)$ by

$$\begin{aligned}\nu(\phi) = \inf\{\nu > 0 : \ & \phi_i(x) \text{ can be expressed as in (3.3)},\\ & \text{and } 0 \leqq r \leqq \nu \text{ for all } (r,\sigma) \in G_i \text{ and for } 1 \leqq i \leqq n\}.\end{aligned} \tag{3.5}$$

If C is a cone in a Banach space X and A and B are bounded linear maps of X to X, we will say that $A \leqq B$ if $(B - A)(C) \subset C$; the ordering depends on C. If $A(C) \subset C$ and $B(C) \subset C$, we will say that A and B are comparable if there exist positive numbers c_1 and c_2 such that

$$c_1 A \leqq B \leqq c_2 A.$$

If $C = K$ and $X = \mathbb{R}^n$, then bounded linear maps are $n \times n$ matrices, $A = (a_{ij}) \leqq B = (b_{ij})$ if and only if $a_{ij} \leqq b_{ij}$ for all i, j. If $a_{ij} \geqq 0$ and $b_{ij} \geqq 0$ for all i, j, A and B are comparable if and only if there exist positive reals c_1 and c_2 such that

$$c_1 a_{ij} \leqq b_{ij} \leqq c_2 a_{ij} \quad \text{for all } i, j.$$

Our next lemma is easy, but we give a proof for completeness.

LEMMA 3.1. *Assume that C is a cone in a Banach space X and that $A, B : X \to X$ are bounded linear operators such that $A(C) \subset C$ and $B(C) \subset C$. Assume that A and B are comparable; then there exist positive reals c_1 and c_2 such that*

$$c_1 A \leqq B \leqq c_2 A. \tag{3.6}$$

If A has finite projective diameter $\Delta(A)$ (see (1.6)), then B has finite projective diameter and

$$\Delta(B) \leqq \Delta(A) + 2 \log (c_2/c_1). \tag{3.7}$$

Proof. If x and y are any two elements of C such that Bx and By are nonzero, then by using (3.6) we see that Ax and Ay are nonzero. By definition of finite projective diameter, it follows that there exist positive reals α and β such that

$$\alpha A(x) \leqq A(y) \leqq \beta A(x) \quad \text{and} \quad \log(\beta/\alpha) \leqq \Delta(A). \tag{3.8}$$

By using (3.6) repeatedly we obtain from (3.8) that

$$\alpha(c_1/c_2)B(x) \leqq B(y) \leqq \beta(c_2/c_1)B(x), \tag{3.9}$$

which implies that

$$d(Bx, By) \leqq 2\log(c_2/c_1) + \log(\beta/\alpha). \tag{3.10}$$

Formulae (3.8) and (3.10) yield (3.7). □

In [26] it is proved that if $f \in \mathcal{M}$ (see § 1 for definitions), then f is C^∞ on $\mathring{K}$ and $f'(x)$ and $f'(y)$ are comparable for all $x, y \in \mathring{K}$ (this is not hard). We need a more precise version of this fact, relating the sizes of $f'(x)$ and $f'(y)$, when $f \in M$.

LEMMA 3.2. *Let K denote the standard cone in $\mathbb{R}^n$* (see (3.1)), *let $v = (1, 1, \cdots, 1)$ be the vector all of whose components are* 1, *and define $\psi \in K^*$ by*

$$\psi(x) = \sum_{i=1}^{n} x_i.$$

Suppose that $f: \mathring{K} \to \mathring{K}$ is homogeneous of degree 1 *and order-preserving. If $x \in \mathring{K}$, $\psi(x) = n$, and $d(x, v) \leqq R$ (d denotes Hilbert's projective metric), then*

$$e^{-R} f(v) \leqq f(x) \leqq e^{R} f(v). \tag{3.11}$$

If $f \in M$ and $\mu(f) < \gamma$ (see Definition 3.1), *then for all $x \in \mathring{K}$ such that $d(x, v) \leqq R$,*

$$\exp(-R(\gamma+1)) f'(v) \leqq f'(x) \leqq \exp(R(\gamma+1)) f'(v), \tag{3.12}$$

where $f'(x)$ denotes the Jacobian matrix at x.

