Math 351

Solutions to review problems for Exam #2 **November 14, 2010**

#1 (a) Is $x^3 + x^2 + x + 1$ irreducible in $\mathbf{Z}_3[x]$? Why or why not?

Solution: No, since $x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$. A quick way to see this is to note that 2 is a root, since $2^3 + 2^2 + 2 + 1 = 8 + 4 + 2 + 1 = 15 = 0$ in $\mathbf{Z}_3$.

(b) Is $x^2 + 1$ irreducible in $\mathbf{Z}_3[x]$? Is it irreducible in $\mathbf{Z}_{17}[x]$? Why or why not?

Solution: It is irreducible in $\mathbf{Z}_3[x]$ since it has no root. Since the polynomial has degree 2 if it were reducible it would have a root. It is not irreducible in $\mathbf{Z}_{17}[x]$ since 4 is a root (because $4^2 + 1 = 17 = 0$ in $\mathbf{Z}_{17)}$.

(c) Is $x^4 + x^2 + 1$ irreducible in $\mathbf{Z}_2[x]$? Why or why not?

Solution: It is not irreducible since $x^4 + x^2 + 1 = (x^2 + x + 1)^2$ in $\mathbf{Z}_2[x]$. Note that since the polynomial has degree 4. showing that it has no root does not guarantee that it is irreducible.

#2 Suppose F is a field and that $f(x) \in F[x]$.

(a) Show that if $f(x)$ has degree 3 and $f(x)$ has no roots in F, then $F[x]/(f(x))$ is a field.

Solution: Since $f(x)$ has degree 3 and has no roots in F, it is irreducible (Corollary 4.18). Then $F[x]/(f(x))$ is a field by Theorem 5.10.

(b) Give an example to show that the result of part (a) is not true if the degree of $f(x)$ is changed to 4.

Solution Let $F = \mathbf{Z}_2$ and let $f(x) = x^4 + x^2 + 1$. Then $f(x)$ has no root in F. As noted in problem #1(c), $f(x) = g(x)^2$ where $g(x) = x^2 + x + 1$.. Then the coset of $g(x)$ in $F[x]/(f(x))$ is nonzero, but the square of this coset in zero. Thus $F[x]/(f(x))$ is not an integral domain and so is not a field.

#3 Let R be a commutative ring with identity.

(a) State the definition of a prime ideal in R

Solution: See page 154 of the text.

(b) State the definition of a maximal ideal in R

Solution: See page 156 of the text.

(c) Prove that an ideal I in R is prime if and only if R/I is an integral domain.

Solution: See Theorem 6.14.

(d) Prove that an ideal I in R is maximal if and only if R/I is a field.

Solution: See Theorem 6.15

(e) Prove that every maximal ideal in R is prime.

Solution: See Corollary 6.16.

(f) Give an example of a ring R and an ideal I in R which is prime but not maximal.

Solution: In any integral domain which is not a field (e.g., $\mathbf{Z}$ or $F[x]$ where F is a field), the ideal (0) is prime but not maximal.

#4 Let G be a group. Prove, using only the defining axioms for groups, that:
(a) If $x, y, z \in G$ and $xy = xz$ then $y = z$.

Solution: Since G is a group, there is an identity element $e \in G$ and an element $a \in G$ such that $ax = e$. Then

$$y = ey = (ax)y = a(xy) = a(xz) = (ax)z = ez = z.$$

(b) If $x, y \in G$, then $(xy)^{-1} = y^{-1}x^{-1}$.

Solution:

$$(y^{-1}x^{-1})(xy) = y^{-1}(x^{-1}x)y = y^{-1}ey = y^{-1}y = e$$

and

$$(xy)(y^{-1}x^{-1}) = x(y(y^{-1})x^{-1} = xex^{-1} = xx^{-1} = e.$$

#5 Let G be a cyclic group of order 24? How many subgroups does G have?

Solution: Let $G =< a >$ and let H be a subgroup of G. Let k be the smallest positive integer such that $a^k \in H$. Then, by Theorem 7.16 and its proof, H is the cyclic subgroup generated by a^k and k divides 24. Thus there is one subgroup for each divisor of 24. Since the divisors of 24 are $1, 2, 3, 4, 6, 12, 24$, there are 7 subgroups.

