Math 351

Review problems for Exam #1 **October 4, 2010**

#1 (a) Find the greatest common divisor of 357 and 756 and write it in the form $a(357) + b(756)$ where a and b are integers.

(b) Find the greatest common divisor of $x^3 - 5x^2 + 7x - 2$ and $x^4 - 2x^3 + x^2 + x - 6$ in $\mathbf{Q}[x]$.

(c) Find the greatest common divisof of $x^4 + x^2 + 1$ and $x^4 + x^3 + x^2 + x + 1$ in $\mathbf{Z}_2[x]$.

#2 Let $n \in \mathbf{Z}, n \geq 1$. Prove that $\mathbf{Z}_n$ is a field if and only if n is a prime. You may use (without proving them) results about the greatest common divisor of two integers.

#3 Let R be a ring and I be an ideal in R. Recall that the coset $a + I$ is defined to be $\{a + x | x \in I\}$.

(a) Prove that if $(a + I) \cap (b + I) \neq \emptyset$ then $a + I = b + I$.

(b) Prove that if $a_1 + I = b_1 + I$ and $a_2 + I = b_2 + I$, then $a_1a_2 + I = b_1b_2 + I$.

#4 Let F be a field and let $f(x), g(x) \in F[x]$. Assume $f(x)$ and $g(x)$ are not both 0.

(a) State (but don't prove) the division algorithm for $F[x]$.

(b) State the definition of the greatest common divisor of $f(x)$ and $g(x)$.

(c) Prove that $f(x)$ and $g(x)$ have a greatest common divisor and that it may be written in the form $a(x)f(x) + b(x)g(x)$ for some $a(x), b(x) \in F[x]$.

#5 Let R and S be commutative rings with identity. Recall that $R \times S$ denotes $\{(r, s) | r \in R, s \in S\}$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2, s_1s_2)$. Recall also that $R \times S$ is a ring.

(a) Let I be an ideal in $R \times S$. Define $J_1 = \{r \in R | (r, 0) \in I\}$ and $J_2 = \{s \in S | (0, s) \in I\}$. Prove that J_1 is an ideal in R and that J_2 is an ideal in S. Then prove that $I = \{(a, b) \in R \times S | a \in J_1, b \in J_2\}$.

(b) Suppose the hypothesis that R and S have identity elements is omitted. Does the result of (a) remain true? Why or why not?

#6 Let W denote $\{\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} | a, b \in \mathbf{R}\} \subseteq M(\mathbf{R})$, Y denote $\{\begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} | a \in \mathbf{R}\} \subseteq M(\mathbf{R})$, and N denote $\{\begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} | b \in \mathbf{R}\} \subseteq M(\mathbf{R})$

(a) Show that W and Y are subrings of $M(\mathbf{R})$.

(b) Define a map g from W to Y by $g(\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix}) = \begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix}$. Show that g is a homomorphism.

(c) Show that N is an ideal in W and that W/N is isomorphic to Y.

#7 Prove that a finite integral domain is a field.

#8 Let R be a ring and $a, b \in R$. Prove, directly from the definition of a ring, that $0_Ra = a0_R = 0R$ and that $-(ab) = (-a)b = a(-b)$.