Proof. If $\psi(x) = n = \psi(v)$ and $\alpha = m(x/v)$ and $\beta = M(x/v)$, we must have $\alpha \leqq 1 \leqq \beta$; and if $d(x, v) \leqq R$, then

$$\beta/\alpha \leqq e^R. \tag{3.13}$$

Formula (3.13) implies that $\beta \leqq e^R$ and $\alpha \geqq e^{-R}$ (since $\alpha \leqq 1 \leqq \beta$), and (3.11) follows from the homogeneity and order-preserving properties of f.

If $f \in M$, $\mu(f) < \gamma$, and f_i denotes the ith component of $f(x)$, then we can write

$$f_i(x) = \sum_{(r,\sigma) \in G_i} c_{ir\sigma} M_{r\sigma}(x), \qquad |r| < \gamma \quad \text{for } (r, \sigma) \in G_i. \tag{3.14}$$

Here G_i is a finite collection of ordered pairs (r, σ) with $r \in \mathbb{R}$ and σ a probability vector and $c_{ir\sigma} > 0$ for $(r, \sigma) \in G_i$, $1 \leqq i \leqq n$. If $d(x, v) \leqq R$ we have

$$e^{-R} \leqq x_j/x_k \leqq e^R \quad \text{for all } j, k. \tag{3.15}$$

A calculation implies that for $d(x, v) \leqq R$,

$$\frac{\partial M_{r\sigma}}{\partial x_j}(x) = \sigma_j[x_j(M_{r\sigma}(x))^{-1}]^{r-1}. \tag{3.16}$$

Recall that $M_{r\sigma}$ is an order-preserving map on $\mathring{K}$ for any real number r and that (3.15) implies that

$$e^{-R} x_j v \leqq x \leqq e^R x_j v.$$

Using this we conclude that for $d(x, v) \leqq R$ we have

$$e^{-R} \leqq x_j (M_{r\sigma}(x))^{-1} \leq e^{R};$$

(3.16) then implies that

$$\sigma_j e^{-R|r-1|} \leqq \frac{\partial M_{r\sigma}}{\partial x_j}(x) \leqq \sigma_j e^{R|r-1|}. \tag{3.17}$$

Because $|r| < \gamma$ for all r such that $(r, \sigma) \in G_i$, we conclude from (3.17) that

$$\exp(-R(\gamma+1)) \sum_{(r,\sigma)\in G_i} c_{ir\sigma}\sigma_j \leqq \frac{\partial f_i}{\partial x_j}(x) \leqq \exp(R(\gamma+1)) \sum_{(r,\sigma)\in G_i} c_{ir\sigma}\sigma_j. \tag{3.18}$$

Of course (3.18) is equivalent to

$$\exp(-R(\gamma+1)) \frac{\partial f_i}{\partial x_j}(v) \leqq \frac{\partial f_i}{\partial x_j}(x) \leqq \exp(R(\gamma+1)) \frac{\partial f_i}{\partial x_j}(v),$$

which implies (3.2). □

We can now state a weak ergodic theorem for functions $f_j \in M$. In the statement of the following theorem recall that a nonnegative $n \times n$ matrix B is called "primitive" if there exists $p \geqq 1$ such that B^p has all positive entries.

THEOREM 3.1. *Let K be the standard cone in $\mathbb{R}^n$ (see (3.1)) and suppose that for $j \geqq 1$, $f_j \in M$, where M is the class of maps of $\mathring{K}$ into itself defined in § 1. Assume that $\mu(f_j) < \gamma < \infty$ for all $j \geqq 1$ (see Definition 3.1). If $v = (1, 1, \cdots, 1)$, suppose also that there exist an $n \times n$ primitive matrix B and an $n \times n$ matrix A such that $B \leqq f_j'(v) \leqq A$ for all $j \geqq 1$. Finally, assume that $\langle f_j \rangle$ satisfies the bounded orbit property with respect to Hilbert's projective metric d (see Definition 2.1). Then if $F_k = f_k f_{k-1} \cdots f_1$,*

$$\lim_{k\to\infty} d(F_k(x), F_k(v)) = 0 \quad \textit{for all } x \in \mathring{K},$$

$$\lim_{k\to\infty} \|F_k(x)\|F_k(x)\|^{-1} - F_k(v)\|F_k(v)\|^{-1}\| = 0 \quad \textit{for all } x \in \mathring{K}.$$