#6 Let

$$U = \{\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} | b \in \mathbf{R}\} \subseteq GL(2, \mathbf{R})$$

and

$$W = \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in \mathbf{R}, a, c \neq 0\} \subseteq GL(2, \mathbf{R}).$$

(a) Show that U and W are subgroups of $GL(2, \mathbf{R})$.

Solution: Let $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} \in W$. Then

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} aa' & ab' + bc' \\ 0 & cc' \end{bmatrix} \in W$$

so W is closed under products. Furthermore

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} a^{-1} & -a^{-1}bc^{-1} \\ 0 & c^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and so W is closed under taking inverses. Thus W is a subgroup. The same argument with $a = c = a' = c' = 1$ shows that U is a subgroup.

(b) Is U a normal subgroup of $GL(2, \mathbf{R})$? Why or why not?

Solution: This topic will not be on the exam. (In fact, U is not normal in $GL(2, \mathbf{R})$.)

(c) Is W a normal subgroup of $GL(2, \mathbf{R})$? Why or why not?

Solution This topic will not be on the exam. (In fact, W is not normal in $GL(2, \mathbf{R})$.)

(d) Is U a normal subgroup of W? Why or why not?

Solution: This topic will not be on the exam. (In fact, U is normal in W.)

(e) Describe the right cosets of U in W. (There are infinitely many.)

Solution: For each pair a, c of nonzero real numbers, we have

$$U \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} = \{ \begin{bmatrix} a & y \\ 0 & c \end{bmatrix} | y \in \mathbf{R} \}.$$

Since the union of these sets is W, these are all the right cosets of U in W.

(f) What is $Z(W)$, the center of W.

Solution: $Z(W) = \{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} | 0 \neq a \in \mathbf{R} \}$.

#7 (a) Does S_4 contain any elements of order 6? Why or why not?

Solution: No. Any element in S_4 must be either a 4-cycle (which has order 4) or a 3-cycle (which has order 3), or a 2-cycle (which has order 2), or a product of two disjoint 2-cycles (which has order 2), or the identity (which has order 1).

(b) Let $H =< (1234) >$ be the cyclic subgroup of S_4 generated by the 4-cycle (1234). Find all the right cosets of H in S_4.

Solution: The 6 cosets of H are:

$$H = \{e, (1234), (13)(24), (1432)\},$$

$$H(12) = \{(12), (134), (1423), (243)\},$$

$$H(13) = \{(13), (14)(23), (24), (12)(34)\},$$

$$H(14) = \{(14), (234), (1243), (132)\},$$

$$H(23) = \{(23), (124), (1342), (143)\},$$

$$H(34) = \{(34), (123), (1324), (142)\}.$$

#8 (a) Let $\sigma \in S_9$ be the permutation given in table form by

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 4 & 7 & 3 & 8 & 5 & 2 & 6 & 1 \end{bmatrix}.$$

Express σ as a product of disjoint cycles.

Solution: $\sigma = (19)(2437)(586)$.

(b) Let $\tau \in S_9$ be the following product of disjoint cycles:

$$\tau = (14)(273)(985).$$

Write τ in table form.

Solution: $\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 2 & 1 & 9 & 6 & 3 & 5 & 8 \end{bmatrix}.$

(c) Are σ and τ (from the two previous parts) conjugate in S_9? Why or why not?

Solution: They are not conjugate since, when expressed as products of disjoint cycles, σ is the product of a 2-cycle, a 3-cycle, and a 4-cycle, while τ is the product of a 2-cycle and two 3-cycles (as well as a 1-cycle).

(d) Let $\mu \in S_9$ be the product of cycles

$$\mu = (146)(925)(38)(427)(6923).$$

Write μ as a product of disjoint cycles.

Solution: $\mu = (145976283)$.

(e) Suppose μ (from the previous part) is written as a product of k transpositions. Is k even or odd? Why?

Solution: k must be even. Any n-cycle can be written as a product of $n-1$ transpositions. Thus, using the original expression for μ, it can be written as the produce of $2+2+1+2+3 = 10$ transpositions. Therefore any way of writing μ as a product of transpositions must have an even number of factors. (You can also use the answer to (d). Since μ is a 9-cycle, it can be written as the product of 8 transpositions.)