Proof. Select an integer $p \geqq 1$ such that B^p has all positive entries, and for this p let G_k and g_k be as defined in Theorem 2.1. The bounded orbit property implies that $\{F_k(x): k \geqq 0\}$ has finite projective diameter for any $x \in \mathring{K}$. In particular, there exists $R > 0$ such that

$$\{F_k(v): k \geqq 0\} \subset B_R(v) = \{z: d(z, v) < R\}. \tag{3.19}$$

For notational convenience, define

$$v_k = F_k(v) \quad \text{and} \quad u_k = G_k(u)/\|G_k(u)\|.$$

Because each f_j is nonexpansive with respect to d we see that if $x \in B_R(v_k)$, then for $j \geqq 1$ we have

$$f_{k+j} f_{k+j-1} \cdots f_{k+1}(x) \in B_R(v_{k+j}) \subset B_{2R}(v). \tag{3.20}$$

If $x \in B_R(u_k) = B_R(G_k(u))$, then by using (3.20) and the chain rule we see that

$$g_{k+1}'(x) = f_{kp+p}'(y_1) f_{kp+p-1}'(y_2) \cdots f_{kp+1}'(y_p), \tag{3.21}$$

where $y_1, y_2, \cdots, y_p$ are points in $B_{2R}(v)$ that depend on x and the maps f_j.

We now use Lemma 3.2 and (3.21) to conclude that

$$\exp(-p(\gamma+1)(2R)) f_{jp}'(v) f_{jp-1}'(v) \cdots f_{jp-p+1}'(v) \leqq g_j'(x),$$

$$g_j'(x) \leqq \exp(p(\gamma+1)(2R)) f_{jp}'(v) f_{jp-1}'(v) \cdots f_{jp-p+1}'(v)$$

for all $x \in B_R(u_{j-1})$, $j \geqq 1$. It follows that for some $c_1 > 0$ we have

(3.22) $$\exp(-2p(\gamma+1)R)B^p \leqq g_j'(x) \leqq c_1 \exp(2p(\gamma+1)R)B^p \quad \text{for all } x \in B_R(u_{j-1}).$$

By using (3.22) we see that there exists a positive real number c, independent of j, such that

$$g_j'(x) \geqq c g_j'(u_{j-1}) \quad \text{for all } j \geqq 1, \quad x \in B_R(u_{j-1}),$$

which is (2.25).

If we define $B_j = g_j'(u_{j-1})$, then (3.22) and Lemma 3.1 imply that

$$\Delta(B_j) \leqq 4p(\gamma+1)R + \Delta(B^p),$$

so

$$\sum_{j=1}^{\infty} \exp(-\Delta(B_j)) = \infty.$$

The conclusions of Theorem 3.1 now follow directly from Corollary 2.1. □

As an immediate consequence of Theorems 3.1 and 2.4 we obtain the following result.

COROLLARY 3.1. *Let the notation and assumptions be as in Theorem* 3.1. *In addition, assume that there exists* $u \in \mathring{K}$ *such that* (2.40) *is satisfied. Then for each* $x \in \mathring{K}$ *there exists* $\gamma = \gamma(x) > 0$ *such that*

$$\lim_{m \to \infty} \|F_m(x) - \gamma F_m(u)\| = 0,$$

and $x \to \gamma(x)$ *is continuous, order-preserving, and homogeneous of degree* 1.

Similarly, by using Lemmas 3.1 and 3.2, we can derive the following corollary of Theorem 2.3. Details are left to the reader.

COROLLARY 3.2. *Let notation and assumptions be as in Theorem* 3.1. *Assume that there exists* $w \in K$ *such that*

$$\lim_{j \to \infty} d(f_j(w), w) = 0.$$

Then it follows that for all $x \in \mathring{K}$,

$$\lim_{k \to \infty} d(F_k(x), w) = 0 \quad \textit{and} \quad \lim_{k \to \infty} \|F_k(x)\|F_k(x)\|^{-1} - w\| = 0.$$

If σ is a probability vector and $x \in \mathring{K}$, we will use the notation

(3.23) $$x^\sigma = \prod_{j=1} x_j^{\sigma_j},$$

where x_j and σ_j are the jth components of x and σ, respectively. If $x, y \in \mathring{K}$ we will also use the notation

$$\log(x) = (\log(x_1), \log(x_2), \cdots, \log(x_n)) \quad \text{and} \quad y \cdot \log(x) = \sum_{j=1}^{n} y_j \log(x_j).$$

If $\langle f_k \rangle$ is a sequence of maps, order-preserving and homogeneous of degree 1, of $\mathring{K}$ into itself, verifying the bounded orbit property may be difficult. However, we now show that if $f_{ki}(x)$ can be bounded below in a suitable way by cx^σ, where $c > 0$, σ is a probability vector, and both c and σ may depend on k and i, then we can prove that $\langle f_k \rangle$ satisfies the bounded orbit property with respect to d. This idea has already been used in § 4 of [23] for the case where $f_k = f$ for all $k \geqq 1$.

Our next lemma is a slight variant of Lemma 4.1 in [23]. Since the argument is the same, the proof is left to the reader.

LEMMA 3.3. *Suppose that K is the standard cone in $\mathbb{R}^n$ and f and g are maps of $\mathring{K}$ into itself. Suppose that A is a nonnegative $n\times n$ matrix with no zero rows; suppose also that if $a_{ij}>0$, then there exist a positive constant c and a probability vector σ (σ depends on i and j) such that $\sigma_j \geqq \eta > 0$ (where σ_j denotes the jth component of σ) and*

$$f_i(x) = \text{the } i\text{th component of } f(x) \geqq cx^\sigma \quad \text{for all } x\in \mathring{K}.$$

The constants c and η are assumed independent of i and j. Similarly, suppose that B is a nonnegative $n\times n$ matrix with no zero rows and that if $b_{ij}>0$, then there exist a positive constant d and a probability vector τ such that $\tau_j \geqq \theta > 0$ and

$$g_i(y) \geqq dy^\tau \quad \text{for all } y\in \mathring{K}.$$

Here d and θ are assumed independent of i and j, but τ may depend on i and j. Then BA is an $n\times n$ nonnegative matrix with no zero rows, and if the entry in row i, column j of BA is nonzero, there exists a probability vector μ such that

$$g_i(f(x)) \geqq \lambda x^\mu \quad \text{for all } x\in \mathring{K},$$

where $\lambda \geqq cd$ and $\mu_j \geqq \theta\eta$.

In the statement of the following theorem, recall that $f_{ki}(x)$ denotes the ith component of a map $f_k: \mathring{K}\to \mathring{K}$.

THEOREM 3.2. *Let K denote the standard cone in $\mathbb{R}^n$ and suppose that $f_k: \mathring{K}\to\mathring{K}$, $k\geqq 1$, is a sequence of maps that are order-preserving and homogeneous of degree 1. Let $A=(a_{ij})$ be an $n\times n$ nonnegative, primitive matrix. If $a_{ij}>0$, assume that there exist $c>0$ and $\delta>0$ (independent of i, j, and $k\geqq 1$) and a probability vector σ with $\sigma_j\geqq\delta$ (σ may depend on i, j, and k) such that*

$$f_{ki}(x) \geqq cx^\sigma \quad \text{for all } x\in \mathring{K}.$$

If $v=(1,1,\cdots,1)$, assume that there exists a constant c_2 such that $f_k(v)\leqq c_2 v$ for all $k\geqq 1$. Then $\langle f_k\rangle$ satisfies the bounded orbit property with respect to Hilbert's projective metric d, and for every $u\in\mathring{K}$, there exists $R>0$ such that

$$d(F_k(u),u)\leqq R \quad \text{for all } k\geqq 1.$$

Proof. Let F_k be as defined in Theorem 2.1 and select $p\geqq 1$ such that all entries of A^p are positive. For any fixed $k\geqq 0$, define $g:\mathring{K}\to\mathring{K}$ by

$$g = f_{k+p}f_{k+p-1}\cdots f_{k+1}. \tag{3.24}$$

By applying Lemma 3.3 $(p-1)$ times and recalling that A^p has all positive entries, we see that for any i, j with $1\leqq i, j\leqq n$ there exists a probability vector σ (depending on i, j, k, and p) with $\sigma_j \geqq \delta^p = \eta$ ($\sigma_j =$ the jth component of σ) and

$$g_i(x) = \text{the } i\text{th component of } g(x) \geqq c^p x^\sigma \quad \text{for all } x\in\mathring{K}. \tag{3.25}$$

Suppose that $x\in\mathring{K}$ and $d(x,v)\leqq R$, and select j so that

$$x_j = M(x/v) = M. \tag{3.26}$$

If $m=m(x/v)$, we obtain from (3.25) and (3.26) that

$$g_i(x)\geqq c^p x^\sigma \geqq c^p M^{\sigma_j} m^{1-\sigma_j} \geqq c^p (M/m)^\eta m = c^p M^\eta m^{1-\eta}. \tag{3.27}$$

In deriving (3.27) we use $M/m\geqq 1$.

On the other hand, we assume that $f_j(v) \leqq c_2 v$ for all j, so

$$g(v) \leqq c_2^p v. \tag{3.28}$$

Because $x \leqq Mv$ we conclude from (3.28) that

$$g_s(x) \leqq c_2^p M \quad \text{for } 1 \leqq s \leqq n. \tag{3.29}$$

Combining (3.27) and (3.29) we conclude that

$$d(g(x), v) \leqq p \log\left(\frac{c_2}{c}\right) + (1-\eta)\log\left(\frac{M}{m}\right) \leqq p \log\left(\frac{c_2}{c_1}\right) + (1-\eta)R. \tag{3.30}$$

It follows from (3.30) that there exist a real number λ, $0 < \lambda < 1$, and a number R_1, both independent of k in the definition of g, such that

$$d(g(x), v) \leqq \lambda R \quad \text{if } d(x, v) \leqq R \quad \text{and} \quad R \geqq R_1. \tag{3.31}$$

We can also assume that R_1 is so large that

$$d(F_j(v), v) < R_1 \quad \text{for } 1 \leqq j < p. \tag{3.32}$$

In general we can write F_m for $m \geqq p$ in the form

$$F_m = g_1 g_2 \cdots g_t F_j, \tag{3.33}$$

where $0 \leqq j < p$ and each g_i in (3.33) is assumed to be of the form given by (3.24) for some $k \geqq 0$. Since every g as in (3.24) maps $B_{R_1}(v)$ into itself and since $F_j(v) \in B_{R_1}(v)$ for $0 \leqq j < p$, we conclude that $F_m(v) \in B_{R_1}(v)$ for all $m \geqq 1$. □

With the aid of Theorems 3.1 and 3.2 we can give a more concrete weak ergodic theorem.

THEOREM 3.3. *Let K denote the standard cone in $\mathbb{R}^n$ (see (3.1)) and let $\langle f_k \rangle$, $k \geqq 1$, be a sequence of maps such that $f_k \in M_+$ for all $k \geqq 1$. Assume that $\mu(f_k) < \rho < \infty$ for all $k \geqq 1$ (see Definition 3.1). Let $v = (1, 1, \cdots, 1)$ denote the vector all of whose components equal 1 and assume that there exists an $n \times n$ nonnegative, primitive matrix B such that*

$$f_k'(v) \geqq B \quad \textit{for all } k \geqq 1. \tag{3.34}$$

Assume that there exists $\beta_2 > 0$ such that

$$f_k(v) \leqq \beta_2 v \quad \textit{for all } k \geqq 1. \tag{3.35}$$

Then we have that for all $x \in \mathring{K}$

$$\lim_{k\to\infty} d(F_k(x), F_k(v)) = 0,$$

$$\lim_{k\to\infty} \|F_k(x)\|F_k(x)\|^{-1} - F_k(v)\|F_k(v)\|^{-1}\| = 0.$$

If there exists $u \in \mathring{K}$ such that

$$\sup\{\|F_k(u)\| : k \geqq 1\} < \infty \quad \textit{and} \quad \inf\{\|F_k(u)\| : k \geqq 1\} > 0,$$

then for every $x \in \mathring{K}$ there exists $\gamma = \gamma(x) > 0$ such that

$$\lim_{k\to\infty} \|F_k(x) - \gamma F_k(u)\| = 0.$$

If there exists $w \in \mathring{K}$ such that

$$\lim_{j\to\infty} d(f_j(w), w) = 0,$$

then for every $x \in \mathring{K}$,

$$\lim_{k\to\infty} d(F_k(x), w) = 0.$$

Proof. The existence of a matrix A as in Theorem 3.1 follows from (3.35) and the equation $f'_k(v)(v)=f_k(v)$. Thus, by virtue of Theorem 3.1 and Corollaries 3.1 and 3.2, it suffices to prove that the sequence $\langle f_k\rangle$ satisfies the bounded orbit property. We establish the bounded orbit property by using Theorem 3.2.

First note that, because f_k is homogeneous of degree 1,

$$f_k(v)=f'_k(v)(v)\geqq Bv;$$

therefore there exist positive constants β_1 and β_2 so that

$$\beta_1 v\leqq f_k(v)\leqq \beta_2 v \quad \text{for all } k\geqq 1. \tag{3.36}$$

If we write

$$f_{ki}(x)=\sum_{(r,\sigma)\in\Gamma_{ki}} c_{kir\sigma}M_{r\sigma}(x), \quad 1\leqq i\leqq n, \quad k\geqq 1,$$

we can assume $r\geqq 0$ for $(r,\sigma)\in\Gamma_{ki}$ (because $f_k\in M_+$). Formula (3.36) implies that

$$\beta_1\leqq f_{ki}(v)=c_{ki}=\sum_{(r,\sigma)\in\Gamma_{ki}} c_{kir\sigma}\leqq\beta_2. \tag{3.37}$$

It is a classical result (see [18]) that $M_{r\sigma}(x)\geqq M_{0\sigma}(x)$ for $r\geqq 0$, so

$$f_{ki}(v)\geqq c_{ki}\sum_{(r,\sigma)\in\Gamma_{ki}} c_{kir\sigma}c_{ki}^{-1}M_{0\sigma}(x). \tag{3.38}$$

If we apply log to both sides of (3.38) and use the concavity of log we obtain

$$\log f_{ki}(x)\geqq(\log c_{ki})+\left(\sum_{(r,\sigma)\in\Gamma_{ki}} c_{kir\sigma}c_{ki}^{-1}\sigma\right)\cdot(\log x). \tag{3.39}$$

If we define a probability vector τ_{ki} by

$$\tau_{ki}=\sum_{(r,\sigma)\in\Gamma_{ki}} c_{kir\sigma}c_{ki}^{-1}\sigma, \tag{3.40}$$

we obtain from (3.37), (3.39), and (3.40) that

$$f_{ki}(x)\geqq c_{ki}x^{\tau_{ki}}\geqq\beta_1 x^{\tau_{ki}}. \tag{3.41}$$

Denote by b_i the ith row of the matrix B. A simple calculation shows that the ith row of the Jacobian matrix $f'_k(v)$ is $c_{ki}\tau_{ki}$, so by the hypotheses of our theorem we have that

$$\tau_{ki}\geqq c_{ki}^{-1}b_i\geqq\beta_2^{-1}b_i. \tag{3.42}$$

If $B=(b_{ij})$, define a positive number δ by

$$\delta=\inf\{b_{ij}\beta_2^{-1}\colon b_{ij}>0\}.$$

Then it follows from (3.41) and (3.42) that if $b_{ij}>0$, the jth component of τ_{ki} in (3.41) is greater than or equal to δ. Since β_1 and δ are independent of i, j and $k\geqq 1$, Theorem 3.2 implies that $\langle f_k\rangle$ satisfies the bounded orbit property. □

Remark 3.1. Note that, for functions f_k that satisfy the hypotheses of Theorem 3.3, it can easily happen that f_k does not map certain nonzero points in the boundary of K into the interior of K and that the diameter (with respect to Hilbert's projective metric d) of $\{f_k(x)\colon x\in \mathring{K}\}$ is *not* finite. Both of these phenomena are illustrated by the simple arithmetic-geometric mean map

$$f_k(x)=f(x)=\left(\frac{x_1+x_2}{2},\ \sqrt{x_1x_2}\right).$$

In addition, both phenomena are typical of many examples of interest. For example, suppose that $x_k = F_k(x)$, where F_k is as in Theorem 3.3. More generally, for $f_k \in M_+$, we can define $\tilde{f}_k$ by

$$\tilde{f}_k(x) = \mu_k(x) f_k(x),$$

where $\mu_k(x)$ is a positive scalar function of x. If we define $\tilde{F}_k = \tilde{f}_k \tilde{f}_{k-1} \cdots \tilde{f}_1$ and $x_k = \tilde{F}_k(x)$ and $x_0 = x$, it is easy to see that

$$\tilde{F}_k(x) = \lambda_k(x) F_k(x), \qquad \lambda_k(x) = \prod_{j=1}^{k} \mu_j(x_{j-1}).$$

Here $\lambda_k(x)$ is a positive scalar, and the presence of $\lambda_k(x)$ does not affect the validity of the first part of Theorem 3.3, because Hilbert's projective metric does not distinguish points on rays. In this terminology, x may represent an initial "population vector" (so that the j-component of x represents the number of members of the population in class j) and $x_k = \tilde{F}_k(x)$ may represent the population vector at time k. Under reasonable assumptions on the biological model, we expect f_k to vanish at certain nonzero points on the boundary of K. Note, however, that if f_k is linear and irreducible, f_k does not vanish on nonzero points of the boundary of K. This point indicates a drawback of linear weak ergodic theorems in population biology.

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